MATH H104: Homework 1

William Guss
26793499
wguss@berkeley.edu

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1 Real Numbers

3. Recast the following English sentences in mathematics, using correct mathematical grammar. Preserve their meaning.

   (a) *2 is the smallest prime number.* Let $P \subset \mathbb{N}$ denote the set of prime numbers. Consider that $t = 2$ is clearly a member of P. Then for all $p \in P$, $t \leqslant P$.

   (b) *The area of any bounded plane region is bisected by some line parallel to x-axis.*

   Before we claim the above we must first rigorize the notion of a bounded plane region. The following uses a notion of open and closed sets in $\mathbb{R}^2$.

   Definition 1. *We say that $B_r(x_0)$ is an open ball of radius $r > 0$ if and only if*

   $$B_r(x_0) = \{x \in \mathbb{R}^2 \mid \|x - x_0\| < r\}.$$

   Furthermore $\bar{B}_r(x_0)$ is a closed ball of radius $r > 0$ if and only if

   $$\bar{B}_r(x_0) = \{x \in \mathbb{R}^2 \mid \|x - x_0\| \leqslant r\}.$$

   Using the above definition we now give our notion of a bounded plane reigon.

   Definition 2. *If A is a subset of $\mathbb{R}^2$ we will say that A is the area of a bounded plane region if and only if for every $x \in A$, there is an open or closed ball centered at x which is a subset of A.*

   Lastly, we give the notion of a parallel line to the x-axis

   Definition 3. *We say that $L_r \subset \mathbb{R}^2$ is a line parallel to the x-axis at radius r if and only if*

   $$L_r = \{(x, y) \in \mathbb{R} \mid y = r\}.$$

   Now it is simple to propose the theorem of symantic equivalence to the question.

   Theorem 1. *Let A be the area of a bounded plane region in $\mathbb{R}^2$. Then, there exists some line parallel to the x-axis of height r, L_r, such that $L_r \cap A \neq \varnothing$ and both*

   $$A_L = \{(x, y) \in A \mid y < r\} \text{ and } A_U = \{(x, y) \in A \mid y \geqslant r\} \tag{1}$$

   are areas of bounded plane regions.

(c) *"All that glitters is mot gold."* Let G be the set of all object which glitter. Then let A be the set of all gold objects. $A \neq G$.

12. *Prove the following.*

Theorem 2. *There exists no smallest positive real number.*

Proof. Suppose that there exists a smallest real number, say $a \in \mathbb{R}$. Clearly $a > 0$ and so is $\frac{a}{2}$. Furthermore $\frac{a}{2} < a$, and hence we reach a contradiciton. Therefore does not exist a smallest postivie real number. □

Theorem 3. *There exist no smallest positive rational number.*

Proof. Suppose that there exists a smallest rational number, say $q \in \mathbb{Q}$. Clearly $q > 0$ and so is $\frac{q}{2}$. Furthermore $\frac{q}{2} < q$, and hence we reach a contradiciton. Therefore does not exist a smallest postivie rational number. □

Theorem 4. *Let $x \in \mathbb{R}$. Then there does not exist a smallest real number y such that $y > x$.*

Proof. Suppose that such a y exists. Now consider $\frac{x+y}{2} = b$. Clearly $b > x$, and remarkably $b < y$. Hence y is not the smallest real number such that $y > x$. This leads to a contradiction, and therefore there is no smallest y satisfying the conditions.

□

22. *Show the following.*

(a) *Fixed points*:

Theorem 5. *The function $f : A \to A$ has a fixed point if and only if the graph of f interesects the diagonal.*

Proof. We first show the right implication. If f has a fixed point, then there is some $a \in A$ such that $f(a) = a$. Now consider the graph of f,

$$f(A) = \{(a, f(a) \in A\}.$$

Since f has a fixed point, $f(A)$ contains (a, a). Hence the intersection of $f(A)$ with the diagonal of $A \times A$,D, must contain (a, a) at the least and hence is nonempty.

