MATH 202A: Homework 1

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1.

Theorem 1. *Let X be a nonempty set and let $f, g : X \to R$ be bounded functions. Show that*

$$\sup(f(x) + g(x)) \le \sup f(x) + \sup g(x)$$

Proof. Since X is non empty and the functions f, g are bounded then the function $(f + g)(x)$ is bounded. Let u_f, u_g be the least upper bound of $f(X)$ and $G(X)$ respectively. These exist since f, g are bounded. Furthermore let u_{f+g} be the upperbound for $(f + g)$.

Suppose $u_f + u_g < u_{f+g}$, let x_n^f and x_n^g be sequences in X which acheive $f(x_n^f) \to u_f$ and $g(x_n^g) \to u_g$ respectively, finally let y_n be the sequence which achieves $(f+g)(y_n) = u_{f+g}$. If $u_{f+g} > u_f + u_g$ then there is an N such that for all $n > N$ $(f + g)(y_n) = f(y_n) + g(y_n) > u_f + u_g$ which is a contraction to u_f and u_g being an upperbound. □

2. Lim suppy goodness.

Theorem 2. *Let (a_n) and (b_n) be sequences in $\mathbb{R}$ suppose neither $\limsup_{n\to\infty} a_n$ nor $\limsup_{n\to\infty} b_n$ equals $-\infty$. Show that*

$$\limsup_{n\to\infty} a_n + b_n \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Proof. Observe that a_n, b_n are isomorphically equivalent to $A : \mathbb{N} \to \mathbb{R}$ and $B : \mathbb{N} \to \mathbb{R}$.

Then if a_n and b_n are bounded above, then by the previous problem

$$\sup_{E\subset\mathbb{N}} A(n) + B(N) \le \sup_{E\subset\mathbb{N}} A(n) + \sup_{E\subset\mathbb{N}} B(N). \tag{1}$$

since A, B bounded below and above by the assumption of the problem. And for the the family of sets $E := E_n = \{n, n + 1, ...\}$ the inequality holds so therefore

$$\limsup_{n\to\infty} a_n + b_n \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Without loss of generality we check the other case by assuming that a_n is unbounded above. Then $a_n \to \infty$ and coorespondingly $\limsup a_n = \infty$. We then have that $\limsup a_n + \limsup b_n = \infty$ and for every $x \in \overline{\mathbb{R}}$, $x \leq \infty$ and the inequality holds since $\limsup_{n\to\infty} a_n + b_n \in \overline{\mathbb{R}}$. □

Consider the following example $a_n = (1, -1, 1, -1, 1, -1, ...)$ and $b_n = (-1, 1, -1, 1, -1, 1, -1, ...)$.

Proof. By construction $\limsup b_n = 1$, $\limsup a_n = 1$. However $a_n + b_n = 0$ for all n by construction. Therefore $\limsup 0 = 0 \leq \limsup b_n + \limsup a_n$.

□

3. We show that the matching metric is a metric.

   Proof. First $d : X \times X \to [0, \infty)$ since it is not possible for there to be a negative number of indices for which $x_n \neq y_n$ (doesn't make sense). Furthermore S^n is finite so there can be at most n indices for which x, y could disagree.

   Second, d is semetric since the number of elements for which $x_j \neq y_j$ is equivalent to the number of elements for which $y_j \neq x_j$ by the symmetry of the equality relation on S.

   Third, $x = y$ if and only if for every $j \in \{1, ..., n\}$, $x_j = y_j$ if and only if the number of indices on which x and y agree is 0 if and only if $d(x, y) = 0$ if and only if $x = y$.

   Fourth and finally, if $x, y, z \in X$ then suppose that $d(x, z) + d(z, y) < d(x, y)$. Then there is a j such that $x_j \neq y_j$ but that $x_j = z_j = y_j$. If there not were such a j in this situation then x_j disagrees with z_j or y_j disagrees with z_j for every j and so the LHS is $2n > n \geq d(x, y)$. So such a j must exist and that is a contradition to $d(x, z) + d(z, y) < d(x, y)$. Therefore the triangle equality hodls for this metric. □

4. Consider $\mathbb{N} \subset \mathbb{R}$.

   Proof. We claim that $\lim_{\mathbb{R}} \mathbb{N} = \mathbb{N}$. Suppose there were $x \notin \mathbb{N}$ that was a limit point, then for every $r > 0$, there exists an $n \in B(r, x)$ with $n \in \mathbb{N}$. Take $r = \frac{\min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}}{2}$ Then $B(r, x)$ cannot contain any n since it is a strict subset of $(\lfloor x \rfloor, \lceil x \rceil)$. So x is not a limit point of $\mathbb{N}$ and all of the limit points of $\mathbb{N}$ are in $\mathbb{N}$.

