

# Honours Algebra Notes

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# 1 Vector Spaces

## 1.1 Solutions of simultaneous linear equations

- **Theorem 1.1.4** *Solution sets of inhomogeneous systems of linear equations*

If the solution set of a linear system of equations is non-empty, then we obtain all solutions by adding component-wise an arbitrary solution of the associated homogenised system to a fixed solution of the system.

## 1.2 Fields & vector spaces

- **Definition 1.2.1.1** *Fields*

A *field*  $F$  is a set with functions

$$\begin{aligned}\text{addition} &= + : F \times F \rightarrow F ; (\lambda, \mu) \mapsto \lambda + \mu \\ \text{multiplication} &= \cdot : F \times F \rightarrow F ; (\lambda, \mu) \mapsto \lambda\mu\end{aligned}$$

such that  $(F, +)$  and  $(F \setminus \{0\}, \cdot)$  are abelian groups, with

$$\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F, \quad \forall \lambda, \mu, \nu \in F$$

The neutral elements are called  $0_F, 1_F$ . In particular

$$\lambda + \mu = \mu + \lambda, \lambda \cdot \mu = \mu \cdot \lambda, \lambda + 0_F = \lambda, \lambda \cdot 1_F = \lambda \in F, \quad \forall \lambda, \mu \in F$$

For every  $\lambda \in F$  there exists  $-\lambda \in F$  such that

$$\lambda + (-) = 0_F \in F$$

For every  $\lambda \neq 0 \in F$  there exists  $\lambda^{-1} \neq 0 \in F$  such that

$$\lambda(\lambda^{-1}) = 1_F \in F$$

- **Definition 1.2.1.2** *Vector space*

A *vector space*  $V$  over a *field*  $F$  is a pair consisting of an abelian group  $V = (V, +)$  and a mapping

$$F \times V \rightarrow V : (\lambda, \mathbf{v}) \mapsto \lambda\mathbf{v}$$

such that for all  $\lambda, \mu \in F$  and  $\mathbf{v}, \mathbf{w} \in V$  the following identities hold:

$$\begin{aligned}\lambda(\mathbf{v} + \mathbf{w}) &= (\lambda\mathbf{v}) + (\lambda\mathbf{w}) && \text{(distributivity)} \\ (\lambda + \mu)\mathbf{v} &= (\lambda\mathbf{v}) + (\mu\mathbf{v}) && \text{(distributivity)} \\ \lambda(\mu\mathbf{v}) &= (\lambda\mu)\mathbf{v} && \text{(associativity)} \\ 1_F\mathbf{v} &= \mathbf{v}\end{aligned}$$

A vector space  $V$  over a field  $F$  is called an  $F$ -*vector space*.

- **Lemma 1.2.2** *Product with the scalar zero*

If  $V$  is a vector space and  $\mathbf{v} \in V$ , then  $0\mathbf{v} = \mathbf{0}$

- **Lemma 1.2.3** *Product with the scalar  $(-1)$*

If  $V$  is a vector space and  $\mathbf{v} \in V$ , then  $(-1)\mathbf{v} = -\mathbf{v}$ .

- **Lemma 1.2.4** *Product with the zero vector*

If  $V$  is a vector space over a field  $F$ , then  $\lambda\mathbf{0} = \mathbf{0}$  for all  $\lambda \in F$ . Furthermore, if  $\lambda\mathbf{v} = \mathbf{0}$ , then either  $\lambda = 0$  or  $\mathbf{v} = \mathbf{0}$ .

### 1.3 Products of sets and of vector spaces

#### 1.4 Vector subspaces

- **Definition 1.4.1** *Vector subspaces*

A subset  $U$  of a vector space  $V$  is called a *vector subspace* or *subspace* if  $U$  contains  $\mathbf{0}$  and

$$\mathbf{u}, \mathbf{v} \in U \text{ and } \lambda \in F \implies \mathbf{u} + \mathbf{v} \in U \text{ and } \lambda \mathbf{u} \in U$$

- **Proposition 1.4.5** Generating a vector subspace from a subset

Let  $T$  be a subset of a vector space  $V$  over a field  $F$ . Then amongst all vector subspaces of  $V$  that include  $T$ , there is a smallest vector subspace

$$\langle T \rangle = \langle T \rangle_F \subseteq V$$

It can be described as the set of all vectors  $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r$  with  $\alpha_1, \dots, \alpha_r \in F$  and  $\mathbf{v}_1, \dots, \mathbf{v}_r \in T$ , together with  $\mathbf{0}$  in the case  $T = \emptyset$ .

- **Definition 1.4.7** *Generating set*

A subset of a vector space is called a *generating set* of our vector space if its span is all of the vector space. A vector space that has a finite generating set is said to be *finitely generated*.

- **Definition 1.4.9**

The set of all subsets  $\mathcal{P}(X) = \{U : U \subseteq X\}$  of  $X$  is the *power set* of  $X$ .

A subset of  $\mathcal{P}(X)$  is a *system of subsets* of  $X$ .

Given such a system  $\mathcal{U} \subseteq \mathcal{P}(X)$  we can create two new subsets of  $X$ , the *union* and the *intersection* of the sets of our system  $\mathcal{U}$ :

$$\begin{aligned} \bigcup_{U \in \mathcal{U}} U &= \{x \in X : \exists U \in \mathcal{U}. x \in U\} \\ \bigcap_{U \in \mathcal{U}} U &= \{x \in X : x \in U \forall U \in \mathcal{U}\} \end{aligned}$$

In particular the intersection of the empty system of subsets of  $X$  is  $X$ , and the union of the empty system of subsets  $X$  is the empty set.

#### 1.5 Linear independence and bases

- **Definition 1.5.1** *Linear independence*

A subset  $L$  of a vector space  $V$  is *linearly independent* if for all pairwise different vectors  $\mathbf{v}_1, \dots, \mathbf{v}_r \in L$  and arbitrary vectors  $\alpha_1, \dots, \alpha_r \in F$ ,

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r = \mathbf{0} \implies \alpha_1 = \cdots = \alpha_r = 0$$

- **Definition 1.5.2** *Linear dependence*

A subset  $L$  of a vector space  $V$  is called *linearly dependent* if it is not linearly independent.

- **Definition 1.5.8** *Basis*

A *basis* of a vector space  $V$  is a linearly independent generating set in  $V$ .

- **Theorem 1.5.11** Linear combinations of basis elements

Let  $F$  be a field,  $V$  be a vector space over  $F$ , and  $\mathbf{v}_1, \dots, \mathbf{v}_r \in V$  vectors. The family  $(\mathbf{v}_i)_{1 \leq i \leq r}$  is a basis of  $V$  if and only if the following “evaluation” mapping

$$\begin{aligned} \Phi : F^r &\rightarrow V \\ (\alpha_1, \dots, \alpha_r) &\mapsto \alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r \end{aligned}$$

is a bijection.

- **Theorem 1.5.12** Characterisation of bases

The following are equivalent for a subset  $E$  of a vector space  $V$ :

1.  $E$  is a basis, i.e. a linearly independent generating set;
2.  $E$  is minimal among all generating sets, meaning that  $E \setminus \{\mathbf{v}\}$  does not generate  $V$ ,  $\forall \mathbf{v} \in E$ ;
3.  $E$  is maximal among all linearly independent subsets, meaning that  $E \cup \{\mathbf{v}\}$  is not linearly independent  $\forall \mathbf{v} \in V$ .

- **Corollary 1.5.13** The existence of a basis

Let  $V$  be a finitely generated vector space over a field  $F$ . The  $V$  has a basis.

- **Theorem 1.5.14** (Useful variant on the Characterisation of bases)

Let  $V$  be a vector space.

1. If  $L \subset V$  is a linearly independent subset and  $E$  is minimal amongst all generating sets of our vector space with the property that  $L \subseteq E$ , then  $E$  is a basis.
2. If  $E \subseteq V$  is a generating set and if  $L$  is maximal amongst all linearly independent subsets of our vector space with the property  $L \subseteq E$ , then  $L$  is basis.

- **Definition 1.5.15**

Let  $X$  be a set and  $F$  a field. The set  $\text{Maps}(X, F)$  of all mappings  $f : X \rightarrow F$  becomes an  $F$ -vector space with the operations of point-wise addition and multiplication by a scalar. The subset of all mappings which send almost all elements of  $X$  to zero is a vector subspace

$$F\langle X \rangle \subseteq \text{Maps}(X, F)$$

This vector subspace is called the *free vector space on the set  $X$* .

