Math 202A — UCB, Fall 2016 — William Guss
Problem Set 11, due Wednesday November 9

(11.1) (Folland problem 4.3) Show that every metric space is normal.

Proof. Let (X, ρ, τ) be a toplogical metric space with the standard open ball toplogy, and $\rho : X \to \mathbb{R}$ the metric. We wish to show that for any disjoint closed sets $A, B \in X$ there are disjoint open sets with $A \subset U$ and $B \subset V$.

Define the following function

$$dist : P(X) \times X \to \mathbb{R}$$

so that $dist(A, x) = \inf_{y \in A} \rho(x, y)$. First we claim that the metric is a continuous function. For any ϵ and any $x, y \in X$ such that if $\rho(x, y) < \epsilon$ then $\rho(x, y) < \epsilon$; that is tautologically speaking, ρ is continuous with respect to itself. Next we claim that $dist$ is a continuous function in its second component; that is dist $_1 = dist \circ (A, \cdot)$ is continuous. Suppose that $\rho(x, y) < \epsilon$, then

$$\begin{aligned}
|dist_1(x) - dist_2(y)| &= \left| \inf_{z \in A} \rho(x, z) - \inf_{z \in A} \rho(y, z) \right| \\
&\leq \sup_{z \in A} |\rho(x, z) - \rho(y, z)| \\
&\leq \sup_{z \in A} |\rho(x, y) + \rho(y, z) - \rho(y, z)| \\
&= \sup_{z \in A} |\rho(x, y)| < \epsilon.
\end{aligned}$$

Which follows by the traingle inequality and the properties of infimum. [1] Thus $dist_1$ is continuous for any set A.

Now consider the set $U = \{x \in X \mid dist(A, x) < dist(B, x)\}$ and $V = \{x \in X \mid dist(B, x) < dist(A, x)\}$. We claim that these sets are open. First consider $g_1 = dist(B, x) - dist(A, x)$; this function is continuous as it is the difference of two continuous functions and the preimage of $(0, \infty)$ is exactly U; that is if $dist(B, x) - dist(A, x) > 0$ then (iff) $dist(B, x) > dist(A, x)$. The pre image of open sets is open and thus U is open. Next consider the function $g_2 = dist(A, x) - dist(B, x)$. The preimage of $(0, \infty)$ is exactly V by the same argument and so V is open.

If $x \in A$ then $dist(A, x) = 0$.Suppose that $dist(B, x) = 0$, then there must be for every ϵ a $y \in B$ so that $dist(x, y) < \epsilon$. Then the ϵ neighboorhood of y without x intersects with B. Since ϵ was arbirary, we have that for every neighboorhood of x, we can find an ϵ ball inside that neighboorhood so that its intersection with B is non-empty, and thus $x \in A$ is an accumulation point of B. But since Proposition 4.1 says that $A = A \cup acc(A)$ and $B = B \cup acc(B)$ by their closedness, $acc(A) \cap acc(B) \neq$ and so $A \cap B \neq$; this is a contradiction. Therefore if $x \in A$ then $dist(B, x) > dist(A, x)$. The same argument holds for $x \in B$. Therefore $A \subset U$ and $B \subset V$.

Lastly we will show that $U \cap V = \emptyset$. If $x \in U$ then $dist(A, x) < dist(B, x)$ so it could not be that $dist(B, x) < dist(A, x)$. In the other direction if $x \in V$ then $dist(B, x) < dist(A, x)$ so it could not be that $dist(A, x) < dist(B, x)$. Therefore $x \in U \iff x \notin V$ and $U \cap V = \emptyset$.

Therefore for every disjoint pair of closed sets $A, B \subset X$ there are open sets U, V disjoint so that $A \subset U$ and $B \subset V$; hence X is normal. □

(11.2) (Folland problem 4.5) Show that every separable metric space is second countable.

