Math 185 — UCB, Fall 2016 — William Guss
Problem Set 9, due November 29th

(86.10) Let m and n be integers, where $0 \leq m \leq n$. Show that

$$\int_0^\infty \frac{z^{2m}}{z^{2n}+1}\, dz = \frac{\pi}{2n} \csc\left(\frac{2m+1}{2n}\pi\right).$$

Proof. We will use Residue theory to evaluate the integral. Therefore we first show that the zeroes of the divisor (and therefore singularities of the integrand), $z^{2n}+1$ lying above the real axis are

$$c_k = exp\left(i\frac{(2k+1)\pi}{2n}\right) \quad (k = 0, 1, 2, \dots, n-1).$$

To see this first observe that c_k are solutions to $z^{2n} = -1 = e^{i\pi}$. Then writing z^{2n} in euler form we get $re^{2ni\theta} = e^{i\pi}$ and therefore $r = 1$ and $2ni\theta = i\pi \mod 2\pi i$. Thus $\theta = \pi/(2n) + 2k\pi/(2n) = (2k+1)\pi/(2n)$. Finally $c_k = exp\left(i\frac{(2k+1)\pi}{2n}\right)$ for all k up to $n-1$, for at $k = n$ we yield $\theta = (2n+1)\pi/(2n) = 1/(2n)\pi + \pi$ which is lies below the real axis. Then at $k = 2n$, we yield the solution for $k = 0$, $\theta = (4n+1)\pi/(2n) = 2\pi + \pi/2n$. Therefore our characterization of the roots hold.

Next observe that $(z^{2n}+1)' = 2n(z^{2n-1}) \neq 0$ when $z = c_k$ and therefore the divisor of the integrand is has simple zeros at all c_k; ie. the integrand has simple poles at all $x = c_k$.

Now we use the method of residue integration along the half semi-circle to finally evaluate the integral. Observe that by the evenness of the integrand

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\, dx = \lim_{R\to\infty} 1/2 \int_{-R}^R \frac{z^{2m}}{z^{2n}+1}\, dz$$

Additionally, that for a upper semicircle of radius R,

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\, dx = \pi i \sum_{k=0}^{n-1} Res_{z=c_k}\left(\frac{z^{2m}}{z^{2n}+1}\right) - \lim_{R\to\infty} \frac{1}{2} \int_{C_R} \frac{z^{2m}}{z^{2n}+1}\, dz.$$

Therefore we use our conclusion that the integrand has simple poles at all c_k to just evaluate the residue as

$$\begin{aligned}
\pi i \sum_{k=0}^{n-1} Res_{z=c_k} \left(\frac{z^{2m}}{z^{2n}+1} \right) &= \pi i \sum_{k=0}^{n-1} \frac{c_k^{2m}}{2nc_k^{2n-1}} \\
&= \frac{\pi i}{2n} \sum_{k=0}^{n-1} \frac{exp\left(i\frac{(2k+1)2m\pi}{2n}\right)}{exp\left(i\frac{(2k+1)(2n-1)\pi}{2n}\right)} \\
&= \frac{\pi i}{2n} \sum_{k=0}^{n-1} exp\left(i\pi \frac{2k+1}{2n}(2m-(2n-1))\right) \\
&= \frac{\pi i}{2n} \sum_{k=0}^{n-1} exp\left(i\pi \frac{2k+1}{2n}(2m+1) - i\pi\frac{2n(2k+1)}{2n}\right) \\
&= \frac{\pi i}{2n} \sum_{k=0}^{n-1} exp\left(i\pi \left(\frac{2k+1}{2n}(2m+1) - 2k+1\right)\right) \\
&= \frac{\pi i}{2n} \sum_{k=0}^{n-1} \frac{exp\left(i\pi\left(\frac{2k+1}{2n}(2m+1)\right)\right)}{exp(i\pi)} \\
&= -\frac{\pi i}{2n} \sum_{k=0}^{n-1} exp\left(i\pi \left(\frac{2k+1}{2n}(2m+1)\right)\right).
\end{aligned}$$

To simplify we use the summation formula $\sum_0^{n-1} w^{2k+1} = w\sum_0^{n-1}(w^2)^k = w(1-(w^2)^n)/(1-(w^2))$ where $z \neq 1$ and apply it to our expression where we let $w = exp(i\pi(2m+1)/(2n))$ and then

$$\begin{aligned}
-\frac{\pi i}{2n} \sum_{k=0}^{n-1} exp\left(i\pi \left(\frac{2k+1}{2n}(2m+1)\right)\right) &= -\frac{w\pi i}{2n} \sum_{k=0}^{n-1} w^{2k}. \\
&= -\frac{\pi i}{2n} \frac{w(1-w^{2n})}{1-w^2} \\
&= -\frac{\pi i}{2n} \frac{w(1-exp(i\pi(2m+1)))}{1-w^2} = -\frac{\pi i}{n} \frac{w}{1-w^2}.
\end{aligned}$$

Warranting that $w^{2n} = 1$, $|w| = 1$ algebra yields

$$\begin{aligned}
-\frac{\pi i}{n} \frac{w}{1-w^2} &= -\frac{\pi i}{n} \frac{1}{w(w^{2n-2}-1)} \\
&= -\frac{\pi i}{n} \frac{1}{w(w^{2n-2}-w^{2n})} = -\frac{\pi i}{n} \frac{1}{w(w^{-2}-1)} \\
&= -\frac{\pi i}{n} \frac{1}{w^{-1}-w} = \frac{\pi i}{n} \frac{1}{w-\overline{w}}
\end{aligned}$$

Now $w - \overline{w} = 2iIm(w) = 2i\sin((2m+1)\pi/2n)$. Therefore

$$\pi i \sum_{k=0}^{n-1} Res_{z=c_k} \left(\frac{z^{2m}}{z^{2n}+1} \right) = \frac{\pi}{2n \sin\left(\frac{(2m+1)\pi}{2n}\right)}.$$

