155. Use the fact of algebra that a polynomial in $(x_1, \ldots, x_n)$, which vanishes when $x_i = x_j$, is divisible by $x_i - x_j$.

156. Divide the kth column by k and apply Vandermonde's identity.

158. $H : \mathbb{R}^7 \to \mathbb{R}^1$.

160. The space spanned by columns of A and B contains columns of $A+B$.

166. $(A^t\mathbf{a})(\mathbf{x}) = \mathbf{a}(A\mathbf{x})$ for all $\mathbf{x} \in \mathbb{K}^n$ and $\mathbf{a} \in (\mathbb{K}^m)^*$.

180. Modify the Gaussian elimination algorithm of Section 2 by permuting unknowns (instead of equations).

181. Apply the LUP decomposition to M^t.

182. Apply LPU, LUP, and PLU decompositions to M^{-1}.

189. Consider the map $\mathbb{R}^n \to \mathbb{R}^{p+q}$ defined by the linear forms.

202. $T = A + iB$ where both A and B are Hermitian.

203. $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{z}^\dagger \mathbf{w}$.

205. $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{z}^\dagger \mathbf{w}$.

206. Use the previous exercise.

208. The normal forms are: $i|Z_1|^2 - i|Z_2|^2,\ |Z_1|^2 - |Z_2|^2,\ |Z_1|^2$.

213. Take the linear form for one of new coordinates.

235. Apply Vieta's formula.

239. Find an eigenvector, and show that its orthogonal complement is invariant.

241. Compute $\langle \mathbf{x}, A\mathbf{x} \rangle$ for $\mathbf{x} = \sum t_i \mathbf{v}_i$, where $\{\mathbf{v}_i\}$ is an orthonormal basis of eigenvectors, and $\sum t_i^2 = 1$.

242. If $AB = BA$, then eigenspaces of A are B-invariant.

247. Construct $v_1, \ldots, \mathbf{v}_n$ by applying the Spectral Theorem to the nonnegative Hermitian operator $A^\dagger A$, and check that $A\mathbf{v}_i \in \mathcal{W}$ corresponding to distinct nonzero eigenvalues μ_i of $A^\dagger A$ are pairwiseperpendicular.

248. Rephrase the previous exercise in matrix form.

251. The equation $\langle \mathbf{x} + t\mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle = 0$ quadratic in t has no real solutions unless $\mathbf{x}$ is proportional to $\mathbf{y}$.

253. $\langle \mathbf{x}, \mathbf{x} \rangle = (\sum x_i)^2 + \sum x_i^2$.

255. $\det(U^t U) = 1$.

264. See Example 1.

274. Use the previous exercise.

276. Prove first that the radius of the circle must be equal to the middle semiaxis of the ellipsoid.

279. For "exactly onne" claim, use the coordinate-less description of the spectrum provided by the minimax principle.

280. When $x \approx 0$, $\mathbf{sin}\, x \approx x$.

283. If T denotes the cyclic shift of coordinates in $\mathbb{C}^n$, then $C = C_0 I + C_1 T + \cdots + C_{n-1} T^{n-1}$.

297. Use Newton's binomial formula.

310. Pick A and B so that $e^A e^B \neq e^B e^A$.

313. Verify it first for a Jordan cell.

318. Make two of the lines the coordinate axes on the plane, and the third one the graph of an isomorphism between them.

319. If the first three lines are $y = 0$, $x = 0$, and $y = x$., then the fourth one is $y = \lambda x$.

320. If $d = 0$, $a = \infty$, $b = 1$, then $\lambda = c$.

322. Assuming that the spaces $\mathcal{U}_i$ have the same dimension, and the maps between them are invertible, consider the determinant (or eigenvalues) of the composition of the maps around the cycle.

325. Use the theory of Bruhat cells.

326. Use Lemma, and apply induction on the length of the shortest of the legs.

327. Use Sylvester's rule together with the previous exercise.

330. On the unit sphere of area 4π, the area of a triangle with the angles $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$ has the area $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} - \pi$. Use this to compute the number of required triangles.

