Computing the determinant

Group 12

Catterwell, A. Smith, M. Wang, R. Watson, K.

University of Edinburgh

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Overview

1. Why calculate determinants?
   - Invertibility of matrices
   - Cramer's Rule
   - Eigenvalues and Eigenvectors
   - Jacobian determinant
2. Algorithms for computing determinants
   - Leibniz formula
   - Laplace expansion
   - LU decomposition
   - Bird's algorithm
3. Epilogue
   - Summary of determinant algorithms
   - How fast is Maple's implementation?

Invertibililty of matrices

Theorem

An $n \times n$ square matrix A is invertible if and only if

$$\det(A) \neq 0.$$

Cramer's Rule

Theorem

Given an equation $A\mathbf{x} = \mathbf{b}$ The solutions for $\mathbf{x}$ are given by

$$x_i = \frac{\det(A_i)}{\det(A)}$$

with A_i being the matrix formed by replacing the ith column of A by $\mathbf{b}$.

It turns out this method has the same runtime complexity as Gaussian elimination for solving systems of linear equations.

Eigenvalues and Eigenvectors

Definition

The Eigenvalues λ of a matrix A are the roots of the characteristic polynomial as defined

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The Eigenvectors of A are the vectors $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Eigenvalues and Eigenvectors (cont.)

The absolute value of the determinant of real vectors is equal to the volume of the parallelepiped spanned by those vectors.
$f : \mathbb{R}^n \to \mathbb{R}^n$: the linear map represented by the A. S: any measurable subset of $\mathbb{R}^n$.

$$\text{volume}(f(S)) = |\det(A^T A)| \times \text{volume}(S).$$

Jacobian determinant

For $f : \mathbb{R}^n \to \mathbb{R}^n$, the Jacobian matrix is the $n \times n$ matrix whose entries are defined as

$$D(f) = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq n.}$$

Its determinant is known as the *Jacobian determinant*.

If the determinant of a continuously differentiable function f at a point p is. . .

- Non-zero, f is invertible near a point p in $\mathbb{R}^n$.
- Positive, then f preserves orientation near p.
- Negative, then f reverses orientation near p.

Big-O notation

We'll be looking at the runtime of algorithms, so this is useful.

Definition

A function f is said to be $\mathcal{O}(g)$, with g a function iff

$$\exists k \in \mathbb{R} \ni f(n) < k \cdot g(n)$$

for sufficiently large n.

Leibniz formula

Definition

The Leibniz formula defines the determinant of $A \in \mathbb{M}(n)$ as

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \left(\operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

where $\mathfrak{S}_n$ is the set of permutations length n.

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where $\mathfrak{S}_n$ is the set of permutations length n.

Computing the determinant using this method is slow with runtime $\mathcal{O}(n \cdot n!)$.

Laplace expansion

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Theorem

The formula for the (1st row) Laplace expansion of $A \in \mathbb{M}(n)$ is given as:

$$\det(A) = \sum_{j=1}^{n} a_{1,j}\, C_{1,j}$$

where $C_{i,j} = (-1)^{i+j} \det\left(A\langle i,j\rangle\right)$ is the (i,j) cofactor of A.

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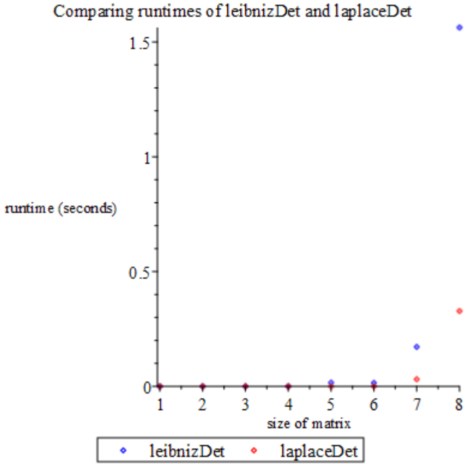
Its runtime complexity of $\mathcal{O}(n!)$ is poor.

Laplace expansion vs Leibniz formula



Runtimes are similar — both run in exponential time.

What is LU decomposition?

Definition

An LU decomposition of an invertible matrix A is a factorization

$$A = LU$$

where L and U are lower and upper triangular matrices, respectively.

Is there always an LU decomposition?

No.

An LU decomposition of A exists if and only if each of its *leading principle minors* (contiguous square submatrices in the top-left corner of A), are also invertible.

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Example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} l_{1,1} & 0 \\ l_{2,1} & l_{2,2} \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix}$$

Here we have $a_{1,1} = l_{1,1} u_{1,1} = 0$.