Lastly to show the statement we must finally show that $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$. When $m < n$, it is clear that $\frac{|z^{2m}|}{|z^{2n}+1|} \leq \frac{R^{2m}}{R^{2n}-1}$ as $|z^{2n}+1| \geq ||z|^{2n}-1| = R^{2n}-1$. Since R^{2n} subsumes R^{2m} in the limit ($m < n$), $\frac{R^{2m}}{R^{2n}-1} \to 0$ and by the Maximul Modulus principle $|\int_{C_R} \frac{z^{2m}}{z^{2n}+1}\,dz| \leq 2\pi R \frac{R^{2m}}{R^{2n}-1}$. Then since $2m+1 < 2n$, there is yet subsumption in the limit and $2\pi R \frac{R^{2m}}{R^{2n}-1} \to 0$ implies that from the original equation

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \pi i \sum_{k=0}^{n-1} Res_{z=c_k}\left(\frac{z^{2m}}{z^{2n}+1}\right) - \lim_{R\to\infty} \frac{1}{2}\int_{C_R} \frac{z^{2m}}{z^{2n}+1}\,dz = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

□

(86.1) Use residues to derive the integration formula

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$$

Proof. Applying the result in **(86.10)** above we get that

$$\int_0^\infty \frac{dx}{x^2+1} = \int_0^\infty \frac{x^{2\cdot 0}dx}{x^{2\cdot 1}+1} = \frac{\pi}{2}\csc(\pi/2) = \frac{\pi}{2}.$$

□

(86.4) Use residues to derive the integration formula

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6}$$

Proof. Applying the result in **(86.10)** above, we get that $m = 1, n = 3$ and so

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \int_0^\infty \frac{x^{2m}dx}{x^{2n}+1} = \frac{\pi}{6}\csc(3\pi/6) = \frac{\pi}{6}.$$

□

(86.6) Use residues to derive the integration formula

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = \frac{\pi}{200}$$

Proof. We will use Residue theory to evaluate the integral. Therefore we first show that the zeroes of the divisor (and therefore singularities of the integrand), $(z^2+9)(z^2+4)^2$ lying above the real axis are

$$c_0 = 3i, c_1 = 2i, c_2 = 2i$$

To see this first observe that

$$\frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{z^2}{(z^2-(3i)^2)(z^2-(2i)^2)^2} = \frac{z^2}{(z-3i)(z+3i)(z-2i)(z+2i)(z-2i)(z+2i)}$$

Then the zeroes of the divisor are clearly $\pm 3i$ and $\pm 2i$ (multiplicity 2), but those quantities only are above the real line in their positive imaginary form, so our characterization of the zeroes of the divisor hold. We want to characterize the residue, so we will therefore characterize the zeroes of the divsor. First let $u = (z^2+9)$, $v = (z^2+4)$. Then we know that $u' = 2z$, $u'' = 2$ and clearly $u', u'' \neq 0$, at c_k. On the other hand $v' = 2z$, $v'' = 2$ and $v', v'' \neq 0$ at $z = c_k$. So we must differentiate the divsior untill it does not assume a zero value at c_k, recalling that if all of the terms contain u or v, the divisors n-derivative is still zero at c_0 or $c_{1,2}$ respectively.. Next, observe that

$$((z^2+9)(z^2+4)^2)'' = (uv^2)'' = (u'v^2 + 2uvv')' = u''v^2 + 2u'vv' + 2u'vv' + 2u(v'v' + vv'')$$

and so the second derivative always has non-zero terms at each c_k. Thus the integrand has a pole of order 1 at $z = c_0$ and a pole of order two at $z = c_{1,2}$.

Now we use the method of residue integration along the half semi-circle to finally evaluate the integral. Observe that by the evenness of the integrand

$$\int_0^\infty \frac{z^2}{(z^2+9)(z^2+4)^2}\ dz = \lim_{R\to\infty} 1/2 \int_{-R}^{R} \frac{z^2}{(z^2+9)(z^2+4)^2}\ dz$$

Additionally, for a upper semicircle of radius R,

$$\int_0^\infty \frac{z^2}{(z^2+9)(z^2+4)^2}\ dz = \pi i \sum_{k=0}^{1} Res_{z=c_k} \left(\frac{z^2}{(z^2+9)(z^2+4)^2}\right) - \lim_{R\to\infty} \frac{1}{2} \int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2}\ dz.$$

Letting $p = z^2, q = (z^2+9)(z^2+4)^2$, then to calculate the residue at $z = c_0$, we know that $q(z)$ has a pole of order 1 at $z = z_0$, thus the residue is equal to $p(c_0)/q'(c_0)$; that is

$$Res_{z=c_0} p(z)/q(z) = \frac{(3i)^2}{[u'v^2 + 2uvv']_{z=c_0}} = -\frac{9}{2(3i)((3i)^2+4)^2 + 2\cdot 0} = -\frac{4}{2i(-9+4)^2} = \frac{3i}{50}.$$

For $c_{1,2}$, let $\phi(z) = \frac{p(z)}{((z^2+9)(z+2i)^2)}$, then $\phi(z)$ is analytic at $z = c_{1,2}$ and $p(z)/q(z)$ has a pole of order two at $z = 2i$ thus letting $g = (z^2+9)(z+2i)^2$, we can calculate $g' = 2z(z+2i)^2 + 2(z+2i)(z^2+9)$ and evaluate at $z = 2i$ yielding $g'(c_{1,2}) = 4i(4i)^2 + 2(4i)((2i)^2+9) = 4i(-16 + 2(-4+9)) = -24i$. Additionally $g(c_{1,2}) = 5\cdot(4i)^2 = -80$.

$$Res_{z=c_{1,2}} \frac{p(z)}{q(z)} = \phi'(c_{1,2}) = \frac{p'(c_{1,2})g(c_{1,2}) - p(c_{1,2})g'(c_{1,2})}{g(c_{1,2})^2} = \frac{4i(-80) - (-4)(-24i)}{(-80)^2} = \frac{-416i}{6400} = \frac{-13i}{200}.$$

