MATH 105: Homework 1

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5 Multivariable Calculus

3. Prove the following.

Theorem 1. *Let $T : V \to W$ be a linear transformation between normed spaces. Then,*

$$\begin{aligned} \|T\| &= \sup\{|Tv| \ : \ |v| < 1\} \\ &= \sup\{|Tv| \ : \ |v| \leq 1\} \\ &= \sup\{|Tv| \ : \ |v| = 1\} \\ &= \inf\{M \ : \ v \in V \implies |Tv| \leq M|v|\} \end{aligned} \tag{1}$$

Proof. Let the following defenitions stand,

$$\begin{aligned} A &= \sup\{|Tv| \ : \ |v| < 1\} \\ B &= \sup\{|Tv| \ : \ |v| \leq 1\} \\ C &= \sup\{|Tv| \ : \ |v| = 1\} \\ D &= \inf\{M \ : \ v \in V \implies |Tv| \leq M|v|\} \end{aligned} \tag{2}$$

Observe that $A \leq B$ and $C \leq B$ since the family considting of the underlying sets is respectiverley ordered by size. By definition we have that,

$$\|T\| = \sup\left\{|Tv|/|v|\right\},$$

and nameley $|Tv|/|v| = |T(v/|v|)|$. Therefore $\|T\| \leq C$. If $|v| \leq 1$ then $|Tv| \leq |Tv|/|v|$ and so $B \leq \|T\|$. We yield that $\|T\| = B = C$.

By the same logic $A \leq \|T\|$ and therefore is equivalent. Lastly $|Tv| \leq \|T\||v|$ and so by the epsilon property $D = A$. □

4. Consider the following theorem.

Theorem 2. *If $T : V \to V$ is a linear transformation on the normed vector space V. Let $A = \sup\{r : B_r(0) \supset TU\}$ and $B = \inf\{r : B_r(0) \subset TU\}$. Then, $A = \|T\|$ and $B = m(T)$.*

Proof. Observe $U \subset V$ is the unit ball induced by $|.|$ and therefore U is compact. T is linear so by its continuity we have that TU is compact and thereby contains all its limit points.

Then there is a sequence in TU so that $v_n \to v \in \partial TU \cap B_r(0)$. In particular $|v| = A$. Likewise there is a sequence $w_n \to w$ in TU so that $|w| = B$.

Suppose that $\|T\| < A$. Then $\|T\| < |v|$. There exists a u so that $Tu = v$ and $v \in \partial TU$ implies that $u \in \partial U$ by continuity and linearity. Thus $\|T\| < |Tu|/|u|$ which is a contradiction.

Suppose that $\|T\| > A$ or equivalently $\|T\| - A = \epsilon > 0$. By the linearity of T we have that for all $z \in V$ $\|T\| - |z| \leq \epsilon$ since $z = \alpha q$ for $\alpha \in \mathbb{R}$ and $q \in TU$. So $\|T\| = \sup\{|Tu|/|u| : u \in U\} + \epsilon$ which is a contradiction.

So $\|T\| = A$.

Suppose that $m(T) < B$. Then $m(T) < |w|$. There exists a u so that $Tu = w$ and $w \in \partial TU$ implies that $u \in \partial U$ by continuity and linearity. Thus $m(T) > |Tu|/|u|$ which is a contradiction.

Suppose that $m(T) > B$ or equivalently $m(T) - B = \epsilon > 0$. By the linearity of T we hawe that for all $z \in V$ $m(T) - |z| \leq \epsilon$ since $z = \alpha q$ for $\alpha \in \mathbb{R}$ and $q \in TU$. So $m(T) = \inf\{|Tu|/|u| : u \in U\} + \epsilon$ which is a contradiction.

So $m(T) = B$. □

Theorem 3. *If $T : V \to V$ is a linear isomorphism then, $m(T) > 0$.*

Proof. In the contrapositive, $m(T) = 0$ implies that the largest ball which is contained in TU is the 0 ball and so the kernel of T is non-triial. Therefore T is note an isomorphism.

