MATH 185: Homework 1

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1. *Show that multiplication of complex numbers satisfies the associative, commutative, and distributive laws.*

Theorem 1. *Given that $\mathbb{C}$ is Abelian under addition, $\mathbb{C}$ is a field.*

Proof. Let $a, b, c \in \mathbb{C}$. Then recall that for any $z \in \mathbb{C}$, $z = |z|e^{i\theta_z}$, where $\theta_z = Arg z$. We show that $\mathbb{C}$ satisfies associative, commutative, and distributive laws.

Using that $\mathbb{R}$ is a field, it follows that

$$\begin{aligned}(ab)c &= (|a|e^{i\theta_a}|b|e^{i\theta_b})|c|e^{i\theta_c} \\ &= |a||b|e^{i(\theta_a+\theta_b)}|c|e^{i\theta_c} \\ &= |a||b||c|e^{i(\theta_a+\theta_b+\theta_c)} \\ &= |a|e^{i\theta_a}|b||c|e^{i(\theta_b+\theta_c)} \\ &= a(bc).\end{aligned} \tag{1}$$

Without the assumption of eulers identity , we have that

$$\begin{aligned}(ab)c &= ((a_1 + ia_2)(b_1 + ib_2))(c_1 + ic_2) \\ &= ((a_1b_1 - a_2b_2) + (a_1b_2 + a_2b_1)i)(c_1 + ic_2) \\ &= ((a_1b_1 - a_2b_2)c_1 - (a_1b_2 + a_2b_1)c_2) \\ &\quad + ((a_1b_1 - a_2b_2)c_2 + (a_1b_2 + a_2b_1)c_1)i \\ &= a_1b_1c_1 - a_2b_2c_1 - a_1b_2c_2 + a_2b_1c_2 \\ &\quad + (a_1b_1c_2 - a_2b_2c_2 + a_1b_2c_1 + a_2b_1c_1)i \\ &= a_1(b_1c_1 - b_2c_2) - a_2(b_2c_1 + b_1c_2) \\ &\quad + (a_1(b_1c_2 + b_2c_1) - a_2(b_2c_2 + b_1c_1))i \\ &= (a_1 + a_2i)((b_1c_1 - b_2c_2) + (b_1c_2 + b_2c_1)i) \\ &= a(bc).\end{aligned} \tag{2}$$

In a similar fashion, consider the following rearrangement which follows by the field properties of $\mathbb{R}$:

$$\begin{aligned}ab &= (a_1b_1 - a_2b_2) + (a_1b_2 + a_2b_1)i \\ &= (b_1a_1 - b_2a_2) + (b_2a_1 + b_1a_2)i \\ &= ba.\end{aligned} \tag{3}$$

Lastly we show the distributive property:

$$\begin{aligned} a(b+c) &= a(b_1 + b_2 i + c_1 + c_2 i) \\ &= a((b_1 + c_1) + (b_2 + c_2)i) \\ &= (a_1(b_1 + c_1) - a_2(b_2 + c_2)) + (a_1(b_2 + c_2) + a_2(b_1 + c_1))i \\ &= (a_1 b_1 - a_2 b_2) + (a_1 c_1 - a_2 c_2) + (a_1 b_2 + a_2 b_1)i + (a_1 c_2 + a_2 c_1)i \\ &= ab + ac \end{aligned} \tag{4}$$

Therefore $\mathbb{C}$ is a ring. □

2. Gamelin Exercise I.1.7 (Chapter I, Section 1, Exercise 7)

Theorem 2. *Let $\rho > 1, \rho \neq 1$ and fix $z_0, z_1 \in \mathbb{C}$. Then*

$$S = \{|z - z_0| = \rho|z - z_1| : z \in \mathbb{C}\}$$

is isometric to some $S^1_r \subset \mathbb{R}^2$ for some r.

Proof. Since all $s \in S$ satisfy the above equation, we have that

$$\sqrt{(s_1 - z_{01})^2 + (s_2 - z_{02})^2} = \rho\sqrt{((s_1 - z_{11})^2 + (s_2 - z_{12})^2}. \tag{5}$$

The form of (5) is identical to a distance meterization in $\mathbb{R}^2$; that is, take the isometry $\phi : \mathbb{C} \to \mathbb{R}^2, ((x + iy) \mapsto (x, y)$ and

$$d(\phi(s), \phi(z_0)) = \rho d(\phi(s), \phi(z_1)) \frac{d(S, Z_0)}{d(S, Z_1)} = \rho, \tag{6}$$

which from high school geometry one might recognize as the equation of the circle of Appolonius. □

The geometric proof of a equivalency between Appolonius' circle and the Euclidean circle is omitted.