Finally

$$\pi i \sum_{k=0}^{1} Res_{z=c_k} \frac{p(z)}{q(z)} = \pi i \frac{(12i - 13i)}{200} = \frac{\pi}{200}.$$

Lastly to show the statement we must finally show that $\int_{C_R} f(z)\ dz \to 0$ as $R \to \infty$. When $m < n$, it is clear that $p(z)/q(z) \leq \frac{R^2}{(R^2-9)(R^2-4)^2}$ as $|(z^2+9)(z^2+4)^2| = |z^2+9||z^2+4|^2 \geq ||z|^2 - 9|||z^2| - 4|^2 = (R^2-9)(R^2-4)^2$. Since R^3 is order three it is subsumed by $(R^2-9)(R^2-4)^2$ in the limit , $\frac{R^2}{(R^2-9)(R^2-4)^2} \to 0$ and by the Maximul Modulus principle

$$\left|\int_{C_R} p(z)/q(z)\ dz\right| \leq 2\pi \frac{R^2}{(R^2-9)(R^2-4)^2} \to 0.$$

Therefore we conclude

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\ dx = \frac{\pi}{200}.$$

□

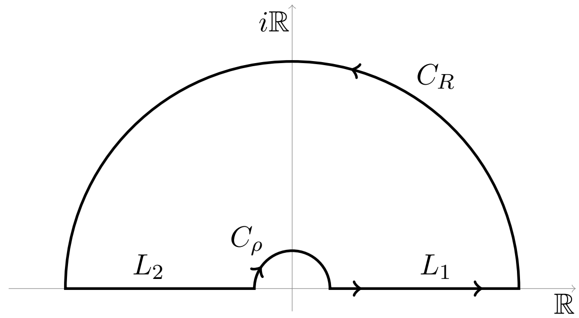
(91.1) Use the function $f(z) = (e^{iaz} - e^{ibz})/z^2$ and the indented contour in Fig 109 (Sec 89) to derive the integration formula

$$\int_0^\infty \frac{cos(ax) - cos(bx)}{x^2}\ dx = \frac{\pi}{2}(b-a) \quad (a \geq 0, b \geq 0).$$

Then show that

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\ dx = \frac{\pi}{2}.$$

Proof. Assuming that $a \geq 0$ and $b \geq 0$ we will use the contours provided below:



and let $f(z) = (e^{iaz} - e^{ibz})/z^2$. Then the function has a pole of order two at $z = 0$ and is analytic elsewhere. Thus

$$\int_{C_R} f(z)\ dz + \int_{C_\rho} f(z)\ dz + \int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz = 0$$

$$\int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz = -\int_{C_R} f(z)\ dz - \int_{C_\rho} f(z)\ dz$$

In particular we manipulate the left hand side by changing the orientation of L_2 so that

$$\begin{aligned}
\int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz &= \int_{L_1} f(z)\ dz - \int_{-L_2} f(z)\ dz \\
&= \int_\rho^R f(re^{i0})(re^{i0})'\ dr - \int_\rho^R f(re^{i\pi})(re^{i\pi})'\ dr \\
&= \int_\rho^R f(r)\ dr + \int_\rho^R f(-r)\ dr \\
&= \int_\rho^R (e^{iar} - e^{ibr})/r^2 + (e^{-iar} - e^{-ibr})/(-r)^2\ dr \\
&= \int_\rho^R ((e^{iar} + e^{-iar}) - (e^{-ibr} + e^{ibr}))/r^2\ dr \\
&= 2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}\ dr.
\end{aligned}$$

Consequently we need only evaluate the semicircular contour integrals; that is,

$$2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}\ dr = -\int_{C_R} f(z)\ dz - \int_{C_\rho} f(z)\ dz.$$

Now observe that $f(z)$ only has a pole of order two at $z = 0$. Therefore

$$f(z) = \frac{\phi(z)}{z^2}; \phi(z) \text{ analytic } \implies f(z) = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + \sum_{n=0}^{\infty} c_n z^n.$$

Now observe that since $\sum_{n=0}^{\infty} c_n z^n$ is analytic and $z = 0$ is an isolated singular point of f, there is an ϵ-ball small enough so that $\sum_{n=0}^{\infty} c_n z^n$ is analytic on a closed subdomain D with no singularities. Thus it achieves a maximum M, so that

$$\left| \int \sum_{n=0}^{\infty} c_n z^n \; dz \right| \leq ML = M\pi\rho \to 0.$$

Therefore

$$\lim_{\rho \to 0} \int_{C_\rho} f(z) \; dz = \lim_{\rho \to 0} \int_{C_\rho} \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} \; dz = \lim_{\rho \to 0} \int_{C_\rho} \frac{c_{-2}}{z^2} \; dz + \pi i Res_{z=0} f(z).$$

Finally we evaluate the remaining integral and yield that

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{c_{-2}}{z^2} \; dz = \lim_{\rho \to 0} \int_0^{\pi} \frac{c_{-2}}{\rho^2 e^{i2\theta}} \rho i e^{i\theta} \; d\theta = \lim_{\rho \to 0} \frac{i c_{-2}}{\rho} \int_0^{\pi} e^{-i\theta} \; d\theta$$

But then because $|\int_0^{\pi} e^{-i\theta} \; d\theta| \leq \pi$ we have that the integral tends to 0, and so

$$\int_{C_\rho} f(z) \; dz \to \pi i Res_{z=0} \; f(z).$$

Now we expand the Larent series representation of $f(z)$ and get

$$\begin{aligned} f(z) = \frac{1}{z^2}(e^{iaz} - e^{ibz}) &= \frac{1}{z^2}\left(1 + \frac{(iaz)}{1!} + \frac{(aiz)^2}{2!} + \cdots\right) - \frac{1}{z^2}\left(1 + \frac{(ibz)}{1!} + \frac{(ibz)^2}{2!} + \cdots\right) \\ &= \left(\frac{1}{z^2} + \frac{(ia)}{z1!} + \frac{(ai)^2}{2!} + \cdots\right) - \left(\frac{1}{z^2} + \frac{(ib)}{z1!} + \frac{(ib)^2}{2!} + \cdots\right) \\ &= \frac{ia - ib}{z} + \frac{b-a}{2!} + \cdots \\ \implies & \pi i Res_{z=0} \; f(z) = \pi(b-a). \end{aligned}$$

