EECS E6720 Fall 2016 HW3
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Problem 1. <u>Setup</u>: $y \overset{ind}{\sim} \text{Normal}(x_i^T w, \lambda^{-1})$ $\quad w \sim \text{Normal}(0, \text{diag}(\alpha_1 \cdots \alpha_d)^{-1})$

$\lambda \sim \text{Gamma}(e_0, f_0) \quad \alpha_k \sim \text{Gamma}(a_0, b_0)$

a) i). Update λ:

We know from general approach that

$$q(\lambda) \propto \exp\{E_{q(\alpha,w)}[\ln P(y|x,\alpha,w,\lambda) + \ln P(\alpha) + \ln P(w|\alpha) + \ln P(\lambda)]\}$$

$$\propto \exp\{E_{q(\alpha,w)}[\ln P(y|x,\alpha,w,\lambda)] + \ln(P(\lambda))\}$$ For the other 2 terms do not ~~now~~ contain λ.

$$\propto P(\lambda) \exp\{E[\ln P(y|x,w,\alpha,\lambda)]\}$$

$$\propto \text{Gamma}(e_0, f_0) \cdot \exp\left\{\sum_{i=1}^{N} E_{q(\alpha,w)}[\ln P(y_i|x_i,\alpha,w,\lambda)]\right\}$$

$$\propto \lambda^{e_0-1} e^{-f_0\lambda} \prod_{i=1}^{N} \left(\lambda^{\frac{1}{2}} \cdot \exp\left\{-\frac{\lambda}{2} E_{q(\alpha,w)}(y_i - x_i^T w)^2\right\}\right)$$

$$= \lambda^{e_0 + \frac{N}{2} - 1} \cdot e^{-\lambda\left[f_0 + \frac{1}{2}\sum_{i=1}^{N} E_{q(\alpha,w)}(y_i - x_i^T w)^2\right]}$$

which is in the form of a gamma distribution with Gamma(e', f'):

$$\begin{cases} e' = e_0 + \frac{N}{2} \\ f' = f_0 + \frac{1}{2}\sum_{i=1}^{N} E_{q(\alpha,w)}[(y_i - x_i^T w)^2] \end{cases}$$

We will further discuss f' later when we talk about $E(w)$ and $E(ww^T)$.

ii) Update α:

$$q(\alpha_k) \propto \exp\{E_{q_{w\lambda}}[\ln(y|x,\alpha.w.\lambda) + \sum_{k=1}^{d} \ln P(\alpha_k) + \ln P(w|\alpha) + \ln P(\lambda)]\}$$

$$\propto \exp\{E_{q_{w.\lambda}}[\ln P(\alpha_k) + \ln P(w|\alpha_1 \dots \alpha_k)]\}$$

$$\propto P(\alpha_k) \exp\{E_{q_{w\lambda}}[\ln P(w|\alpha_1 \dots \alpha_k)]\}$$

$$\propto \mathrm{Gamma}(a_0, b_0) \cdot \sqrt{\alpha_k} \cdot \exp\{-\frac{\mathrm{diag}(\alpha_1 \dots \alpha_k)}{2} E_{q_{w\lambda}}(w^T w)\}$$

$$\propto \alpha_k^{a_0+\frac{1}{2}-1} \cdot e^{-\alpha_k[b_0 + \frac{1}{2} E_{q_{w\lambda}}(w^T w)]}$$

which follows a form of Gamma (a_0', b') where

$$\begin{cases} a_k' = a_0 + \frac{1}{2} \\ b_k' = b_0 + \frac{1}{2} E_{q_{w\lambda}}[(w^T w)] = b_0 + \frac{1}{2} E_{q_{w.\lambda}}[ww^T]_{kk} \end{cases}$$

Again, we'll come back to this later.

iii) Update w.

$$q(w) \propto \exp\{E_{q_{\alpha.\lambda}}[\ln P(y|x.\alpha.w.\lambda) + \sum_{i=1}^{d} \ln P(\alpha_k) + \ln P(w|\alpha) + \ln P(\lambda)]\}$$

$$\propto \exp\{E_{q_{\alpha.\lambda}}[\ln P(y|x.\alpha.w.\lambda)] + \ln P(w)\}$$

$$\propto P(w) \cdot \exp\{E_{q_{(\lambda)}}[\ln P(y|w.\lambda)]\}$$

$$\propto P(w)\, E_{q_\lambda}[\lambda]^{\frac{N}{2}} \cdot \prod_{i=1}^{N} e^{-\frac{E(\lambda)}{2}(y_i - x_i^T w)^2}$$

$$\propto \exp\{-\tfrac{1}{2} w^T \Sigma_0^{-1} w\} \cdot \exp\{-\frac{E(\lambda)}{2} \sum_{i=1}^{N} (y_i - x_i^T w)^2\} \quad \text{where} \quad \Sigma = \mathrm{diag}(\frac{a_1'}{b_1'}, \dots \frac{a_k'}{b_k'})$$

$$\propto \exp\{-\tfrac{1}{2}[(\Sigma_0^{-1} + E(\lambda)\sum_{i=1}^{N} x_i x_i^T)(w^T w) - 2E(\lambda)\sum_{i=1}^{N} x_i^T y_i w]\}$$

Complete squares and we have. $q(w)$ in the Normal (μ', Σ'^{-1}) form.