- **Theorem 1.5.16** (Useful variant on Linear combinations of basis elements)

Let  $F$  be a field,  $V$  an  $F$ -vector space, and  $(\mathbf{v}_i)_{i \in I}$  a family of vectors from the vector space  $V$ . The following are equivalent:

1. The family  $(\mathbf{v}_i)_{i \in I}$  is a basis for  $V$ ;
2. For each vector  $\mathbf{v} \in V$  there is precisely one family  $(a_i)_{i \in I}$  of elements of our field  $F$ , almost all of which are zero and such that

$$\mathbf{v} = \sum_{i \in I} a_i \mathbf{v}_i$$

## 1.6 Dimension of a vector space

- **Theorem 1.6.1** Fundamental estimate of linear algebra

No linearly independent subset of a given vector space has more elements than a generating set. Thus if  $V$  is a vector space,  $L \subset V$  a linearly independent subset, and  $E \subseteq V$  a generating set, then:

$$|L| \leq |E|$$

- **Theorem 1.6.2** Steinitz exchange theorem

Let  $V$  be a vector space,  $L \subset V$  finite linearly independent subset, and  $E \subseteq V$  generating set. Then there is an injection  $\Phi : L \rightarrow E$  such that  $(E \setminus \Phi(L)) \cup L$  is also a generating set for  $V$ .

We can swap out some elements of a generating set by the elements of our linearly independent set, and still keep a generating set.

- **Lemma 1.6.3** Exchange lemma

Let  $V$  be a vector space,  $M \subseteq V$  a linearly independent subset, and  $E \subseteq V$  a generating subset,

such that  $M \subseteq E$ . If  $\mathbf{w} \in V \setminus M$  is a vector set not belonging to  $M$  such that  $M \cup \{\mathbf{w}\}$  is linearly independent, then there exists  $\mathbf{e} \in E \setminus M$  such that  $\{E \setminus \{\mathbf{e}\}\} \cup \{\mathbf{w}\}$  is a generating set for  $V$ .

- **Corollary 1.6.4** Cardinality of bases

Let  $V$  be a finitely generated vector space.

1.  $V$  has a finite basis;
2.  $V$  cannot have an infinite basis;
3. Any two bases of  $V$  have the same number of elements.

- **Definition 1.6.5** *Dimension*

The cardinality of one (and each) basis of a finitely generated vector space  $V$  is called the *dimension* of  $V$  and is denoted  $\dim V$ . If the vector space is not finitely generated, then  $\dim V = \infty$  and  $V$  is *infinite dimensional*.

- **Corollary 1.6.8** Cardinality criterion for bases

Let  $V$  be a finitely generated vector space.

1. Each linearly independent subset  $L \subset V$  has at most  $\dim V$  elements, and if  $|L| = \dim V$ , then  $L$  is actually a basis;
2. Each generating set  $E \subseteq V$  has at least  $\dim V$  elements, and if  $|E| = \dim V$  then  $E$  is actually a basis.

- **Corollary 1.6.9** Dimension estimate for vector subspaces

A proper vector subspace of a finite dimensional vector space has itself a strictly smaller dimension.

- **Theorem 1.6.11** The dimension theorem

Let  $V$  be a vector space containing vector subspaces  $U, W \subseteq V$ . Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

## 1.7 Linear mappings

- **Definition 1.7.1** *Linear mapping*

Let  $V, W$  be vector spaces over a field  $F$ . A mapping  $f : V \rightarrow W$  is called *linear* if for all  $\mathbf{v}_1, \mathbf{v}_2 \in V$  and  $\lambda \in F$  we have

$$\begin{aligned} f(\mathbf{v}_1 + \mathbf{v}_2) &= f(\mathbf{v}_1) + f(\mathbf{v}_2) \\ f(\lambda \mathbf{v}_1) &= \lambda f(\mathbf{v}_1) \end{aligned}$$

A bijective linear mapping is called an *isomorphism* of vector spaces. If there is an isomorphism of vector spaces, we call them *isomorphic*. A homomorphism from one vector space to itself is called an *endomorphism*. An isomorphism of a vector space to itself is called an *automorphism*.

- **Definition 1.7.5** *Fixed point*

A point that is sent to itself by a mapping is called a *fixed point* of the mapping. Given a mapping  $f : X \rightarrow X$ , we denote the set of fixed points by

$$X^f = \{x \in X : f(x) = x\}$$

- **Definition 1.7.6** *Complementary*

Two vector subspaces  $V_1, V_2$  of a vector space  $V$  are *complementary* if addition defines a bijection

$$V_1 \times V_2 \rightarrow V$$

- **Theorem 1.7.7** Classification of vector spaces by their dimension

Let  $n \in \mathbb{N}$ . Then a vector space over a field  $F$  is isomorphic to  $F^n$  if and only if it has dimension  $n$ .

- **Lemma 1.7.8** Linear mappings and bases

Let  $V, W$  be vector spaces over  $F$  and let  $B \subset V$  be a basis. Then restriction of a mapping gives a bijection

$$\begin{aligned}\text{Hom}_F(V, W) &= \text{Hom}(V, W) \subseteq \text{Maps}(V, W) \\ f &\mapsto f|_B\end{aligned}$$

In other words, each linear mapping determines and is completely determined by the values it takes on a basis.

- **Proposition 1.7.9**

1. Every injective linear mapping  $f : V \rightarrow W$  has a *left inverse*, in other words a linear mapping  $g : W \rightarrow V$  such that  $g \circ f = \text{id}_V$
2. Every surjective linear mapping  $f : V \rightarrow W$  has a *right inverse*, in other words a linear mapping  $g : W \rightarrow V$  such that  $f \circ g = \text{id}_W$

## 1.8 Rank-Nullity theorem

- **Definition 1.8.1**

The *image* of a linear mapping  $f : V \rightarrow W$  is the subset  $\text{im}(f) = f(V) \subseteq W$ . It is a vector subspace of  $W$ . The pre-image of the zero vector of a linear mapping  $f : V \rightarrow W$  is denoted by

$$\ker(f) \equiv f^{-1}(0) = \{v \in V : f(v) = 0\}$$

and is called the *kernel* of the linear mapping  $f$ . The kernel is a vector subspace of  $V$ .

- **Lemma 1.8.2**

A linear mapping  $f : V \rightarrow W$  is injective if and only if  $\ker f = 0$ .

- **Theorem 1.8.4** Rank-Nullity theorem

Let  $f : V \rightarrow W$  be a linear mapping between vector spaces. Then

$$\begin{aligned}\dim V &= \dim(\ker f) + \dim(\text{im} f) \\ &= \text{nullity} + \text{rank}\end{aligned}$$

- **Corollary 1.8.5** (Dimension theorem, again)

Let  $V$  be a vector space, and  $U, W \subseteq V$  vector subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

- **Definition** *Idempotent*

An element  $f$  of a set with composition or product is called *idempotent* if  $f^2 = f$ .

## 2 Linear Mappings and Matrices

### 2.1 Linear mappings $F^m \rightarrow F^n$ and matrices

- **Theorem 2.1.1** Linear mappings  $F^m \rightarrow F^n$  and matrices

Let  $F$  be a field and let  $m, n \in \mathbb{N}$ . There is a bijection between the space of linear mappings  $F^m \rightarrow F^n$  and the set of matrices with  $n$  rows and  $m$  columns and entries in  $F$

$$\begin{aligned}M : \text{Hom}_F(F^m, F^n) &\rightarrow \text{Mat}(n \times m; F) \\ f &\mapsto [f]\end{aligned}$$

This attaches to each linear mapping  $f$  its *representing matrix*  $M(f) \equiv [f]$ . The columns of this matrix are the images under  $f$  of the standard basis elements of  $F^m$

$$[f] \equiv (f(\mathbf{e}_1) | f(\mathbf{e}_2) | \cdots | f(\mathbf{e}_m))$$

- **Definition 2.1.6** *Product*

Let  $n, m, l \in \mathbb{N}$ ,  $F$  and field, and let  $A \in \text{Mat}(n \times m; F)$  and  $B \in \text{Mat}(m \times l; F)$  be matrices. The *product*  $A \circ B = AB \in \text{Mat}(n \times l; F)$  is the matrix defined by

$$(AB)_{ik} = \sum_{j=1}^m A_{ij} B_{jk}$$

Matrix multiplication produces a mapping

$$\begin{aligned} \text{Mat}(n \times m; F) \times \text{Mat}(m \times l; F) &\rightarrow \text{Mat}(n \times l; F) \\ (A, B) &\mapsto AB \end{aligned}$$

