Math 202A — UCB, Fall 2016 — William Guss
Problem Set 6, due Wednesday October 5

Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space. $\int f$ is shorthand for $\int_X f\, d\mu$, where μ is a measure which may not be explicitly specified. m denotes Lebesgue measure on either $\mathcal{B}_{\mathbb{R}}$ or $\mathcal{L}_{\mathbb{R}}$. Unless otherwise indicated, "f is measurable" means that $f : X \to \mathbb{C}$ and f is measurable with respect to $\mathcal{M}$. L^1 refers to functions, rather than to equivalence classes of functions, unless otherwise indicated. proof

(6.1) (Folland problem 2.33) Let f_n, f be measurable. Suppose that $f_n \geq 0$ and $f_n \to f$ in measure. Show that $\int f \leq \liminf \int f_n$.

Proof. First we can always construct a sequence f_{n_k} so that $\int f_{n_k} \to \liminf \int f_{n_k}$. Now $f_{n_k} \to f$ in measure by basic analysis $((\mu(|f_{n_k} - f| \geq \epsilon))_k$ is a subsequence of $(\mu(|f_n - f| \geq \epsilon))_n$). There is a sub subsequence $f_{n_{k_j}}$ which converges to f almost everywhere by Proposition 2.30. So $\int f = \int \lim f_{n_{k_j}} = \int \liminf f_{n_{k_j}} \leq \liminf \int f_{n_{k_j}}$ by Fatou's lemma. Now $\liminf \int f_{n_{k_j}} \leq \lim \int f_{n_{k_j}}$. Now $\int f_{n_{k_j}}$ is a subsequence of the convergent sequence $\int f_{n_k}$, so it converges to that same limit. Therefore $\liminf \int f_{n_{k_j}} \leq \liminf \int f_n$. Therefore $\int f \leq \liminf f_n$. $\square$

(6.2) (Folland problem 2.36) Suppose that $E_n \in \mathcal{M}$ and $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$. Suppose that $\mathbf{1}_{E_n} \to f$ in L^1 (that is, f is measurable and $\mathbb{C}$–valued, and $\int |\mathbf{1}_{E_n} - f| \to 0$). Show that there exists a measurable set E such that $f = \mathbf{1}_E$ almost everywhere.

Proof. If $\chi_{E_n} \to f$ in the L^1 sense then $\chi_{E_n} \to f$ in measure since $\chi_{E_n} \in L^1$ $(\mu(E_n) < \epsilon)$. Let

$$\Delta_{n,\epsilon} = \{x : |\chi_{E_n}(x) - f| \geq \epsilon\}.$$

Now for all ϵ, $\mu(\Delta_{n,\epsilon}) \to 0$ as $n \to \infty$ implies that

$$\mu(\{x : x \in E_n, |1 - f| \geq \epsilon\} \cup \{x : x \notin E_n\, |0 - f| \geq \epsilon\}) \to 0$$

for all ϵ. So $\mu(\{x : f(x) \notin \{0,1\}\} = 0$, and call that set B_f. Partition X so that $X = B_f \cup E \cup (X \setminus (E \cup B_f))$, where $E = \{x : f(x) = 1\}$ then $f(x) = \chi_{B_f} g(x) + \chi_E \cdot 1 + \chi_{E^c \setminus B_f} \cdot 0$, where $g(x)$ is some crazy cooky measurable function. Now

$$\int |f - \chi_E| = \int |\chi_{B_f} g(x)| = 0$$

so $f = \chi_E$ up to a null set and χ_E is measurable so $\chi_E^{-1}(\{1\}) = E$ is measurable by measurability of the a.e. limit of measurable functions. $\square$

(6.3) (Folland problem 2.38) Assume that f_n, g_n, f, g are measurable $\mathbb{C}$–valued functions. Suppose that $f_n \to f$ in measure, and $g_n \to g$ in measure.
(a) Show that $f_n + g_n \to f + g$ in measure.

