GATE CSE NOTES

by

UseMyNotes

DM

Set, Relation, Functions.

- $A-(B\cup C) = (A-B)\cap(A-C)$ | $(A\cup B)\cap C = A\cup(B\cap C)$ iff $A\subseteq C$
  $A-(B\cap C) = (A-B)\cup(A-C)$ | $(A\cap B)\cup C = A\cap(B\cup C)$ iff $A\supseteq C$

- $\lambda\in A \iff A^+ = A^*$ | $(A^*)^+ = (A^+)^* = A^*$
  $(A^*B^*)^* = (A\cup B)^* = (A^*\cup B^*)^*$
  $(A^*)^* = A^*A^* = A^*$ | $A^*A^+ = A^+A^* = A^*$

- $A\oplus B = A\,\Delta\,B = (A\cup B)-(A\cap B)$
  $= (A-B)\cup(B-A)$
  $= (A'\cap B)\cup(A\cap B')$

- $|P(S)| = 2^{|S|}$, $|P(P(S))| = 2^{2^{|S|}}$

- Reflexive: R^{-1}, $R\cup S$, $R\cap S$, superset [not subset]
- Irreflexive: R', $R\cup S$, $R\cap S$
- Symmetric: R^{-1}, $R\cup S$, $R\cap S$, R_1-R_2 [not superset, subset] need not be
- Antisymmetric $R\cap S$, $R-S$, subset [not superset, $R\cup S$, R']
- Transitive R^{-1}, $R\cap S$, (not $R\cup S$, R')
- Equivalence rel^n $R\cap S$ [not $R\cup S$]
- Asymmetric subset, intersection, difference (not superset, union, complement)
- Irreflexive and antisym $\rightarrow$ Asymmetric.

If R is antisym, $R\cap R^{-1}\subseteq\Delta_A$ (diagonal rel^n)

Diagonal elements can be present in antisym, not in asymmetric rel^n.

Reflexive cannot be asymmetric.

Every asymmetric rel^n is irreflexive, not vice versa. $\{(1,2),(2,1)\}$.

Every asym. rel^n is antisym., not vice versa.

R on A is transitive iff $R^n\subseteq R$.

A transitive rel^n is asymmetric iff it is irreflexive.

#	Smallest	Largest
Reflexive	$\lvert\Delta_A\rvert = n$	$\lvert A\times A\rvert = n^2$
Irreflexive	$\lvert\phi\rvert = 0$	$\lvert A\times A\rvert - \lvert\Delta_A\rvert = n^2-n$
Symmetric	$\lvert\phi\rvert = 0$	$\lvert A\times A\rvert = n^2$
Antisym.	$\lvert\phi\rvert = 0$	~~$\lvert A\times A\rvert = n^2$~~ $\frac{n}{2}(n+1)$
Asymmetric	$\lvert\phi\rvert = 0$	$\frac{n^2-n}{2}$
Transitive	$\lvert\phi\rvert = 0$	n^2
Equivalence	$\lvert\Delta_A\rvert = n$	n^2
Partial order	n	

* There's a path of length n from a to b iff $(a,b)\in R^n$. $R^{n+1} = R^n\circ R$

$R^* = \bigcup_{n=1}^{\infty} R^n$ | $S\circ R$: $R(a,b)$, $S(b,c)$ $\rightarrow$ (a,c)

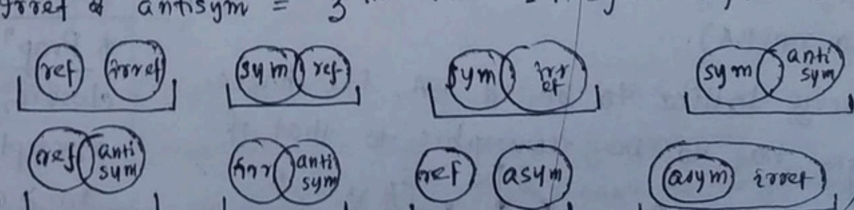
Closure of relations: Reflexive: $R\cup\Delta_A$, Symmetric: $R\cup R^{-1}$,
Transitive: Connectivity rel^n $R^* = \{(a,b)\mid\exists$ a path (in digraph) of length at least one in $R\}$

