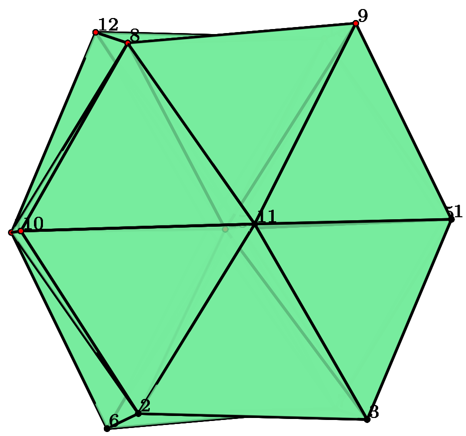
Math 215A — Homework 13— UCB, Spring 2017 — William Guss
Partners: Alekos, Chris
Selected Problems: 1,2,3

(13.1) *(Automated homology computations)*: Use `polymake` to compute the homology of the 2-skeleton of the 4-cube. Furthermore use `polymake` to comptue the homology of the suspension of the orientable genus 5 surface.

Solution. First we will compute the homology of the 2-skeleton of the 4-cube. Polymake convieniently provides functions for building such a construction.

```
polytope > application 'topaz';
topaz > $cc=cube_complex(1,1,1,1); # Makes a 4-cube
topaz > $2skel = k_skeleton($cc, 2);
```

The above code produces the following TIkz visualization.



Finally we can compute the homology

```
topaz > print $skel->HOMOLOGY;
({} 0)
({} 0)
({} 60)
```

Therefore the simplicial homology is a long exact sequence

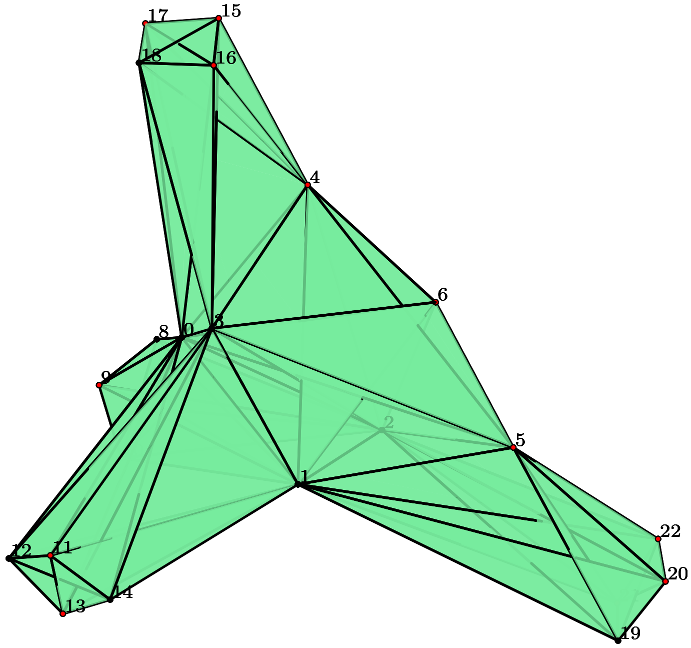
$$\cdots \to 0 \to \mathbb{Z}^{60} \to 0 \to 0$$

with $H_2(\text{2skel};\mathbb{Z}) = \mathbb{Z}^{60}$. Note that this is the simplicial 2-skeleton, which has a H_2 depending on how many simplices were used to approximate the 4-cube.

Next we compute the homology of the suspension of the orientable genus 5 surface.

```
topaz > $cc = surface(5);
topaz > $sus = suspension($cc, 1);
topaz > print $sus->HOMOLOGY;
({} 0)
({} 0)
({} 10)
({} 1)
```

Before suspension, we have the following surface



Using topaz, we yield that the homology is simply a long exact seuence

$$0 \to \mathbb{Z} \to \mathbb{Z}^{10} \to 0 \to 0$$

with $H_2(\text{sus}; \mathbb{Z}) = \mathbb{Z}^{10}$ and $H_1(\text{sus}; \mathbb{Z}) = \mathbb{Z}$. This completes the exercise.

