

GATE CSE NOTES

by

UseMyNotes

Graph Theory

NPTEL - IISER Pune.

- In a graph G , avg degree is $\frac{2e}{n}$. $\left| \sum d_v = 2e \right.$

$$\delta(G) \leq \frac{2e}{n} \leq \Delta(G)$$

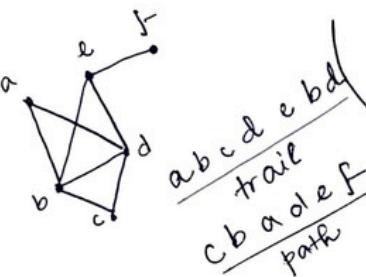
- # vertices with odd degree is even.

order - # vertices

size - # edges

distinct vertices in a walk \Rightarrow distinct edges

Walk



edges distinct \Rightarrow trail
(vertex can be repeated)

vertices distinct \Rightarrow path
(also automatically edges distinct too)

If all vertices distinct except source of sink, then \Rightarrow cycle.

Theorem

G has n vertices (order n). If

$d(u) + d(v) \geq n-1$ for every 2 non-adj

vertices u, v , then G is connected.

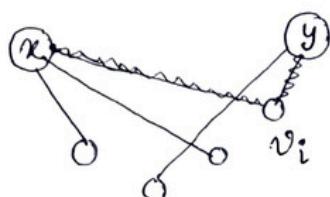
Proof Let $x, y \in V$

if $(x, y) \in E \Rightarrow x, y$ adjacent

Now, when $(x, y) \notin E$

$d(x) + d(y) \geq n-1$ implies there must be a v_i that adj. to

both x, y . [By PHP, $n-1$ pigeons, $n-2$ holes]



$n = 6$

$d(x) + d(y) \geq 5$

3 2

Theorem If G is of order n with $\delta(G) \geq \frac{n-1}{2}$,
 then G is connected.

Proof For every 2 non-adj vertices u, v

$$d(u) + d(v) \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1$$

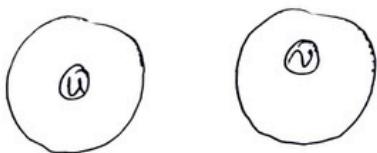
G is connected.

Theorem If G is disconnected, \bar{G} is connected.

Proof If $u, v \in V(G)$, we find a path in
 \bar{G} joining u, v .

case 1

u, v in diff components



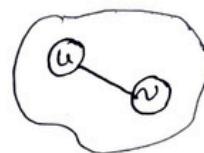
u, v not adj. in G



u, v adj. in \bar{G}

case 2

u, v in same component
 u, v may or may not be
 adj.



w in another

component. In \bar{G} , w adj to u, v .
 Thus, there's path (u, w, v) in \bar{G} .



Thus, \bar{G} is connected.

Trail Walk - edges distinct.

Path Walk - vertices \Rightarrow edges distinct.

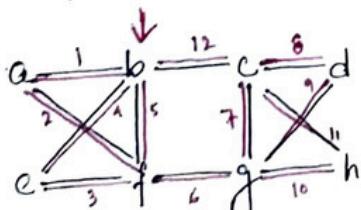
Eulerian trail : Closed trail that includes all the edges of G .

Theorem A connected graph or multigraph G is Eulerian iff every vertex is of even degree.

Proof In the trail, for each entry to a vertex there's an exit. This contributes 2 to the degree of the vertex.
 $d(v) = 2k \rightarrow$ even

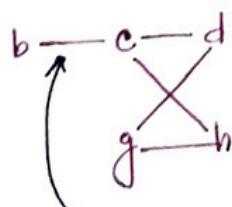
Given G with all $d(v) \rightarrow$ even

only if Fleury's algo : At each step, we move across an edge whose deletion does not result in > 1 component, unless we have no choices. At the end, there are no edges left.



Eulerian trail.

b a f c b f g c d g h c b.



can't use as increases the # components

⑦ \rightarrow ⑧
Step

• Hamiltonian path

Path that includes all vertices of G .

→ A graph is Hamiltonian if it contains

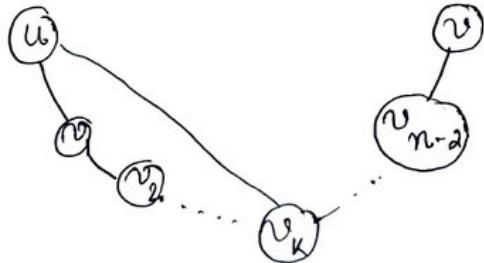
a Hamiltonian cycle.

→ No known tests for Hamiltonian graph.

 **Theorem :** If G is a simple graph with $n (\geq 3)$ vertices & if $d(v) \geq \frac{n}{2}$ then G is Hamiltonian.

Proof By contradiction,

u, v be 2 non-adj. vertices. If we join u, v there's Hamiltonian path joining $u \& v$.



Let $d(u) = k \geq \frac{n}{2}$

If u is adj. to v_k then v_{k-1} cannot be adj to v .

If possible, let v be adj.

to v_{k-1} . Then we have Hamiltonian cycle $v, v_{n-2}, \dots, v_k, u, v_1, v_2, \dots, v_{k-1}, v$

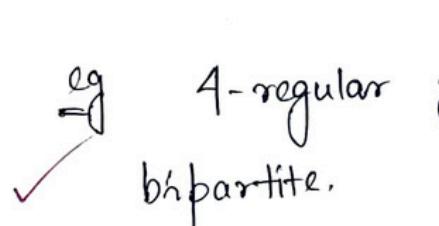
Since, we assumed it is not Hamiltonian graph, this is not possible.

Thus $d(v) \leq n-1-k \leq n-1-\frac{n}{2} = \frac{n}{2}-1$
So, G is Hamiltonian. ↑ contradiction

• Bipartite Graph

V set divided into A, B .

Each $e \in E$ connects a vertex in A to a vertex in B .