332. Compute $R_{\mathbf{v}_1} R_{\mathbf{v}_2}$ to show that it is a Jordan cell of size 2 with the eigenvalue 1.

333. They correspond to positive roots $\mathbf{e}_i - \mathbf{e}_j = \mathbf{v}_{i+1} + \cdots + \mathbf{v}_j$, where $0 \leq i < j \leq n$.

344. $\overrightarrow{AA'} = 3\overrightarrow{AM}/2$.

350. $\cos^2\theta \leq 1$.

355. Rotate the regular n-gon through $2\pi/n$.

358. Take $X = A$ first.

359. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

360. If $a+b+c+d = 0$, then $2(b+c)(c+d) = a^2 - b^2 + c^2 - d^2$.

361. $XA^2 = |\overrightarrow{OX} - \overrightarrow{OA}|^2 = 2R^2 - 2\langle \overrightarrow{OX}, \overrightarrow{OA} \rangle$ if O is the center.

362. Consider projections of the vertices to any line.

363. Project faces to an arbitrary plane and compute signed areas.

372. Compute $(\cos\theta + i\sin\theta)^3$.

374. Divide P by $z - z_0$ with a remainder.

377. $(z - z_0)(z - \bar{z}_0)$ is real.

378. Apply Vieta's formula.

382. $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$.

383. See Figure spl2-5.

387. Show first that $\binom{n}{k}$ equals the number of k-element subsets in the set of n objects.

393. Examine the kernel of the map $\mathbb{R}[x] \to \mathbb{R}^2$ associating to a polynomial P the pair $(P(1), P(-1))$ of its values at $x = \pm 1$.

397. To a linear form $f : \mathcal{V}/\mathcal{W} \to \mathbb{K}$, associate $\mathcal{V} \xrightarrow{\pi} \mathcal{V}/\mathcal{W} \xrightarrow{f} \mathbb{K}$.

399. Given $B : \mathcal{V} \to \mathcal{W}^*$, show that evaluation $(B\mathbf{v})(\mathbf{w})$ of the linear form $B\mathbf{v} \in \mathcal{W}^*$ on $\mathbf{w} \in \mathcal{W}$ defines a bilinear form.

404. Compute $q_1 q_2 q_2^* q_1^*$.

407. Introduce multiplication by scalars $q \in \mathbb{H}$ as $A \mapsto q^* A$.

409. $|q|^2 = |-1| = 1$.

411. Prove that -1 is a non-square.

414. There are 1 even and 3 odd non-degenerate forms.

418. $\frac{1}{3} = 1 - \frac{2}{3}$.

419. Start with inverting 2-*adic units* $\cdots * * * 1$. ($*$ is a wild card).

421. Use the previous exercise.

Answers

7. For normal forms, X^2 and 0 can be taken.

12. $(n+1)(n+2)/2$.

15. $x(t) = x(0)e^{\lambda t}$.

16. $x(t) = x(0)e^{3t}, y(t) = y(0)e^{-t}, z(t) = z(0)$.

19. $\dot{X}_1 = iX_1, \quad \dot{X}_2 = -iX_2$.

22. Semiaxes $2/\sqrt{3}$ and 2.

25. (a) $\pm(\sqrt{\alpha^2 - \beta^2}, 0)$, (b) $\pm(\sqrt{\alpha^2 + \beta^2}, 0)$.

29. Yes, $k(x^2 + y^2)$.

30. $y = \pm x$; level curves of (a) are ellipses, of (c) hyperbolas.

31. $2X^2 - Y^2 = 1$.

34. 2nd ellipse, 1st & 4th hyperbolas.

38. A pair of intersecting lines, $x - 1 = \pm(2y - 1)$.

45. Yes, some non-equivalent equations represent the empty set.

50. $\frac{n(n+1)}{2}$.

65. Points, lines, and planes.

74. $\dim = 2$.

76. $x^k = x_1^k L_1(x) + \cdots + x_n^k L_n(x)$, $k = 0, \ldots, n-1$.

80. yes, no, no, yes, no, yes, yes, no.

