Determinants

Group 12

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Overview

1. Using LU decomposition to compute determinants
   - What is LU decomposition?
   - How it helps us compute determinants
2. Other algorithms for computing determinants
   - Leibniz formula
   - Laplace expansion
   - Gaussian elimination
   - Bareiss algorithm
   - Bird's algorithm
3. Epilogue
   - Summary of determinant algorithms
   - How fast is Maple's implementation?

What is LU decomposition?

Somebody else do this section

- What LU decomposition is
- Its limitations
- How PLU decomposition addresses these limitations

How it helps us compute determinants

some maths.

Leibniz formula

Definition

The Leibniz formula defines the determinant of $A \in \mathbb{M}(n)$ as

$$|A| = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

where $\mathfrak{S}_n$ is the set of permutations length n.

Computing the determinant using this method is slow with runtime $\mathcal{O}((N+1)!)$.

Laplace expansion

The Laplace (1st row) expansion for computing determinants is usually the first method taught for computing determinants of 3×3 matrices and larger.

Theorem

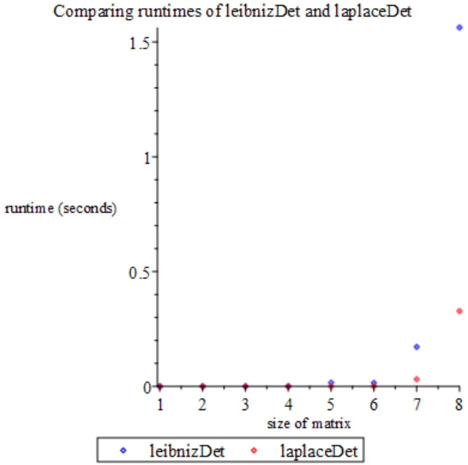
The formula for the (1st row) Laplace expansion of $A \in \mathbb{M}(n)$ is given as:

$$|A| = \sum_{j=1}^{n} a_{1j} \cdot C_{1j}$$

where C_{ij} is the (i, j) cofactor of A.

Its runtime complexity of $\mathcal{O}(N!)$ is poor.

Laplace expansion vs Leibniz formula



Runtimes are similar — both run in exponential time.

Laplace expansion vs LU decomposition

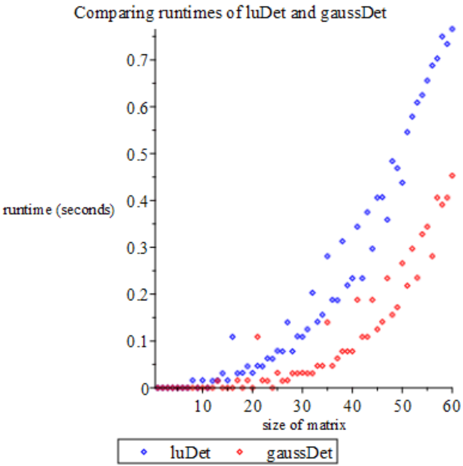
The difference between exponential and polynomial-time functions is clear.

Gaussian elimination

- The determinant of a triangular matrix can be computed by taking the product of its diagonal entries (which is a quick $\mathcal{O}(N)$ operation).
- Any invertible square matrix can be transformed into echelon form by performing Gaussian elimination, which takes $\mathcal{O}(N^3)$ time.

So how does it compare to LU decomposition?

Gaussian elimination vs LU decomposition



The difference in runtimes is small (a constant factor).

Gaussian elimination (cont.)

Conventional Gaussian elimination requires division, meaning that solutions maybe inexact, so precision is lost.
This is addressed by...

Bareiss algorithm

- Addresses the issue of precision-loss by performing *integer-preserving* Gaussian elimination on integer matrices.
- The runtime complexity is $\mathcal{O}(N^3)$ which is the same as conventional Gaussian Elimination, whilst preserving exactness.

Bird's algorithm

Define $\mu : \mathbb{M}(n) \to \mathbb{M}(n)$:

$$\mu(X) = \begin{bmatrix} \mu_{2,2} - x_{2,2} & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & \mu_{3,3} - x_{3,3} & \cdots & x_{2,n-1} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{n,n} - x_{n,n} & x_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and $F_A : \mathbb{M}(n) \to \mathbb{M}(n)$, with $A \in \mathbb{M}(n)$

$$\begin{aligned} F_A(X) &= \mu(X) \cdot A \\ F_A^2(X) &= \mu(F_A(X)) \cdot A \\ &\vdots \\ F_A^n(X) &= \mu(F_A^{n-1}(X)) \cdot A \end{aligned}$$

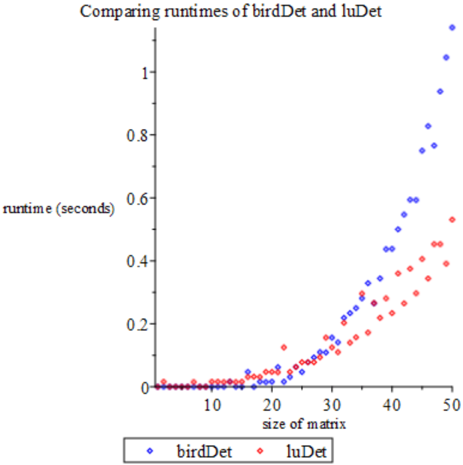
Bird's algorithm (cont.)

Bird's Theorem

$$F_A^{n-1}(A) = \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ with } d = \begin{cases} |A| & \text{odd } n \\ -|A| & \text{even } n \end{cases}$$

- Enables the *division-free* computation of determinants in $\mathcal{O}(n \cdot M(n))$ where $M(n)$ is the runtime complexity of the matrix multiplication algorithm used.
- Given a good matrix multiplication algorithm with runtime $\mathcal{O}(n^{2.376})$ (*Coppersmith-Winograd*), this algorithm runs in $\mathcal{O}(n^{3.376})$.

Bird's algorithm vs LU decomposition

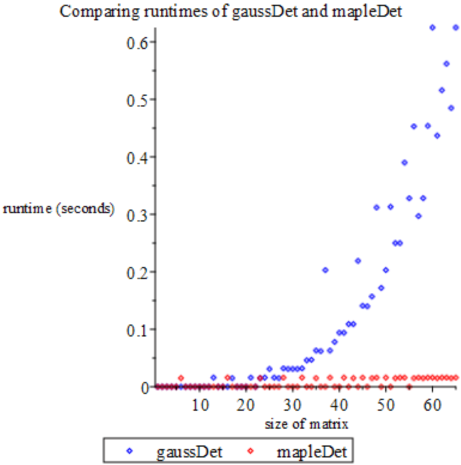


Bird's runtimes increase noticeable more rapidly than LU decomposition, but it's still polynomial.

Summary of determinant algorithms

Algorithm	*Runtime*	*Exact*
Leibniz formula	$\mathcal{O}((N+1)!)$	Yes
Laplace expansion	$\mathcal{O}(N!)$	Yes
LU decomposition	$\mathcal{O}(N^3)$	No
Gaussian elimination	$\mathcal{O}(N^3)$	No
Bareiss' algorithm	$\mathcal{O}(N^3)$	Yes
Bird's algorithm	$\mathcal{O}(N^{3.376})$	Yes

How fast is Maple's built-in determinant function?



Very. Maple's optimisation means a fair comparison cannot be made.

Thanks!