GATE CSE NOTES

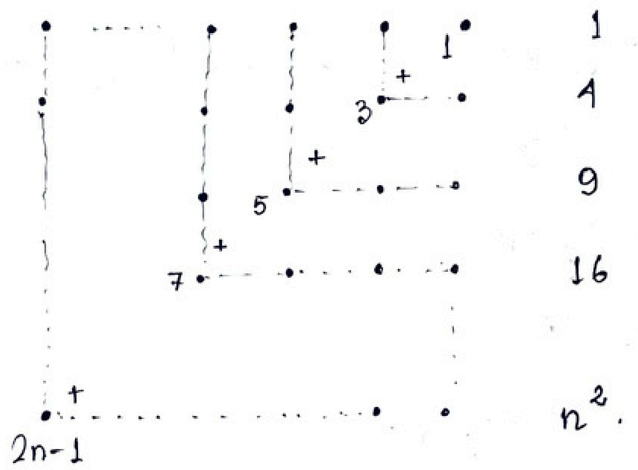
by

UseMyNotes

Counting.

Eg. Sum of odd numbers.

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$



* The Product Rule
* The Sum Rule.
* Inclusion - Exclusion Principle / Sieve Principle / Subtraction Principle.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Generally,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{K=1}^{n} (-1)^{K+1} \left(\sum_{1 \leq i_1 < \cdots < i_K \leq n} |A_{i_1} \cap \cdots \cap A_{i_K}| \right)$$

* The Pigeonhole Principle / Dirichlet drawer Principle
    * Theorem 1 : If K is a positive integer & $K+1$ or more objects are placed into K boxes, then there is at least one box containing two or more of the objects.

→ Corollary : A function f from a set with $K+1$ or more elements to a set with K elements is not one-to-one

- <u>Theorem 2.</u> : <u>The Generalised Pigeonhole Principle.</u>

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

→ $N = k(r-1)$ is the smallest integer satisfying $\lceil N/k \rceil \geq r$.

- <u>Theorem 3.</u> : Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

* <u>Binomial Theorem.</u>

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

→ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ | Σ

→ $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$ | Alternating sum Σ.

→ $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$

- $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$ | $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

- Vandermonde's Identity : m, n, r → nonnegative integer

$r \not> m, n$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

| m girls
| n boys
| group of r

Cor.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \qquad \Big| \qquad \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

- Theorem: n, r nonnegative integers $r \leq n$

Hockey-stick identity

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}. \qquad \Big| \qquad \text{eg } \binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}$$

* Generating Permutations

→ lexicographic ordering

* Generating Combinations

* Pigeonhole Principle (Strong form).

If $n(r-1)+1$ objects are put into n boxes, at least one of the boxes contains r or more of the objects.

$\rightarrow N = K(r-1)+1$

$N \rightarrow$ Pigeons

$K \rightarrow$ pigeonholes.

$r \rightarrow$ no. of pigeons or more than it in at least one ~~box~~. pigeonhole.

Advanced Counting Techniques.

* Recurrence Relation.

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence for all integers n with $n \geq n_0$, where n_0 is a non-negative integer.
A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

* Linear Homogeneous Recurrence Relation.

A linear homogeneous recurrence relation of degree k with constant co-efficient is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

$$\begin{cases} c_1, c_2, \dots, c_n \longrightarrow \text{real numbers} \\ c_k \neq 0. \end{cases}$$

eg. $f_n = f_{n-1} + f_{n-2}$ of degree two.

* Solving Linear Homogeneous Recurrence Relations with constant coefficients.

Solution of the form $a_n = r^n$.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}.$$

$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0.$$

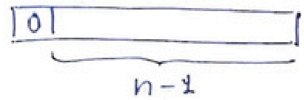
The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if & only if r is a solution of the equation (Characteristic equation).

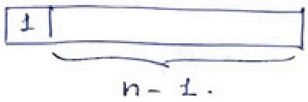
→ Theorem. : Let c_1 & c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 & r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$ where α_1 & α_2 are constants.

→ Theorem. : Let c_1 & c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \dots$ where α_1 & α_2 are constants.

Recurrence Relⁿ.

1. $T(n)$ = no. of binary strings of length n.

C-1. 

C-2 

$$T(n) = \underbrace{T(n-1)}_{C-1} + \underbrace{T(n-1)}_{C-2}$$

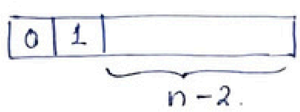
$$= 2T(n-1)$$

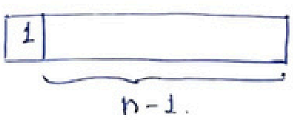
$$T(1) = 2 \begin{cases} 0 \\ 1 \end{cases}$$

$$T(2) = 2T(1) = 4.$$

$$T(3) = 2T(2) = 8$$

2. $T(n)$ = # binary strings without consecutive zeros.

C-1 

C-2 

$$T(n) = T(n-1) + T(n-2).$$

$$T(1) = 2 \begin{cases} 0 \\ 1 \end{cases}$$

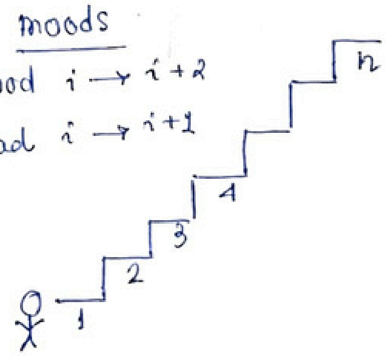
$$T(2) = 3 \begin{cases} 01 \\ 10 \\ 11 \end{cases}$$

$$T(3) = T(2) + T(1) = 5.$$

3. moods

good $i \to i+2$

bad $i \to i+1$



how many ways a person can reach nth step?

$n = 1$. ⇝ 1 way

$n = 2$ ⇝ 2 ways
- 1 - 2
- 2.

$n = 3$ ⇝ 3 ways
- 1 - 2 - 3
- 2 - 3
- 1 - 3

$n = 4$ ⇝ 5 ways
- 1 - 2 - 3 - 4
- 2 - 4
- 1 - 2 - 4
- 1 - 3 - 4
- 2 - 3 - 4

C-1 If a person has good mood, find answer for $(n-2)$ places.

C-2 If has bad mood, find for $(n-1)$ places.

$$T(n) = T(n-1) + T(n-2). \qquad T(1) = 1 \qquad T(2) = 2.$$

T(1)	T(2)	T(3)	T(4)	T(5)	T(6)	T(7)	T(8)
1	2	3	5	8	13	21	34.

4. Count no. of ways of the person can climb up to ~~m~~ n stairs for a given value m?

$$T(n, m) = T(n-1, m) + T(n-2, m) + \dots + T(n-m, m).$$

5. G'15. Let a_n be the # of bit strings of length n containing 2 consecutive 1's. What is the recurrence relⁿ for a_n?

a) $a_{n-2} + a_{n-1} + 2^{n-2}$

b) $a_{n-2} + 2a_{n-1} + 2^{n-2}$

c) $2a_{n-2} + a_{n-1} + 2^{n-2}$

d) $2a_{n-2} + 2a_{n-1} + 2^{n-2}$

→ $T(1) = 0.$ $a_1 = 0.$

$a_2 = 1.$ — 11.

$a_3 = 3$ — $\left\{ \begin{array}{l} 011 \\ 110 \\ 111 \end{array} \right.$

$a_3 = a_1 + a_2 + 2^1$ — a.

$= 3.$ ✓

$a_3 = 2a_1 + \cancel{4}a_2 + 2^1$ — c.

