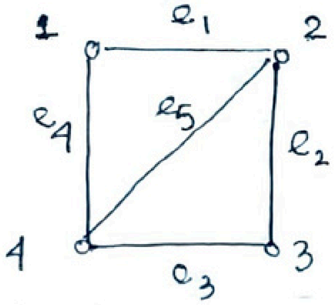
# GATE CSE NOTES

by

UseMyNotes

* **Graph:** A graph $G$ is a triple consisting of vertex set $V(G)$, an edge set $E(G)$ and the relation that associates with each edge two vertices (not necessarily distinct) called its end points.




$V(G) = \{1, 2, 3, 4\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$

$e_1 \to \{1, 2\}$ $\quad e_3 \to \{3, 4\}$ $\quad e_5 \to \{2, 4\}$

$e_2 \to \{2, 3\}$ $\quad e_4 \to \{4, 1\}$

Undirected graph
(unordered pairs in relation $e_i \to (v_j, v_k)$,
$e_i \to (v_k, v_j)$ )

- Loop : $e_i \to (v_j, v_j)$
- Multiple edges : $e_i \to (v_l, v_m)$
  $e_j \to (v_l, v_m)$
- Graph without loop or multiple edges is simple graph.

* Adjacency matrix as graph representation:
$n \times n$ matrix in which entry $a_{ij}$ is the number of edges in $G$ with end points $\{v_i, v_j\}$

→ In simple graph the diagonal elements are

→ If graph is undirected, it is symmetric

$$A = A^T.$$

→ Sum of all 1's (/k's if k edges between $v_i, v_j$) in a row gives the degree of the corresponding vertex. (As symmetric, also column wise)

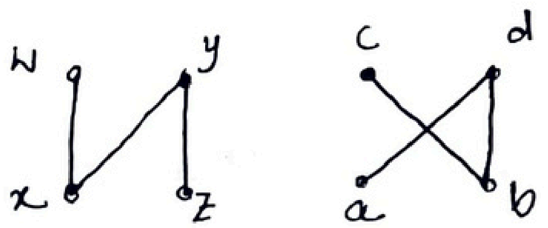
$$A(G) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left[\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}\right] & \\ 3 & & & \end{array}$$

$$A(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Degree of $v_1 = 1 + 1$

$= 2$

* Isomorphism : An isomorphism from a simple graph G to a simple graph H is a bijection $f: V(G) \to V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$




G H

$f: V(G) \to V(H)$

$w \to c$

$x \to b$

$y \to d$

$z \to a.$

By swapping rows of G, we can get H. (adj. matrix)

• No. of simple undirected graphs (labeled) with n vertices is $2^{({}^nC_2)}$.

no. of edges possible → ${}^nC_2$

[ Choosing 2 vertices among n to form an edge − ${}^nC_2$

$e_1 \quad e_2 \quad e_3 \quad \cdots\cdots \quad e_{{}^nC_2}$

$2 \times 2 \times 2 \times \cdots \times 2$

[each edge 2 possibilities − include or not]

$\Rightarrow 2^{{}^nC_2}$ ]

eg. $n = 3$ $\quad {}^3C_2 = 3.$ = No. of edges.

| $e_1$ | $e_2$ | $e_3$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |
| 1 | 1 | 1 |

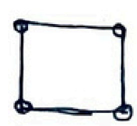
- A simple graph that is complete will contain ${}^nC_2$ edges. [n - # vertices]

* **Degree Sequence**: The arrangement of degrees in non-ascending or non-descending order.

→ $\boxed{\text{Havel Hakimi procedure}}$ *

Whether there is any graph having the degree sequence:

eg. 2, 2, 2, 2

 ✓

eg. 3, 2, 1, 1, 0 $\qquad 3+2+1+1+0 = 7$ ✗

Not even

eg. 7, 6, 5, 4, 4, 3, 2, 1.

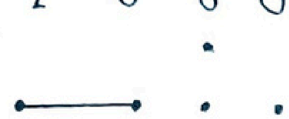
$\Sigma = 32$

$\cancel{7}$ 6 5 4 4 3 2 1

$\cancel{5}$ 4 3 3 2 1 0 $\quad \Sigma$ even

$\cancel{3}$ 2 2 1 0 0 $\quad \Sigma$ even

1 1 0 0 0



Graph exists ✓

If a vertex has degree k, then k edges are contributing to it (k adjacent vertices)

By deleting one vertex of degree k, it will cause other k vertices' degree to decrease by 1 (Simple graph)

Th.: Let us order the degrees decreasingly with the exception of one vertex, & write the degree sequence as

$$d_1, d_2 \geq d_3 \geq \cdots \geq d_n.$$

This degree sequence is graphical iff

$$d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2} - 1, \ldots, d_n$$

is also graphical.

Algorithm

* 1. Sort the sequence of non-negative integers in non-increasing order.

2. Delete the first element (say V). Subtract 1 from the next V elements.

3. Repeat 1 & 2 until one of the stopping conditions is met.

Stopping conditions:

i) All the elements remaining are equal to 0.

$\Rightarrow$ Simple graph exists.

ii) Negative number encountered after subtraction.

$\Rightarrow$ No simple graph exists.

iii) Not enough elements remaining for the subtraction step.

$\Rightarrow$ No simple graph exists.

e.g. (5, 4, 3, 3, 3)

eg. 6 6 6 6 3 3 2 2

5 5 5 2 2 1 2

5 5 5 2 2 2 1 sort

4 4 1 1 1 1

3 0 0 0 1

3 1 0 0 0

0 -1 -1 0 ✗

Simple graph not possible.

eg. 7 6 6 4 4 3 2 2

$\downarrow$

1 1 0 0 0

0 0 0 0 ✓

Simple graph possible.

## * Minimum & maximum degree.

✓ Min degree ( $\delta$ ) Max degree ( $\Delta$ ).

✓ $\delta \leq \dfrac{2|E|}{|V|} \leq \Delta$.

$\dfrac{2|E|}{|V|} = \dfrac{\text{Sum of all degr}}{\text{No. of vertices.}}$

$=$ avg. degree.

eg. G is a graph with 11 edges & minimum degree is 3. What's the max. number of vertices?

$\rightarrow \quad \delta = 3.$

$$\delta \le \frac{2|E|}{|V|} \Rightarrow |V| \le \frac{2|E|}{\delta} = \left\lfloor \frac{2 \times 11}{3} \right\rfloor = 7.$$

- Subgraph: A subgraph of G is a graph H such that $V(H) \subseteq V(G)$ & $E(H) \subseteq E(G)$.

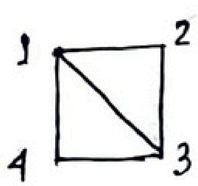
The assignment of end points to edges in H is same as in G.

- Induced subgraph: Subgraph obtained by deleting a set of vertices.

- In a cycle, no. of edges & no of vertices are same. Each node has degree 2.
- If there's a cycle, there is a path.
  If there's a path, there need not be a cycle.

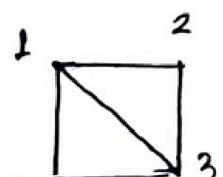
- Clique: A clique in a graph is a set of pairwise adjacent vertices.

Clique of G is the complete subgraph of G.



{1, 2, 3} & {1, 3, 4} are cliques
{1, 2}, {2, 3}, {3, 4}, {4, 1}, {1, 3}.

- Independent set: Set of pairwise non-adjacent vertices.



{2, 4} is an independent set.

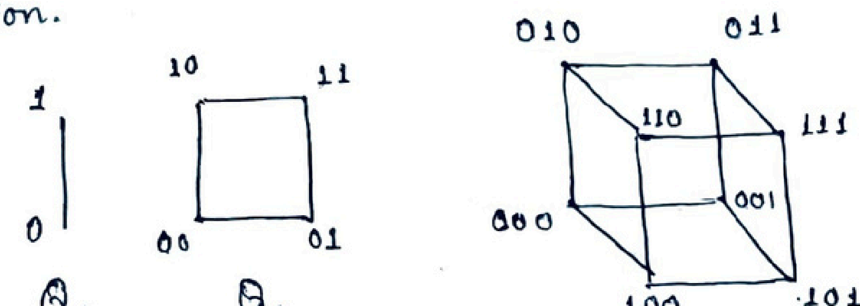
In a complete graph, no independent sets.