where $\mu' = \Sigma' \cdot E(\lambda) \sum_{i=1}^{N} x_i^T y_i = \Sigma' \cdot \frac{e'}{f'} \cdot \sum_{i=1}^{N} x_i^T y_i$

$$\Sigma' = (\Sigma_0^{-1} + E(\lambda)\sum_{i=1}^{N} x_i x_i^T)^{-1} = (\Sigma_0^{-1} + \frac{e'}{f'}\sum_{i=1}^{N} x_i x_i^T)^{-1} = [\mathrm{diag}(\frac{a_1'}{b_1}, \frac{a_2'}{b_2} \dots \frac{a_d'}{b_d'})^{-1} + \frac{e'}{f'}\sum_{i=1}^{N} x_i x_i^T$$

where $\Sigma_0 = \mathrm{diag}(\frac{a_1'}{b_1'} \dots \frac{a_d'}{b_d'})$

Then we have $E(w) = \mu'$

$$E(ww^T) = \mu'\mu'^T + \Sigma'$$

going back to part i): $f' = f_0 + \frac{1}{2}\sum_{i=1}^{N} E_{q_{\alpha,w}}[(y_i - x_i^T w)^2] = f_0 + \frac{1}{2}\sum_{i=1}^{N}\left\{\left[E_{q_{\alpha,w}}(y_i - x_i^T w)\right]^2 + Var(y_i - x_i^T w)\right\}$

$$= f_0 + \frac{1}{2}\sum_{i=1}^{N}\left[(y_i - x_i^T\mu')^2 + x_i^T\Sigma' x_i\right]$$

and part ii): ~~$b' = b_0 + \frac{1}{2} E_{q_{w,\lambda}}[w^T w]$~~

$$b' = b_0 + \frac{1}{2}E_{q_{w,\lambda}}[w^T w] = b_0 + \frac{1}{2}E[ww^T]_{kk} = \left[(\mu'\mu'^T)_{kk} + \Sigma'_{kk}\right]\cdot\frac{1}{2} + b_0$$

b) Pseudo-Code: Initialize $a_0, b_0, \mu_0, \Sigma_0, e_0, f_0$.

For iteration $t = 1 \cdots T$,

step 1: Update λ by setting $e'_t = e_0 + \frac{N}{2}$

$$f'_t = f_0 + \frac{1}{2}\sum_{i=1}^{N}\left[(y_i - x_i^T\mu'_{(t-1)})^2 + x_i^T\Sigma'_{t-1}x_i\right]$$

step 2: Update α_k by setting $a'_{kt} = a_0 + \frac{1}{2}$

$$b'_{kt} = b_0 + \frac{1}{2}\left[(\mu'\mu'^T)_{kk} + \Sigma'_{kk}\right]$$

step 3: Update w by setting $\mu'_t = \frac{e'_t}{f'_t}\cdot\Sigma'\cdot\sum_{i=1}^{N} x_i^T y_i$

$$\Sigma'_t = \left(diag\left(\frac{a'_1}{b'_1}, \frac{a'_2}{b'_2} \cdots \frac{a'_d}{b'_d}\right)^{-1} + \frac{e'}{f'}\sum_{i=1}^{N} x_i x_i^T\right]^{-1}$$

~~End~~ Step 4: Evaluate $\mathcal{L}(a'_t, b'_{t1} \cdots b'_{td}, e'_t, f'_t, \mu'_t, \Sigma'_t)$ to assess convergence.

c)

$$\mathcal{L}(a', b', e', f', \mu', \Sigma') = \int_0^\infty \underbrace{\int_0^{\alpha_1'} \cdots \int_0^{\alpha_d'}}_{\alpha_1' \cdots \alpha_d'} \int_{\mathbb{R}^d} q(w, \lambda, \alpha_1' \cdots \alpha_d') \ln \frac{P(y, w, \lambda, \alpha_1 \cdots \alpha_d \mid x)}{q(w, \alpha_1' \cdots \alpha_d', \lambda)} \, dw \, d\alpha_1' \cdots d\alpha_d' \, d\lambda$$

$$= \int_0^\infty \int_0^{\alpha_1'} \cdots \int_0^{\alpha_d'} \int_{\mathbb{R}^d} q(w, \lambda, \alpha_1' \cdots \alpha_d') \ln P(y, w, \lambda, \alpha_1' \cdots \alpha_d' \mid x) \, dw \, d\alpha_1' \cdots d\alpha_d' \, d\lambda \quad \text{(Term 1)}$$

$$- \int_0^\infty \int_0^{\alpha_1'} \cdots \int_0^{\alpha_d'} \int_{\mathbb{R}^d} q(w, \lambda, \alpha_1' \cdots \alpha_d') \ln q(w, \alpha_1' \cdots \alpha_d', \lambda) \, dw \, d\alpha_1' \cdots d\alpha_d' \, d\lambda \quad \text{(Term 2)}$$

$$= \text{Term 1} - \text{Term 2}.$$

$$\text{Term 1} = \text{const} + \int_0^\infty \int_0^{\alpha_1'} \cdots \int_0^{\alpha_d'} \int_{\mathbb{R}^d} q(w) q(\lambda) \prod_{k=1}^d q(\alpha_k') \underbrace{\left[\ln P(y \mid w, \lambda, x) + \ln P(\lambda) + \ln P(w \mid \alpha_1' \cdots \alpha_d') + \sum_{k=1}^d P(\alpha_k') \right]}_{dw \, d\alpha_1 \cdots d\alpha_d \, d\lambda}$$

$$= \text{const} + \int_0^\infty \int_0^{\alpha_1'} \cdots \int_0^{\alpha_d'} \int_{\mathbb{R}^d} q(w) q(\lambda) \prod_{k=1}^d q(\alpha_k') \ln P(y \mid w, \lambda, x, \alpha_1' \cdots \alpha_d') \, dw \, d\alpha_1' \cdots d\alpha_d' \, d\lambda$$

$$+ \int q(\lambda) \ln P(\lambda) \, d\lambda + \sum_{i=1}^d \int q(\alpha_i') \ln P(\alpha_i') \, d\alpha_i'$$