Proof. If $f_n \to f$ and $g_n \to g$ in measure let

$$\Delta_{\epsilon/2,n}(f) = \{x : |f_n - f| \geq \epsilon/2\}$$

$$\Delta_{\epsilon/2,n}(g) = \{x : |g_n - g| \geq \epsilon/2\}$$

Now observe that $|f_n + g_n - f - g| \leq |f_n - f| + |g_n - g|$ and so in the worst case $\Delta(f)$ and $\Delta(g)$ are large enough in measure that (say $\delta/2 + \delta/2$, $\delta \to 0$) that the x satisifying $|f_n + g_n - f - g| > \epsilon$ are either $x \in \Delta_{\epsilon,n}(g)$ or $x \in \Delta_{\epsilon,n}(f)$ or both then

$$x \in \Delta_{\epsilon,n}(f,g) = \{x : |g_n + f_n - g - f| \geq \epsilon\} \subset \Delta_{\epsilon,n}(g) \cup \Delta_{\epsilon,n}(f)$$

Then since these difference sets go to sets with null measure the difference sets for $f_n + g_n$ go to a null set in the limit.

□

(b) Show that if $\mu(X) < \infty$ then $f_n g_n \to fg$ in measure.

Proof. If $f_n \to f$ and $g_n \to g$ in measure then using the previous notation, recall $\mu(\Delta_{\epsilon,n}(f)) \to 0$, $\mu(\Delta_{\epsilon,n})(g) \to 0$. Now $|f_n g_n - fg| \geq \epsilon$ can be ammenable using the triangle inequality using the trick from homework one: $|f_n g_n - f_n g + f_n g - fg| = |f_n g_n - fg| \geq \epsilon$ has triangle inequality expansion $|f_n g_n - f_n g + f_n g - fg| \leq |g||f_n - f| + |f_n||g - g_n|$. So we need to show that $|g - g_n|$ is eventually small enough so that the set $|f_n||g - g_n| > \epsilon$ is small. First we know that $|f_n| > r$ on some set say D_r. Then $|f_n||g - g_n| > \epsilon$ on $D_r \cup \{x : |g - g_n| > \epsilon/r\} := D_r \cup G_{r,\epsilon}$. If $p > q$ then $D_p \subset D_q$. and since $m(X) < \infty$ we have that $\bigcap D_r = \emptyset$ implies that $\mu(D_r) \to 0$ for large enough r and furthermore that D_r has finite measure. Additionally we have that $\mu(G_{r,\epsilon}) \to 0$ as $n \to \infty \to 0$. So the set for which $|f_n||g - g_n| > \epsilon$ with fixed r for all r has measure 0 as $n \to \infty$. A similar argument applies to $|g||f_n - f| > \epsilon$. Therefore $f_n g_n \to fg$ in measure. □

(c) Show by example that the conclusion of (b) need not hold if $\mu(X)$ is not finite.

Proof. Take any $g_n \to g$ on $\mathbb{R}$ in measure Take any increasing f measurable such that $f > x$. Let $f_n == f + 1/n = g_n$. Both functions tend to f in measure since $|f_n - f| = 1/n$ and the set of x so that $|f_m(x) - f(x)| \geq 1/n$ is infinite until $m > n$ at which case $|f_m - f| < 1/n$ for all x so the difference set for $\epsilon = 1/n$ is a zeroset. Then for any ϵ find the closest $1/n$ and wait for $m > n$ and the same argument gives that for every ϵ the difference set by ϵ becomes a zeroset.

Now consider $f_n g_n$. Claim that $f_n g_n \not\to f^2$ in measure. The set of x so that $|f_n(x)f_n(x) - f^2| > 1/m$ is the set $|f(x)^2 + 2f(x)/n + 1/n^2 - f^2| > 1/m$. The LHS becomes $|2f(x)/n + 1/n^2| > 1/m$. By the monotonicity of f, $2f(x)/n > 1/n^2$ eventually and then for all x more than that eventuallity and increasing by more than $1/m$ since $f > x$, so that the set of x for which $x > 1/m$ always has infinite measure. Therefore $f_n^2 \not\to f^2$ in measure.