82. $E = \mathbf{va}$, i.e. $e_{ij} = v_i a_j, \ i = 1, \ldots, m, \ j = 1, \ldots, n$.

83. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

84. Check your answers using associativity, e.g. $(CB)A = C(BA)$.

`Other answers:` $BAC = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}, \quad ACB = \begin{bmatrix} 0 & -2 \\ 3 & 6 \end{bmatrix}$.

91. $\begin{bmatrix} \cos 1^\circ & -\sin 1^\circ \\ \sin 1^\circ & \cos 1^\circ \end{bmatrix}$.

92. If $B = A^k$, then $b_{i,i+k} = 1$, and all other $b_{ij} = 0$.

93. The answer is the same as in the previous problem.

94. (a) If A is $m \times n$, B must be $n \times m$. (b) Both must be $n \times n$.

96. Exactly when $AB = BA$.

103. No; yes (of $x, y \in \mathbb{R}^1$); yes (in $\mathbb{R}^2$).

104. I.

106. $S = 2x_1y_1 + x_1y_2 + x_2y_1$, $A = x_1y_2 - x_2y_1$.

107. $(\sum x_i)(\sum y_i)$ and $\sum_{i \neq j} x_iy_j/2$.

108. No, only if $AB = BA$.

112. $T^\dagger = \overline{z_2}\, w_1$; $S_1 = (\overline{z_1}\, w_2 + \overline{z_2}\, w_1)/2$, $S_2 = (\overline{z_1}\, w_2 - \overline{z_2}\, w_1)/2i$; $H_1 = (\overline{z_1}\, z_2 + \overline{z_2}\, z_1)/2$, $H_2 = (\overline{z_1}\, z_2 - \overline{z_2}\, z_1)/2i$.

113. $(\overline{z_1}\, w_2 + \cdots + \overline{z_{n-1}}\, w_n + \overline{z_2}\, w_1 + \cdots + \overline{z_n}\, z_{n-1})/2$.

115. No, yes, n^2.

118. 1; 1; $\mathbf{cos}(\mathbf{x} + y)$.

119. 2, 1; -2.

121. $k(k-1)/2$.

123. $n(n-1)/2 - l$.

127. Transpositions $\tau^{(i)}$ of nearby indices $(i, i+1)$, $i = 1, \dots, n-1$.

129. E.g. $\tau_{14}\tau_{34}\tau_{25}$.

130. No; yes.

131. + (6 inverted pairs); + (8 inverted pairs).

132. $-1\,522\,200$; $-29\,400\,000$.

134. 0.

135. Changes sign, if $n = 4k + 2$ for some k, and remains unchanged otherwise.

136. $x = a_1, \dots, a_n$.

138. Leave it unchanged.

141. (a) $(-1)^{n(n-1)/2} a_1a_2 \cdots a_n$, (b) $(ad - bc)(eh - fg)$.

142. (a) 9, (b) 5.

143. (a) $\begin{bmatrix} -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \\ \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

144. (a) $\mathbf{x} = [-\frac{2}{9}, \frac{1}{3}, \frac{1}{9}]^t$, (b) $\mathbf{x} = [1, -1, 1]^t$.

146. $(\det A)^{n-1}$, where n is the size of A.

147. Those with determinants ± 1.

148. (a) $x_1 = 3, x_2 = x_3 = 1$, (b) $x_1 = 3, x_2 = 4, x_3 = 5$.

149. (a) $-x_1^2 - \cdots - x_n^2$, (b) $(ad-bc)^3$.

153. $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$.

154. 1.

156. $1!3!5!\cdots(2n-1)!$.

158. 1.

161. The system is consistent whenever $b_1 + b_2 + b_3 = 0$.

165. (a) Yes, (b) yes, (c) yes, (d) no, (e) yes, (f) no, (g) no, (h) yes.

169. $0 \le \text{codim} \le k$.

171. Two subspaces are equivalent if and only if they have the same dimension.

172. The equivalence class of an ordered pair $\mathcal{U}, \mathcal{V}$ of subspaces is determined by $k := \dim U$, $l := \dim \mathcal{V}$, and $r := \dim(\mathcal{U} + \mathcal{V})$, where $k, l \le r \le n$ can be arbitrary.

