# Computing the determinant

## Group 12

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# Overview

# Invertibililty of matrices

**Theorem**

An $n \times n$ square matrix $A$ is invertible if and only if

$$\det(A) \neq 0.$$

# Cramer's Rule

**Theorem**

Given an equation $A\mathbf{x} = \mathbf{b}$ The solutions for $\mathbf{x}$ are given by

$$x_i = \frac{\det(A_i)}{\det(A)}$$

with $A_i$ being the matrix formed by replacing the $i$th column of $A$ by $\mathbf{b}$.

It turns out this method has the same runtime complexity as Gaussian elimination for solving systems of linear equations.

# Eigenvalues and Eigenvectors

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The Eigenvectors of $A$ are the vectors $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

# Eigenvalues and Eigenvectors (cont.)

The absolute value of the determinant of real vectors is equal to the volume of the parallelepiped spanned by those vectors.
$f : \mathbb{R}^n \to \mathbb{R}^n$: the linear map represented by the $A$. $S$: any measurable subset of $\mathbb{R}^n$.

$$\text{volume}(f(S)) = |\det(A^T A)| \times \text{volume}(S).$$

# Jacobian determinant

For $f : \mathbb{R}^n \to \mathbb{R}^n$, the Jacobian matrix is the $n \times n$ matrix whose entries are defined as

$$D(f) = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq n.}$$

Its determinant is known as the *Jacobian determinant*.

If the determinant of a continuously differentiable function $f$ at a point $p$ is...

- Non-zero, $f$ is invertible near a point $p$ in $\mathbb{R}^n$.
- Positive, then $f$ preserves orientation near $p$.
- Negative, then $f$ reverses orientation near $p$.

# Big-O notation

We'll be looking at the runtime of algorithms, so this is useful.

## Definition

A function $f$ is said to be $\mathcal{O}(g)$, with $g$ a function iff

$$\exists k \in \mathbb{R} \ni f(n) < k \cdot g(n)$$

for sufficiently large $n$.

# Leibniz formula

## Definition

The Leibniz formula defines the determinant of $A \in \mathbb{M}(n)$ as

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \left( \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

where $\mathfrak{S}_n$ is the set of permutations length $n$.

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where $\mathfrak{S}_n$ is the set of permutations length $n$.

Computing the determinant using this method is slow with runtime $\mathcal{O}(n \cdot n!)$.

# Laplace expansion

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**Theorem**

The formula for the (1st row) Laplace expansion of $A \in \mathbb{M}(n)$ is given as:

$$\det(A) = \sum_{j=1}^{n} a_{1,j}\, C_{1,j}$$

where $C_{i,j} = (-1)^{i+j} \det\left(A\langle i, j\rangle\right)$ is the $(i, j)$ cofactor of $A$.

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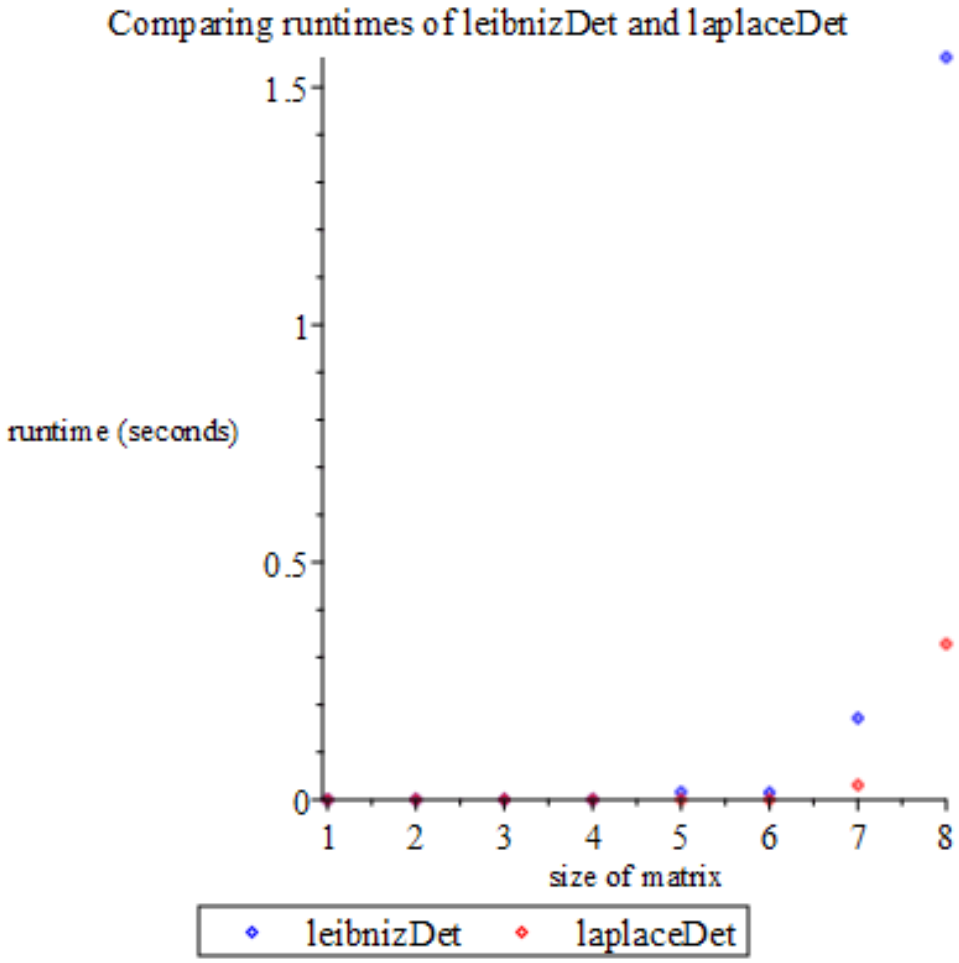
Its runtime complexity of $\mathcal{O}(n!)$ is poor.

