GATE CSE NOTES

by

UseMyNotes

Logic.

* **Assertion** ~ An assertion is a statement.

* **Proposition** ~ A proposition is an assertion that is either true or false but not both.

eg. 4 is a prime number.

Proposition will have truth value.

eg. $x + y > 4$ is not a proposition.

eg. This statement is false. [Liar Paradox]

It is an assertion, but not a proposition.

* **Propositional Variable**: A propositional variable denotes an arbitrary proposition with unspecified truth value P, Q, R, ...

- Logical Connectives - P and Q
  P or Q
  Not P, $P \oplus Q$.

And - $\wedge$ Conjunction

Or - $\vee$ Disjunction.

Not - $\neg$ / ~

Exclusive Or - $\oplus$

- Well formed formula (WFF).

* **Implication**: $P \Rightarrow Q$. P implies Q.

P Premise, Hypothesis, Antecedent

Q Conclusion, Consequence.

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

If P then Q

P only if Q

P is sufficient condition for Q

Q is a necessary condition for P

Q if P

Q follows from P.

Q provided P.

Q is a logical consequence of P.

Q whenever P. • $\neg P \to \neg Q$ Inverse

• $Q \Rightarrow P$ is called the converse.

• $\neg Q \Rightarrow \neg P$ is called the contrapositive.

* Equivalence : $P \Leftrightarrow Q$. P is equivalent to Q.

P if & only if Q.

P is a necessary & sufficient condition for Q.

P	Q	$P \Leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

Eg. $(Q \wedge \neg P) \Rightarrow P$.

P	Q	$\neg P$	$Q \wedge \neg P$	$(Q \wedge \neg P) \Rightarrow P$
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	0	0	1

Eg. $[(P \wedge Q) \vee \neg R] \Leftrightarrow P. \equiv X$

P	Q	R	$P \wedge Q$	$\neg R$	$(P \wedge Q) \vee \neg R$	X
0	0	0	0	1	1	0
0	0	1	0	0	0	1
0	1	0	0	1	1	0
0	1	1	0	0	0	1
1	0	0	0	1	1	1
1	0	1	0	0	0	0
1	$\cancel{0}$1	0	1	1	1	1
1	1	1	1	0	1	1

* **Tautology:** A tautology is a propositional form whose truth value is true for all possible values of its propositional variables.

Eg. $P \vee \neg P$.

* **Contradiction/Absurdity:** A propositional form which is always false.

Eg. $P \wedge \neg P$.

* **Contingency:** A propositional form that is neither a tautology nor a contradiction is called contingency.

* **Logical Identities:**

1. Idempotence of $\vee$: $P \Leftrightarrow P \vee P$.
2. Idempotence of $\wedge$: $P \Leftrightarrow P \wedge P$.
3. Commutativity of $\vee$: $(P \vee Q) \Leftrightarrow (Q \vee P)$.
4. Commutativity of $\wedge$: $(P \wedge Q) \Leftrightarrow (Q \wedge P)$.
5. Associativity of $\vee$:

$$[(P \vee Q) \vee R] \Leftrightarrow [P \vee (Q \vee R)]$$

6. Associativity of $\wedge$:

$$[(P \wedge Q) \wedge R] \Leftrightarrow [P \wedge (Q \wedge R)]$$

7. Demorgan's Law :

$$\neg (P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg (P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q).$$

8. Distributivity of $\wedge$ over $\vee$:

$$[P \wedge (Q \vee R)] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$$

9. Distributivity of $\vee$ over $\wedge$:

$$[P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$$

✓10. $(P \vee 1) \Leftrightarrow 1$ $\quad (P \vee \neg P) \Leftrightarrow 1$

$(P \wedge 1) \Leftrightarrow P$ $\quad (P \wedge \neg P) \Leftrightarrow 0$

$(P \vee 0) \Leftrightarrow P$ $\quad P \Leftrightarrow \neg(\neg P)$ Double Negation.

$(P \wedge 0) \Leftrightarrow 0$

✓11. Implication : $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

12. Equivalence :

$$(P \Leftrightarrow Q) \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$$

✓13. Exportation :

$$[(P \wedge Q) \Rightarrow R] \Leftrightarrow [(P \Rightarrow (Q \Rightarrow R))]$$

✓14. Absurdity :

$$[(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)] \Leftrightarrow \neg P.$$

✓15. Contrapositive :

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).$$

Eg. $[(A \Rightarrow B) \vee (A \Rightarrow D)] \Rightarrow (B \vee D)$

$\equiv [(\neg A \vee B) \vee (\neg A \vee D)] \Rightarrow (B \vee D)$

$\equiv [\neg A \vee (B \vee D)] \Rightarrow (B \vee D)$

$\equiv \neg[\neg A \vee (B \vee D)] \vee (B \vee D)$

$\equiv [A \wedge \neg(B \vee D)] \vee (B \vee D)$

$\equiv [A \vee (B \vee D)] \wedge [\neg(B \vee D) \vee (B \vee D)]$

$\equiv (A \vee (B \vee D)) \wedge 1$

$\equiv A \vee B \vee D.$

Eg, P : It is snowing.

Q : I will go to town.

R : I have time.

$P \Rightarrow Q$

P only if Q

Q if P

(i) If it is not snowing & I have time, I will go to town.

$$(\neg P \wedge R) \Rightarrow Q$$

(ii) I will go to town only if I have time.

* $$Q \Rightarrow R.$$

(iii) It is not snowing. $\neg P$.

(iv) It is snowing & I will not go to town.

$$P \wedge \neg Q$$

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Eg. $Q \Leftrightarrow (R \wedge \neg P)$

I will go to town if & only if I have time & it's not snowing.

* Logical Implications :

1. Addition : $P \Rightarrow (P \vee Q)$.
2. Simplification : $(P \wedge Q) \Rightarrow P$
3. Modus Ponens : $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$
4. Modus Tollens : $[(P \Rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$.

5. Disjunctive Syllogism:

$$[\neg P \wedge (P \vee Q)] \Rightarrow Q.$$

6. Hypothetical Syllogism:

$$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R).$$

7. $(P \Rightarrow Q) \Rightarrow [(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)]$

8. $[(P \Rightarrow Q) \wedge (R \Rightarrow S)] \Rightarrow [(P \wedge R) \Rightarrow (Q \wedge S)]$

9. $[(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)] \Rightarrow (P \Leftrightarrow R).$

- Modus Ponens. P, $P \Rightarrow Q$
  then Q.

- Modus Tollens $P \Rightarrow Q$, $\neg Q$
  then $\neg P$

* Predicates $P(x_1, x_2, \ldots, x_n)$.

Predicate variables – x_i

Predicate constants – 2, 3 ...
make the predicate a proposition.

- If $P(x_1, x_2, \ldots, x_n)$ is true for all values $(c_1, \ldots, c_n)$ from the universe U then P is valid in U.

If $P(x_1, \ldots, x_n)$ is true for some values from U, then P is satisfiable in U.