$= 3.$ ✓

$a_4 = 8$

0011
0110
0111
1011
1100
1101
1110
1110

Check with options.

$a_4 = a_2 + a_3 + 2^2$ — a

$= 1 + 3 + 4 = 8$ ✓

$a_4 = 2a_2 + a_3 + 2^2$

$= 9$ ✗

6. a_n be the # of bit strings of length n don't containing 2 consecutive 1's.

a) $a_n = a_{n-1} + 2a_{n-2}$

b) $a_{n-1} + a_{n-2}$

c) $2a_{n-1} + a_{n-2}$

d) $2a_{n-1} + 2a_{n-2}$

→ $a_1 = 2$ $\left\{ \begin{array}{l} 0 \\ 1 \end{array} \right.$

$a_2 = 3$ $\left\{ \begin{array}{l} 00 \\ 01 \\ 10 \end{array} \right.$

$a_3 = 5$ $\left\{ \begin{array}{l} 000 \\ 001 \\ 010 \\ 100 \\ 101 \end{array} \right.$

Check with options.

$a_3 = a_1 + a_2$ — b

$= 5$ ✓

(-1) $0\underline{\quad n-1 \quad}$

(-2) $10\underline{\quad n-2 \quad}$

7. G'08. Let x_n denote # of binary strings of length n that contain no consecutive 0's.

- Which recurrence rel^n for x_n?

a) $2x_{n-1}$ b) $x_{n/2} + 1$ c) $x_{n/2} + n$ ✓d) $x_{n-1} + x_{n-2}$

- Value of x_5 -

a) 5   b) 7   c) 8   ✓d) 13.

$x_3 = x_2 + x_1 = \cancel{3+0=3} \quad 3 + 2 = 5$

$x_4 = \cancel{3+3=6} \quad 3 + 5 = 8$

$x_5 = \cancel{6+3=9}. \quad 8 + 5 = 13.$

8. G'02. Sol^n to the recurrence rel^n

$T(2^K) = 3T(2^{K-1}) + 1. \quad T(1) = 1.$ is

a) 2^K ✓b) $\dfrac{3^{K+1} - 1}{2}$ c) $3^{\log_2 K}$ d) $2^{\log_3 K}$.

×

→ $K = 1$ $T(2) = 3T(2^{1-1}) + 1 = 4.$

$2^1 = 2$ ×

$3^{\log_2 1} = 1$ ×

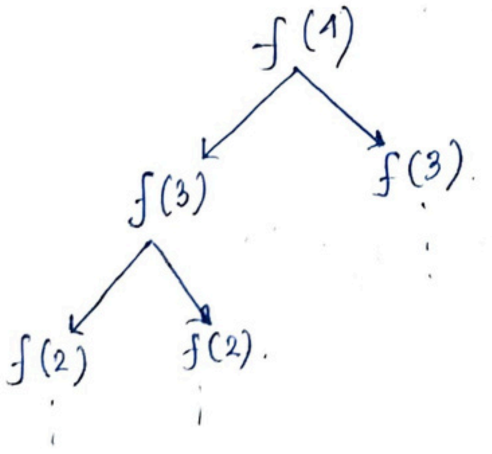
$2^{\log_3 1} = 1$ ×.

b) $\dfrac{3^2 - 1}{2} = 4$ ✓

- Void f(int n)

```
if (n ≤ 1) return;
f(n-1);
f(n-1);
```

No. of recursive calls
Returning value
Time complexity
No. of comparisons
Growth of stack



No. of recursive calls.

$T(1) = 0.$

$T(n) = T(n-1) + T(n-1) + 2$

$= 2T(n-1) + 2$

$$T(n) = 2T(n-1) + 2$$

$$T(n-1) = 2T(n-2) + 2 \qquad \times 2$$

$$T(n-2) = 2T(n-3) + 2 \qquad \times 2^2$$

$$\times 2^3$$

$$\vdots$$

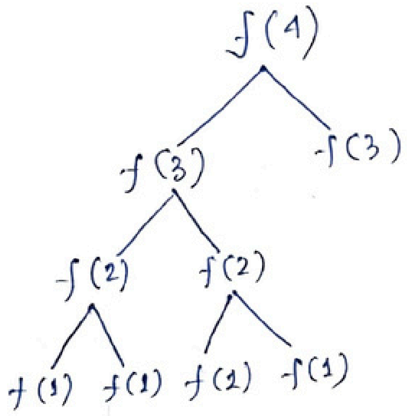
$$T(3) = 2T(2) + 2 \qquad \times 2^{n-3}$$

$$T(2) = 2T(1) + 2 \qquad \times 2^{n-2}$$

$$T(1) = 0 \qquad \times 2^{n-1}$$

$$T(n) = 2^1 + 2^2 + 2^3 + \dots + 2^{n-2} + 2^{n-1} = \frac{a(r^{n+1} - 1)}{r-1}$$

$$= \frac{2^n - 1}{2-1} - 1 = 2^n - 2$$



$\Rightarrow$ no. of recursive calls $= 7 \times 2 = 14$.

$$2^4 - 2 = 14$$

- f (int n)
  {
  if $(n \leq 1)$ return;
  f(n/2);
  }

$$T(1) = 0$$

$$T(n) = T(n/2) + 1$$

$$T(n) = T(n/2) + 1$$

$$T(n/2) = T(n/2^2) + 1$$

$$T(n/2^2) = T(n/2^3) + 1$$

$$\vdots$$

$$T(1) = 0$$

$$T(2^m) = T(2^{m-1}) + 1$$

$$T(2^{m-1}) = T(2^{m-2}) + 1$$

$$\vdots$$

$$T(2^2) = T(2^1) + 1$$

$$T(2^1) = T(2^0) + 1$$

$$T(2^0) = 0.$$

$$n = 2^k$$

$$k = \log_2 n$$

$$T(2^m) = m \Rightarrow T(n) = \log n$$

→ Theorem : Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation *

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

iff

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_n r_k^n$$

for $n = 0, 1, 2, \ldots$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

→ Theorem : Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ & $m_1 + m_2 + m_3 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation *

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

if & only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n +$$

$$(\alpha_{2,0} + \alpha_{2,1} n + \dots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \dots +$$

$$(\alpha_{t,0} + \alpha_{t,1} n + \dots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2 \dots$ where $\alpha_{i,j}$ are constants for $1 \le i \le t$ & $0 \le j \le m_i - 1$.

* Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

$$a_n = \underbrace{c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}}_{\text{Associated homogeneous recurrence relation.}} + f(n).$$

→ **Theorem.** : If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}.$$

→ Theorem. : Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + f(n)$$

and

$$f(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ & s are real numbers.

When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation & its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

* Generating Functions. :

The generating function for the sequence of real numbers $a_0, a_1, a_2, \ldots, a_k, \ldots$ is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

→ **Theorem.:** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and

$g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

✓ $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$ &

✓ $f(x)\, g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$.

→ **Extended Binomial Coefficient.**

Let u be a real number & k be a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0. \end{cases}$$

→ **Theorem:**

Extended Binomial Theorem. —

Let x be a real number with $|x| < 1$ & let u be a real number.

Then $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$.

$\rightarrow \dfrac{1}{1-x} = 1 + x + x^2 + \dots$ for $|x| < 1$

$\rightarrow \dfrac{1}{1-ax} = 1 + ax + a^2x^2 \dots$ for $|ax| < 1$.

$\rightarrow \binom{-n}{r} = (-1)^r \, C(n+r-1, r)$.

$\rightarrow$ Useful generating functions.