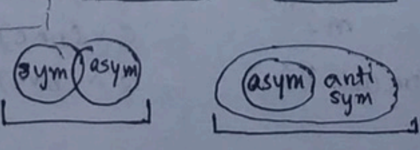
Counting # rel^ns. (total $= 2^{mn}$ or 2^{n^2})

1. # ref = # irref = 2^{n^2-n}
2. # sym = $2^{n(n+1)/2}$
3. # antisym = $2^n\times 3^{(n^2-n)/2}$
4. # asym = $3^{(n^2-n)/2}$
5. # transitive = 2, 13, 171, 3994, .. (from $n=1$)
6. # eqv. rel^ns = Bell(n). $=\sum_{k=1}^{n} S(n,k)$
7. Either ref or irr $= 2\cdot 2^{n^2-n}$
8. neither ref nor irref $= 2^{n^2}-(2\cdot 2^{n^2-n})$
9. Both sym & ref (compatibility rel^n) $= 2^{(n^2-n)/2}$
10. Both sym & irr. $= 2^{(n^2-n)/2}$
11. Both sym, antisym $= 2^n$ $\subseteq\Delta_A$
12. Ref & Antisym $= 3^{(n^2-n)/2}$
13. Irref & antisym $= 3^{(n^2-n)/2}$
14. Sym & asym $= 1$.

#partitions possible for a set into nonempty subsets.

$n=$	
1	1
2	1 2
3	2 3 5
4	5 7 10 15





15. # total order $= n!$
16. # partial order = 1, 3, 19, 219, .. (from $n=1$)

* <u>Partial order relⁿ (PO)</u>: Ref, Antisym, Trans. * <u>POSET</u> [A; PO]

* <u>Total order</u> Partial order such that aRb or bRa $\forall a,b \in A$ [Connexity]

<u>Connex order</u> ↳ or, Binary rel^n which is antisym, transitive & a connex rel^n.

* <u>TOSET</u> ([A; TO]) Toset's hasse diagram is a chain, [$\#v = n$, $\#e = n-1$].

* [Maximal (a) / Minimal (b) / Maximum (1) (c) / Minimum (0) (d) / LUB ($\vee$, Join, Supremum) (g) / GLB ($\wedge$, Meet, Infimum) (h) / LB (e) / UB (f) / Join Semi Lattice (i) / Meet Semi Lattice (j) / Lattice (k) / Sublattice (l), Bounded Lattice (m), Complement (n), Complemented lattice (o), Distributive lattice (p), Boolean algebra (q).

$(S, \leq)$ be the poset or (S, R)

(a) $\nexists b \in S$, aRb or $a \leq b$ → a maximal

(b) $\nexists b \in S$, bRa or $b \leq a$ → a minimal

(c) a maximal & bRa $\forall b \in S$

(d) a minimal & aRb $\forall b \in S$.

(e) LB of $A \subseteq S$, $a \in LB(A)$, iff aRb $\forall b \in A$

(f) UB of $A \subseteq S$, $a \in UB(A)$, iff bRa $\forall b \in A$.

(g) $LUB(A) = a$, iff $a \in UB(A)$ & aRb $\forall b \in UB(A)$

(h) $GLB(A) = a$, iff $a \in LB(A)$ & bRa $\forall b \in LB(A)$

(i) Poset for which every pair of elems has LUB.

(j) Poset for which every pair of elems has GLB.

(k) Poset, every pair has both GLB & LUB.

(l) Subset of a lattice such that the subset also is a lattice and <u>for any pair GLB, LUB are same as the original lattice</u>.

(m) Lattice for which both LB, UB exists.

(n) (of an elem) for a $\exists$ a' s.t. $a \vee a' = 1$, $a \wedge a' = 0$

(o) Each elem has a complement.

(p) That follows distributive $prop^y$.
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(q) Lattice that is distributive & Complemented (thus bounded).

* Every TOSET is a lattice.

