Math 202A — UCB, Fall 2016 — M. Christ
Problem Set 11, due Wednesday November 9

Please study §4.1 and §4.2 of our text.

(11.1) (Folland problem 4.3) Show that every metric space is normal. (See text for a hint.) □

(11.2) (Folland problem 4.5) Show that every separable metric space is second countable. □

(11.3) (Folland problem 4.6) Let $\mathcal{E}$ be the collection of all intervals $(a, b] \subset \mathbb{R}$ such that $a, b \in \mathbb{R}$ and $a < b$. (Thus $\pm\infty$ are excluded.) **(a)** Show that $\mathcal{E}$ is a base for a topology $\mathcal{T}$ on $\mathbb{R}$ in which each element of $\mathcal{E}$ is both open and closed. **(b)** Show that $\mathcal{T}$ is first countable but not second countable. (Hint in text.) **(c)** Show that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, with respect to $\mathcal{T}$. (Thus this furnishes an example of a separable topological space that is not second countable.) □

(11.4) (Folland problem 4.7) Let X be a topological space. Let $S = (x_n : n \in \mathbb{N})$ be a sequence of elements of X. Show that if X is first countable, then a point $x \in X$ is a cluster point of the sequence S if and only if some subsequence of S converges to x. (See problem statement in text for definition of a cluster point. Note that S is a sequence, not a set; the corresponding notion for sets is that of an *accumulation point*. See text, p. 114. The crucial distinction is that for x to be a cluster point, every neighborhood of x must contain x_n for infinitely many *indices* n. In particular, if $x_n = x$ for every n, then x is a cluster point of the sequence. But x is not an accumulation point of the underlying set $\{x_n : n \in \mathbb{N}\}$, which is equal to $\{x\}$.) □

(11.5) (Folland problem 4.9) See text for definition of the order topology. Let X be a linearly ordered set, equipped with the order topology, $\mathcal{T}$. **(a)** Show that if $a, b \in X$ and $a < b$ then there exist pairwise disjoint open sets U, V containing a, b respectively, such that $x < y$ whenever $x \in U$ and $y \in V$. **(a′)** Show that the order topology is the weakest topology with this property. **(b)** Show that the order topology on a subset $Y \subset X$ is never stronger than the relative topology on Y induced by the order topology on X. **(b′)** Give an example in which the order topology on $Y \subset X$ is strictly weaker than this relative topology. **(c)** Show that the order topology on $\mathbb{R}$ is equal to the standard topology on $\mathbb{R}$. □

(11.6) (Folland problem 4.10) An important topological concept is connectivity, also known as connectedness. See problem statement in text for definition. Let $(X, \mathcal{T})$ be a topological space.
(a) Show that X is connected if and only if $\emptyset$ and X are the only subsets of X that are both open and closed. **(b)** Suppose that E_α are connected subsets of X, and that the intersection of all of these sets is nonempty. Show that the union of all of them is connected. **(c)** Show that the closure of any connected set is connected. **(d)** Show that each point $x \in X$ is contained in a unique maximal connected subset of X, and that this subset is closed. (This maximal connected subset is called the connected component of x.) **(e)** (I added this part.) Show that any two connected components of X are either identical, or disjoint. □

(11.7) (Folland problem 4.11) Show that the closure of a union of finitely many subsets is equal to the union of their closures. □

(11.8) (Folland problem 4.13) Let $(X, \mathcal{T})$ be a topological space. Let U be open in X, and let A be dense in X. Show that $\overline{U} = \overline{U \cap A}$. □