$G(x)$	a_k.
$(1+x)^n = \sum_{k=0}^{n} C(n,k)\, x^k$	$C(n,k)$.
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)\, a^k x^k$	$C(n,k)\, a^k$.
$\dfrac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \dots + x^n$	1 if $k \le n$; 0 otherwise.
$\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$	1.
$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)\, x^k = 1 + 2x + 3x^2 + \dots$	$k+1$.
$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{n} C(n+k-1, k)\, x^k$	$C(n+k-1, k) = C(n+k-1, n-1)$.
$e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots$	$1/k!$
$\ln(1+x) = \sum_{k=0}^{\infty} \dfrac{(-1)^{k+1}}{k}\, x^k = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots$	$(-1)^{k+1}/k$.

Recurrence relation.

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one one or more of the previous terms of the sequence, namely $a_0, a_1, \dots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is non-negative integer.

Q. $T(n) = 2T(n-1) - T(n-2)$

Solution ~

✓ a) $T(n) = 3n$ b) 2^n ✓ c) 5

→ a) $T(0) = 0$, $T(1) = 3$, $T(2) = 6$, $T(3) = 9$

$T(2) = 2T(1) - T(0) = 2 \times 3 - 0 = 6$ ✓

$T(3) = 2T(2) - T(1) = 9$ ✓

$T(n) = 2 \times 3(n-1) - 3(n-2) = 6n - 6 - 3n + 6 = 3n$ ✓

b) $T(0) = 1$, $T(1) = 2$, $T(2) = 4$, $T(3) = 8$

$T(2) = 2(T(1)) - T(0) = 2 \times 2 - 1 = 3$ ✗

$T(n) = 2 \cdot 2^{n-1} - 2^{n-2} = 2^{n-2}(2^2 - 1) = 3 \times 2^{n-2}$ ✗

c) $T(0) = T(1) = \dots = T(n) = 5.$

$T(2) = 2 \times 5 - 5 = 5.$ ✓

$T(3) = 2 \times 5 - 5 = 5$ ✓

$T(n) = 2 \times 5 - 5 = 5$ ✓

2 solutions – $T(n) = 3n$
$T(n) = 5.$

Q. G'04 $T(1) = 1.$
$T(n) = 2T(n-1) + n$ $n \geq 2.$ evaluates to

✓ a) $2^{n+1} - n - 2$ c) $2^{n+1} - 2n - 2$

– b) $2^n - n$ d) $2^n + n$

→ a) $2^{1+1} - 1 - 2 = 1.$ ✓ b) $2^1 - 1 = 1$ ✓

c) $2^2 - 2 - 2 = 0$ ✗ d) $2^1 + 1 = 3$ ✗.

$T(2) = 2 + 2 = 4.$

✓ a) $2^3 - 2 - 2 = 4.$ ✓ b) $2^2 - 2 = 2.$ ✗.

Q. G'02. $T(2^k) = 3T(2^{k-1}) + 1$.

$T(1) = 1$.

a) 2^k c) $3^{\log_2 k}$

✓ b) $\frac{3^{k+1} - 1}{2}$ d) $2^{\log_3 k}$

→ $2^k = n$

$T(n) = 3T(n/2) + 1$.

By master's theorem,

$n^{\log_b a}$ $O(n^{\log_2 3})$.

$n^{\log_2 3}$ $\Rightarrow O(2^{k \log_2 3})$

$\Rightarrow O(3^k)$.

a) ~~T(1) ...~~

$T(2^k) = 2^k$ ✓ $T(1) = 1$

b) $T(2^k) = \frac{3^{k+1} - 1}{2}$ | $2^k = 1 \Rightarrow k = 0$

$= \frac{3-1}{2} = 1$ ✓

c) $T(2^k) = $ ✗

d) $T(2^k) = $ ✗

$T(2) = 3T(2^0) + 1$

$= 3T(1) + 1 = 4$.

a) $T(2) = 2$ ✗

b) $T(2) = \frac{3^2 - 1}{2} = 4$ ✓

Q. G'08 $n = 2^{2k}$ for some $k \geq 0$, the recurrence relation $T(n) = \sqrt{2}\, T(n/2) + \sqrt{n}$, $T(1) = 1$. evaluates to

✓ a) $\sqrt{n}(\log n + 1)$ c) $\sqrt{n} \log \sqrt{n}$

b) $\sqrt{n} \log n$ d) $n \log \sqrt{n}$.

→ $T(1) = 1$ a) $T(1) = 1$ c) 0

$T(2) = 2\sqrt{2}$. b) $T(1) = 0$ d) 0.

a) $\sqrt{2}(\log_2 2 + 1) = 2\sqrt{2}$

Q. G'12. Recurrence relⁿ capturing the optimal execution time of the Towers of Hanoi problem with n discs is —

a) $T(n) = 2T(n-2) + 2$ c) $T(n) = 2T(n/2) + 1$

b) $T(n) = 2T(n-1) + n$ ✓ d) $T(n) = 2T(n-1) + 1$.

→ $T(n) = 2T(n-1) + 1$.

No. of movements required.

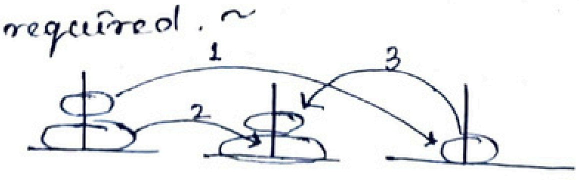
1 disc $T(1) = 1$

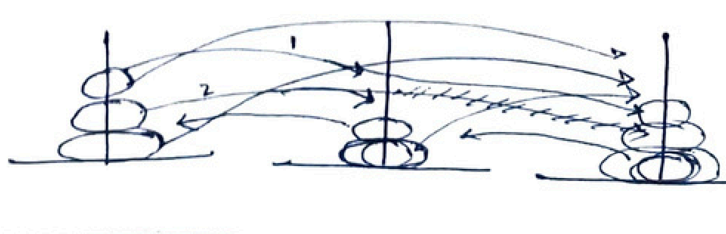
2 discs $T(2) = 3$

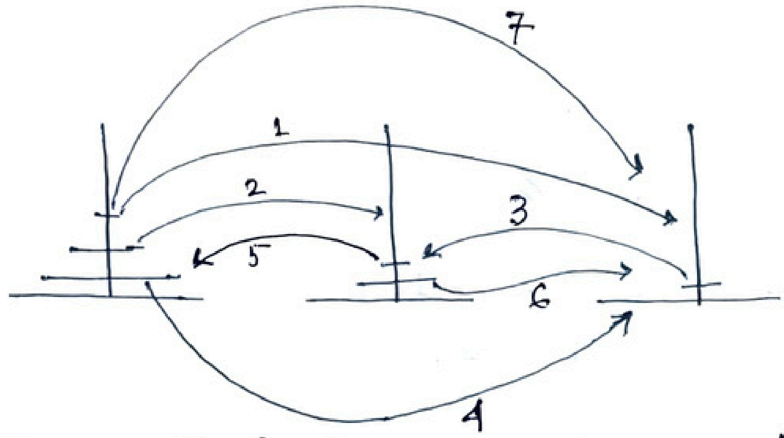
3 discs $T(3) = 7$

d) $2T(2) + 1$

$= 7$







Q. G'08 $x_n \to$ # of binary strings of length n that contain no consecutive 0's.

a) $x_n = 2x_{n-1}$ c) $x_n = x_{n/2} + n$

b) $x_n = x_{n/2} + 1$ ~~d)~~ $x_n = x_{n-1} + x_{n-2}$

$\longrightarrow$ $x_1 = 2$ a) $x_2 = 2 \times 2 = 4$. ×

$x_2 = 3$ b) $x_2 = 2 + 1 = 3$ ✓ $\Rightarrow x_3 =$ ×.

