Math 215A — UCB, Spring 2017 — William Guss

Partners: Alekos, Chris

Selected Problems: 3 (Depending on that which wasn't submitted by Alekos or Chris.)

(7.3a) (Categorical kernels and cokernels): Let \mathcal{C} be a category with an object $0 \in \mathcal{C}$ that is initial and terminal. A categorical kernel of $g \in \mathcal{C}(B, \mathcal{C})$ is a pullback

$$\begin{array}{ccc}
K & \longrightarrow 0 \\
\downarrow^k & \downarrow \\
B & \stackrel{g}{\longrightarrow} C
\end{array}$$

Show that k is a monomorphism.

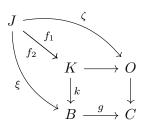
Proof. Suppose that $J \in \mathcal{C}$ and $\xi = k \circ f_1 = k \circ f_2$ for $f_i \in \mathcal{C}(J,K)$. Now let $\gamma : K \to 0$ be the unique terminal map for K and let ξ be the unique terminal map for J. Then $\gamma \circ f_2, \gamma \circ f_1 \in \mathcal{C}(J,0) \Longrightarrow \gamma \circ f_2 = \gamma \circ f_1 := \zeta$. There fore the following two diagrams commute, summerizing the situation.

$$J \xrightarrow{f_1} K \qquad 0$$

$$\downarrow k \qquad \downarrow k \qquad \uparrow \uparrow$$

$$B \qquad J \xrightarrow{f_1} K$$

By the universality of the pullback there is a unique $u: J \to K$ which is the mediating map for the categorical pullback (J, ξ, ζ) , and so because f_1, f_2 are mediating maps, $f_1 = f_2$. Therefore the following diagram commutes



Therefore k is a monomorphism.

(7.3b) Given a commutative diagram

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C'$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

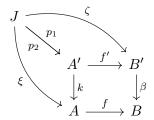
where the left square is a pullback, show that

- If f is a monomorphism then so if f'.
- If f is a kernel of g then f' is a kernel of g'.

Proof. We will first show that if f is a monomorphism then f' is a monomorphism. Take some $J \in \mathcal{C}$ and maps p_1, p_2 so that $f' \circ p_1 = f' \circ p_2 := \zeta$. Then using the commutativity of the diagram above,

$$f \circ (\alpha \circ p_1) = \beta \circ f' \circ p_1$$
$$= \beta \circ \zeta$$
$$= \beta \circ f' \circ p_2$$
$$= f \circ (\alpha \circ p_2).$$

Since f is a monomorphism, $\alpha \circ p_1 = \alpha \circ p_2 = \xi$. Therefore the following diagram commutes, and using the universal property of pullbacks, p_1, p_2 are the same unique mediation map between the pullback in the first diagram and the new pullback (J, ξ, ζ) .



Therefore $p_1 = p_2$ and f' is a monomorphism.

Dual Definitions: A categorical cokernel of $g \in \mathcal{C}(B,C)$ is a pushforward so that the diagram commutes

$$\begin{array}{ccc} K & \longleftarrow & 0 \\ k \uparrow & & \uparrow \\ B & \longleftarrow & C \end{array}$$

We say that $f: X \to Y$ is an *epimorphism* if and only if $T \in \mathcal{C}$ and $f_1, f_2: Y \to T$ so that $f_1 \circ f = f_2 \circ f$ implies that $f_2 = f_1$. Furthermore Given a commutative diagram

$$A' \leftarrow G' \qquad B' \leftarrow G' \qquad C'$$

$$\alpha \uparrow \qquad \beta \uparrow \qquad \gamma \uparrow \qquad A \leftarrow G \qquad B \leftarrow G \qquad C$$

where the left square is a pushforward, show that if f is an epimorphism then so is f'. If f is a cokernel of g then f' is a cokernel of g'