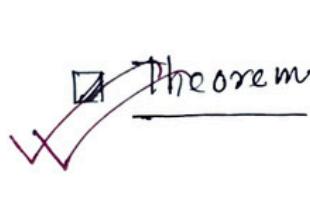
 eg 4-regular graph having 15 vertices can't be bipartite.

$$|V| = 15 \quad V = A \cup B \quad \text{Let } |A| = x$$

$$\sum_{A} d(v) = \sum_{B} d(v)$$

$$\Rightarrow 4x = (15-x)4$$

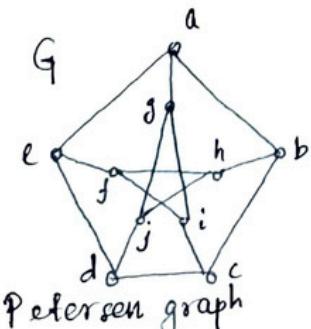
$\Rightarrow x$ = not an integer.

 theorem A graph is bipartite iff it does not have any odd cycles.

• Diameter of G

$$\text{diam}(G) = \max \{ d(u,v) \} \quad u, v \in V$$

$d(u,v) \rightarrow$ length of shortest path b/w $u \& v$



$$\begin{aligned} d(u,v) &= 1 && \text{adj } u, v \\ &= 2 && \text{non-adj } u, v \end{aligned}$$

$$\text{diam}(G) = 2$$

Theorem (~~G~~ G is a simple graph)

$$\boxed{\text{diam}(G) \geq 3 \Rightarrow \text{diam}(\bar{G}) \leq 3}$$

Theorem If $\text{diam}(G) \geq 1 \Rightarrow \text{diam}(\bar{G}) \leq 2$



Isomorphism

$G = (V, E)$, $G' = (V', E')$. $G \cong G'$ if there exists a bijection $f: V \rightarrow V'$ with $(u, v) \in E \Leftrightarrow (f(u), f(v)) \in E'$ for all $u, v \in V$.

eg

 G	 G'	$f(A) = 1$ $f(B) = 3$ $f(C) = 2$ $f(D) = 4$.
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$(A, B) \in E \Leftrightarrow (1, 2) \in E'$ & so on.

eg

 G	 G'	$f(u) = 1$ $f(v) = 3$ $f(w) = 5$ $f(x) = 2$ $f(y) = 4$ $f(z) = 6$
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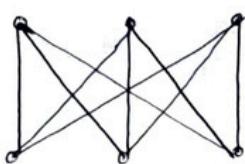
$\rightarrow (1, 3, 5)$
 $\quad (2, 4, 6)$

If 2 graphs are isomorphic, they have same number of vertices, edges, components, same degree sequence, diameter, length of longest path. If two graph differ in any of these, they are not isomorphic. However, having all these

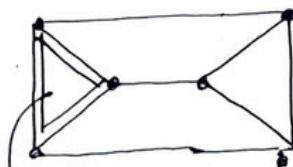
same does not imply that they are isomorphic.

✓ If two graphs are isomorphic & one of them contains a cycle of particular length, then the same must be true of the other graph.

e.g.



$K_{3,3}$



G

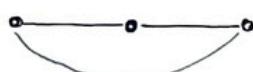
cycle of length 3

not isomorphic.

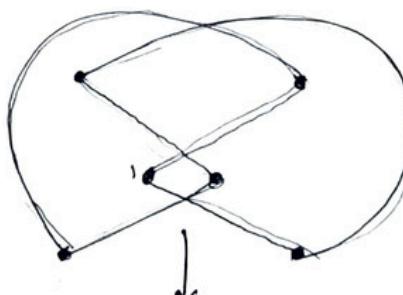
Can't be a cycle of odd length in $K_{3,3}$.

✓ G & G' are isomorphic iff $\bar{G} \simeq \bar{G}'$.

e.g.



$\bar{K}_{3,3}$



\bar{G}

cycle of length 6

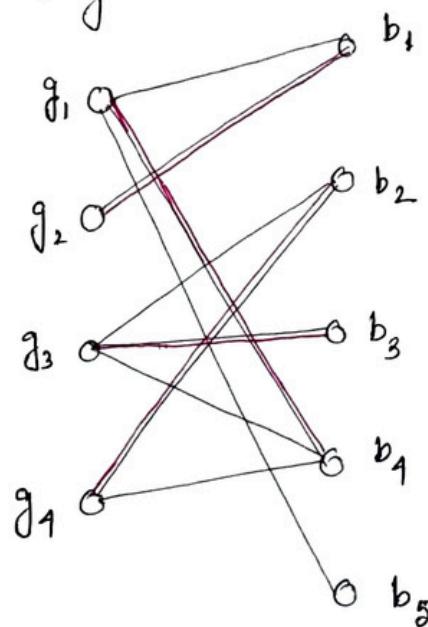
does not exist in $\bar{K}_{3,3}$

$K_{3,3}$ & G are not isomorphic.

• Matching

- Set of disjoint edges.
- Every vertex of G is incident to at most one edge in the matching.
- Size of matching is the # edges in the matching M .
- Matching is maximum when it has the largest possible size.
- A perfect matching is that matches every vertex.
- Alternating path is a path that alternates b/w matched & unmatched edges.
- $\text{e.g. } b_5 g_1 b_1 g_3 b_3$
- Augmenting path is an alternating path that starts & ends on unmatched vertices.

Th: A matching is maximum iff there's no augmenting path wrt M .



$$M = \{(g_1, b_1), (g_2, b_1), (g_3, b_3), (g_1, b_2)\}$$

Every graph with $S(G) \geq 2$ has a cycle of length at least $S(G) + 1$.

- Each vertex in matching M has degree one.
- Maximum matching for a G is not unique.

\Rightarrow Matching number
(of a graph) Size of a maximum matching of the graph.

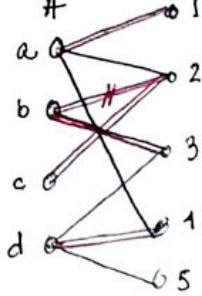
Matching in Bipartite Graphs

$G = ((A, B), E)$ be a bipartite graph. If $|A| \leq |B|$, size of maximum matching is at most $|A|$. We want to decide whether there exists a matching saturating A . If there is such a matching, M , then for any subset S of A , the edges of M link the vertices of S to as many vertices of B .

