Math 215A — UCB, Spring 2017 — William Guss

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Selected Problems: 3 (Depending on that which wasn't submitted by Alekos or Chris.)

(7.3a) *(Categorical kernels and cokernels):* Let $\mathcal{C}$ be a category with an object $0 \in \mathcal{C}$ that is initial and terminal. A *categorical kernel* of $g \in \mathcal{C}(B, C)$ is a pullback

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "k"] & 0 \arrow[d] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

Show that k is a monomorphism.

Proof. Suppose that $J \in \mathcal{C}$ and $\xi = k \circ f_1 = k \circ f_2$ for $f_i \in \mathcal{C}(J, K)$. Now let $\gamma : K \to 0$ be the unique terminal map for K and let ξ be the unqiue terminal map for J. Then $\gamma \circ f_2, \gamma \circ f_1 \in C(J, 0) \implies \gamma \circ f_2 = \gamma \circ f_1 := \zeta$. There fore the following two diagrams commute, summerizing the situation.

$$\begin{tikzcd}
J \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[dr, "\xi"'] & K \arrow[d, "k"] \\
& B
\end{tikzcd}
\qquad
\begin{tikzcd}
& 0 \\
J \arrow[ur, "\zeta"] \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & K \arrow[u]
\end{tikzcd}$$

By the universality of the pullback there is a unique $u : J \to K$ which is the mediating map for the categorical pullback (J, ξ, ζ), and so because f_1, f_2 are mediating maps, $f_1 = f_2$. Therefore the following diagram commutes

$$\begin{tikzcd}
J \arrow[dr, "f_1", "f_2"'] \arrow[drr, bend left, "\zeta"] \arrow[ddr, bend right, "\xi"'] & & \\
& K \arrow[r] \arrow[d, "k"] & O \arrow[d] \\
& B \arrow[r, "g"] & C
\end{tikzcd}$$

Therefore k is a monomorphism. $\square$

(7.3b) Given a commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "\alpha"] & B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[d, "\gamma"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

where the left square is a pullback, show that

- If f is a monomorphism then so if f'.
- If f is a kernel of g then f' is a kernel of g'.

Proof. We will first show that if f is a monomorphism then f' is a monomorphism. Take some $J \in \mathcal{C}$ and maps p_1, p_2 so that $f' \circ p_1 = f' \circ p_2 := \zeta$. Then using the commutativity of the diagram above,

$$\begin{aligned} f \circ (\alpha \circ p_1) &= \beta \circ f' \circ p_1 \\ &= \beta \circ \zeta \\ &= \beta \circ f' \circ p_2 \\ &= f \circ (\alpha \circ p_2). \end{aligned}$$

Since f is a monomorphism, $\alpha \circ p_1 = \alpha \circ p_2 = \xi$. Therefore the following diagram commutes, and using the universal property of pullbacks, p_1, p_2 are the same unique mediation map between the pullback in the first diagram and the new pullback (J, ξ, ζ).

$$\begin{tikzcd} J \arrow[dr, "p_1", "p_2"'] \arrow[drr, bend left, "\zeta"] \arrow[ddr, bend right, "\xi"'] & & \\ & A' \arrow[r, "f'"] \arrow[d, "k"] & B' \arrow[d, "\beta"] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

Therefore $p_1 = p_2$ and f' is a monomorphism. $\square$

Dual Definitions: A *categorical cokernel* of $g \in \mathcal{C}(B, C)$ is a pushforward so that the diagram commutes

$$\begin{tikzcd} K & 0 \arrow[l] \\ B \arrow[u, "k"] & C \arrow[l, "g"] \arrow[u] \end{tikzcd}$$

We say that $f : X \to Y$ is an *epimorphism* if and only if $T \in \mathcal{C}$ and $f_1, f_2 : Y \to T$ so that $f_1 \circ f = f_2 \circ f$ implies that $f_2 = f_1$. Furthermore Given a commutative diagram

$$\begin{tikzcd} A' & B' \arrow[l, "f'"] & C' \arrow[l, "g'"] \\ A \arrow[u, "\alpha"] & B \arrow[l, "f"] \arrow[u, "\beta"] & C \arrow[l, "g"] \arrow[u, "\gamma"] \end{tikzcd}$$

where the left square is a pushforward, show that if f is an epimorphism then so is f'. If f is a cokernel of g then f' is a cokernel of g'