# Laplace expansion vs Leibniz formula




Runtimes are similar — both run in exponential time.

# What is LU decomposition?

**Definition**

An LU decomposition of an invertible matrix $A$ is a factorization

$$A = LU$$

where $L$ and $U$ are lower and upper triangular matrices, respectively.

# Is there always an LU decomposition?

No.

An LU decomposition of $A$ exists if and only if each of its *leading principle minors* (contiguous square submatrices in the top-left corner of $A$), are also invertible.

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## Example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} l_{1,1} & 0 \\ l_{2,1} & l_{2,2} \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix}$$

Here we have $a_{1,1} = l_{1,1}u_{1,1} = 0$.

This matrix is invertible but has no LU decomposition.

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# PLU decomposition

*Partial pivoting*.
We can pivot the matrix into the correct form by multiplication with an orthogonal, permutation matrix $P$ (representing a permutation $\sigma_P$) which gives us the PLU decomposition:

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This technique works on *any* matrix.

# How it helps us compute determinants

Now that we have $PA = LU$, it follows that

$$\begin{aligned} A &= P^{-1}LU \\ &= P^{T}LU \end{aligned}$$

since $P^{-1} = P^{T}$ by the definition of orthogonal matrices.

# How it helps us compute determinants (cont.)

Now that we have $A = P^T LU$, it follows that

$$\begin{aligned} \det(A) &= \det(P^T LU) \\ &= \det(P^T) \cdot \det(L) \cdot \det(U) && \text{(Thm. 4.4.1)} \\ &= \det(P) \cdot \det(L) \cdot \det(U) && \text{(Lem. 4.4.4)} \end{aligned}$$

Given that

- the determinant of a triangular matrix is the product of its diagonal elements
- the determinant of a permutation matrix ($P$) is the parity of the permutation it represents ($\sigma_P$)

it follows that

$$\det(A) = \operatorname{sgn}(\sigma_P) \cdot \left( \prod_{i=1}^{n} l_{i,i} \right) \cdot \left( \prod_{i=1}^{n} u_{i,i} \right)$$

# How do we find the PLU decomposition?

**Input:** $\mathbf{A} \in \mathbb{R}^{n \times n}$

**Output:** $\mathbf{L}, \mathbf{U}, \mathbf{P} \in \mathbb{R}^{n \times n}$, with $\mathbf{PA} = \mathbf{LU}$, $\mathbf{L}$ unit lower triangular, $\mathbf{U}$ non-singular upper triangular, and $\mathbf{P}$ a permutation matrix