□

(6.4) (Folland problem 2.40) Show that in Egoroff's theorem, the hypothesis that $\mu(X)$ is finite can be replaced by the existence of $g \in L^1$ such that $|f_n| \leq g$ for all n.

Proof. We adapt Egoroff's proof. Without loss of generality assume that $f_n \to f$ everywhere on X. Then since f_n dominated we get that $\int f_n \to \int f$ and therefore $f_n \to f$ in the L^1 sense. Now consider the sets

$$E_n(k) = \bigcup_{m=n}^{\infty} \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \right\}.$$

When fixing k these sets decrease as $n \to \infty$ since there are less elements in the union. Additionally by the convergence of f_n to f pointwise we have $\bigcap_{n=1}^{\infty} E_n(k) = \emptyset$. Let $\epsilon > 0$ be given. Since f_n

converges in L^1 it therefore converges in measure and so $\mu(E_n(k)) \to 0$ for every k as $n \to \infty$ and so for each k take n_k large enough that $\mu(E_{n_k}(k)) < 2^{-k}\epsilon$.[1] Now

$$\mu\left(\bigcup_{k=1}^{\infty} E_{n_k}(k)\right) < \sum_{k=1}^{\infty} 2^{-k}\epsilon < \epsilon.$$

So the set $X \setminus E$, the compliment of the union above, has the property that for every $k > 0$ there is an N so that $\|f_m - f\| < k^{-1}$ in the sup, uniform norm. Additionaly $\mu(E) < \epsilon$ so f is uniformly continuous (as $\epsilon \to 0$) except for a zero set. $\square$

(6.5) (Folland problem 2.44) Let $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{L}_{\mathbb{R}}, m)$. Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval. Let $f : [a, b] \to \mathbb{C}$ be Lebesgue measurable. Show that for any $\varepsilon > 0$ there exist a compact set $K \subset [a, b]$ such that $m([a, b] \setminus K) < \varepsilon$ and the restriction of f to K is a continuous function.

Proof. Let $B_r \subset \mathbb{C}$ be a ball of radius r centered at the origin in $\mathbb{C}$. Now consider the *finite restriction* of f, say f_r, which is clipped by the ball in its imaginary and compex parts. Then the preimage of this ball is a measurable set, which is contained by a compact say K_r so that $\mu(K_r \cap f_r^{pre}(B_r)) = 0$, by the measurability of $f_r^{pre}(B_r)$. On this compact set f_r is measurable and L^1. Therefore there is a continuous g_r so that $\int |g_r - f_r| = 0$ and g_r vanishes outside of $[a, b]$ by a proposition of our book.

Now take a sub family of those such continuous functions say $(g_n)_{n\in\mathbb{N}}$. Every continuous function is itself L^1. Now we have a sequence of g_n continuous L^1 functions converging to f almost everywhere by transitivity of the approximation on $f_n \to f$ and $g_n \to f_n$. By Egoroff's Theorem for every ϵ there is a set $E \subset [\eth,]$ measurable such that g_n converges to f uniformly on E and $\mu(E) = \mu([a, b]) - \epsilon$. By uniform convergence and continuity of g_n it must be that f is continuous on the set E. Then by measurability of E there is a set $K \supset E$ $\mu(K \cap E) = 0$. Lastly if f continuous on E then there is a unique continuous extension of f to K which is also continuous (from basic undergraduate real analysis, Royden Prop 35 Section 7.9). Therefore $f|K$ is continuous and $\mu([a, b] \setminus K) < \epsilon$. $\square$

[1] To see that these measures are really decreasing, convergence in measure gives $E_n(k) \cap E_{n+1}(k)$ is eventually as small as we like. And so we sum infinitely many elements as small as we like (and possibly smaller as the sum continues), so eventually we sum infintieley many zeroes which must be zero ($\infty \cdot 0 = 0$). This is a high level explaination of the phenomena occuring, for a detailed and rigourous treatment of decreasing infinite series of convergent sequences, see Charles Pugh's undergraduate text, *Real Mathematical Analysis*.