$\downarrow$

tautology / contingency

If P is not true for any values, then P is <u>unsatisfiable</u> in U.
Contradiction

- Binding variables by giving values.

$$P(x,y) \longrightarrow \text{give values of } x, y$$

- Binding using quantifiers :

(i) $\forall x \, P(x)$ $\forall \rightarrow$ For all.
<u>Universal quantifier.</u>

Eg. $\forall x \, [x < x+1]$

(ii) $\exists x \, P(x)$ $\exists \rightarrow$ There exists.
For some.
<u>Existential quantifier.</u>

Eg. $\exists x \, [x = 3]$

(iii) $\exists ! x \, P(x)$ $\exists !$ There exists a unique x

Eg. $\exists ! x \, [x = x+1]$

- $\exists ! x \, P(x) \equiv$

$$\exists x \left[P(x) \wedge \forall y \{ P(y) \Rightarrow x = y \} \right]$$

- $\exists x \forall y \, P(x,y) \not\Leftrightarrow \forall y \exists x \, P(x,y)$.

$\forall x \forall y \, P(x,y) \Leftrightarrow \forall y \forall x \, P(x,y)$

$\exists x \exists y \, P(x,y) \Leftrightarrow \exists y \exists x \, P(x,y)$

- $\forall x \left[\underbrace{\forall y \, \underbrace{P(x,y)}_{\text{Scope of } y} \Rightarrow x = 0}_{\text{Scope of } x} \right]$

- <u>Summary</u> : A predicate is an expression of one or more variables defined on some specific domain.
A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable. Predicate – $E(x,y) \equiv x = y$

※ Properties of Predicate Logic :

1. $\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee P(x_3) \vee \cdots$

   $\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \cdots$

2. $\forall x\, P(x) \rightarrow \exists x\, P(x)$

3. De Morgan's $\quad \neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$

   $\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$

4. $\exists x\, \forall y\, P(x,y) \rightarrow \forall y\, \exists x\, P(x,y)$

   $\forall x\, \exists y\, P(x,y) \not\rightarrow \exists y\, \forall x\, P(x,y)$

5. $\forall x\, \forall y\, P(x,y) \leftrightarrow \forall y\, \forall x\, P(x,y)$

   $\exists x\, \exists y\, P(x,y) \leftrightarrow \exists y\, \exists x\, P(x,y)$

6. $\forall x\, \forall y\, P \rightarrow \exists x\, \forall y\, P$

   $\forall x\, \forall y\, P \rightarrow \forall x\, \exists y\, P$

7. $\forall x\, (P(x) \wedge Q(x)) \equiv (\forall x\, P(x)) \wedge (\forall x\, Q(x))$

   $\exists x\, (P(x) \vee Q(x)) \equiv (\exists x\, P(x)) \vee (\exists x\, Q(x))$

Essential   Job to do   Phone No.

8. $\forall x\, (P(x) \vee Q(x)) \not\Rightarrow (\forall x\, P(x)) \vee (\forall x\, Q(x))$

   $(\forall x\, P(x)) \vee (\forall x\, Q(x)) \Rightarrow \forall x\, (P(x) \vee Q(x))$

9. $\exists x\,(P(x) \wedge Q(x)) \rightarrow (\exists x\, P(x)) \wedge (\exists x\, Q(x))$

✓ $(\exists x\, P(x)) \wedge (\exists x\, Q(x)) \not\rightarrow \exists x\,(P(x) \wedge Q(x))$

10. $\forall x\,(P(x) \wedge Q) \equiv (\forall x\, P(x)) \wedge Q$

$\exists x\,(P(x) \wedge Q) \equiv (\exists x\, P(x)) \wedge Q$

$\forall x\,(P(x) \vee Q) \equiv (\forall x\, P(x)) \vee Q$

$\exists x\,(P(x) \vee Q) \equiv (\exists x\, P(x)) \vee Q.$

eg. Predicate translation.

a) Every student in the class has studied DM.

$x \in$ Set of persons $\quad C(x)$: x is in the class

$S(x)$: x studied DM

$\forall x\,(C(x) \rightarrow S(x))$ $\quad$ not $\forall x\,(C(x) \wedge S(x))$

b) Some students in the class has studied DM.

$\exists x\,(C(x) \wedge S(x))$ $\quad$ not $\exists x\,(C(x) \rightarrow S(x))$

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DM₂ - 3

Translation

1. p unless q. $p \vee q$

eg. You will not get A grade unless you work hard.
$\neg p$ ... q

$\neg p \vee q \equiv (p \rightarrow q)$.

2. p or q is true but not both → $p \oplus q$
Exactly one of p or q →

3. At least one of p or q → $p \vee q$
Either p or q →

4. Neither p nor q $p \downarrow q$

5. At least one of p or q is false
Either p or q is false
$p \uparrow q$
$\neg p \vee \neg q$

6. p is necessary for q
p is required for q
p is must for q
$q \rightarrow p$

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7. p is sufficient for q } $p \to q$
p is enough for q
p only if q : $p \to q$

* $\oplus$ is commutative & associative.

$$p \wedge (Q \oplus R) = (P \wedge Q) \oplus (P \wedge R)$$

$\uparrow, \downarrow$ are both commutative, not associative.

* Tautology, Contradiction, Contingency

* A wff is satisfiable iff it's either a tautology or a contingency.
Unsatisfiable – contradiction.

* $p \Leftrightarrow q$ equivalence iff $p \leftrightarrow q$ is a tautology.

* PCNF ( ) ∧ ( ) ∧ … Product of sums (maxterms)
PDNF ( ) ∨ ( ) ∨ … Sum of products (minterms)

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* Set of wffs are consistent if their product (∧) is satisfiable.
Inconsistent – unsatisfiable.

* Valid – Always true

- **Well formed formula:**

WFF is a predicate holding any of the following ~

i) All propositional constants & propositional variables are wffs.

ii) If x is a variable & Y is a wff, $\forall x Y$ & $\exists x Y$ are also wff.

iii) Truth value & false values are wffs.

iv) Each atomic formula is a wff.

v) All connectives connecting wffs are wffs.

- **Examples.:** Let the universe be the integers & let $N(x)$ denotes "x is a nonnegative integer", $E(x)$ denotes "x is even", $P(x)$ denotes "x is prime". $O(x)$ denotes "x is odd".

Then,

a) There exists an even integer.

$$\exists x\, E(x).$$

b) Every integer is even or odd.

$$\forall x\, [\, E(x) \vee O(x) \,].$$

c) All prime integers are nonnegative.

$$\forall x\, [\, P(x) \Rightarrow N(x) \,]$$

d) The only even prime is two.

$$\forall x [(E(x) \wedge P(x)) \Rightarrow x = 2]$$

e) There is one & only even prime.

$$\exists! x [E(x) \wedge P(x)]$$

f) Not all integers are odd.

$$\neg \forall x\, O(x) \quad \text{or} \quad \exists x \neg O(x).$$

g) Not all primes are odd.

$$\neg \forall x [P(x) \Rightarrow O(x)] \quad \text{or}$$

$$\exists x [P(x) \wedge \neg O(x)]$$

h) If an integer is not odd, then it's even.

$$\forall x [\neg O(x) \Rightarrow E(x)]$$

- Predicate logic
  - Valid (True for all universes)
  - Satisfiable (True for some)
  - Unsatisfiable (False for everyone)

- $\exists x [P(x) \Rightarrow Q(x)]$

$$\not\Leftrightarrow \exists x\, P(x) \Rightarrow \exists x\, Q(x).$$

- $\neg \forall x\, P(x) \Leftrightarrow \exists x \neg P(x)$

$$\neg \exists x\, P(x) \Leftrightarrow \forall x \neg P(x).$$

Eg. $\neg \forall x \forall y \exists z\, P(x, y, z)$

$$\Leftrightarrow \exists x \neg \forall y \exists z\, P(x, y, z)$$

$$\Leftrightarrow \exists x \exists y \neg \exists z\, P(x, y, z).$$

$$\Leftrightarrow \exists x \exists y \forall z \neg P(x, y, z).$$

- $\forall$ distributes over $\wedge$.

