**Math 113 — Problem Set 7 — William Guss**

**(P94. 2)** Find all orbits of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}.$$

*Proof.* (Solution *really*). Consider the following orbit of 1,

$$1 \to 5 \to 8 \to 7 \to 1$$

which gives $O^1 = \{1, 5, 7, 8\}$. We now consider the orbit of 2,

$$2 \to 6 \to 3 \to 2$$

which gives $O^2 = \{2, 3, 6\}$. We now consider the orbit of 4,

$$4 \to 4$$

which gives $O^4 = \{4\}$ as a fixed point. We conclude that $\bigsqcup O^n = G$ gives a partition so all the orbits are found. $\square$

**(P94. 5)** Find all the orbits of $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 2$.

*Proof.* (Solution *really*). First consider that $\sigma^{-1}(n) = n - 2$. Now the first orbit $O^0 = \{2n \mid n \in \mathbb{Z}\}$ is the set of all even integers. Then $O^1 = \{1 + 2n \mid n \in \mathbb{Z}\}$ is exactly the set of odd integers. Clearly $O^1 \sqcup O^0 = \mathbb{Z}$ and so the orbits form a partition of the space therefore all orbits are found. $\square$

**(P94. 7)** Compute the product of cycles $(1, 4, 5)(7, 8)(2, 5, 7)$ on $\{1, \dots, 8\}$.

*Proof.* (Solution *really*). We iterate on each cycle through composition. First

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 3 & 4 & 7 & 6 & 2 & 8 \end{pmatrix}$$

which is composed into

$$\sigma_2\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 3 & 4 & 8 & 6 & 2 & 7 \end{pmatrix}.$$

Finally we take the last composition, thus giving

$$\sigma_3\sigma_2\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 8 & 6 & 2 & 7 \end{pmatrix}.$$

$\square$

**(P94. 10)** Write the following permutation as a product of disjoint cycles and then transpositions when

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}.$$

*Proof.* (Solution *really*). First we gather the constituent cycles. This can be done by obtaining each orbit and composing the cyclic product as follows. Take $O^1$ and yield

$$1 \to 8 \to 1.$$

Take $O^2$ and yield

$$2 \to 2$$

Take $O^3$ and yield

$$3 \to 6 \to 4 \to 3.$$

Take $O^5$ and yield

$$5 \to 7 \to 5.$$

Now we get the following orbits $O^1 = \{1, 8\}$, $O^2 = \{2\}$, $O^3 = \{3, 6, 4\}$, $O^5 = \{5, 7\}$ such that $\bigsqcup O^n = G$. Ignoring fixed point orbits given is thus a disjoint product of cycles $\sigma = (1, 8)(3, 6, 4)(5, 7)$.

Next the decomposition of $\sigma$ into transpositions is achieved as $(3, 6, 4) = (3, 4)(3, 6)$ giving

$$(3 \mapsto 4 \mapsto 3) \circ (3 \mapsto 6 \mapsto 3) = \begin{pmatrix} 3 & 4 & 6 \\ 4 & 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & 4 & 6 \\ 6 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 6 \\ 6 & 3 & 4 \end{pmatrix}.$$

Therefore $\sigma = (1, 8)(3, 4)(3, 6)$. $\square$