Hall's condⁿ (existence of matching saturating A)

$$|N(S)| \geq |S| \quad \forall S \subseteq A.$$

$N(S)$ is set of vertices of $G \setminus S$ adjacent to at least one vertex of S . $N(S)$ is the neighbourhood of S



$$\begin{aligned} S &= \{a, b, c\} \\ G \setminus S &= \{d, 1, 2, 3, 4\} \\ N(S) &= \{1, 2, 3, 4\} \end{aligned}$$

$|N(S)| \geq |S|$

Hall's matching theorem

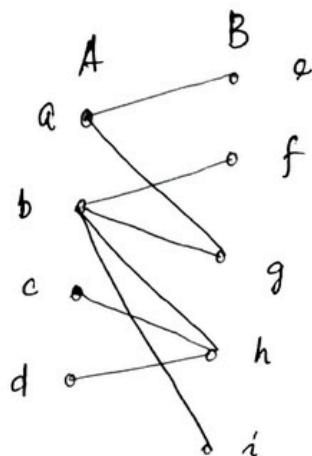
$G = ((A, B), E)$ has a matching saturating A iff

$|N(S)| \geq |S|$ for all $S \subseteq A$.

e.g. for $S = \{a, b, c, d\}$

$$G \setminus S = \{e, f, g, h, i\}$$

$$N(S) = \{e, f, g, h, i\} \quad |N(S)| \geq |S|$$



for $S = \{c, d\}$

$$G \setminus S = \{a, b, e, f, g, h, i\}$$

$$N(S) = \{h\} \quad |N(S)| \neq |S|$$

So, no matching exists saturating A .

Stable Marriage Problem

N men, N women.

Rogue couple. A man & a woman such that they prefer each other to the person they are currently paired with in current matching.

Stable matching no rogue couples.

\Rightarrow There's always an stable matching.

\Rightarrow Gale - Shapley Algorithm for stable matching

- 1) In the beginning of each round, every man not engaged proposes to his favorite girl not yet crossed off his preference list.
- 2) Then each woman looks at her new proposals, & also the man she is provisionally engaged to (if her one) & then picks favorite from them. They become engaged (as of now) while all other men get permanently rejected from her.
- 3) All rejected men cross off their lists the woman who rejected them.

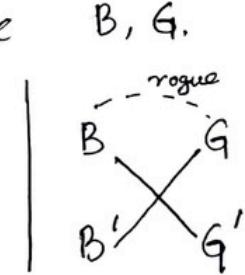
Cycle repeats until no man is rejected in a round.

\Rightarrow This alg guarantees that everyone gets married. \Rightarrow Algo is male optimal, female pessimal.

\Rightarrow It produces a stable matching (no rogue couples).

Proof Suppose there's rogue couple

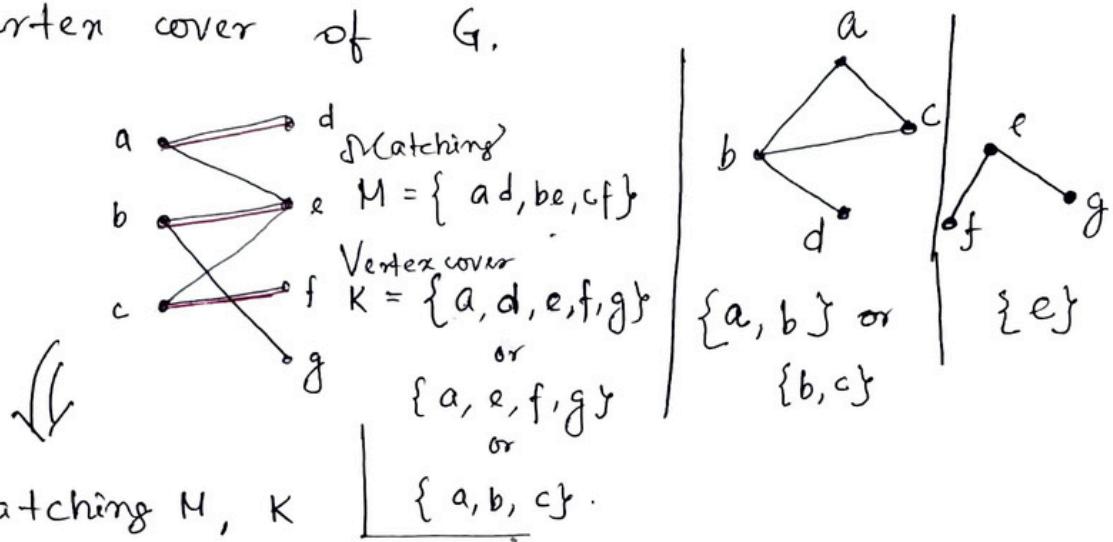
B must have proposed to G & G rejected B.
Only way to reject B for G is that she got someone better. But woman can only trade up.
So, she must like B' better than B. Contradiction! So stable.



Vertex cover

$G = (V, E)$. A set $K \subseteq V$ is a vertex cover of E if any edge of G is incident to a vertex in K .

The vertex cover number of G , $\tau(G)$, is the min. size of a vertex cover of G .



For any matching M , K

contains at least one end vertex of each edge of M .

So, $|M| \leq |K|$. So, the max size of a matching is at most the min. size of a vertex cover for any graph.

König's theorem

Let G be a bipartite graph.
Then the size of a max.

matching equals the size of a min. vertex cover.

i.e. ✓ $\mu(G) = \tau(G)$.

↓
matching no. ↓ vertex cover no.

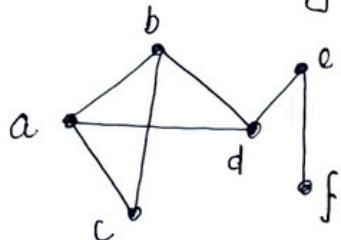
Edge cover.