* <u>Regular Graph.</u> G is K-regular graph if its every vertex has degree K.

- A complete graph with n vertices ($K_n$) is (n-1) regular.
- Every cycle ($C_n$) is 2-regular.
- No. of edges of a K-regular graph with n vertices $= \frac{nK}{2}$. [from Handshaking Lemma]

* <u>Hyper cube or K-dimensional cube ($Q_K$)</u>

A simple graph whose vertices are the K-tuple with entries in $\{0,1\}$ & whose edges are the pairs of K-tuples that differ in exactly one position.




$Q_1$ $Q_2$ $Q_3$.

No. of vertices in K-dimensional cube $= 2^K$

No. of edges in a K-dimensional cube $= K \cdot 2^{K-1}$

[In $Q_K$, each vertex can be adjacent to K vertices. Sum of all degrees $= K \times 2^K$.

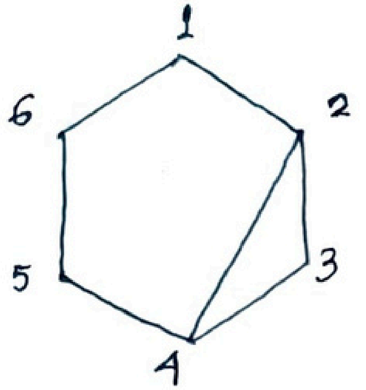
Now, $2|E| = K \times 2^K \Rightarrow |E| = K \cdot 2^{K-1}$ ]

## * Diameter, Radius, Eccentricity.

If G has a uv path, then the distance from u to v is the least length of a uv path.

If G has no path from u to v, $d(u,v)$ is infinite.

eg.



$d(1,2) = 1$
$d(1,3) = 2$
$d(1,4) = 2$
$\vdots$

- Diameter is $\max_{u,v \in V(G)} d(u,v)$
- Radius is $\min_{u \in V(G)} \varepsilon(u)$.

[ Eccentricity of vertex u,
$$\varepsilon(u) = \max_{v \in V(G)} d(u,v)$$ ]

- We can reach to any vertex from one vertex in less than or equal to the diameter edges.
- Eccentricity - If we want to reach to any vertex from u, ~~it~~ it will not take more than the eccentricity of u edges.

Q. Let G be an undirected graph (complete) $K_6$.
** If vertices of G are labeled, then the no. of ~~vertices~~ distinct cycles of length 4 in G is –


Out of 6 vertices, choose 4 vertices first. $\Rightarrow {}^6C_4$.

* Now, the vertices between starting & ending vertices (same) can be arranged in 3! ways.

2 3 4 — 3 4 2
2 4 3 — 4 2 3
3 2 4 — 4 3 2

For forward & backward arrangement divide by 2.

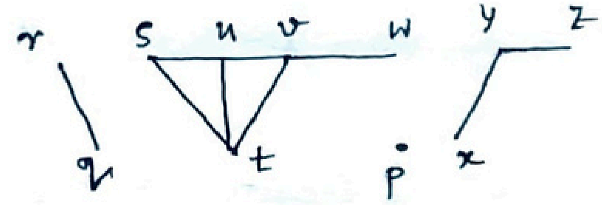
$$\left( {}^6C_4 \times \frac{3!}{2} \right) \text{ Ans.}$$

• Complete graph $K_n$. Vertices labeled. No. of distinct cycles of length $m$ is –

* $$\left[ {}^nC_m \times \frac{(m-1)!}{2} \right]$$

* <u>Connected graph</u>: A graph $G$ is connected if it has a $u,v$-path whenever $u,v \in V(G)$.

* <u>Component</u>: The component of a graph is a subgraph in which any two vertices are connected to each other by paths, & which is connected to no additional vertices in the graph.




$G$.

components.
$\{p\}$, $\{r,q\}$, $\{x,y,z\}$
$\{t,s,u,v,w\}$

isolated component

All components are pairwise disjoint.

- Adding an edge decreases the number of components by 0 or 1.
- Deleting an edge increases the no. of components by 0 or 1.
- Every graph with $n$ vertices & $k$ edges has at least $n-k$ components.

Q G'03 The $2^n$ vertices of a graph G corresponds to all subsets of a set of size n [$n \geq 6$]. 2 vertices of G are adjacent if & only if the corresponding sets intersect in exactly 2 elements. The number of vertices with zero degree in G is $\underline{n+1}$, & no. of connected components $\underline{n+2}$.

→ $2^n$ vertices.

Different level of vertices (based on no. of elements of set that represents vertex).

$\phi$. – $\phi$ not adjacent to any other vertex as intersection of $\phi$ with any subset will not yield exactly 2 elements.

{a} {b} {c} ... – Not adjacent to any

{a, b} {b, c} {c, d} ... – Will be adjacent to {a, b, c}, {b, c, d}, {c, d, e} etc.

So, no. of vertices with zero degree is $n+1$.

[subsets of cardinality 1 and 0]

Any subset having cardinality > 2, will be adjacent to any of the subsets of cardinality 2. So, they have positive degrees.

Now, every subset having cardinality $\geq 2$, forms a big component having all subsets of $c \geq 2$.

$\{a, b, c, d, e\} \ldots$ All are connected among themselves.

$\{a, b\}$ $\{b, c\}$ $\{c, d\} \ldots$

As any subset of $c > 2$ is adjacent with any of the subsets of $c = 2$, the subsets of $c = 2$ are connected among them too. So, like this all subsets of $c \geq 2$ are connected.

So, no. of connected components =

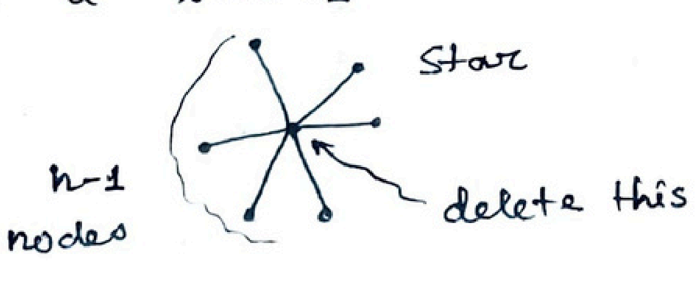
$(n+1) + 1 = n+2.$

$\phi, \{a\}, \{b\}, \ldots$ $\quad$ $c \geq 2$

Q. G'03 Let G be an arbitary graph with n nodes & k components. If a vertex is removed from G, the no. of components in the resultant graph must lie between $\underline{k-1}$ & $\underline{n-1}$.

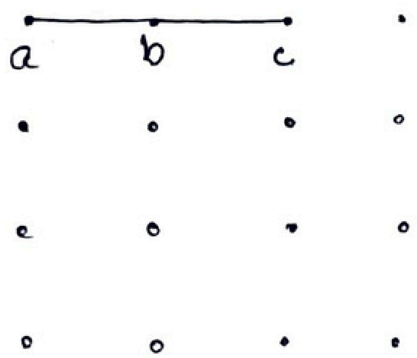
→ By deleting a vertex at most one component can get deleted. [When one isolated vertex is deleted] $\Rightarrow k-1$

Maximum can be $n-1$ [as 1 vertex is deleted and it being the center of a star.]



Q. G'03 If G is a graph with $n$ vertices & $K$ connected components then what is the minimum & maximum number of edges does G have?

→ **Minimum:** $n$ vertices All are isolated vertices. We have to add edges to get $k$ components.