* <u>LUB</u>: $\subseteq$ union of sets; $\leq$ maximum elem; $\div$ LCM

* <u>GLB</u>: $\subseteq$ intersection; $\leq$ least elem; $\div$ GCD.

* Finite lattice is always bounded.

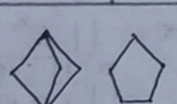
* A complemented lattice is always bounded.

* If a complemented lattice is distributive then each elem has unique complement.
↳ In BA each elem has unique complement.

* In BA, if $a \vee b = a \vee c$, and $a \wedge b = a \wedge c$, then $b = c$.

* In bounded lattice, $a \vee 1 = 1$, $a \wedge 1 = a$, $a \vee 0 = a$, $a \wedge 0 = 0$.

* <u>Well ordered set</u>: $(S, \leq)$ if it is a poset s.t. $\leq$ is a total order & every non empty subset of S has a least element.

*  If in some lattice, any of these is a sublattice, then it can't be distributive.

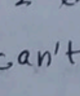
* A lattice with 4 or fewer elems is distributive.

* Every TOSET is a distributive lattice.

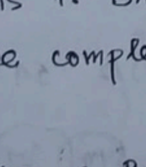
* Every sublattice of a distributive lattice is also a dis. lattice.

* TOSET with 2 elems is BA. TOSET with ≥ 3 elems can't be complemented lattice (so no BA).

* Every lattice to be a BA, its hasse dgm has to be isomorphic to that of [P(A), $\subseteq$].



A=2



A=3

$\#v = 2^n$, $\#e = n\,2^{n-1}$ } for any BA.

* [D_n, $\div$] is a BA iff n is a square free number (unique primes in prime factorizⁿ).
↳ If BA, then compl. (x) = n/x

* Examples of POSET: $(R, \subseteq)$, $(N, \leq)$, $(D_n, \div)$.

* Examples of Distributive lattice: Every BA, Every TOSET, $(N, \leq)$, $(D_n, /)$ — set of +ve divisors

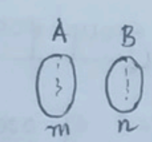
* Examples of BA: $(P(A), \subseteq)$, $(D_n, /)$

* A TOSET can't be a complemented lattice if $|S| \geq 3$.

* <u>Prop^y of lattice</u> (op^ns $\vee, \wedge$) (DMAb)
closure, commutative, associative, idempotent, absorption ($a \vee (a \wedge b) = a$), consistency ($a, b \leq a \vee b$, $a, b \geq a \wedge b$); distributive inequality —
$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

$|A| > |B|$ & $|B| = 2$ then $\frac{\#\text{ onto } f^n s}{A \to B} = 2^{|A|} - 2$

* **Function** $\forall x \in A,\ \exists (x,y) \in R,\ y \in B.$ | Domain
$(x,y) \in R \wedge (x,z) \in R \Rightarrow y = z$ | Codomain ~ Range



$\# f^n s = n^m$

> **Partial f^n** : Subset of elems in A has mapping to B $\searrow$ $\# = (n+1)^m$

> **One to one** (Injective) : $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ | $\# = {}^nP_m$ $\downarrow$ $0,\ n<m$
Choosing m elems to map: $n(n-1)(n-2)\cdots(n-m+1)$

Fun^n: Subset of $A \times B$ such that each elem of A is related to only one elem of B.

> **Many to one** $\#\ n^m - {}^nP_m$

> **Onto** Codomain = range
(Surjective) $\# = n!\left\{ {m \atop n} \right\}$
$= n!,\ n = m$

> **Into** Codomain $\supset$ Range
$\#\ n^m - n!\left\{ {m \atop n} \right\}$

> **Bijection** One to one, onto. $|A| = |B| = n$
$\# = n!$

• $\left\{ {m \atop n} \right\} = S(m,n)$ Stirling's number of 2nd kind
$=$ #ways to partition set of m elements into n subsets