- ✓ Set of edges $F \subseteq E(G)$ such that every vertex $v \in V(G)$ is incident to an edge $e \in F$. [Edge cover can exist if there's no isolated vertices.]
- $\rho(G)$, edge cover number Min size of edge cover.

✓ \Rightarrow Stable set, C Subset of V such that for each edge e of G , $e \notin C$.

or Independent set

↓
no two vertices
are adjacent



$\{a, e\}$ or $\{b, f\}$ or

$\{c, d, f\}$

↑
max stable set.

Stable set number $\alpha(G)$. / Independence no:

$$\alpha(G) = \max \{ |C| : C \text{ is a stable set} \}$$

Gallai's theorem

For G w/o isolated vertices,

$$\boxed{\alpha(G) + \gamma(G) = |V| = \rho(G) + \rho(G)}$$

✓ for bipartite graph with no

isolated vertices,

$$\boxed{\alpha(G) = \rho(G)}$$

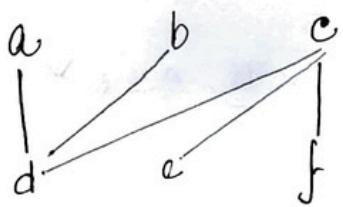
$$\mu(G) = \gamma(G).$$

✓ $\gamma(G) \approx$ min size of vertex cover

$\mu(G) \approx$ max. matching size.

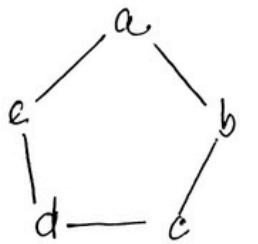
$\rho(G) \approx$ min. size of edge cover

e.g Independent set.



$$S = \{a, b, e, f\}$$

$$\bar{S} = \{c, d\} \rightarrow \text{min vertex cover}$$



$$S = \{a, d\}$$

$$\bar{S} = \{b, c, e\} \rightarrow \text{min vertex cover}$$

Theorem In a graph G , $S \subseteq V$ is an independent set iff \bar{S} is a vertex cover of G hence

$$\alpha(G) + \tau(G) = |V|$$

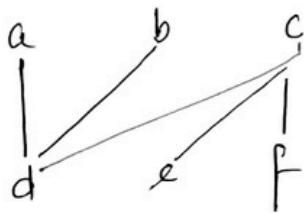
Proof: \Rightarrow If S is a stable set, every edge is incident to at most one vertex in S . So, every edge is incident to at least one vertex of \bar{S} . So, \bar{S} covers all edges.
 \Leftarrow \bar{S} being vertex cover, there are no edges joining vertices of S . So, S is a stable set.

Hence, every max. independent set is complement

of min. vertex cover.

$$\alpha(G) + \tau(G) = |V|.$$

e.g Edge cover, L

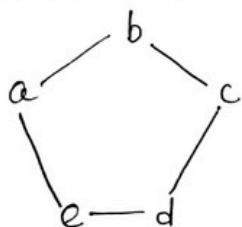


L, edge cover =

$$\{ ad, bd, ec, fc \}$$

min edge cover

Max. matching = { ad, cf }



$$L = \{ ae, bc, cd \}$$

$$M = \{ ab, de \}$$

- 1. M Independent vertex set $\alpha(G)$
- 2. X Independent edge set (or Matching) $\mu(G)$
- 3. N Vertex cover $\tau(G)$
- 4. N Edge Cover $\rho(G)$

$$\left\{ \begin{array}{l} S \rightarrow \text{stable set} \\ \bar{S} \rightarrow \text{vertex cover} \end{array} \right.$$

$$\boxed{\alpha(G) + \tau(G) = |V|}$$

$$\boxed{\mu(G) + \rho(G) = |V|}$$

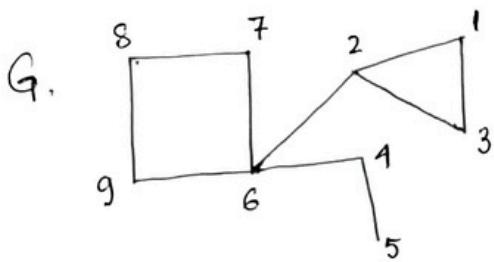
stable set ~ vertex cover
matching ~ edge cover

Theorem For graph G, with no isolated vertices,

$$\boxed{\mu(G) + \rho(G) = |V|}$$

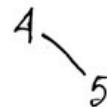
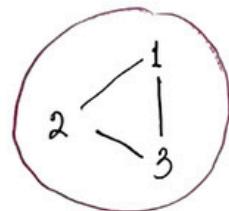
Tutte's theorem

$G = (V, E)$ & S be subset of V, $S \subseteq V$. Let $\text{odd}(G \setminus S)$ denote the number of odd components that get generated if S is removed from G.



$$S = \{6, 8\}$$

$G \setminus S$



odd components = 3

~~Tutte's Theorem~~: G has a perfect matching iff
for every subset $S \subseteq V$,
 $\text{odd}(G \setminus S) \leq |S|$

• Maximal Matching If we cannot add any edge to the existing matching.

→ If M is maximum \Rightarrow it is maximal.

→ Max. matching isn't unique.

• If G is a k -regular bipartite graph with $k > 0$,
 G has a perfect matching.

Social Networks & Stable Marriage.

- $\sum d(v) = 2e$.

- Ramsey's Theorem (3,4 version)

Any group of 10 people must contain 3 mutual friends or 4 mutual strangers.

K_3

K_4

\Rightarrow If every edge of K_{10} is colored red or blue, then it must contain a red K_3 or a blue K_4 .

adj. to

Proof

Each vertex has 9 edges leaving it.

