Linear Programming, Modelling & Solution

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Derivation

We have

$$\begin{aligned} f &= \overline{\mathbf{c}}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N \\ \overline{A}\mathbf{x} &= \mathbf{b} = B\mathbf{x}_B + N\mathbf{x}_N \end{aligned}$$

combining the two gives us:

$$\begin{aligned} f &= \widehat{f} + \widehat{\mathbf{c}}_N^T\mathbf{x}_N \\ &= \{\mathbf{c}_B^T\widehat{\mathbf{b}}\} + \{\mathbf{c}_N - N^TB^{-T}\mathbf{c}_B\}\,\mathbf{x}_N \\ &= \{\mathbf{c}_B^T\,B^{-1}\mathbf{b}\} + \{\mathbf{c}_N - N^TB^{-T}\mathbf{c}_B\}\,\mathbf{x}_N \end{aligned}$$

where

$$\begin{aligned} \widehat{\mathbf{c}}_N &= \mathbf{c}_N - N^TB^{-T}\mathbf{c}_B \\ \widehat{f} &= \mathbf{c}_B^T\widehat{\mathbf{b}} \end{aligned}$$

$\widehat{f}$ is the objective value when $\mathbf{x}_N = \mathbf{0}$ (so $\mathbf{x}_B = \widehat{\mathbf{b}}$)

LP Theory: Analysis of LPs in standard form

Definition: Feasible Vertex I

The vertex of the feasible region K is a point $\mathbf{x} \in K$ which does not lie strictly within any line segment joining two points in K.

Theorem 1: A unique optimal solution is a vertex

If an LP has a unique optimal solution then it is a vertex.

Theorem 2: Non-unique optimal solution at a vertex

If an LP has a non-unique optimal solution then there is an optimal solution at a vertex.

Definition: Feasible Vertex II

A vertex of the feasible region K is a point $\mathbf{x} \in K$ with

- n zero components
- m non-negative components uniquely defined by $\overline{A}\mathbf{x} = \mathbf{b}$

LP Theory: Basic feasible solutions and optimality conditions for LP problems

Definition: A basic solution

The point $\mathbf{x} \in \mathbb{R}^{n+m}$ is a **basic solution** of an LP problem in standard form if there is a **partition** of $\{1, 2, \ldots, n+m\}$ into

- A set $\mathcal{N}$ of n indices of **non-basic variables** with value zero
- A set $\mathcal{B}$ of n indices of **basic variables** whose values are then uniquely defined by the m equations.

Why can't we just work with vertices?

- A vertex is also a point with n zero components and m non-negative components uniquely defined by $\overline{A}\mathbf{x} = \mathbf{b}$
- A **degenerate** vertex has more than n zero components
  - More than on partition of $\{1, 2, \ldots, n+m\}$ into sets $\mathcal{N}$ and $\mathcal{B}$ is possible
  - There may be more than one basic solution at a degenerate vertex.

Theorem 3: A sufficient optimality condition for LP problems

A point $\mathbf{x} \in K$ is an optimal solution of an LP problem if it is a basic feasible solution with non-positive **reduced costs** $\widehat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B \leq 0$

The simplex algorithm

Description of the simplex algorithm

1. If the reduced costs are non-positive then **stop**
   The solution is optimal
2. Determine the non-basic variable $x_{q'}$ with the most positive reduced cost
3. Determine the feasible direction $\mathbf{d}$ when $x_{q'}$ is increased from zero
4. If no basic variable is zeroed on $\mathbf{x} + \alpha\mathbf{d}$ then **stop**
   The LP is unbounded
5. Determine the first basic variable $x_{p'}$ to be zeroed on $\mathbf{x} + \alpha\mathbf{d}$
6. Make $x_{p'}$ non-basic and $x_{q'}$ basic
7. Go to 1

Definition of the simplex algorithm

Given a basic feasible solution $\mathbf{x}$ with $\mathcal{B}$ and $\mathcal{N}$

1. If $\widehat{\mathbf{c}}_N \leq \mathbf{0}$ then **stop** (with $\widehat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$)
   The solution is optimal
2. Determine the index $q' \in \mathcal{N}$ of the variable $x_{q'}$ with the most positive reduced cost $\widehat{c}_q$
   q' is the qth entry in $\mathcal{N}$.
3. Let $\widehat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q$, where $\mathbf{a}_q$ is column q of N
4. If $\widehat{\mathbf{a}}_q \leq \mathbf{0}$ then **stop**
   The LP is unbounded