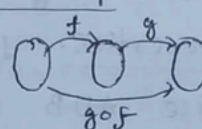
$S(m,n) = S(m-1,n-1) + n\,S(m-1,n).$ $= \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m$
$\Downarrow$
$\#\text{ onto } f^n s = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m$

Building table:

m \ n	0	1	2	3	4
0	1				
1	0	1			
2	0	1	1		
3	0	1	3	1	
4	0	1	7	6	1
⋮					

> Inverse exists only for bijection

> **Composition of f^n** : $f \circ g(x) = f(g(x))$



$g \circ f() = g(f())$

> $g \circ f \neq f \circ g$ > $g \circ (f \circ h) = (g \circ f) \circ h$ > If $g \circ f$ is one to one, then f is one to one but g may not be.

> If f & g i) one to one $\rightarrow$ i) $g \circ f$ one to one
ii) onto $\rightarrow$ ii) $g \circ f$ onto
iii) bijection $\rightarrow$ iii) $g \circ f$ bijection.

> If $g \circ f$ is onto, then g is onto but f may not be.

> $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

> $f: A \to B$, X, Y be non-empty subsets of A, then
1. If $X \subseteq Y$, $f(X) \subseteq f(Y)$, 2. $f(X \cup Y) = f(X) \cup f(Y)$, 3. $f(X \cap Y) \subseteq f(X) \cap f(Y)$
(equality when f is one-one)

Group Theory (DM$_1$)

CAIdIC

* **Semigroup** : Closure, Associative
* **Monoid** : Closure, Associative, Identity
* **Group** : Closure, Associative, Identity, Inverse
* **Abelian group** : Closure, Associative, Identity, Inverse, Commutative.

* **Binary operation (*)**
– $a*b$ is closed.
– $a*b$ is unique.

* **Algebraic structure** (wrt some bin. operation).
eg. $(P(A), \cup)$
$(\{1,-1\}, \times)$

* **Identity** (Unit) : $a*e = e*a = a$

* **Inverse** : $a*a^{-1} = a^{-1}*a = e$

* **Group properties** : i) e is unique. ii) $a*e = e*a$ iii) a^{-1} unique
iv) $(a*b)^{-1} = b^{-1}*a^{-1}$ v) $(a^{-1})^{-1} = a$ vi) $ab = ac \Leftrightarrow b = c$
vii) $ba = ca \Leftrightarrow b = c$ viii) $ax = b \Rightarrow x = a^{-1}b$ is unique, $xa = b \Rightarrow x = ba^{-1}$ is unique. ix) Cayley's table of a finite group has no repetition, in any row or column.

* **Subgroup** : $(H,*)$ is a subgroup of $(G,*)$ iff i) $H \subseteq G$, ii) $(H,*)$ is also a group. or $a*b^{-1} \in H$ $\forall a,b \in H$.

* **Cyclic group** : $(G,*)$ such that $\exists g \in G$, $\forall a \in G$, $a = g^n$, $n \in \mathbb{Z}$
(multiple g's may exist) g ~ generator
• e can't be the generator unless only elem is e.
• if g is generator, g^{-1} too.
• A cyclic group is always abelian. (contrative also)

* **Power of an element**
$a^0 = e$, $a^1 = a$, $a^2 = a*a$, ...
a^{-1} = inverse(a)
a^{-2} = inverse(a^2)
$(a^m)^n = a^{mn}$
$(a^m \cdot a^n) = a^{m+n}$

* **Order of group** : $(G,*)$ #elems in G. Can be ∞.

* **Order of elem** : $o(a)$ = smallest $n \in \mathbb{Z}$ s.t. $a^n = e$

* Abelian group properties. 1. $a*b = b*a$, 2. $\forall a \in G$, $a^{-1} = a$, 3. $(ab)^2 = a^2b^2$
4. Every cyclic group is abelian.

$\boxed{a*b \approx ab}$

* Properties of order of elem 1. order of an elem $o(a)$ $\leq$ order of group $o(G)$
2. $o(a)$ divides $o(G)$ for finite group. 3. for finite group $o(a)$ exists & is finite.
4. $o(G)$ = prime(p) $\Rightarrow$ $o(a) = 1$ or p, $o(e) = 1$; $o(a) = p$ $\forall a \neq e$, 5. $o(ab) = n \Leftrightarrow (ba)^n = e$
$\Leftrightarrow (ab)^n = e$ 6. $o(ab) = o(ba)$ for all groups.