$x_3 = 5$. c) $x_2 = 2 + 2 = 4$ ×.

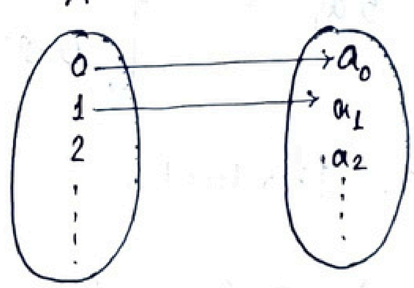
d) $x_3 = x_2 + x_1 = 5$ ✓.

Generating Functions.

Sonendra Gupta

A function whose domain is the set $\{0, 1, 2, 3, \dots\}$ of non-negative integers & whose range is a subset of R, is called a discrete numeric function. (denoted as a_r).

A R



$a_r : A \to R$.

eg. $\{1, 9, 28, \dots\}$

$\Rightarrow \{1, 2^3+1, 3^3+1, \dots\}$

$a_r = r^3 + 1$

eg. $\{0, 3, 6, 7, 15, 31, \dots\}$

$$a_r = \begin{cases} 3r & ; 0 \le r \le 2 \\ 2^r - 1 & ; r \ge 3 \end{cases}$$

eg. $\{0, 3, 6, 9, 12, 15, 7, 8, 9, \dots\}$

$$a_r = \begin{cases} 3r & ; 0 \le r \le 5 \\ r+1 & ; r \ge 6 \end{cases}$$

→ Operations on numeric functions :

$$a_r = \begin{cases} 0 & ; \ 0 \le r \le 2 \\ 2^{-r} + 5 & ; \ r \ge 3 \end{cases}$$

$$b_r = \begin{cases} 3 - 2^r & ; \ 0 \le r \le 1 \\ r + 2 & ; \ r \ge 2. \end{cases}$$

1. Summation.

	0	1	2	3	4	5	
a_r	0	0	0	$2^{-3}+5$	$2^{-4}+5$	$2^{-5}+5$	....
b_r	$3-2^0$	$3-2^1$	$2+2$	$3+2$	$4+2$	$5+2$	....
$c_r = a_r + b_r$	$3-2^0$	$3-2^1$	$2+2$	$2^{-3}+3+7$	$2^{-4}+4+7$	$2^{-5}+5+7$	

$$c_r = \begin{cases} 3 - 2^r & ; \ 0 \le r \le 1 \\ 4 & ; \ r = 2 \\ 2^{-r} + r + 7 & ; \ r \ge 3 \end{cases}$$

2. Multiplication by a real number :

$$a_r = \begin{cases} 0 & ; \ 0 \le r \le 2 \\ 2^{-r} + 5 & ; \ r \ge 3 \end{cases}$$

$$5a_r = \begin{cases} 0 & ; \ 0 \le r \le 2 \\ 5 \cdot 2^{-r} + 25 & ; \ r \ge 3 \end{cases}$$

3. Product.

a_r	0	0	0	$2^{-3}+5$	$2^{-4}+5$	$2^{-5}+5$	....
b_r	$3-2^0$	$3-2^1$	$2+2$	$3+2$	$4+2$	$5+2$	....
$c_r = a_r \cdot b_r$	0	0	0	$(2^{-3}+5)(3+2)$	$(2^{-4}+5)(4+2)$	$(2^{-5}+5)(5+2)$	...

$$c_r = \begin{cases} 0 & ; \ 0 \le r \le 2 \\ (2^{-r} + 5)(r+2) & ; \ r \ge 3. \end{cases}$$

4. Value of a Numeric function.

$|a_r|$ denotes a value at r.

eg. $a_r = (-1)^r \frac{7}{r3}$, $r \geq 0$

$|a_r| = b_r = \frac{7}{r3}$, $r \geq 0$

5. Forward & Backward difference.

$\Delta a_r = a_{r+1} - a_r$

$\nabla a_r = a_r - a_{r-1}$; $r \geq 1$

$\nabla a_0 = a_0$

eg. $a_r = \begin{cases} 0 & ; 0 \leq r \leq 2 \\ 2^{-r}+7 & ; r \geq 3. \end{cases}$

Calculating. $b_r = \Delta a_r = \cancel{\frac{1}{2^r+1}} - \frac{1}{2^{r+1}}$

$c_r = \nabla a_r = -\frac{1}{2^r}.$

6. Convolution :

$c_r = a_r * b_r = \sum_{K=0}^{r} a_K \cdot b_{r-K}$ such that $c_0 = a_0 b_0$.

eg. $a_r = \begin{cases} 1 & ; 0 \leq r \leq 2 \\ 0 & ; r \geq 3 \end{cases}$ $b_r = \begin{cases} 1 & ; 0 \leq r \leq 2 \\ 0 & ; r \geq 3 \end{cases}$

$c_0 = a_0 b_0 = 1$

$c_1 = \sum_{K=0}^{1} a_K b_{1-K} = a_0 b_1 + a_1 b_0 = 2.$

$c_2 = \sum_{K=0}^{2} a_K b_{2-K} = a_0 b_2 + a_1 b_1 + a_2 b_0 = 3$

$c_3 = 2$ $c_4 = 1$ $c_5 = 0$

$c_r = \begin{cases} 1 & ; r = 0,4 \\ 2 & ; r = 1,3 \\ 3 & ; r = 2 \\ 0 & ; r \geq 5 \end{cases}$

• If $a_r = (a_0, a_1, a_2, \ldots, a_r, \ldots)$ is a discrete numeric function, then an infinite series in terms of a parameter z written as

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_r z^r + \ldots$$

is said to be a generating function of the numeric function a_r & also written in the form

$$A(z) = \sum_{r=0}^{\infty} a_r z^r.$$

eg. $a_r = (1, 3, 9, 27, \ldots)$

$(1+\alpha)^{-1} = 1 - \alpha + \alpha^2 - \alpha^3 + \ldots$

$a_0 = 1 \quad a_1 = 3 \quad a_2 = 9 \quad \ldots$

$$A(z) = 1 + 3z + 9z^2 + 27z^3 + \ldots$$
$$= 1 + 3z + 3^2 z^2 + 3^3 z^3 + \ldots$$
$$= 1 + 3z + (3z)^2 + (3z)^3 + \ldots$$
$$= (1 - 3z)^{-1}$$

– Operations on generating functions.

1. Sum. $c_r = a_r + b_r$

$$C(z) = A(z) + B(z) = \sum_{r=0}^{\infty} (a_r + b_r) z^r$$

2. Scalar multiplication

$$b_r = K a_r \Rightarrow B_z = K A_z = K \sum_{r=0}^{\infty} a_r z^r.$$

3. Product

$$c_r = a_r \cdot b_r \Rightarrow C(z) = A(z) \cdot B(z) = c_r z^r$$

$$c_r = \sum_{k=0}^{r} a_k b_{r-k}$$

4. Multiplication of α^r with a_r

$$b_r = \alpha^r a_r \Rightarrow B(z) = A(\alpha z) = \sum_{r=0}^{\infty} a_r (\alpha z)^r$$

Numeric functions. $\qquad\qquad A(z)$.

$a_r = 1 \qquad 1 + z + z^2 + \dots + z^r + \dots = \frac{1}{1-z}$.

$a_r = k \qquad k + kz + \dots + kz^r + \dots = \frac{k}{1-z}$.

$a_r = \alpha^r \qquad 1 + \alpha z + \alpha^2 z^2 + \dots + \alpha^r z^r + \dots = \frac{1}{1-\alpha z}$.

