MATH 105: Homework 6

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1. Show some things.

   (a) *Show that the definition of linear outer measure is unaeffected if we demand that the intervals I_k in the coverings be closed instead of open.*

   Definition 1. *The linear outer measure of a set $A \subset \mathbb{R}$ is given by*

   $$m^*A = \inf\left\{\sum_k |I_k| : \{I_k\} \textit{ is a covering of } A \textit{ by open intervals}\right\}. \tag{1}$$

   Definition 2. *The closed linear outer measure of a set $\subset \mathbb{R}$ is given by*

   $$\bar{m}^*A = \inf\left\{\sum_k |\bar{I}_k| : \{\bar{I}_k\} \textit{ is a covering of } A \textit{ by } \textbf{\textit{closed}} \textit{ intervals}\right\}. \tag{2}$$

   Theorem 1. *Definition 1 and definition 2 give equivalent measures.*

   Proof. Take some set A and obtain its linear outer measure m^*A. By the definition of infimum, m^*A is the limit of outer measures of finer and finer countable coverings of A. The same argument can be made for $\bar{m}^*A$, except for $\bar{I}_k$ closed.

   Let the two respective sequences of coverings be given by $\mathcal{C}_i$ and $\bar{\mathcal{C}}_i$. Clearly

   $$m^*A \leftarrow m_i^*A = \sum_{C \in \mathcal{C}_i} |C| = \bar{m}_i^*A = \sum_{\bar{C} \in \bar{\mathcal{C}}_i} |\bar{C}| \to \bar{m}^*A \tag{3}$$

   And so $m^*A = \bar{m}^*A$. This follows subtly from $m(I) = m(\bar{I}) = b - a$. The proof is complete. □

   (b) The middle thirds cantor set has a covering by closed intervals C_i whose constituent area is $1/3^i$ and so the infimum has area 0.

   (c) How open should I really be?

   Theorem 2. *The outer measure of an interval can be taken without without conditions onc closedness/openess.*

Proof. Consider that any other covering of A besides that depicted in definition 1 and definition 2, has area in between those two coverings by monotonicity of outer measure. Therefore $m^*A \leq \nu A \leq \bar{m}^*A \implies \nu A = m^*A$. $\square$

(d) The same thing holds for planar outer measure, since effectiveley S as a rectangle is the product of n intervals. Furthermore, we can approximate any recatngle (open, closed, clopen, or neither) $\pm\epsilon$ by a bunch of squares.

3.

Theorem 3. *All lines are zero sets.*

Proof. Recall that (from the book) all rigid transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ are meseometries. Take any rotation and translation ϕ. By the exercise $m(\mathbb{R} \times \{a\}) = 0$ implies that $m(\phi(\mathbb{R} \times \{a\})) = m(\mathbb{R} \times \{a\}) = 0$. $\square$

Theorem 4. *All $n-1$ hyperplanes are zero sets in $\mathbb{R}^n$.*

Proof. Recall proposition 2 (from the book) then without loss of generality apply the meaeomorphism in the previous proof. $\square$

4. Higher dimensional Lemmas!

Lemma 1. *The boundary of an n-dimensional ball is an n-dimensional zero set.*

Proof. If Δ is the closed unit ball in $\mathbb{R}^n$, then $0 < m\Delta < \infty$ since $[-1/\sqrt{2}, 1/\sqrt{2}]^n \subset [-1,1]^0 n$. The unit sphere S^{n-1} is the boundry of Δ. It is sandwhiched between balls Δ_- of radius $1-\epsilon$ and Δ_+ of radius $1+\epsilon$. Corollary 8 implies

$$m(\Delta_-) = (1-\epsilon)^n m\Delta < m\Delta < (1+\epsilon)^n m\Delta = m(\Delta_+). \tag{4}$$

Measurability implies that $m(\Delta_+ \setminus \Delta_-) = m(\Delta_+) - m(\Delta_-) = ((1+\epsilon)^n - (1-\epsilon)^n)m\Delta$. This gives us

$$m\left(S^{n-1}\right) \leq ((1+\epsilon)^n - (1-\epsilon)^n)m\Delta = 2\left(\sum_{i=0}^{n} \binom{n}{i} \epsilon^{n-i}\right) m\Delta. \tag{5}$$

Since $\epsilon > 0$ is arbitrary, we get $m(S^{n-1}) = 0$. $\square$

Lemma 2. *Every open cube is a countable disjoint union of open balls plus a zero set.*

Proof. Let $S \subset \mathbb{R}^n$ be an open cube. It contains a compact ball Δ whose volume is greater than $1/2^n$ of the volume of the cube. This follows from

$$\frac{m(\Delta)}{m(S)} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} > \frac{1}{2^n}. \tag{6}$$

The difference $U_1 = S \setminus \Delta$ is an open subset of S with $m(U_1) < m(S)((2^n-1)/2^n)$. It is therefore the disjoint countable union of small open cubes S_i plus a zero set. Each cube contains a ball is volume is greater than $1/2^n$ of the volume of each cube, and so the total volume of the small balls are more than $1/2^n$ the volume of the small cubes. So we get that the difference is U_2 whose total volume is less than $m(U_1)(((2^n-1)/2^n)) = ((2^n-1)^2/2^{2n})$.