This matrix is invertible but has no LU decomposition.

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PLU decomposition

Partial pivoting.
We can pivot the matrix into the correct form by multiplication with an orthogonal, permutation matrix P (representing a permutation σ_P) which gives us the PLU decomposition:

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This technique works on *any* matrix.

How it helps us compute determinants

Now that we have $PA = LU$, it follows that

$$\begin{aligned} A &= P^{-1}LU \\ &= P^{T}LU \end{aligned}$$

since $P^{-1} = P^{T}$ by the definition of orthogonal matrices.

How it helps us compute determinants (cont.)

Now that we have $A = P^T LU$, it follows that

$$\begin{aligned} \det(A) &= \det(P^T LU) \\ &= \det(P^T) \cdot \det(L) \cdot \det(U) && \text{(Thm. 4.4.1)} \\ &= \det(P) \cdot \det(L) \cdot \det(U) && \text{(Lem. 4.4.4)} \end{aligned}$$

Given that

- the determinant of a triangular matrix is the product of its diagonal elements
- the determinant of a permutation matrix (P) is the parity of the permutation it represents (σ_P)

it follows that

$$\det(A) = \operatorname{sgn}(\sigma_P) \cdot \left(\prod_{i=1}^{n} l_{i,i}\right) \cdot \left(\prod_{i=1}^{n} u_{i,i}\right)$$

How do we find the PLU decomposition?

Input: $\mathbf{A} \in \mathbb{R}^{n\times n}$

Output: $\mathbf{L}, \mathbf{U}, \mathbf{P} \in \mathbb{R}^{n\times n}$, with $\mathbf{PA} = \mathbf{LU}$, $\mathbf{L}$ unit lower triangular, $\mathbf{U}$ non-singular upper triangular, and $\mathbf{P}$ a permutation matrix

1: $\mathbf{U} \leftarrow \mathbf{A}, \mathbf{L} \leftarrow \mathbf{I}, \mathbf{P} \leftarrow \mathbf{I}$
2: **for** $k \leftarrow 1, \dots, n-1$ **do** ▷ Loop over columns
3: Choose $i \in \{k, \dots, n\}$ which maximises $|u_{ik}|$
4: Exchange row $(u_{kk}, \dots, u_{kn})$ with $(u_{ik}, \dots, u_{in})$ ▷ Col. 1 to k have zeros below the diagonal
5: Exchange row $(l_{k1}, \dots, l_{k,k-1})$ with $(l_{i1}, \dots, l_{ik-1})$ ▷ $\mathbf{L}$ has unit diagonal, zeros above and below the diagonal in columns $k+1$ to n
6: Exchange row $(p_{k1}, \dots, p_{kn})$ with $(p_{i1}, \dots, p_{in})$
7: **for** $j \leftarrow k+1, \dots, n$ **do**
8: $l_{jk} \leftarrow u_{jk}/u_{kk}$ ▷ u_{kk} now largest possible
9: $(u_{jk}, \dots, u_{jn}) \leftarrow (u_{jk}, \dots, u_{jn}) - l_{jk}(u_{kk}, \dots, u_{kn})$
10: **end for**
11: **end for**

This algorithm only works on invertible matrices (line 8 division).

How do we find the PLU decomposition? (cont.)

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{pmatrix}$$
$$PA = LU$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$$

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$$\frac{U(2,1)}{U(1,1)} = -\frac{1}{2}. \text{ So we do } R_2 - (-\frac{1}{2})R_1.$$

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 0 & \frac{5}{2} \end{pmatrix}$$

How do we find the PLU decomposition? (cont.)

So we have

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}, U = \begin{pmatrix} -4 & 3 \\ 0 & \frac{5}{2} \end{pmatrix} \text{ and } \operatorname{sgn}(\sigma_P) = -1.$$

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Thus we have

$$\det(A) = \det \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \operatorname{sgn}(\sigma_P) \cdot \left(\prod_{i=1}^{n} l_{i,i} \right) \cdot \left(\prod_{i=1}^{n} u_{i,i} \right)$$

$$= -1 \cdot -4 \cdot \frac{5}{2} = 10.$$

Runtime analysis

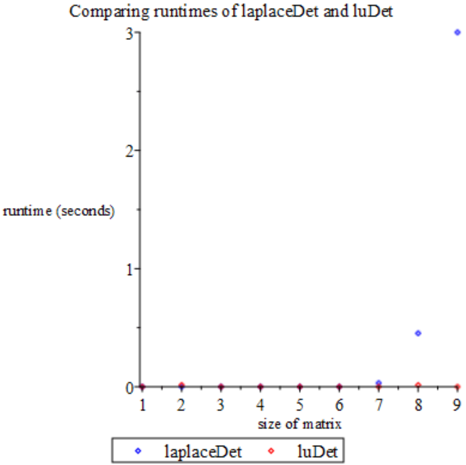
How quick is it?