* Cyclic group properties. 1. Group having order = prime#, it's cyclic. generators = $G - \{e\}$.
2. $(G,*)$ is cyclic $\Leftrightarrow$ $\exists a$, s.t. $o(a) = o(G)$. 3. #generators of a cyclic group of order n = $\phi(n)$ Euler's totient f^n. $\phi(n) = n-1$ when n prime. If n not prime, use prime factorization to determine $\phi(n)$. $n = p_1^{m_1} p_2^{m_2} p_3^{m_3} \ldots$, $\phi(n) = (p_1^{m_1} - p_1^{m_1-1})(p_2^{m_2} - p_2^{m_2-1}) \ldots$ 4. If g_i, g_j are generators of cyclic group, $g_i = g_j^x$, 5. $o(g_i)$ is relatively prime (gcd=1) to $o(G)$. | $\phi(n) \to$ # integers (i) from 1 to n s.t. $\gcd(i,n) = 1$

* Subgroup properties. 1. A, B be subgroups of G. Then, $A \cap B$ is also a subgroup, while $A \cup B$ need not be. 2. Lagrange's th^m: $o(H)$ divides $o(G)$ when H is a subgroup of G. 3. If $|G|$ = prime, $o(G) = p$ [no proper subgroup],
4. H, K be subgroups of G then $HK = \{h*k \mid h \in H, k \in K\}$ is also subgroup.
5. $o(HK) = \dfrac{o(H) \cdot o(K)}{o(H \cap K)}$ • $o(H \cap K) \leq o(H), o(K)$, $o(HK) \mid o(H), o(K)$,
$o(H \cap K) \mid o(H), o(K)$ (where H, K are subgroups of G). 6. $a * b^{-1} \in H$ $\forall a, b \in H$.

* $(M, *, e)$ be a monoid. If element x in M is invertible, then there is a unique inverse element. $xx' = x'x = e \wedge xx'' = x''x = e \Rightarrow x' = x''$.

* Ring: Algebraic structure consisting of a nonempty set R & 2 binary operations $(+, \cdot)$ such that i) $R(+)$ is abelian group, ii) $R(\cdot)$ is a semigroup, iii) $R(+)$ & $R(\cdot)$ satisfy: $x \cdot (y+z) = x \cdot y + x \cdot z$, $(x+y) \cdot z = x \cdot z + y \cdot z$.
> If R is a ring with at least 2 elements (different), then additive & multiplicative identities are not equal.

* Field: $R(\cdot, +)$ such that i) $(R, +)$ is abelian group, ii) $(R - \{0\}, \cdot)$ is an abelian group. iii) $\cdot$ is right & left distributive over $+$.

• For a finite cyclic group, $o(G) = o(g)$ ~ $g = gen^r$.

For any infinite cyclic group, there are only 2 generators.

• (n^{th} roots of unity, $\times$) is a finite cyclic group, Generator = $e^{2\pi/n}$

• Every finite group of order <6 must be abelian.
(non abelian has order ≥ 6).

Graph Theory

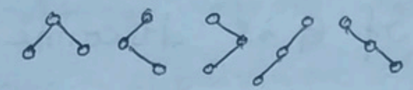
* $\delta_G \le \frac{2e}{v} \le \Delta_G$
* Order # vertices
  Size # edges

| # graphs with n vertices possible = $2^{{}^nC_2}$ ↓ |

* $\sum d(v) = 2e$
* Ramsey's Theorem (3,3) (3,4)
* Cayley's Theorem: # labeled trees $= n^{n-2}$ $(n \ge 2)$
* Tree → n ver. ↓ $n-1$ edges ↳ Cor.: (↓)
* Total # trees possible with n nodes $= 2^n - n$.
* # unlabeled binary trees $= C_n$ | # labeled binary trees $= n! \cdot C_n$