- **Theorem 2.1.8** *Composition of linear mappings and products of matrices*

Let  $g : F^l \rightarrow F^m$  and  $f : F^m \rightarrow F^n$  be linear mappings. The representing matrix of their composition is the product of their representing matrices

$$[f \circ g] = [f] \circ [g]$$

- **Proposition 2.1.9** *Calculating with matrices*

Let  $k, l, m, n \in \mathbb{N}$ ,  $A, A' \in \text{Mat}(n \times m; F)$ ,  $B, B' \in \text{Mat}(m \times l; F)$ ,  $C \in \text{Mat}(l \times k; F)$  and  $I = I_m$ . Then the following hold for matrix multiplication

$$\begin{aligned} (A + A')B &= AB + A'B \\ A(B + B') &= AB + AB' \\ IB &= B \\ AI &= A \\ (AB)C &= A(BC) \end{aligned}$$

## 2.2 Basic properties of matrices

- **Definition 2.2.1** *Invertible*

A matrix  $A$  is called *invertible* if there exist matrices  $B$  and  $C$  such that  $BA = I$  and  $AC = I$ .

- **Definition 2.2.2** *Elementary matrix*

An *elementary matrix* is any square matrix that differs from the identity matrix in at most one entry.

- **Theorem 2.2.3**

Every square matrix can be written as a product of elementary matrices.

- **Definition 2.2.4** *Smith Normal Form*

Any matrix whose only non-zero entries lie on the diagonal, and which has first 1s on along the diagonal followed by 0s is in *Smith Normal Form*.

- **Theorem 2.2.5** *Transformation of a matrix into Smith-Normal form*

For each matrix  $A \in \text{Mat}(n \times m; F)$  there exist invertible matrices  $P$  and  $Q$  such that  $PAQ$  is a matrix in Smith Normal Form and  $Q$  such that  $PAQ$  is a matrix in Smith Normal Form.

- **Definition 2.2.6** *Rank*

The *column rank* of a matrix  $A \in \text{Mat}(n \times m; F)$  is the dimension of the subspace of  $F^n$  generated by the columns of  $A$ . Similarly, the *row rank* of  $A$  is the dimension of the subspace of  $F^m$  generated by the rows of  $A$ .

- **Theorem 2.2.7**

The column rank and the row rank of any matrix are equal.

- **Definition 2.2.8** *Full rank*

Whenever the rank of a matrix is equal to the number of rows (or columns — whichever is smaller), it has *full rank*.

## 2.3 Abstract linear mappings and matrices

- **Theorem 2.3.1** Abstract linear mappings and matrices

Let  $F$  be a field,  $V$  and  $W$  vector spaces over  $F$  with ordered bases  $\mathcal{A} = (\mathbf{v}_1, \dots, \mathbf{v}_m)$  and  $\mathcal{B} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ . Then to each linear mapping  $f : V \rightarrow W$  we associated a *representing matrix*  ${}_B[f]_A$  whose entries  $a_{ij}$  are defined by the identity

$$f(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + \dots + a_{nj}\mathbf{w}_n \in W$$

This produces a bijection, which is even an isomorphism of vector spaces

$$\begin{aligned} M_B^A : \text{Hom}_F(V, W) &\rightarrow \text{Mat}(n \times m; F) \\ f &\mapsto_B [f]_A \end{aligned}$$

- **Theorem 2.3.2** The representing matrix of a composition of linear mappings

Let  $F$  be a field and  $U, V, W$  finite-dimensional vector spaces over  $F$  with ordered bases  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ . If  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are linear mappings, then the representing matrix of the composition  $g \circ f : U \rightarrow W$  is the matrix product of the representing matrices of  $f$  and  $g$

$${}_C[g \circ f]_A = {}_C[g]_B \circ_B [f]_A$$

- **Definition 2.3.3**

Let  $V$  be a finite-dimensional vector spaces with an ordered basis  $\mathcal{A} = (\mathbf{v}_1, \dots, \mathbf{v}_m)$ . We denote the inverse to the bijection  $\Phi_A : F^m \rightarrow V, (\alpha_1, \dots, \alpha_m)^T \mapsto \alpha_1\mathbf{v}_1 + \dots + \alpha_m\mathbf{v}_m$  by

$$\mathbf{v} \mapsto_A [\mathbf{v}]$$

The column vector  ${}_A[\mathbf{v}]$  is called the *representation of the vector  $\mathbf{v}$  with respect to the basis  $\mathcal{A}$* .

- **Theorem 2.3.4** Representation of the image of a vector

Let  $V, W$  be finite-dimensional vector-spaces over  $F$  with ordered bases  $\mathcal{A}, \mathcal{B}$  and let  $f : V \rightarrow W$  be a linear mapping. The following holds for  $\mathbf{v} \in V$ :

$${}_B[f(\mathbf{v})] = {}_B[f]_A \circ_A [\mathbf{v}]$$

## 2.4 Change of a matrix by change of basis

- **Definition 2.4.1** *Change of basis matrix*

Let  $\mathcal{A} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$  and  $\mathcal{B} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$  be ordered bases of the same  $F$ -vector space  $V$ . Then the matrix representing the identity mapping with respect to these bases

$${}_B[\text{id}_V]_A$$

is called a *change of basis matrix*. By definition, its entries are given by the equalities  $\mathbf{v}_j = \sum_{i=1}^n a_{ij}\mathbf{w}_i$ .

- **Theorem 2.4.3** Change of basis

Let  $V$  and  $W$  be finite-dimensional vector-spaces over  $F$  and let  $f : V \rightarrow W$  be a linear mapping. Suppose that  $\mathcal{A}, \mathcal{A}'$  are ordered bases of  $V$  and  $\mathcal{B}, \mathcal{B}'$  are ordered bases of  $W$ . Then

$${}_{B'}[f]_{A'} = {}_{B'}[\text{id}_W]_B \circ_B [f]_A \circ_A [\text{id}_V]_{A'}$$



- **Corollary 2.4.4** Let  $V$  be a finite-dimensional vector-space and let  $f : V \rightarrow V$  be an endomorphism of  $V$ . Suppose that  $\mathcal{A}, \mathcal{A}'$  are ordered bases of  $V$ . Then

$${}_{\mathcal{A}'}[f]_{\mathcal{A}'} = {}_{\mathcal{A}'}[\text{id}_V]_{\mathcal{A}'}^{-1} \circ {}_{\mathcal{A}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\text{id}_V]_{\mathcal{A}'}$$

- **Theorem 2.4.5** Smith Normal Form

Let  $f : V \rightarrow W$  be a linear mapping between finite-dimensional  $F$ -vector spaces. There exist an ordered basis  $\mathcal{A}$  of  $V$  and an ordered basis  $\mathcal{B}$  of  $W$  such that the representing matrix  ${}_{\mathcal{B}}[f]_{\mathcal{A}}$  has zero entries everywhere except possibly on the diagonal, and along the diagonal there are 1s first, followed by 0s.

- **Definition 2.4.6** *Trace*

The *trace* of a square matrix is defined to be the sum of its diagonal entries. We denote this by

$$\text{tr}(A)$$

## 3 Rings and Modules

### 3.1 Rings

- **Group Axioms**

1. Closure
2. Associativity
3. Existence of identity
4. Existence of inverses

- **Definition 3.3.1** *Ring*

A *ring* is a set with two operations  $(R, +, \cdot)$  that satisfy

1.  $(R, +)$  is an abelian group;
2.  $(R, \cdot)$  is a *monoid*; this means that the second operation  $\cdot : R \cdot R \rightarrow R$  is associative and that there is an *identity element*  $1 = 1_R \in R$ .
3. The distributive laws hold.

The two operations are called *addition* and *multiplication* in our ring.

A ring in which multiplication is commutative is a *commutative ring*.

- **Proposition 3.1.7** Divisibility by sum

A natural number is divisible by 3 (respectively 9) precisely when the sum of its digits is divisible by 3 (respectively 9).

- **Definition 3.1.8** *Field*

A *field*  $F$  is a non-zero commutative ring in which every non-zero element  $a \in F$  has an inverse  $a^{-1} \in F$ .

