MATH H104: Homework Explorations

William Guss
26793499
wguss@berkeley.edu

November 28, 2015

2 Some Interesting Problems

Professor Pugh, as per your suggestion. Here are some of the difficult problems I've done (attempted).

Theorem 1. *Let $S, R \subset \mathbb{R}$ be closed intervals and $\Sigma_l : C(R) \to C(S)$ be a linear operator such that*

$$\xi \mapsto \int_R \xi(i)w(i,j)\ di$$

then for every $\epsilon > 0$ and every $\xi : R \to Q \subset S$, there exists a weight function $w(i,j)$ such that the supremum norm over S

$$\|K\xi - \Sigma_{l+1}\xi\|_\infty < \epsilon \tag{1}$$

Proof. Let $\zeta_t : C(R) \to S$ be a linear form which evaluates its arguments at $t \in R$; that is, $\zeta_t(f) = f(t)$. Then because ζ_t is bounded on its domain, $\zeta_t \circ K = K^\star \zeta_t$ is a bounded linear functional. Then from the Riesz Representation Theorem we have that there is a unique regular Borel measure μ_t on R such that

$$\begin{gathered}(K\xi)(t) = K^\star\zeta_t(\xi) = \int_R \xi(s)\ d\mu_t(s), \\ \|\mu_t\| = \|K^\star\zeta_t\|\end{gathered} \tag{2}$$

Then if there exists a regular Borel measure μ such that μ_t is significantly smaller that μ for all t, then we have that, by the Radon-Nikodim derivative, $d\mu_t(s) = K_t(s)d\mu(s)$ under the assumption that K_t is L^1 integrable over R with the measure μ. Thus it follows that

$$K[\xi](t) = \int_R \xi(s)K_t(s)\ d\mu(s) = \int_R \xi(s)K(t,s)\ d\mu(s). \tag{3}$$

Therefore, for any bounded linear operator $K : C(X) \to C(X)$ there exists a unique $K(t,s)$ such that $K[f] = \int_X f(s)K(t,s)d\mu(s)$. Now we show that the operation of Σ_l can approximate any such operator. Because K is of the form of Σ_l where the only difference is the weighting function, so the proof follows. $\square$