$n-1 \leftarrow (n-2)+1$

$n-2 \leftarrow (n-3)+1$

$n - x = K$

$x = \boxed{n-K}$ Ans.

**Maximum:**

$n$

$K-1$ isolated vertices

$n-(K-1)$

$= n-K+1$

↓

Making a complete graph with $n-k+1$ vertices.

No. of edges ${}^{n-k+1}C_2$.

$\boxed{{}^{n-K+1}C_2}$ Ans.

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→ Adding 1 edge, decreases the no. of components by 1. Adding a & b produces $n-2$ isolated vertices and 1 connected component ab. ($n-1$ components). → $n-2$ isolated vertices & 1 component). And so on (till we get K components).

For minimum, we can also think like - the total vertices are divided into 2 parts -

$$
\begin{array}{ccc} & n & \\ \swarrow & & \searrow \\ K-1 & & n-(K-1) = n-K+1 \end{array}
$$

| | |
|---|---|
| $K-1$ vertices that are all ~~connected~~ isolated vertices | Remaining 1 component we have to connect with min no of edges. To connect them we need at least $n-k$ edges |

$\Downarrow$

Min. no of edges G has $\underline{n-K}$.

For maximum, suppose at first all $n$ vertices are connected ($K_n$). Now, we have to remove edges (min) so that we get $K$ connected components. In $K_n$ every vertex is connected to other $n-1$ vertices. So, to isolate one vertex we need to delete $n-1$ edges. Then we get 2 components (1 isolated vertex, 1 $K_{n-1}$). Then in $K_{n-1}$ we delete ~~$K_n$~~ $n-2$ edges to get total 3 components (2 isolated vertices, 1 $K_{n-2}$). & so on (till we get $K$ components)

$$
\#\text{edges}\Big|_{max} = {}^{n}C_2 - \Big( \underset{\substack{\uparrow \\ 2 \text{ components}}}{(n-1)} + \underset{\substack{\downarrow \\ 3 \text{ components}}}{(n-2)} + \overset{\substack{4 \text{ components} \\ \uparrow}}{(n-3)} + \cdots \underset{\substack{\downarrow \\ \text{for } K \text{ components}}}{(n-K+1)} \Big)
$$

$$
= \frac{n(n-1)}{2} - \left( (K-1)n - \frac{(K-1)K}{2} \right) = {}^{n-K+1}C_2
$$

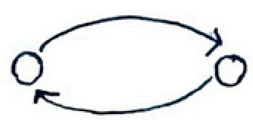
## * A directed graph or digraph G.

G is a triple consisting of a vertex set V(G), an edge set E(G) & function assigning each edge an ordered pair of vertices.

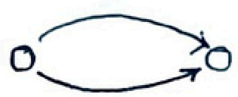
first vertex of ordered pair tail.

Second " " " " head.

- Directed graph can be simple and multi-graph too.



Simple digraph

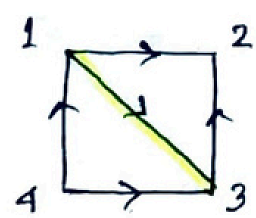


Not simple digraph

• Path in case of digraph : A path is a simple digraph whose vertices can be linearly ordered so that there is an edge with tail u & head v iff v immediately follows u in the vertex ordering.


path from 1 to 4

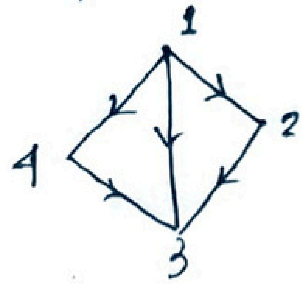
$1 \to 2 \to 3 \to 4$.




path 1 to 3,

$1 \to 3$

* The underlying graph : Underlying graph of a digraph D is the graph G obtained by treating the edges of D as unordered pairs.

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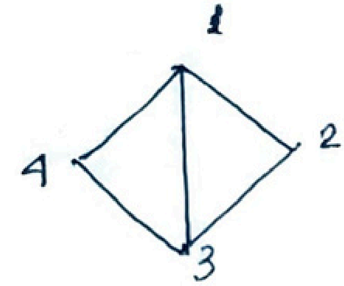
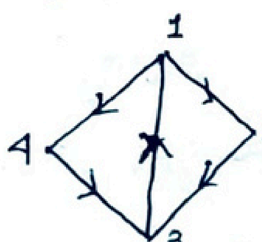



- Connectedness in digraph → for every pair there should be adirected path. [2 → 1 , 2 → 4 no directed path].

# ✓ Sometimes underlying graph of some digraph may be connected even if the digraph is not.

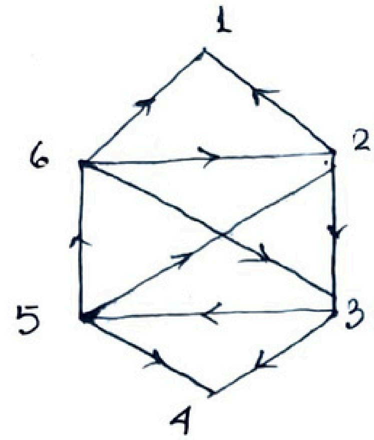
- A digraph is weakly connected if its underlying graph is connected.
- A digraph is strongly connected if for each ordered u, v vertices, there's a path from u to v.




Strongly connected.

# ✓ If digraph is strongly connected, it must be weakly connected.

* Strong components : Strong components of a digraph are its maximal strong subgraphs.



$\{2,3,6,5\}$ strongly connected.

• Vertex degrees (Digraph).

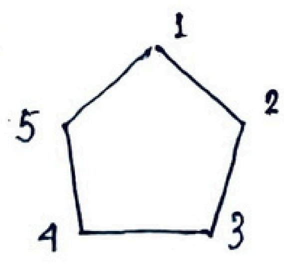
Let $v$ be a vertex in a digraph. The out degree $d^+(v)$ is the no. of edges with tail $v$. The in degree $d^-(v)$ is the no. of edges with head $v$.

$$\sum_{v \in V(G)} d^+(v) = |E(G)| = \sum_{v \in V(G)} d^-(v).$$

| | Min | Max |
|---|---|---|
| In | $\delta^-(G)$ | $\Delta^-(G)$ |
| Out | $\delta^+(G)$ | $\Delta^+(G)$. |

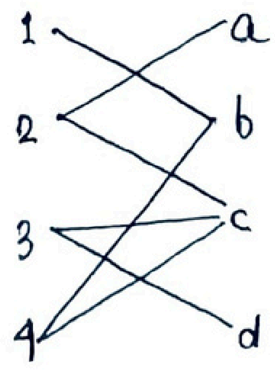
* Matching. A matching in a graph $G$ is a set of non-loop edges with no shared end-points.

Independent edge set.



$M_1$ (1, 5) and (2, 3) | 1, 2, 3, 5 saturated, 4 unsaturated.

$M_2$ (1, 2), and (3, 4).



$M$ (1, b), (2, a), (3, d), (4, c)

All nodes are saturated.

