Math 215A — UCB, Spring 2017 — William Guss
Partners: Alekos, Chris
Selected Problems: 2

(3.1) *Show that the following sets form smooth submanifolds of the real vector space of $n \times n$ matrices with real coefficents:*

$$GL(n), SL(n), O(n), SO(n) \subset Mat(n, n).$$

Determine[1] *their dimensions, the number of components and show that multiplication and inversion are smooth maps for these groups.*

Solution. Before we consider the subsets given, we verify an assumption of the problem.

Lemma 0.1. *$Mat(n, n)$ is a smooth manifold.*

Proof. We will first that this space is a smooth manifold. To do so, consider that $Mat(n, n) = \mathbb{R}^n \otimes \mathbb{R}^n$ and we can give the diffeomorphism:

$$Mat(n, n) \ni M \mapsto (M_{floor(i/n),(i \mod n)})_{i=1}^{n^2} \in \mathbb{R}^{n^2}$$

by rearrangement of coordinates. Since $\mathbb{R}^{n^2}$ is a smooth manifold, we have that $Mat(n, n)$ is a smooth manifold of dimension n^2. □

Lemma 0.2. *If M is a smooth manifold and $N \subset M$ is an open subset then N is a smooth submanifold of M and $dim(M) = dim(N)$.*

Proof. First if $N \subset M$ then in the inherited subspace topology N is also Hausdorff and second countable from standard topology. Consider the following collection of charts

$$\mathcal{A}_N = \{U \cap N, \phi|_{U \cap N} \;\Big|\; U \cap N \neq \emptyset (U, \phi) \in \mathcal{A}_M\}.$$

Since $U \cap N$ is open $\phi|_{U \cap N}$ is a diffeomorphism to $\mathbb{R}^{\dim(M)}$. Furthermore N is covered as the atlas $\mathcal{A}_M$ covers M with the union of its domain collection. Therefore $\mathcal{A}_N$ forms a smooth atlas on N. Thereafter yield the smooth structure under the equivalence class of compatible atlases, and then N is a smooth manifold with dimension $\dim(M)$. □

We will now address each group by case.

- $GL(n)$. Recall that the general linear group is defined such that

  $$GL(n) = \{M \in Mat(n, n) \;\Big|\; det(M) \neq 0\},$$

  We claim that this space is a submanifold. To see this, take the map $\det : M \to \mathbb{R}$ and under the standard topology of $Mat(n, n)$, det is continuous map. This is true as det is the composition of multiplication and addition operations on components of elements of M. Since $GL(n) = \det^{-1}[(-\infty, 0) \cup (0, \infty)]$ we have that $GL(n)$ is open by definition of continuity. Therefore $GL(n)$ is a smooth submanifold of $Mat(n, n)$ inheriting the subspace topology and intersecting smooth structure of $Mat(n, n)$ by the previous lemma.

  We will now show that $GL(n)$ is a Lie group. First consider matrix multiplication as an operation in $GL(n)$, we will recall from linear algebra that this operation is closed and $GL(n)$

[1] Not sure if you also want us to prove this.

forms a group. As the product of smooth manifolds is smooth, it suffices to show that $mul : GL(n) \times GL(n) \to GL(n)$ is a smooth map. Take (U, ϕ) from the smooth structure on $GL(n) \times GL(n)$ so that $\phi : GL(n) \times GL(n) \to \mathbb{R}^{n^2} \times \mathbb{R}^{n^2}$. Then without loss of generality there is a $V = mul(U)$ and ψ so that (V, ψ) is in the smooth structure of th codomain $GL(n) = mul(GL(n), GL(n))$. We will consider the composition of these maps in local coordinates; that is

$$\psi \circ mul \circ \phi^{-1} : (x^1, x^2) \mapsto \psi \circ \left(\sum_{k=1}^{m} \phi_1^{-1}(x^1_{ik}) \cdot \phi_2^{-1}(x^2_{kj}) \right)_{i,j}$$

is the composition of smooth operations. In particular multiplication and summation in $\mathbb{R}^{n^2}$ are smooth and continuous. Therefore mul is smooth.