- **Proposition 3.1.11**

Let  $m \in \mathbb{Z}^+$ . The commutative ring  $\mathbb{Z}/m\mathbb{Z}$  is a field if and only if  $m$  is prime.

### 3.2 Properties of rings

- **Lemme 3.2.1** Additive inverses

Let  $R$  be a ring and let  $a, b \in R$ . Then

1.  $0a = 0 = a0$

$$2. (-a)b = -(ab) = a(-b)$$

$$3. (-a)(-b) = ab$$

• **Definition 3.2.3**

Let  $m \in \mathbb{Z}$ . The  $m$ -th multiple  $ma$  of an element  $a$  in abelian group  $R$  is

$$ma = \underbrace{a + a + \cdots + a}_{m \text{ terms}} \quad \text{if } m > 0$$

$0a = 0$ , and negative multiples are defined by  $(-m)a = -(ma)$ .

• **Lemma 3.2.4** Rules for multiples

Let  $R$  be a ring, let  $a, b \in R$  and let  $m, n \in \mathbb{Z}$ . Then

$$1. m(a + b) = ma + mb;$$

$$2. (m + n)a = ma + na;$$

$$3. m(na) = (mn)a;$$

$$4. m(ab) = (ma)b = a(mb);$$

$$5. (ma)(nb) = (mn)(ab);$$

• **Definition 3.2.6** Unit

Let  $R$  be a ring. An element  $a \in R$  is called a *unit* if it is invertible in  $R$  or (in other words) has a multiplicative inverse in  $R$ .

• **Proposition 3.2.10**

The set  $R^\times$  of units in a ring  $R$  forms a group under multiplication.

• **Definition 3.2.13** Integral domains

An *integral domain* is a non-zero commutative ring that has no zero-divisors.

• **Proposition 3.2.16** Cancellation law for integral domains

Let  $R$  be an integral domain and let  $a, b, c \in R$ .

$$ab = ac \text{ and } a \neq 0 \implies b = c$$

• **Proposition 3.2.17**

Let  $m \in \mathbb{N}$ . Then  $\mathbb{Z}/m\mathbb{Z}$  is an integral domain if and only if  $m$  is prime.

• **Theorem 3.2.18**

Every *finite* integral domain is a field.

### 3.3 Polynomials

• **Definition 3.1.1**

Let  $R$  be a ring. A *polynomial over  $R$*  is an expression of the form

$$P = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m$$

for some  $m \in \mathbb{N}$  and elements  $a_i \in R$  for  $i \in [0, m]$ .

The set of all polynomials over  $R$  is denoted by  $R[X]$ .

In case  $a_m$  is non-zero, the polynomial  $P$  has *degree  $m$* , written  $\deg(P)$ , and  $a_m$  is its *leading coefficient*.

When the leading coefficient is 1, the polynomial is a *monic polynomial*.

A polynomial of degree one is called *linear*, a polynomial of degree two is called *quadratic*, and a polynomial of degree three is called *cubic*.

- **Definition 3.3.2** *Ring of polynomials*

The set  $R[X]$  is a ring called the *ring of polynomials over  $R$* . The zero and the identity of  $R[X]$  are the zero and identity of  $R$ , respectively.

- **Lemma 3.3.3**

1. If  $R$  is ring with no zero-divisors, then  $R[X]$  has no zero-divisors and  $\deg(PQ) = \deg(P) + \deg(Q)$  for non-zero  $P, Q \in R[X]$ .
2. If  $R$  is an integral domain, then so is  $R[X]$

- **Theorem 3.3.4** *Division and remainder*

Let  $R$  be an integral domain, and let  $P, Q \in R[X]$  with  $Q$  monic. Then there exists unique  $A, B \in R[X]$  such that  $P = AQ + B$  and  $\deg(B) < \deg(Q)$  or  $B = 0$ .

- **Definition 3.3.6**

Let  $R$  be a commutative ring and  $P \in R[X]$  a polynomial. Then the polynomial  $P$  can be *evaluated* at  $\lambda \in R$  to produce  $P(\lambda)$  by replacing the powers of  $X$  in the polynomial  $P$  by the corresponding powers of  $\lambda$ . This gives a mapping

$$R[X] \rightarrow \text{Maps}(R, R)$$

An element  $\lambda \in R$  is a *root* of  $P$  if  $P(\lambda) = 0$ .

- **Proposition 3.3.9**

Let  $R$  be a commutative ring, let  $\lambda \in R$  and  $P(X) \in R[X]$ . Then  $\lambda$  is a root of  $P(X)$  if and only if  $(X - \lambda)$  divides  $P(X)$ .

- **Theorem 3.3.10**

Let  $R$  a ring, or more generally, an integral domain. Then a non-zero polynomial  $P \in R[X] \setminus \{0\}$  has at most  $\deg(P)$  roots in  $R$ .

- **Definition 3.3.11** *Algebraically closed*

A field  $F$  is *algebraically closed* if each non-constant polynomial  $P \in F[X] \setminus F$  with coefficients in  $F$  has a root in  $F$ .

- **Theorem 3.3.13** *Fundamental theorem of algebra*

If  $F$  is an algebraically closed field, then every non-zero polynomial  $P \in F[X] \setminus \{0\}$  *decomposes into linear factors*

$$P = c(X - \lambda_1) \cdots (X - \lambda_n)$$

with  $n \geq 0, c \in F^\times$  and  $\lambda_1, \dots, \lambda_n \in F$ . This decomposition is unique up to reordering of the factors.

### 3.4 Homomorphisms, Ideals, and Subrings

- **Definition 3.4.1** *Ring homomorphism*

Let  $R$  and  $S$  be rings. A mapping  $f : R \rightarrow S$  is a *ring homomorphism* if the following hold  $\forall x, y \in R$

$$\begin{aligned} f(x + y) &= f(x) + f(y) \\ f(xy) &= f(x)f(y) \end{aligned}$$

- **Prelude to ideals**

Let  $f : R \rightarrow S$  be a ring homomorphism with  $\ker f = \{r \in R : f(r) = 0_S\}$ . Then  $\ker f$  is:

- a subgroup of  $R$  under addition
- $0_R \in \ker f$
- closed under multiplication

- closed under left and right multiplication by arbitrary elements of  $R$   
i.e.  $x \in \ker f \implies rx, xr \in \ker f \forall r \in R$

• **Lemma 3.4.5**

Let  $R$  and  $S$  be rings and  $f : R \rightarrow S$  a ring homomorphism. Then  $\forall x, y \in R$  and  $m \in \mathbb{Z}$

1.  $f(0_R) = 0_S$
2.  $f(-x) = -f(x)$
3.  $f(x - y) = f(x) - f(y)$
4.  $f(m \cdot x) = m \cdot f(x)$

Where  $mx$  denotes the  $m$ -th multiple of  $x$ .

• **Definition 3.4.7 Ideal**

A subset  $I$  of a ring  $R$  is an *ideal*, written  $I \trianglelefteq R$ , if the following hold:

1.  $I \neq \emptyset$
2.  $I$  is closed under subtraction (it's a subgroup)
3.  $\forall i \in I$  and  $\forall r \in R$  we have  $ri, ir \in I$  ( $I$  is closed under multiplication by elements of  $R$ )

Ideals satisfy the properties of rings, except possibly the existence of a multiplicative identity.

Ideals are subrings which are closed under multiplication with elements from the *ring* — not just elements from within the ideal!

• **Definition 3.4.11 Generated ideal**

Let  $R$  be a commutative ring and let  $T \subset R$ . Then the *ideal of  $R$  generated by  $T$*  is the set

$${}_R\langle T \rangle = \{r_1t_1 + \cdots + r_mt_m : t_1, \dots, t_m \in T, r_1, \dots, r_m \in R\}$$

together with the zero element in the case  $T = \emptyset$ .

• **Proposition 3.4.14**

Let  $R$  be a commutative ring and let  $T \subseteq R$ . Then  ${}_R\langle T \rangle$  is the smallest ideal of  $R$  that contains  $T$ .

• **Definition 3.4.15 Principle ideal**

Let  $R$  be a commutative ring. An ideal  $I \trianglelefteq R$  is called a *principle ideal* if  $I = \langle t \rangle$  for some  $t \in R$ .