  $\forall$ doesn't distribute over $\vee$.

* Logical relationships involving quantifiers:

1. $\forall x\, P(x) \Rightarrow P(c)$, where c is an arbitary element of the universe.
2. $P(c) \Rightarrow \exists x\, P(x)$, where c is an arbitary element of the universe.
3. $\forall x\, \neg P(x) \Leftrightarrow \neg \exists x\, P(x)$
4. $\forall x\, P(x) \Rightarrow \exists x\, P(x)$.
5. $\exists x\, \neg P(x) \Leftrightarrow \neg \forall x\, P(x)$

   predicate not involving x.
6. $[\forall x\, P(x) \wedge Q] \Leftrightarrow \forall x\, [P(x) \wedge Q]$
7. $[\forall x\, P(x) \vee Q] \Leftrightarrow \forall x\, [P(x) \vee Q]$
8. $[\forall x\, P(x) \wedge \forall x\, Q(x)] \Leftrightarrow \forall x\, [P(x) \wedge Q(x)]$
9. $[\forall x\, P(x) \vee \forall x\, Q(x)] \Rightarrow \forall x\, [P(x) \vee Q(x)]$
10. $[\exists x\, P(x) \wedge Q] \Leftrightarrow \exists x\, [P(x) \wedge Q]$
11. $[\exists x\, P(x) \vee Q] \Leftrightarrow \exists x\, [P(x) \vee Q]$
12. $\exists x\, [P(x) \wedge Q(x)] \Rightarrow [\exists x\, P(x) \wedge \exists x\, Q(x)]$
13. $[\exists x\, P(x) \vee \exists x\, Q(x)] \Leftrightarrow \exists x\, [P(x) \vee Q(x)]$

Eg. The limit of $f(x)$ as x approaches c is K $\left(\lim_{x \to c} f(x) = K\right)$ iff for every $\epsilon > 0$ there exists $\delta > 0$ s.t. for all x, if $|x - c| < \delta$ then $|f(x) - K| < \epsilon$

Using logical expressions,

$$\forall_{\epsilon > 0} \ \exists_{\delta > 0} \ \forall_x \left[|x-c| < \delta \Rightarrow |f(x) - K| < \epsilon \right]$$

Eg. $\lim_{x \to c} f(x) \neq K$ iff there is an $\epsilon > 0$

s.t. for every $\delta > 0$, there is some x s.t.

$|x-c| < \delta$ & yet $|f(x) - K| \geq \epsilon$.

$$\rightarrow \quad \neg \forall_{\epsilon > 0} \ \exists_{\delta > 0} \ \forall_x \left[|x-c| < \delta \Rightarrow |f(x)-K| < \epsilon \right]$$

$$P \rightarrow Q \leftrightarrow \neg P \vee Q$$

or $\exists_{\epsilon > 0} \ \forall_{\delta > 0} \ \exists_x \neg \left[\neg |x-c| < \delta \vee |f(x)-K| < \epsilon \right]$

or $\exists_{\epsilon > 0} \ \forall_{\delta > 0} \ \exists_x \left[|x-c| < \delta \wedge \neg |f(x)-K| < \epsilon \right]$

$\exists_\epsilon \ \forall_\delta \ \exists_x \left[|x-c| < \delta \wedge |f(x) - K| \geq \epsilon \right]$

Eg. Let the universe consists of all integers

& let

$P(x)$: x is a prime.

$Q(x)$: x is positive.

$E(x)$: x is even.

$N(x)$: x is divisible by 9.

$S(x)$: x is a perfect square.

$G(x)$: x is greater than 2.

Then,

a) x is even or x is a perfect square.

$$E(x) \vee S(x).$$

b) $P(x) \wedge N(x)$

c) $P(x) \wedge G(x)$

d) $P(x) \Rightarrow G(x)$.

e) $P(x) \Rightarrow [Q(x) \wedge \neg E(x)]$

Eg. Translate each of the following into symbols, first using no existential quantifiers & second using no universal quantifiers.

a) Not all cars have carburetors.

b) No dogs are intelligent.

c) Some numbers are not real.

→ Car(x) : x is a car

carbu(x) : x has carburetors.

a) $\neg \forall x \, (\text{car}(x) \Rightarrow \text{carbu}(x))$

$\exists x \, \neg (\text{car}(x) \Rightarrow \text{carbu}(x))$

or $\exists x \, \neg (\neg \text{car}(x) \vee \text{carbu}(x))$

or $\exists x \, (\text{car}(x) \wedge \neg \text{carbu}(x))$

dog(x) : x is a dog.

int(x) : x is intelligent.

b) $\forall x \, (\text{dog}(x) \Rightarrow \neg \text{int}(x))$

$\forall x \neg\neg (\neg dog(x) \vee \neg int(x))$.

or $\neg \exists x (dog(x) \wedge int(x))$

$\forall \to \Rightarrow$

$\exists \to \wedge$

$num(x)$: x is a number.

$real(x)$: x is real.

c) $\exists x (num(x) \wedge \neg real(x))$.

$\exists x \neg\neg (num(x) \wedge \neg real(x))$

or $\neg \forall x (\neg num(x) \vee real(x))$

or $\neg \forall x (num(x) \Rightarrow real(x))$.

* Logical Inference :

Every system has some axioms associated with it. We derive theorems from the axioms by using rules of inference.

- Rules of inference related to the language of propositions :

Name	Tautological form	Rule of inference
1. Addition	$P \Rightarrow (P \vee Q)$	$\dfrac{P}{\therefore P \vee Q}$
2. Simplification	$(P \wedge Q) \Rightarrow P$	$\dfrac{P \wedge Q}{\therefore P.}$
3. Modus Ponens	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$	$\dfrac{P \quad P \Rightarrow Q}{\therefore Q}$
4. Modus Tollens	$[\neg Q \wedge (P \Rightarrow Q)] \Rightarrow \neg P$	$\dfrac{\neg Q \quad P \Rightarrow Q}{\therefore \neg P.}$

5. Disjunctive Syllogism $[(P \vee Q) \wedge \neg P] \Rightarrow Q$

$$\begin{array}{l} P \vee Q \\ \neg P \\ \hline \therefore Q \end{array}$$

6. Hypothetical Syllogism $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow [P \Rightarrow R]$

$$\begin{array}{l} P \Rightarrow Q \\ Q \Rightarrow R \\ \hline P \Rightarrow R \end{array}$$

7. Conjunction

$$\begin{array}{l} P \\ Q \\ \hline \therefore P \wedge Q \end{array}$$

8. Constructive Dilemma. $[(P \Rightarrow Q) \wedge (R \Rightarrow S) \wedge (P \vee R)] \Rightarrow (Q \vee S)$

$$\begin{array}{l} (P \Rightarrow Q) \wedge (R \Rightarrow S) \\ P \vee R \\ \hline \therefore Q \vee S \end{array}$$

9. Destructive Dilemma $[(P \Rightarrow Q) \wedge (R \Rightarrow S) \wedge (\neg Q \vee \neg S)] \Rightarrow (\neg P \vee \neg R)$

$$\begin{array}{l} (P \Rightarrow Q) \wedge (R \Rightarrow S) \\ \neg Q \vee \neg S \\ \hline \therefore \neg P \vee \neg R. \end{array}$$

Eg. If horses fly or cows eat grass, then the mosquito is the national bird. If the mosquito is the national bird, the peanut butter tastes good on hot dogs. But, peanut butter tastes terrible on hot dogs. Therefore, cows don't eat grass.

