MATH H105: Homework 2

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15.

Theorem 1. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a function so that*

$$\begin{aligned} (x, y) &\mapsto \frac{xy}{x^2 + y^2} \\ (0, 0) &\mapsto 0. \end{aligned} \tag{1}$$

Then, f *has partial derivatives at* $(0, 0)$ *but is not differentiable there.*

Proof. By definition we take the partial derivative to be the limit

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{t \to 0} \frac{f(x + h, y) - f(x, y)}{h} \\ &= \lim_{t \to 0} \frac{(h)^2}{h} \\ &= 0. \end{aligned} \tag{2}$$

Since the closed form for f is identical, we have the same definition for $\partial f / \partial y$.

However, if f is differentiable, then it is continuous at $(0, 0)$. But the limik along $y = x$, does not exist unabiguously

$$\lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2} \neq 0.$$

Since $(Df)_0(0, 1) = 0, (Df)_0(1, 0) = 0$, and the derivative must be linear, it follows that $Df_0(1, 1) = 0 \neq \frac{1}{2}$. So f couldn't be differentiable. □

16. Yass!!!!!!!!!!!!!!!!!!!!!!!!

We build the matrix of partials accordingly! Using partial differnetiation we get

$$(Df)_p = \begin{bmatrix} 1 & 0 \\ \cos 1 & 0 \\ \sin 1 & 0 \end{bmatrix}. \tag{3}$$

$$(Dg)_q = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

$$(Dg \circ f)_p = (Dg)_q \circ (Df)_p = 0. \tag{5}$$

We get that

$$g \circ f = w(s,t) = (st)(s\cos t) + (s\cos t)(s\sin t) + (s\sin t)(st) \tag{6}$$

and so the pial derivatives at least contain s in every term:

$$\begin{aligned} D_s w &= 2st(\cos t) + 2s\cos t\sin t + 2st\sin t \\ D_t w &= s^2(\cos t - t\sin t) + s^2(\cos t\cos t - \sin t\sin t) + s^2(\sin t + t\cos t) \end{aligned} \tag{7}$$

These partials evaluate to 0 and so are 0.

The statement of multivariable chain rule for functions $g : \mathbb{R} \to \mathbb{R}^m, f : \mathbb{R}^m \to \mathbb{R}$ is that $d/dtf \circ g = \sum \partial f/\partial g_i \partial g_i/\partial t$ which is the row vector matrix Df with the column vector Dg.

17. Multidimensional Mean Value Theorem

(a) Vector valued functions!

Theorem 2. *Let $n = 1, m = 2$. Then if*

$$f(t) = (\cos t, \sin t) \tag{8}$$

for $\pi \leq 2\pi$ and $p = \pi, q = 2\pi$, then there is no $\theta \in [p,q]$ which satisfies

$$f(p) - f(q) = (Df)_\theta(q-p) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} (q-p) \tag{9}$$

Proof. Take $f(p) - f(q)$. This value is

$$f(p) - f(q) = \begin{bmatrix} \cos 2\pi \\ \sin 2\pi \end{bmatrix} - \begin{bmatrix} \cos\pi \\ \sin\pi \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{10}$$

□

It could not be that there is a θ such that $-\sin\theta$, the first component of the derivative, is 2. So the theorem holds.

(b) Convex derivative set.

Theorem 3. *If the set of derivatives of $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$,*

$$S = \{(Df)_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) : rx \in [p,q]\} \tag{11}$$

is convex, then there is a $\theta \in [p,q]$ satisfying

$$f(p) - f(q) = (Df)_\theta(q-p). \tag{12}$$

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Proof. Since the set of derivatives is convex, given any two points in S every point on the line segment between them is also in S.

□

18. Directional derivative:

(a) By Theorem 5 we know that if f is differentiable, then its derivative is given by

$$(Df)_p = \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t}, \tag{13}$$

and since partial derivatives are given by letting u be the basis to which the partial is tied, it follows that a 'directional' derivative would be given by a projection of each partial contribution of a component of f onto a direction u.

See the proof of theorem 5, and corollary 7.

However if f is not differentiable, one must intuit from the formula. The limit observes the 'slope' in the component f directions as $u \to p$ in th u direction. That is if the hyrpersurface, $f(U)$ was sliced along the u direction at p, the following formula follows

$$g(t) = f(p+tu), g(0) = f(p). \tag{14}$$

So, $g'(t) = \nabla_p f(u)$ since $g' = \lim(g(t) - g(0))/t$.

(b)

Theorem 4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ so that,*

$$x \mapsto \frac{x^3 y}{x^2+y^2}. \tag{15}$$

The f has directional derivatives, but is not differentiable.

Proof. Take any $u = \begin{bmatrix} a \\ b \end{bmatrix}$. Then the limit

$$\lim_{t\to 0} \frac{\frac{(ta)^3(tb)^2}{(ta)^4+(tb)^2}}{t} = \frac{t^5a^3b^2}{t^3(t^2a^4+b^2)} = 0 \tag{16}$$

when $b \neq 0$. In the case that $b = 0$, we have

$$\lim_{t\to 0} \frac{0}{t^5a^4} = 0. \tag{17}$$

To show that f is not differentiable, we must show that for all suitable T

$$f(p+v) = f(p) + T(v) + R(v) \wedge \lim_{|v|\to 0} \frac{R(v)}{|v|} \neq 0. \tag{18}$$

Suppose that f was differentiable. The only derivative could be the 0 transformation since it unambiguously determines $\nabla_0 f$.

So it follows that

$$f(p+v) = f(p)+R(v) \implies \lim_{|v|\to 0} \frac{R(v)}{|v|} = 0 \implies \lim_{|v|\to 0} \frac{f(p+v)}{|v|} = \lim_{|v|\to 0} \frac{f(v)+R(v)}{|v|} = 0. \tag{19}$$

and f is sublinear! Now consider any approach, say $y = x^2$. The limit had better be sublinear.

$$\lim_{x \to 0} \frac{\frac{x^5}{2x^4}}{x} = 1, \tag{20}$$

so f is not sublinear along that curve. A contradiction to f differentiable!

$\square$

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