

Computing the determinant

Group 12

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Overview

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 - Cramer's Rule
 - Eigenvalues and Eigenvectors
 - Jacobian determinant
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 - Laplace expansion
 - LU decomposition
 - Bird's algorithm
- 3 Epilogue
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Invertibility of matrices

Theorem

An $n \times n$ square matrix A is invertible if and only if

$$\det(A) \neq 0.$$

Cramer's Rule

Theorem

Given an equation $A\mathbf{x} = \mathbf{b}$ The solutions for \mathbf{x} are given by

$$x_i = \frac{\det(A_i)}{\det(A)}$$

with A_i being the matrix formed by replacing the i th column of A by \mathbf{b} .

It turns out this method has the same runtime complexity as Gaussian elimination for solving systems of linear equations.

Eigenvalues and Eigenvectors

Definition

The Eigenvalues λ of a matrix A are the roots of the characteristic polynomial as defined

$$\chi_A = \det(A - \lambda I) = \mathbf{0}.$$

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The Eigenvectors of A are the vectors \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Eigenvalues and Eigenvectors (cont.)

The absolute value of the determinant of real vectors is equal to the volume of the parallelepiped spanned by those vectors.

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$: the linear map represented by the A . S : any measurable subset of \mathbb{R}^n .

$$\text{volume}(f(S)) = |\det(A^T A)| \times \text{volume}(S).$$

Jacobian determinant

For $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the Jacobian matrix is the $n \times n$ matrix whose entries are defined as

$$D(f) = \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i, j \leq n}.$$

Its determinant is known as the *Jacobian determinant*.

If the determinant of a continuously differentiable function f at a point p is...

- Non-zero, f is invertible near a point p in \mathbb{R}^n .
- Positive, then f preserves orientation near p .
- Negative, then f reverses orientation near p .

Big-O notation

We'll be looking at the runtime of algorithms, so this is useful.

Definition

A function f is said to be $\mathcal{O}(g)$, with g a function iff

$$\exists k \in \mathbb{R} \ni f(n) < k \cdot g(n)$$

for sufficiently large n .

Leibniz formula

Definition

The Leibniz formula defines the determinant of $A \in \mathbb{M}(n)$ as

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \left(\operatorname{sgn}(\sigma) \cdot \prod_{i=1}^n a_{i, \sigma(i)} \right)$$

where \mathfrak{S}_n is the set of permutations length n .

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where \mathfrak{S}_n is the set of permutations length n .

Computing the determinant using this method is slow with runtime $\mathcal{O}(n \cdot n!)$.

Laplace expansion

The Laplace (1st row) expansion for computing determinants is usually the first method taught for computing determinants of 3×3 matrices and larger.

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Theorem

The formula for the (1st row) Laplace expansion of $A \in \mathbb{M}(n)$ is given as:

$$\det(A) = \sum_{j=1}^n a_{1,j} C_{1,j}$$

where $C_{i,j} = (-1)^{i+j} \det(A\langle i,j \rangle)$ is the (i,j) cofactor of A .

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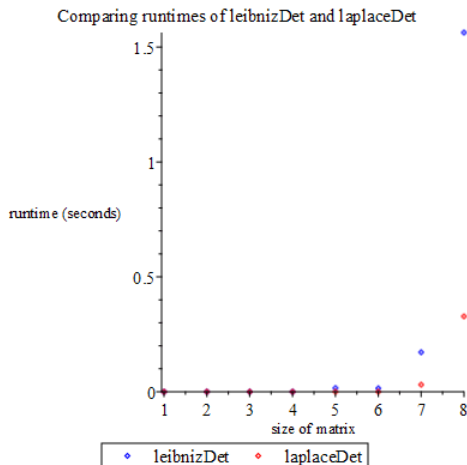
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Its runtime complexity of $\mathcal{O}(n!)$ is poor.

Laplace expansion vs Leibniz formula



Runtimes are similar — both run in exponential time.

What is LU decomposition?

Definition

An LU decomposition of an invertible matrix A is a factorization

$$A = LU$$

where L and U are lower and upper triangular matrices, respectively.

Is there always an LU decomposition?

No.

An LU decomposition of A exists if and only if each of its *leading principle minors* (contiguous square submatrices in the top-left corner of A), are also invertible.

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Example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \textcolor{red}{a}_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \textcolor{red}{l}_{1,1} & 0 \\ l_{2,1} & l_{2,2} \end{pmatrix} \begin{pmatrix} \textcolor{red}{u}_{1,1} & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix}$$

Here we have $a_{1,1} = l_{1,1}u_{1,1} = 0$.

