Math 202A— UCB, Fall 2016 — M. Christ
Problem Set 5, due Wednesday September 28 - William Guss

(5.1) Let $f_n \in L^+$, $f_n \to f$ pointwise, and $\int f = \lim_{n\to\infty} \int f_n < \infty$. Then $\lim_{n\to\infty} \int_A f_n = \int_A f$ for every $A \in \mathcal{M}$.

Proof. First we show that if $g \in L^+$ then $\chi_A g \in L^+$ if A is measurable and also that $\int_A g = \int \chi_A g$. First χ_A is measurable by A measurable and the product of pointwise functions is measurable. Indeed $\chi_A g \leq g$ so through the suprememum of simple functions there approximating $\int \chi_A g \leq \int g$, and $\chi_A g \in L^+$. Lastly

$$\int \chi_A g = \sup_{\phi \text{ simple } \phi\leq g} \int \phi \chi_A = \sup_{\phi \text{ simple } \phi\leq g} \int_A \phi = \int_A g$$

by proposition 2.13.

Now observe that $\chi_A f_n \to \chi_A f$ pointwise, since on A, $f_n \to f$ by our hypothesis and off A, $0 \to 0$. Now for every ϵ 2.13 also gives $|\chi_A f - \chi_A f_n| \leq |f - f_n|$ for every $n > N$ implies that $\left|\int_A f_n - \int_A f\right| \leq \left|\int f - \int f_n\right| < \epsilon$. This completes the proof. □

Notation. I wil use ϕ to denote a simple function, in the proceeding examples.

(5.2) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^+$. It follows that $\lambda(E) = \int_E f \, d\mu$ defines a measure on $\mathcal{M}$ and if $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$.

Proof. First recall that for any ϕ simple the map $E \mapsto \int_E \phi \, d\mu$ is a measure on $\mathcal{M}$. Then recall that since $f \in L^+$, $\int f \, d\mu = \sup_{\phi\leq f} \int \phi \, d\mu$. Take a sequence of increasing ϕ so that $\lim \int \phi_n \, d\mu = \int f \, d\mu$. Then by our previous exercise $\int_E \phi_n \, d\mu \to \int_E \; f d\mu$. It follows that the measures $\lambda_n(E) = \int_E \phi_n \, d\mu \to \lambda(E) = \int_E \phi \, d\mu$.

We claim that λ is a measure. First $\lambda(\emptyset) = 0$ since $\lambda_n(\emptyset) = 0$ for all n. So λ is nonnegative. Furthermore $\lambda_n(E) \leq \lambda_n(F)$ if $E \subset F$ for every n so $\lambda(E) \leq \lambda(F)$. Finally for (E_k) pairwise disjoint family of $\mathcal{M}$

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{k=1}^{m} \lambda_n(E_k) = \sup_n \sup_m \sum_{k=1}^{m} \lambda_n(E_k)$$

since λ_n is nondecreasing and nonnegative. Therefore

$$\sup_n \sup_m \sum_{k=1}^{m} \lambda_n(E_k) = \sup_m \sup_n \lambda_n\left(\bigcup_{k=1}^{m} E_k\right) = \lim_{m\to\infty} \lim_{n\to\infty} \lambda_n\left(\bigcup_{k=1}^{m} E_k\right),$$

again applying λ_n non decreasing and λ_n countably addative. This gives

$$\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \lambda(E_k)$$

and so λ is a measure. Now applying the definition of simple functions we get

$$\int f \, d\lambda = \sup_{\phi\leq g} \sum_y y\lambda(J_y) = \sup_{\phi\leq g} \sum_y y \int_{J_y} g \, d\mu = \sup_{\phi\leq g} \int \sum_y \chi_{J_y} g \, d\mu = \sup_{\phi\leq g} \int \phi g \, d\mu$$

and thus $\int f \, d\lambda = \int fg \, d\mu$. □

(5.3) Let f_n, g_n, $f, g \in L^1$. Suppose that $f_n \to f$ a.e. and $g_n \to g$ a.e. Suppose that $|f_n| \leq g_n$, and that $\int g_n \to \int g$. Show that $\int f_n \to \int f$.

Proof. The function f is measurable and in L^1. Without loss of generality we assume that f_n, g_n, f, g are real valued since the proof is the same for real and imaginary parts. Now define the following sequence $\mathfrak{g}_n = \sup_k g_k$ Then $|f_n| \leq \mathfrak{g}_n$ and $\mathfrak{g}_n + f_n \geq 0$ a.e. and $g_n - f_n \geq 0$ a.e.

Thus by Fatou's lemma,

$$\int g + \int f \leq \int \liminf \mathfrak{g}_n + f_n \leq \liminf \left(\int \mathfrak{g}_n + \int f_n \right) = \int g + \liminf \int f_n,$$

$$\int g - \int f \leq \int \liminf \mathfrak{g}_n - f_n \leq \liminf \left(\int \mathfrak{g}_n - \int f_n \right) = \int g - \limsup \int f_n,$$

Therefore $\liminf \int f_n \geq \int f \geq \limsup \int f_n$ and the result follows. $\square$

(5.4) (Folland problem 2.22) Let X $= \mathbb{N}$, let $\mathcal{M} = P(\mathbb{N})$, and let μ be counting measure on $\mathbb{N}$Interpret Fatou's Lemma, the MCT, and the DCT for this measure space $(X, \mathcal{M}, \mu)$ as statements about infinite series.

In this case, $\int f \; d\mu = \sum f(n) \; \mu(n) = \sum f(n)$. Then Fatou's lemma says that $\sum \liminf f_k(n) \leq \liminf \sum f_k(n)$. Let us consider f_k so that $f_k(n) = a_n$ if $n \leq k$ and 0 after, so that $\sum \liminf f(k) = \sum \lim_j \inf_{k \geq j} f_k(n) = \sum a_n \leq \lim_j \inf_{k \geq j} \sum f_k(n)$. Using $\sum f_k(n) = \sum_{n=1}^{k} a_n$ we have $\sum a_n \leq \lim_j \inf_{k \geq j} \sum_{n=1}^{k} a_n = \lim_j \sum_{n=1}^{j} a_n = \sum a_n$. So for functions of this form the equality holds. Now for any sequence b_n we have that $\sum \liminf b_n \leq \liminf \sum b_n$ which says that if $\lim b_n \to a > 0$ then $a\mu(X) \leq \sum b_n$. Therefore if $\sum b_n \neq \infty$ then $a = 0$ so that $b_n \to 0$.

Fatou's lemmma gives us the Convergence $\implies$ the limit of the series tends to 0.

(5.5) Suppose $f_n \in L^1(\mu)$ and $f_n \to f$ uniformly.

(a) Show that if $\mu(X) < \infty$ then $\int f_n \to \int f$.

Proof. If $f_n \to f$ uniformly then for every ϵ there is a N so that for all $n > N$ $\sup_x d(f_n, f) < \epsilon$. Suppose that f_n and f are both finite on the domain, since for the infinite points, convergence could only be uniform ifd $f_n(x) = f(x) = \infty$ for all n unless $\epsilon = \infty$. WE can handle that case since $\int f = \infty$ implies that there is a set with postive measure along which f is infinite and so the same goes for all f_n and thus $\int f_n = \infty \to \int f = \infty$.

For the finite case, then f_n is bounded by $M = \sup_n \sup_x \sup\{|f_n|, |f|\}$ for every x on the domain and so $\mu(X)M = \int M < \infty$ so by the dominated convergence theorm $\int f_n \to \int f$. $\square$