On the otherhand if the graph of f intersects the diagonal, then there exists some $(a, a) \in D$ such that $(a, a) \in f(A)$. Then by definition of the graph of f, $(a, a) = (a, f(a))$, which implies that $f(a) = a$. This completes the proof. □

(b) *Intermediate fixed point*

Theorem 6. *Every continuous function $f : [0, 1] \to [0, 1]$ has at least one fixed-point.*

Proof. To show this we recall the intermediate value theorem. More specifically we want to find a function whose 0 exists on $[0,1]$ which implies the theorem. Consider that $f(x) = x$ implies that $0 = f(x) - x$, so let's simply let $g(x) = f(x) - x$. By definition of the bound on the codomain, $g(0) \geqslant 0$ and $g(1) \leqslant 0$. Then application of the intermediate value theorem yields that there exists at $c \in [0,1]$ with $g(c) = 0$. Hence, $f(a) = a$. This completes the proof. ☐

(c) No, consider the case of some function for which $f(x) > x$ on $(0,1)$. Such a function need not attain the value $f(0) = 0, f(1) = 1$ because such values could not possiblt exist on its graph. Hence, $f(x) \neq x$ for all x.

(d) No, consider the function $f(x) = x + 0.5$ when $0 \leqslant x < 0.5$, and $f(x) = x - 0.5$ when $0.5 \leqslant x \leqslant 1$. This function never is equivalent to $g(x) = x$.

23. *Show the following.*

(a) *Dyadic squares:*

Theorem 7. *If x and y are two dyadic squares, then they are either identical, intersect along a common edge, intersect at a common vertex, or do not intersect at all.*

Proof. Since we must show all cases, let us consider them with respect to the general definition of a planar dyadic cube. In particular, $x, y \in \mathbb{Q}_2^2$. Let us fix x such that

$$x = \left[\frac{p}{2^k}, \frac{p+1}{2^k}\right]^2 \text{ and } y = \left[\frac{q}{2^k}, \frac{q+1}{2^k}\right]^2$$

for some $p, k, q \in \mathbb{Z}$.

If $q = p$, then $y = x$ naturaly. In the case that $q > p+1$ or $q+1 < p$, we have that $x \cap y = \varnothing$. Next consider intersections along different edges. If

$$y = \left[\frac{p}{2^k}, \frac{p+1}{2^k}\right] \times \left[\frac{p+1}{2^k}, \frac{p+2}{2^k}\right],$$

then $y \cap x = [(\frac{p}{2^k} \frac{p+1}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k})]$. In general,

$$y = \left[\frac{p+r}{2^k}, \frac{p+r+1}{2^k}\right] \times \left[\frac{p+s}{2^k}, \frac{p+s+1}{2^k}\right]$$

implies the following intersections.

If $r = 1, s = 0$, then $x \cap y = \left[(\frac{p+1}{2^k}, \frac{p}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k})\right]$. If $r = -1, s = 0$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p}{2^k}), (\frac{p}{2^k}, \frac{p+1}{2^k})\right]$. If $r = 0, s = 1$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p+1}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k})\right]$. If $r = 0, s = -1$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p}{2^k}), (\frac{p+1}{2^k}, \frac{p}{2^k})\right]$.

Lastly we need to consider the vertex edge cases. If $r = 1, s = 1$, then $x \cap y = \{(\frac{p+1}{2^k}, \frac{p+1}{2^k})\}$. If $r = -1, s = 1$, then $x \cap y = \{(\frac{p}{2^k}, \frac{p+1}{2^k})\}$. If $r = -1, s = -1$, then $x \cap y = \{(\frac{p}{2^k}, \frac{p}{2^k})\}$. If $r = 1, s = -1$, then $x \cap y = \{(\frac{p+1}{2^k}, \frac{p}{2^k})\}$.

Furthermore if r and s attain other values, we have those cases previously considered. Hence the proof is complete.

☐

(b) *For the following problem we adopt the following notation.*

Definition 4. *We say that say that some $X \subset \mathbb{R}^n$ is a dyadic hyper-interval of partition $2^{-\gamma}$ if and only if*

$$X \in \overline{\Delta_n^k} = \left\{ Y \subset \mathbb{R}^n \mid Y = \bigtimes_{i \in \delta_k} 2^{-\gamma} \left[(m_1, \ldots, m_n), (m_1, \ldots, m_i + 1, \ldots, m_n) \right] \right\},$$

where δ_k is the index set of dimensions in which the interval is non-empty and non-singular. Furthermore, $|\delta_k| = k$, and $m_i \in \mathbb{Z}$.

So now we need to operationalize this proof. If x and y are two dyadic squares, then they are either identical, intersect along a common edge, intersect at a common vertex, or do not intersect at all.