• **Definition 3.4.17 Kernel**

Let  $R$  and  $S$  be rings, and let  $f : R \rightarrow S$  be a ring homomorphism. Since  $f$  is in particular a group homomorphism from  $(R, +)$  to  $(S, +)$ , the *kernel* of  $f$  already has a meaning:

$$\ker f = \{r \in R : f(r) = 0_S\}$$

• **Proposition 3.4.18**

Let  $R$  and  $S$  be rings and  $f : R \rightarrow S$  a ring homomorphism. Then  $\ker f$  is an ideal of  $R$ .

• **Lemma 3.4.20**  $f$  is injective if and only if  $\ker f = \{0\}$

• **Lemma 3.4.21** The intersection of any collection of ideals of a ring  $R$  is an ideal of  $R$ .

• **Lemma 3.4.22** Let  $I$  and  $J$  be ideals of a ring  $R$ . Then

$$I + J = \{a + b : a \in I, b \in J\}$$

is an ideal of  $R$ .

- **Definition 3.4.23** *Subring*

Let  $R$  be a ring. A subset  $R' \subseteq R$  is a *subring* of  $R$  if  $R'$  is itself a ring under the operations of addition and multiplication defined in  $R$ .

- **Proposition 3.4.26** *Test for a subring*

Let  $R$  be a ring, and  $R' \subseteq R$ . Then  $R'$  is a subring if and only if

1.  $R'$  has a multiplicative identity, and
2.  $R'$  is closed under subtraction, and
3.  $R'$  is closed under multiplication.

- **Proposition 3.4.29** Let  $R$  and  $S$  be rings and  $f : R \rightarrow S$  a ring homomorphism.

1. If  $R'$  is a subring of  $R$  then  $f(R')$  is a subring of  $S$ . In particular,  $f$  is a subring of  $S$ .
2. Assume that  $f(1_R) = 1_S$ . Then if  $x$  is a unit in  $R$ ,  $f(x)$  is a unit in  $S$  and  $(f(x))^{-1} = f(x^{-1})$ . In this case  $f$  restricts to a group homomorphism  $f|_{R^\times} : R^\times \rightarrow S^\times$ .

### 3.5 Equivalence Relations

- **Definition 3.5.1** *Relation*

A *relation*  $R$  on a set  $X$  is a subset  $R \subseteq X \times X$ .  $R$  is an *equivalence relation* on  $X$  when  $\forall x, y, z \in X$  the following hold:

1. *Reflexivity*:  $xRx$
2. *Symmetry*:  $xRy \iff yRx$
3. *Transitivity*:  $xRy$  and  $yRz \implies xRz$

- **Definition 3.5.3**

Suppose that  $\sim$  is an equivalence relation on a set  $X$ . For  $x \in X$  the set  $E(x) \equiv \{z \in X : z \sim x\}$  is called the *equivalence class* of  $x$ .

A subset  $E \subseteq X$  is called an *equivalence class* for  $\sim$  if  $\exists x \in X \ni E = E(x)$ .

An element of an equivalence class is called a *representative* of the class.

A subset  $Z \subseteq X$  containing precisely one element from each equivalence class is called a *system of representatives* for the equivalence relation.

- **Definition 3.5.5** *Set of equivalence classes*

Given an equivalence relation  $\sim$  on the set  $X$ , the *set of equivalence classes*, which is a subset of  $\mathcal{P}(X)$ , is

$$(X/\sim) \equiv \{E(x) : x \in X\}$$

There is a canonical mapping  $\text{can} : X \rightarrow (X/\sim)$ ,  $x \mapsto E(x)$ . It is obviously a surjection.

- **Remark**

Suppose that  $\sim$  is an equivalence relation on  $X$ . If  $f : X \rightarrow Z$  is a mapping with the property that  $x \sim y \implies f(x) = f(y)$ , then there is a unique mapping  $\bar{f} : (X/\sim) \rightarrow Z$  with  $f = \bar{f} \circ \text{can}$ . Its definition is easy:  $f(E(x)) = f(x)$ . This property is called the *universal property of the set of equivalence classes*.

- **Definition 3.5.7** *Well-defined*

$g : (X/\sim) \rightarrow Z$  is *well-defined* if there is a mapping  $f : X \rightarrow Z$  such that  $f$  has the property  $x \sim y \implies f(x) = f(y)$  and  $g = \bar{f}$ .

### 3.6 Factor Rings and the First Isomorphic Theorem

- **Prelude**

Let  $f : R \rightarrow S$  be a ring homomorphism.

$$x \sim y \iff f(x) = f(y) \iff f(x - y) = 0 \iff x - y \in \ker f$$

Then:

$$E(x) = x + \ker f \equiv \{x + k : k \in \ker f\}$$

So we have that:

- the rule  $x \sim y \iff x - y \in \ker f$  is an equivalence relation;
- the equivalence classes are the sets  $x + \ker f$  for  $x \in R$ ;
- the set of equivalence classes  $(R / \sim)$  is a ring, isomorphic to a subring of  $S$ .

- **Definition 3.6.1 Cosets**

Let  $I \trianglelefteq R$  be an ideal in a ring  $R$ . The set

$$x + I \equiv \{x + i : i \in I\} \subseteq R$$

is a *coset of  $I$  in  $R$* , or *the coset of  $x$  with respect to  $I$  in  $R$* .

- **Definition 3.6.3 Factor ring**

Let  $R$  be a ring,  $I \trianglelefteq R$  be an ideal, and  $\sim$  the equivalence relation defined by  $x \sim y \iff x - y \in I$ . Then  $R/I$ , the *factor ring of  $R$  by  $I$*  or the *quotient of  $R$  by  $I$* , is the set  $(R / \sim)$  of cosets of  $I$  in  $R$ .

$$R/I = \{r + I : r \in R\}$$

- **Theorem 3.6.4**

Let  $R$  be a ring, and  $I \trianglelefteq R$  an ideal. Then  $R/I$  is a ring, where the operation of addition is defined by

$$(x + I) + (y + I) = (x + y) + I \quad \forall x, y \in R$$

and multiplication is defined by

$$(x + I) \cdot (y + I) = xy + I \quad \forall x, y \in R$$

- **Theorem 3.6.7 Universal Property of Factor Rings**

Let  $R$  be a ring, and  $I \trianglelefteq R$ .

1. The mapping  $\text{can} : R \rightarrow R/I$  with  $\text{can}(r) = r + I$  is a surjective ring homomorphism with kernel  $I$ .
2. If  $f : R \rightarrow S$  is a ring homomorphism with  $f(I) = \{0_S\}$ , so that  $I \subseteq \ker f$ , then there is a unique ring homomorphism  $\bar{f} : R/I \rightarrow S$  such that  $f = \bar{f} \circ \text{can}$ .

- **Theorem 3.6.9 First Isomorphic Theorem for Rings**

Let  $R$  and  $S$  be rings. Then every ring homomorphism  $f : R \rightarrow S$  induces a ring isomorphism

$$\bar{f} : R / \ker f \xrightarrow{\sim} \text{im } f$$

### 3.7 Modules

- **Definition 3.7.1** A (*left*) *module  $M$  over a ring  $R$*  is a pair consisting of an abelian group  $M = (M, +)$  and a mapping

$$\begin{aligned} R \times M &\rightarrow M \\ (r, a) &\mapsto ra \end{aligned}$$

such that  $\forall r, s \in R$  and  $a, b \in M$  the following identities hold:

$$\begin{aligned} r(a+b) &= (ra) + (rb) && \text{(distributivity)} \\ (r+s)a &= (ra) + (sa) && \text{(distributivity)} \\ r(sa) &= (rs)a && \text{(associativity)} \\ 1_R a &= a \end{aligned}$$

i.e. a vector space, but with a *ring* instead of a *field*.

- **Lemma 3.7.8** Let  $R$  be a ring, and  $M$  an  $R$ -module.

1.  $0_R a = 0_M \quad \forall a \in M$
2.  $r 0_M = 0_M \quad \forall r \in R$
3.  $(-r)a = r(-a) = -(ra), \quad \forall r \in R, a \in M$ . (Here, the first negative is in  $R$ , and the last two negatives are in  $M$ .)

- **Definition 3.7.11**

Let  $R$  be a ring, and let  $M, N$  be  $R$ -modules. A mapping  $f : M \rightarrow N$  is an  $R$ -homomorphism if the following hold  $\forall a, b \in M$  and  $r \in R$ :

$$\begin{aligned} f(a+b) &= f(a) + f(b) \\ f(ra) &= rf(a) \end{aligned}$$

The *kernel* of  $f$  is  $\ker f = \{a \in M : f(a) = 0_N\} \subseteq M$  and the *image* of  $f$  is  $\operatorname{im} f = \{f(a) : a \in M\} \subseteq N$ .