[1] Note: The property that for two PSD functions $\inf f - \inf g \leq \sup(f - g)$ follows from Proposition 2.18 of Foundations of Analysis. (https://www.math.ucdavis.edu/ hunter/m125b/ch2.pdf)

Proof. Let (X, ρ, τ) be a separable metric space with the standard open ball toplogy. We would like to show that there is a $\mathcal{B} \subset \tau$ so that $\mathcal{B} \sim \mathbb{N}$ and for every $x \in X$ there is a $\mathcal{N}_X \subset \mathcal{B}$ which is a neighborhood base of x.

Since X is separable let A be a countable dense subset. Then take $\mathcal{B}$ to be the set of balls centered at $\alpha \in A$ with ρ-radius in $\mathbb{Q}$. Now we claim that for any $x \in X$ and any $U \in \tau$ with $x \in U$ there is a $V \in \mathcal{B}$ so that $x \in V \subset U$. By A dense, $\overline{A} = X = A \cup acc(A)$. We will first examine the accumulation points of A.

If $x \in acc(A)$ then for every neighboorhood of x, say U, we have $U^o \in \tau$ with $x \in U^o$ and $(U \setminus \{x\}) \cap A \neq \emptyset$. In particular for every U, U^o is also a neighboorhood with $x \in V^o \subset V$, and so its intersection with A (minus $\{x\}$) is non-empty by x an accumulation point. Thus WLOG assume U is open.

Now by U open and $U \in \tau$ we have that there is an $r > 0$ so that the ball of radius r, $B_r(x)$, centered at x is contained in U, furthermoire there is a rational $q < r$ so that $B_q(x) \subset B_r(x)$ obviously. Now $B_{q/2}(x)$ is a neighboorhood of x and its intersection with A is non-empty (and not x). Thus there is an $\alpha \in A$ so that $\alpha \in B_{q/2}(x)$. Since $\rho(\alpha, x) < q/2$ and if $y \in X$ is such that $\rho(\alpha, y) < q/2$ then

$$\rho(x, y) \leq \rho(x, \alpha) + \rho(\alpha, y) < q.$$

Therefore $B_{q/2}(\alpha) \subset B_q(x)$ and $x \in B_{q/2}(\alpha)$. Finally by $B_q(x) \subset U$ we have that $B_{q/2}(\alpha) \subset U$.

Now take the following collection let $\mathcal{N}_x \subset \mathcal{B}$ be the collection of all such balls; that is, for every neighboorhood of x, U, there is a $B_{q/2}(\alpha) = V \in \mathcal{N}_x$ so that $V \subset U$. And for all $V \in \mathcal{N}_x$, $x \in V$. This collection exists by the above constructive proof and is countable since $\mathcal{B}$ is countable. $\square$

(11.3) (Folland problem 4.6) Let $\mathcal{E}$ be the collection of all intervals $(a, b] \subset \mathbb{R}$ such that $a, b \in \mathbb{R}$ and $a < b$. (Thus $\pm\infty$ are excluded.)

(a) Show that $\mathcal{E}$ is a base for a topology τ on $\mathbb{R}$ in which each element of $\mathcal{E}$ is both open and closed.

Proof. If $\mathcal{E}$ is a base for a toplogy τ on $\mathbb{R}$ then for every $U \in \tau$ which contains a point x there is a $E \in \mathcal{E}$ so that $x \in E \subset U$. Additionally $\mathcal{E} \subset \tau$, thus every E is open.

Now for any $E = (a, b]$, $\mathbb{R} \setminus E = (-\infty, a] \cup (b, \infty)$. Consider the families of opensets

$$\mathcal{L}_E = \{(a - n, a - n + 1] \ : \ n = 1, 2, \dots\}$$
$$\mathcal{R}_E = \{(b + n - 1, b + n] \ : n = 1, 2, \dots\}$$

Then it is immediate that

$$\mathbb{R} \setminus E = \left(\bigcup_{F \in \mathcal{L}_E} F \right) \cup \left(\bigcup_{G \in \mathcal{R}_E} G \right)$$

and so $\mathbb{R} \setminus E$ is the aribrary union of open sets and so must it must be open. By definition the compliment of an open set is closed so $\mathbb{R} \setminus (\mathbb{R} \setminus E)) = E$ is closed.