→ Horses fly : H

Cows eat grass : G

Mosquito is N.B. : M.

PB tastes good on hot dogs. : P

Arguments	Conclusion
$(H \vee G) \Rightarrow M$ — (I)	$\neg G$
$M \Rightarrow P$ — (II)	
$\neg P$ — (III)	

From (i) & (ii) by Hypothetical syllogism,

$H \vee G \Rightarrow P$. —(iv).

From (iii) & (iv) by Modus Tollens,

$\neg(H \vee G)$. —(v).

From (v) by de Morgan's Law,

$\neg H \wedge \neg G$. —(vi)

From (vi) by simplification,

we get $\neg G$.

So, arguments are valid.

Eg. It is not the case that IBM or Xerox will take over the copier market. If RCA returns to the computer market, then IBM will take over the copier market. Hence, RCA will not return to the computer market.

→ IBM will take over copier market : I

Xerox " " " " : X

RCA " return to the computer " : R

$\neg(I \vee X)$. —(i)

$R \Rightarrow I$ —(ii)

$\therefore \neg R$.

From (i), $\neg I \wedge \neg X$ —(iii) [de Morgan's]

From (iii), $\neg I$ —(iv) [Simplification]

From (ii) & (iv), $\neg R$ [Modus Tollens]

Hence, valid argument,

Eg. If today is tuesday, then I have a test in CS or a test in Econ. If my Econ professor is sick, then I will not have a test in Econ. Today is Tuesday & my Econ. professor is sick. Therefore, I have a test in CS.

→ Today is Tuesday : T

I have test in CS : C

I have test in Econ : E

Econ. professor is sick : S

$T \Rightarrow (C \vee E)$ — (i)

$S \Rightarrow \neg E$ — (ii)

$T \wedge S$ — (iii)

$\therefore C$

from (iii), T — (iv) [Simplification]

S — (v)

from (i) & (iv), $C \vee E$ — (vi) [Modus Ponens]

from (ii) & (v), $\neg E$ — (vii) ["]

from (vi) & (vii), C. [Disjunctive Syllogism]

Hence, arguments are valid.

* Fallacies :

1. Fallacy of affirming the consequence.

$$\begin{array}{l} P \Rightarrow Q \\ Q \\ \hline \therefore \times \end{array}$$

We can't conclude anything from the ~~pro~~ premises.

2. Fallacy of denying the antecedent.

$$\begin{array}{l} P \Rightarrow Q \\ \neg P \\ \hline \therefore \times \end{array}$$

* Rules of Inference related to Quantifiers:

1. If $P(c)$ for an arbitary element c of U.
then $\forall x\, P(x)$. [Universal generalisation]

2. If $P(c)$ for some ~~arbitary~~ element c of U
then $\exists x\, P(x)$ [Existential generalisation]

3.
$$\begin{array}{l} \exists x\, P(x) \\ \hline \therefore P(c) \end{array}$$
where c is some element of U.
[Existential instantiation]

4.
$$\begin{array}{l} \forall x\, P(x) \\ \hline \therefore P(c) \end{array}$$
where c is an arbitary element of U.
[Universal instantiation].

Eg. All men are mortal. Socrates is a man.
So, Socrates is mortal.

→ man (x) : x is a man.
mortal (x) : x is mortal.

$\forall x\, [\text{man}(x) \Rightarrow \text{mortal}(x)]$ —(i)

man (Socrates) —(ii)

$\therefore$ mortal (Socrates).

From (i) using Uni. Ins.,
man (Socrates) ⇒ mortal (Socrates) —(iii)

From (iii) & (ii) by Modus Ponens,
mortal (Socrates).

Hence, arguments are valid.

Translation

- Nothing $\forall x \neg \ldots$
- Every cat $\forall x\ (Cat(x) \rightarrow \ldots$
- Some cats $\exists x\ (Cat(x) \wedge \ldots$
- All white cats $\forall x\ ((Cat(x) \wedge White(x)) \rightarrow \ldots$

- At least $\exists x\ (\ldots$
  - ↳ At least one animal is human.

$$\exists x\ (Animal(x) \wedge Human(x))$$

- At least two
  - ↳ There are at least two cats.

$$\exists x\, \exists y\ (Cat(x) \wedge Cat(y) \wedge x \neq y)$$

  - ↳ There are at least two things.

$$\exists x\, \exists y\ (x \neq y)$$

- At most one
  - ↳ Illus. Universe containing cats, dogs, pandas. There is at most one cat in the universe.

First we pick a cat out of the universe. Then put it back in the universe (say a cage where all cats, dogs, pandas are there). If again we pick a cat from the universe, then we can be sure that this cat is same as the one picked last time (as there's at most one cat).

So, if we pick a cat x, and (after putting it back) again pick a cat y, then x must be same as y.

↳ Using quantifiers.

There's at most one cat.

$$\forall x \forall y \; ((\text{cat}(x) \land \text{Cat}(y)) \rightarrow x = y).$$

↳ At most one thing is present.

$$\forall x \forall y \quad x = y$$

- Exactly one.

  ↳ Illus. Conjunction of at least one & at most one.

  There's exactly one cat.

  Conjoining the FOL translations for at least one & at most one:

$$\exists x \; \text{Cat}(x) \land \forall x \forall y \; ((\text{Cat}(x) \land \text{Cat}(y)) \rightarrow x = y).$$

  But, there are more compact ways.

  ↳ There is an x such that x is a cat & no matter which y we pick, if y is a cat, then y & x are one & same object.

  In FOL symbols,

$$\exists x \; (\text{Cat}(x) \land \forall y \; (\text{Cat}(y) \rightarrow y = x))$$

⋟ More compact way : we can delete the clause $Cat(x)$, but get the effect of including it by changing the $\to$ to a $\leftrightarrow$.

$$\exists x \, \forall y \; (Cat(y) \leftrightarrow x = y).$$

⤳ There is an x such that no matter which y we pick, y is a cat iff y and x are one & same.

↳ There's exactly one thing.

$$\exists x \, \forall y \;\; y = x.$$

- At least three (Logic same as at least two)

↳ There are at least three cats.