If  $f$  is a bijection then it is an *isomorphism*.

- **Definition 3.7.15**

A non-empty subset  $M'$  of an  $R$ -module  $M$  is a *submodule* if  $M'$  is an  $R$ -module with respect to the operations of the  $R$ -module  $M$  restricted to  $M'$ .

- **Proposition 3.7.20** Test for a submodule

Let  $R$  be a ring and let  $M$  be an  $R$ -module. A subset  $M' \subseteq M$  is a submodule if and only if

1.  $0_M \in M'$
2.  $a, b \in M' \implies a - b \in M'$
3.  $r \in R, a \in M' \implies ra \in M'$

- **Lemma 3.7.21**

Let  $f : M \rightarrow N$  be an  $R$ -homomorphism. Then  $\ker f$  is a submodule of  $M$  and  $\operatorname{im} f$  is a submodule of  $N$ .

- **Lemma 3.7.22**

Let  $R$  be a ring, let  $M$  and  $N$  be  $R$ -modules and let  $f : M \rightarrow N$  be an  $R$ -homomorphism. Then  $f$  is injective if and only if  $\ker f = \{0_M\}$ .

- **Definition 3.7.23**

Let  $R$  be a ring,  $M$  an  $R$ -module, and let  $T \subseteq M$ . Then the *submodule of  $M$  generated by  $T$*  is the set

$${}_R\langle T \rangle = \{r_1 t_1 + \cdots + r_m t_m : t_1, \dots, t_m \in T, r_1, \dots, r_m \in R\},$$

together with the zero element in case  $T = \emptyset$ .

The module  $M$  is *finitely generated* if it is generated by a finite set:  $M = {}_R\langle \{t_1, \dots, t_n\} \rangle$ .

It is *cyclic* if it is generated by a singleton:  $M = {}_R\langle t \rangle$ .

- **Lemma 3.7.28** Let  $T \subseteq M$ . Then  ${}_R\langle T \rangle$  is the smallest submodule of  $M$  that contains  $T$ .

- **Lemma 3.7.29** The intersection of any collection of submodules of  $M$  is a submodule of  $M$ .

- **Lemma 3.7.30** Let  $M_1$  and  $M_2$  be submodules of  $M$ . Then

$$M_1 + M_2 = \{a + b : a \in M_1, b \in M_2\}$$

is a submodule of  $M$ .

- **Definition 3.7.31.1 Coset**

Let  $R$  be a ring,  $M$  an  $R$ -module, and  $N$  a submodule of  $M$ . For each  $a \in M$ , the *coset of  $a$  with respect to  $N$  in  $M$*  is

$$a + N = \{a + b : b \in N\}.$$

It is a coset of  $N$  in the abelian group  $M$  and is an equivalence class for the equivalence relation  $a \sim b \iff a - b \in N$ .

- **Definition 3.7.31.2 Factor**

$M/N$ , the *factor of  $M$  by  $N$*  or the *quotient of  $M$  by  $N$* , is the set  $(M / \sim)$  of all cosets of  $N$  in  $M$ .

$$M/N = \{a + N : a \in M\}$$

This becomes an  $R$ -module by introducing the operations of addition and multiplication as follows:

$$\begin{aligned} (a + N) + (b + N) &= (a + b) + N \\ r(a + N) &= ra + N \end{aligned}$$

for all  $a, b \in M, r \in R$ .

- **Theorem 3.7.31.3 Factor module**

- The zero of  $M/N$  is the coset  $0_{M/N} = 0_M + N$ .
- The negative of  $a + N \in M/N$  is the coset  $-(a + N) = (-a) + N$ .
- The  $R$ -module  $M/N$  is the *factor module* of  $M$  by the submodule  $N$ .

- **Theorem 3.7.32 The Universal Property of Factor Modules**

Let  $R$  be a ring, and let  $L$  and  $M$  be  $R$ -modules, and  $N$  a sub-module of  $M$ .

1. The mapping  $\text{can} : M \rightarrow M/N$  sending  $a$  to  $a + N$ ,  $\forall a \in M$  is a surjective  $R$ -homomorphism with kernel  $N$ .
2. If  $f : M \rightarrow L$  is an  $R$ -homomorphism with  $f(N) = \{0_L\}$ , so that  $N \subseteq \ker f$ , then there is a unique homomorphism  $\bar{f} : M/N \rightarrow L$  such that  $f = \bar{f} \circ \text{can}$ .

- **Theorem 3.7.33 First Isomorphism Theorem for Modules**

Let  $R$  be a ring and let  $M$  and  $N$  be  $R$ -modules. Then every  $R$ -homomorphism  $f : M \rightarrow N$  induces a  $R$ -isomorphism

$$\bar{f} : M / \ker f \rightarrow \text{im } f$$

## 4 Determinants & Eigenvalues Redux

### 4.1 The sign of a permutation

- **Definition 4.1.1 Transposition**

The group of all permutations of the set  $\{1, 2, \dots, n\}$ , also known as bijections from  $\{1, 2, \dots, n\}$  to itself, is denoted by  $\mathfrak{S}_n$  and called the  *$n$ -th symmetric group*. It is a group under composition and has  $n!$  elements.

A *transposition* is a permutation that swaps two elements of the set and leaves all the others unchanged.



- **Definition 4.1.2** *Inversion & Sign*

An *inversion* of a permutation  $\sigma \in \mathfrak{S}_n$  is a pair  $(i, j)$  such that  $1 \leq i < j \leq n$  and  $\sigma(i) > \sigma(j)$ . The number of inversions of the permutation  $\sigma$  is called the *length* of  $\sigma$  and written  $\ell(\sigma)$ . In formulas:

$$\ell(\sigma) = |\{(i, j) : i < j \text{ but } \sigma(i) > \sigma(j)\}|$$

The *sign* of  $\sigma$  is defined to be the parity of the number of inversions of  $\sigma$ . In formulas:

$$\text{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

A permutation whose sign is  $+1$ , in other words which has even length, is called an *even permutation*, while a permutation whose sign is  $-1$ , in other words which has odd length, is called an *odd permutation*.

- **Lemma 4.1.5** (Multiplicativity of the sign)

For each  $n \in \mathbb{N}$  the sign of a permutation produces a group homomorphism  $\text{sgn} : \mathfrak{S}_n \rightarrow \{+1, -1\}$  from the symmetric group to the two-element group of signs. In formulas:

$$\text{sgn}(\sigma\tau) = \text{sgn}(\sigma) \text{sgn}(\tau) \quad \forall \sigma, \tau \in \mathfrak{S}_n$$

- **Definition 4.1.7** *Alternating group*

For  $n \in \mathbb{N}$ , the set of even permutations in  $\mathfrak{S}_n$  forms a subgroup of  $\mathfrak{S}_n$  because it is the kernel of the group homomorphism  $\text{sgn} : \mathfrak{S}_n \rightarrow \{+1, -1\}$ . This group is the *alternating group* and is denoted  $A_n$ .

## 4.2 Determinants & what they mean

- **Definition 4.2.1** Let  $R$  be a commutative ring and  $n \in \mathbb{N}$ .

The *determinant* is a mapping  $\det : \text{Mat}(n; R) \rightarrow R$  from square matrices with coefficients in  $R$  to the ring  $R$  that is given by the following formula:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mapsto \det(A) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

This formula is called the *Leibniz formula*.

The degenerate case  $n = 0$  assigns the value 1 as the determinant of the “empty matrix”.

- *The connection between determinants and volumes*

The determinant of a matrix is equal to the scaling factor it performs.

- *The connection between determinants and orientation*

The sign of the determinant determines the orientation:  $\det = +1$  preserves the orientation;  $\det = -1$  reverses the orientation.

## 4.3 Characterising the determinant

- **Definition 4.3.1** *Bi-linear forms*

Let  $U, V, W$  be  $F$ -vector spaces.

A *bi-linear form* on  $U \times V$  with values in  $W$  is a mapping  $H : U \times V \rightarrow W$  which is a linear mapping in both of its entries.