$$+ \iint q(w) \ln P(w \mid \alpha_1', \alpha_2' \cdots \alpha_d') \prod_{i=1}^d q(\alpha_1' \cdots \alpha_d') \, dw \, d\alpha_1' \cdots d\alpha_d'$$

$$= \text{const} + \underbrace{\sum_{i=1}^N E_q[\ln(P(y_i \mid x_i, w, \lambda))]}_{①} + \underbrace{E_q[\ln P(\lambda)]}_{②} + \underbrace{E_q[\ln P(w \mid \alpha_1' \cdots \alpha_d')]}_{③} + \underbrace{\sum_{i=1}^d E_q[\ln P(\alpha_i')]}_{④}$$

where

$$① = \sum_{i=1}^N \left[\frac{E_q[\ln \lambda]}{2} - \text{const} - \frac{E_q[\lambda]}{2} (y_i^2 - 2 E_q(w) x_i y_i + x_i^T E_q[ww^T] x_i) \right]$$

$$\cancel{= \frac{N}{2} E_q(\lambda) + \text{const} =}$$

$$= \frac{N}{2} E_q[\ln \lambda] + \text{const} - \frac{E_q[\lambda]}{2} \sum_{i=1}^N \left[y_i^2 - 2 E_q(w) x_i y_i + x_i^T E_q[ww^T] x_i \right]$$

$$= \text{const} + \frac{N}{2} (\psi(e') - \ln f') - \frac{e'}{2f'} \sum_{i=1}^N \left[y_i^2 - 2\mu'^T x_i y_i + x_i^T [\Sigma' + \mu' \mu'^T] x_i \right]$$

$$② = E_q[\ln P(\lambda)] = E_q[e_0 \ln f_0 - \ln \Gamma(e_0) + (e_0 - 1) \ln \lambda - f_0 \lambda]$$

$$= \text{const} + (e_0 - 1)(\psi(e') - \ln f') - f_0 \frac{e'}{f'}$$

$$③ = \frac{1}{2} \sum_{i=1}^d E_q[\ln \alpha_i] - \text{const} - \frac{1}{2} (w^T E_q[\text{diag}(\alpha_1 \cdots \alpha_d)] w)$$

$$= \text{const} + \frac{1}{2} \sum_{i=1}^d (\psi(a_i') - \ln(b_i')) - \frac{1}{2} \sum_{i=1}^d \left[\frac{a_i'}{b_i'} [(\mu' \mu'^T)_{ii} + \Sigma_{ii}'] \right]$$

$$④ = \sum_{k=1}^{d} E_q[\ln P(\alpha_k)] = \sum_{k=1}^{d} E_q[a_0 \ln b_0 - \ln\Gamma(a_0) + (a_0-1)\ln\alpha_k - b_0\alpha_k]$$

$$= \text{const} + (a_0-1)\sum_{k=1}^{d}(\psi(a_k') - \ln b_k') - b_0 \sum_{k=1}^{d} \frac{a_k'}{b_k'})$$

※ Note: since $a_k' = a_0 + \frac{1}{2}$, a_k' are the same regardless of k.

$$\text{Term2} = \underbrace{\int q(w)\ln q(w)\,dw}_{⑤} + \underbrace{\int q(\lambda)\ln q(\lambda)\,d\lambda}_{⑥} + \underbrace{\sum_{k=1}^{d}\int q(\alpha_k)\ln q(\alpha_k)\,d\alpha_k}_{⑦}$$

$$⑤ = E_q[\ln q(w)] = \frac{1}{2}\ln(\Sigma'] + \text{const} + \frac{1}{2}E_q[(w-\mu')^T \Sigma'^{-1}(w-\mu')]$$

From our Lecture notes we know $⑤ = \frac{1}{2}\ln|\Sigma'| + \text{const}$.

$$⑥ = E_q[\ln q(\lambda)] = e'\ln f' - \ln\Gamma(e') + (e'-1)(\psi(e') - \ln f') - f' \cdot \frac{e'}{f'}$$

$$\cancel{= e' + \ln f' - \ln\Gamma(e') + (1-e')}$$

$$= -e' + \ln f' - \ln\Gamma(e') + (e'-1)\psi(e')$$

$$⑦ = \sum_{k=1}^{d} E_q[\ln(q\alpha_k)] = \sum_{k=1}^{d}[a_k'\ln b_k' - \ln\Gamma(a_k') + (a_k'-1)(\psi(a_k') - \ln b_k') - b_k'\frac{a_k'}{b_k'}]$$

$$= \sum_{k=1}^{d}[-a_k' + \ln b_k' - \ln\Gamma(a_k') + (a_k'-1)\psi(a_k')]$$

Putting all together,

$$\mathcal{L}(a', b', e', f', \mu', \Sigma') = \text{Const} + \frac{N}{2}[\psi(e') - \ln f'] - \frac{e'}{2f'}\sum_{i=1}^{N}[y_i^2 - 2\mu'^T x_i y_i + x_i^T(\Sigma' + \mu'\mu'^T)x_i]$$

$$+ \text{constant} + (e_0-1)(\psi(e') - \ln f') - f_0\frac{e'}{f'}$$

$$+ \text{constant} + \frac{1}{2}\sum_{k=1}^{d}(\psi(a_k') - \ln(b_k')) - \frac{1}{2}\sum_{k=1}^{d}(\frac{a_k'}{b_k'}[(\mu'\mu'^T)_{kk} + \Sigma'_{kk}])$$

$$+ \text{constant} + (a_0-1)\sum_{k=1}^{d}(\psi(a_k) - \ln(b_k)) - b_0\sum_{k=1}^{d}\frac{a_k'}{b_k}$$