Theorem 8. *In other words, if $X, Y \in \overline{\Delta_n^n}$ are of the same partition, $2^{-\gamma}$, let*

$$Y = \bigtimes_{i=1}^{k} 2^{-\gamma} \left[(m_1 + r_1, \ldots, m_n + r_n), (m_1 + r_1, \ldots, m_i + 1 + r_i, \ldots, m_n + r_n) \right],$$

where the m_j are those which define X, and $r_j \in \mathbb{Z}$. Then, if $|r_j| \leqslant 1$ for all j, the following two results hold. If $k = n - \sum_i |r_i| > 0$, $X \cap Y \in \overline{\Delta_n^k}$. If $k = 0$, $X \cap Y \subset \mathbb{Q}_2^n$ with $|X \cap Y| = 1$. Otherwise if there exists some j such that $|r_j| > 1$, then $X \cap Y = \varnothing$.

Proof. We denote X_j, Y_j as the j^{th} interval composing X and Y. In the above definition of Y we wish to explore a multitude of different r_j values so as to express the theorem.

In the simplest case, $|r_j| > 1$ for some j then

$y_j = 2^{-k}[(m_1+r_1, \ldots, m_j+r_j \ldots, m_1+r_1), (m_1+r_1, \ldots m_j+r_j+1, \ldots, m_n+r_n)]$.

Clearly $m_j + 1 < m_j + r$ or $m_j > m_j + r_j + 1$, and thus $y_j \cap x_j = \varnothing$, we have that the whole cartesian product,

$$X \cap Y = \varnothing \times \left(\bigtimes_{i \neq j}^{n} x_j \cap y_j \right) = \varnothing,$$

because $\varnothing \times B$ cannot form any pair (a, b) as there is no $a \in \varnothing$.

We claim that when $|r_i| \leqslant 1$, $X \cap Y \in \overline{\Delta_n^k}$ for $k = n - \sum_{i=1}^n |r_i| > 0$. Let (n_p) denote the finite (possibly empty) list of indices for which $|r_j| = 1$. In other words, for all p, $|r_{n_p}| = 1$, else $|r_j| = 0$. The intersection as aforementioned is the cartesian product of all x_j, y_j. Hence for $j \notin \{n_p\}$, $x_j \cap y_j \in \overline{\Delta_n^1}$ with $\delta_1 = j$. Hence, the cartesian product of all such j is $X^* \cap Y^* \in \overline{\Delta_n^c}$ with $\delta_c = \{j \neq n_p \forall p\}$, and $c = n - |\{n_p\}|$. We claim that $X \cap Y$ cannot exist in any higher dimenisonality than $X^* \cap Y^*$.

Suppose $X \cap Y \in \overline{\Delta_n^d}$, with $n \geqslant d > c$. This implies that there exists a $q \in \{n_p\}$ such that $x_q \cap y_q = z_q$ is non-singular and non-empty. We have that

$$\begin{aligned} z_q &= [(m_1, \ldots, m_q, \ldots, m_n), (m_1, \ldots, m_q + 1, \ldots, m_n)] \\ &\quad \cap [(m_1, \ldots, m_q \pm 1, \ldots, m_n), (m_1, \ldots, m_q + 1 \pm 1, \ldots, m_n)] \\ &= \left\{ \left(m_1, \ldots, m_q + \frac{1 \pm 1}{2}, \ldots, m_n \right) \right\} \end{aligned}$$

is singular. Hence we reach a contradiction and $X \cap Y \in \overline{\Delta_n^c}$.

□

24. *Show the following*

   (a) *Dyadic squares in the unit ball.*

   Theorem 9. *Given* $\epsilon > 0$, *show that the unit disc contains finitely many dyadic squares whose total area exceeds* $\pi - \epsilon$, *and which intersect with eachother only along their boundries.*

   Proof. Let B_c^2 be a disk of radius $\sqrt{\frac{\epsilon}{\pi}} \leqslant c < 1$. Then consider the finite set S_k of all dyadic squares of partition $2^{-\gamma} = \frac{1-c}{2}$ such that $B^2 \supset \bigcup S_k \supset B_c^2$. Clearly the area of $\bigcup S_k > \pi - \epsilon$ but less that π. Hence for any $\epsilon > 0$, take S_k as aforementioned, and these satisfying squares do not intersect. The proof is complete. □

   (b) *Disjoint dyadic squares.*

   Theorem 10. *Given* $\epsilon > 0$, *show that the unit disc contains finitely many dyadic squares whose total area exceeds* $\pi - \epsilon$, *and which are disjoint.*