Therefore for any $E \in \mathcal{E}$, E is clopen. $\square$

(b) Show that τ is first countable but not second countable.

Proof. First for any $x \in \mathbb{R}$ take the subfamily $\mathcal{N}_x \subset \mathcal{E}$ of shrinking rational intervals around x; that is,

$$\mathcal{N}_x = \{(x - p, x] \ : \ p \in \mathbb{Q} \setminus \{0\}\}.$$

Then for any $U \in \tau$ containg x, from $\mathcal{E}$ a base for τ we have that there is an $(a, b] \subset U$ so that $x \in (a, b]$. Then $a < x \leq b$, but by the construction of $\mathbb{R}$ there is a rational q so that $a < x - q < x$ (equivalently there is a rational so that $a - x < -q < 0$) so there is a $V \in \mathcal{N}_x$ so that $x \in V \subset (a, b] \subset U$, and so $\mathcal{N}_x$ is a countable base. Since x was arbitrary, X is first countable.

Importantly, for every x and $a < x$ the set $(a, x]$ is an open neighboorhood of x so every neighboorhood base of x must contain some $U_x \subset (a, x]$. That is any base of the topology must therefore contain all such U_x. Since there are uncountably many U_x then a subset of any base of the topology is bijective to $\mathbb{R}$ and so any base must be uncountable. Therefore $(\mathbb{R}, \tau)$ is not second countable. $\square$

(c) Show that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$ with respect to τ.

Proof. We need show that the closure of $\mathbb{Q}$ is the wholse space $\mathbb{R}$. Nameley $\overline{Q} = Z = \mathbb{Q} \cup acc(\mathbb{Q}$.

If $x \in \mathbb{R}$ and $x \notin \mathbb{Q}$ we need show that $x \in acc(\mathbb{Q})$. Take any open neighborhood of x, say U. Then by $\mathcal{E}$ a base, there is a unique[2] $(a, b] \subset U$ so that $x \in (a, b]$. Then we claim that $((a, b] \setminus \{x\}) \cap \mathbb{Q} \neq \emptyset$. Since $x \notin \mathbb{Q}$ and $a < x \leq b$ then $(a, b] \setminus \{x\} = (a, b) \neq \emptyset$. Then by the properties of $\mathbb{R}$ there is a rational between any two numbers [3], thus $(a, b) \cap \mathbb{Q} \neq \emptyset$. This holds for all neighborhoods and $x \in acc(\mathbb{Q})$.

Therefore $x \in \mathbb{R} \implies x \in \mathbb{Q} \cup x \in acc(\mathbb{Q})$ and $\overline{\mathbb{Q}} = \mathbb{R}$. $\square$

(11.4) (Folland problem 4.7) Let X be a topological space. Let $S = (x_n : n \in \mathbb{N})$ be a sequence of elements of X. Show that if X is first countable, then a point $x \in X$ is a cluster point of the sequence if and only if some subsequence of S converges to x.

Proof. If there is a subsequence x_{n_j} so that x_{n_j} converges to x then for every open neighborhood of X say U there is a J so that for all $j \geq J$, $x_{n_j} \in U$. Since $\{x_{n_j} \; : j \geq J\}$ is infinite then for every neighborhood U of x, $x_k \in U$ for infiniteley many k (take $k = n_j$, $j \geq J$), and x is a cluster point of the sequence.

In the other direction, suppose that x is a cluster point of X. By the first countability of X take the countable neighborhood base of x, $\mathcal{N}_x$, with members V_j. Without loss of generality we can let V_j be a nested sequence by letting each V_j be the finite intersection of the all of the previous j in the sequence[4]. Then for every neighborhood of X, say U there $x_k \in U$ for infiniteley many k. Additionally we have that there exists an j so that $x \in V_j \subset U$.

Pick x_{n_1} to be an element of the sequence S in V_1. We can do this since there infintely many such x_j in V_1. Pick $x_{n_2} \in V_2$ with the restriction that $n_2 > n_1$. We can again do this because there are infiniteley many j so that $j \geq n_1$ and $x_j \in V_2$. In general pick x_{n_p} so that $n_p > n_{p-1}$ and $x_{n_p} \in V_p$. This is possible because there are infintieley many j so that $j \geq n_{p-1}$ and $x_j \in V_p$. Form a subsequence x_{n_p} under this process.