This matrix is invertible but has no LU decomposition.

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PLU decomposition

Partial pivoting.

We can pivot the matrix into the correct form by multiplication with an orthogonal, permutation matrix P (representing a permutation σ_P) which gives us the PLU decomposition:

$$\sigma_P(A) = PA = LU$$

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We can pivot the matrix into the correct form by multiplication with an orthogonal, permutation matrix P (representing a permutation σ_P) which gives us the PLU decomposition:

$$\sigma_P(A) = PA = LU$$

This technique works on *any* matrix.

How it helps us compute determinants

Now that we have $PA = LU$, it follows that

$$\begin{aligned} A &= P^{-1}LU \\ &= P^T LU \end{aligned}$$

since $P^{-1} = P^T$ by the definition of orthogonal matrices.

How it helps us compute determinants (cont.)

Now that we have $A = P^T L U$, it follows that

$$\begin{aligned}\det(A) &= \det(P^T L U) \\ &= \det(P^T) \cdot \det(L) \cdot \det(U) && \text{(Thm. 4.4.1)} \\ &= \det(P) \cdot \det(L) \cdot \det(U) && \text{(Lem. 4.4.4)}\end{aligned}$$

Given that

- the determinant of a triangular matrix is the product of its diagonal elements
- the determinant of a permutation matrix (P) is the parity of the permutation it represents (σ_P)

it follows that

$$\det(A) = \operatorname{sgn}(\sigma_P) \cdot \left(\prod_{i=1}^n l_{i,i} \right) \cdot \left(\prod_{i=1}^n u_{i,i} \right)$$

How do we find the PLU decomposition?

Input: $A \in \mathbb{R}^{n \times n}$

Output: $L, U, P \in \mathbb{R}^{n \times n}$, with $PA = LU$, L unit lower triangular, U non-singular upper triangular, and P a permutation matrix

```
1:  $U \leftarrow A, L \leftarrow I, P \leftarrow I$ 
2: for  $k \leftarrow 1, \dots, n-1$  do                                ▷ Loop over columns
3:   Choose  $i \in \{k, \dots, n\}$  which maximises  $|u_{ik}|$ 
4:   Exchange row  $(u_{kk}, \dots, u_{kn})$  with  $(u_{ik}, \dots, u_{in})$     ▷ Col. 1 to  $k$  have zeros below the diagonal
5:   Exchange row  $(l_{k1}, \dots, l_{k,k-1})$  with  $(l_{i1}, \dots, l_{i,k-1})$     ▷  $L$  has unit diagonal, zeros above and
                                                below the diagonal in columns  $k+1$  to  $n$ 
6:   Exchange row  $(p_{k1}, \dots, p_{kn})$  with  $(p_{i1}, \dots, p_{in})$ 
7:   for  $j \leftarrow k+1, \dots, n$  do
8:      $l_{jk} \leftarrow u_{jk}/u_{kk}$                                 ▷  $u_{kk}$  now largest possible
9:      $(u_{jk}, \dots, u_{jn}) \leftarrow (u_{jk}, \dots, u_{jn}) - l_{jk}(u_{kk}, \dots, u_{kn})$ 
10:  end for
11: end for
```

This algorithm only works on invertible matrices (line 8 division).

How do we find the PLU decomposition? (cont.)

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{pmatrix}$$

$$PA = LU$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$$

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$$R_2 - (-\frac{1}{2})R_1.$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 0 & \frac{5}{2} \end{pmatrix}$$

How do we find the PLU decomposition? (cont.)

So we have

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}, U = \begin{pmatrix} -4 & 3 \\ 0 & \frac{5}{2} \end{pmatrix} \text{ and } \text{sgn}(\sigma_P) = -1.$$

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Thus we have

$$\det(A) = -1 \cdot -4 \cdot \frac{5}{2} = 10.$$

Runtime analysis

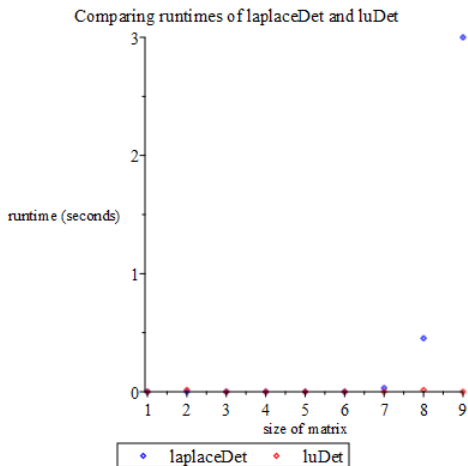
How quick is it?