$a_r = r \qquad z + 2z^2 + 3z^3 + \dots + rz^r + \dots = \frac{z}{(1-z)^2}$

$a_r = r^2 \qquad z + 2^2 z^2 + 3^2 z^3 + \dots + r^2 z^r + \dots = \frac{z(1+z)}{(1-z)^3}$

$a_r = r(r+1) \qquad 1.2z + 2.3z^2 + 3.4z^3 + \dots + r(r+1)z^r + \dots$

$= \frac{2z}{(1-z)^3}$.

eg. Find generating function of 5.2^{r+2}.

$a_r = 5 \cdot 2^{r+2}$.

$A(z) = 5 \cdot 2^2 + 5 \cdot 2^3 z + 5 \cdot 2^4 z^2 + 5 \cdot 2^5 z^3 + \dots$

$A(z) = 5 \cdot 2^2 (1 + 2z + 2^2 z^2 + 2^3 z^3 + \dots)$ $\qquad (-2)^r$

$= \frac{20}{1-2z}$.

eg. $a_r = 2^r + 3^r$.

$b_r = 2^r \qquad c_r = 3^r$

$\downarrow \qquad\qquad \downarrow$

$B(z) = \frac{1}{1-2z} \qquad C(z) = \frac{1}{1-3z}$.

$A(z) = B(z) + C(z) = \frac{1}{1-2z} + \frac{1}{1-3z}$.

eg. $a_r = \begin{cases} 2^r & ; \text{if } r \text{ is even} \\ -2^r & ; \text{if } r \text{ is odd}. \end{cases}$

$A(z) = 1 - 2z + 2^2 z^2 - 2^3 z^3 + 2^4 z^4 + \dots$ $\qquad$ | CR = $(-2z)$

$= \frac{1}{1+2z}$.

eg. $(2, 3, 5, 9, 17, 33, \ldots)$

$$A(z) = 2 + 3z + 5z^2 + 9z^3 + 17z^4 + 33z^5 + \ldots$$

$$= (1 + z + z^2 + z^3 + z^4 + z^5 + \ldots) + (1 + 2z + 4z^2 + 8z^3 + 16z^4 + 32z^5 + \ldots)$$

$$= \frac{1}{1-z} + \frac{1}{1-2z} = \frac{2-3z}{(1-z)(1-2z)}$$

eg. $r(r+1) = a_r$

$$A(z) = 0 + 2z + 6z^2 + 12z^3 + 20z^4 + \ldots$$

$$= 2z(1 + 3z + 6z^2 + 10z^3 + \ldots)$$

$$= 2z(1-z)^{-3}$$

$(1-\alpha)^{-3} = 1 + 3\alpha + 6\alpha^2 + 10\alpha^3 + \ldots$

eg. $a_r = r \cdot 5^r$

$$A(z) = 0 + 5z + 2 \cdot 5^2 z^2 + 3 \cdot 5^3 \cdot z^3 + 4 \cdot 5^4 z^4 + \ldots$$

$$= 5z[1 + 2(5z) + 3(5z)^2 + 4(5z)^3 + \ldots]$$

$$= 5z(1-5z)^{-2}$$

$(1-\alpha)^{-2} = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \ldots$

Recurrence Relations Solution

※ Types of recurrence relⁿs

1. Divide & Conquer $T(n) = aT(n/b) + f(n)$.

(Can be solved by Master Method)

2. Linear recurrence $T(n) = aT(n-b) + f(n)$

(solvable by substitution method)

Linear homogeneous recurrence ~

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

※ eg. Solve $x_1 = 3$, $x_n = 3x_{n-1}$

$x_n = 3^n$

Solving Linear Recurrence

⇒ Recurrence $a_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$

& solⁿ be $x_n = ar^n$

$$ar^n = c_1 ar^{n-1} + c_2 ar^{n-2} + \cdots + c_k ar^{n-k}$$

Since, $a, r \neq 0$, dividing by ar^{n-k}

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0.$$ (Characteristic polynomial)

$x_n = ar^n$ satisfies if r is a root of this polynomial eqn.

Note: If $a_1 r_1^n$ & $a_2 r_2^n$ satisfy the recurrence their sum will also satisfy it.

So, solving linear recurrence involves:

1. Find characteristic polynomial
2. Find roots $r_1, r_2, \dots, r_k$ of the char. poly.
3. Assuming no multiple roots, the closed form:

$$x_n = a_1 r_1^n + a_2 r_2^n + \dots + a_k r_k^n$$

4. Use initial values to find values of a_i's.

eg $x_n = 5x_{n-1} + 6x_{n-2}$ $\quad x_0 = 1$

$n \geq 2$ $\quad x_1 = 3$

Char. poly. is found by letting the lowest indexed term be x_k^*, & replacing all x_i's by a^{i-k}.

$$x_n = 5x_{n-1} + 6x_{n-2}$$

$n-k = n-2$

$k = 2$

$$\Rightarrow ar^n = 5 \cdot ar^{n-1} + 6ar^{n-2}$$

$$\Rightarrow r^2 = 5r + 6$$

$r_1 = 6 \quad r_2 = -1$

$$\Rightarrow x_n = a_1 (-1)^n + a_2 (6)^n$$

$n = 0 \Rightarrow$

$x_0 = a_1 + a_2 = 1.$

$\Rightarrow \quad a_1 = 3/7$

$n = 1 \rightarrow$

$x_1 = -a_1 + 6a_2 = 3.$

$a_2 = 4/7$

Closed form $\Rightarrow$

$$x_n = \frac{3}{7}(-1)^n + \frac{4}{7}(6)^n.$$

※ Repeated roots in the Characteristic polynomial

eg. One repeated root.

$x_n = 4x_{n-1} + 4x_{n-2} \quad n \geq 2 \quad | \quad x_0 = 0 \quad x_1 = 1.$

$ar^n - 4ar^{n-1} + 4ar^{n-2}$

$\Rightarrow r^2 - 4r - 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0 \quad r = 2, 2.$

(2^n) is one of the sol^{n}s.

Other sol^n $\quad \beta_n = n \cdot 2^n.$

$$x_n = a_1 2^n + a_2 \cdot n \cdot 2^n = 2^n(a_1 + na_2)$$

Solving, $\quad x_n = n\,2^{n-1}.$

※ In the process of solving recurrences with repeated roots, if r is a root of the char. poly. with multiplicity $k > 1$, then we need to consider the sequences

$$r^n, nr^n, n^2r^n, \ldots, n^{k-1}r^n$$ as part of the closed form of the sol^n of the recurrence.

※ General Method

1. Characteristic polynomial.
2. Find roots $r_1, r_2, \ldots, r_m$ where $m \leq k$ with multiplicities $l_1, l_2, \ldots, l_m$.
3. Closed form can be expressed as a linear combination of the sequences of form
   $n^j r_i^n$, $0 \leq j \leq l_i - 1$ & $1 \leq i \leq m$.
4. Use init. values to find constants.

eg 3rd degree ~~recu~~ polynomial:

$$x_n = 6x_{n-1} + 8x_{n-3} - 12x_{n-2}$$

$x_0 = -1$
$x_1 = 0$
$x_2 = 1$

$$r^3 = 6r^2 + 8 - 12r$$

$$\Rightarrow r^3 - 6r^2 + 12r - 8 = 0$$

$(r-2)^3 = 0$

$r = 2$ with multiplicity 3.

$x_n = a_1 2^n + a_2 \cdot n 2^n + a_3 n^2 2^n$

$x_n = -\frac{2^n}{8}\left(3n^2 - 11n + 8\right).$

eg 3rd deg. polynomial 2 roots, one has mul. 2.

