Equivalent metrics

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Overview

1. Metric spaces
    - What constitutes a metric space?
    - Some metrics of R^2
    - Are they equivalent?
2. Equivalence
    - Definition
    - Other ways of expressing equivalence
    - Strong equivalence
3. Epilogue

What constitutes a metric space?

Definition

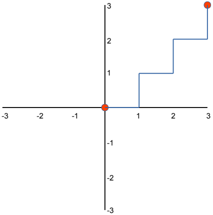
A set X with function $d : X \times X \to \mathbb{R}$ which $\forall x, y, z \in X$ satisfy

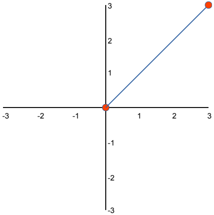
$$\begin{aligned}
\textsc{Positive Definite:}\quad & d(x,y) \geq 0 \text{ and} \\
& d(x,y) = 0 \iff x = y \\
\textsc{Symmetric:}\quad & d(x,y) = d(y,x) \\
\textsc{Triangle Inequality:}\quad & d(x,y) \leq d(x,z) + d(z,y)
\end{aligned}$$

Some metrics of R^2

Manhattan distance $d_1(\mathbf{x}, \mathbf{y}) := |x_1 - y_1| + |x_2 - y_2|$

Euclidean distance $d_2(\mathbf{x}, \mathbf{y}) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$





Are they equivalent?

$$\mathrm{d}_1(\mathbf{x},\mathbf{y}) := |x_1 - y_1| + |x_2 - y_2|$$
$$d_2(\mathbf{x},\mathbf{y}) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

	$O : (0,0)$	$(4,0)$	$(15,18)$	$(57,42)$	$(900,600)$
$d_1(O,x)$	0	4	23	99	1500
$d_2(O,x)$	0	4	33	71	1082

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- d_1 and d_2 seem to increase proportionally with each other.
- Characteristic of equivalent metrics.

Definition

There are several ways to express metric equivalence.

Definition

d_1 and d_2 in X are said to be *equivalent* if $\forall \epsilon > 0 \ \exists \delta > 0$ such that

$$d_1(x, y) < \delta \implies d_2(x, y) < \epsilon$$

$\forall x, y \in X$, and vice-versa.

Open balls

Definition

An open ball $B_r(x; d)$ in a metric space (X, d) is defined as

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Theorem

The metrics d_1 and d_2 on a set X are equivalent iff

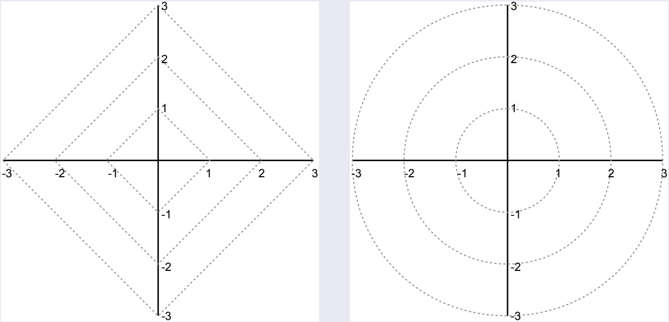
$$\forall \epsilon > 0 \; \exists \delta > 0 \;\; \ni \;\; B_\delta(x; d_1) \subseteq B_\epsilon(x; d_2)$$

in their corresponding metric spaces (X, d_1) and (X, d_2).
And vice-versa.

Balls (cont.)

Relating back to our example with Manhattan metric d_1 and Euclidean metric d_2 in $\mathbb{R}^2$:

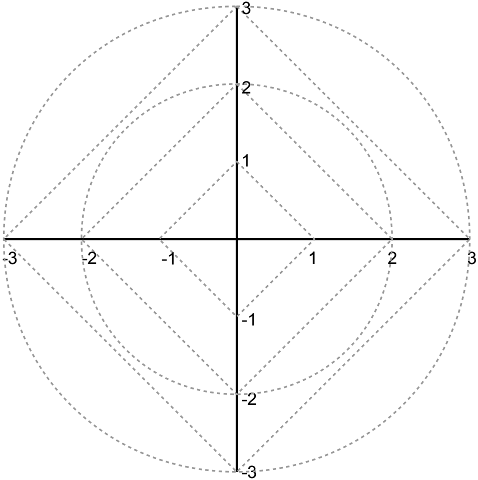
$B_r(O, d_1)$ $\qquad$ $B_r(O, d_2)$



$$\forall r \in \{0, 1, 2, 3\}$$

Balls (cont. 2)

Overlaying them, you can see how d_1 and d_2 could be equivalent.



Open sets

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A set $V \subseteq X$ is said to be *open* if and only if

$$\forall x \in V \; \exists \epsilon > 0 \;\; \ni \;\; B_\epsilon(x) \subseteq V$$

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Theorem

Metrics d_1 and d_2 on a set X are equivalent if and only if

$$A \subseteq X \text{ is open under } d_1 \iff A \text{ is open under } d_2.$$

Strong equivalence

Strong equivalence is a *sufficient* condition for equivalence.

Definition

Two metrics d_1 and d_2 are said to be *strongly equivalent* if $\exists \alpha, \beta \in \mathbb{R}^+$such that

$$d_1(x, y) \leq \alpha \cdot d_2(x, y) \quad \text{and} \quad d_2(x, y) \leq \beta \cdot d_1(x, y)$$

Somewhat analogous to continuity vs uniform continuity.

Manhattan and Euclidean distance are strongly equivalent

$$d_1(\mathbf{x}, \mathbf{y}) := |x_1 - y_1| + |x_2 - y_2|$$
$$d_2(\mathbf{x}, \mathbf{y}) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Proof.

Let $a, b = x_1 - y_1,\ x_2 - y_2$. Then

$$d_1(\mathbf{x}, \mathbf{y})^2 = a^2 + b^2 \quad \text{and} \quad d_2(\mathbf{x}, \mathbf{y})^2 = a^2 + b^2 + 2ab$$
$$\implies d_1(\mathbf{x}, \mathbf{y}) \leq 1 \cdot d_2(\mathbf{x}, \mathbf{y})$$

Since we have $ab \leq \max(a, b)^2$

$$\implies 2ab \leq 2(a^2 + b^2)$$
$$\implies a^2 + b^2 + 2ab \leq 3(a^2 + b^2)$$
$$\implies d_2(\mathbf{x}, \mathbf{y}) \leq \sqrt{3} \cdot d_1(\mathbf{x}, \mathbf{y})$$

Epilogue

Might be tempting to think that

If d_1 and d_2 are equivalent then $\forall a, b, c \in X$

$$d_1(a, b) \leq d_1(a, c) \implies d_2(a, b) \leq d_2(a, c)$$

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Take (2,2) and (3,0) with Manhattan and Euclidean distance metrics d_1 and d_2.

$$d_1(O,(3,0)) = 3 < d_1(O,(2,2)) = 4$$

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but

$$d_2(O,(3,0)) = 3 > d_2(O(2,2)) = 2.8$$

Thanks!