✓ $$\exists x \, \exists y \, \exists z \; (Cat(x) \wedge Cat(y) \wedge Cat(z) \wedge x \neq y \wedge y \neq z \wedge z \neq x).$$

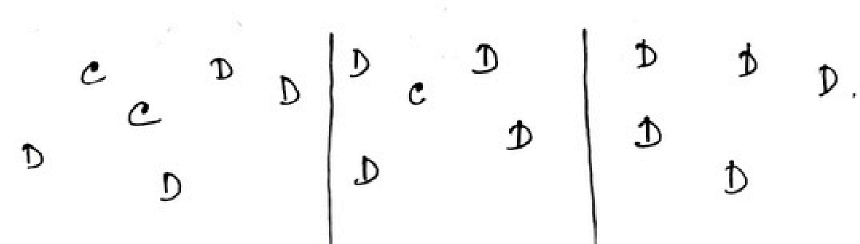
↳ There are at least three things.

$$\exists x \, \exists y \, \exists z \; (x \neq y \wedge y \neq z \wedge z \neq x).$$

- At most two.

↳ Illus. Universe containing cats, dogs. If there are at most 2 cats, if we pick

* a cat 3 times in a row (with replacement), then we must have picked a same cat more than once.



↳ In FOL,

✓ $\forall x \, \forall y \, \forall z \; ((Cat(x) \land Cat(y) \land Cat(z)) \Rightarrow$

$(x = y \lor y = z \lor z = x)$

↳ At most 2 things.

$\forall x \, \forall y \, \forall z \; (x = y \lor y = z \lor z = x).$

- Exactly two.

↳ Not so compact way : Conjoining

$\exists x \, \exists y \; (Cat(x) \land Cat(y) \land x \neq y) \land$

$\forall x \, \forall y \, \forall z \; ((Cat(x) \land Cat(y) \land Cat(z)) \rightarrow (x = y \lor y = z \lor z = x))$

↳ Compact : Logic same as exactly one - there exist 2 cats and for all othere things, if they are cat then it has to be one of those two.

✓ $\exists x \, \exists y \; (Cat(x) \land Cat(y) \land x \neq y \land$

$\forall z \; (Cat(z) \Rightarrow (z = x \lor z = y)))$

or

✓ $\exists x \, \exists y \; (\cancel{Cat(x)} \; x \neq y \land \forall z \; (Cat(z) \leftrightarrow$

$(z = x \lor z = y))).$

↳ There are exactly two things.

$\exists x \, \exists y \; (x \neq y \land \forall z \; (z = x \lor z = y)).$

* Examples.

1. There are at least two cubes.

$\rightarrow$ $\exists x \exists y \, (\text{Cube}(x) \wedge \text{Cube}(y) \wedge x \neq y).$

2. There are at most two cubes.

$\rightarrow$ $\forall x \forall y \forall z \, ((\text{Cube}(x) \wedge \text{Cube}(y) \wedge \text{Cube}(z)) \rightarrow$
$(x = y \vee y = z \vee z = x))$

3. There is exactly one large cube.

$\rightarrow$ $\exists x \forall y \, (x = y \leftrightarrow (\text{Cube}(y) \wedge \text{Large}(y)))$

or

$\exists x \, \cancel{\forall y} \, ((\text{Cube}(x) \wedge \text{Large}(x)) \wedge$
$\forall y \, ((\text{Cube}(y) \wedge \text{Large}(y)) \rightarrow x = y).)$

4. There are exactly two small cubes.

$\rightarrow$ $\exists x \exists y \, (x \neq y \wedge \forall z \, ((\text{Cube}(z) \wedge \text{Small}(z)) \leftrightarrow$
$(z = x \vee z = y)))$

5. At least three things are small.

$\rightarrow$ $\exists x \exists y \exists z \, (\text{small}(x) \wedge \text{small}(y) \wedge \text{Small}(z) \wedge$
$x \neq y \wedge x \neq z \wedge y \neq z).$

* Abbreviations for numerical claim.

$\exists^{\geq n} P(x)$ at least n objects satisfying $P(x)$

$\exists^{\leq n} P(x)$ at most n " " "

$\exists^{!n} P(x)$ exactly " " " "

↳ For $n = 1$, $\exists ! x\, P(x)$ as $\exists^{!1} P(x)$.

There is a unique x s.t. $P(x)$.

- Superlative.

↳ Illus. b is the largest cube.

We may come up with

$$\text{Cube}(b) \wedge \text{Largest}(b).$$

But, this tells b is the largest thing in the universe & b is a cube. Whereas, b might be the largest cube without being the largest thing.

✓ So, we have to infer that b is a cube and every cube that is not b is smaller than b. [Use of comparative predicate 'larger']

In FOL,

✓ $$\text{Cube}(b) \wedge \forall x\, ((\text{Cube}(x) \wedge x \neq b) \rightarrow \underbrace{\text{Larger}(b, x)}_{\text{b is larger than } x})$$

- Exceptive.
  - ↳ Everything is a cube except b.

$$\forall x \; Cube(x) \land \neg Cube(b)$$ × Wrong!

Contradiction!

So, we want to show that b is not a cube, but everything else is a cube.

✓ $\neg Cube(b) \land \forall x \; (x \neq b \Rightarrow Cube(x))$.

↳ More compact: No cube is b.

$$\forall x \; (Cube(x) \to x \neq b).$$

Replacing $\neg Cube(b)$ with

$$\forall x \; (Cube(x) \to x \neq b) \land \forall x \; (x \neq b \to Cube(x))$$

or $\forall x \; ((Cube(x) \to x \neq b) \land (x \neq b \to Cube(x)))$

or

✓ $\forall x \; (Cube(x) \leftrightarrow x \neq b)$.

↳ Exceptive sentences go into FOL as negative biconditional.

✓ Phrase 'except b' is rendered as $\underline{\leftrightarrow x \neq b}$.

- at least one

$\exists x\,(C(x))$

- at most one

$\forall x \forall y\,((C(x) \land C(y)) \to x = y)$

- exactly one

$\exists x\,(C(x) \land$

$\forall y\,(C(y) \to y = x)$.

or

$\exists x \forall y\,(C(y) \leftrightarrow y = x)$.

- at least 2

$\exists x \exists y\,(C(x) \land C(y) \land x \neq y)$

- at most 2

$\forall x \forall y \forall z\,((C(x) \land C(y) \land C(z)) \to$

$(x = y \lor y = z \lor z = x)$

- exactly 2

$\exists x \exists y\,(C(x) \land C(y) \land$

$x \neq y \land \forall z\,(C(z) \to$

$(z = x \lor z = y)))$.

- at least 3

$\exists x \exists y \exists z\,(C(x) \land C(y) \land C(z) \land$

$x \neq y \land y \neq z \land z \neq x)$

- at most 3

$\forall x \forall y \forall z \forall u\,((C(x) \land C(y) \land C(z) \land C(u)) \to$

$(x = y \lor x = z \lor x = u \lor y = z \lor y = u$

$\lor z = u))$

exactly 3

$\exists x \exists y \exists z \forall w\,((C(w) \leftrightarrow$

$(w = x \lor w = y \lor w = z)) \land$

$(x \neq y, y \neq z, z \neq x))$

~~3~~

- b largest cube.

$Cube(b) \wedge \forall x\ ((Cube(x) \wedge b \neq x) \rightarrow larger(b, x))$

- every thing cube except b.

$\neg cube(b) \wedge \forall x\ (x \neq b \rightarrow cube(x))$.

same as

No cube is b.

$\Downarrow$

$\forall x\ (Cube(x) \leftrightarrow x \neq b)$.