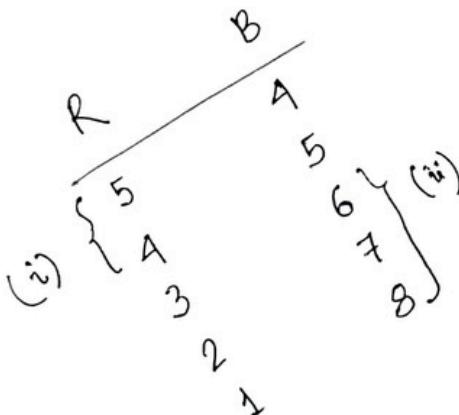
9 edges have to be colored with blue or red.

\Rightarrow It must have at least 5 red edges or 5 blue edges. (PHP)

Also, vertex 10 must have either

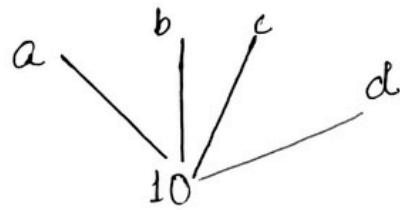
at least 4 red edges or at least 6 blue edges. (If not $\rightarrow 3R + 5B = 8 \times$)

(i) B, R interchangeable.



if none is correct
then $3R + 5B$

case a) K_{10} has at least 1 red edges.

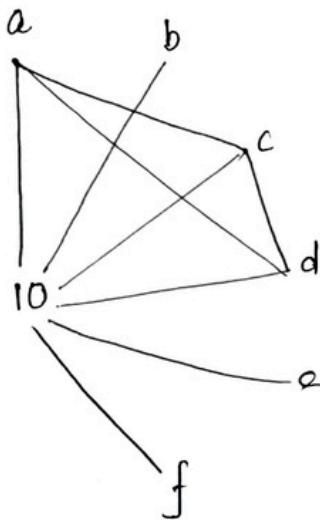


If one of $a-b, b-c, c-d$,
 ~~$c-d$~~ is red, that
gives a red K_3 .

If none of $a-b, b-c, c-d$ is red, then
 $a-c, a-d, b-d$

it gives a blue K_4 .

case b) K_{10} has at least 6 blue edges.



Among a, b, c, d, e, f , we must
have a red K_3 or a blue K_3 .
(Ramsey $(3,3)$).

If a red K_3 is present, then done.

If a blue K_3 is there, we get a
blue K_4 in K_{10} .

• Stable Marriage

$K_{N,N}$ - N vertices for men, N vertices for
women, edge connecting each man to
each woman.

Complete
bipartite.

- A TreeSet (Chain) can + be a ~~conformated~~ ~~affice~~ if

Perfect matching

Selection of N edges such that every vertex is included on exactly one edge.

There are $N!$ perfect matchings. (for $K_{n,n}$).

\Rightarrow Stable matching

Problem: Each person has an ordered list of some members of the opposite sex as his/her preference list.

If the resulting set of marriages (matching) contains no pairs of the form $\{m_i, w_j\}$, $\{m_k, w_\ell\}$ such that m_i prefers w_ℓ to w_j and w_ℓ prefers m_i to m_k , the marriage is stable.

(Stable matching can always be achieved)

		Men			Women		
		1	A	B	C		
		2	B	A	C		
		3	A	B	C		
i)	A	(1), 3	ü)	A	1		
	B	(2)		B		(2)	
	C	-		C		-	

		Women	Men		
			1	2	3
	A	2	1	3	
	B	1	2	3	
	C	1	2	3	

	3	A	B	C		
i)	A B C	(1) (2) -	ii) A B C	1 (2), 3 -	iii) A B C	(1) (2) (3)

Stable matching

② Tree : Acyclic, connected graph.

✓ \Rightarrow #~~labeled~~ tree graphs with n nodes = n^{n-2}
 [Cayley's theorem]

binary trees with n nodes
 (unlabeled) = $\frac{(2n)!}{(n+1)! n!}$

BSTs with n nodes = c_n

\Rightarrow Every tree with n vertices has $n-1$ edges.

\Rightarrow Any 2 vertices are connected by a unique path in a tree.

③ Spanning tree. Any tree inside a graph, that uses all vertices of G is a spanning tree.

\Rightarrow # spanning trees are calculated using

Kirchoff's matrix tree theorem.

$$L_{ij} = \begin{cases} d(i) & i=j \\ -1 & \text{if } i \neq j; i, j \text{ are adj.} \\ 0 & \text{otherwise.} \end{cases}$$

delete one row + one column

$$L^* \rightarrow \text{find } \det(L^*)$$

span. trees

• Planar graph

$$v + f - e = 2$$

Proof

Induct on e .

$$\rightarrow e=0, v=1, f=1$$

$$v + f - e = 2$$

IHOP True for $e=k$

$$\rightarrow e=k+1$$

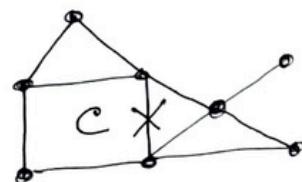
case I G is a tree.

$$e = v - 1 \quad f = 1$$

$$v + f - e = v + 1 - v + 1 = 2.$$

case II G not a tree.

G must contain a cycle.



	v	f	e
G	v	f	$k+1$

	v'	$f-1$	k
G'			*

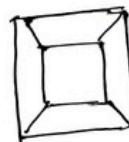
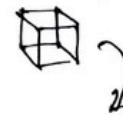
* G' has k edges. So, using IHOP,

$$v + (f-1) - k = 2$$

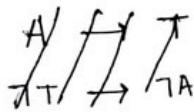
$$v - (k+1) + f = 2$$

————— proved for $k+1$ edges

• For a cube



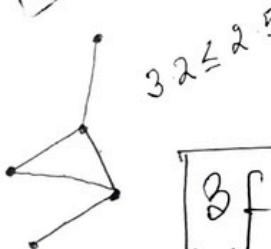
$$8 + 6 - 12 = 2$$



Remove any edge from cycle C to obtain a smaller graph G' .