(13.2) *(Simplicial Approximation):* Use the simplicial approximation theorem to prove that S^n is $(n-1)$-connected, ie. that the homotopy groups vanish below dimension n.

Proof. First consider the $(n-1)$-sphere, S^{n-1}. Any simplicial triangulation of S^{n-1} cannot contain an n-simplex because S^{n-1} is locally homeomorphic to $\mathbb{R}^{n-1}$. Let T^n denote some simplicial triangulation of the n-sphere. To show that S^n is $n-1$ connected we will show that the homotopy groups vanish below dimension n; that is any continuous map $S^m \to S^n$ is homotopic to the constant map when $m < n$.

For any continuous map $F : |T^m| = S^m \to S^n = |T^n|$, there exists an r and a simplicial approximation $f : Bd^r(T^m) \to T^n$ so that $|f| \simeq F$ where $Bd^r(\cdot)$ denotes the rth Barycentric subdivision. Since f is a simplicial map, $|F|(S^m)$ is not contained in the interior of $k > m$ simplices of T^n. To see this, note that the Barycentric subdivision of T^m (to any degree) does contain higher dimensional simplices by definition and additionally that T^m does not contain any k-simplices inductively when $k > m$. Therefore $|f|$ must miss at least one point in the interior of some n simplex in T^n and therefore $F \simeq |f| : S^m \to S^n \setminus \{pt\}$.

Since $S^n \setminus \{pt\}$ is homeomorphic to $\mathbb{R}^{n-1}$, say through h, then by $\mathbb{R}^{n-1}$ contractible, $h \circ f \simeq c$ where c is the constant map. Composing this homotopy with h^{-1} we yield that $h^{-1} \circ h \circ f \simeq h^{-1} \circ c = f : S^m \to \{pt\} \subset S^n$; that is $F \simeq f$ is nullhomotopic.

Therefore $\pi_m(S^n) = 0$ when $m < n$ and so S^n is $(n-1)$-connected. This completes the proof. ☐

(13.3) [**Already Graded**] *(Vector-fields and Euler characteristic):* Let $\mathcal{M}$ be a compact smooth manifold with $\chi(\mathcal{M}) \neq 0$. If F is a mapping from $\mathcal{M}$ onto the tangent bindle so that $\pi \circ F = id$ where $\pi : T\mathcal{M} \to M$, then there exists a $x \in \mathcal{M}$ so that $F(x) = 0$.

Proof. Suppose that for all x $F(x) \neq 0$. Then by Corollary 11.5 of Bredon, there exists a map $f : M \to M$ without fixed points and $f \simeq id$. By the Whitney embedding theorem $\mathcal{M}$ is a compact smooth manifold and so it is embeddable in $\mathbb{R}^k$ for some k. Therefore it is homeomorphic to a retract of the bounding open ball of the compact image. Thus $\mathcal{M}$ is a Euclidean neighboorhood retract and Corollary 23.5 gives that if $\mathcal{M}$ is a compact ENR, then $L(f) \neq 0$ implies f has a fixed point.

It remains to show that $L(f) \neq 0$. Recall that

$$L(f) = \sum_i (-1)^i \mathrm{tr}_i(f_*)$$

where $f_* : H_i(\mathcal{M}; \mathbb{Z}) \to H_i(\mathcal{M}; \mathbb{Z})$ is the homomorphism of homology induced by f. Since $f \simeq \mathrm{id}$ then the homotopy invariance of homology homomorphisms[1] gives that $0 = L(f) = L(\mathrm{id})$. But then

$$L(\mathrm{id}) = \sum_i \mathrm{tr}_i(\mathrm{id}_*) = \sum_i \mathrm{rank}(H_i(\mathcal{M}; \mathbb{Z})) = \chi(\mathcal{M})$$

which contradicts $\chi(\mathcal{M}) = 0$. Therefore there is an x so that $F(x) = 0$. This completes the proof. □

[1]See Concise Alg. Top. Chapter 12 for a nice proof!