This means that it must satisfy the following properties for all  $u_1, u_2 \in U$ ;  $v_1, v_2 \in V$ ;  $\lambda \in F$ :

$$H(u_1 + u_2, v_1) = H(u_1, v_1) + H(u_2, v_1)$$

$$H(u_1, v_1 + v_2) = H(u_1, v_1) + H(u_1, v_2)$$

$$H(u_1, \lambda v_1) = \lambda H(u_1, v_1)$$

$$H(\lambda u_1, v_1) = \lambda H(u_1, v_1)$$

The first two conditions state that for any fixed  $v \in V$  the mapping  $H(-, v) : U \rightarrow W$  is linear.  $H$  is a *bi-linear form*. A bi-linear form  $H$  is *symmetric* if  $U = V$  and

$$H(u, v) = H(v, u) \quad \forall u, v \in U$$

while it is *alternating* or *antisymmetric* if  $U = V$  and

$$H(u, u) = 0 \quad \forall u \in U$$

- **Definition 4.3.3** *Multi-linear forms*

Let  $V_1, \dots, V_n, W$  be  $F$ -vector spaces. A mapping  $H : V_1 \times V_2 \times \dots \times V_n \rightarrow W$  is a *multi-linear form* or *multi-linear* if for each  $j$ , the mapping  $V_j \rightarrow W$  defined by  $v_j \mapsto H(v_1, \dots, v_j, \dots, v_n)$ , with  $v_i \in V_i$  arbitrary fixed vectors of  $V_i$  for  $i \neq j$ , is linear. In the case  $n = 2$ , this is exactly the definition of a bi-linear mapping.

- **Definition 4.3.4** *Alternating*

Let  $V$  and  $W$  be  $F$ -vector spaces. A multi-linear form  $H : V \times \dots \times V \rightarrow W$  is *alternating* if it vanishes on every  $n$ -tuple of elements of  $V$  that has at least two entries equal, in other words if:

$$(\exists i \neq j \text{ with } v_i = v_j) \implies H(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = 0$$

In the case  $n = 2$ , this is exactly the definition of an alternating or anti-symmetric bi-linear mapping.

- **Theorem 4.3.6** Characterisation of the determinant

Let  $F$  be a field. The mapping

$$\det : \text{Mat}(n; F) \rightarrow F$$

is the unique, alternating, multi-linear form on  $n$ -tuples of column vectors with values in  $F$  that takes the value  $1_F$  on the identity matrix.

1. Is it a multi-linear form?
2. Does it go from  $F^n \times \dots \times F^n \rightarrow F$ ?
3. Is it alternating?
4. Does it take the value 1 on the identity?

If (and only if) answered *yes* to all, then we have a determinant.

## 4.4 Rules for calculating with determinants

- **Theorem 4.4.1** Multiplicativity of the determinant

Let  $R$  be a commutative ring and let  $A, B \in \text{Mat}(n; R)$ . Then

$$\det(AB) = \det(A) \det(B)$$

- **Theorem 4.4.2** Determinantal criterion for invertibility

The determinant of a square matrix with entries in a field  $F$  is non-zero if and only if the matrix is invertible.

- **Lemma 4.4.4**

The determinant of a square matrix and the transpose of the square matrix are equal, that is, for all  $A \in \text{Mat}(n; R)$  with  $R$  a commutative ring

$$\det(A^T) = \det(A)$$

- **Definition 4.4.6** *Cofactor*

Let  $A \in \text{Mat}(n; R)$  for some commutative ring  $R$  and  $n \in \mathbb{N}$ . Let  $i, j \in (1, n) \subset \mathbb{N}$ . Then the  $(i, j)$  cofactor of  $A$  is  $C_{ij} = (-1)^{i+j} \det(A\langle i, j \rangle)$  where  $A\langle i, j \rangle$  is the matrix obtained by deleting the  $i$ -th row and the  $j$ -th column.

- **Theorem 4.4.7** Laplace's expansion of the determinant

Let  $A = (a_{ij})$  be an  $(n \times n)$  matrix with entries from a commutative ring  $R$ .

For a fixed  $i$ , the  $i$ -th row expansion of the determinant is

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij}$$

and for a fixed  $j$ , the  $j$ -th column expansion of the determinant is

$$\det(A) = \sum_{i=1}^n a_{ij} C_{ij}$$

- **Definition 4.4.8** *Adjugate matrix*

Let  $A$  be an  $(n \times n)$  matrix whose entries are  $\text{adj}(A)_{ij} = C_{ji}$  where  $C_{ji}$  is the  $(j, i)$  cofactor.

- **Theorem 4.4.9** Cramer's rule

Let  $A$  be an  $(n \times n)$  matrix with entries in a commutative ring  $R$ . Then

$$A \cdot \text{adj}(A) = (\det A) I_n$$

- **Corollary 4.4.11** Invertibility of matrices

A square matrix with entries in a commutative ring  $R$  is invertible if and only if its determinant is a unit in  $R$ . That is,  $A \in \text{Mat}(n; R)$  is invertible if and only if  $\det(A) \in R^\times$ .

## 4.5 Eigenvalues & Eigenvectors

- **Definition 4.5.1** *Eigenvalue*

Let  $f : V \rightarrow V$  be an endomorphism of an  $F$ -vector space  $V$ . A scalar  $\lambda \in F$  is an *eigenvalue* of  $f$  if and only if there exists a non-zero vector  $\mathbf{v} \in V$  such that  $f(\mathbf{v}) = \lambda \mathbf{v}$ .

Each such vector is called an *eigenvector* of  $f$  with *eigenvalue*  $\lambda$ .

For any  $\lambda \in F$ , the *eigenspace* of  $f$  with *eigenvalue*  $\lambda$  is

$$E(\lambda, f) = \{\mathbf{v} \in V : f(\mathbf{v}) = \lambda \mathbf{v}\}$$

When  $\lambda = 1$ , this is equivalent to having a *fixed-point mapping*.

When  $\lambda = 0$ , this is equivalent to the *kernel* of the mapping.

The corresponding *eigenvectors* are the null-space of  $(A - \lambda I_n)$

- **Theorem 4.5.4** Existence of Eigenvalues

Each endomorphism of a non-zero finite-dimensional vector space over an algebraically closed field has an eigenvalue.

- **Definition 4.5.6** *Characteristic polynomial*

Let  $R$  be a commutative ring and let  $A \in \text{Mat}(n; R)$  be a square matrix with entries in  $R$ . The polynomial  $\det(A - xI_n) \in R[x]$  is called the *characteristic polynomial* of the matrix  $A$ . It is denoted by

$$\chi_A(x) \equiv \det(A - xI_n)$$

where  $\chi$  stands for *characteristic*.

- **Theorem 4.5.8** Eigenvalues and characteristic polynomials

Let  $F$  be a field and  $A \in \text{Mat}(n; F)$  a square matrix with entries in  $F$ . The eigenvalues of the linear mapping  $A : F^n \rightarrow F^n$  are exactly the roots of the characteristic polynomial  $\chi_A$ .

## 4.6 Triangularisable, Diagonalisable, & the Cayley-Hamilton theorem

- **Proposition 4.6.1** Triangularisability

Let  $f : V \rightarrow V$  be an endomorphism of a finite-dimensional  $F$ -vector space  $V$ . The following two statements are equivalent:

1. The vector space  $V$  has an ordered basis  $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$  such that

$$\begin{aligned} f(\mathbf{v}_1) &= a_{11}\mathbf{v}_1 \\ f(\mathbf{v}_2) &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 \\ &\vdots \\ f(\mathbf{v}_n) &= a_{1n}\mathbf{v}_1 + a_{2n}\mathbf{v}_2 + \dots + a_{nn}\mathbf{v}_n \in V \end{aligned}$$

(so that the first basis vector  $\mathbf{v}_1$  is an eigenvector, with eigenvalue  $a_{11}$ ) or equivalently such that the  $n \times n$  matrix  ${}_{\mathcal{B}}[f]_{\mathcal{B}} = (a_{ij})$  representing  $f$  with respect to  $\mathcal{B}$  is upper triangular.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

When this happens,  $f$  is *triangularisable*.

2. The characteristic polynomial  $\chi_{f(x)}$  of  $f$  decomposes into linear factors in  $F[x]$ .

- **Definition 4.6.5** *Diagonalisable*

An endomorphism  $f : V \rightarrow V$  of an  $F$ -vector space  $V$  is *diagonalisable* if and only if there exists a basis of  $V$  consisting of eigenvectors of  $f$ .