Converse not true

If G is planar with $n \geq 3$ vertices, e edges then,



$$e \leq 3n - 6$$

$$3f \leq 2e$$

$$n + f - e = 2$$

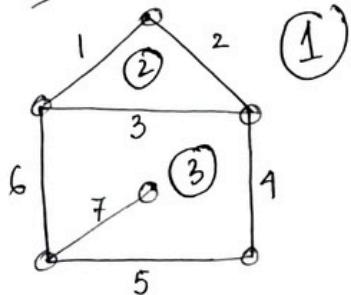
If $e > 3n - 6$ for a G with $n \geq 3$ then G is non-planar.

K_5 non-planar

$$3(e - n + 2) \leq 2e$$

$$\Rightarrow e \leq 3n - 6$$

Proof



Edge-face matrix

if edge i borders face j , then $M_{ij} = 1$

$e \setminus f$	1	2	3
1	1	1	0
2	1	1	0
3	0	1	1
4	1	0	1
5	1	0	1
6	1	0	1
7	0	0	1

Counting col & row wise.

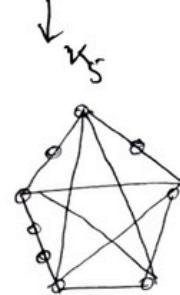
$\alpha = \# 1's$ in M

row wise addition - how many faces does an edge border.
 $\Rightarrow \alpha \leq 2e$.

col wise - how many edges border a face.
 $\Rightarrow \alpha \geq 3f$ as min 3 edges required to form a face.

So, $3f \leq \alpha \leq 2e$

Every non-planar G contains either K_5 or a subdivision of K_5 or $K_{3,3}$.



Kuratowski's theorem

Connectivity

\rightarrow k vertices removed \Rightarrow graph disconnected
 $K(G) = k$

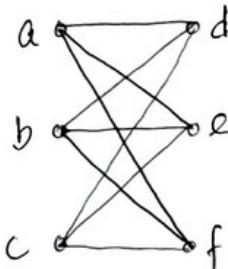
- A graph is k -connected if there does not exist a set of $k-1$ vertices whose removal disconnects the graph. (how many vertices we can remove such that G stays connected)

- Vertex connectivity $\kappa(G)$ of G is the largest k such that G is k -connected.

$\checkmark \boxed{K(G) \leq \delta(G)}$ min degree

[Size of minimal vertex cut]

e.g. $K_{3,3}$



1-connected

$$K(K_{3,3}) = 3$$

2 - "

3 - "

not 4-connected.

(removal of a, b, c. disconnects the graph).

or vertex cut.

- Vertex set (of a connected graph) cutset

Set $S \subseteq V$ whose removal disconnects the graph.
 $\{a, b, c\}$

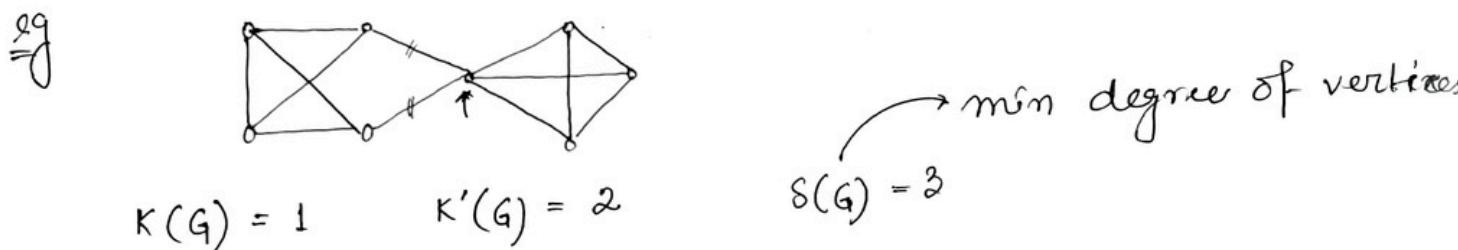
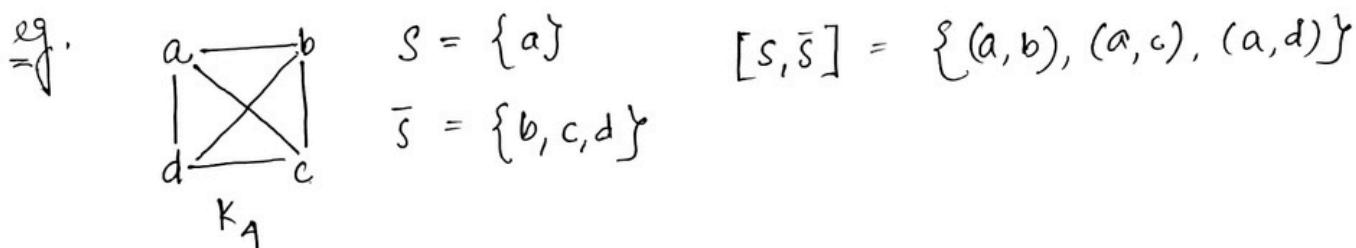
- Edge set / cut set

Minimal set of edges whose removal disconnects the graph.

- Edge connectivity $\kappa'(G)$: size of smallest edge set/cut set.

Min # edges to be removed to disconnect G .

- Edge cut : Set of edges of the form $[S, \bar{S}]$ where S is a non-empty proper subset of V & $\bar{S} = V - S$.

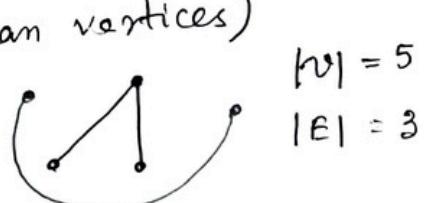


Theorem : If G is a simple graph,
 $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

Expansion Lemma

Let G be a k -connected graph. If G' is obtained from G by adding a new vertex x adj. to at least k vertices in G . then G' is k -connected.

Theorem Every graph with n vertices & e edges has at least $n - e$ connected components.
 (fewer edges than vertices)



• Articulation point / Cut Vertex

Removing it or edges incident on it disconnects G.
(or produce more connected components)

• Bridge / Cut edge

Removal of this edge disconnects G.
(or produce more conn. components)

• Strongly connected components - in case of digraphs.