Then for every neighborhood U of x there is a V_p so that $x \in V_p \subset U$ and additionally for all $q \geq p$ $x_{n_q} \in V_q$ by our nested construction of $\mathcal{N}_x$. Hence $x_{n_q} \in U$. Therefore the subsequence converges to x. $\square$

(11.5) (Folland problem 4.9) (Partnered With Lucas) Let X be a linearly ordered set, equipped with the ordered topology, τ.
(a) Show that if $a, b \in X$ and $a < b$ then there exist pairwise disjoint open sets containing a, b respectiveley, such that $x < y$ wherenever $x \in U$ and $y \in V$.

[2] The subset must have a supremum, x and because τ contains all of the open intervals on $\mathbb{R}$, τ is hausdorf and this supremum is unique.

[3] This is not a toplogical fact and can be proven using Dedekind cuts and the ordering of $\mathbb{R}$.

[4] See page 116 of Folland

Proof. Let $a < b$. We will break this up into two cases. First suppose that there is no c so that $a < c < b$. Then take $V = \{x : x > a\}$ and $U = \{x : x < b\}$ open then $a < b$ so $a \in U$ and $b > a$ so $b \in V$. since there are no c with $a < c < b$ and thus $U \cap V = \emptyset$. Furthermore if $x \in U$ then $x < b$ and if $y \in V \setminus \{x\}$ we have that $b < y$. Thus $x < y$ for all $x \in U$ and for all $y \in V$ using transitivity.

Suppose there is such a c. Then let $U = \{x : x < c\}$ and $V = \{x : x > c\}$. We know that $U \cap V = \{x : x < c \text{ and } x > c\} = \emptyset$. These sets are open and $a < c$ implies $a \in U$ and $b > c$ implies $b \in V$. Take any $x \in U$ and any $y \in V$ then $x < c$ and $c < y$ and by transitivity $x < y$.

Thus for any $a, b \in X$ with $a < b$ there are disjoint opensets containing a, b respectively.

□

(a') ..

Proof. Suppose that τ' is another topology with the property listed adn $\tau' \subset \tau$. Then we will show that $\mathcal{E} \in \tau'$ where $\mathcal{E}$ are the generating sets in the problem statement, but then since τ is generated by $\mathcal{E}$, τ is the smallest topology containing $\mathcal{E}$, which will contradict $\tau' \subset \tau$.

First let b be given in $\mathbb{R}$. Then for every $c < b$ there are disjoint open sets (in τ') , $L(c)$, $R(c)$ containing c and b respectiveley so that $L(c) \cap R(c) = \emptyset$. Then for every c, and for every $x \in L(c)$ and for every $y \in R(c)$, $x < y$; in particular every element of every $L(c)$ is strictly less than b. Thus

$$\bigcup_{c<b} L(c) = \{x : x < b\}$$

is an open set in τ'. Applying the symmetric argument, for every $c > b$ there are disjoint open sets (in τ') , $L(c)$, $R(c)$ containing b and c respectiveley so that $L(c) \cap R(c) = \emptyset$. Then for every c, and for every $x \in R(c)$ and for every $y \in L(c)$, $y < x$; in particular every element of every $R(c)$ is strictly greater than b. Thus

$$\bigcup_{b<c} R(c) = \{x : x > b\}$$

is an open set in τ'. Since b was arbitrary, we have that $\mathcal{E} \subset \tau'$ which contradicts $\tau' \subset \tau$. Thus $\tau \subset \tau'$. □

(b) ..

Proof. If $Y \subset X$, then let $\mathcal{E}_Y$ be the family of sets $\{x \in Y \mid x > b, b \in Y\}$ and $\{x \in \ Y \mid x < b, b \in Y\}$. Then let τ^Y be the order topology generated by $\mathcal{E}_Y$, andτ^X be theorder topology generated on the whole space by the family $\mathcal{E}_X$. Next let $\tau_Y = \{U \cap Y \ : U \in \tau^X\}$ be the relative topology on Y with respect to τ_X.