Now consider the group inverse $inv : GL(n) \to GL(n)$ and again take an arbitrary (U, ϕ) but this time in $GL(n)$. Then let $V = inv(V)$ so that (V, ψ) is without loss of generality a chart in the smooth structure of $GL(n)$. We will consider the the composition of these maps in local coordinates; that is,

$$\psi \circ inv \circ \phi^{-1} : x \mapsto \psi \circ \left(\frac{det((\phi^{-1}(x)_{[ij]})}{det(\phi_1^{-1}(x))} \right)_{i,j}$$

is the composition of smooth operations. In particular the determinant of matrices $\phi^{-1}(x)_{[ij]})^2$ is a polynomial function of the elements and so it is a smooth function. Therefore the map is smooth. This makes $GL(n)$ a Lie group.

$GL(n)$ has two connected components.

- $O(n)$. To show that the orthogonal group is a manifold we will use theorems presented in class. In particular, consider the smooth map $\phi(M) = M^T M - id$ to $Mat(n, n)$. Then clearly $\phi^{-1}(0) = O(n)$ so using that the inverse of a regular value is a submanifold (smooth) we get that $O(n)$ is a manifold. Since $O(n)$ is an embedded submanifold of $GL(n)$ so we yield that the group operations on $O(n)$ are aslo smooth and $O(n)$ is a Lie group. The key is here is that $mul(O(n), O(n)) \subset O(n)$ and $inv(O(n)) \subset O(n)$.

  $O(n)$ has two connected components which are necissary in the classification of $GL(n)$.

  As for the dimension of $O(n)$, we need consider the kernal dimension of the linear map $D\phi$ of tangent spaces in concordance with the lectures. In this case the matrix derivative of ϕ is $D(M^T M) = M^T \cdot + (M^T \cdot)^T$ and thus the operator on tangent spaces is a symmetric operator by the invertibility of M. Therefore the dimnension of $O(n)$ is that of the set of difference between $dim(GL(n))$ and that of $n \times n$ symmetric matrices. We consider combinations of free elements in matrices and yield

$$dim(O(n)) = n^2 - \sum_{k=1}^{n} k = n(n+1)/2.$$

- $SL(n)$. Consider the special linear group. We will show that is a closed subanifold of $GL(n)$ by showing that 1 is a regular value of $det : Mat(n, n) \to \mathbb{R}$. First by definitio we have $SL(n) = f^{-1}(1)$. To show that 1 is a regular value of det, suppose there is some matrix $A \in SL(n)$ which is a critical point. Then every matrix $A_{[ij]}$ where we adopt notation from earlier must have determinant 0. But the only possible such matrix herefore has determinant

[2] Removing the ith row and jth column.

0, and so all points of $SL(n)$ are not critical and since $SL(N) = f^{-1}(1)$, the special linear group is a closed submanifold of $GL(n)$.

Furthermore, $SL(n)$ inherits the the smoothness of matrix multiplication and inversion from $GL(n)$

As for connected components, if $det(A) > 0, A \in GL(n)$ then A path connected to $O^{+}(n)$. Since path components and components coincide on smooth manifolds, the connectedness of $O^{+}(n)$ yields $SL(n)$ conncted. This space has dimension $n^2 - 1$.

- $SO(n)$. Since $SO(n)$ is a connected subset of $O(n)$, and the inverse image of $\{1\} \subset \{\pm 1\}$ through det we have that it is a submanifold of $O(n)$. Again $SO(n)$ inherits the smoothness of the group operations from $GL(n)$.

(3.2) *Show that the Mobiusband M* admits a non-vanishing vecotr field but is not parallelizable.

This map is clearly smooth as $\binom{0}{1}$ is a constant vector.