1: $\mathbf{U} \leftarrow \mathbf{A}, \mathbf{L} \leftarrow \mathbf{I}, \mathbf{P} \leftarrow \mathbf{I}$
2: **for** $k \leftarrow 1, \ldots, n-1$ **do** ▷ Loop over columns
3: Choose $i \in \{k, \ldots, n\}$ which maximises $|u_{ik}|$
4: Exchange row $(u_{kk}, \ldots, u_{kn})$ with $(u_{ik}, \ldots, u_{in})$ ▷ Col. 1 to $k$ have zeros below the diagonal
5: Exchange row $(l_{k1}, \ldots, l_{k,k-1})$ with $(l_{i1}, \ldots, l_{ik-1})$ ▷ $\mathbf{L}$ has unit diagonal, zeros above and below the diagonal in columns $k+1$ to $n$
6: Exchange row $(p_{k1}, \ldots, p_{kn})$ with $(p_{i1}, \ldots, p_{in})$
7: **for** $j \leftarrow k+1, \ldots, n$ **do**
8: $l_{jk} \leftarrow u_{jk}/u_{kk}$ ▷ $u_{kk}$ now largest possible
9: $(u_{jk}, \ldots, u_{jn}) \leftarrow (u_{jk}, \ldots, u_{jn}) - l_{jk}(u_{kk}, \ldots, u_{kn})$
10: **end for**
11: **end for**

This algorithm only works on invertible matrices (line 8 division).

# How do we find the PLU decomposition? (cont.)

$$A = \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$$

$$PA = LU$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$$

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$$R_2 - (-\frac{1}{2})R_1.$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 0 & \frac{5}{2} \end{pmatrix}$$

# How do we find the PLU decomposition? (cont.)

So we have

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}, U = \begin{pmatrix} -4 & 3 \\ 0 & \frac{5}{2} \end{pmatrix} \text{ and } \operatorname{sgn}(\sigma_P) = -1.$$

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Thus we have

$$\det(A) = -1 \cdot -4 \cdot \frac{5}{2} = 10.$$

# Runtime analysis

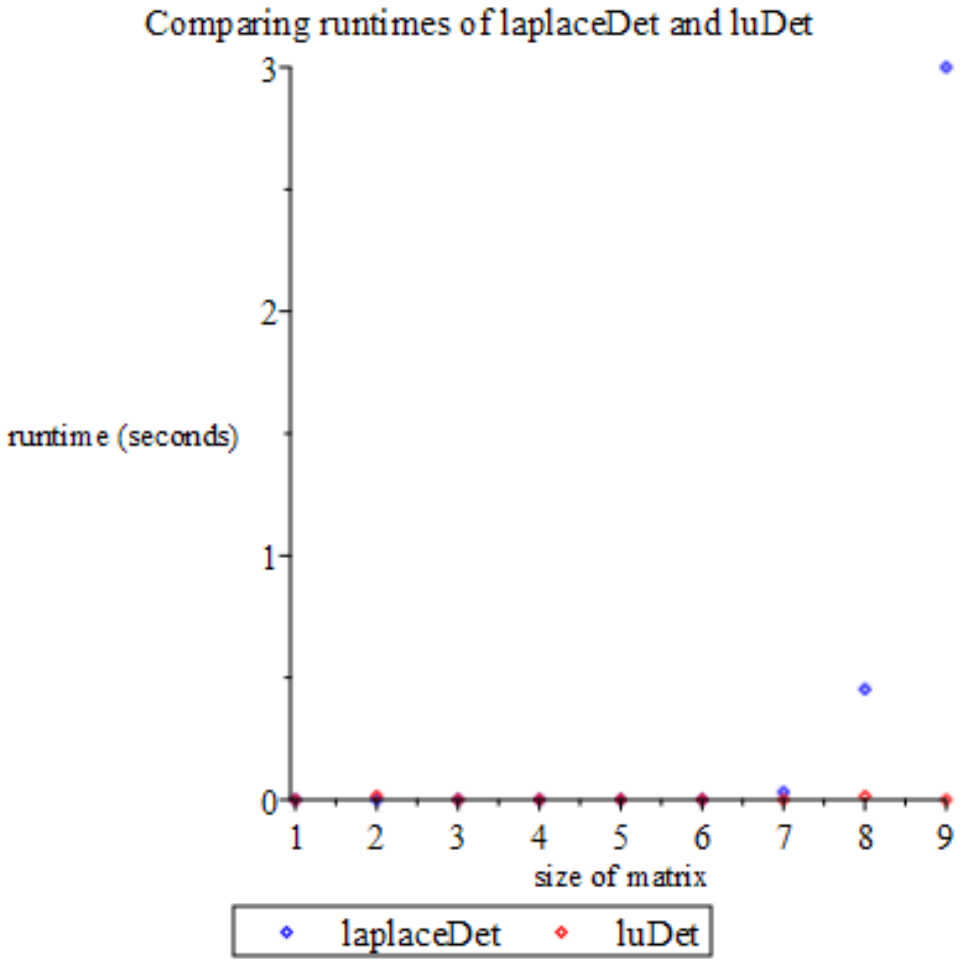
How quick is it?