$$- \frac{1}{2}\ln|\Sigma'| + \text{constant}$$

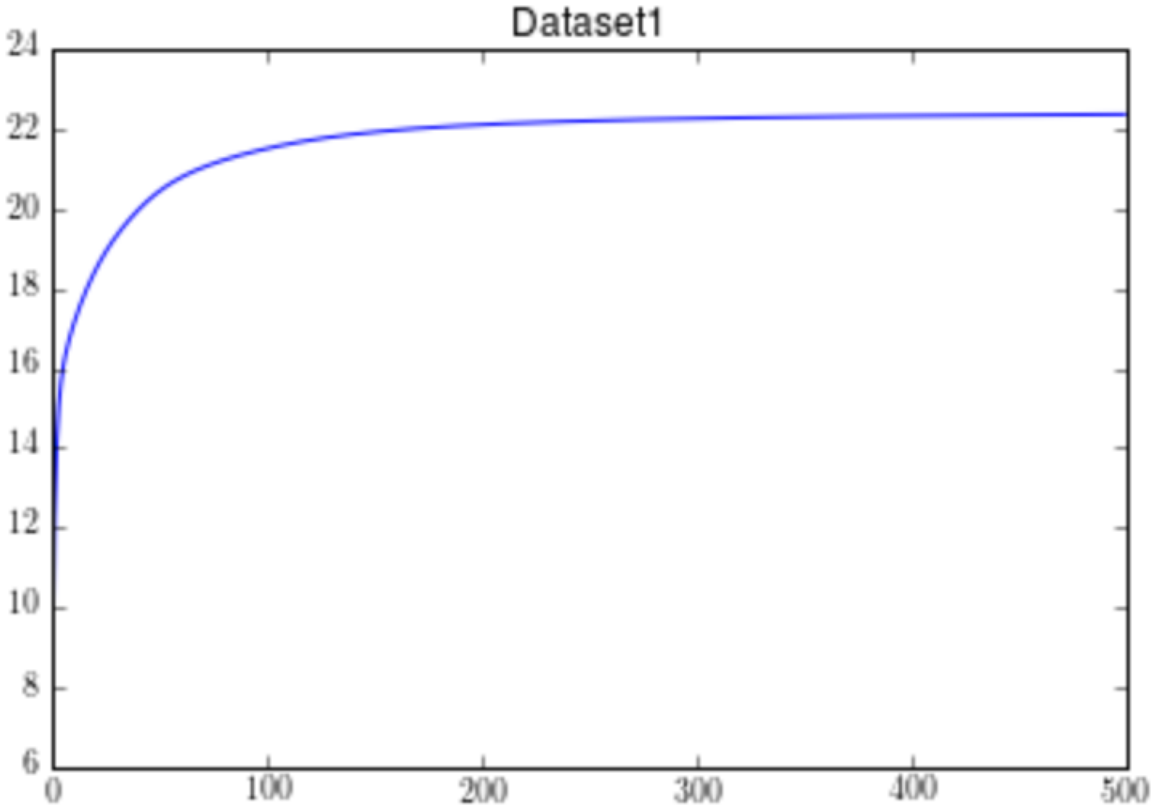
$$+ e' - \ln f' + \ln(\Gamma(e')) + (1-e')\psi(e') + \text{constant}$$

$$+ \sum_{k=1}^{d}[a_k' - \ln b_k' + \ln\Gamma(a_k) + (1-a_k')\psi(a_k)]$$

which can be further reduced to

$$\mathcal{L}(a', b', e', f', \mu', \Sigma') = const + \frac{N}{2}[\psi(e') - \ln f'] - \frac{e'}{f'}(f' - f_0)$$
$$- f_0 \frac{e'}{f'}$$
$$+ (e_0 - 1)(\psi(e') - \ln f')$$
$$+ \frac{1}{2}\sum_{k=1}^{d}(\psi(a'_k) - \ln(b'_k)) - \frac{a'_k}{b'_k}(b'_k - b_0)$$
$$+ (a_0 - 1)\sum_{k=1}^{d}(\psi(a'_k) - \ln(b'_k)) - b_0 \sum_{k=1}^{d}\frac{a'_k}{b'_k}$$
$$- \frac{1}{2}\ln|\Sigma'|$$
$$+ e' - \ln f' + \ln(\Gamma(e')) + (1 - e')\psi(e')$$
$$+ \sum_{k=1}^{d}[a'_k - \ln b'_k + \Gamma(a'_k) + (1 - a'_k)\psi(a'_k)]$$