However, if we take the euclidean distance on $\mathbb{R}^2$, we have the following theorem.

Theorem 3. *Suppose that $P, Q \in \mathbb{R}^2$ and S such that*

$$\frac{\overline{PS}}{\overline{QS}} = k \in (0, 1)[WLOG],$$

then S is a point on a circle.

Proof. Observe the following algebraic derivation using the parallelagram law inspired by J Wilson at the University of Georgia:

$$\begin{aligned} \frac{|P - S|^2}{|Q - S|^2} &= k^2 \\ |P|^2 + |S|^2 - 2\langle P, S\rangle &= k^2(|Q|^2 + |S|^2 - 2\langle Q, S\rangle) \\ 0 &= |P|^2 + |S|^2 - 2\langle P, S\rangle - k^2(|Q|^2 + |S|^2 - 2\langle Q, S\rangle) \\ &= (1 - k^2)|S|^2 + |P|^2 - k^2|Q|^2 - 2\langle P - Q, k^2 S\rangle \qquad = |S|^2 + \frac{|P|^2}{1 - k^2} - \frac{}{k}^2|Q| \end{aligned} \tag{7}$$

□

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3. Gamelin Exercise I.2.5

Theorem 4. *For $n \geq 1$ and $z \in \mathbb{C}$ such that $z \neq 1$, we have that*

$$1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1 - z). \tag{8}$$

Proof. Observe that for $z \in \mathbb{C}$ we have that, $z = e^{i\theta}$. Therefore,

$$e^{i0} + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = 1 + z + z^2 + \cdots + z^n \tag{9}$$

Multiplication by $(1 - z)$ gives,

$$\begin{aligned}(1 - e^{i\theta})e^{i0} + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= e^{i0} + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} \\ &\quad - e^{i(0+\theta)} + e^{i(\theta+\theta)} + e^{i(2\theta+\theta)} + \cdots + e^{i(n\theta+\theta)} \\ &= e^{i0} - e^{i(n\theta+\theta)} \\ &= 1 - z^{n+1}.\end{aligned} \tag{10}$$

Reducing using eulers identity it follows that,

$$\begin{aligned}(1 - z)(1 + z + z^2 + \cdots + z^n) &= (1 - z^{n+1}) \\ 1 + z + z^2 + \cdots + z^n &= (1 - z^{n+1})/(1 - z),\end{aligned} \tag{11}$$

when $z \neq 1$. This completes the proof. $\square$

Theorem 5. *For $n \geq 1$ and $z \in \mathbb{C}$ such that $z \neq 1$, we have that*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})}{2\sin\theta/2} \tag{12}$$

Proof. Recall that $z = rcis\theta$. Take in particular all such z whose absolute magnitude is unity. Then Theorem 4 implies that

$$1 + cis\theta + cis2\theta + \cdots + cisn\theta = (1 - z^{n+1})/(1 - z). \tag{13}$$

A little algebra gives us

$$\begin{aligned}\frac{Re(1 - cis(n+1)\theta}{Re(1 - cis\theta)} &= \frac{Re(1 - e^{(n+1)\theta})Re(1 - e^{-i\theta})}{Re(1 - e^{i\theta})Re(1 - e^{-\theta i})} \\ &= \frac{Re(1 - e^{i\theta} - e^{i(n+1)\theta} + e^{in\theta})}{Re(2 - 2cos\theta)} \\ &= \frac{1 - \cos\theta - \cos(n+1)\theta + \cos n\theta}{4\sin^2(\theta/2)} \\ &= \frac{2\sin^2\theta/2 - \sin(n + 1/2)\sin(\theta/2)}{4\sin^2(\theta/2)} \\ &= \frac{1}{2} - \frac{\sin(n + 1/2)}{2\sin(\theta/2)}\end{aligned} \tag{14}$$

Since the above was the real part of $1 + z + z^2 + \cdots + z^n$, the theorem holds. $\square$