Lastly we will show that $\int_{C_R} \; f(z) \; dz \to 0$. Consider that $e^{iaz} - e^{ibz}$. Additonally $|z| = R^2$ and so $|1/z^2| \leq \frac{1}{R^2}$. Therefore by Jordans lemma, $a, b \geq 0$, and the triange inequality

$$\left| \int_{C_R} f(z) \; dz \right| \leq \left| \int_{C_R} \frac{e^{az}}{z^2} \; dz \right| + \left| \int_{C_R} \frac{e^{bz}}{z^2} \; dz \right| \to 0.$$

Therefore $\int_{C_R} f(z) \; dz \to 0$, and we have the following statement.

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} \; dx = \lim_{\rho \to 0, R \to \infty} \int_{\rho}^{R} \frac{\cos(ar) - \cos(br)}{r^2} \; dr = \frac{\pi(b-a)}{2} + 0.$$

It follows immediately that $\int_0^{\infty} \frac{\sin^2(x)}{x^2} \; dx = \pi/2$ because $\sin^2(x) = \frac{\cos(0x) - \cos(2x)}{2}$; in other words

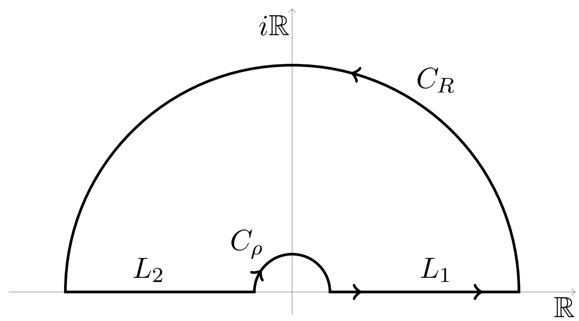
$$\int_0^{\infty} \frac{\sin^2(x)}{x^2} \; dx = \frac{1}{2} \int_0^{\infty} \frac{\cos(0) - \cos(2x)}{x^2} \; dx = \frac{1}{2} \frac{\pi(2-0)}{2}.$$

This completes the proof. $\square$

(91.2) Derive the integral formula

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

Proof. Assuming that $\rho < 1$ and $R > 1$ we will use the contours provided below:



and let $f(z) = \frac{z^{-1/2}}{z^2+1}$. Where the particular branch in contention is $|z| > 0$, and $-\pi/2 < \arg z < 3\pi/2$.

Then the function has a singularity inside the domain of the contour (at i) since $i^2 + 1 = 0$. Thus Cauchy's residue theorem gives us

$$\int_{C_R} f(z)\ dz + \int_{C_\rho} f(z)\ dz + \int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz = 2\pi i Res_{z=i}\ f(z)$$
$$\int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz = 2\pi i Res_{z=i}\ f(z) - \int_{C_R} f(z)\ dz - \int_{C_\rho} f(z)\ dz$$

In particular we manipulate the left hand side by chaning the orientation of L_2 so that

$$\begin{aligned}
\int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz &= \int_{L_1} f(z)\ dz - \int_{-L_2} f(z)\ dz \\
&= \int_\rho^R f(re^{i0})(re^{i0})'\ dr - \int_\rho^R f(re^{i\pi})(re^{i\pi})'\ dr \\
&= \int_\rho^R f(r)\ dr + \int_\rho^R f(-r)\ dr \\
&= \int_\rho^R \frac{e^{-i0/2} + e^{-i\pi/2}}{\sqrt{r}(r^2+1)}\ dr \\
&= (1-i)\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\ dr
\end{aligned}$$

Consequently we need only evaluate the semicircular contour integrals and the residue; that is,

$$(1-i)\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\ dr = 2\pi i Res_{z=i}\ f(z) - \int_{C_R} f(z)\ dz - \int_{C_\rho} f(z)\ dz.$$

Next observe that $|z^2| = r^2$ when $z = re^{i\theta}$. Thus $|z^2+1| \geq ||z|^2 - 1| = 1 - \rho^2$. When $\rho < 1$. Now the contour integral is bounded using the maximum modulus principle and we get that

$$\left| \int_{C_\rho} f(z)\ dz \right| \leq \frac{\rho^{-1/2}}{1-\rho^2}\pi\rho \leq \frac{\pi\rho^{1/2}}{1-\rho^2} \to 0$$

since $1-\rho^2 \to 1$ and $\sqrt{\rho} \to 0$ as $\rho \to 0$. Therefore $\int_{C_\rho} f(z)\ dz \to 0$.

In the case that z is on C_R then $|z^2+1| \geq ||z|^2 - 1| = |R^2 - 1| = R^2 - 1$ when $R > 1$. Now we can bound the contour integral using the maximum modulo principle and we get

$$\left| \int_{C_R} f(z)\ dz \right| \leq \frac{R^{-1/2}}{R^2-1}\pi R \leq \frac{\pi R^{1/2}}{R^2-1}.$$

Since $R^{1/2}$ is subsumed by R^2 as $R \to \infty$ we have that the ML bound tends to 0. In fact $\int_{C_R} f(z)\ dz \to 0$ as $R \to \infty$.