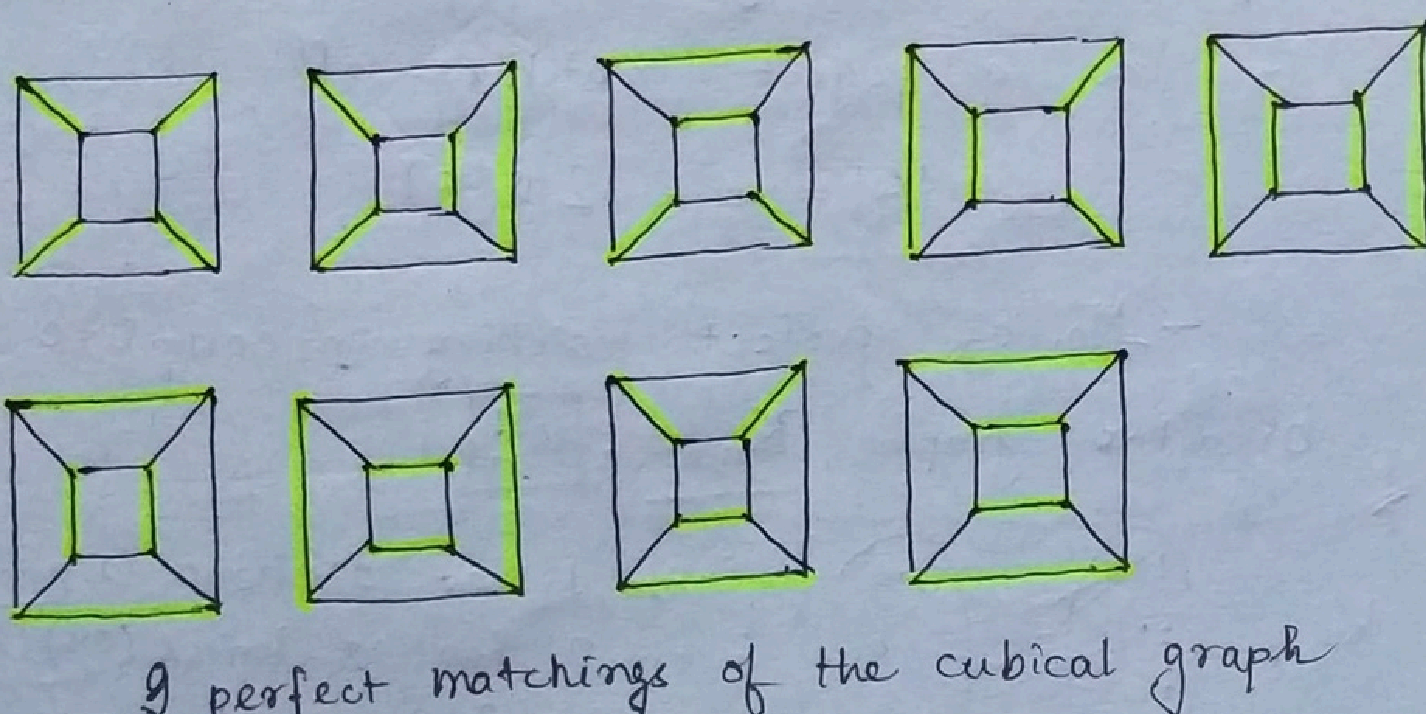
└ Perfect matching.

- The vertices incident to the edges of a matching M are saturated by M, the others are unsaturated.

- A perfect matching in a graph is a matching that saturates every vertex.

✓ Perfect matching is an independent edge set in which every vertex of the graph is incident to exactly one edge of the matching.

- A perfect matching is therefore a matching containing $n/2$ edges (the largest possible), meaning perfect matchings are only possible on graphs with an even number of vertices. A perfect matching is sometimes called a complete matching or 1-factor.



9 perfect matchings of the cubical graph

- While not all graphs have a perfect matching, all graphs do have a maximum independent edge set (maximum matching). Every perfect matching is a maximum independent edge set. A graph either has the same number of perfect matchings as maximum matchings (for a perfect matching graph) or else no perfect matching (for a no perfect matching graph).

- <u>Matching number ( $\nu(G)$ )</u>

✓ $\nu(G)$ of $G$, sometimes known as the edge independence number, is the size of a maximum independent edge set.

✓ $\nu(G) \leq \lfloor n/2 \rfloor$ $n$ is the vertex count.

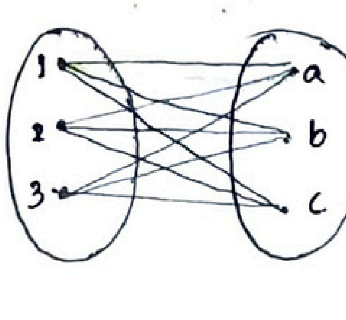
Equality occurs only for a perfect matching & $G$ has a perfect matching iff

$$|G| = n = 2\,\nu(G).$$

- No. of perfect matching in complete bipartite graph, <u>$K_{n,n}$</u> : $\boxed{n!}$

*

$K_{3,3}$




From 1 has 3 choices (1, a) (1, b) (1, c)
From 2 has 2 choices (2, b) (2, c) if (1,
From 3 has only 1 choice. (3, c) if (2,

$\Rightarrow 3 \times 2 \times 1 = 3!$

- Number of perfect matchings in complete graph : $K_n$

→ $K_{2n+1}$ has no perfect matching.

→ No. of perfect matchings in $K_{2n}$ is the no. of ways to pair up $2n$ distinct people.

For 1st person $2n-1$ choices.
For 2nd person $2n-3$ choices.
For 3rd " $2n-5$ ".

$$\Rightarrow \boxed{(2n-1)(2n-3)(2n-5)\cdots 1}$$

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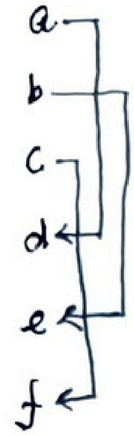



$a \to 5$ choices
$b \to 3$ choices
$c \to 1$ choice

$$= \frac{(2n)!}{(n!)\,2^n}$$

$(1\ 2)\ (3\ 4)\ (5\ 6)\ \ldots\ ((2n-1)\ (2n))$

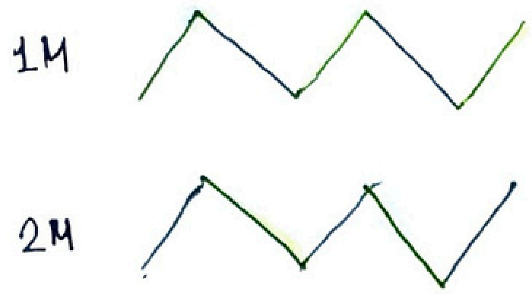
2 2 2

Rearranging of them $\Rightarrow n!$

rearranging of these $\Rightarrow 2^n$

- Maximal matching : Can't be enlarged by adding an edge.

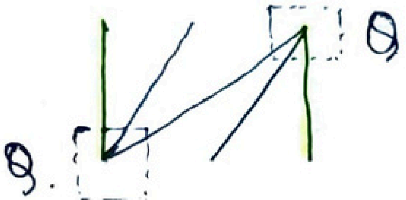
Maximum matching : Maximum size ~~of~~ among all matchings in the graph.




Both are maximal.

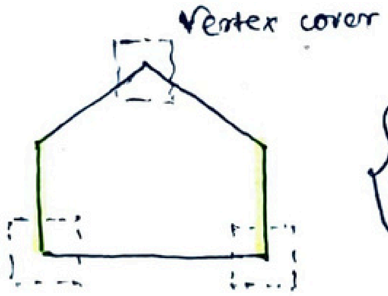
1M is maximum.

* <u>Vertex cover</u> : A vertex cover of a graph is a set $Q \subseteq V(G)$ that contains at least one end point of every edge. The vertices in $Q$ cover $E(G)$.




$\begin{cases} \text{matching } 2 \\ \text{vertex covers } 2. \end{cases}$

Since no vertex can cover two edges of matching the size of every vertex cover is at least the size of every matching.




$\begin{cases} \text{matching } 2 \\ \text{vertex cover } 3. \end{cases}$

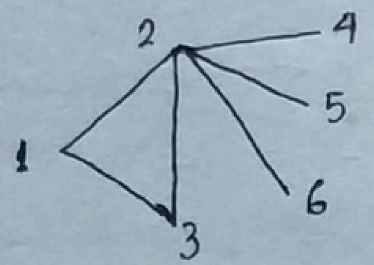
$$\boxed{|\text{vertex cover}| \geq |\text{matching}|}$$

* **Minimal Vertex Cover**: Vertex cover from which we can't remove any vertex.

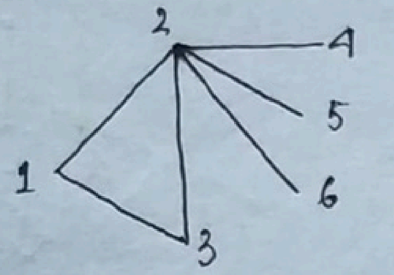
→ not unique.