Theorem In any G , if \exists exactly 2 vertices of odd degree, namely x, y , then \exists a way.

Theorem In a simple graph, e edges, n vertices, k components (note: 1 component \Rightarrow G is connected)

then

$$n-k \leq e \leq \frac{(n-k)(n-k+1)}{2}$$

→ complete K_{n-k+1} (to maximise #edges)

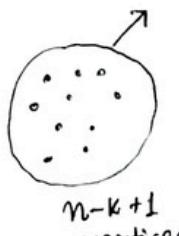
∴ When $k=1$,

$$n-1 \leq e \leq \frac{n(n-1)}{2}$$

min. $n-1$ edges
for n vertices.
to be connected

for complete
graph

\leq \dots
 $k-1$ vertices
/components



$$\Rightarrow e = |E(K_{n-k+1})| \\ = \frac{(n-k)(n-k+1)}{2}$$

\geq min #edges = $(n-k+1) - 1 = n-k$
(at least has to be a tree; hence, #e = #v - 1)

$$Q. |V| = 10, \kappa = 3$$

$$\min \# \text{edges} = 10 - 3 = 7$$

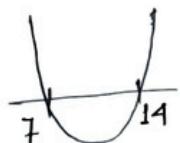
$$\max \# \text{edges} = \frac{7 \cdot 8}{2} = 28$$

$$Q. n = 10, \ell = 6, \kappa = ?$$

*

$$\begin{aligned} 10 - \kappa &\leq 6 & G &\leq \frac{(10-\kappa)(10-\kappa+1)}{2} \\ \Rightarrow \kappa &\geq 4 & \Rightarrow (10-\kappa)(11-\kappa) &\geq 12 \\ && \Rightarrow \kappa^2 - 21\kappa + 130 &\geq 12 \\ && \Rightarrow \kappa^2 - 21\kappa + 98 &\geq 0 \\ && \Rightarrow (\kappa-7)(\kappa-14) &\geq 0 \end{aligned}$$

$\kappa \leq 7$ or $\kappa \geq 14$ as a parabola.



Q. Max #edges in a disconnected graph G with n vertices.

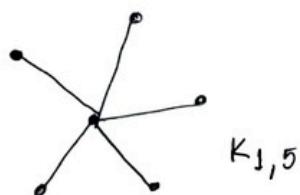
*

$$\rightarrow \kappa \geq 2$$

$$n - \kappa \leq e \leq \frac{(n-\kappa)(n-\kappa+1)}{2}$$

$$e \leq \frac{(n-2)(n-1)}{2}$$

• Star graph. $K_{1,n}$



$$\# \text{cut edges} = 5 \quad \text{or } n$$

$$\# \text{cut vertices} = 1$$

Produces max #connected components by removing cut vertex.

$\text{diam } (K_{3,n}) = 2 \text{ if } m+1 = |V| \geq 3$

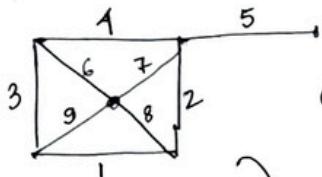


- For cycle graph
- Wheel graph
- complete graph, there's no cut vertices.



- Cut Set : Minimal set of edges that if removed disconnects the graph.

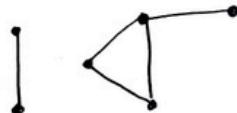
eg



$$\text{cut-set} = \{e_1, e_9, e_6, e_4\}$$

cut

$$\underline{\text{Minimal cut-set}} = \{e_5\}$$



- Cut : Partition of vertices into 2 disjoint subsets.
Any cut determines a cut-set, set of edges that have one end in each subset of partition.

- Connected component : Maximal connected subgraph of und. graph.

- Vertex connectivity $\kappa(G)$

$$\kappa(G) \leq \delta(G) \leq \frac{2e}{n} \leq \Delta(G)$$

avg deg.

$$\sum d = 2e$$

• Edge connectivity $\kappa'(G)$.

$$\checkmark \quad \kappa'(G) \leq \delta(G)$$

(with each vertex we remove,
we remove at least one
edge)

$$\checkmark \quad \kappa(G) \leq \kappa'(G) \leq \delta \leq \frac{2e}{n} \leq \Delta$$

- G is separable iff $\kappa(G) = 1$.

↳ 1 connected

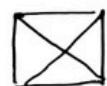
↳ if has a cut vertex.

eg 

- K -line connected if $\kappa'(G) = K$

Δ

For K_n , $\kappa(K_n) = n-1$



$K_{n,n}$



$$\kappa'(K_n) = \cancel{n-1}$$

$$\kappa(K_{n,n}) = n$$

for C_n (cycle), $\kappa(C_n) = \kappa'(C_n) = 2$

$$\kappa(N_3) = 3$$

$$\kappa'(N_3) = 3$$

$$\kappa(K_{n,n}) = n$$

$$\kappa'(K_{n,n}) = n$$

$$\kappa(K_{m,n}) = \min(m,n)$$

$$\kappa'(K_{m,n}) = \min(m,n)$$

- For directed graphs,

unilateral

$\rightarrow \forall x, y \quad x \rightsquigarrow y$

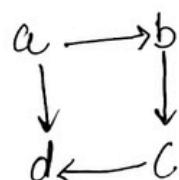
$x \rightsquigarrow y$
or $y \rightsquigarrow x$

connectivity \rightarrow strong

Weak

\rightarrow undir. graph is connected

Unilateral



for $a, c \quad a \rightsquigarrow c$

but $c \not\rightsquigarrow a$



Theorem If G has at least 3 vertices, the following are equivalent:

1. G is 2-connected.
2. G is connected & has no cut vertex.
3. For all distinct vertices u, v, w in G , there's a path from u to v that does not contain w .

Proof $1 \Rightarrow 3$ Since G is 2-connected, G with w removed is a connected graph G' . Thus, in G' , $u \sim v$ exists, which in G is path $u \sim v$ avoiding w .