Eg. Some trigonometric functions are continuous. Some continuous functions are periodic. So, some trigonometric functions are periodic.

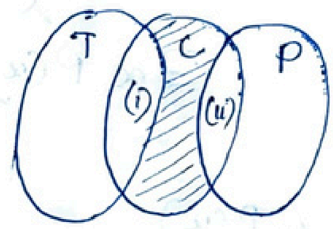
→ $T(x)$: x is trig. funⁿ.

$C(x)$: x is continuous funⁿ.

$P(x)$: x is periodic funⁿ.

$\exists x \, (T(x) \wedge C(x))$ — (i)

$\exists x \, (C(x) \wedge P(x))$ — (ii)



$\therefore \exists x \, [T(x) \wedge P(x)]$ — conclusion

For the shaded region conclusion is not true.

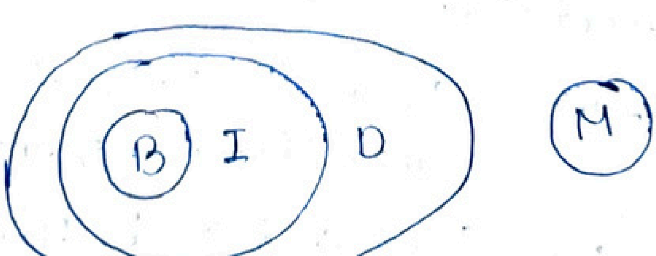
Eg. Babies are illogical. Illogical people are despised. Nobody who can manage a crocodile is despised. So, babies cannot manage crocodiles.

→ $B(x)$: x is a baby.

$I(x)$: x is illogical.

$D(x)$: x is despised.

$M(x)$: x can manage a crocodile.



B & M are disjoint.

$\forall x\ [\ B(x) \Rightarrow I(x)\ ]$ — (i)

$\forall x\ [\ I(x) \Rightarrow D(x)\ ]$ — (ii)

$\neg \exists x\ [\ M(x) \wedge D(x)\ ]$ — (iii) or $\forall x\ (\ M(x) \rightarrow \neg D(x))$

$\therefore \forall x\ (\ B(x) \Rightarrow \neg M(x))$

From (iii) $\Leftrightarrow$ $\forall x \neg [\ M(x) \wedge D(x)\ ]$

$\forall x\ [\ \neg D(x) \vee \neg M(x)\ ]$ — (iv).

~~From~~ $\forall x\ (\ D(x) \Rightarrow \neg M(x)\ )$

By using Universal instantiation,
for an arbitrary c in universe,

$B(c) \Rightarrow I(c)$ — (v)

$I(c) \Rightarrow D(c)$ — (vi)

$D(c) \Rightarrow \neg M(c)$ — (vii)

From (v) & (vi) by using Hypothetical Syllogism, $B(c) \Rightarrow D(c)$. — (viii)

From (vii) & (viii), by using Hypothetical Syllogism, $B(c) \Rightarrow \neg M(c)$ — (ix)

By using universal generalisation, from (ix) we get,

$\forall x\ (\ B(x) \Rightarrow \neg M(x)\ )$.

- Duality — $\wedge$ replaced with $\vee$ & vice versa.
  T " " F & " "

eg. u : $(p \wedge q) \vee (r \wedge T)$

dual $\rightarrow u^d$: $(p \vee q) \wedge (r \vee F)$

1. $(u^d)^d \Leftrightarrow u$
2. $(u \Leftrightarrow v) \rightarrow (u^d \Leftrightarrow v^d)$

- ↑ Principle of duality

Eg, Premises.

1. $\forall x \, (L(x) \Rightarrow (P(x) \vee A(x)))$
2. $L(d) \wedge \neg P(d)$
3. $\forall x \, (A(x) \Rightarrow H(x))$.

Conclusion $H(d)$.

From 1. using universal instantiation,

4. $L(d) \Rightarrow (P(d) \vee A(d))$

From 2. using simplification,

5. $L(d)$
6. $\neg P(d)$

From 3. using universal instantiation,

7. $A(d) \Rightarrow H(d)$

From 4 & 5 by Modus Ponens,

8. $P(d) \vee A(d)$.

from, 6 & 8, by disjunctive syllogism,

9. $A(d)$.

From 7. & 9. by Modus Ponens,

$H(d)$.

Arguments are correct.

* CNF, DNF

- A variable or negation of a variable is called a literal. Eg. P, $\neg P$ etc.
- Disjunction of literals is called a sum. $A \vee B \vee C \vee \neg D$ Maxterm
- Conjunction of literals is called a product. $A \wedge B \wedge C \wedge \neg D$. Minterm.

- CNF : A product of sums is called CNF.

Eg. $(P_1 \vee P_2) \wedge (P_3 \vee \neg P_1)$.

Eg. $P \wedge (P \Rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$.

- DNF : A sum of products is called DNF.

Eg. $(P_1 \wedge P_2) \vee (P_3 \wedge \neg P_1)$.

Eg. $P \wedge (P \Rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$.

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$.

NB. PROLOG.

- Functionally Complete Set of Boolean operators.

F : $\{\wedge, \vee, \neg\}$

Minimally FC Set : $\{\wedge, \neg\}$, $\{\vee, \neg\}$, $\{\rightarrow, \neg\}$, $\{\uparrow\}$, $\{\downarrow\}$

$\{\leftrightarrow, \neg\}$ not FC.

Q $\boxed{\{\uparrow\} \text{ is FC.}}$

$\neg$: $p \uparrow p = \neg(p \wedge p) = \neg p$

$\vee$: $(p \uparrow p) \uparrow (q \uparrow q)$

$= \neg(p \wedge p) \uparrow \neg(q \wedge q)$

$= \neg p \uparrow \neg q$

$= \neg(\neg p \wedge \neg q)$

$= p \vee q$

$\wedge$: $(p \uparrow q) \uparrow (p \uparrow q)$

$= \neg(p \wedge q) \uparrow \neg(p \wedge q)$

$= \neg(\neg(p \wedge q) \wedge \neg(p \wedge q))$

$= (p \wedge q) \vee (p \wedge q)$

$= p \wedge q$.

We are able to simulate $\{\wedge, \vee, \neg\}$ which is FC.

Q $\boxed{\{\downarrow\} \text{ is FC.}}$

$\neg$: $p \downarrow p = \neg(p \vee p) = \neg p$

$\vee$: $(p \downarrow q) \downarrow (p \downarrow q) = p \vee q$

$\wedge$: $(p \downarrow p) \downarrow (q \downarrow q) = p \wedge q$

Q| There are exactly 2 apples :

* P_1 : $\exists x \exists y\ (x \neq y \wedge \forall z\ (Apple(z) \leftrightarrow ((z = x) \vee (z = y)))$

P_2 : $\exists x \exists y\ (Apple(x) \wedge Apple(y) \wedge x \neq y) \wedge \forall x \forall y \forall z$
$((Apple(x) \wedge Apple(y) \wedge Apple(z)) \rightarrow (x = y \vee x = z \vee y = z))$

P_3 : $\exists x \exists y\ (Apple(x) \wedge Apple(y) \wedge (x \neq y) \wedge \forall z\ (Apple(z) \rightarrow (z = x) \vee (z = y)))$.

P_1, P_2, P_3 all represent same.

Essential | Job to do | Phone No.