So it remains to evaluate the residue of f at $z = i$. Since $f(z) = z^{-1/2}/(z+i)(z-i)$, f has a pole of order 1 at $z = i$ and so we may evaluate the residue as $\phi(i)$ where $\phi(i)$ removes the singularity. Therefore

$$Res_{z=i}f(z) = \frac{i^{-1/2}}{[(z+i)]_{z=i}} = \frac{i^{-1/2}}{[(z+i)]_{z=i}} = -\frac{e^{-1/2(Log|1|+iArg(i))}}{2i} = \frac{1-i}{2i\sqrt{2}}.$$

Then $2\pi i Res_{z=i}\ f(z) = (1-i)\pi/\sqrt{2}$. Putting everything together we yield that in the limit

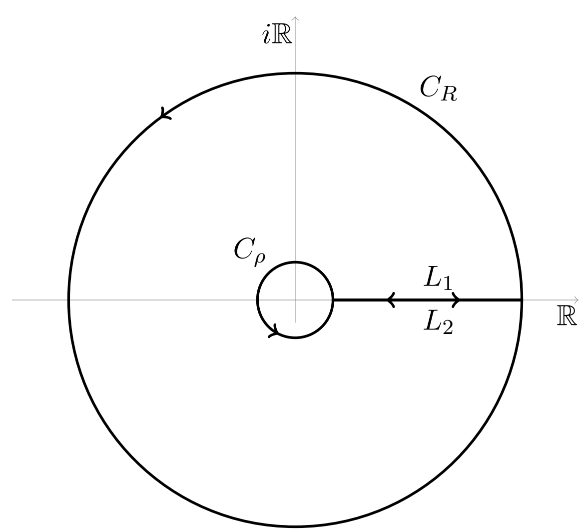
$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

□

(91.4) Derive the integration formula

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\ dx = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \quad (a > b > 0).$$

Proof. Assuming that $\rho < b$ and $R > a$ we will use the contours provided below:



and let $f(z) = \frac{z^{1/3}}{(z+b)(z+a)}$. Where the particular branch in contention is $|z| > 0$, and $0 < \arg z < 2\pi$. (We referr to Problem 6 for a rigorous derivation of integration along L_1.)

Then the function has a singularity inside the domain of the contour (at i) since $i^2 + 1 = 0$. Thus Cauchy's residue theorem gives us

$$\int_{C_R} f(z)\ dz + \int_{C_\rho} f(z)\ dz + \int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz = 2\pi i \left(Res_{z=-a}\ f(z) + Res_{z=-b}\ f(z)\right)$$
$$\int_{L_1} f(z)\ dz + \int_{L_2} f(z)\ dz = 2\pi i \left(Res_{z=-a}\ f(z) + Res_{z=-b}\ f(z)\right) - \int_{C_R} f(z)\ dz - \int_{C_\rho} f(z)\ dz$$

Recall that in the principle branch $f(z) = \frac{exp[1/3(\ln r + i0)]}{(r+a)(r+b)}$ when $z = re^{i0}$. Then when $z = re^{i2\pi}$, $f(z) = \frac{exp[1/3 \ln r + i2\pi]}{(r+a)(r+b)}$, since L_2 is the reverse parameterization then we get

$$\int_{L_1} f(z)\ dz - \int_{-L_2} f(z)\ dz = \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\ dr - \int_\rho^R \frac{\sqrt[3]{r}e^{i2\pi/3}}{(re^{i2\pi}+a)(re^{i2\pi}+b)}\ dr$$
$$= (1 - e^{i2\pi/3}) \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\ dr.$$

Consequently we need only evaluate the semicircular contour integrals and the residue; that is,

$$(1 - e^{i2\pi/3}) \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\ dr = 2\pi i \left(Res_{z=a}\ f(z) + Res_{z=b}\ f(z)\right) - \int_{C_R} f(z)\ dz - \int_{C_\rho} f(z)\ dz.$$

Next observe that $|z^{1/3}| = R^{1/3}$ when $z = re^{i\theta}$ is on C_R. Thus when $b < R$, $|(z+a)(z+b)| \geq |z+a||z+b| \geq ||z|-a|||z|-b| = (R-a)(R-b)$. Now we can bound the contour integral using the maximum modulo principle and we get

$$\left|\int_{C_R} f(z)\ dz\right| \leq \frac{R^{1/3}}{(R-a)(R-b)}\pi R \leq \frac{R^{4/3}\pi}{(R-a)(R-b)}.$$

Since $R^{4/3}$ is subsumed by R^2 as $R \to \infty$ we have that the ML bound tends to 0. In fact $\int_{C_R} f(z)\ dz \to 0$ as $R \to \infty$.

Next observe that when $z = re^{i\theta}$ is on C_ρ, $r = \rho$ so $|z^{1/3}| \leq \rho^{1/3}$ and when $\rho < a$, $|(z+a)(z+b)| \geq |z+a||z+b| \geq ||z|-a|||z|-b| = |a-|z|||b-|z|| = (a-\rho)(b-\rho)$. Now we can bound the contour integral using the maximum modulo principle and we get

$$\left|\int_{C_\rho} f(z)\ dz\right| \leq \frac{\rho^{1/3}}{(a-\rho)(b-\rho)}\pi\rho \leq \frac{\rho^{4/3}\pi}{(a-\rho)(b-\rho)}.$$

Since $\rho^{4/3} \to 0$ and the divisor tends to ab as $\rho \to 0$ we have that the ML bound tends to 0. In fact $\int_{C_\rho} f(z)\ dz \to 0$ as $\rho \to 0$.

So it remains to evaluate the residue of f at $z = -a$ and $z = -b$. Since f has a pole of order 1 at both $z = -a, -b$ we may evaluate the residue as

$$Res_{z=-a} f(z) = \phi_1(-a);\ \ \phi_1(z) = \frac{z^{1/3}}{(z+b)}.\ \ Res_{z=-b} f(z) = \phi_2(-b);\ \ \phi_2(z) = \frac{z^{1/3}}{(z+a)}$$

Since a, b are positive real numbers we get that in the principle branch.

$$\phi_1(-a) = \frac{e^{i\pi/3}\sqrt[3]{a}}{(b-a)}; \quad \phi_1(-b) = \frac{e^{i\pi/3}\sqrt[3]{b}}{(a-b)};$$