$x_n = 4x_{n-1} + 3x_{n-2} - 18x_{n-3}$

$x_0 = 1$
$x_1 = 1$
$x_2 = 2.$

$x_n = ar^n$

$\Rightarrow r^3 - 4r^2 - 3r + 18 = 0.$

$r^3 + 2r^2 - 6r^2 - 12r + 9r + 18$

$\Rightarrow r^2(r+2) - 6r(r+2) + 9(r+2) = 0$

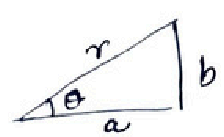
$-8 - 4\cdot 4 + 6 + 18 = 0.$

$\Rightarrow (r-3)^2(r+2) = 0$

$x_n = a_1 3^n + a_2 n 3^n + a_3 (-2)^n$

$\Rightarrow x_n = \frac{1}{15}\left[(12-5n)3^n + 3(-2)^n\right]$

eg Recurrence with complex roots.



$x_n = x_{n-1} - 4x_{n-2}$ $\quad x_0 = 0 \quad x_1 = 1$

$r = \sqrt{1/4 + 15/4} = 2$

$r^2 - r + 4 = 0$

$a \pm ib = r\cos\theta \pm i r\sin\theta = r e^{\pm i\theta}$

$r = \frac{1 \pm i\sqrt{15}}{2} = 2e^{\pm i\theta}$ $\quad \theta = \arctan\sqrt{15}$

$$x_n = a_1 2^n \frac{e^{in\theta} + e^{-in\theta}}{2} + a_2 2^n \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

$r = 2e^{\pm i\theta}$

$$= a_1 2^n \cos n\theta + a_2 2^n \sin n\theta$$

$$x_n = \frac{2^{n+1}}{\sqrt{15}} \sin(n\theta).$$

$$\boxed{(r \cdot e^{i\theta})^n = r^n e^{in\theta}}$$

$\boxed{a_1 2^n e^{\pm in\theta} + a_2 2^n e^{\mp in\theta}}$

Generally, $r^2 - pr - q = 0$.

When $p^2 + 4q < 0$,

2 complex conjugate sol^{n}s $(a+bi), (a-bi)$

For real-valued sol^{n}s,

$$c_n = \frac{(a+bi)^n + (a-bi)^n}{2}$$

$$d_n = \frac{(a+bi)^n - (a-bi)^n}{2i}$$

if $\rho = \sqrt{a^2+b^2}$, θ is such that

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}.$$

then

$$c_n = \rho^n \cos n\theta$$

$$d_n = \rho^n \sin n\theta$$

$$x_n = c_1 \rho^n \cos n\theta + c_2 \rho^n \sin n\theta$$

* Finding recurrence relⁿ for a sequence given in Closed form

Closed form $n^j r^n$

$\downarrow$

Find the roots, multiplicities

$\downarrow$

Characteristic polynomial

$\downarrow$

Recurrence relⁿ

eg $x_n = (-3)^n (3+4n) + n^2 2^n$

roots are $-3, 2$ with multiplicities 2 and 3.

$\downarrow$ $(1+1)$ $\quad$ $\downarrow$ $(2+1)$

Chara. poly. $(r+3)^2 (r-2)^3$ $\quad | k=5$

*
$$= r^5 - 15r^3 + 10r^2 + 60r - 72$$

$$\therefore \underset{(5-5)}{x_n} = 15\underset{(5-3)}{x_{n-2}} - 10\underset{(5-2)}{x_{n-3}} - 60\underset{(5-1)}{x_{n-4}} + 72\underset{(5-0)}{x_{n-5}}$$

eg $x_n = a\sin n\theta + b\cos n\theta$

$x_1 = 15,\ x_2 = 3,\ x_3 = -12.$ $\boxed{\text{find } x_7}$

$$\sin n\theta + \sin(n-2)\theta = 2\sin\frac{2(n-1)\theta}{2}\cos\theta$$

$$\cos n\theta + \cos(n-2)\theta = 2\cos\frac{2(n-1)\theta}{2}\cos\theta$$

$$x_n = 2\cos\theta\, x_{n-1} - x_{n-2} \quad \left[x_n + x_{n-2} = \ldots 2\cos\theta \cdot x_{n-1}\right]$$

$x_n + x_{n-2} = 2a\sin(n-1)\theta\cos\theta + 2b\cos(n-1)\theta\cos\theta$

$x_3 = -12 = 2\cos\theta \cdot 3 - 15$

$\Rightarrow \cos\theta = 1/2.$

$x_n = x_{n-1} - \underline{x_{n-2}}$ $\quad x_7 = 15.$

or

$x_1 + x_3 = a\sin\theta + b\cos\theta + a\sin 3\theta + b\cos 3\theta$

$3 = a \cdot 2\sin 2\theta\cos\theta + b \cdot 2\cos 2\theta\cos\theta$

$\Rightarrow 2\cos\theta\,[\underbrace{a\sin 2\theta + b\cos 2\theta}_{x_2}] = 3$

$\Rightarrow x_2 = 1 \Rightarrow \cos\theta = 1/2.$

$\theta = 2n\pi \pm \pi/3$

$7\theta = 14n\pi \pm 7\pi/3$

$7\theta = 2\pi(7n \pm 1) \pm \pi/3$

$7\theta = \theta \Rightarrow x_7 = x_1 = 15.$

eg Solve recurrence $a_n = a_{n-1} + n$, $a_0 = 4$.

$\longrightarrow$ 4, 5, 7, 10, 14, 19, ...

(Telescoping)

$a_1 - a_0 = 1$

$a_2 - a_1 = 2$

$a_3 - a_2 = 3$

$\vdots$

$a_n - a_{n-1} = n$

$(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = \frac{n(n+1)}{2}$

$\Rightarrow -a_0 + a_n = \frac{n(n+1)}{2}$

$\Rightarrow a_n = \frac{n(n+1)}{2} + 4$

(by iteration)

$a_1 = a_0 + 1$

$a_2 = (a_0 + 1) + 2$

$a_3 = ((a_0 + 1) + 2) + 3$

$\vdots$

$a_n = a_0 + \frac{n(n+1)}{2} = 4 + \frac{n(n+1)}{2}$

$$\boxed{\begin{aligned} a_n &= \cancel{a_{n-1}} + n \\ \cancel{a_{n-1}} &= \cancel{a_{n-2}} + n - 1 \\ &\vdots \\ \cancel{a_1} &= a_0 + 1 \\ \hline a_n &= (1 + 2 + \cdots + n) + a_0 \end{aligned}}$$

eg (By iteration) $a_n = 3a_{n-1} + 2$; $a_0 = 1$

$a_1 = 3a_0 + 2$

$a_2 = 3(3a_0 + 2) + 2 = 3^2 a_0 + 3 \cdot 2 + 2$

$a_3 = 3\{3(3a_0 + 2) + 2\} + 2 = 3^3 a_0 + 3^2 \cdot 2 + 3 \cdot 2 + 2$

$\vdots$

$a_n = 3(3^{n-1} a_0 + 2 \cdot 3^{n-2} + \cdots + 2 \cdot 3 + 2) + 2$

$= 3^n a_0 + (2 \cdot 3^{n-1} + \cdots + 2 \cdot 3 + 2)$

$= 3^n + \frac{2(1 - 3^n)}{1 - 3} = 3^n + \frac{2 - 2 \cdot 3^n}{-2}$

$= 2 \cdot 3^n - 1$.