   Every point and only every point of $\mathbb{N}$ is a limit point and $\mathbb{N}$ is countable. □

   The set $[0, 1]$ is a closed set and so every point is a limit point and $[0, 1]$ is uncountable.

5. The set $E = \{x | x^2 < 2, \} \subset \mathbb{Q}$ is clopen.

   Proof. If $E = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$ then it is open since any rational $r \in E$ has the property that $\epsilon = \min\{d(r, -\sqrt{2}), d(r, \sqrt{2})\}$ gives a ball $B(\epsilon/2, r) \cap \mathbb{Q}$ which contains every element in E since it could not possibly contain $l > \sqrt{2}$ or $l < \sqrt{2}$ by definition of ϵ.

   Now take a sequence of convergent rational numbers in E (which may converge outside of E. Suppose that it does converge outside of E. It must be the case that there is a $r > \sqrt{2}$ or $r < \sqrt{2}$ to which the sequence converges. Without loss of generality assume that $r > \sqrt{2}$. Then take $\epsilon = r/2 + \sqrt{2}/2$. There must be an N so that all elements of the sequence with index greater than n are more than $\sqrt{2}$ since there exists rationals within ϵ of $r > \sqrt{2}$, but this contradicts the sequence being in E. Therefore E is closed. $\square$

6. The set $E = \{(x, y) \in \mathbb{R}^2 \ : \ x > 0, y > 0, xy > 1\}$ is open.

   Proof. We show that $f(x, y) = xy$ is continuous. It is obvious that the identity map $id(x) = x, id(y) = y$ is continuous (take $\delta = \epsilon$). Furthermore it is obvious that $f(x, y) = x, = y$ is continuous (take $\delta = \epsilon$) by effectivelty the same argument. Then the product of $f(x, y) = x, f(x, y) = y$ is continuous.

   Then the set $f(E) = (1, \infty)$ is open in $\mathbb{R}$ and by continuity of f the preimage is open. That is E is open. $\square$

7. Graph goodness.

   Theorem 3. *If $f : [0, 1] \to \mathbb{R}$ and f continuous then $G(f) = \{(x, y) \in [0, 1] \times \mathbb{R} : y = f(x)\}$ is closed.*

   Proof. Since f is continuous, for any sequence (x_n) in $[0, 1]$, $x_n \to x$ implies $f(x_n) \to f(x)$. Let (x_n, y_n) be any convergent sequence from $G(f)$. We wish to show that $(x_n, y_n) \to (x, y) \in G(f)$.

   Since (x_n, y_n) a convergent sequence in $\mathbb{R}^2$ then x_n must be a convergent sequence in $\mathbb{R}$ (it is not hard to see this since $|x_n - x|^2 < |x_n - x|^2 + |y_n - y|^2 < \epsilon$). However since $x_n \in [0, 1]$ and $[0, 1]$ closed $x_n \to x \in [0, 1]$ and by the continuity of f, $y_n = f(x_n) \to f(x) = y$ such that $(x, y) \in G$.

   This completes the proof. $\square$

8.

   Theorem 4. *Let (x_n) be a sequence of points in a metric space $(X, \rho$, and let $z \in X$. Suppose that any subsequence of (x_n) has a sub-subsequence which converges to z. Then $x_n \to z$.*

   Proof. Suppose not, then there exists an $\epsilon > 0$ such that for all N, there exists an $n > N$ such that $\rho(x_n, z) > \epsilon$. Take the subsequence n_j such that n_j is the first $n > j$ where $\rho(x_n, z) > \epsilon$.

   This sequence has a convergent subsequence j_p such that there exists an N for which all $p > N$ gives $\rho(x_{n_{j_p}}, z) < \epsilon$. This is a contradiction, and therefore the theorem holds. $\square$

9.

   Theorem 5. *Let (x_n) be a Cauchy sequence in (X, ρ). Show that if some subsequence (x_{n_k}) converges, then (x_n) also converges.*

Proof. If (x_n) is cauchy then for all $\epsilon > 0$ there exists an M such that fora all $p, q > M$ $\rho(x_p, x_q) < \epsilon/2$. Take M to be large enough that $\rho(x_{n_q}, x) < \epsilon/2$ by $x_{n_k} \to x$. By the triangle inequality, $\rho(x_m, x) \leq \rho(x_m, x_{n_q}) + \rho(x_{n_q}, x) < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore $x_n \to x$. □

10.

Theorem 6. *Any cauchy sequence is bounded.*

Proof. Let (x_n) be a cauchy sequence. Pick any ϵ, then take N large enough such that for all $n, m > N$, $d(x_n, x_m) < \epsilon$. Then fix n. Let

$$R = \max\{d(x_1, x_n), \ldots d(x_{n-1}, x_n), \epsilon\}. \tag{2}$$

It is obvious that $\{x_l\} \subset B(R, x_n)$. This completes the proof. □

11.