a)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8
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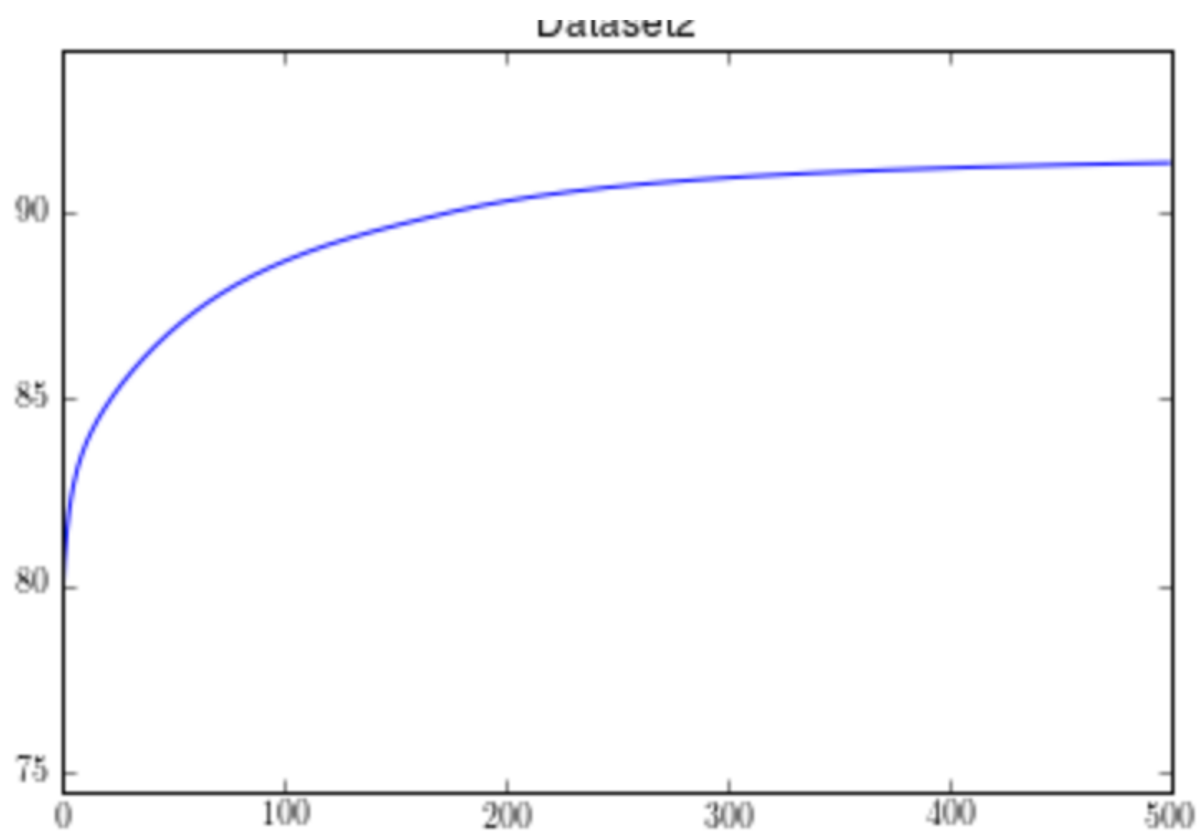
300
400
500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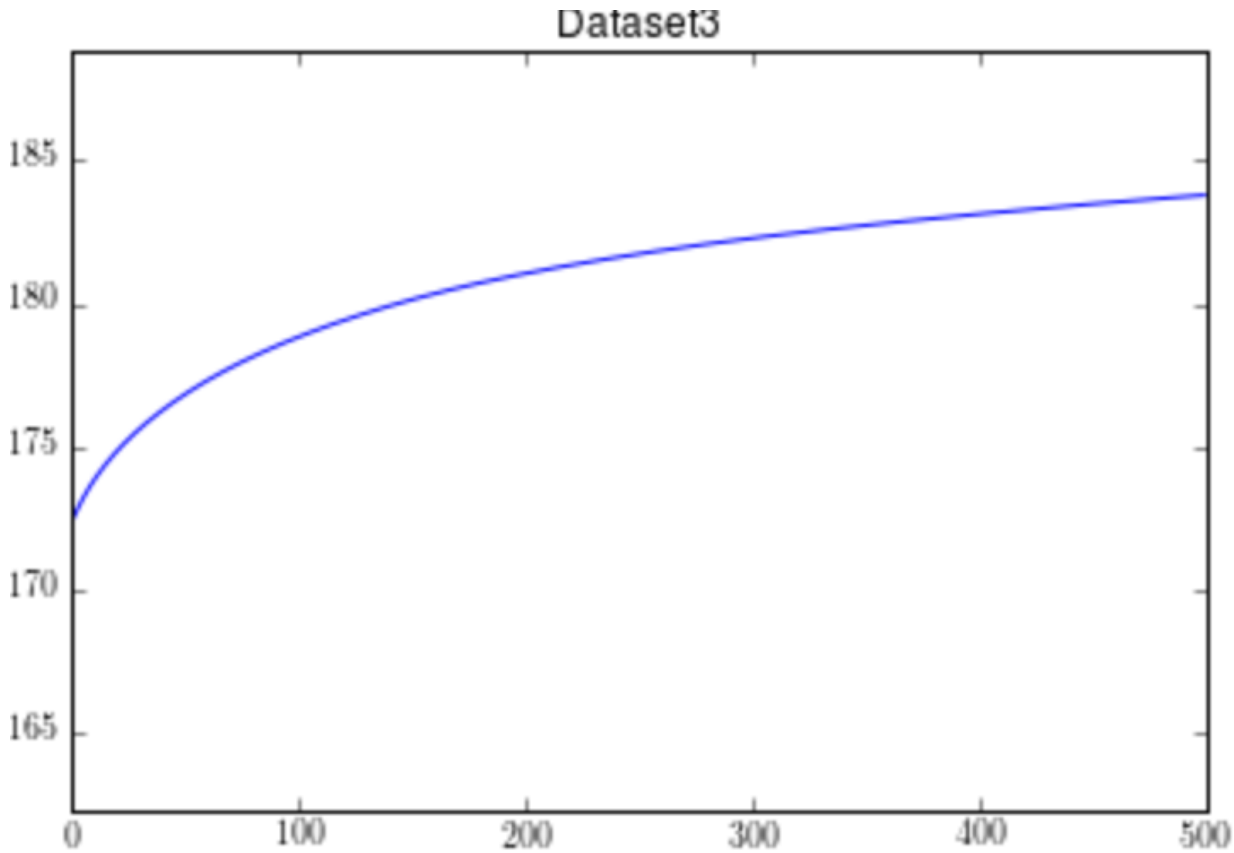
Dataset3
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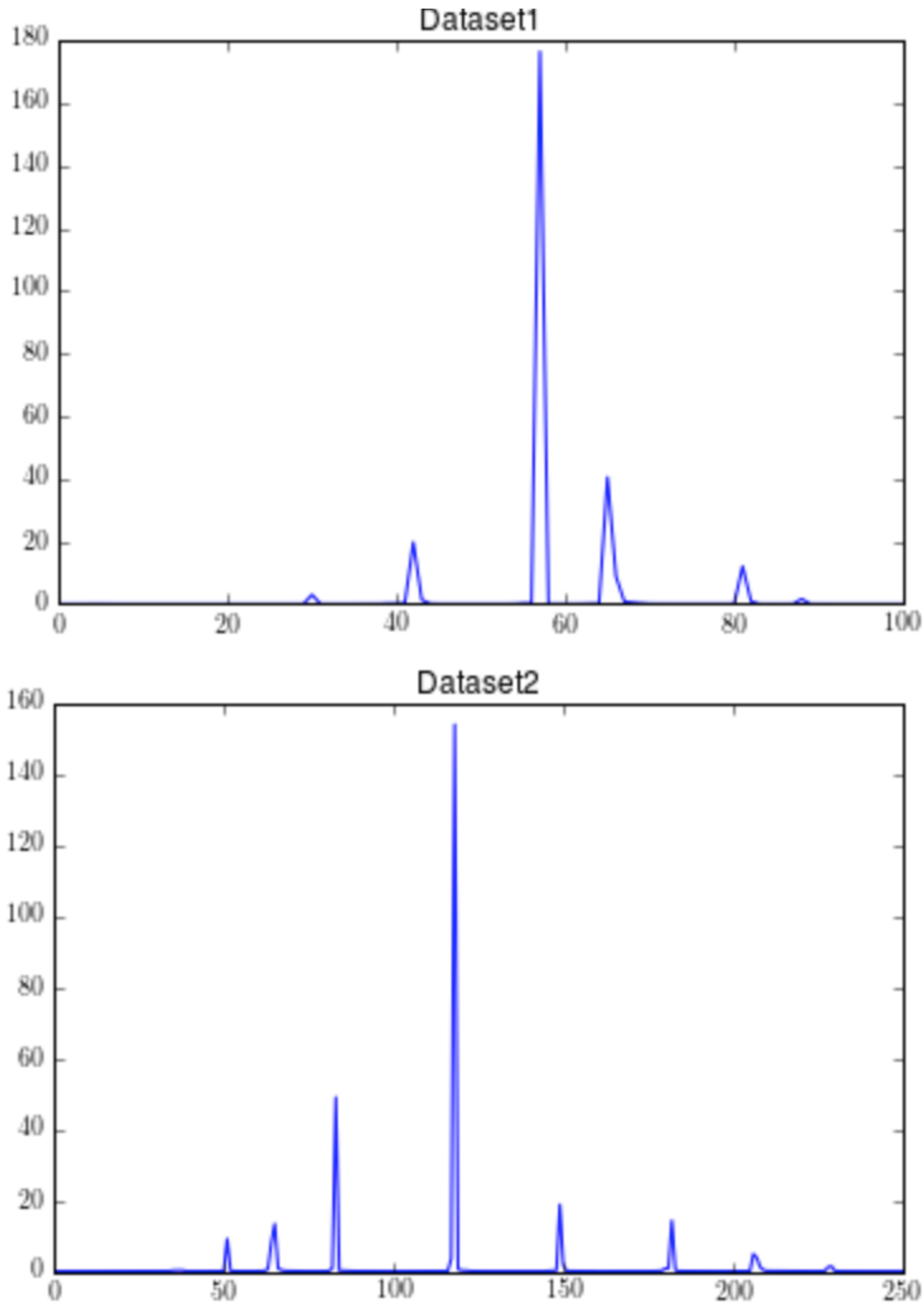
175
170
165
0
100
200
300
400
500