BSTs with n keys $= C_n$

\- # structurally diff binary trees $= C_n$ ↗ $n = 3$ $C_n = 5$



* Cayley's formula: # trees on n labeled vertices $= n^{n-2}$
  ↳ isomorphic not counted | $n = 3 \Rightarrow \# = 3$.
  ↳ Also, # spanning trees in K_n on n labeled nodes.
* # different labeled binary trees of n nodes (where $k_1, k_2, \ldots, k_m$ are the repetitions of $a_1, a_2, \ldots, a_m$ values of the tree, $m \le n$). $=$ (unique or non-unique values at nodes)

$$\frac{n!}{k_1!\, k_2! \cdots k_m!} \cdot C_n.$$

| * every tree is bipartite |

| for K_n, # STs $= n^{n-2}$ |

* Any 2 vertices in a tree - unique path b/w them.
* # spanning trees: Kirchoff's Matrix tree Theorem

$$L_{ij} = \begin{cases} d(i) & i = j \\ -1 & i \ne j \text{ ; } i,j \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Delete one row & one col. → get L^*

sp. trees $= \det(L^*)$

* Planar graphs (4-colorable): $v + f - e = 2$ $(= k+1)$ | $3f \le 2e$ & $e \le 3v - 6$ for $v \ge 3$
  * If $e > 3v - 6$ for $v \ge 3$, then G is nonplanar. | $kf = 2e$
* Kuratowski's theorem: Every non-planar G contains either K_5, or $K_{3,3}$ or a subdivision of K_5 or $K_{3,3}$.
* (↑) Forest F of order n & k conn. comp.s, $e = n - k$.
* For every connected graph, $e \ge v - 1$ (↑)

| # Connected simple graphs with n labelled vertices $= n\,2^{\binom{n}{2}}$ |

* Walk → neither edges, ver's repeat → Path $\xrightarrow{\text{Closed}}$ Cycle
  → no edges repeat → Trail $\xrightarrow{\text{Close}}$ Circuit

(Path ⇓ Trail, Cycle ⇓ Circuit)

(If vertices don't repeat, automatically edges can't repeat.)

* **Connectivity :** Th| If for every 2 non-adj vertices (u, v), degree $d_u + d_v \geq |V| - 1$ then G is connected.

Th| If in G, $\delta(G) \geq \frac{|V|-1}{2}$, then G is connected.

Th| If G is disconnected, G' is connected.

* **Euler trail :** Includes all the edges of G. (Circuit if closed)
  ↳ Th| A connected graph or multigraph G is Eulerian iff each vertex is of even degree. (In case of a trail except start & end vertex.)
  ↳ Fleur's algo (finding Euler trail) At each step we move across an edge whose deletion does not result in >1 components, unless we have no choices (visiting an edge ⇒ delete from G for further consideration). In the end, no edges are left.

K_{2n+1}, C_n, K(even) regular ✓ Q_n, W_n ✗ (DM 4a)

* **Hamiltonian path :** Includes all vertices in G. (If G is hamiltonian, no pendant vertex.)
  (No known tests for Hamiltonian).
  If a graph is hamiltonian it has a hamiltonian cycle.
  ↳ Dirac's Th| If G is a simple graph with $(n \geq 3)$ & if $d(v) \geq n/2$ then G is Hamiltonian. ← (Last pg)
  ↳ # hamiltonian cycles in K_n $= \frac{(n-1)!}{2}$ (circulare arrangement of beads)

K_n, C_n, W_n ✓ $K_{n,m}$ ✓ Q_n, K-regular ... Q_n ✗ K_m^n

* **Bipartite graph.** V divided into A, B. for each $e \in E$, connects A & B's vertex.
  ↳ $\sum_A d_v = \sum_B d_v$
  ↳ A graph is bipartite iff it not have any odd cycle.
  ↳ A non-null graph is bipartite iff it is bichromatic or 2-colorable.
  ↳ Any acyclic graph is bipartite.