- The PLU decomposition can be computed in $\mathcal{O}(n^3)$ time.
- The determinants of the triangular matrices computed in $\mathcal{O}(n)$ time.
- The parity of the permutation matrix in $\mathcal{O}(n^2)$ time.

Therefore the total runtime for computing the determinant using the method is

$$\mathcal{O}(n^3) + \mathcal{O}(n^2) + \mathcal{O}(n) = \mathcal{O}(n^3).$$

# Laplace expansion vs LU decomposition




The difference between the exponential and polynomial-time function is clear.

# Limitations of LU decomposition

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Let's try something else...

# Bird's algorithm

Define $\mu : \mathbb{M}(n) \to \mathbb{M}(n)$:

$$\mu(X) = \begin{pmatrix} \mu_{2,2} - x_{2,2} & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & \mu_{3,3} - x_{3,3} & \cdots & x_{2,n-1} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{n,n} - x_{n,n} & x_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

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and $F_A : \mathbb{M}(n) \to \mathbb{M}(n)$, with $A \in \mathbb{M}(n)$

$$\begin{aligned} F_A(X) &= \mu(X) \cdot A \\ F_A^2(X) &= \mu(F_A(X)) \cdot A \\ &\vdots \\ F_A^n(X) &= \mu(F_A^{n-1}(X)) \cdot A. \end{aligned}$$

# Bird's algorithm (cont.)

## Bird's Theorem

$$F_A^{n-1}(A) = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } d = \begin{cases} \det(A) & \text{odd } n \\ -\det(A) & \text{even } n. \end{cases}$$

# Bird's algorithm (cont.)

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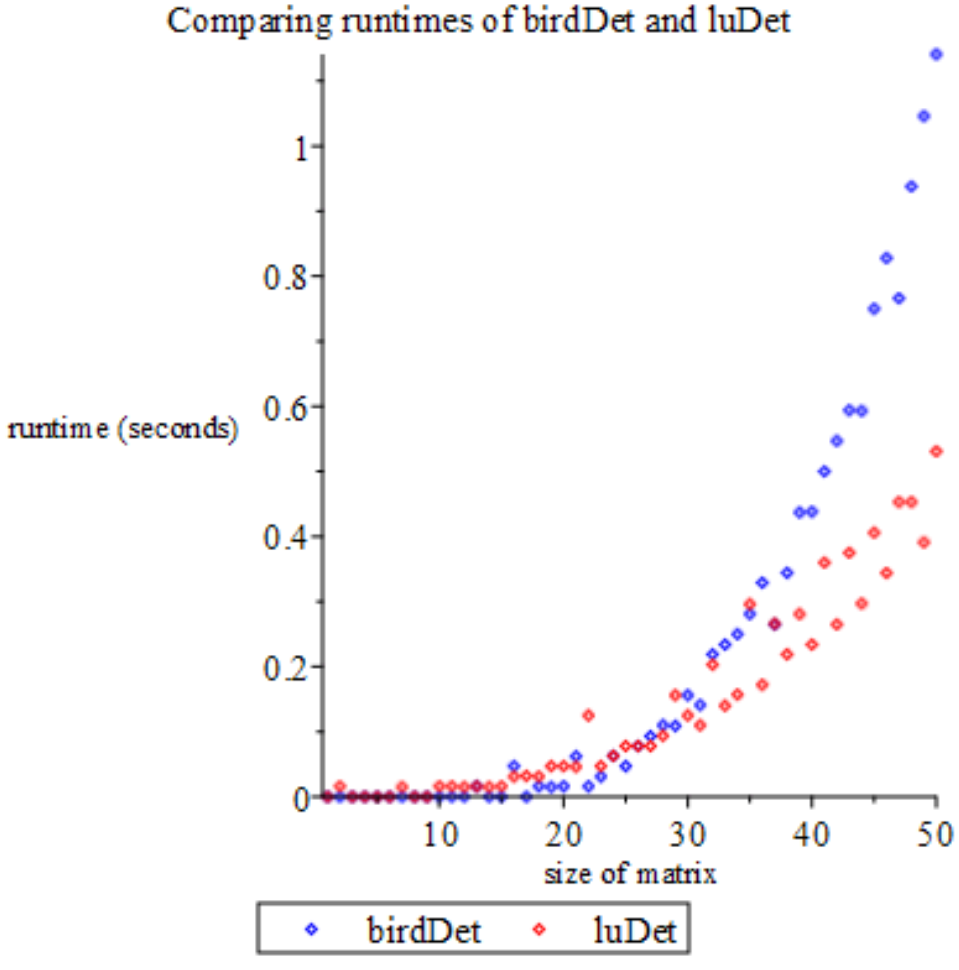
- Enables the *division-free* computation of determinants in $\mathcal{O}(n \cdot M(n))$ where $M(n)$ is the runtime complexity of the matrix multiplication algorithm used.
- If the conventional $\mathcal{O}(n^3)$ matrix multiplication algorithm is used, then Bird's algorithm will run in $\mathcal{O}(n^4)$ time.
- But this can be reduced to $\mathcal{O}(n^{3.8})$ by using a faster (e.g. *Strassen*) algorithm for matrix multiplication.