174. $x_1 = 3, x_2 = 1, x_3 = 1$; inconsistent; $x_1 = 1, x_2 = 2, x_3 = -2$;
$x_1 = 2t_1 - t_2, x_2 = t_1, x_3 = t_2, x_4 = 1$;
$x_1 = -8, x_2 = 3 + t, x_3 = 6 + 2t, x_4 = t$; $x_1 = x_2 = x_3 = x_4 = 0$;
$x_1 = \frac{3}{17}t_1 - \frac{13}{17}t_2, x_2 = \frac{19}{17}t_1 - \frac{20}{17}t_2, x_3 = t_1, x_4 = t_2$;
$x_1 = -16 + t_1 + t_2 + 5t_3, x_2 = 23 - 2t_1 - 2t_2 - 6t_3, x_3 = t_1, x_4 = t_2, x_5 = t_3$.

175. $\lambda = 5$.

176. (a) $\text{rk} = 2$, (b) $\text{rk} = 2$, (c) $\text{rk} = 3$, (d) $\text{rk} = 4$, (e) $\text{rk} = 2$.

177. Inverse matrices are:

$$\begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 31 & -19 & 3 & -4 \\ -23 & 14 & -2 & 3 \end{bmatrix}.$$

185. $\binom{n}{2} - l(\sigma)$, where $l(\sigma)$ is the length of the permutation.

186. Inertia indices $(p, q) = (1, 1), (2, 0), (1, 2)$.

191. Empty for $p = 0$, has 2 components for $p = 1$, and 1 for $p = 2, 3, 4$.

195. $z_1^2 + z_2^2 = 1, z_1^2 + z_2^2 = 0, z_2 = z_1^2, z_1^2 = 1, z_1^2 = 0$.

196. Two parallel lines.

201. Those all of whose entries are imaginary.

202. $T(\mathbf{z}, \mathbf{w}) =$
$\frac{1}{2}\left[T(\mathbf{z} + \mathbf{w}, \mathbf{z} + \mathbf{w}) + iT(i\mathbf{z} + \mathbf{w}, i\mathbf{z} + \mathbf{w}) - (1 + i)\left(T(\mathbf{z}, \mathbf{z}) + T(\mathbf{w}, \mathbf{w})\right)\right]$.

209. $d_{ii} = \Delta_i / \Delta_{i-1}$.

210. $(p, q) = (2, 2), (3, 1)$.

214. No.

215. The sum of the squares of all sides of a parallelogram is equal to the sum of the squares of the diagonals.

228. id.

252. $\theta(\mathbf{e}_i, \mathbf{e}_{i+1}) = 2\pi/3$, while all other pairs are perpendicular.

254. One can take $\mathbf{f}_i = \mathbf{e}_i - \mathbf{e}_0$, and $\mathbf{h}_i = \mathbf{e}_i - \mathbf{e}_{i-1}$, where $i = 1, \dots n$.

256. Rotations about the origin, and reflections about a line passing thorough the origin, which have determinants 1 and -1 respectively.

262. ± 1.

269. The operator is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

295. $(a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $(b) \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(c) \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
$(e) \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $(f) \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $(k) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $(l) \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$.

306. $(a_0 - n(a_0 - a_1/3))3^n$.

307. (a) $(1 + \sqrt{10})/3$.

310. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ will do.

317. Pairs of $m \times n$-matrices; pairs of matrices of transposed sizes.

321. $\mathbb{K}^1$.

323. The restriction to the eigenline.

328. (a) $(2,2,r)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$; (b) $(3,3,3)$, $(2,4,4)$, $(2,3,6)$.

330. Symmetry planes of regular polyhedron (tetrahedron in case (b), octahedron or cube in case (c), and icosahedron or dodecahedron in case (d)) partition the surface of the sphere circumscribed around the polyhedron into (respectively, 24, 48, and 120) required spherical triangles.