   Proof. For any $\epsilon > 0$, let $r = \frac{1+\sqrt{\frac{\epsilon}{\pi}}}{2}$. Clearly such a point is the average radfius of the unit ball and the unit ball with radius r. Now as before, divide the inside into pices of side length $2^{-n+1} = 1 - \sqrt{\frac{\epsilon}{\pi}}$. If only every second square in every direction is selected, that set, say S_1, is clearly disjoint. Furthermore the total area of this set is at least

$$a_1 = \frac{\alpha_0}{4} = \frac{\pi r^2}{4}.$$

Now for those dyadics not selected, subdivide those sets into 8 pieces in basis direction, and choose every other dyadic which is disjoint from S_1 and dyadics of the same class. Let S_2 be the set of S_1 union with this new set. The area of S_2 is at least

$$a_2 = a_1 + \frac{\alpha_0 - a_1}{4}.$$

Upon repeating this process we yield the following recurrence relation,

$$a_n = a_{n-1} + \frac{\alpha_0 - a_{n-1}}{4}.$$

Hence, we apply the methods of non-homogeneous recurrence relations and find that the general solution is clearly $a_n = c_1 \left(\frac{3}{4}\right)^n$. Then we solve for the particular solution, and yield that $a_n^p = \pi r^2$. So we simply solve $a_1 = \frac{\alpha_0}{4} = c_1 \frac{3}{4} + \alpha_0$ for c_1. Upon yielding $c_1 = -\alpha_0$, we find the total solution to the area upon n repitions of the process is

$$a_n = -\alpha_0 \left(\frac{3}{4}\right)^n + \alpha_0.$$

Now we show that there exists an N such that for some $n > N$, $a_n > \pi - \epsilon$. Observe,

$$\begin{aligned}\alpha_0 - \alpha_0\left(\frac{3}{4}\right)^n &< \pi - \epsilon \\ \left(\frac{3}{4}\right)^n &> \frac{\epsilon - \pi}{\alpha_0} + 1 \qquad (2) \\ n\ln\left(\frac{3}{4}\right) &> \ln\left(\frac{\epsilon-\pi}{\alpha_0} + 1\right).\end{aligned}$$

Hence, let $N(\epsilon) = \frac{\ln\left(\frac{\epsilon-\pi}{\alpha_0}+1\right)}{\ln\left(\frac{3}{4}\right)}$. By the logic of derivation for N, for every $\epsilon > 0$ and for all $n > N(\epsilon)$, $a_n > \pi - \epsilon$.

Take the first such n. Then the set of disjoint dyadics, S_n, which induce the area a_n is finite, and the proof is complete. □

(c) *Dyadic hypercubes filling a ball.*

Theorem 11. *Given* $\epsilon > 0$, *show that the unit ball contains finitely many dyadic hypercubes whose total hypervolume exceeds* $V_m(1) - \epsilon = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2}+1)} - \epsilon$, *and which intersect with eachother only along their boundries.*

Proof. Let B_c^m be a ball of hypervolume $\epsilon < v < V_m(1)$, and therefore radius $\frac{\left(\epsilon\Gamma(\frac{m}{2}+1)\right)^{1/m}}{\sqrt{\pi}} \leqslant c < 1$. Then consider the finite set $S_k \subset \overline{\Delta_m^m}$ of all dyadic hypercubes of partition $2^{-\gamma} = \frac{1-c}{2}$ such that $B^m \supset \bigcup S_\gamma \supset B_c^m$. These cubes will fill the ball of radius $\frac{\left(\epsilon\Gamma(\frac{m}{2}+1)\right)^{1/m}}{\sqrt{\pi}}$ at least. Clearly the hypervolume of $\bigcup S_k >$ $V_m(1) - \epsilon$ but less than $V_m(1)$. Hence for any $\epsilon > 0$, take S_γ as aforementioned, and these satisfying hypercubes do not intersect except along common edges (as proved in 23. The proof is complete. □

(d) *Prove theorem 10 when the unit ball is replaced with the unit square, and circles are inscribed on the dyadic hypercube lattice.*

Proof. Given $\epsilon > 0$. Let $B^2_{-\gamma/2}$ denote the disk inscribed in the dyadic square $\delta \in \overline{\Delta_2^2}$ of partition $2^{-\gamma}$ at some position in $\mathbb{Q}_2^2$. Now consider the unit square and the square at the origin of area ϵ_2 and sidelength $\sqrt{\epsilon} + c$. Define γ to be the rounded solution of $2^{-\gamma} = \frac{1-\sqrt{\epsilon}+c}{2}$. Then let S_1 be the family of every other dyadic square of partition $2^{-\gamma}$ filling the square of area ϵ_2 completeley and then some. The area of such squares is at least $a_1 = \frac{\epsilon_2}{4}$. Then the area of union of the family of ball inscribing all dyadic squares in S_1 is $b_1 = \frac{\epsilon_2}{8}$.