It is given that τ^Y is the smallest topology which contains $\mathcal{E}_Y$. Now take consider $\mathfrak{s} = \{U \cap Y \ : U \in \mathcal{E}_X\}$. Then if $E \in \mathfrak{s}$, we have that for some $b \in Y$ either $E = \{x \in X \cap Y = Y \ : \ x > b\}$ or $E = \{x \in X \cap Y \ : \ x < b\}$. Thus $\mathfrak{s} \subset \mathcal{E}_Y$. In the reverse direction, if $E \in \mathcal{E}_Y$, we have that for some $b \in Y$ either $E = \{x \in Y \ : \ x > b\}$ or $E = \{x \in Y \ : \ x < b\}$. Thus $\mathfrak{s} \supset \mathcal{E}_Y$. Thus $\mathfrak{s} = \mathcal{E}_Y$. But then $\mathcal{E}_Y \subset \tau_Y$ and so $\tau_Y \supset \tau^Y$; this completes the proof. □

(b') ..

Proof. Let $X = \mathbb{R}^2$ be endowed with the following ordering. We say $a < b$, $a, b \in \mathbb{R}^2$ iff $\pi_1(a) < \pi_1(b)$ or $(\pi_1(a) = \pi_1(b) \wedge \pi_2(a) < \pi_2(b).)$ This ordering is a complete, transitive, and antisemetric so we can generate the order topology τ_X on X. Then we let Y be unit square in this space, and $\mathcal{E}_Y$ be the family of sets of the form $\{x \in Y : x > b, b \in Y\}$ and $\{x \in Y : x < b, b \in Y\}$ which generate the order topology τ^Y.

Suppose $\gamma \in [0,1]$ is a constant and $U_Y = \{\gamma\} \times [0, 1/2)$. We can express U_Y as $\{\gamma\} \times (-1/2, 1/2) \cap Y$ and $\{\gamma\} \times (-1/2, 1/2) = \{\gamma\} \times (-1/2, \infty) \cap \{\gamma\} \times (-\infty, 1/2)$ is the interesection of open sets in τ_X. Therefore U_Y is in the relative topology τ_Y.

Suppose U_Y were in the order topology then there must be an open set in the base of τ^Y, say $(\gamma, 0) \in O$, so that $O \subset U_Y$ by Proposition 4.3. Furthermore as Proposition 4.4 gives that O is the union of finiteley many interesections of $\mathcal{E}_Y$, but this could not because U_Y can only be constructed using infiniteley many intersections.

Therefore the order topology os strictly weaker than the relative topology.

$\square$

(c) ..

Proof. Let $a < b, a, b \in \mathbb{R}$. Then we will show that the order topology τ contains the generating sets of the standard topology κ and visa versa. Thus they are the smallest two sets which contain their complimentary generating sets so the must be equal.

First recall that $\mathcal{E}_\kappa = \{(a,b) : a, b \in \mathbb{R}\} \subset \kappa$ generates κ, and

$$\mathcal{E}_\tau = \{(a,b) : a \in \mathbb{R} \wedge b = \infty \vee a = -\infty \wedge b \in \mathbb{R}\}$$

Then we have for any $c \in \mathbb{R}$, $(-\infty, c) = \bigcup_{d<c}(d, c)$ and $(c, \infty) = \bigcup_{d>c}(c, d)$; thus for any $E \in \mathcal{E}_\gamma$ we have that $E \in \kappa$. Otherwise, if $E \in \mathcal{E}_\kappa$ we know that $E = (a,b) = (-\infty, b) \cap (a, \infty) \in \tau$.

Therefore $\tau \subset \kappa \subset \tau$ and $\tau = \kappa$. $\square$

(11.6) (Folland problem 4.10) Let (X, τ) be a topological space.
(a) Show that X is connected if and only if $\emptyset$ and X are the only clopen subsets of X.