**Minimum vertex cover**: Minimal vertex cover with least no. of vertices.

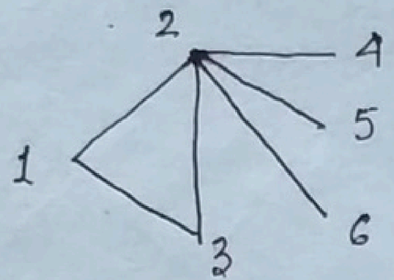
$\boxed{\text{Not unique}}$



$\{1,2,4,5,6\}$ is not minimal. We can delete 4, 5, 6.



$\{1,3,4,5,6\}$ is minimal.

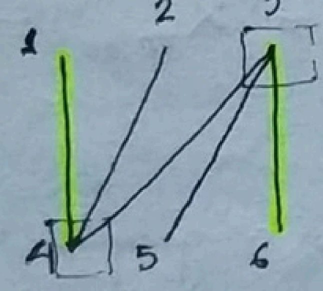


$\{2,3\}$ is minimum.
$\{1,2\}$ is minimum.

– Finding the minimum vertex cover is a np-complete problem.

– Minimum vertex cover is always minimal, vice versa need not be true.

• Obtaining a matching & a vertex cover of same size proves that each is optimal.



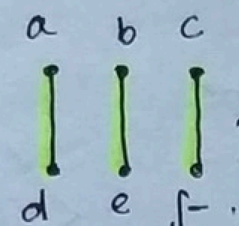
matching of size 2
vertex cover of size 2.

⇒ maximum matching
minimum vertex cover.

• |Maximal matching| ≤ |minimal vertex cover|

≤ 2 × |Maximal matching|

In worst case, we should be able to cover all the edges by taking all the vertices which are present in the matching.

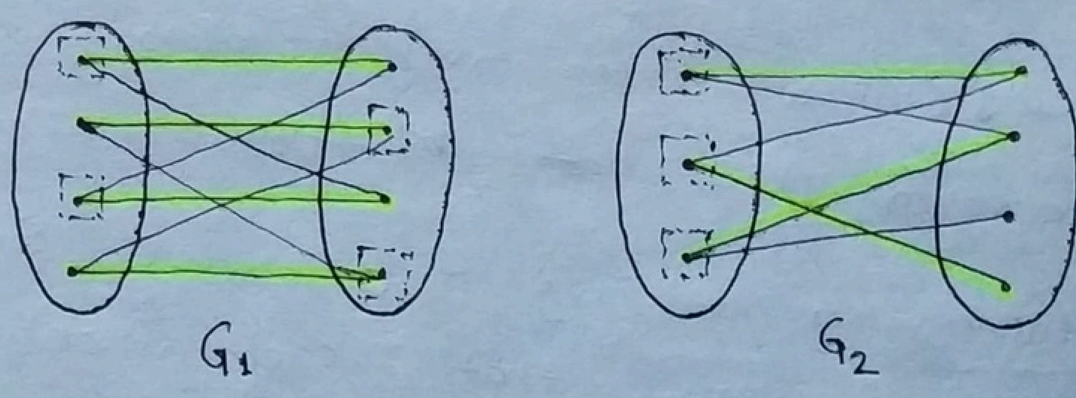



← Worst case.

⇓

|minimal vc| = 2 × |maximal matching|

• Th. If G is a bipartite graph, then the max. size of a matching in G is equal to the minimum size of vertex cover.




• Minimum vertex cover in $K_{m,n}$ = min (m, n)

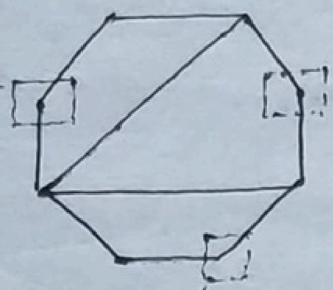
* <u>Independent sets & covers.</u>

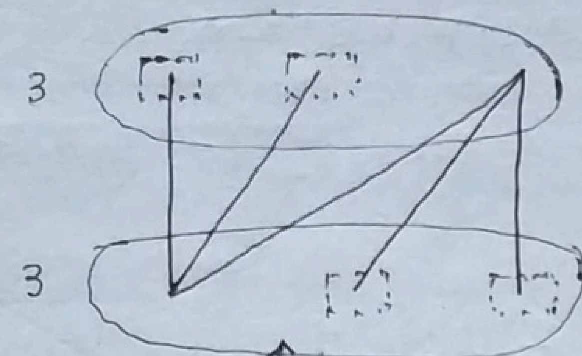
An independent set in a graph is a set of pairwise non-adjacent vertices.

Independent sets need not be unique.

The independence number of a graph is the maximum size of the independent set of vertices.

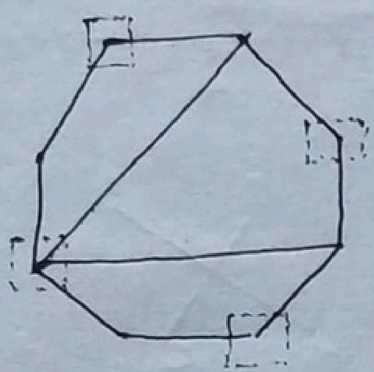
Independence no. = 4



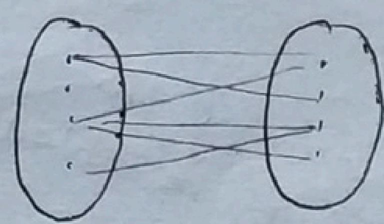



Independence no. = 4.

between the 3 vertices, no 2 vertices have an edge between them.

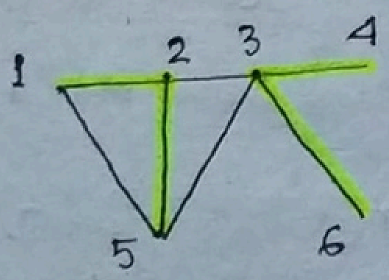


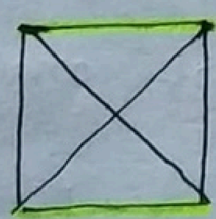
- The independence no. of a bipartite graph does not always have the size of a partite set.



independent set     independent set

Edge cover : An edge cover of G is a set L of edges such that every vertex of G is incident in L.

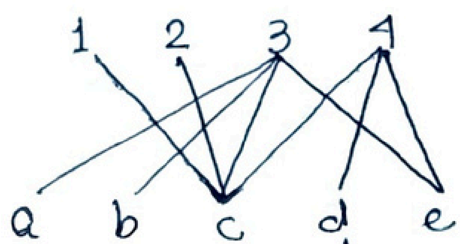


Edge covers need not be unique.

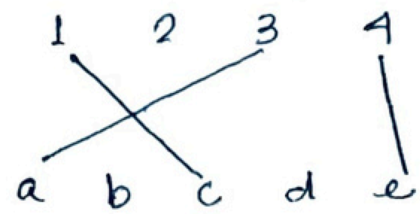
A perfect matching forms an edge cover with $\frac{|V|}{2}$. | $|V| \to$ no. of vertices.

- We can obtain an edge cover by adding edges to maximum matching.



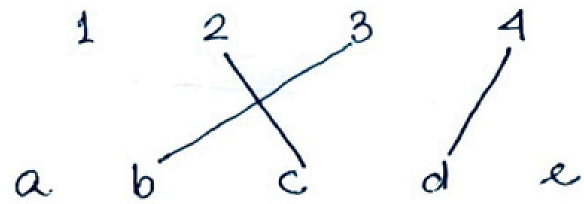

Graph

$\Rightarrow$




Maximum matching

+




Additional edges.

Each vertex of degree 1, $\Rightarrow$ edge containing that vertex must be added in an edge cover.