* Resolution Principle.

- A clause is a disjunction of literals.

- Resolvent –

For any 2 clauses C_1 & C_2, if there is a literal L_1 in C_1 that is complementary to a literal L_2 in C_2, then delete L_1 & L_2 from C_1 & C_2 respectively, & construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 & C_2.

Eg. $\cancel{P} \vee Q \vee \neg R$

$\cancel{\neg P} \vee \neg S \vee T$

Resolvent $Q \vee \neg R \vee \neg S \vee T$.

Eg. Modus Ponens

$\cancel{P}$; $\cancel{\neg P} \vee Q$ ☆ Disjunctive Syllogism

Resolvent is Q.

Eg. Modus Tollens

$P \Rightarrow \cancel{Q}$ $\neg \cancel{Q}$

Resolvent is P.

- **Theorem:** Given the two clauses C_1 & C_2, a resolvent C of C_1 & C_2 is a logical consequence of C_1 & C_2.

- **The Principle:** Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, C_2, \dots, C_k$ of clauses such that each C either is a clause in S or a resolvent of clauses preceding C; & $C_k = C$. A deduction of the empty class $\square$ from S is called a refutation or a proof of S.

Eg. If today is tuesday (T), then I have a test in CS (C) or a test in Eco (E). If my Eco prof is sick (S), then I will not have a test in Eco. Today is Tuesday. & my Eco prof is sick. Therefore, I have a test in CS.

$\longrightarrow$

$$T \Rightarrow (C \vee E)$$
$$S \Rightarrow \neg E$$
$$T \wedge S.$$
$$\overline{\therefore C}$$

1. $\neg T \vee C \vee E$
2. $\neg S \vee \neg E$
3. T
4. S.
5. $\neg C$.

write in class form ($\vee$).

From (1) & (3), $C \vee E$. 6
From (2) & (4), $\neg E$. 7
From (6) & (7), C 8.
From (5) & (8), $\square$ Empty class.

Argument is correct.

* Methods of Proof

(i) Vacous Proof.

Eg. $P \Rightarrow Q$

prove P is false.

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

(ii) Trivial Proof.

Eg. $P \Rightarrow Q$

prove Q is true.

(iii) Direct Proof.

Assuming P & then proving Q.

(iv) Indirect Proof

Proving the contrapositive.

$$\neg Q \Rightarrow \neg P$$

- $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \Rightarrow Q$.

$\neg Q \Rightarrow \neg(P_1 \wedge P_2 \wedge \dots \wedge P_n)$

$\neg(\neg Q) \vee \neg(P_1 \wedge P_2 \wedge \dots \wedge P_n)$.

$Q \vee (\neg P_1 \vee \neg P_2 \vee \dots \vee \neg P_n)$

$Q \vee Q \vee Q \vee \dots \vee (\neg P_1) \vee (\neg(P_2)) \vee \dots \vee (\neg P_n)$

$(Q \vee \neg P_1) \vee (Q \vee \neg P_2) \vee \dots \vee (Q \vee \neg P_n)$.

$(\neg Q \Rightarrow \neg P_1) \vee (\neg Q \Rightarrow \neg P_2) \vee \dots$

$\neg Q \Rightarrow \neg P_i$ for some i $\quad 1 \leq i \leq n$

If we can prove $\neg Q \Rightarrow \neg P_i$ for some i,
we can prove the statement..

- $(P_1 \vee P_2 \vee \dots \vee P_n) \Rightarrow Q$.

$\neg(P_1 \vee P_2 \vee \dots \vee P_n) \vee Q$

$(\neg P_1 \wedge \neg P_2 \wedge \dots \wedge \neg P_n) \vee Q$

$(\neg P_1 \vee Q) \wedge (\neg P_2 \vee Q) \wedge \dots \wedge (\neg P_n \vee Q)$.

$(P_1 \Rightarrow Q) \wedge (P_2 \Rightarrow Q) \wedge \dots \wedge (P_n \Rightarrow Q)$.

So, we have to prove $P_i \Rightarrow Q$ for all i.
(Proof by cases).

---

v) Proof by Contradiction.

Prove P. $\longrightarrow$ Assume $\neg P$.
$\vdots$
$Q \wedge \neg Q$. [Wrong].

Eg. $\sqrt{2}$ is an irrational number. P

$\sqrt{2}$ is a rational number. $\neg P$.

✓ $\neg P \Rightarrow Q$ $\quad Q : \sqrt{2} = p/q \begin{cases} p, q \in I \\ \text{relatively prime} \end{cases}$

$$2 = \frac{p^2}{q^2}$$

$$\Rightarrow p^2 = 2q^2$$

So, p^2 is even.

Hence, p is even. $p = 2r$

$$\Rightarrow (2r)^2 = 2q^2$$

$$\Rightarrow q^2 = 2r^2$$

q is even. $q = 2s$.

$p = 2r$, $q = 2s$. $r, s \in I$.

So, p & q have a common factor 2. So, they are not relatively prime. So, $Q \Rightarrow \neg Q$.

$\neg P \Rightarrow \neg Q$ [Hypothetical syllogism]

$\neg P \Rightarrow Q$.

So, we get $\neg Q \wedge Q$. & this is a contradiction.

So, P is true.

Eg. The Halting Problem. [Undecidability]

Can you write a program (input free) that will consist of a procedure named 'Halt' that will return 'yes' if P will halt & 'no' if will not?

→ Assume it is possible to write a procedure halt.

```
Program.                  procedure
  contradict
   begin
  If halt(contradict) = 'yes'
          then begin  while true
                       print x.
               end
          else stop.
end.
```

Will contradict halt?

- halts
  - Keeps on printing x
  - does not halt
  - Paradox
- doesn't halt
  - halts
  - Contradiction

∴ Halt does not exist.

* Normal Forms:

- A product of the variables & their negations in a formula is called an elementary product. $P \wedge Q \wedge \neg R$

- A sum of the variables & their negations is called an elementary sum.
$P \vee Q \vee \neg R$

- A necessary & sufficient condition for an elementary product to be identically false is that it contains at least one pair of factors in which one is the negation of the other. $P \wedge Q \wedge \neg Q \wedge \neg R$ ×
Contradiction

- A necessary & sufficient condition for an elementary sum to be identically true is that it contains at least one pair of factors in which one is the negation of the other. $P \vee Q \vee \neg P$ ✓
Tautology.

- Disjunctive Normal forms:

A formula which is equivalent to a given formula & which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

$(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Eg. $(\neg P \vee \neg Q) \Rightarrow (P \Leftrightarrow \neg Q)$

or $\neg(\neg P \vee \neg Q) \vee (\underline{P \Leftrightarrow \neg Q})$

or $\neg(\neg P \vee \neg Q) \vee \underline{(P \wedge \neg Q) \vee (\neg P \wedge \neg(\neg Q))}$

or $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$.

Q Eg. $(P \Rightarrow (Q \wedge R)) \wedge (\neg P \Rightarrow (\neg Q \wedge \neg R))$

or $(\neg P \vee (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R))$

or $((\neg P \vee (Q \wedge R)) \wedge P) \vee [(\neg P \vee (Q \wedge R)) \wedge (\neg Q \wedge \neg R)]$

or $[(\neg P \wedge P) \vee (Q \wedge R \wedge P)] \vee [(\neg P \wedge (\neg Q \wedge \neg R)) \vee ((Q \wedge R) \wedge (\neg Q \wedge \neg R))]$.

or $(\neg P \wedge P) \vee (Q \wedge R \wedge P) \vee (\neg P \wedge \neg Q \wedge \neg R) \vee (Q \wedge R \wedge \neg Q \wedge \neg R)$.