Proof. For the sake of contradiciton, suppose that X is not connected. Then there are non-empty open sets so that $U \cap V = \emptyset$ and $U \cup V = X$. Since X is clopen, then $X \cap U$ is open. additionally $C \cap V$ is open. Next $X \setminus V = (V \sqcup U) \setminus V = (V \setminus V) \sqcup (U \setminus V) = U \setminus V = U$ by disjoitness; that is U is V compliment and so U is closed. Finally U is a clopen set which is non-empty and not the whole set X which contradicts our hypothesis that $X, \emptyset$ are the only clopen sets; thus X must be connected. $\square$

(b) Suppose that E_α are connected subsets of X indexed by some set $\mathcal{A} \ni \alpha$. If the intersection of them is not empty, then the union of all of them is connected.

Proof. Given $\{E_\alpha\}$ as above we would like to show

$$\bigcap_{\alpha \in \mathcal{A}} E_\alpha = F \neq \emptyset \implies \bigcup_{\alpha \in \mathcal{A}} E_\alpha = G \text{ connected}$$

Suppose for the sake of contradiction that G is not connected. Then there exist $U, V \in \tau$ so that $A \sqcup B = G$ and $A, B \neq \emptyset, A \cap B = \emptyset$.

We claim that there does not exist an $\alpha \in \mathcal{A}$ so that E_α intersects both A and B. If this were true then $E_\alpha \cap A$ is a non-empty open set in the relative topology τ_{E_α}, since the relative topology inherits its open sets from the topology via intersection. Additionally $E_\alpha \cap B$ is a non-empty open set in τ_{E_α} and thus its compliment is closed. Thus $E_\alpha \cap A$ is a non-empty clopen strict subset of E_α, but this contradicts E_α connected by (a). Therefore $E_\alpha \subset A$ or $E_\alpha \subset B$ but both cannot be true.

Let $\mathcal{A}_A = \{\alpha : E_\alpha \subset A, E_\alpha \cap B = \emptyset\}$ and $\mathcal{A}_B = \{\alpha : E_\alpha \subset B, E_\alpha \cap A = \emptyset\}$. By the above logic $\mathcal{A} = \mathcal{A}_A \sqcup \mathcal{A}_B$ and additionally

$$G = \left(\bigcup_{\alpha \in \mathcal{A}_A} E_\alpha\right) \sqcup \left(\bigcup_{\alpha \in \mathcal{A}_B} E_\alpha\right) = A \sqcup B.$$

Now if $E_\alpha \in \mathcal{A}_A$ and $E_\beta \in \mathcal{A}_B$ then $E_\alpha \subset A$ and $E_\beta \subset B$ implies that $E_\alpha \cap E_\beta \subset A \cap B = \emptyset$. Thus

$$\bigcap_{\gamma \in \mathcal{A}} E_\gamma \subset E_\alpha \cap E_\beta = \emptyset$$

therefore $F = \emptyset$ which contradicts $F \neq \emptyset$. Thus G must be connected. □

(c) Show that the closure of any connected set is connected.

Proof. Let (E, τ_E) be a connected topological subspace with τ_E the subspace topology. Suppose that $(cl(E), \tau_{cl(E)})$ is not connected, for the sake of contradiction.

There are then clopen sets $A \sqcup B = cl(E)$ so that $A \cap B = \emptyset$. Then in the global topology τ, there are U, V open so that $U \cap cl(E) = A$ and $V \cap cl(E) = B$. Then $U \cap E \subset A$ and $V \cap E \subset B$. Furthermore $U \cap E \in \tau_E$ and $V \cap E \in \tau_E$ because U, V open with respect to the global topology. We then use A, B disjoint to yield $(V \cap E) \cap (U \cap E) = \emptyset$. Furthermore

$$\begin{aligned}
(V \cap E) \cup (U \cap E) &= ((V \cap E) \cup U) \cap ((V \cap E) \cup E), \\
&= ((V \cup U) \cap (E \cup U)) \cap ((V \cup E) \cap (E \cup E)), \\
&= (V \cup U) \cap (E \cup U) \cap (V \cup E) \cap E, \\
&= ((V \cup U) \cap E) \cap (E \cup U) \cap (V \cup E), \\
&= (cl(E) \cap E) \cap (E \cup U) \cap (V \cup E), \\
&= E \cap (E \cup U) \cap (E \cup V), \\
&= E \cap E \cap (U \cup V), \\
&= E \cap (U \cup V), \\
&= E \cap cl(E), \\
&= E.
\end{aligned}$$