* Relation ~~between~~ among Edge Cover ($\beta'$), Vertex cover ($\beta$), independent sets ($\alpha$) & matching ($\alpha'$):

Maximum size of independent set $\alpha(G)$

" " of matching $\alpha'(G)$

Minimum size " vertex cover $\beta(G)$

" " " edge cover $\beta'(G)$

For every bipartite graph, $\alpha'(G) = \beta(G)$.

For every bipartite graph with no isolated vertices, $\alpha(G) = \beta'(G)$.

no. of vertic

In a graph G, $\alpha(G) + \beta(G) = n(G)$.

If G is a graph without isolated vertices,

$$\alpha'(G) + \beta'(G) = n(G).$$

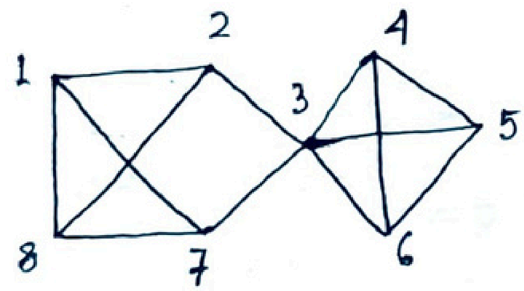
- For a graph with isolated vertex, there's no edge cover.

K(G) is the minimum size of a cut set of G.

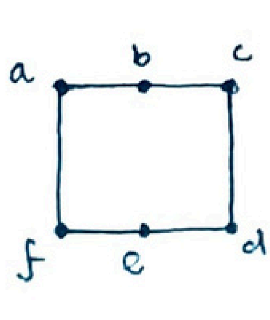
## * Cuts and Connectivity.

A separating set or vertex cut (cut set) of a graph G is set $S \subseteq V(G)$ such that $G-S$ has more than one components.

The connectivity of G ($K(G)$) is the minimum size of a vertex set S such that $G-S$ is disconnected or has only one vertex.

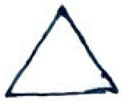



Cuts -
{3}
{2, 7}
{3, 2}
Connectivity = 1




Connectivity = 2
{b, e}

- In a complete graph, connectivity is $n-1$.



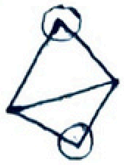

Connectivity, $\kappa = 2$

- A graph is <u>$\kappa$-connected</u> if its connectivity is at least $\kappa$.

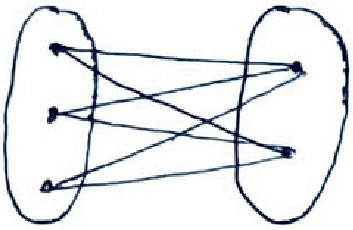
- A clique has no separating set.

- $\kappa(K_n) = n-1 = \delta(G)$. Each vertex is adjacent to other $n-1$ vertices.

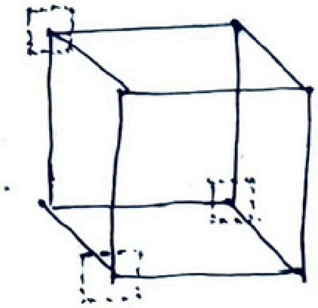
- $\kappa(G) \leq n(G) - 2$ <u>when $G$ is not a complete graph.</u>



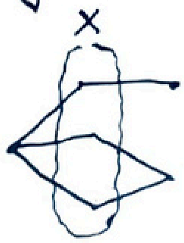
$\kappa(K_{m,n}) = \min(m,n)$



- Hypercube $Q_K$, for $K \geq 2$ the neighbours of one vertex in $Q_K$ form a separating set, so $\kappa(Q_K) \leq K$.



- Deleting the neighbours of a vertex disconnects a graph, so $\kappa(G) \leq \delta(G)$.



- Complete graphs do not have any cut sets, since $G-S$ is connected for all proper subsets $S$ of the vertex set.

Every non-complete graph has a cut-set.

- If $G$ is a connected, non-complete graph of order $n$, then

$$1 \leq \kappa(G) \leq n-2.$$

If $G$ is complete of order $n$, then

$$\kappa(G) = n-1.$$

- For a +ve integer $k$, we say that a graph is $k$ connected if $k \leq \kappa(G)$.

Connectivity is at least $k$.

- 1-connected means connected.

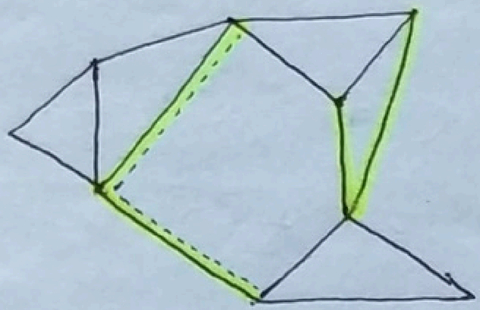
- A graph is connected if & only if $\kappa(G) \geq 1$.
- $\kappa(G) \geq 2$ iff $G$ is connected & has no cut vertices.
- Every 2-connected graph contains at least one cycle.
- For every graph, $\kappa(G) \leq \delta(G)$.

## * Edge connectivity.

A disconnecting set of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one components.

The edge connectivity of $G$ $(\kappa'(G))$ is the minimum size of disconnecting set.

Isolate vertex with least degree.



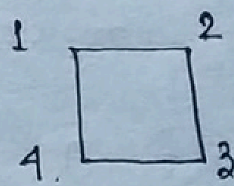
$\kappa'(G) = 2$

2-edge connected

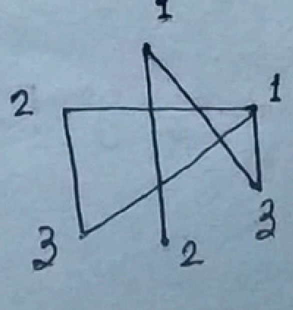
A graph is $\kappa$-edge connected if every disconnecting set has at least $\kappa$ edges.

## * Vertex Coloring.

A $K$-coloring of a graph $G$ is a labeling (coloring) $f: V(G) \to S$, where $S$ is a set of labels & $|S| = k$.




✓ A $K$-coloring is proper if adjacent vertices have different labels.




3-colorable.

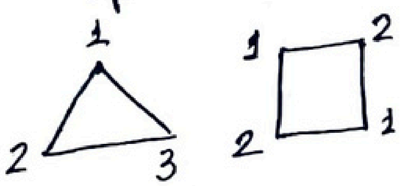
Not 2-colorable

- A graph is k-colorable if it has a proper k-coloring. [ Graph with $\boxed{CN \le K}$ → k colorable ]

• <u>Chromatic Number.</u> [ $\chi(G)$ ]

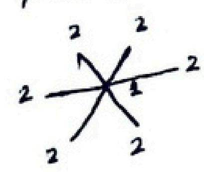
CN of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color i.e. the smallest value of k possible to obtain a k-coloring.

- CN of bipartite graph is 2.
- CN of odd length cycle is 3,
  of even length cycle is 2.



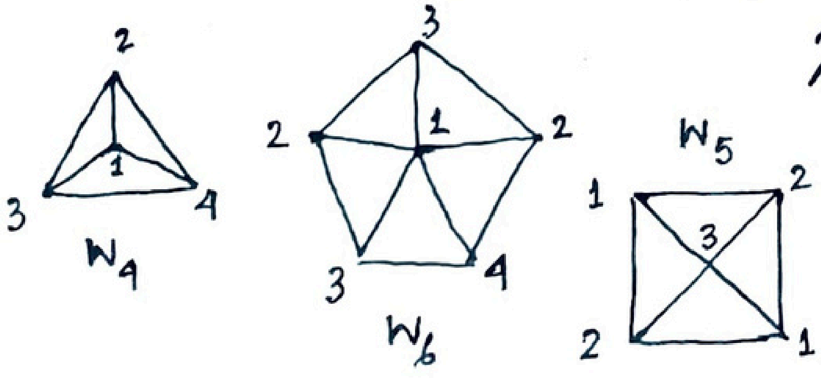

- CN of a complete graph $K_n$, is $n$.
- CN of star graph $S_n$, $n > 1$

$$\chi = 2$$




- CN of wheel graph, $W_n$, $n > 1$

$$\chi = 3, \quad n \text{ odd}$$
$$= 4, \quad n \text{ even}$$

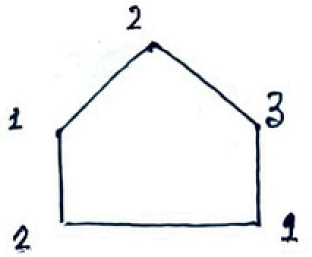


- Graph with CN K is said to be an K-chromatic graph.