* **Eccentricity (of vertex)**

$\varepsilon(u) = \max_v \{d(u,v)\}$

$d(u,v)$ being shortest distance

Radius

$r(G) = \min_v \varepsilon(v)$

Diameter

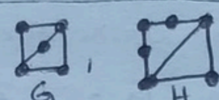
$d(G) = \max_v \varepsilon(v)$

largest of the shortest paths b/w any pair of vertices.

$A_G = PA_HP^T$ P - Permutation matrix

* If diameter of a simple graph is ≥ 3, then diam$(\bar{G}) \leq 3$.
Also, " " a graph $\geq 4 \Rightarrow$ diam$(\bar{G}) \leq 2$.

* **Isomorphism**: If there's a bijection b/w 2 vertex sets.
(Checking: # v's, e's, same deg. seq., same # of circuits of particular length, # comp^{nt}s., diameter, radius, length of longest path, checking isomorphism on $\bar{G}_1$ & $\bar{G}_2$., checking planarity, bipartite or not

eg.  $\rightarrow$ subdivision 

* **Homeomorphism**: If there's graph isomorphism from some $\boxed{\text{subdivision}}$ of G to some subdivision of G'.
(subdivision of edges - u —— v $\rightarrow$ u — w — v, reverse op^n is smoothing an edge).

* **Matching** (max): Set of disjoint edges. (Perfect matching) (Max^M & max^l matching, matching no. $\mu(G)$) • Hall's cond^n/Matching cond^n (in Bipartite):
Bipartite (A, B) Existence of matching saturating A. $|N(S)| \geq |S| \ \forall\ S \in A$. N(S) neighbourhood of S. (vertices of G\S adj. to S's at least one vertex)
• Stable matching (no rogue couples, algorithm terminates at most $n(n-1)+1$ rounds (n men, n women) - repeats until no man is rejected in a round - algo is male optimal, female pessimal

* **Vertex cover** (min) (Q): $Q \subseteq V(G)$, at least one endpoint of each edge in Q. Vertex cover number $\tau(G)$. $\mu \leq \tau$ as one end vertex of each edge $\in$ Matching is in vertex cover. For bipartite G, $\mu = \tau$.
* **Edge cover** (min) (F): $F \subseteq E(G)$, each v incident to an edge in F, Edge cover number $(\rho(G))$. $\rho \geq \lceil n/2 \rceil$
* **Stable set/Independent set** (max) (I): $I \subseteq V$, no 2 vertices in I are adjacent. Independence no. $\alpha(G)$.

* **Gallai's theorem**. $\alpha + \tau = \lvert V \rvert = \mu + \rho$ (max I, min Q, max M, min F)	* For bipartite graph with no isolated ver's, $\alpha = \rho$, $\mu = \tau$	* $S \subseteq V$, is vertex cover (independent set iff $\bar{S}$ is a) $\Rightarrow \alpha + \tau = \lvert V \rvert$ $\Rightarrow$ max I is complement of min Q.

* Tutte's theorem: G has a perfect matching iff for every $S \subseteq V$, odd$(G \setminus S) \leq |S|$
* If G is a k-regular bipartite graph with $k > 0$, G has a perfect matching
* Connectivity: k-connected (there doesn't exist k-1 vertices whose removal disconnects G), Vertex connectivity (κ): Largest k s.t. G is k-connected. $\kappa \leq \delta(G)$, Vertex set/cut: Whose removal disconnects G. Size of minimal vertex cut $= \kappa$, Edge set/cutset: Set of edges, edge connectivity (κ'): size of smallest edge set; If G is k-connected, it's k-1 connected; $\kappa(K_{n,n}) = n$.

If G is a simple G, $\kappa \leq \kappa' \leq \delta$. **Th** Every G has at least $|V|-|E|$ connected components. <u>Articulation pt./ Cut vertex</u>, <u>Bridge</u>, <u>Cut edge</u>. **Th** If $\exists$ exactly 2 vertices of odd degree (x,y), then $\exists$ $x \rightsquigarrow y$. **Th** In a simple graph, (k components) $n-k \leq e \leq \frac{(n-k)(n-k+1)}{2}$ • $\underbrace{\kappa \leq \kappa'}_{}\leq \delta \leq \frac{2e}{n} \leq \Delta$. → with each v we remove at least one e.