The above set algebra follows using the distributive law and that $U \cup V \supset (U \cap E) \cup (V \cap E) = A \cup B = cl(E)$. Finally since $(V \cap E), (U \cap E)$ form a partition of E and are open sets, they are clopen (intersecting with E, taking complimets, ... the usual). □

(d) Show that each point $x \in X$ is contained ina unique maximal connected subset of X, and that subset is closed with respect to τ.

Proof. We would like to show that for every x there is a unique maximal connected subspace which contains it; that is, x cannot be contained in two maximal connected components. Intuively this makes perfect sense since if $x \in E_1, E_2$ maximally connected then $E_1 \cap E_2 \neq \emptyset$ which would imply that $E_1 \cup E_2$ is maximally connected, contradictinct the maximality of E_1 and E_2.

More formally, we will adopt a methodology from Charles Pugh. First let $\sim$ be the realtion on x, so that $x, y \in X$ have the property that $x \sim y$ if and only if there is a connected subspace of X, say E so that $x, y \in E$.

We will first show that this is an equivalence relation. First if $x \sim y$ then there is an E so that $x, y \in E$ so for the same E, $y, x \in E$, and thus $y \sim x$. Secondly let $E = \{x\}$ for some fixed x, then τ_E is a conntected topology; that is E is clopen and $\emptyset$ is clopen, and by (a), $x \sim x$. Finally if $x \sim y$ and $y \sim z$ then there are connected subspaces E, F so that $x, y \in E$ and $y, z \in F$. By (b), $E \cap F \supset \{y\}$ implies that $E \cup F$ is connected. Thus $x, y, z \in E \cup F$ implies $x \sim z$.

Now we claim that every equivalence class is a unique maximal connected subspace of X. Suppose not, then take the offending equivalence class on $[x]_\sim$. Suppose there is another connected subspace, say $B \supset [x]_\sim$ which is strictly larger. Then there is a $y \in B$ so that $y \notin [x]_\sim$, but then for every $x \in [x]_\sim$, $x \in B$, so then $x \sim y$ which implies $y \in [x]_\sim$; a contradiction to $y \in B$. Therefore such a B does not exist and $[x]_\sim$ is maximal. Now for every $x \in X$, $[x]_\sim$ is unique because equivalence classes form a unique partition of the space by the fundamental theorem of equivalence relations.

Finally we would like to show that $[x]_\sim$ is closed for every x. By the previous (c) $cl([x]_\sim)$ is a connected, closed subspace containing $[x]_\sim$, but because $[x]_\sim$ is maximal and unique, $cl([x]_\sim) = [x]_\sim$ and so $acc([x]_\sim) \subset [x]_\sim$ which implies that $[x]_\sim$ is closed.

□

(e) Show that any two connected components of X are either identical or disjoint.

Proof. Observe that $\{[x]_\sim\}_{x\in X} = X/\sim$ and by the fundamental theorem of equivalence relations $\sim$ forms a partition of X, having the properties of (e). This completes the proof. □

(11.7) (Folland problem 4.11) Show that the closure ($cl(\cdot) = \bar{\cdot}$)[5] of a union of finitely many subsets is equal to the union of their closures.

Proof. Let (X, τ) be a topological space. We first show that for any two sets A, B, $cl(A) \cup cl(B) = cl(A \cup B)$. Then we will use induction to generalize the calim to finiteley many sets.

First we know that $A \subset A \cup B \subset cl(A \cup B)$ and $cl(A \cup B)$ is a closed set containing A. Since $cl(A)$ is the smallest such closed set, $cl(A) \subset cl(A \cup B)$. Symmetrically $B \subset A \cup B \subset cl(A \cup B)$ and $cl(A \cup B)$ is a closed set containing B. Since $cl(B)$ is the smallest such closed set, $cl(B) \subset cl(A \cup B)$.