- Conjunctive Normal Forms:

A formula which is equivalent to a given formula & which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

$(P \vee Q) \wedge (P \vee \neg Q)$..

- Every PCNF/PDNF corresponds to some boolean exp^n.
- If for some expression, PCNF has m, PDNF has n terms then # variables in the exp^n = $\log_2(m+n)$.

Eg. $(\neg P \vee \neg Q) \Rightarrow (P \Leftrightarrow \neg Q)$

or $\neg(\neg P \vee \neg Q) \vee [(P \Rightarrow \neg Q) \wedge (\neg Q \Rightarrow P)]$

or $(P \wedge Q) \vee [(\neg P \vee \neg Q) \wedge (Q \vee P)]$

D or $(P \vee [(\neg P \vee \neg Q) \wedge (P \vee Q)]) \wedge$
$(Q \vee [(\neg P \vee \neg Q) \wedge (P \vee Q)])$

or $(P \vee \neg P \vee \neg Q) \wedge (P \vee P \vee Q) \wedge (Q \vee \neg P \vee \neg Q)$
$\wedge (Q \vee P \vee Q)$. CNF.

- Principal Disjunctive Normal forms

Let P & Q be two propositional variables.

Consider $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$
These formulas are called minterms.

→ For a given formula, an equivalent formula consisting of disjunctions of minterms only is known as its principal disjunction normal form. Such a normal form is also called the sum-of-products canonical form.

Eg. $P = P \wedge (Q \vee \neg Q) \wedge (R \vee \neg R)$ When 3 var's

$= (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$

$(P \wedge Q \wedge R)$: 1 1 1 → 7
$(P \wedge Q \wedge \neg R)$: 1 1 0 → 6
$(P \wedge \neg Q \wedge R)$: 1 0 1 → 5
$(P \wedge \neg Q \wedge \neg R)$: 1 0 0 → 4

Σ 4, 5, 6, 7

- **Principal Conjunctive Normal Forms.**

→ For a given number of variables, the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once. $P \vee Q$, $\neg P \vee Q$, $P \vee \neg Q$, $\neg P \vee \neg Q$

→ For a given formula, an equivalent formula consisting of conjunctions of the maxterms only is known as its principal conjunctive normal form. This is also called product of sums canonical form.

Eg. $(Q \Rightarrow P) \wedge (\neg P \wedge Q)$.

or $(\neg Q \vee P) \wedge (\neg P \wedge Q)$

or $(\neg Q \vee P) \wedge (\neg P) \wedge Q$

$P \vee F = P$
$P \wedge T = P$
$P \wedge \neg P = F$
$P \vee \neg P = T$

$\neg P = \neg P \vee (Q \wedge \neg Q)$

$= (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$

$Q = Q \vee (P \wedge \neg P)$

$= (P \vee Q) \wedge (\neg P \vee Q)$.

So, the main expression,

$(P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee Q)$

$\wedge \cancel{(\neg P \vee Q)}$

— PCNF

$\rightarrow (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R)$

1 1 0   0 1 1   0 0 1

6   3   1

Π 1, 3, 6

* PRENEX NORMAL FORMS IN THE FIRST ORDER LOGIC :

- **Definition:** A formula F in the first-order logic is said to be in prenex normal form iff the formula F is in the form of $(Q_1 x_1) \dots (Q_n x_n)(M)$ where every $(Q_i x_i) \dots i = 1, \dots, n$ is either $(\forall x_i)$ or $(\exists x_i)$ & M is a formula containing no quantifiers.

  $(Q_1 x_1) \dots (Q_n x_n)$ is called the prefix & M is called the matrix of the formula F.

- **Transforming Formulas into prenex normal forms :**

  Step 1: Use the laws.

  $$F \Leftrightarrow G = (F \Rightarrow G) \wedge (G \Rightarrow F)$$

  $$F \Rightarrow G = \neg F \vee G$$

  to eliminate the logical connectives $\Leftrightarrow$ & $\Rightarrow$.

Step 2: Repeatedly use the laws

$$\neg(\neg F) = F$$

De Morgan's Law

$$\neg(F \vee G) = \neg F \wedge \neg G.$$

$$\neg(F \wedge G) = \neg F \vee \neg G.$$

& the laws

$$\neg((\forall x)\, F[x]) = (\exists x)(\neg F[x])$$

$$\neg((\exists x)\, F[x]) = (\forall x)(\neg F[x])$$

to bring the negation signs immediately before atoms.

Step 3: Rename bound variables if necessary.

Step 4: Use these laws.

$$(Qx)\, F[x] \vee G = (Qx)(F[x] \vee G)$$

$$(Qx)\, F[x] \wedge G = (Qx)(F[x] \wedge G).$$

$$(\forall x)\, F[x] \wedge (\forall x)\, H[x] = (\forall x)(F[x] \wedge H[x]).$$

$$(\exists x)\, F[x] \vee (\exists x)\, H[x] = (\exists x)(F[x] \vee H[x]).$$

$$(Q_1 x)\, F[x] \vee (Q_2 x)\, H[x] = (Q_1 x)(Q_2 z)(F[x] \vee H[z]).$$

$$(Q_3 x) F[x] \wedge (Q_4 x) H[x] =$$

$$(Q_3 x)(Q_4 z)(F[x] \wedge H[z]).$$

to move the quantifiers to the left of the entire formula to obtain prenex normal form.

Eg. $\forall x\, P(x) \Rightarrow \exists x\, Q(x)$

or $\neg \forall x\, P(x) \vee \exists x\, Q(x)$

or $\exists x\, \neg P(x) \vee \exists x\, Q(x)$.

or $\underbrace{\exists x}_{\text{prefix}} \underbrace{(\neg P(x) \vee Q(x))}_{\text{matrix}}$. PNF

Eg. $\forall x \forall y\, [\exists z\, (P(x,z) \wedge P(y,z)) \Rightarrow \exists u\, Q(x,y,u)]$

or $\forall x \forall y\, [\neg \exists z\, (P(x,z) \wedge P(y,z)) \vee \exists u\, Q(x,y,u)]$

or $\forall x\, \forall y\, [\forall z\, \{\neg (P(x,z) \wedge P(y,z))\} \vee \exists u\, Q(x,y,u)]$

or $\forall x\, \forall y\, [\forall z\, (\neg P(x,z) \vee \neg P(y,z)) \vee \exists u\, Q(x,y,u)]$

or $\underbrace{\forall x\, \forall y\, \forall z\, \exists u}_{\text{prefix}}\, \underbrace{[\neg P(x,z) \vee \neg P(y,z) \vee Q(x,y,u)]}_{\text{matrix}}$

- $p \to q$

$q \to p$ converse

$\neg p \to \neg q$ inverse

$\neg q \to \neg p$ contrative

$(p \to q) \Leftrightarrow (\neg q \to \neg p)$

$(q \to p) \Leftrightarrow (\neg p \to \neg q)$.