MATH 113: Notes

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Recall multiplying a complex number in trigonometric form

$$\begin{aligned} z_k &= r(\cos\theta_k + i\sin\theta_k), \quad k = 1, 2 \\ z_1 z_2 &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ z_k^n &= r_k^n(\cos(n\theta_k) + i\sin(n\theta_k)) \end{aligned}$$

The advanced student might observe that the trigonometric parameterizaiton of z is *homomorphic* under complex number multiplication. Furthermore $z_1 = z_2$ if and only if $|z_1| = |z_2|$, $\theta_1 \equiv \theta_2 mod 2\pi$. Using the complex conjugate we also get $z\overline{z} = |z|^2$. Notationally we denote the real part and the imaginary part of a complex number z by $Re(z), Im(z)$ respectively.

Complex numbers also have the property that for any z, there exist r, θ such that

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta), \quad r = |z|, \theta = Arg(z).$$

We can derive the rleation as follows. Consider the taylor series of exp.

$$e^z = \sum_{n=1}^{\infty} \frac{z^n}{n!} := \sum_{n=1}^{\infty} \frac{Re(z^n)}{n!} + i\sum_{n=1}^{\infty} \frac{Im(z^n)}{n!}.$$

Definition 1. *A complex series $\sum_k z_k$ is absolutely convergent iff*

$$\sum_{k=1}^{\infty} |Re(z_k)| < \infty, \sum_{k=1}^{\infty} |Im(z_k)| < \infty$$

Fact 1. *A complex series converges if it absolutely converges.*

Proposition 1. *For any $z \in \mathbb{C}$ the series e^z converges absolutely.*

Proof. Recall that $|a| \leq |a+bi|$ and $|b| \leq |a+bi$. Now consider that

$$\begin{aligned} \left|\frac{Re(z^n)}{n!}\right| &\leq \left|\frac{z^n}{n!}\right| \\ \left|\frac{Im(z^n)}{n!}\right| &\leq \left|\frac{z^n}{n!}\right| \end{aligned},$$

Therefore we need show that the series $\sum_n |z|^n/n!$ is convergent which implies that e^z is absolutely convergent. Recall that $\sum_n |z|^n/n!$ is just $e^|z|$ which converges since $|z| \in \mathbb{R}$. Therefore e^z converges to a compelx number. $\square$

Fact 2. *If $z_1, z_2 \in \mathbb{C}$ then $e^{z_1}e^{z_2} = e^{z_1+z_2}$; that is, exp is a homomorphism.*

Proposition 2. *If $a, b \in \mathbb{R}$, then $e^{a+bi} = e^a(\cos(b) + i\sin(b))$.*

Proof. By fact 1, we have that $e^{a+bi} = e^a e^{bi}$. We claim that $e^{ib} = \cos b + i \sin b$. Recall the series definition of e^z,

$$e^{ib} = \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n b^n}{n!}$$

Using that $i^2 = -1$, we have

$$\begin{aligned} e^{ib} &= 1 + \frac{ib}{1} + \frac{-b^2}{2!} + \frac{-ib^3}{3!} + \frac{b^4}{4!} + \frac{ib^5}{5!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k b^{2k+1}}{(2k+1)!} \\ &= \cos(b) + i\sin(b). \end{aligned}$$

This completes the proof. □

What are the complex numbers of $|\cdot| = 1$? They must be $z = \cos\theta + i\sin\theta = e^{i\theta}$, $\theta \in \mathbb{R}$. We can use such an intuition to compute roots of complex numbers.

Proposition 3. *For a complex number z, let $R_z = \{w : \ ^n = z \ \in \mathbb{C}\}$ be th set of n^{th} roots of z. We claim that*

$$R_z = \left\{ |z|^{1/n} \exp\left(\frac{i(Arg(z) + 2\pi k)}{n}\right) \ \middle| \ k \in \mathbb{Z} \right\}$$

and $R_z \cong \mathbb{Z}/n$ if $|z| = 1$.

Proof. Take any $w \in R_z$, then there is a $k \in \mathbb{Z}$ such that

$$w^n = \left(|z|^{1/n} \exp\left(\frac{i(Arg(z) + 2\pi k)}{n}\right)\right)^n = |z| \exp\left(i(Arg(z) + 2k\pi\right) = z.$$

Furthermore define an index set for R_z, such that $w_j = |z|^{1/n} \exp\left(\frac{i(Arg(z)+2\pi j)}{n}\right)$. Then define the mapping $\phi : \mathbb{Z}_n \to R_z$ such that $j \mapsto w_j$. Clearly such a map is injective since there are n elements of both $\mathbb{Z}_n$ and R_z, and each j, w_j in those sets respectively are unique up to n. Therefore ϕ is bijective.

Now we claim that if $|z| = 1$ $\phi^{-1} = \gamma$ is a *homomorphism*, that is $\gamma(w_k w_j) \equiv \gamma(w_k) + \gamma(w_j) \mod n$. We do simple algebra

$$w_k w_j = |z|^{1/n}|z|^{1/n} e^{\frac{i(Arg(z)+2\pi k)}{n}} e^{\frac{i(Arg(z)+2\pi j)}{n}} = 1e^{\frac{i(Arg(z)+2\pi(k+j)}{n}} = e^{\frac{i(Arg(z)+2\pi(k+j))}{n}} \mod 2\pi$$

so it follows that

$$\gamma(w_k w_j) \equiv (2\pi(k+j) \mod 2\pi) \mod n \equiv (k+j) \mod n.$$

□