- Brooks's theorem.

CN of a graph is at most the maximum vertex degree $\Delta$, unless the graph is complete or an odd cycle, in which case $\Delta + 1$ colors are required.

- If $\chi(H) < \chi(G) = K$ for every proper subgraph $H$ of $G$, then $G$ is a K-critical.




3-colorable

$\chi(G) = 3$

3-critical.

- The clique number of a graph $G$, $(\omega(G))$ is the max size of set of pairwise adjacent vertices (clique) in $G$.

Clique - maximum possible complete subgraph inside a graph

- For every graph

$$\chi(G) \geq \omega(G) \longrightarrow \chi(K_n) = n$$

$$\chi(G) \geq \frac{n(G)}{d(G)}$$

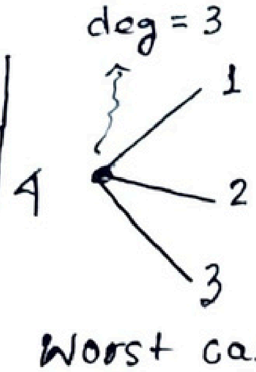
independence number

## * Greedy Coloring.

Obtained by coloring vertices in order $v_1, v_2, \ldots, v_n$ assigning to $v_i$ the smallest indexed color not already used on its lower-indexed neighbours.

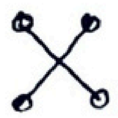
- $\chi(G) \leq \Delta(G) + 1$

$\begin{cases} \text{if max degree in } G \text{ is } d, \\ \chi \leq d+1 \end{cases}$



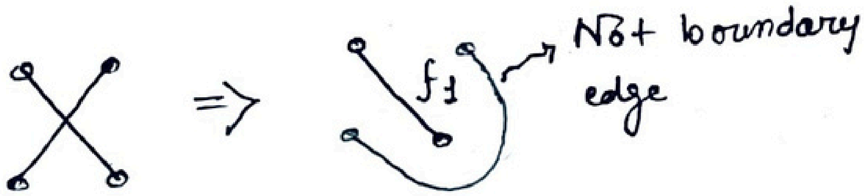

Worst case.

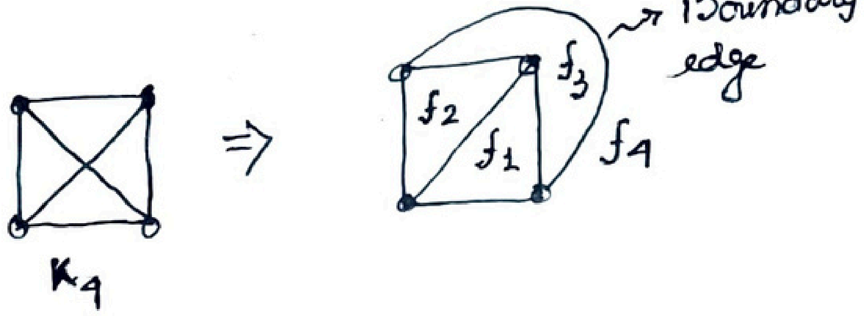
## * Planarity :

- Cross over : 

- The planar representation of a graph is drawing the graph on a plane without cross-over.

- A graph having planar representation is called a planar graph.



- Faces/region.

Planar representation of a planar graph divides entire plane into faces.

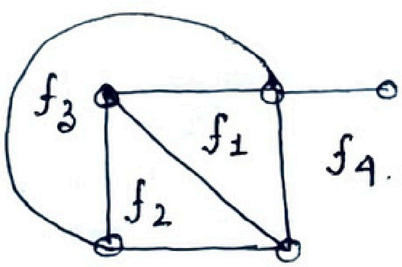
- Degree of region.

Degree of interior region:

No. of edges enclosing the region/face.

Degree of exterior region:

No. of edges exposed to the region/face.




$\deg(f_1) = 3$
$\deg(f_2) = 3$
$\deg(f_3) = 3$
$\deg(f_4) = 5$

- For any planar graph with n vertices

$$\sum_{i=1}^{n} \deg(f_i) = 2 \times e.$$

- In a planar graph,

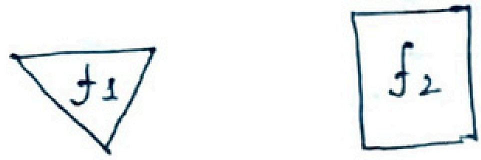
$K \times f = 2e$ if $\deg(f) = K$

$K \times f \leq 2e$ if $\deg(f) \geq K$

$K \times f \geq 2e$ if $\deg(f) \leq K$

- A planar graph with no loops & no parallel edges is a simple planar graph.

- In a simple planar graph with at least 2 edges, the degree of every face is at least 3.
interior.




$\deg(f_1) = 3$ $\quad\quad$ $\deg(f_2) = 4$.

- $3f \leq 2e$.

- <u>Euler's formula</u>.
  If $G$ is connected graph with $n$ vertices, $e$ edges, $f$ faces,

$$\boxed{n - e + f = 2}$$

- If $G$ is a simple planar connected graph with $n$ vertices, $e$ edges, $f$ faces with at least three vertices

$$\boxed{e \leq 3n - 6}$$

$\rightsquigarrow$ $n - e + f = 2$

Min degree of any face $= 3$

$\sum d(f_i) = 2e \quad \Big| \quad 3f \leq 2e$

$$f \leq \frac{2e}{3}$$

$$\Rightarrow n - e + \frac{2e}{3} \geq 2.$$

$$\Rightarrow e \leq 3n - 6.$$

Also, $\boxed{f \leq 2n-4.}$

$$\rightsquigarrow \quad 3f \leq 2e$$

$$\Rightarrow e \geq \frac{3f}{2}$$

$$n - \frac{3f}{2} + f \geq 2$$

$$\Rightarrow f \leq 2n-4.$$

$$n+f = 2+e.$$

$$n+f \geq 2 + \frac{3f}{2}$$

$$f \leq 2n-4.$$

Q. The minimum number of edges & vertices required to form 10 faces whose degree is 3, in a simple connected planar graph.

$$3f \leq 2e \Rightarrow e \geq 15 \text{ Ans } e_{min}$$

$$n - e + f = 2 \Rightarrow n = 2 + e - f$$

$$\Rightarrow n \geq 2 + 15 - 10 = 7 \text{ Ans}$$

Q. The maximum number of faces $[\deg(f) \geq 3]$ that are possible for a simple connected planar graph with 10 vertices.

$$f \leq 2n - 4. \Rightarrow f \leq 16$$

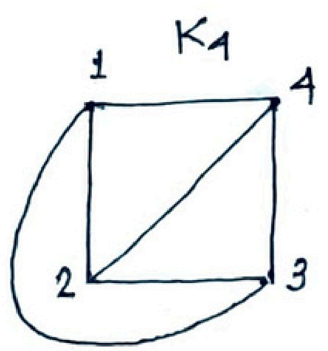
* <u>Euler's formula for disconnected graphs.</u>

If $G$ is a simple planar graph with $K$ components

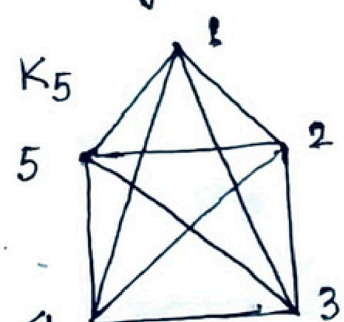
$$n - e + f = K + 1.$$

* <u>Four color theorem.</u>

Every planar graph is 4-colorable.




planar



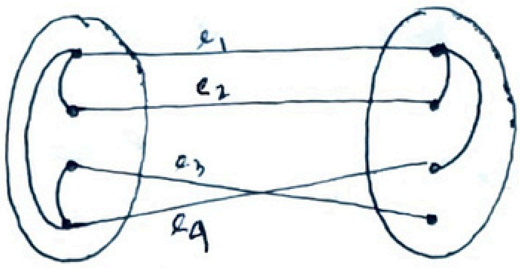

Non-planar

$\Rightarrow$ not 4-colorable

## * Edge cut

Every minimal disconnecting set of edges is an edge cut [when $n(G) > 1$].