$a_n = 3\cancel{a_{n-1}} + 2 \quad \times 1 \qquad a_0 = 1,$

$\cancel{a_{n-1}} = 3\cancel{a_{n-2}} + 2 \quad \times 3^1$

$\cancel{a_{n-2}} = 3\cancel{a_{n-3}} + 2 \quad \times 3^2$

$\times 3^3$

$\boxed{\begin{matrix} n - n + 1 \\ n - (n-1) \end{matrix}}$

$\vdots$

$\cancel{a_2} = 3\cancel{a_1} + 2 \quad \times 3^{n-2}$

$\cancel{a_1} = 3a_0 + 2 \quad \times 3^{n-1}$

$a_n = 3^n a_0 + 2(3^0 + 3^1 + 3^2 + \cdots + 3^{n-1})$

$= 3^n + \cancel{2} \cdot \dfrac{-1 + 3^n}{\cancel{-1+3}}$

$= 2 \cdot 3^n - 1.$

Using general sol^n method for linear recurrence:

$$a_n = 3a_{n-1} + 2 \qquad \Big| \qquad a_n = pr^n$$

$$\Rightarrow pr^n = 3 \cdot pr^{n-1} + 2$$

eg $\quad a_n = 7a_{n-1} - 10a_{n-2} \quad ; \quad a_0 = 2, \; a_1 = 3$

$$pr^n = 7pr^{n-1} - 10pr^{n-2}$$

$$\Rightarrow r^2 - 7[r] + 10 = 0 \qquad r = 5, 2.$$

$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot 2^n.$$

$$a_0 = \alpha_1 + \alpha_2 = 2 \qquad \Rightarrow \quad a_n = \frac{7}{3}2^n - \frac{1}{3}3^n$$

$$a_1 = 5\alpha_1 + 2\alpha_2 = 3$$

eg $\quad F_n = F_{n-1} + F_{n-2} \qquad F_0 = 0 \qquad F_1 = 1$

$$\Rightarrow pr^n = pr^{n-1} + pr^{n-2}$$

$$\Rightarrow r^2 - r - 1 = 0. \qquad r = \frac{1 \pm \sqrt{5}}{2}$$

$$F_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$$F_0 = 0 = \alpha_1 + \alpha_2 \qquad \Rightarrow F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_1 = 1 = \frac{1+\sqrt{5}}{2}\alpha_1 + \frac{1-\sqrt{5}}{2}\alpha_2$$

eg. $a_n = 6a_{n-1} - 9a_{n-2}$ | $a_0 = 1,\ a_1 = 4$

$$r^2 - 6r + 9 = 0$$

$$\Rightarrow (r-3)^2 = 0. \qquad r = 3, 3$$

$$a_n = an\,3^n + b\,3^n$$

$$\Rightarrow a_n = 3^n + \tfrac{1}{3} n 3^n.$$

※ Recurrence of the form

$$A a_n = B a_{n-1} + C.$$

| Method of Summation Factors.

(Brilliant.org)

Summation factor s_n

$$A s_n a_n = B s_n a_{n-1} + C s_n$$

s_n chosen such that

$$s_n = \frac{A_{n-1} A_{n-2} \cdots A_2 A_1}{B_n B_{n-1} \cdots B_2 B_2}$$

$$b_n = A s_n a_n \qquad b_{n-1} = B s_n a_{n-1}$$

$$b_n = b_{n-1} + C s_n$$

$$b_n = b_0 + \sum_{k=1}^{n} C s_k$$

$$a_n = \frac{1}{A s_n}\left(A_{n=0}\, s_0 a_0 + \sum_{k=1}^{n} C s_k\right)$$

eg

$$T_n = 2T_{n-1} + 1 \qquad T_0 = 0$$

$$A_n = 1, \quad B_n = 2$$

$$S_n = \frac{1 \times 1 \times \cdots \times 1}{2 \times 2 \times \cdots \times 2} = \frac{1}{2^{n-1}}$$

$$\frac{1}{2^{n-1}} T_n = \frac{1}{2^{n-2}} T_{n-1} + \frac{1}{2^{n-1}}$$

Let $b_n = \dfrac{T_n}{2^{n-1}}$, $b_{n-1} = \dfrac{T_{n-1}}{2^{n-2}}$, $b_0 = \dfrac{T_0}{2^{0-1}} = 0$

$$b_n = b_{n-1} + \frac{1}{2^{n-1}}$$

$$b_n = 0 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = \frac{1\left(\frac{1}{2}\right)^{n} - 1}{\frac{1}{2} - 1}$$

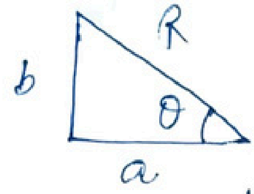
$$= 2 - \frac{1}{2^{n-1}}$$

$$b_n = T_n / 2^{n-1}$$

$$\Rightarrow T_n = \underline{2^n - 1}$$

- Complex roots $a \pm ib$, corresponding general sol^n $\underline{R^n \left[p_1 \cos n\theta + p_2 \sin n\theta \right]}$, $R = \sqrt{a^2+b^2}$, $\theta = \tan^{-1} \frac{b}{a}$

$$a_n = c_1 (a+ib)^n + c_2 (a-ib)^n.$$



$$= c_1 (R\cos\theta + iR\sin\theta)^n + c_2 (R\cos\theta - iR\sin\theta)^n$$

$\sin\theta = b/\sqrt{a^2+b^2}$

$\cos\theta = a/\sqrt{a^2+b^2}$

$$= c_1 R^n (\cos\theta + i\sin\theta)^n + c_2 R^n (\cos\theta - i\sin\theta)^n$$

$$= c_1 R^n (\cos n\theta + i\sin n\theta) + c_2 R^n (\cos n\theta - i\sin n\theta)$$

$$* = R^n \left[\underbrace{(c_1 + c_2)}_{p_1} \cos n\theta + \underbrace{(c_1 - c_2)}_{p_2} \sin n\theta \right]$$

1. $a_{n+2} + a_n = 0$

$r^{n+2} + r^n = 0$

$r^2 + 1 = 0$

$r = \pm\sqrt{-1}$

$r = 0 \pm i.1 \rightarrow$ Check a, b, to determine coordinate quadrant

$R = \sqrt{1} = 1$

$\theta = \tan^{-1}(b/a)$

$= \pi/2$

$$a_n = p_1 \cos n\theta + p_2 \sin n\theta$$

$$= p_1 \cos \frac{n\pi}{2} + p_2 \sin n\pi/2$$

eg $a_n - 2a_{n-1} + 2a_{n-2} - a_{n-3} = 0$

$\Rightarrow r^3 - 2r^2 + 2r - 1 = 0$

$r^3 - r^2 - r^2 + r + r - 1 = 0$

$\Rightarrow r^2(r-1) - r(r-1) + 1(r-1) = 0$

$\Rightarrow (r-1)(r^2 - r + 1) = 0$

$r = 1, \frac{1 \pm \sqrt{-3}}{2}$

$\frac{1}{2} \pm i\left(\frac{\sqrt{3}}{2}\right)$

$R = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\theta = \tan^{-1}(\sqrt{3})$

$= \pi/3$

$a_n = \alpha(1)^n + R^n[a \cos n\theta + b \sin n\theta]$

$= \alpha + a \cos n\pi/3 + b \sin n\pi/3$

- **Non-homogeneous Linear RR**

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$

degree/order = k

- $f(n) = (b_0 n^0 + b_1 n^1 + b_2 n^2 + \dots + b_t n^t) k^n$ where $b_0, b_1, \dots, b_t$ & k are real numbers. Particular sol^n of the form

$(p_0 n^0 + p_1 n^1 + p_2 n^2 + \dots + p_t n^t) k^n$ if k is not a root of the characteristic equ^n of associated linear homogeneous recurrence rel^n.