# Bird's algorithm vs LU decomposition

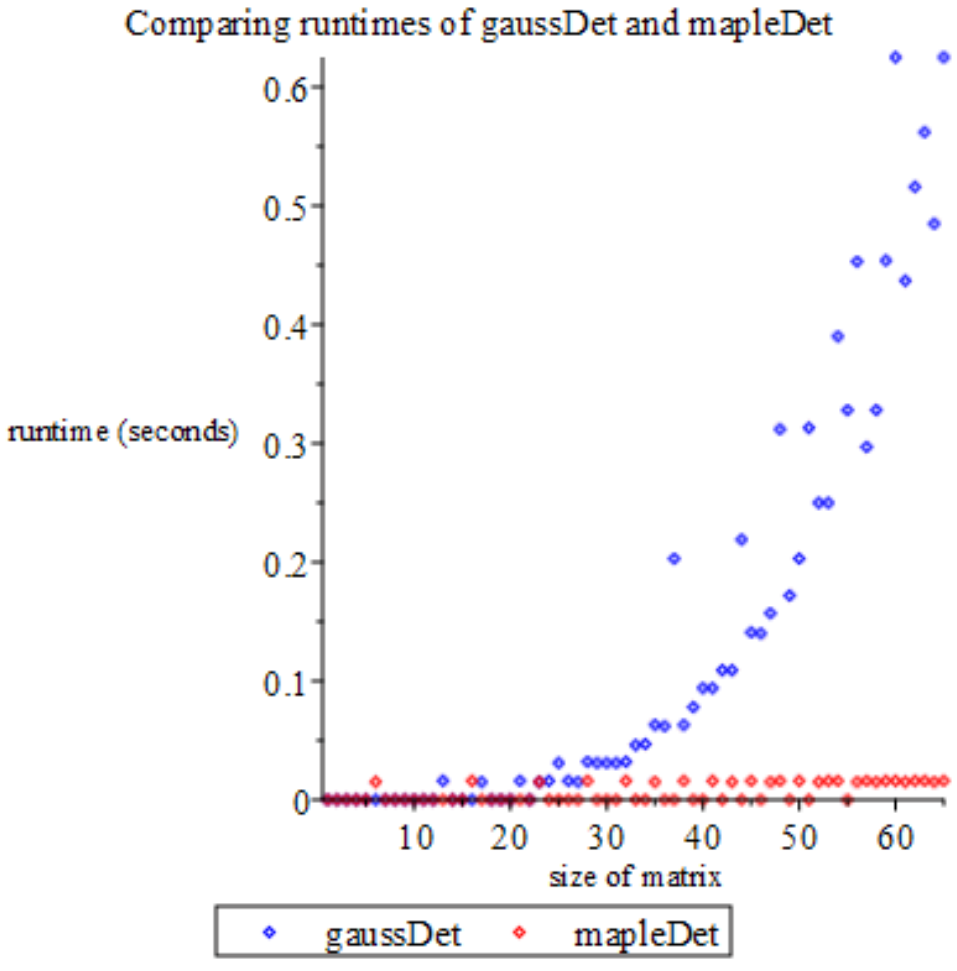



Bird's runtimes increase noticeably more rapidly than LU decomposition, but it's still polynomial.

# Summary of determinant algorithms

| *Algorithm* | *Runtime* | *Exact?* |
|---|---|---|
| Leibniz formula | $\mathcal{O}(n \cdot n!)$ | Yes |
| Laplace expansion | $\mathcal{O}(n!)$ | Yes |
| LU decomposition | $\mathcal{O}(n^3)$ | No |
| Bird's algorithm | $\mathcal{O}(n^{3.8})$ | Yes |

# How fast is Maple's built-in determinant function?




Very. Maple's optimisation means a fair comparison cannot be made.

# Thanks!