- The PLU decomposition can be computed in $\mathcal{O}(n^3)$ time.
- The determinants of the triangular matrices computed in $\mathcal{O}(n)$ time.
- The parity of the permutation matrix in $\mathcal{O}(n^2)$ time.

Therefore the total runtime for computing the determinant using the method is

$$\mathcal{O}(n^3) + \mathcal{O}(n^2) + \mathcal{O}(n) = \mathcal{O}(n^3).$$

Laplace expansion vs LU decomposition



The difference between the exponential and polynomial-time function is clear.

Limitations of LU decomposition

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Let's try something else. . .

Bird's algorithm

Define $\mu : \mathbb{M}(n) \to \mathbb{M}(n)$:

$$\mu(X) = \begin{pmatrix} \mu_{2,2} - x_{2,2} & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & \mu_{3,3} - x_{3,3} & \cdots & x_{2,n-1} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{n,n} - x_{n,n} & x_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

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and $F_A : \mathbb{M}(n) \to \mathbb{M}(n)$, with $A \in \mathbb{M}(n)$

$$\begin{aligned} F_A(X) &= \mu(X) \cdot A \\ F_A^2(X) &= \mu(F_A(X)) \cdot A \\ &\vdots \\ F_A^n(X) &= \mu(F_A^{n-1}(X)) \cdot A. \end{aligned}$$

Bird's algorithm (cont.)

Bird's Theorem

$$F_A^{n-1}(A) = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } d = \begin{cases} \det(A) & \text{odd } n \\ -\det(A) & \text{even } n. \end{cases}$$

Bird's algorithm (cont.)

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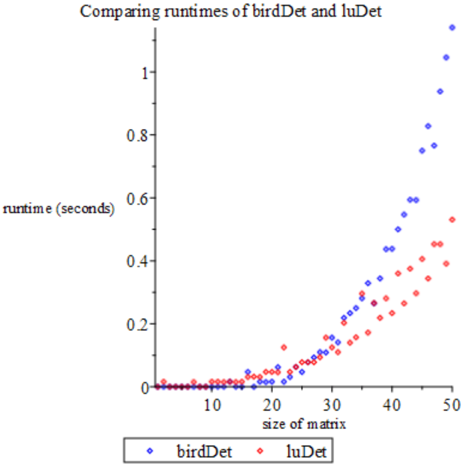
- Enables the *division-free* computation of determinants in $\mathcal{O}(n \cdot M(n))$ where $M(n)$ is the runtime complexity of the matrix multiplication algorithm used.
- If the conventional $\mathcal{O}(n^3)$ matrix multiplication algorithm is used, then Bird's algorithm will run in $\mathcal{O}(n^4)$ time.
- But this can be reduced to $\mathcal{O}(n^{3.8})$ by using a faster (e.g. *Strassen*) algorithm for matrix multiplication.

Bird’s algorithm vs LU decomposition

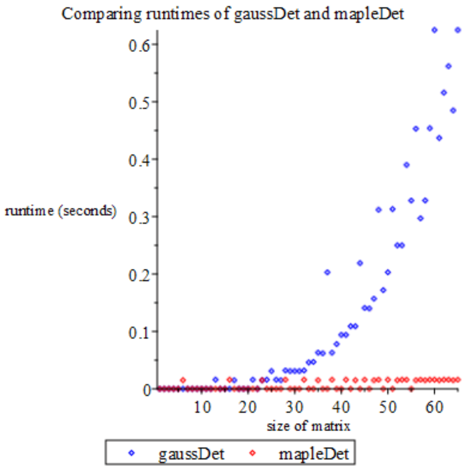


Bird’s runtimes increase noticeably more rapidly than LU decomposition, but it’s still polynomial.

Summary of determinant algorithms

Algorithm	*Runtime*	*Exact?*
Leibniz formula	$\mathcal{O}(n \cdot n!)$	Yes
Laplace expansion	$\mathcal{O}(n!)$	Yes
LU decomposition	$\mathcal{O}(n^3)$	No
Bird's algorithm	$\mathcal{O}(n^{3.8})$	Yes

How fast is Maple's built-in determinant function?



Very. Maple's optimisation means a fair comparison cannot be made.

Thanks!