Edge cut $([S,\bar{S}])$ • G <u>separable</u> iff 1-connected. $(\kappa = 1)$

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• <u>k-line connected</u> $(\kappa' = k)$ • **Th** If G has at least 3 vertices, following are equivalent: 1. G 2-connected, 2. G connected & has no cut vertex.

Th G has at least 3 vertices, then G is 2-connected iff every 2 vertices u & v are contained in a cycle. • <u>Menger's th</u>: If G has at least $k+1$ vertices then G is k-connected iff b/w every 2 vertices u, v there are k pairwise internally disjoint paths.

* <u>Coloring</u>: Chromatic no. $\chi(G) = \max\{\chi(C) ;\ C \text{ is conn. component of } G\}$

$\chi \geq \omega$ (clique no.), $\underbrace{\Delta + 1 \geq \chi}_{} \geq \frac{|V|}{\alpha}$ indep. no. → [for K_n, $\chi = \Delta + 1 = n$].

DM4a

<u>Brook's th</u> If G is not K_n or C_{2n+1}, then $\chi \leq \Delta$. <u>Edge coloring</u>: $\chi'(G) = \Delta$ or $\Delta + 1$ for simple graph. ($= \Delta$ for bipartite G). • Each color class is an independent set. • If G is 2-colorable it is bipartite. • If G is not regular, $\chi \leq \Delta$. • Every k-chromatic graph has at least $\binom{k}{2}$ edges. • $\chi_{planar} \leq 4$

• Degree seq. of $G = (a,b,c,d,\ldots)$ and Δ = max deg., deg. seq of $\bar{G} = (\Delta-a, \Delta-b, \Delta-c, \ldots)$
• Complement of disconnected G is connected, (not vice versa). [A graph or its complement must be connected). • <u>Self complementary graphs</u> ($G, \bar{G}$ isomorphic. $|E|_G = |E|_{\bar{G}} = \frac{n(n-1)}{4}$, $n = 4p$ or $4p+1$, $p \in I^+$) → always connected.
• $G_1 - G_2$: $V = V(G_1)$, $E = E(G_1) - E(G_2)$ • $G_1 \oplus G_2$: $V = V(G_1) \cup V(G_2)$, $E = E(G_1) \oplus E(G_2)$ • <u>Fundamental cycle</u>: Cycle obtained by adding one edge • # f. cycles in a tree $= {}^nC_2$ • k-ary tree : $h+1 \leq n \leq (k^{h+1}-1)/(k-1)$, $h \leq \#i \leq \frac{k^h-1}{k-1}$, $\#l \leq k^h$
• Perfect matching possible only when $|V|$ = even • # perfect matchings in $K_{2n} = \frac{(2n)!}{2^n \cdot n!}$ • # edges in perfect matching $= \frac{|V|}{2}$ • An edge covering is minimal, iff every component of it is a star graph. • # perfect matchings in $K_{2n+1} = 0$.

* <u>Branch set</u> * <u>Chord set</u> [* Enumeration of graphs : DM4b]
→ # edges in the spanning forest   rank + nullity = $|E|$
* <u>Rank</u>: $n-k$ * <u>Nullity</u>: $|E|$ − rank/# edges to be removed to break all cycles
* <u>Directed Eulerian</u> : $\forall v \in V$, indegree = outdegree.
* <u>Traversable/Tracable</u> : If G is unicursal or Eulerian
* <u>Unicursal graph</u>: Only 2 vertices of odd degree, have Eulerian path.

* <u>Ore's th</u> $n \geq 3$ $deg_v + deg_w \geq n$ for each pair of distinct non-adj vertices, then G is <u>Hamiltonian</u>. 

* <u>Clique</u>: Complete subgraph. <u>Clique no $\omega(G)$</u>: Size of max clique. • <u>Girth</u>: Smallest cycle.
* Every G with $\delta \geq 2$, has a cycle of length at least $\delta + 1$.