For those squares not selected subdivide them into 16 dyadic squares and choose every other such that these squares are disjoint from one another and their family is disjoint from S_1. Take the union of their family and S_1 to produce S_2 whose area is at least $a_2 = a_1 + \frac{\epsilon_2 - a_1}{8}$. Taking those circles inscribed yields that $b_2 = b_1 + \frac{\epsilon_2 - a_1}{32}$.

Repeating this process yields a geometric series b_n similar to a_n in part (b). By the same logic in part (b), there will exist an n such that $b_n > \epsilon$ and hence a finite disjoint dyadic partitioning of the unit square such that the area of disk inscription of this partitioning has area greater than ϵ which approaches ϵ_2. This completes the proof. $\square$

32. *Suppose that E is a convex region in the plane bounded by a curve C.*

    (a) *Show the following*

    Theorem 12. *The curve C has a unique tangent line except at a countable number of points.*

    Proof. We first show that their exists a tangent line for every point $c \in C$. Let

$$T_c = \left\{x \in \mathbb{R}^2 \mid x = c + rt, t \in \mathbb{R}\right\},$$

for some slope vector r such that $T_c \cap (E \backslash C) = \varnothing$. We show that $\forall c, T_c \cap E \neq \varnothing$. Take some $c \in C$ and fix it. Then for some sequence of points on the curve, q_n, which start at some other point c' and increase monotonically with respect to angle from the center of E such that $q_n \to c$. Let the secant line to c at some point q be denoted,

$$S_q = \left\{x \in \mathbb{R}^2 \mid q + \frac{(c-q)}{\|c-q\|}t, t \in \mathbb{R}\right\}.$$

Consider that $[q_n, c] \subset S_{q_n}$, and $S_{q_n} \backslash [q_n, c] \cap E = \varnothing$. For all n, $[q_n, c]$ is clearly non-empty (it contains at least, c), so $\bigcap_n [q_n, c]$ is also non-empty. Therefore, as $q_n \to c$, $S_{q_n} \to S_c \supset \bigcap_n S_{q_n} = c$. S_c could not possibly contain an element of $E \backslash C$. Suppose it contains, $e \in E \backslash C$, for the purpose of reaching a contradiction. Then $e \in [c, c]$ such that $e \neq c$, which leads to a contradiction. Therefore, $S_c = T_c$ for some tangent line satisfying the definition.

Now we will show that T_c is unique except at countably many points. Let us define the function $\tau : C \to [0, 2\pi]$ which assigns to every point on the curve C the angle of its tangent line ,with respect to some starting angle of a point $q \in C$. By the logic above, for every p $\tau(p)$ exists. Let $\phi : \mathbb{R} \to C$ be a bijective paramaterization of C starting at some point q such that one walks counter clockwise with respect to q a distance t and yields $\phi(t)$.

It is easy to see that $\phi(t)$ is a continuous function because the region E is connected. We claim that $\tau \circ \phi(\mathbb{R})$ has countably many discontinuities. For every p, $\tau(p)$ is the angle of the tangent to p. If no angle exists, there must exist more than one tangent by the previous logic. Furthermore angle for which there is no tangent lies in some open interval. In other words, $\tau(\phi(S \subset \mathbb{R}))$ experiences discontinuities in disjoint intervals.

Given a family of open disjoint intervals in $\mathbb{R}$, we claim that those disjoint families are countable. This holds because for each interval there exists a unique rational number therein, and since the rationals are countable, the family must be countable. Therefore, the discontinuities in $\tau(\phi(S \subset \mathbb{R}))$ are countably many. Geometrically speaking, the points at which there exist no unique tangent in the convex curve C surrounding a convex region E are therefore countably many.