b)


Dataset1
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140
120
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80
60
40
20
0
0
20
40
60
80
100
Dataset2
160
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120
100
80
60
40
20
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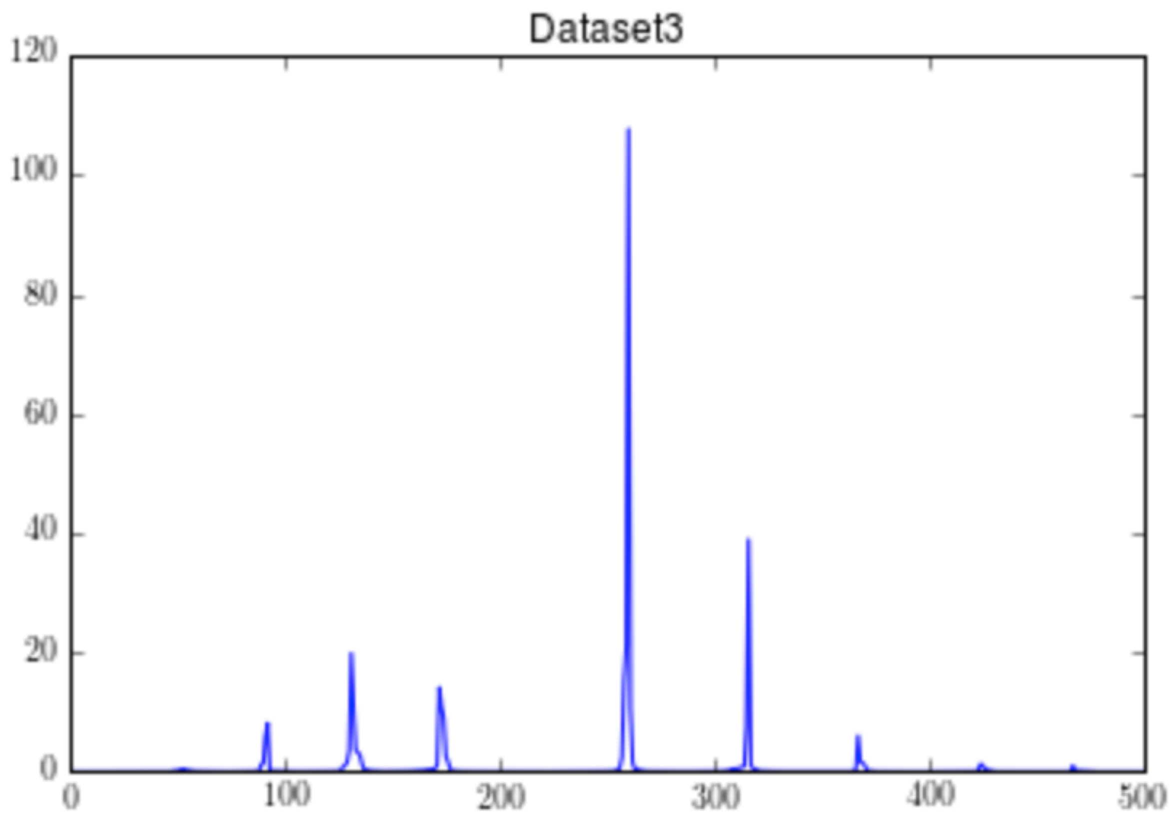
0
50
100
150
200
250



