MATH 185: Homework 2

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1.

Definition 1. *A set $S \subset \mathbb{C}$ is bounded if and only if there exists $z \in \mathbb{C}$ such that for every $s \in S$, $|s| \leq |z|$*

Definition 2. *Alternativeley, a set $S \subset \mathbb{C}$ is bounded if and only if there is an r such that $S \subset B_r(0)$, where $B_s(z)$ is the ball of radius s with center z.*

Theorem 1. *If $(z_n)_{n=1}^{\infty}$ is a convergent sequence of compex numbers, then the sequence is bounded.*

Proof. Take the value set $S = \{z_n\}$. Then suppose there were no r such that $S \subset B_r(0)$. If this is the case, the countability of S implies that for every n, $S \cap B_n(0)$ is finite. Since $z_n \to z$, take $N \in \mathbb{N}$ such that $N > |z|$. Such an n exists by the archimedian principle of $\mathbb{R}$. Then $S \cap B_N(0)$ must be finite.

Take $\epsilon = N - |z|$, then there is an M such that for all $m > M$, $d(z_n, z) < \epsilon$. That is there are infinite elements within ϵ of z, and thereby there are infinite elements in $S \cap B_N(0)$. This is a contradiction to its finiteness.

Therefore it must be that the value set is contained within the N ball, and therefore, (z_n) is bounded. □

2. Exercise II.1.11

Theorem 2. *The function $Arg : \mathbb{C} \to \mathbb{R}$ is continuous except for along the line $L = \{z : Im(z) = 0 \wedge Re(z) < 0\}$.*

Proof. A function is continuous if and only if it preserves limits. Specifically, if $\lim_{h \to x} f(h) = f(x)$ implies that f is continuous at h. Consider the restricted Arg function, say $A : \mathbb{C} \backslash L \to \mathbb{R}$. Then it is clear that $\lim_{\mathbb{C} \backslash L} A(h) = (-\pi, \pi)$, since if a point is within an ϵ neighborhood of another point, its gradial distance is proportionate to $\sin^{-1}$ of its ϵ distance, (a continuous function).

However consider any $z \in L$ Such that $h \to z$ approaches from the upper half plane and $g \to z$ from the lower. Clearly $Arg(h) \to \pi$ and $Arg(g) \to -\pi$, so no limit exists and the function is not continuous at z. This completes the proof. □

3. Exercise II.1.16

Theorem 3. *The punctured plane $\mathbb{C} \setminus L = \mathbb{C}_P$ is star shaped but not convex.*

Proof. Take any $z \in \mathbb{C}_P$. Then for any $r \geq 1$, z/r is clearly in $\mathbb{C}_P$ since r is always positive and the imaginary part of z is always non-zero or its real part is non-negative. In the first case z/r is never in L for all finite r, and when $r \to \infty$, then $r = 0 \in \mathbb{C}_P$. In the second case, its real part is always positive or 0 until it reaches 0 by the same logic. In the case that both are true, we consider again the same logic. If $z = 0$, we are done.

Clearly, $\mathbb{C}_P$ is not convex when considering the line, $B = \{x + iy : x = -1\}$ which contains $-1 \in L$. □

Definition 3. *A space X is contractible if the identity map is homotopic to some constant map.*

Definition 4. *A homotopy between two continuous functions f, g from a topological space X to a topological space Y is a continuous function $H : X \times [0, 1] \to Y$, such that if $x \in X$ then, $H(x, 0) = f(x), H(x, 1) = g(x)$. [Wikipedia]*

Theorem 4. *Every homeomorphism is a homotopy equivalence.*

Theorem 5. *A star-shaped space X is homotopic to a point.*

Proof. Let $H(x, t) = x(1 - t) + z_0 t$, then $H(x, 0) = id_X$, and $H(x, 1)$ is the constant identity. H is continuous by the definition of H as a star shaped space. Therefore, the star-shaped space is homotopic to a point. □

Theorem 6. *The space $\gamma = \mathbb{C} \setminus [-1, 1]$ is not star shaped.*

Proof. The set γ is not homeomorphic to the open unit ball B^2 since it is homeomorphic to the open annalus. Therefore, γ is not homotopic to B^2 which is homotopic to a point since B^2 is star shaped. The space γ could not be star shaped since if it were it would be homotopic to a point which it is not. Therefore, γ is not star shaped. □

Theorem 7. *The punctured disk is not star shaped.*

Proof. The punctured disk is not homeomorphic to B^2 for the same reason as the previous proof. Therefore it is not homeotopic, and by the logic of the above proof, it is not homeotopic to a point, and so it could not possibly be star shaped as that would lead to a contradiction. This completes the proof. □

4.

