MATH 202A: Notes

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Definition 1. *A topology on X is $\tau \subset P(X)$ that contains ϕ, X and is closed with respect to arbitrary unions and finite intersection; that is closed under an intersection of finitely many sets in τ.*

Example 1. *Take any metric space (X, ρ) and let τ be the collection of all ρ-open subsets of X, then ρ is a topology.*

Example 1 is a very good example. We'll generalize this example by removing the notion of distance but keeping the notion of infintesmally close.

Example 2 (Naive)**.** *Take any X and let $\tau = \{\phi, X\}$, this is a topology. Take any X and let $\tau = P(X)$.*

Example 2 is stupid.

Definition 2. *A set $E \subset X$ is closed if $X \setminus E$ is open.*

Definition 3. *The interior of a set $E \subset X$ is the largest open set $O \in \tau$ contained in E; that is $int(E) = \{x \in E \ : \ \exists V^{open} \ : x \in V \subset E\}$.*

Definition 4. *The closure of E is the smallest closed set containing E; that is, $cl(E) = X \setminus \int(X \setminus E)$.*

If E_α is closed then $\cap E_\alpha$ is closed by DeMorgans law.

Definition 5. *The boundary of E, $\partial E = cl(E) \setminus \int(E)$.*

Example 3. *Let $X = \mathbb{R}^2$ with the usual metric. Let $E = \{x = (x_1, x_2) : x_1 \geq 0$ and $|x| \leq 1$ or $x_1 < 0$ and $|x| < 1\}$. Then $\partial E = S^1, int(E) = B_1^{open}(0), cl(B) = B_1(0)$.*

Definition 6. *A set $E \subset X$ is dense if the closure of $E = X$.*

Definition 7. *A point $x \in X$ is an accumulation point of $E \subset X$ if any openset $V \ni x$ intersects $E \setminus \{x\}$. We denote $acc(E)$ as the set of all accumulation points of E.*

Example 4. *If $E = \mathbb{Z} \subset \mathbb{R}$, then there are no accumulation points at all; hence, $acc(\mathbb{Z}) = \emptyset$.*

Proposition 1. *For any $E \subset X$, $cl(E) = E \cup acc(E)$.*

Proof. We would like to show that $X \setminus (E \cup acc(E))$ is open. Suppose that $y \notin E$, and $y \notin acc(E)$. There is an openset which contains why such that $V \cap E \setminus \{y\} = \emptyset$. Therefore $V \cap E = \emptyset = \emptyset$. The union of all such V is the compliment of $E \cup acc(E)$. Thus is open. Therefore $E \cup acc(E)$ is closed.

Consider any closed set A containig E. Claim $A \supset acc(E)$. If $y \in acc(E)$ and $y \notin A$, then $y \in X \setminus A = V^{open}$. Since $V^{open} \cap E = \emptyset$ because $V = X \setminus A \subset X \setminus E$. Therefore $y \notin acc(E)$. And thats a contradiction. :)

On the one hand, $E \cup acc(E)$ is closed and contains E and is contained in any other closed set which contains E and thus it is the smallest closed set containg E or equivalently it is the closure of E. $\square$

The following are examples of topological spaces that are not metric spaces.

Example 5 (Simple)**.** *Take $X = \{a \neq b\}$. Take the topology to be $\tau = \{\emptyset, X, \{a\}\}$. It is obviously closed under intersections and unions. We claim that this topology is not* compatible *with any metric space structure.*

Proof. If X is a metric space, then $\rho(a, b) = r$. Then $B(a, r/2) \cap B(b, r/2) = \emptyset$. any open set in τ which contains b is X itself and must contain A and so must intersect the openset containing a. BUT THIS CANNOT BE! □

So toplogy is more general than metric spaces.

Example 6 (Zariski Toplogy)**.** *Let $X = \mathbb{C}^n$, then $V \subset \mathbb{C}^N$ is open if V is the union of finite intersections of sets of form $\{z \ : \ P(z) \ \neq 0\}$ where P is any polynomial. For example when $n = 1$, there are finiteley many 0s and so for any finite collection of points we can construct a polynomial.*

Disatisfied.

Example 7. *Set X of all functions $f : \mathbb{R} \to \{0, 1\}$. Consider all sets $V \subset X$ of this form: Choose $S \subset \mathbb{R}$ to be a finite set. (Mark finiteley many points on the axis.) For each element of the set choose t_s to be a 0 or a 1. Then $V = \{f : \mathbb{R} \to \{0, 1\} : f(s) = t_s \forall s \in S\}$. The topology is the set of all sets that can be written as unions of sets V of this type.*

Definition 8. *If $\mathcal{E} \subset P(X)$, the toplogy generated bu $\mathcal{E}$ is the smallest topology which containes $\mathcal{E}$. That is if $E \in \mathcal{E}$ then $E \in \tau$. This is the same as the collection of all unions of finite intersections of elements of $\mathcal{E}$.*

Proposition 2. *Any intersection of two (induct for finite) ... is equal to a union of finite intersections of elements of $\mathcal{E}$. Equivalently, for any index set A and associated finite indexsets A_α*

$$\left(\bigcup_{a \in A} \bigcap_{j \in A_\alpha} V_{\alpha,j} \in \mathcal{E} \right) \cap \left(\bigcup \beta \in B) \bigcap_{k \in B_\beta} W_{\beta,k} \in \mathcal{E} \right) = \bigcup_{(\alpha,\beta) \in A \times B} \left(\bigcap_{j \in A_\alpha, k \in B_\beta} V_{j,\alpha} \cap W_{k,\beta} \right).$$

DANK.