If  $V$  is finite-dimensional, then this is the same as saying that there exists an ordered basis  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  such that the corresponding matrix representing  $f$  is diagonal, that is  ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_n)$ . In this case, of course,  $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ .

A square matrix  $A \in \text{Mat}(n; F)$  is *diagonalisable* if and only if the corresponding linear mapping  $F^n \rightarrow F^n$  given by the left multiplication of  $A$  is diagonalisable. This just means that  $A$  is conjugate to a diagonal matrix: there exists an invertible matrix  $P \in \text{GL}(n; F)$  such that  $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$ . In this case, the columns of  $P$  are the vectors of a basis of  $F^n$  consisting of eigenvectors of  $A$  with eigenvalues  $\lambda_1, \dots, \lambda_n$ .

- **Lemma 4.6.8** Linear independence of Eigenvectors

Let  $f : V \rightarrow V$  be an endomorphism of a vector space  $V$  and let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be eigenvectors of  $f$  with pairwise different eigenvalues  $\lambda_1, \dots, \lambda_n$ .

Then the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly independent.

- **Theorem 4.6.9** Cayley-Hamilton Theorem

Let  $A \in \text{Mat}(n; R)$  be a square matrix with entries in a commutative ring  $R$ . Then evaluating its characteristic polynomial  $\chi_A(x) \in R[x]$  at the matrix  $A$  gives zero.

## 4.7 Google's PageRank Algorithm

## 5 Inner Product Spaces

### 5.1 Inner Product Spaces: Definitions

- **Definition 5.1.1** *Real inner product space*

Let  $V$  be a vector space over  $\mathbb{R}$ . An *inner product* on  $V$  is a mapping

$$(-, -) : V \times V \rightarrow \mathbb{R}$$

that satisfies the following for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$  and  $\lambda, \mu \in \mathbb{R}$ :

1.  $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \lambda(\mathbf{x}, \mathbf{z}) + \mu(\mathbf{y}, \mathbf{z})$  (associative)
2.  $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$  (commutative)
3.  $(\mathbf{x}, \mathbf{x}) \geq 0$ , with equality if and only if  $\mathbf{x} = \mathbf{0}$ . (positive definite)

A *real inner product space* is a real vector space endowed with an inner product.

• **Definition 5.1.3** *Complex inner product space*

Let  $V$  be a vector space over  $\mathbb{C}$ . An *inner product* on  $V$  is a mapping

$$(-, -) : V \times V \rightarrow \mathbb{C}$$

that satisfies the following for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$  and  $\lambda, \mu \in \mathbb{C}$ :

1.  $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \lambda(\mathbf{x}, \mathbf{z}) + \mu(\mathbf{y}, \mathbf{z})$  (associative)
2.  $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$  (commutative)
3.  $(\mathbf{x}, \mathbf{x}) \geq 0$ , with equality if and only if  $\mathbf{x} = \mathbf{0}$ . (positive definite)

Here  $\bar{z}$  denotes the complex conjugate of  $z$ . A *complex inner product space* is a complex vector space endowed with an inner product.

• **Definition** *Skew-linear*

A mapping  $f : V \rightarrow W$  between complex vector spaces is *skew-linear* if  $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$  and  $f(\lambda\mathbf{v}_1) = \bar{\lambda}f(\mathbf{v}_1)$  for all  $\mathbf{v}_1, \mathbf{v}_2 \in V$  and all  $\lambda \in \mathbb{C}$ .

• **Definition** *Sesquilinear*

A complex form that is *skew-linear* in its second variable. When such a form is commutative, it is *hermitian*.

• **Terminology**

- A finite-dimensional real inner product space is a *Euclidean vector space*.
- A complex inner product space is a *unitary space* or *pre-Hilbert space*.
- A finite-dimensional inner product space is a *finite-dimensional Hilbert space*.

• **Definition 5.1.5** *Length or Inner Product Norm*

In a real or complex inner product space the *length* or *inner product norm* or *norm*  $\|\mathbf{v}\| \in \mathbb{R}$  of a vector  $\mathbf{v}$  is defined as the non-negative square root

$$\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$$

Vectors whose length is 1 are called *units*. Two vectors  $\mathbf{v}, \mathbf{w}$  are *orthogonal* and we write

$$\mathbf{v} \perp \mathbf{w}$$

if and only if  $(\mathbf{v}, \mathbf{w}) = 0$ .

• **Definition 5.1.7** *Orthonormal family*

A family  $(\mathbf{v}_i)_{i \in I}$  for vectors from an inner product space is an *orthogonal family* if all the vectors  $\mathbf{v}_i$  have length 1 and if they are pairwise orthogonal to each other, which, using the Kronecker delta, means

$$(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}$$

An orthonormal family that is a basis is an *orthonormal basis*.

• **Theorem 5.1.10**

Every finite dimensional inner product space has an orthonormal basis.

## 5.2 Orthogonal Complements and Orthogonal Projections

- **Definition 5.2.1** *Orthogonal*

let  $V$  be an inner product space and let  $T \subseteq V$  be an arbitrary subset. Define

$$T^\perp = \{\mathbf{v} \in V : \mathbf{v} \perp \mathbf{t}, \forall \mathbf{t} \in T\},$$

calling this set the *orthogonal* to  $T$ .

- **Proposition 5.2.2**

Let  $V$  be an inner product space and let  $U$  be a finite dimensional subspace of  $V$ . Then  $U$  and  $U^\perp$  are complementary (*Definition 1.7.6*). In other words

$$V = U \oplus U^\perp$$

- **Definition 5.2.3** *Orthogonal complement*

Let  $U$  be a finite dimensional subspace of an inner product space  $V$ . The space  $U^\perp$  is the *orthogonal complement* to  $U$ . The *orthogonal projection from  $V$  onto  $U$*  is the mapping

$$\pi_U : V \rightarrow V$$

that sends  $\mathbf{v} = \mathbf{p} + \mathbf{r}$  to  $\mathbf{p}$ .

- **Proposition 5.2.4**

Let  $U$  be a finite-dimensional subspace of an inner product space  $V$  and let  $\pi_U$  be the orthogonal projection from  $V$  to  $U$ .

1.  $\pi_U$  is a linear mapping with  $\text{im}(\pi_U) = U$  and  $\ker(\pi_U) = U^\perp$ .
2. If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is an orthonormal basis of  $U$ , then  $\pi_U$  is given by the following formula for all  $\mathbf{v} \in V$

$$\pi_U(\mathbf{v}) = \sum_{i=1}^n (\mathbf{v}, \mathbf{v}_i) \mathbf{v}_i$$

3.  $\pi_U^2 = \pi_U$ , that is  $\pi_U$  is an idempotent.

- **Theorem 5.2.5** Cauchy-Schwarz Inequality

Let  $\mathbf{v}, \mathbf{w}$  be vectors in an inner product space. Then

$$|(\mathbf{v}, \mathbf{w})| \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

with equality if and only if  $\mathbf{v}$  and  $\mathbf{w}$  are linearly dependent.

- **Corollary 5.2.6**

The norm  $\|\cdot\|$  on an inner product space  $V$  satisfies, for any  $\mathbf{v}, \mathbf{w} \in V$  and scalar  $\lambda$ :

1.  $\|\mathbf{v}\| \geq 0$  with equality if and only if  $\mathbf{v} = \mathbf{0}$
2.  $\|\lambda \mathbf{v}\| = |\lambda| \|\mathbf{v}\|$
3.  $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ , the *triangle inequality*.

- **Theorem 5.2.7**

Let  $\mathbf{v}_1, \dots, \mathbf{v}_k$  be linearly independent vectors in an inner product space  $V$ . Then there exists an orthonormal family  $\mathbf{w}_1, \dots, \mathbf{w}_k$  with the property that for all  $1 \leq i \leq k$

$$\mathbf{w}_i \in \mathbb{R}_{<0} \mathbf{v}_i + \langle \mathbf{v}_{i-1}, \dots, \mathbf{v}_1 \rangle$$

## 5.3 Adjoints and Self-Adjoints

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## 6 Reference

### 6.1 Terminology of Algebraic Structures

	<i>Associativity</i>	<i>Identity</i>	<i>Inverses</i>
Group	Yes	Yes	Yes
Monoid	Yes	Yes	No
Semi-group	Yes	No	No
Magma	No	No	No

Ring = (Group, Monoid)

Field = (Group, Group)