5. Determine the index $p' \in \mathcal{B}$ of the variable $x_{p'}$ corresponding to $p = \text{argmin}_{i=1,\widehat{\mathbf{a}}_{iq}>0}^{m} \frac{\widehat{b}_i}{\widehat{a}_{iq}}$ (with $\widehat{\mathbf{b}} = B^{-1}\mathbf{b}$)
   p' is the pth entry in $\mathcal{B}$
6. Exchange indices p' and q' between $\mathcal{B}$ and $\mathcal{N}$ to yield a new basic feasible solution
7. Go to 1

Obtaining the initial basic feasible solution

As the initial basic feasible solution, try the "all slack" basis (i.e. starting at the origin)

$$\mathcal{B} = \{n+1, \dots, n+m\} \text{ and } \mathcal{N} = \{1, \dots, n\}$$

So we have:

- $\widehat{\mathbf{b}} = \mathbf{b}$
- $\widehat{\mathbf{c}}_N = \mathbf{c}$
- Basis is feasible iff $\mathbf{b} \geq \mathbf{0}$

How to start if $\mathbf{b} \ngeq \mathbf{0}$

Can't use the "all-slack" basis (because the origin is not in the feasible region)

- If $\mathbf{b} \ngeq \mathbf{0}$ then, for each constraint i, subtract an **artificial variable** $x_{n+m+i} \geq 0$
- Replace the objective $f = \overline{\mathbf{c}}^T\mathbf{x}$ with the **Phase I** objective $f = -\sum_{i=1}^{m} x_{n+m+i}$ (i.e. the negated sum of infeasibilities)

The Phase I problem

Construct an initial basic feasible solution as follows: For $i = 1, \dots, m$

If $b_i \geq 0$

- Slack $x_{n+i} = b_i \geq 0$ is basic
- Artificial $x_{n+m+i} = 0$ is non-basic
- Column i of B is $\mathbf{e}_i$

If $b_i < 0$

- Slack x_{n+i} is non-basic
- Artificial $x_{n+m+i} = -b_i > 0$ is basic
- Column i of B is $-\mathbf{e}_i$

Basis matrix B is non-singular and, by construction, $\widehat{\mathbf{b}} \geq \mathbf{0}$

At an optimal solution of the Phase I problem

The simplex algorithm drives f up towards zero.
At an optimal basic feasible solution x of the Phase I problem:

If $f = 0$

- The values of the original and slack variables at x yield a basic feasible solution for the original LP

If $f < 0$

- The artificial variables cannot all be driven to zero
- The original LP is **infeasible**

If the Phase I problem is solved with $f = 0$ (and all artificial variables being in $\mathcal{N}$)

1. Remove the artificial variables from the problem (they are now zero)
2. Revert to the original objective function
3. Solve the original **Phase II** problem

Does the algorithm terminate?

- If $\widehat{\mathbf{b}}$ has any zero components then $\mathbf{x}$ is a **degenerate** vertex
- There may be several basic feasible solutions at $\mathbf{x}$
- If $\widehat{b}_p = 0$ then $\overline{\alpha} = 0$ so the simplex algorithm does not move to a new vertex
- It may never leave!

Sensitivity and fair prices

Sensitivity theory and weak duality

- **LP Duality**
  For the following LP problem
  $$\text{maximise } f = \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}$$
  The *dual* LP problem is
  $$\text{minimise } f = \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c}$$
- **Theorem 6** Weak duality theorem
  If x and y are feasible solutions of the primal and dual problems respectively, then
  $$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$
- **Weak duality theorem — consequences**
  - If the primal problem (P) is unbounded, then the dual problem (D) is infeasible
  - If the dual problem (D) is unbounded, then the primal problem (P) is infeasible

Strong duality and the tableau simplex method

For the primal LP
$$\text{maximise } f = \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

If $\mathbf{x}^*$ is an optimal basic feasible solution for the problem in standard form then

- The optimal solution of
  $$\text{minimise } f = \mathbf{b}^T\mathbf{y} \text{ subject to } \mathbf{A}^T\mathbf{y} \geq \mathbf{c}$$
  is $\mathbf{y}^* = \pi = \mathbf{B}^{-T}\mathbf{c}_B$
- The optimal objective values are equal: $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$
- **Conclusions**
  - Only one of (P) and (D) need ever be solved
  - Leads to the (practically preferred) *dual simplex method.*