Theorem 8. *The functions $x = Rez$ and $y = Imz$ are not complex differentiable at any point.*

Proof. Suppose those functions were differentiable. Then it follows that there partials as functions of $\mathbb{R}^2$ should be

$$Dx(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, Dy(p) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{1}$$

This is a contradiction to the Cauchy-Rieman equations. □

5. We take the derivative as follows

$$f' = a2z + b\bar{z} + bz(\bar{z})' + 2c\bar{z}(\bar{z})'. \tag{2}$$

This only makes sense where the terms containing $\bar{z}'$ are not a part of the equation, since the complex conjugate is not complex differentiable. This occurs when $\bar{z}'(bz + 2c\bar{z}) = 0$. So it must be that $bx + 2cx = 0, by - 2cy = 0$ which implies that the differentiability of f does not depend on z but on b, c.

Setting up a linear system, we get

$$Aa = 0 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} a \\ b \end{bmatrix} \in Nul(A). \tag{3}$$

And f is analytic every where in this case!

6. Exercise II.2.5

Theorem 9. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic on D. Then let $g = \overline{f}$ is analytic on $D^* = \{\bar{z} : z \in \mathbb{D}\}$. It follows that, $g'(w) = \overline{f(\bar{w})}$.*

Proof. Take $w \in D^*$, so that there exists a $z \in D$ so that $\bar{z} = w, \bar{w} = z$. Consider the standard limit definition of the derivative.

$$\begin{aligned} \lim_{\Delta w \to w} \frac{g(w + \Delta w) - g(w)}{\Delta w} &= \lim_{\Delta w \to w} \frac{\overline{f(\overline{w + \Delta w})} - \overline{f(\bar{w})}}{\Delta \bar{w}} \\ &= \lim_{\Delta z \to z} \frac{\overline{f(z + \Delta z)} - \overline{f(z)}}{\Delta z} \\ &= \lim_{\Delta z \to z} \frac{\overline{f(z + \Delta z) - f(z)}}{\Delta z} \\ &= \overline{\lim_{\Delta z \to z} \frac{f(z + \Delta z) - f(z)}{\Delta z}} \\ &= \overline{f'(z)} \qquad\qquad = \overline{f'(\bar{w})}, \end{aligned} \tag{4}$$

which is the statement of the theorem. $\square$

7. Validation:

$$Df(p) = \begin{bmatrix} \cos x \sinh y & \sin x \cosh y \\ -\sin x \cosh y & \cos x \sinh y \end{bmatrix} \tag{5}$$

The complex function is $f = = sinx \sinh y, v = \cos x \cosh y$. It follows that, $f = ie^z + e^{-z}$; So I estimate this fucntion is a rotated cosine.

8. Exercise II.3.3

Theorem 10. *If f and $\bar{f}$ are analytic on D, then f is constant.*

Proof. If f and $\bar{f}$ are analytic for all $z \in D \subset \mathbb{C}$,

$$Df(z) = \begin{bmatrix} u_x(z) & u_y(z) \\ -u_y(z) & u_x(z) \end{bmatrix}, D\bar{f}(z) = \begin{bmatrix} u_x(z) & -v_y(z) \\ u_y(z) & -v_y(z) \end{bmatrix} \tag{6}$$

So by the cauchy riemann equations we have that, $\bar{f}$ and f analytic implies

$$-v_y = u_y, -v_y = u_x \tag{7}$$
$$v_x = u_y, v_y = u_x \tag{8}$$

So, $-v_y = v_y$ and $-v_x = v_x$ gives

$$Df(z) = \begin{bmatrix} 0 & 0 \\ 0 & 0. \end{bmatrix} \tag{9}$$

Which from multivariable calculus, we know is true if and only if f is a constant map! □

9. Exercise II.3.8

Theorem 11. *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Then if $z = re^{i\theta}$ f satisfies*

$$u_r = \frac{1}{r} v_\vartheta, u_\vartheta = -rv_r. \tag{10}$$

Proof. By the chain rule of partial derivatives,

$$u_r = u_x x_r + u_y y_r = u_x \cos\vartheta + u_y \sin\vartheta \tag{11}$$

and identically by the Cauchy-Riemann equations,

$$\begin{aligned} v_\vartheta = v_x x_\vartheta + v_y y_\vartheta &= -v_x r \sin\vartheta + v_y r \cos\vartheta \\ &= r(-v_x \sin\vartheta + v_y \cos\vartheta) \\ &= r(u_x \cos\vartheta + u_y \sin\vartheta) \\ &= ru_r. \end{aligned} \tag{12}$$

So it follows that $u_r = \frac{1}{r} v_\vartheta$. Likewise,

$$u_\vartheta = r(u_x x_\vartheta + u_y y_\vartheta) = r(-u_x \sin\vartheta + u_y \cos\vartheta)) \tag{13}$$

and identically by the Cauchy-Riemann equations,

$$v_r = v_x x_r + v_y y_r = v_x \cos\vartheta + v_y \sin\vartheta = -u_y \cos\vartheta + u_x \sin\vartheta \tag{14}$$

So it follows that $-rv_r = u_\vartheta$. This completes the proof. □