Returning to the derivation, we have then that

$$\begin{aligned}
\lim_{\rho\to 0, R\to\infty}(1-e^{i2\pi/3})\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\ dr &= 2\pi i\left(Res_{z=-a}\ f(z) + Res_{z=-b}\ f(z)\right)\\
&= 2\pi i e^{i\pi/3}\left(\frac{\sqrt[3]{a}}{(b-a)} + \frac{\sqrt[3]{b}}{(a-b)}\right)\\
\implies \int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\ dx &= \frac{2\pi i e^{i\pi/3}}{1-e^{i\pi 2/3}}\left(\frac{\sqrt[3]{a}}{(b-a)} + \frac{\sqrt[3]{b}}{(a-b)}\right)\\
&= \frac{2\pi i^2}{\sqrt[2]{3}}\left(\frac{\sqrt[3]{a}}{(b-a)} + \frac{\sqrt[3]{b}}{(a-b)}\right)\\
&= -\frac{2\pi}{\sqrt[2]{3}}\left(\frac{-\sqrt[3]{a}}{(a-b)} + \frac{\sqrt[3]{b}}{(a-b)}\right)\\
&= \frac{2\pi}{\sqrt[2]{3}}\left(\frac{\sqrt[3]{a}-\sqrt[3]{b}}{(a-b)}\right)
\end{aligned}$$

This completes te derivation. □

(91.5) The beta function is this function of two real variables:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\ dt \quad (p>0, q>0).$$

Make the subsitution $t = 1/(x+1)$ and use the result obtained in the example in Sec. 91 to show that

$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)} \quad (0<p<1)$$

Proof. As stated above, let $t = 1/(x+1)$, then $\int \phi(t)\ dt = \int \phi(t(x))\ t'(x)\ dx$, via the change of variable formula. Tne $t'(x) = -1/(x+1)^2$ and thuis

$$\begin{aligned}
B(p,1-p) = \int_{t^{-1}(0)}^{t^{-1}(1)} -\frac{1}{(x+1)^{p-1}}\left(1-\frac{1}{x+1}\right)^{1-p-1}\frac{1}{(x+1)^2}\ dx &= \int_{t^{-1}(1)}^{t^{-1}(0)} \frac{1}{(x+1)^{p+1}}\left(\frac{x}{x+1}\right)^{-p}\ dx\\
&= \int_0^\infty \frac{x^{-p}}{x+1}\ dx.
\end{aligned}$$

Then since $0<p<1$ then by the exaple of the previous seciton we yield

$$B(p,1-p) = \int_0^\infty \frac{x^{-p}}{x+1}\ dx = \frac{\pi}{\sin(p\pi)}.$$

□

(88.1) Use residues to derive the integration formula

$$\int_{-\infty}^{\infty} \frac{\cos x\ dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \quad (a > b > 0).$$

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ so that

$$f : z \mapsto \frac{1}{(z^2+a^2)(z^2+b^2)}$$

Then on the real axis, the real part of the function fe^{iz} assumes values $Re(e^{iz}f(x+0i)) = Re(e^{ix}/((x^2+a^2)(x^2+b^2))) = (\cos x)/(x^2+x^2)(z^2+b^2)$. Therefore we adopt the methodology of the example in seciton 88 and thus calculate the principle value integral (and by eveness, this is equivalent to the sum of both indefnite integrals.)

By the Cauchy Residue theorem under the standard semi-circle contour of radius R we have that

$$\int_{-R}^{R} f(z)e^{iz}\ dz + \int_{C_R} f(z)e^{iz}\ dz = 2\pi i \sum_{c_k \in S(f)\cap C_R^o} Res_{z=c_k}\ e^{iz}f(z)$$

where $S(f)$ is the set of singularities of f in $\mathbb{C}$.

First to calculate the residues at $S \cap \mathbb{C}^H = \{ai, bi\}$, we observe that f has poles of order 1 at both singulariites in $S \cap \mathbb{C}^H$. Therefore we need only evaliuate $\phi_1(z) = e^{iz}/((x^2+b)(x+ia)), \phi_2(z) = e^{iz}/((x^2+a)(x+ib))$ at the respective c_k. First we yield $\phi_1(ia) = e^{-a}/((b^2-a^2)(2ia))$, then we yield $\phi_2(ib) = e^{-b}/((a^2-b^2)(2ib))$. Returning to the derivation

$$\int_{-R}^{R} f(z)e^{iz}\ dz + \int_{C_R} f(z)e^{iz}\ dz = 2\pi i\left(\frac{e^{-a}}{(2ia)(b^2-a^2)} - \frac{e^{-b}}{(2ib)(a^2-b^2)}\right) = -\frac{-2\pi i}{a^2-b^2}\left(\frac{e^{-b}}{2ib} - \frac{e^{-a}}{2ia}\right)$$

Lastly we need show that the C_R contour integral tens to 0. In particular we will bound $f(z)$ on C_R and then apply Jordan's lemma. We know that $|(z^2+a^2)(z^2+b^2)| = |z^2+a^2||z^2+b^2| \geq ||z^2|-a^2|||z^2|-b^2|$ Then for R large enough $||z^2|-a^2|||z^2|-b^2| = (R^2-a)(R^2-b)$. Therefore $|f(z)| \leq 1/((R^2-a)(R^2-b))$ and $1/((R^2-a)(R^2-b)) \to 0$ as $R \to \infty$. Therefore under Jordan's lemma $(a, b > 0)$ the integral is bounded by

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iz}\ dz = 0.$$

Finally we yield that

$$\int_{-\infty}^{\infty} \frac{\cos x\ dx}{(x^2+a^2)(x^2+b^2)} = (P.V) \int_{-\infty}^{\infty} \frac{\cos x\ dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

and this completes the derivation. □

(88.2) Use residues to derive the integration formula

$$\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\ dx = \frac{\pi}{2}e^{-a}. \quad (a > 0)$$

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ so that

$$f : z \mapsto \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

Then on the real axis, the real part of the function fe^{iaz} assumes values $Re(e^{iz}f(x+0i)) = Re(e^{ix}/(x^2+1) = \cos(ax)/(x^2+1)$. Therefore we adopt the methodology of the example in seciton 88 and thus calculate the principle value integral (and by eveness, this is equivalent to the sum of both indefnite integrals.) We do not presume the existence of the integral untill we have shown the formula is valid.