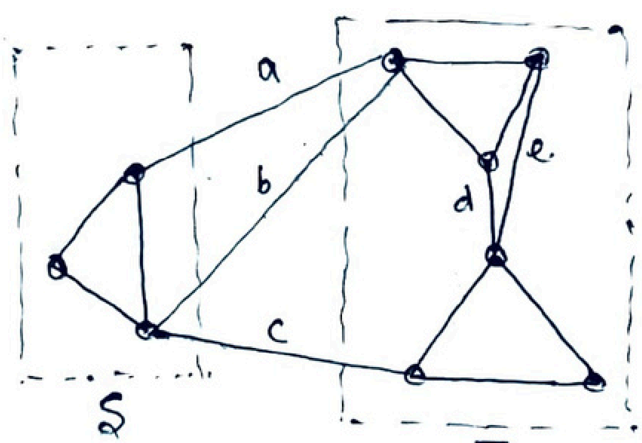
Given $S, T \subseteq V(G)$, we write $[S,T]$ for the set of edges having one end point in $S$ & other in $T$.




$[S, T]$

$e_1, e_2, e_3, e_4$ edge cut

An edge cut is an edge set of the form $[S, \bar{S}]$ where $S$ is non-empty proper subset of $V(G)$ & $\bar{S}$ denotes $G - S$.




$\{a, b, c\}$ both are $\{c, d, e\}$ minimal disconnecting edg sets & edge cuts

- Every edge cut is a disconnecting set.

- Deleting one endpoint of each edge in an edge cut $F$ deletes every edge of $F$.

vertex connectivity $\le$ edge connectivity

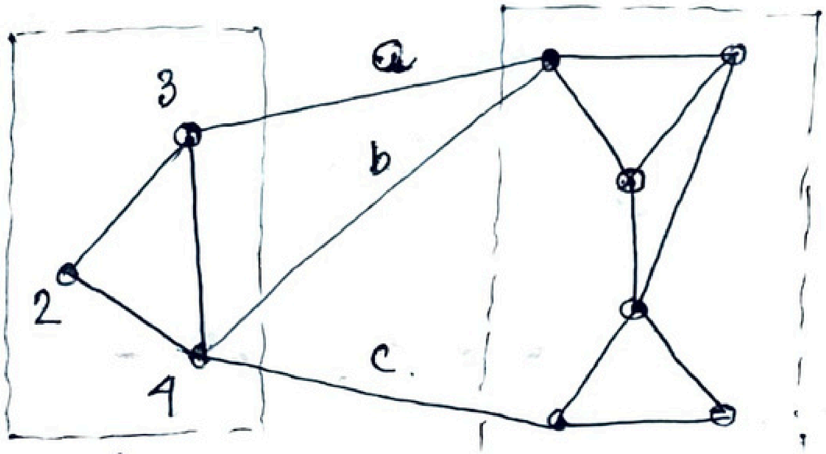
$$\kappa(G) \le \kappa'(G).$$

- If $G$ is a simple graph,

$$\kappa(G) \le \kappa'(G) \le \delta(G).$$

- If $S$ is a set of vertices in $G$,

$$\boxed{|[S, \bar{S}]| = \left[\sum_{v \in S} d(v)\right] - 2e(G(S))}$$




$\sum_{v \in S} d(v) = 4 + 3 + 2 = 9$

$2 \times e(G(S)) = 2 \times 3 = 6$

$\Rightarrow |[S, \bar{S}]| = 9 - 6 = 3.$

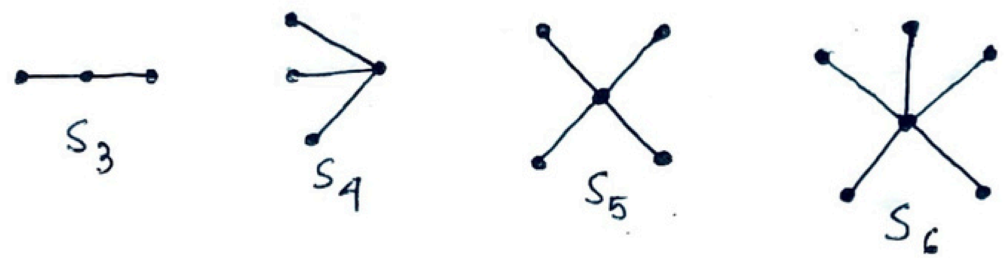
## * Different Named Graphs.

1. <u>Star graph</u>, $S_n$

Tree on $n$ nodes with one node having vertex degree $n-1$ and the other $n-1$ having vertex degree 1.

$S_n$ is isomorphic to $K_{1,n-1}$.

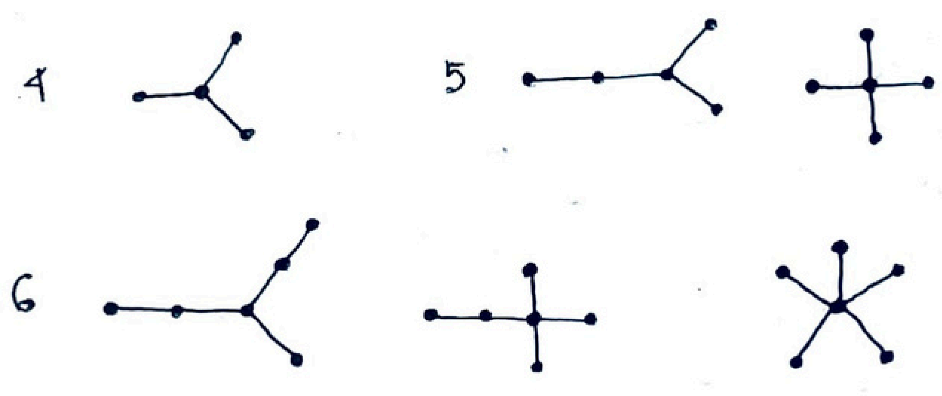
$$\chi(S_n) = 1 \quad \text{for } n = 1$$
$$= 2 \quad \text{otherwise.}$$




2. <u>Tree.</u>

Set of straight linesegments connected at their ends containing no closed loops.

It's a simple, undirected, connected, acyclic graph. A tree with $n$ nodes has $n-1$ edges.
All trees are bipartite.



## 3. Wheel Graph, $W_n$

Graph that contains a cycle of order $n-1$ & for which every graph vertex in the cycle is connected to one other graph vertex (hub).

$W_n$ can be defined as $K_1 + \underbrace{C_{n-1}}_{\text{Cycle graph}}$

No. of graph cycles in $W_n$ =

$$n^2 - 3n + 3$$

$$(7, 13, 21, 31, \ldots)$$

In $W_n$, hub has degree $n-1$, other nodes have 3.

Wheel graphs are 3-connected. $W_4 = K_4$

$$\chi(W_n) = \begin{cases} 3 & n \text{ odd} \\ 4 & n \text{ even} \end{cases}$$