Theorem 7. *Let (X, ρ) be a metric space and let $Y \subset X$. Let ρ' be the metric on Y defined by restricting ρ to Y. Show that if (Y, ρ') is complete then Y is a closed subset of X.*

Proof. Suppose that Y does not contain all of its limit points. Then there is a sequence such that $y_n \to x \in X \setminus Y$. Then for every $\epsilon > 0$ there is an N such that for all $n, m > N$ $\rho(y_n, x) < \epsilon/2$ and $\rho(y_m, x) < \epsilon/2$.

It follows that $\rho'(y_m, y_n) < \rho(y_n, x) + \rho(x, y_m) < \epsilon$ so y_n is cauchy in Y. Therefore by y complete, $y \to y \in Y$ which is a contradiction to $y_n \to x \in X \setminus Y$. □

12.

Theorem 8. *Let $f : X \to Y$. If G is the graph of f show that if f continuous then G is closed.*

Proof. Define $F : X \to G$ as the function which takes x to $(x, f(x))$. Such a map is a bijection since every element of x is uniqueley indexed in G by $(x, .)$ and the definition of G says that for every $(x, f(x)) \in G$ there is an y in X namely x which maps to $(x, f(x))$ under X.

Then for any sequence in G there exists a $x_n \in X$ which cooresponds through F uniquely. So let x_n which converges then $F(x_n)$ converges in G since $F = id \times f$ is continuous. Therefore every sequence in G converges. □

13.

Theorem 9. *Let $d : (x, y) \mapsto |x - y|^{1/2}$. Then d is a metric and other things in the assignment.*

Proof. The function $d = \sqrt{\circ}\rho$ where ρ is a metric. Therefore $d(a,b) = \sqrt{\circ}\rho(a,b) = \sqrt{\circ}\rho(b,a) = d(b,a)$. Furthermore $\sqrt{:}\mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ so $\sqrt{\circ}\rho$ is still positive definite. Finally we show the triangle inequality,

$$d(a,c) = \sqrt{\rho(a, b - b + c)} \leq \sqrt{\rho a, b + \rho(b,c)} \tag{3}$$

and so we show $d(a,c)^2 = \rho(a,c) \leq \rho(a,b) + \rho(b,c)$ implies by monotonicity of ρ that $d(a,c) \leq d(a,b) + d(b,c)$.

Now we show that the metrics are not strongly equivalent. Suppose there were constants α, β such that for every $x, y \in X$ $\alpha d(x,y) \leq \rho(x,y) \leq \beta(x,y)$. THen $\alpha|\gamma| \leq |\gamma|^2 \leq \beta|\gamma|$ but clearly there exixts no β such that γ^2 never exceeds the line $\gamma\beta$ so the metrics are not strongly equivalent (although they are topologically equivalent.)

Now we show that cauchy in ρ if and only if cauchy in d. Pick $\epsilon > 0$ and $\delta = \epsilon^2 > 0$ then $d(x_m, x_n) < \delta$ if and only if $\rho(x_m, x_n) = d(x_m, x_n)^2 < \delta^2 = \epsilon$.

The set $\mathbb{R} \setminus A$ is closed under d and contains all of its limit points if and only if it is cauchy under d if and only if it is cauchy under ρ if and only if it contains it limits under ρ if and only if it is closed under ρ. Therefore A is open under ρ if and only if it is open under d. □

14.

Theorem 10. *If $f : X \to Y$ continuous and $K \subset X$ compact then $f(K)$ compact.*

Proof. If K compact then every sequence has a convergent subsequence. Take any sequence $y_n \in f(K)$ then clearly there is a sequence in K such that $f(x_n) = y_n$. Then take the subsequence of x_n which converges, say n_j. Then $f(x_{n_j}) \to f(x) \in f(K)$ (as $x \in X$) by continuity of f and $f(x_{n_j})$ is a subsequence of y_n. This completes the proof. □

15.

Theorem 11. *Let $f : K \to Y$ be continuous and $K \subset X$ compact, then f is uniformly continuous.*

Proof. Since f is continuous then for any $\epsilon > 0$ for every x there is a $\delta(x)$ such that $\rho(x,y) < \delta(x)$ implies that $\rho'(fx, fy) < \epsilon$. Let $\mathcal{V}$ be the family defined as

$$\mathcal{V} = \left(B(\delta(x), x)\right)_{x \in K}. \tag{4}$$

This is clearly an open cover of K and by K compact there is a finite subcover indexed by a finite $\mathcal{F} \subset K$. Let

$$\delta = \min_{x \in \mathcal{F}} \delta(x). \tag{5}$$

It follows that any for every $x, y \in K$ such that $\rho(x,y) < \delta$, $\rho(x,y) < \delta(x)$ and $\rho(x,y) < \delta(y)$ and so $\rho'(fx, fy) < \epsilon$. Therefore$f$ is uniformly continuous. □