Dataset3
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60
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20
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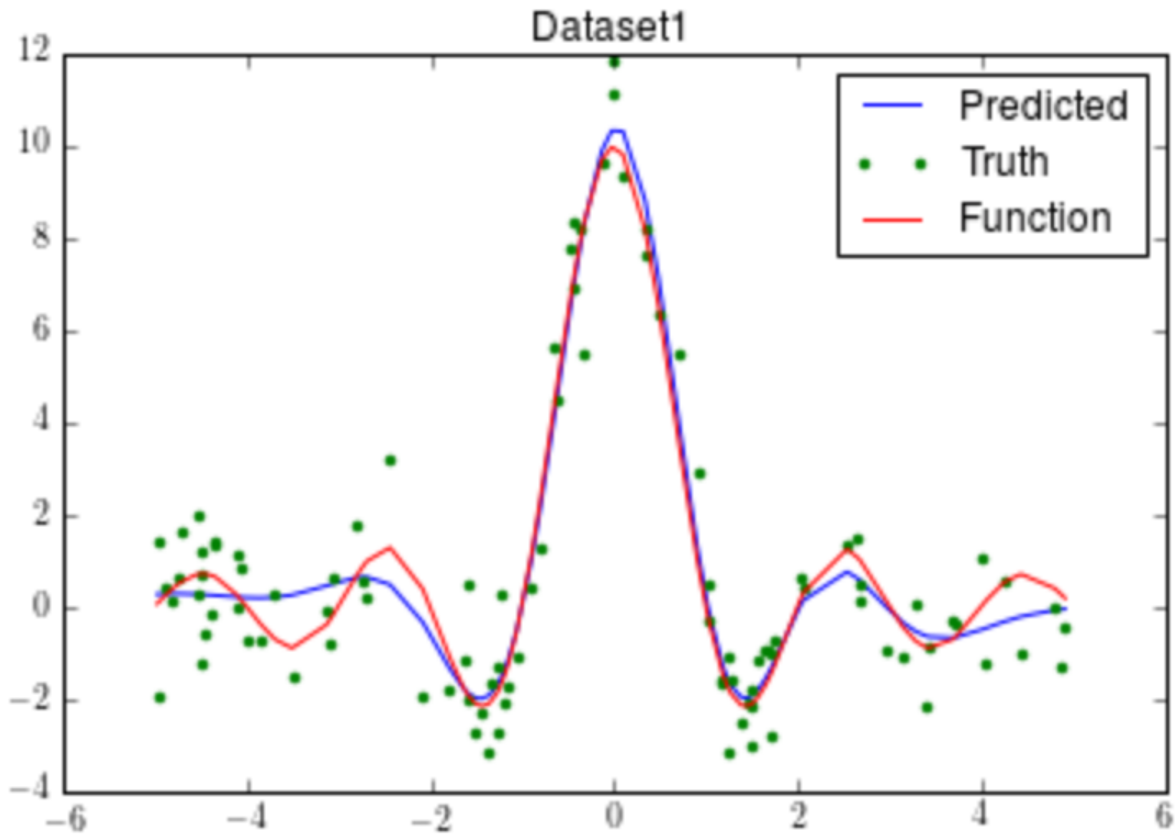
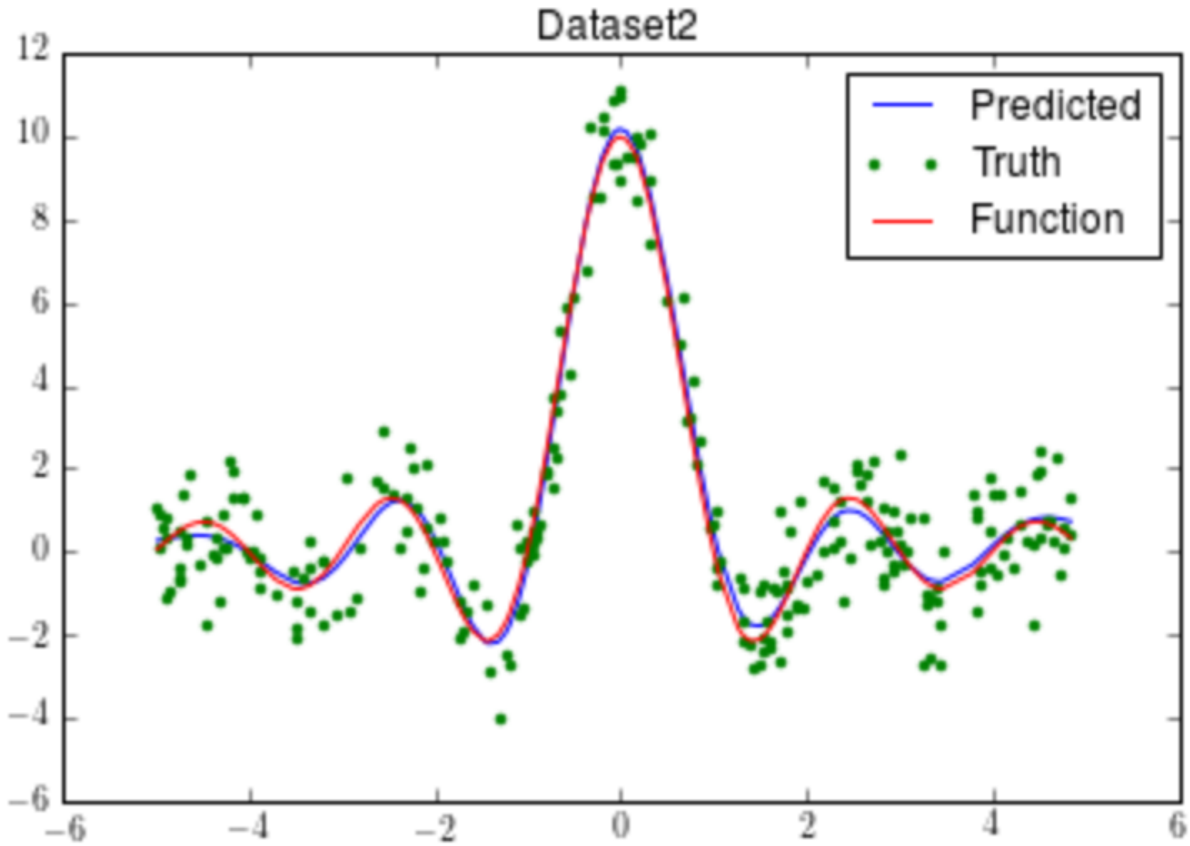
0
100
200
300
400
500