$3 \Rightarrow 2$ 3 implies G is connected as $u \sim v$ always exists if 3 holds true. Suppose, w is a cut vertex, so that $G' = G - w$ is disconnected. Let, u, v be vertices in 2 different components in G' so that $u \sim v$ does not exist. Then every path joining u to v must use w . Contradiction!

$2 \Rightarrow 1$ Since, G has ≥ 3 vertices & has no cut vertex its connectivity is at least 2.



Th: If G has at least 3 vertices, then G is 2-connected iff every 2 vertices u & v are contained in a cycle.

Cor. If G has ≥ 3 vertices, then G is 2-connected iff between every 2 vertices u, v there are 2 internally disjoint paths, i.e., paths that share only the vertices u, v .

■ Menger's Theorem If G has at least $k+1$ vertices then G is k -connected iff b/w every 2 vertices u, v there are k pairwise internally disjoint path.

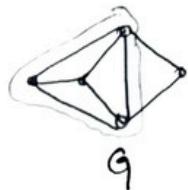


Coloring

- ✓ If H is a subgraph of G , then $\chi(H) \leq \chi(G)$.
Chromatic no.
- ✓ $\chi(G) = \max \{ \chi(c) ; c \text{ is conn. component of } G \}$

- $\chi(K_n) = n$
- Clique : Complete subgraph.

Clique number $\omega(G)$: size of max. clique



$$\omega(G) = 4.$$

• Lower bounds for χ

$$\checkmark \Rightarrow \boxed{\chi(G) \geq w(G)} \rightsquigarrow \text{size of max clique (clique number)}$$

$$\checkmark \boxed{\chi(G) \leq \Delta(G) + 1}$$

$$\checkmark \Rightarrow \chi(G) \geq \frac{|V(G)|}{\alpha(G)} \quad | \quad \alpha(G) \rightsquigarrow \text{size of max indep. set}$$

↓ proof

Let G be colored with $1, 2, \dots, \chi(G)$

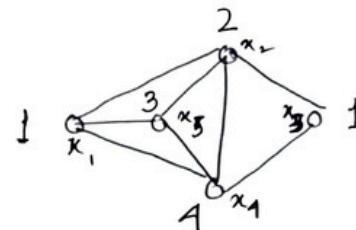
$S_i = \text{Set of vertices colored with } i$

$$|S_i| \leq \alpha(G)$$

Which vertices we color with the same colour? → Ones that are not adj. to each other

$$|V(G)| = \sum_{i=1}^{\chi(G)} |S_i| \leq \sum_{i=1}^{\chi(G)} \alpha(G)$$

$$\Rightarrow \chi(G) \alpha(G).$$



$$S_1 = \{x_1, x_3\}$$

$$S_2 = \{x_2\} \quad S_3 = \{x_5\}$$

$$S_4 = \{x_1\}$$

$$\Rightarrow \boxed{\chi(G) \geq \frac{|V(G)|}{\alpha(G)}}.$$

$$\boxed{\chi \geq \frac{n}{\alpha}}$$

$$\boxed{\chi \geq w}$$

- Greedy coloring can't force to use more than $\Delta(G) + 1$ colors.

$$\boxed{\chi(K_n) = n = \Delta(K_n) + 1}$$

Ordering of vertices determines $\chi(G)$. # colors.

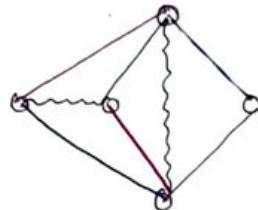
• Brooks theorem

If G is not K_n or C_{2n+1} then

$$\chi(G) \leq \Delta(G).$$

• Edge Coloring

Ex



4-edge colorable.

$\chi'(G)$ chromatic index
for edge coloring

\Rightarrow Since edges sharing an end vertex need diff. colors.

Vertex having Δ degree, has Δ edges incident on it. All Δ edges have to be colored with diff. colors.

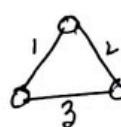


$$\checkmark \boxed{\chi'(G) \geq \Delta(G)}$$

Th If G is a simple graph, $\chi'(G) \leq \Delta(G) + 1$

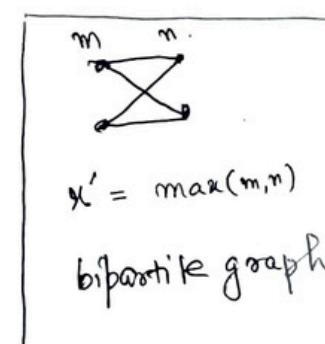
$$\text{So, } \chi'(G) = \frac{\Delta(G)}{\text{Class 1}} \text{ or } \frac{\Delta(G) + 1}{\text{Class 2}}$$

Regular graphs of odd order are class 2 graphs.



$$\Delta(G) = 2$$

$$\chi'(G) = 3.$$



✓ Every bipartite graph is of class 1.

~~✓ If G is bipartite, $\chi'(G) = \Delta(G)$.~~

- If a graph is properly colored, the vertices that are assigned a particular color form an independent set.
(Each color class is $n \times n$)

- Girth: Smallest cycle in the graph.

~~✓ If G is 2-colored, it is bipartite. (2 set of indep. vertices)~~

- ~~✓ Ordering in Greedy algorithm~~

Pick a vertex v_n & list the vertices of G as v_1, v_2, \dots, v_n so that if $i < j$, $d(v_i, v_n) \geq d(v_j, v_n)$, that is, we list the vertices from v_n first.



~~✓ If G is not regular (all vertices having same degree Δ), $\chi \leq \Delta$.~~

~~✓ If G is not an odd cycle (C_{2n+1}) or a complete graph (K_n), $\chi \leq \Delta$.~~

Every k -chromatic graph has at least $\binom{k}{2}$ edges.

Proof For each pair of colors i, j ($1 \leq i < j \leq k$) there must be at least 1 edge joining a vertex of color i to a vertex of color j (otherwise we could collapse those 2 color classes to one).
pair

Since, at least one edge for each of color classes,
 g has at least $\binom{k}{2}$ edges.