This completes the proof. $\square$

(b) *Show the following.*

Theorem 13. *If f is a convex function, then it has a derivative except at countably many points.*

Proof. If f is a convex function, then for any $x, y \in [a, b] = I$, $m = \min_I f$, $M = \max_I f$, the line segment $\overline{f(x), f(y)}$ is contained in the region

$$S = \{(x, y) \in I \mid m \leqslant b \leqslant f(a)\}.$$

Let C be the curve, $C = \{(x, f(x))\}_{x \in I} \cup [(a, m), (a, f(a))] \cup [(a, m), (b, m)] \cup [(b, m), (b, f(b))]$. Since S is convex and bounded by C, C has tangents everywhere except at countably many points. Since $G = \{(x, f(x))\}_{x \in I} \subset C$ then the graph G has tangents except at countably many points. In other words, in any $[a, b]$, f has derivatives at except at countably many points. The proof is complete. □

45. *Let (a_n) be a sequence of real numbers. It is bounded if the set $A = \{a_1, a_2, \dots\}$ is bounded. The limit supremum, or lim sup, of a bounded sequence (a_n) as $n \to \infty$ is*

$$\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \left(\sup_{k \geqslant n} a_k \right)$$

(a) *Why does lim sup exist?* It exists because if A is bounded, so is any subset. Therefore the set of a_k with $k \geqslant n$ must be a bounded set in $\mathbb{R}$ with an upper bound $\sup_{k \geqslant n} a_n$.

(b) *If* $\sup a_n = \infty$*, how should we define* $\limsup_{n \to \infty} a_n$*?*. One should define $\limsup a_n = \infty$, as every subsequence diverges.

(c) *If* $\lim a_n = -\infty$*, how should we define* $\limsup_{n \to \infty} a_n$*?*. One should define $\limsup a_n = -\infty$, as every subsequence diverges.

(d) Prove the following.

Definition 5. *We say (s_n) is a sequence of supremums for a sequence (a_n) if and only if,*$s_n = \sup\{a_k : k \geqslant n\}$.

Theorem 14. *Let $\{a_n\}, \{b_n\}$ be sequences in $\mathbb{R}$. If* $\limsup a_n$ *and* $\limsup b_n$ *are finite, then* $\limsup (a_n + b_n) \leqslant \limsup a_n + \limsup b_n$.

Proof. Consider the sequences of supremums for both a_n and b_n respectively, $\{s_{a,k}\}, \{s_{b,k}\}$. Also consider the sequence of supremums for $a_n + b_n$, that is $\{s_{c,k}\}$.

We have that for fixed k, $a_n + b_n \leqslant s_{a,k} + s_{b,k}$ for all $n \geqslant k$ by adding the inequalities defined for $s_{a,k}$ and $s_{b,k}$.

Then notice that $\sup_{n \geqslant k} a_n + b_n$ is less than or equal to $s_{a,k} + s_{b,k}$ for k fixed. So

$$\sup s_{c,k} \leqslant s_{a,k} + s_{b,k} \tag{3}$$

holds as $k \to \infty$. Therefore $\limsup (a_n + b_n) \leqslant \limsup a_n + \limsup b_n$. This completes the proof. □

(e) *Define the limit infinum and relate it to the limit supremum.*

Definition 6. *We say* (i_n) *is a sequence of infinums for a sequence* (a_n) *if and only if,*$i_n = \inf\{a_k : k \geqslant n\}$.

Definition 7. *If* (a_n) *is a sequence and* (i_n) *is defined as above, then*

$$\liminf a_n = \lim i_n.$$

Theorem 15. *Every limit of subsequences* (a_n) *is between* $\liminf a_n$ *and* $\limsup a_n$.

Proof. If some subsequence (a_{m_k}) converges, it follows that because $(i_n), (s_n)$ are both non-decreasing/non-increasing sequences for all n, $i_n \leqslant a_n \leqslant s_n$. Therefore $i_{n_k} \leqslant a_{n_k} \leqslant s_{n_k}$. Because (i_n) and (s_n) both have limits, then each of their subsequences converge. Then by squeeze theorem we have that

$$\liminf a_n \leqslant \lim a_{n_k} \leqslant \limsup a_n.$$

□

(f) *Show that a sequence has a limit if its limit supremum and limit infimum are the same.*

Theorem 16. *A sequence* (a_n) *has a limit iff*

$$\limsup a_n = \liminf a_n = a.$$

Proof. As above we have that $i_n \leqslant a_n \leqslant s_n$. Since $\lim i_n = \lim s_n = b$, it follows from the squeeze theorem that $\lim a_n = b$. On the other hand if $\lim a_n = a$, we have that every subsequence of (a_n) has a limit, say b. Since $\limsup a_n, \liminf a_n$ are limits of subsequences of a_n, then they must both be b. This completes the proof. □

48. *Deform the treifoil knot in* 4 *dimensions.*