By the Cauchy Residue theorem under the standard semi-circle contour of radius R we have that

$$\int_{-R}^{R} f(z)e^{iaz}\ dz + \int_{C_R} f(z)e^{iaz}\ dz = 2\pi i \sum_{c_k \in S(f)\cap C_R^o} Res_{z=c_k}\ e^{iaz} f(z)$$

where $S(f)$ is the set of singularities of f in $\mathbb{C}$.

First to calculate the residues at $S \cap \mathbb{C}^H = \{i\}$, we observe that f has a pole of order 1 at the element in $S \cap \mathbb{C}^H$. Therefore we need only evaliuate $\phi_1(z) = e^{iaz}/(z+i)$ at $z = i$. We yield $\phi_1(i) = e^{-a}/2i$. Returning to the derivation

$$\int_{-R}^{R} f(z)e^{iz}\ dz + \int_{C_R} f(z)e^{iz}\ dz = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a}$$

Lastly we need show that the C_R contour integral tends to 0. In particular we will bound $f(z)$ on C_R and then apply Jordan's lemma as $a > 0$. We know that $|z^2+1| \geq ||z|^2 - 1|$. Then for R large enough $||z|^2-1| = R^2 - 1$. Therefore $|f(z)| \leq 1/(R^2-1)$ and $1/(R^2-1) \to 0$ as $R \to \infty$. Therefore under Jordan's lemma the integral is bounded by

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iz}\ dz = 0.$$

Finally we yield that through the evenness of the integrand in the original statement,

$$\int_0^\infty \frac{\cos ax}{x^2+1}\ dx = \frac{1}{2}\int_{-R}^{R} f(z)e^{iz}\ dz = \frac{\pi}{2}e^{-a}$$

and this completes the derivation. □

(88.3) Use residues to derive the integration formula

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab}$$

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ so that

$$f : z \mapsto \frac{1}{(x+bi)^2(x-bi)^2}$$

Then on the real axis, the real part of the function fe^{iaz} assumes values $Re(e^{iz}f(x+0i)) = Re(e^{ix}/((x+bi)^2(x-bi)^2)) = \cos(ax)/(x^2+b^2)^2$. Therefore we adopt the methodology of the example in seciton 88 and thus calculate the principle value integral (and by eveness, this is equivalent to the sum of both indefnite integrals.) We do not presume the existence of the integral untill we have shown the formula is valid.

By the Cauchy Residue theorem under the standard semi-circle contour of radius R we have that

$$\int_{-R}^{R} f(z)e^{iaz}\ dz + \int_{C_R} f(z)e^{iaz}\ dz = 2\pi i \sum_{c_k \in S(f)\cap C_R^o} Res_{z=c_k}\ e^{iaz} f(z)$$

where $S(f)$ is the set of singularities of f in $\mathbb{C}$.

First to calculate the residues at $S \cap \mathbb{C}^H = \{bi\}$, we observe that f has a pole of order 2 at the element in $S \cap \mathbb{C}^H$. Therefore we need only evaliuate $\phi_1'(bi)$ where $\phi_1(z) = e^{iaz}(z+bi)^{-2}$. We yield via differntiation that $\phi'(z) = iae^{iaz}(z+bi)^{-2} - 2e^{iaz}(z+bi)^{-3}$. Therefore $\phi'(bi) = iae^{-ab}(2bi)^{-2} - 2e^{-ab}(2bi)^{-3}$. This is just $\phi'(bi) = e^{-ab}i/(4b^3)\cdot(-ab-1)$Returning to the derivation

$$\int_{-R}^{R} f(z)e^{iz}\ dz + \int_{C_R} f(z)e^{iz}\ dz = 2\pi i \frac{e^{-ab}i}{4b^3}\cdot(-ab-1) = 2\frac{\pi}{4b^3}(ab+1)e^{-ab}$$

Lastly we need show that the C_R contour integral tends to 0. In particular we will bound $f(z)$ on C_R and then apply Jordan's lemma as $a > 0$. We know that $|(z^2+b^2)^2| = |z^2+b^2||z^2+b^2| \geq ||z|^2 - |b^2||||z|^2 - |b^2||$. Then for R large enough $||z|^2 - |b^2|| = R^2 - b^2$. Therefore $|f(z)| \leq 1/(R^2-b^2)^2$ and $1/(R^2-b^2)^2 \leq C/R^4 \to 0$ as $R \to \infty$ for some C. Therefore under Jordan's lemma the integral is bounded by

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iz}\ dz = 0.$$

Finally we yield that through the evenness of the integrand in the original statement,

$$\int_0^\infty \frac{\cos ax}{x^2+1}\ dx = \frac{1}{2}\int_{-R}^{R} f(z)e^{iz}\ dz = \frac{\pi}{4b^3}(ab+1)e^{-ab}$$

and this completes the derivation. □

(88.12) Evaluate the Fresnel integrals

$$\int_0^\infty cos(x^2)\ dx = \int_0^\infty \sin(x^2)\ dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Proof. First consider the contour from Figure 106. Then we will integrate $exp(iz^2)$ arround the positiveley oriented vboundary of the sector $0 \leq r \leq R.0 \leq \theta \leq \pi/4$ and appeal to the Cauchy-Goursat theorem which states that since the interiror of this curve is a domain on which $exp(iz^2)$ is analyitic, the closed path integral on this domain of $exp(iz^2)$ is null. Therefore

$$0 = \int_0^R exp(ix^2)e^{i0}\ dx + \int_0^R exp\left(r^2e^{5\pi/2}\right)e^{5\pi/4}\ dr + \int_{C_R} e^{iz^2}\ dz$$

$$\int_0^R exp(ix^2)e^{i0}\ dx = (1+i)\int_0^R \frac{exp\left(r^2e^{5\pi/2}\right)}{\sqrt{2}}\ dr - \int_{C_R} e^{iz^2}\ dz$$

$$\implies \int_0^R \cos(x^2)\ dx = Re\left((1+i)\int_0^R \frac{exp\left(r^2e^{5\pi/2}\right)}{\sqrt{2}} - \int_0^R exp(ix^2)e^{i0}\ dx\right)$$

$$\implies \int_0^R \cos(x^2)\ dx = Im\left((1+i)\int_0^R \frac{exp\left(r^2e^{5\pi/2}\right)}{\sqrt{2}} - \int_0^R exp(ix^2)e^{i0}\ dx\right)$$