331. Since $\langle \mathbf{v}, \mathbf{v} \rangle = 2$, we have:
$\langle \mathbf{x} - \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{v},\ \mathbf{y} - \langle \mathbf{y}, \mathbf{v} \rangle \mathbf{v} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{v} \rangle \langle \mathbf{x}, \mathbf{v} \rangle + 2 \langle \mathbf{x}, \mathbf{v} \rangle \langle \mathbf{y}, \mathbf{v} \rangle$
$= \langle \mathbf{x}, \mathbf{y} \rangle$.

333. $\mathcal{V}^{i,j} : \dots \to 0 \to \mathbb{K}^1 \overset{\cong}{\to} \cdots \overset{\cong}{\to} \mathbb{K}^1 \to 0 \to \dots$ (i zeroes followed by $j - i$ copies of $\mathbb{K}^1$, followed by $n - j$ zeroes).

334. $\mathcal{V}^{0,n} \oplus \mathcal{V}^{1,n} \oplus \cdots \oplus \mathcal{V}^{n-1,n}$.

335. $X(t) = e^{tA} X(0) e^{-tA}$.

336. The eigenvalues are $\lambda_i - \lambda_j$, $i, j = 1, \dots, n$. In the space of linear forms, they form the root system of type A_{n-1}. The eigenvectors are the elementary matrices E_{ij} (with the entry 1 at the ith row and jth column, and 0 everywhere else). Upper-triangular E_{ij} correspond to positive roots, and lower-triangular negative.

337. $mg/2$, $mg\sqrt{3}/2$.

338. 18 *min* (reloading time excluded).

342. It rotates with the same angular velocity along a circle centered at the barycenter of the triangle formed by the centers of the given circles.

346. 3/4.

349. $7\overrightarrow{OA} = \overrightarrow{OA'} + 2\overrightarrow{OB'} + 4\overrightarrow{OC'}$ for any O.

351. 3/2.

353. $2\langle \mathbf{u}, \mathbf{v}\rangle = |\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2$.

354. (b) No.

364. Yes; Yes (0); Yes.

365. (a) $\frac{1+5i}{13}$; (b) $\frac{1-i\sqrt{3}}{2}$.

367. $|z|^{-2}$.

368. $\pm i\sqrt{3}$.

370. (a) $\sqrt{2}, -\pi/4$; (b) $2, -\pi/3$.

371. $\frac{-1+i\sqrt{3}}{2}$.

375. $2 \pm i$; $\quad i\frac{1\pm\sqrt{5}}{2}$; $\quad 1+i,\ 1+i$; $\quad 1 \pm \frac{\sqrt{6}-i\sqrt{2}}{2}$.

376. $-2, 1 \pm i\sqrt{3}$; $i, \frac{\pm\sqrt{3}-i}{2}$; $\pm i\sqrt{2}, \pm i\sqrt{2}$; $\frac{\sqrt{3}\pm i}{2}, \frac{\sqrt{3}\pm i}{2}$; $\pm i,\ \pm\frac{\sqrt{3}\pm i}{2}$.

379. $a_k = (-1)^k \sum_{1\le i_1<\cdots<i_k\le n} z_{i_1}\cdots z_{i_k}$.

382. $\cos\theta = (e^{i\theta}+e^{-i\theta})/2,\ \sin\theta = (e^{i\theta}-e^{-i\theta})/2i$.

384. The upper halp-plane $\operatorname{Im} w > 0$.

386. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

389. p^n; p^{mn}.

390. $p^{n(n-1)/2}$.

392. $\operatorname{Ker} D = \mathbb{K}[x^p] \subset \mathbb{K}[x]$.

394. (a) n; (b) $n(n+1)/2$.

395. 2, if $n > 1$.

401. $\operatorname{Ker} A^t = A(\mathcal{V})^\perp$, $A^t(\mathcal{W}^*) = (\mathcal{V}/\operatorname{Ker} A)^* \subset \mathcal{V}^*$.

403. -1; $-q$.

408. $\begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}$.

409. $\{bi + cj + dk \mid b^2 + c^2 + d^2 = 1\}$.

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