MATH H104: Homework 9

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59.

Theorem 1. *If $\sum a_n$ converges and $a_n \geq 0$, then show $\sum \sqrt{a_n}/n$ converges.*

Proof. Let $x = (\sqrt{a_n})_n$, $y = \left(\frac{1}{n}\right)_n$. Clearly $y \in \ell_1$, and since $\sum a_n \to c$, $a_n \to 0$ implies that $\sqrt{a_n} \to 0$. Therefore, $x \in \ell_1$. Since ℓ_1 is an inner product space, the cauchy schwartz inequality gives,

$$0 \leq \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} = \langle x, y \rangle \leq |x||y| = \sqrt{\sum_{n=1}^{\infty} a_n} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \sqrt{\frac{c}{6}}\pi$$

and so the series is bounded and therefore converges. □

61. Consider the following $\{a_n\} \in \ell_1$. We say that $a_n = 1/4^n$ if n odd and $a_n = 1/2^n$ otherwise. Clearly

$$0 < \sum_{n \in \mathbb{N}} a_n = \sum_{n \text{ odd}} \frac{1}{4^n} + \sum_{n \text{ even}} \frac{1}{2^n} < \sum \frac{1}{2^n} < \sum \frac{1}{n^2} = \frac{\pi^2}{6}.$$

So the series converges. Let $\rho_N = \sup_{n>N} |a_{n+1}|/|a_n| = \sup_{n>N} 2^n = \infty$. So clearly $\rho = \lim \rho_N = \infty$, and yet the series converges. If we were to suppose that $\lambda = \rho$ then the test would be wrong since $\lambda > 1$ implies divergence. So it must be the case that the test is inconclusive when $\rho \geq 1$.

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68. Here is confirmation of the existence of an unrelation between the convergence of a series and the related infinite product,.

(a) Suppose a series is defined as the infinite sum of a sequence $a_k = (-1)^k/\sqrt{k}$. We first show that such a series converges. Consider that $a_k \leq f(x) = (-1)^x x^{-0.5}$ for all $x = k$. So we simply must show convergence of the improper integral of $f(x)$. Recall that $(-1)^x = e^{i\pi x}$, then

$$\int_1^\infty (-1)^x x^{-0.5}\ dx \sim \int_{\mathbb{R}+} e^{i\pi x} x^{-0.5}\ dx = \mathcal{L}\{t^{-0.5}\} = \frac{\Gamma(1/2)}{(\pi)^{\frac{1}{2}}}$$

So at least the series converges. Consider the infinite product in terms of its partial products. Specifically we consider the partial products in pairs, $c_n * c_{n+1} = (1 + k^{-0.5} + (k+1)^{-0.5} + (k^2+k)^{-0.5})$ and so

$$\prod_{k=1}^{\infty}(1 + a_k) = \prod_{k=1}^{\infty}(1 + (2k)^{-0.5} + (2k+1)^{-0.5} + (2k(2k+1))^{-0.5})$$

which converges if and only if $\sum_{k=1}^\infty b_k + c_k + \frac{1}{\sqrt{k}}$ converges, which it does not. Therefore the infinite product can't converge.

(b) Let $b_k = e_k/k + (-1)^k/\sqrt{k}$. Clearly $\sum b_k$ diverges since $\sum b_k = \sum e_k/k + \sum(-1)^k/\sqrt{k} \geq 0.5 \sum 1/n + (1-\sqrt{2})\zeta(1/2) = \infty$. However, by performing the same grouping of two test on the infinite product we get that the product converges if and only if

$$\sum \frac{1}{n} + \frac{1}{\sqrt{n}} - \frac{1}{n\sqrt{n+1}} - \frac{1}{\sqrt{n}\sqrt{n+1}} - \frac{1}{\sqrt{n+1}}$$

converges. This is true if and only if $\sum n^{-3/2}$ converges (which it does by the integral test.) Thus, here is an example where the infinite product converges and the sum does not.

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