- The PLU decomposition can be computed in $\mathcal{O}(n^3)$ time.
- The determinants of the triangular matrices computed in $\mathcal{O}(n)$ time.
- The parity of the permutation matrix in $\mathcal{O}(n^2)$ time.

Therefore the total runtime for computing the determinant using the method is

$$\mathcal{O}(n^3) + \mathcal{O}(n^2) + \mathcal{O}(n) = \mathcal{O}(n^3).$$

Laplace expansion vs LU decomposition



The difference between the exponential and polynomial-time function is clear.

Limitations of LU decomposition

The main problem with the LU decomposition algorithm used is that it often requires division.

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Let's try something else...

Bird's algorithm

Define $\mu : \mathbb{M}(n) \rightarrow \mathbb{M}(n)$:

$$\mu(X) = \begin{pmatrix} \mu_{2,2} - x_{2,2} & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & \mu_{3,3} - x_{3,3} & \cdots & x_{2,n-1} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{n,n} - x_{n,n} & x_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

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and $F_A : \mathbb{M}(n) \rightarrow \mathbb{M}(n)$, with $A \in \mathbb{M}(n)$

$$F_A(X) = \mu(X) \cdot A$$

$$F_A^2(X) = \mu(F_A(X)) \cdot A$$

$$\vdots$$

$$F_A^n(X) = \mu(F_A^{n-1}(X)) \cdot A.$$

Bird's algorithm (cont.)

Bird's Theorem

$$F_A^{n-1}(A) = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } d = \begin{cases} \det(A) & \text{odd } n \\ -\det(A) & \text{even } n. \end{cases}$$

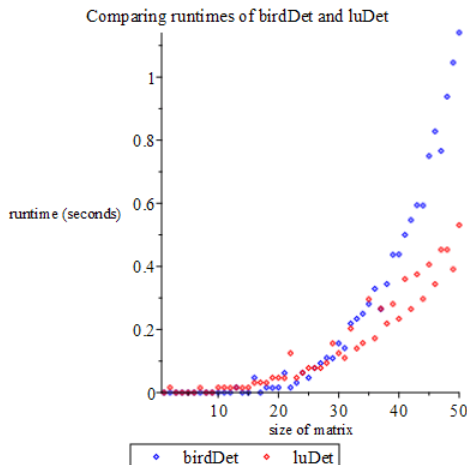
Bird's algorithm (cont.)

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- Enables the *division-free* computation of determinants in $\mathcal{O}(n \cdot M(n))$ where $M(n)$ is the runtime complexity of the matrix multiplication algorithm used.
- If the conventional $\mathcal{O}(n^3)$ matrix multiplication algorithm is used, then Bird's algorithm will run in $\mathcal{O}(n^4)$ time.
- But this can be reduced to $\mathcal{O}(n^{3.8})$ by using a faster (e.g. *Strassen*) algorithm for matrix multiplication.

Bird's algorithm vs LU decomposition

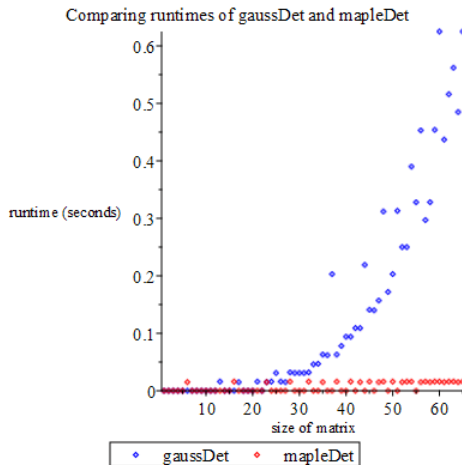


Bird's runtimes increase noticeably more rapidly than LU decomposition, but it's still polynomial.

Summary of determinant algorithms

<i>Algorithm</i>	<i>Runtime</i>	<i>Exact?</i>
Leibniz formula	$\mathcal{O}(n \cdot n!)$	Yes
Laplace expansion	$\mathcal{O}(n!)$	Yes
LU decomposition	$\mathcal{O}(n^3)$	No
Bird's algorithm	$\mathcal{O}(n^{3.8})$	Yes

How fast is Maple's built-in determinant function?



Very. Maple's optimisation means a fair comparison cannot be made.

Thanks!