Secondly, $cl(A \cup B)$ is the smallest closed set containing $A \cup B$. Furthermore $cl(A) \cup cl(B) = A \cup B \cup acc(A) \cup acc(B) \supset A \cup B$. But then since $cl(A) \cup cl(B)$ closed, the minimality of $cl(A \cup B)$ gives that $cl(A) \cup cl(B) \supset cl(A \cup B)$; thus $cl(A \cup B) = cl(A) \cup cl(B)$.

Now let $\{K_j\}$ be a finite family of subsets of X. Then clearly $cl(K_1) = cl(K_1)$. Now suppose that

$$cl\left(\bigcup_{j=1}^{k} K_j\right) = \bigcup_{j=1}^{k} cl(K_j) = cl(\mathfrak{K}_k).$$

where $\mathfrak{K}_k$ is the union of the K_j up to k. Then $cl(\mathfrak{K}_k) \cup cl(K_{k+1}) = cl(\mathfrak{K}_k \cup K_{k+1})$ by the previous part of the proof. Finally

$$\bigcup_{j=1}^{k+1} cl(K_j) = cl(\mathfrak{K}_k \cup K_{k+1}) = cl\left(\bigcup_{j=1}^{k} K_j \cup K_{k+1}\right) = cl\left(\bigcup_{j=1}^{k+1} K_j\right).$$

Thus the proof is complete. □

[5] I switched notation mid-assignment. Sorry–

(11.8) (Folland problem 4.13) Let (X, τ) be a topological space. Let U be open in X and let A be dense in X. Show that $cl_X(U) = cl_X(U \cap A)$.

Proof. We will show the statement using inclusions.

First we will show that for any non-empty open (w.r.t the subspace topology) subset of U, say V , that $V \cap (U \cap A)$ is non-empty; that is $U \cap A$ is dense in U equipped with the subspace topology. First because U is open and V is open in the subspace topology, V is open with respect to the global topology. Therefore $V \cap A$ is non-empty by the density of A. Finally since $V \subset U$, we have that $V \cap (U \cap A) \neq \emptyset$. Thus since V was arbitrary, $U \cap A$ is a dense subset of U equipped with the subspace topology. Next the closure of $A \cap U$ in the subspace topology $cl_U(U \cap A) = U$.

Now for any subset T of U we have

$$cl_X(T) = \bigcap_{\substack{E \subset X \text{ closed w.r.t } \tau \\ E \supseteq U}} E$$

and cooorespondingly the closure in the subspace topology is

$$cl_U(T) = \bigcap_{\substack{E \subset X \text{ closed w.r.t } \tau_U \\ E \supseteq U}} E = \bigcap_{\substack{E \subset X \text{ closed w.r.t } \tau \\ E \supseteq U}} E \cap U = U \cap cl_X(U)$$

Thus $U \cap cl_X(U \cap A) = cl_U(U \cap A) = U \subset cl_X(U)$. But then since $cl_X(U \cap A) = U \cap A \cup (acc(U \cap A))$, we have that $U \cap cl_X(U \cap A) = cl_X(U \cap A)$, and thus $\mathrm{cl}_X(U \cap A) \subset cl_X(U)$.

In the other direction observe that $cl_X(U) \cap U = cl_X(U) = U = cl_U(U \cap A)$ and so if $x \in U$ then $x \in cl_X(U \cap A)$. It remains to show that if $x \in acc(U)$ then $x \in cl_X(U \cap A)$. If $x \in acc(U)$ then for all neighborhoods of x, say V, the interesection of $V \setminus \{x\}$ with U is not trivial. Therefore $V^o \cap U$ is an open set in U and so $V \cap U \cap A$ is not trivial; and so $x \in acc(U \cap V)$ and $x \in cl_X(U \cap A)$. Therefore $cl_X(U) \subset cl_X(U \cap A)$.

Thus $cl_X(U) = cl_X(A \cap U)$. $\square$