NN-body: Approximating Solutions to the n-body Problem using Neural Networks

William H. Guss
Unviersity of California, Berkeley
Berkeley, CA 94720
wguss@ml.berkeley.edu

Abstract

The n-body problem, although simple in its formulation, is one of the most classic and unsolved problems in Newtonian mechanics. In this paper, we work towards a deeper understanding of the underlying dynamics of solutions by utilizing neural networks to approximate target states from initial conditions. Unlike previous numerical methods, neural networks are universal approximators which can capture and learn dynamics across variable timescales, without solving for intermediate solutions. We take advantage of one-shot prediction in order to analyze eigen-states **TODO: Finish abstract**

1 Introduction

2 Background

The n-body problem.   Formally, the n-body problem considers the graviational dynamics of n different point masses in $\mathbb{R}^3$. Let m_i, p_i, and v_i denote the mass, position, and velocity of the ith point mass respectively. For every mass pair $i \neq j$, the force induced on mass m_i by mass m_j is given by Newton's law of gravitation

$$F_{ij} = Gm_im_j \frac{p_j - p_i}{\|p_j - p_i\|^3} \tag{2.1}$$

where G is the graviational constant. Intuitively, F_{ij} describes a force in the direction of p_j (relative to p_i) whose magnitude is dependent on both the distance and combined mass potential of the two point-masses.

Integrating all force data from (2.1) we yield that the acceleration applied to mass m_i is

$$\frac{d^2p_i}{dt^2} = G \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{m_j(p_j - p_i)}{\|p_j - p_i\|^3}, \tag{2.2}$$

where the acceleration is yielded by means of Newton's first law. In vector form,

$$\begin{aligned}
\frac{d^2p}{dt^2} &= G \left(\left\langle \left(p^{(i)} - p_i \right) \odot \frac{1}{\|p - p_i\|^3} \,\middle|\, m^{(i)} \right\rangle \right)_{i=1}^{n} \\
&= G\tilde{M} \cdot \left(\left(p^{(i)} - p_i \right) \odot \frac{1}{\|p - p_i\|^3} \right)_{i=1}^{n} =: G\tilde{M}\Pi(t)
\end{aligned} \tag{2.3}$$

where $m^{(i)}, p^{(i)}$ are the full mass and position vectors with ith element removed, and $\tilde{M}$ is a matrix whose ith row is $m^{(i)}$.

In its full statement, the n-body problem can be stated as follows: given arbitrary initial position and velocity vectors, p_0 and v_0, does there exist an analytical solution $\mathcal{P}$ to $\frac{d^2p}{dt^2} = G\tilde{M}\Pi(t)$?

Submitted as the final project for ASTRO 9.

3 Learning the n-body problem

As of yet, there is no analytical solution to the arbitrary n-body problem, which given its simple formulation, raises questions as to our understanding of the complex dynamical patterns which emerge therefrom. Although numerical methods for solving (2.3) are sufficiently powerful for visualizing solutions, they are limited in their capacity for analysis; that is, on face value, such methods attempt to yield the temporally global solution by repeatedly and accurately modeling the local one.

In this paper, we approach numerical approximation from the alternative perspective of machine learning. Instead of iteratively arriving at a global solution, we will *learn* the map $\pi : (p_0, v_0, t) \mapsto (p(t), v(t))$ using a family of *hypothesis* function approximators whose analytical forms are condusive to statistical and spectral analysis globally in time. Formally we will solve the following optimization problem, usually called a *classification problem*.

Let $\mathcal{D}$ be the distribution induced by the jointly uniform random variables P_0, V_0, T_0. Given some hypothesis class $\mathcal{H} = \{h_\theta : \mathbb{R}^{3n} \times \mathbb{R}^{3n} \times \mathbb{R} \times [t_0, \infty) \to \mathbb{R}^{3n} \times \mathbb{R}^{3n}\}$ parameterized smoothly by $\theta \in \Theta$, we wish to find θ^* such that

$$\theta^* = \arg\min_{\theta \in \Theta} \mathbb{E}_{(p_0, v_0, t) \sim \mathcal{D}} \mathcal{L}\Big(\pi(p_0, v_0, t), h_\theta(p_0, v_0, t)\Big), \tag{3.1}$$

where $\mathcal{L}$ is some monotonic function called a *loss*. Usually we take $\mathcal{L}$ to be the ℓ_2 norm. For intuition, note that if $h_{\theta^*} = \pi \in \mathcal{H}_\theta$ then the minimal expectation above is zero.

In the foregoing regime, if we find a hypothesis class with which is sufficiently expressive, then the solution h_{θ_*} will approximate solutions to the n-body problem with arbitrary accuracy. It remains, however, to find such a hypothesis class $\mathcal{H}$ which also has the desired interpritability conditions and can yield novel statistical insights into the n-body problem. Luckily, artifical neural networks come close.

3.1 Deep Learning

For the purpose of this work, define the hypothesis class of all ℓ-layer neural networks, $\mathcal{N}_\ell$ as