2. Let $U = (-1,1) \times [(\frac{1}{2}, 1] \cup [-1, \frac{1}{2})] / \sim$ and φ ~~bees~~ a local coordinate chart s.t.

$$\varphi : [(x,y)] \longmapsto \begin{cases} (x, y+1) & \text{if } y \in [-1, -\frac{1}{2}) \\ (-x, y-1) & \text{if } y \in (\frac{1}{2}, 1] \end{cases}$$

We know φ is a diffeomorphism from Hw3. We now consider the coordinate charts on $F(U)$, say ψ so that $\psi : F(U) \to \mathbb{R}^2 \times \mathbb{R}^c$ w/

$$\psi = (\varphi_1 \circ \pi, \varphi_2 \circ \pi, d\varphi_1, d\varphi_2).$$

Then $\psi \circ F \circ \varphi^{-1} = \psi \circ F((x,y))$

$$= \psi \circ (x, y, 0, 1)$$

$$= (\varphi(x), \varphi(y), 0, d\varphi_2(\begin{smallmatrix}0\\1\end{smallmatrix}))$$

which is clearly a smooth map of (x,y) and using smoothness of φ we get that F is a smooth vector field U.

Problem 2 Show that the ~~geom~~ Möbius band $\mathcal{M}$ admits a non-vanishing vector field but is not parallelizable.

Prf Recall that a vector field on $\mathcal{M}$ is a (smooth) map $F : \mathcal{M} \to T\mathcal{M}$ st. $\pi \circ F = id_{\mathcal{M}}$. We say that F is not vanishing if $\forall x \in \mathcal{M}$, ~~$F(x) \neq 0(x, 0)$~~ $F(x) \neq (x, \vec{0})$.

Define the following vector field. $F : \mathcal{M} \to T\mathcal{M}$ st $[x] \mapsto (x, e_2)$ where e_2 is the 2^{nd} elementary vector in $\mathbb{R}^2$. F is trivially a vector field and it vanishes nowhere. To show that F is smooth, we'll consider two charts on $\mathcal{M}$:

1. Let $U = [(-1,1) \times (-1,1) / \sim]$, then $U \overset{\varphi}{\cong} \mathbb{R}^2$ via the identity map. Then take the charts $\psi = (x_1 \circ \pi, x_2 \circ \pi, dx_1, dx_2)$ on $F(U)$ and yield that

$$\psi \circ F \circ \varphi^{-1} = \psi \circ (x_1^{-1}, x_2^{-1}, 0, 1)$$
$$= (x_1, x_2, 0, dx_2(1))$$

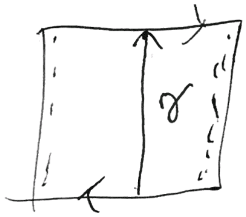
Claim $\mathcal{M}$ is not parallelizable.

Take the vector field from before and consider the map.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[dr, "\psi"'] & T\mathcal{M} \arrow[d, "i"] \\ & GL(\mathbb{R}) \end{tikzcd}$$

so that $i(\vec{x}) = [\vec{a}\ \vec{dx}]$.

As the loop $\gamma : [0,1] \to \mathcal{M}$



moves on the manifold consider the push forward $\psi \circ \gamma = \gamma^*$ in $GL(n, \mathbb{R})$

It is clear from the following pic. that $\det(\psi \circ \gamma[0])$ has different sign than $\det(\psi \circ \gamma[1])$, it is ~~clear that~~ $\exists\, y$ st $\det(\psi \circ \gamma[1]) = 0$.

diffable ~~continuity of det~~

If there existed two ~~frames~~ loops F_1, F_2 such that at every point they were independent, then using this loop at one point $\det(\psi \circ F_1) = \det(\psi \circ F_2) = 0$.

det is a polynomial function of smooth coordinates

$\perp$. Therefore $\mathcal{M}$ not para