- sol^n = Particular sol^n + Homogeneous sol^n

eg $a_{n+2} - a_{n+1} - a_n = 4$

$f(n) = 4. = (4n^0)\, 1^n.$

$r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{5}}{2}$

$\downarrow$

Homogeneous solⁿ.

Particular solⁿ $= (p_0 n^0) \cdot 1^n = p_0$

Substituting,

$p_0 - p_0 - p_0 = 4 \Rightarrow p_0 = -4.$

$\therefore a_n = HS + PS$

$= c_1 \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2}\right) - 4.$

eg $a_{n+2} - 4a_n = n$

HS

$a_{n+2} - 4a_n = 0$

$r^2 - 4 = 0$

$r = \pm 2$

$HS = c_1 2^n + c_2 (-2)^n$

PS

$f(n) = n = (1 \cdot n^1)\, 1^n$

$PS = (p_0 n^0 + p_1 n)\, 1^n$

$= p_0 + p_1 n.$

Substituting,

$p_0 + p_1(n+2) - 4\{p_0 + p_1 n\} = n$

$\Rightarrow n(p_1 - 4p_1) + (p_0 + 2p_1 - 4p_0) = n$

$p_1 - 4p_1 = 1$

$\Rightarrow -3p_1 = 1$

$p_1 = -1/3$

$-3p_0 - 2/3 = 0$

$\Rightarrow p_0 = -2/9$

$PS = -1/3 - 2/9\, n$

Solⁿ. $a_n =$

$(-1/3 - 2/9\, n) +$

$(c_1 2^n + c_2 (-2)^n).$

eg $\quad a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

HS

$$r^2 + 5\cdot r + 6 = 0$$

$$\Rightarrow r = -3, -2$$

$$HS = c_1(-3)^n + c_2(-2)^n$$

PS

$$f(n) = 3n^2$$

$$= (3\cdot n^2)\, 1^n.$$

$$PS = (p_0 n^0 + p_1 n^1 + p_2 n^2)\, 1^n$$

$$= p_0 + p_1 n + p_2 n^2$$

substituting,

$$p_0 + p_1 n + p_2 n^2 + 5p_0 + 5p_1(n-1) + 5p_2(n-1)^2$$

$$+ 6p_0 + 6p_1(n-2) + 6p_2(n-2)^2 = 3n^2$$

$-24p_2$

$$\Rightarrow n^2(p_2 + 5p_2 + 6p_2) + n(p_1 + 5p_1 - 10p_2 + 6p_1 - 12p_2)$$

$$+ (6p_0 + 5p_1 + 5p_2 + 6p_0 - 12p_1 + 24p_2) = 3n^2$$

72

$$\Rightarrow 12p_2 = 3$$

$$p_2 = 1/4$$

$$12p_1 - 22p_2 = 0$$

$$\Rightarrow 6p_1 = 11\cdot 1/4$$

$$\Rightarrow p_1 = \frac{11}{24}$$

$$12p_0 - 7p_1 + 29p_2 = 0$$

$$\cancel{\Rightarrow 3 - \frac{77}{24} + 29p_2 = 0}$$

$$\cancel{\Rightarrow p_2 = \frac{5}{24\cdot 29}}$$

$$\Rightarrow 12p_0 - \frac{77}{24} + \frac{29}{4} = 0$$

$$\Rightarrow 12p_0 + \frac{-77+174}{24} = 0$$

$$\Rightarrow p_0 = -\frac{97}{12\cdot 24}$$

$$PS = -97/_{12\cdot 24} + 11/_{24}\, n + 1/_4\, n^2$$

Sol^n: $a_n = HS + PS$

eg $a_n + 5a_{n-1} + 6a_{n-2} = 42(4^n)$.

HS

$r^2 + 5r + 6 = 0$

$\Rightarrow r = -2, -3$

$HS = \alpha(-2)^n + \beta(-3)^n$

PS $f(n) = 42(4^n) = (1 \cdot 42 \cdot n^0) \cdot 4^n$

$PS = (p_0)\, 4^n$.

$p_0 \cdot 4^n + 5 \cdot p_0 \cdot 4^{n-1} + 6 \cdot p_0 \cdot 4^{n-2} = 42(4^n)$

$\Rightarrow p_0 \left(1 + 5 \cdot \frac{1}{4} + 6 \cdot \frac{1}{4^2}\right) = 42$

$\Rightarrow p_0 \cdot \frac{16 + 20 + 6}{16} = 42 \quad \Rightarrow p_0 = 16$

$\therefore a_n = \alpha(-2)^n + \beta(-3)^n + 16(4^n)$.

eg $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$.

HS

$r^2 - 6r + 9 = 0$

$\Rightarrow r = 3, 3$

$HS = \alpha \cdot 3^n + \beta \cdot n \cdot 3^n$

PS $f(n) = 3^n = (1 n^0)\, 3^n$ root same

$ps = (p_0)\, 3^n \cdot n^2$

substituting,

$p_0 \cdot 3^n \cdot n^2 - 6 \cdot p_0\, 3^{n-1} (n-1)^2 + 9\, p_0 (3^{n-2})(n-2)^2 = 3^n$

$\Rightarrow p_0 n^2 - 6 p_0 \frac{1}{3}(n-1)^2 + 9 p_0 \frac{1}{3^2}(n-2)^2 = 1$.

$\Rightarrow p_0 n^2 - 2p_0 (n-1)^2 + p_0 (n-2)^2 = 1$

$-2p_0 + 4p_0 = 1$

$p_0 = \frac{1}{2}$ | $PS = \frac{1}{2} 3^n \cdot n^2$.

$\boxed{k \text{ root of char. eq}^n}$

$\Downarrow$

$(b_0 n^0 + b_1 n^1 + \dots + b_t n^t)\, k^n$

$\downarrow$

$(p_0 n^0 + p_1 n^1 + \dots + p_t n^t)\, k^n \cdot n^m$

multiplicity m of root k

※ Non-linear Recurrence Relⁿ

eg $\sqrt{a_n} - \sqrt{a_{n-1}} - 2\sqrt{a_{n-2}} = 0$; $a_0 = a_1 = 1$

$a_n = x_n^2$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$\Rightarrow x_n - x_{n-1} - 2x_{n-2} = 0$

$x_n = c_1(2)^n + c_2(-1)^n$. $x_0 = 1, x_1 = 1$.

$\sqrt{a_n} = -\frac{1}{2}(2^n) + \frac{3}{2}(-1)^n$

$a_n = \left(-2^{n-1} + \frac{3}{2}(-1)^n\right)^2$.

eg $a_n = 3a_{n/2} + n$. , $n = 2^k$ for $k \geq 1$, $a_1 = 1$

$a_{2^k} = 3a_{2^{k-1}} + 2^k$

$x_k = 3x_{k-1} + 2^k$

$2^k = n$

$k = \log_2 n$

HS

$r = 3$

PS

$PS = \alpha 2^k$

$\alpha 2^k = 3 \cdot \alpha \cdot 2^{k-1} + 2^k$

$(-1+\alpha)(2^k) = 3\alpha 2^{k-1}$

$\Rightarrow \frac{\alpha - 1}{3\alpha} = \frac{1}{2}$

$\Rightarrow \alpha = -2$

$x_k = c \cdot 3^k - 2^{k+1}$. $x_0 = 1$

$c \to 3$

$a_n = 3 \cdot 3^k - 2 \cdot 2^k$

$= 3 \cdot 3^{\log_2 n} - 2 \cdot n = 3n^{\log_2 3} - 2n$.