c)

d)





Dataset2
Predicted
Truth
Function
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6
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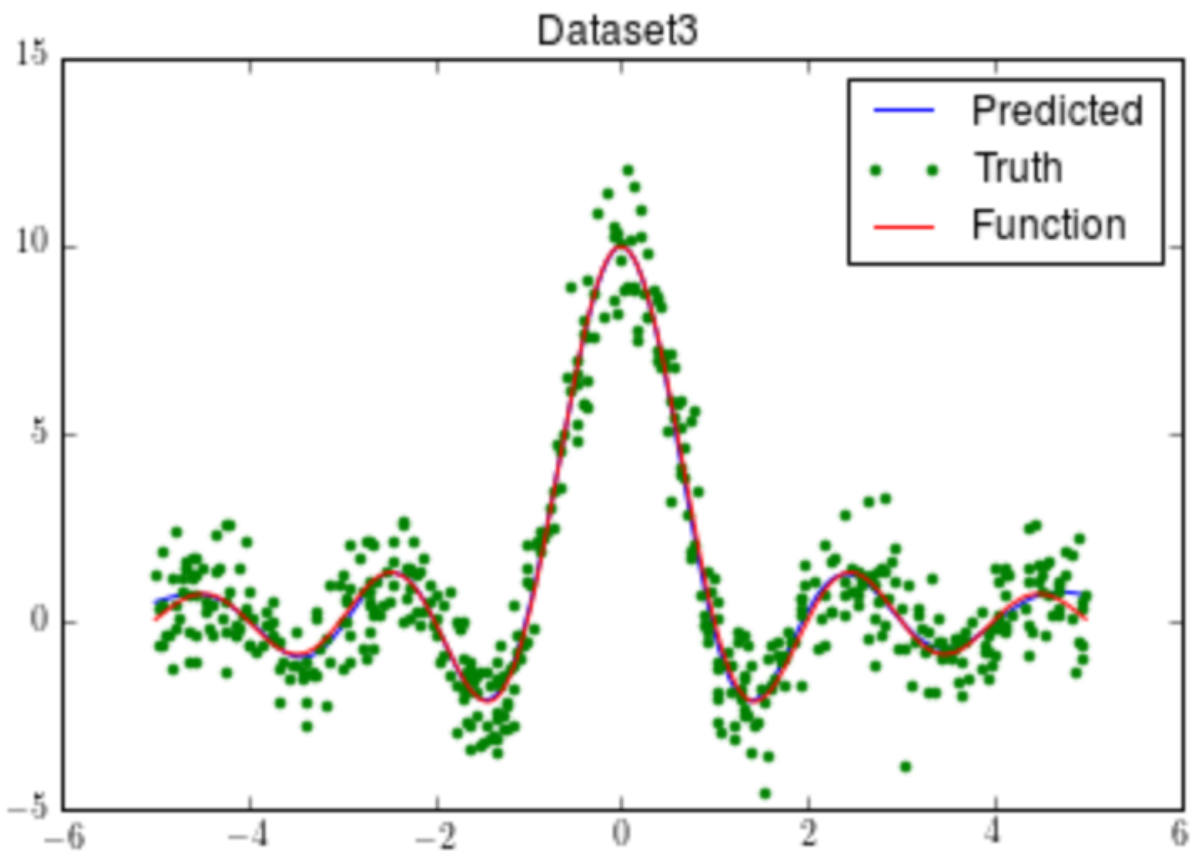
−2
−4
−6
−6
−4
−2
0
2
4
6


Dataset3
Predicted
Truth
Function
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0
2
4
6