Therefore we will show that the arc integral tends to 0 as $R \to \infty$. In particular

$$\int_{C_R} e^{iz^2}\ dz = \int_0^{\pi/4} iexp\left(iR^2e^{i\theta}\right)Re^{i\theta}\ d\theta = \frac{R}{1}i\int_0^{\pi/4} exp\left(R^2(i\cos(\theta))\right)exp(-R^2\sin(\theta))e^{i\theta}\ d\theta$$

Taking absolute values we yield that

$$\begin{aligned}
\left|\int_{C_R} e^{iz^2}\ dz\right| &= R\left|\int_0^{\pi/4} exp\left(i(R^2\cos(\theta)+\theta)\right) exp(-R^2\sin(\theta))\ d\theta\right| \\
&= R\left|\int_0^{\pi/4} cis\left(i(R^2\cos(\theta)+\theta)\right) exp(-R^2\sin(\theta))\ d\theta\right| \\
&\le R\int_0^{\pi/4} \left|cis\left(i(R^2\cos(\theta)+\theta)\right)\right|\left|exp(-R^2\sin(\theta))\right|\ d\theta \\
&\le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin(\theta)}\ d\theta
\end{aligned}$$

Then in Jordan's Lemma's proof, we observe by automorphism on R

$$\left|\int_{C_R} e^{iz^2}\ dz\right| \le \frac{R}{2}\frac{\pi}{2R^2} \le \frac{\pi}{4R} \to 0$$

Then reclaling that

$$\int_0^\infty e^{-r^2}\ dr = \frac{\sqrt{\pi}}{2}$$

we get finally that

$$\int_0^\infty cos(x^2)\ dx = \int_0^\infty \sin(x^2)\ dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

□

(92.1) Show that $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi/3$.

Proof. We let the integrand be denoted $f(\theta)$ and we make the substitutions from Chapter 92, thus yielding

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{-dz}{2z^2-5zi-2)}.$$

Then the divisor $2z^2-5zi-2$ has roots $z = i/2, 2i$. Therefore $z = i/2$ is the only root of the polynomial contained in C_R, so we apply the theory of residues and recall that this is just a simple pole, so we evaluate $-1/(2z+i)$ at $z = i/2$ yielding a residue of $-i/3$. Applying the logic from the exercise and omitting the full application of the residue we get

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i/3(-1) = \pi 2/3$$

□

(92.6) Show that $\int_0^\pi \sin^{2n}(\theta)\ d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi$.

Proof. We make a a first automorphic substitution, $\theta = \theta/2$ but also observe symmetry in the graph as in Example 2 thus

$$\int_0^\pi \sin^{2n}(\theta)\ d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}(\theta/2)\ d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}(\theta)\ d\theta$$

Next we substitute again and yield that

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi} \sin^{2n}(\theta)\ d\theta &= \frac{1}{2}\int_{C_R}\left(\frac{z+z^{-1}}{2i}\right)^{2n}\frac{dz}{iz}\\ &= \frac{1}{(2i)^{2n+1}}\int_{C_R}\frac{(z+z^{-1})^{2n}}{z}\ dz\\ &= \frac{1}{(2i)^{2n+1}}\int_{C_R}\frac{z^{2n}}{z^{2n}}\frac{(z+z^{-1})^{2n}}{z}\ dz\\ &= \frac{-i}{(2)^{2n+1}}\int_{C_R}\frac{(z^2+1)^{2n}}{z^{2n+1}}\ dz\end{aligned}$$

Therefore we need only calculate the reisdue of $\frac{(z^2+1)^{2n}}{z^{2n+1}}$ at its pole of order $2n+1$ at $z=0$. The bionial expansion of ϕ is $\sum_{k=0}^{2n}\frac{(2n)!}{(2n-k)!k!}z^{4n-2k}$ therefore the $(2n)^{th}$ derivative $\phi^{(2n)}$ is $\sum_{k=0}^{n}\frac{(4n-2k)!(2n)!}{(2n-2k)!(2n-k)!k!}z^{2n-2k}$ by applying the nth derivative power rule. Finally evaluation at 0, implies that when $2n-2k>0$ then the term is null; therefore $k=n$ and we get $\phi^{(2n)}(0)=\frac{(2n)!(2n)!}{(n)!n!}$. Finally residue theorem has us divide this evclaution by the order of the pole subract one factorial, and thus we have

$$\int_0^{\pi} \sin^{2n}(\theta)\ d\theta = 2\pi i\cdot\frac{-i}{(2)^{2n+1}}\cdot\frac{(2n)!(2n)!}{n!n!(2n)!} = \frac{2n!}{2^{2n}n!^2}\pi.$$

□

(95.1) Find the inverse Laplace transfor of

$$F(s)=\frac{2s^3}{s^4-4}.$$

In this case we deferr to the book which says that only a formal treatment is necissary. Observe $F(s)$ has singularities at $s^4-4=0$ and so $s^4=4$ has fourth roots with magnitue $4^{1/4}=2^{1/2}$. Now we evaluate $\phi(s)=e^{st}/(z\pm\sqrt{2})(z\pm i\sqrt{2})$ whilst removing one of the factors. Then we get as residues $\phi_1=e^{\sqrt{2}t}/(2\sqrt{2})(i+1)2(1-i), \phi_2=-e^{-\sqrt{2}t}/(2\sqrt{2})2(i+1)(1-i), \phi_3=e^{i\sqrt{2}t}/(2i\sqrt{2})(i+1)(i-1), \phi_4=-e^{-i\sqrt{2}t}/(2i\sqrt{2})(i+1)(i-1)$. Multipling the summand by $2\pi i$ we get $\cosh\sqrt{2}+\cos\sqrt{2}t$.