□

Theorem 4. *If $T : V \to V$ has positive conorm and is linear, then it is an isomorphism.*

Proof. Positive conorm impliesn that T has a trivial kernel and so by the invertible matrix theorem, $T \equiv A$ where A is invertible and so T is invertible. □

Theorem 5. *If $T : V \to V$ and T is linear, then T is the identity.*

Proof. The conorm is equal to the norm if and only if $U \mapsto U$. Then by linearity $v/|v| \mapsto v/|v|$ implies $v \mapsto v$. □

6. Consider the following theorem.

Theorem 6. *$\mathcal{L}_n$ and $\mathcal{M}_n$ are rings where the abelian operator is pointwise and componentwise respectveley, and where the monoid law of composition is multiplication and functional composition respectiveley.*

Proof. The set of linear transformations $\mathcal{L}_n$ is Abelian with respect to addition since it occurs over the field $\mathbb{R}$; that is,

$$+_{\mathcal{L}} : \mathcal{L}_n \times \mathcal{L}_n \to \mathbb{R} \times \mathbb{R} \to \mathbb{R} \to \mathcal{L}_n.$$

As for monoid laws of composition, we show the distributive properties. First, $f, id \in \mathcal{L}_n$ imples that $f \circ id : V \to W$ with the mapping $x \mapsto x \mapsto f(x) \equiv x \mapsto f(x)$. So, $f \circ id \equiv f$. Now consider $g, h \in \mathcal{L}_n$. The composition $f \circ (g +_{\mathcal{L}} h) : V \to W$ has the mapping

$$x \mapsto h(x) + g(x) \mapsto f(h(x) + g(x)).$$

By linearity, we equivelently have $x \mapsto f(h(x)) + f(g(x))$. So in total $f \circ (g +_{\mathcal{L}} h) \equiv f \circ g +_{\mathcal{L}} f \circ h$. Lastly, $(f \circ g) \circ h \equiv f \circ (g \circ h)$ by the same logic. Therefore, $\mathcal{L}_n$ is a ring.

Matrices have the following result. Take $M, N, L \in \mathcal{M}_n$. gain the addition operator is Abelian since it maps to $\mathbb{R}^n$; that is

$$+_M : \mathcal{M}_n \times \mathcal{M}_n \to \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \to \mathcal{M}_n.$$

Then it follows that, $MI = M$ by the rules matrix multiplication. Furthermore matrix multiplication is associative and distributive. Therefore $\mathcal{M}_n$ is a ring.

$\square$

Theorem 7. *There exists a ring isomorphism between $\mathcal{M}_n$ and $\mathcal{L}_n$.*

Proof. Let $\tau : \mathcal{M}_n \to \mathcal{L}_n$ be defined by the mapping $A \mapsto (x \mapsto Ax)$. Clearly this mapping is a surjection since given any $f \in \mathcal{L}_n$ there is at least a corresponding matrix in $\mathcal{M}_n$ by the following construction. Take the standard basis of V and produce

$$A = [f(e_1) \dots f(e_n)].$$

Then $f(v) = f(e_1)v_1 + \cdots + f(e_n)v_n = Av$. Suppose there were another matrix B such that $\tau(B) = f = \tau(A)$. Then $\tau(B - A) = \tau(B) - \tau(A) = f - f = 0$ but this contradicts the fact that $B \neq A$. Therefore τ is bijective.

Finally let $C \in \mathcal{M}_n$. Then $\tau(A(B + C)) = (x \mapsto A(B + C)x)$. By linearity this is equivalent to $(x \mapsto ABx + ACx) = \tau(A) \circ \tau(B) + \tau(A) \circ (C) = \tau(A) \circ (\tau(B) + \tau(C))$. So, τ is a homomorphism.

Hence τ is an isomorphism. $\square$

12. Prove the following.

Theorem 8. *If V is a normed finite dimensional vector space, then the unit ball, $B = \{v : |v| = 1\}$ is compact.*

Proof. $\dim V = n \in \mathbb{N} \implies V \cong \mathbb{R}^n \implies B \cong S^{n-1} \implies B$ compact. $\square$

13. Prove the following.

Theorem 9. *The set of invertible $n \times n$ matrices is not dense in $\mathcal{M}$.*

Proof. Consider the set of matrix all of whose entries are the same $((a_{ij} = r \forall i \forall j)$. They create a linear subspace which is a connected open subset of $\mathcal{M}$ disjoint from the set of invertible matrices. Therefore the set of invertible matrices could not possibly be dense in $\mathcal{M}$. $\square$