Repeating this process we get

$$m(U_k) = \frac{(2^n-1)^k}{2^{kn}} \implies \ln(m(U_k)) = \ln((2^n-1)^k) - \ln(2^{kn}) = k(\ln(2^n-1)) - n\ln(2)) \to 0$$

since $\ln(2^n-1) \to n\ln(2)$. In other words, repition gives smaller and smaller compact balls with total measure equal to $m(S)$. Lemma 10 implies that the measure of a closed ball is the same as the measure of its interor, which completes the proof that S consists of countably many disjoint open cubes plus a zero set. □

Theorem 5. *An affine motion $T : \mathbb{R}^n \to \mathbb{R}^n$ is a meseomorphism. It multiplies measure by $|\det T|$.*

Proof. Assume that $Tv = Mv$ where M is an invertible matrix. We first claim that if Z is a zero set then so is TZ. Given any $\epsilon > 0$ there is a countable covering of Z by boxes R_k with total volume $< \epsilon$. Each R_k can be covered by cubes with total volume $m(R_k) + \epsilon/2^k$. Hence Z can be covered by countably many cubes S_i with volume 2ϵ. The T image of each S_i is contained in a cube with edge length $\|T\| diam S_i$. This finally gives, TZ contained by cubes whose total volume is

$$\sum (\|T\| diam S_i)^n = \sum n^{n/2} \|T\|^n |S_i| \leq 2n^{n/2} \|T\|^2 \epsilon. \tag{7}$$

Since $\epsilon > 0$ is as small as we like, we have $m(TZ) = 0$.

Next we claim that orthogonal transformations are meseometries. Let $O : \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal. It carries the ball $B(r,p)$, to the ball $B(r, Op)$, which is a translate of $B(r,p)$. Let S be a cube. The previous lemma implies that $S = \bigsqcup B_i \cup Z$ where B_i are n-balls and Z is a zero set. The O-image of B_i is a ball of equal measure, and the O-image of Z is a zeroset. Hence, $m(OS) = mS$. Given $\epsilon > 0$, there is a countable covering of A by cubes S_i with $\sum |S_i| < m^*A + \epsilon$. Thus $\{O(S_i)\}$ covers OA and has total area $< m^*A + \epsilon$. We therefore get

$$m^*(OA) \leq m^*A. \tag{8}$$

Since O^{-1} is also orthogonal, it too does not increase outer measure. Theorem 7 implies that O is a meseometry.

Finally, we use Polar Form to write

$$M = O_1 D O_2 \tag{9}$$

where O_1, O_2 are orthogonal and D is diagonal. Since O_1 and O_2 are meseometries and by Corrolary 8 D is a meseomorphism which multiplies measire bt $|detD| = |detT|$, the proof is complete. □

5. Interesting general stuff for $\mathbb{R}$!

Theorem 6. *Every closed set in $\mathbb{R}^n$ is a G_δ set, furthermore every open set is a F_σ set.*

Proof. Take $S \subset N$ to be some closed set. Then for every $n \in \mathbb{N}$ let

$$O_n = \bigcup_{x \in S} B\left(x, \frac{1}{n}\right), \tag{10}$$

where $B(p, r)$, is the open ball of radius r at p. Then clearly

$$\bigcap_{n=1}^{\infty} O_n = S, \tag{11}$$

and S is a G_δ set. Let Y be some open set in N. Then Y^c is closed and therefore is an G_δ set. That is, there exist some open family $\{O_n\}$ so that

$$Y^c = \bigcap_{n=1}^{\infty} O_n \implies Y^{cc} = \bigcup_{n=1}^{\infty} O_n^c \tag{12}$$

and Y is an F_σ set. □

7. *Prove that inner measure is translation invariant.* Observe that trnaslation, $T : \mathbb{R}^n \to \mathbb{R}^n$ is an affine motion with $|detT| = 1$. This can be seen from linear algebra class using an augmented matrix! Furthermore all dialations are affine motions so we propose the following theorems warranting that

Theorem 7. *A set E is measurable if and only if $m^*E = m_*E$.*

which will be shown in Question 9.

Lemma 3. *The boundary of an n-dimensional ball is an n-dimensional zero inner measure set.*

Proof. If the outer measure of a set is 0, then the inner measure must be 0. □

Lemma 4. *Every open cube is a countable disjoint union of open balls plus a zero inner measure set.*

Proof. If it is true for a zero outer measure set, then it must be that the inner measure of such a set is a zero set. Furthermore, a ball is measurable so inner measure is outer measure. □

Theorem 8. *An affine motion $T : \mathbb{R}^n \to \mathbb{R}^n$ is a inner meseomorphism. It multiplies inner measure by $|\det T|$.*

Proof. Assume that $Tv = Mv$ where M is an invertible matrix. We first claim that if Z is a zero inner measure set then so is TZ. Given any $\epsilon > 0$ there is a countable covering of Z by boxes R_k with total inner volume $< \epsilon$. Each R_k can be covered by cubes with total volume $m_*(R_k) + \epsilon/2^k$. Hence Z can be covered by countably many cubes S_i with innern volume 2ϵ. The T image of each S_i is contained in a cube with inner edge length $\|T\| diamS_i$. This finally gives, TZ contained by cubes whose total inner volume is

$$\sum (\|T\| diamS_i)^n = \sum n^{n/2} \|T\|^n |S_i| \leq 2n^{n/2} \|T\|^2 \epsilon. \tag{13}$$

Since $\epsilon > 0$ is as small as we like, we have $m_*(TZ) = 0$.

Next we claim that orthogonal transformations are inner meseometries. Let $O : \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal. It carries the ball $B(r,p)$, to the ball $B(r,Op)$, which is a translate of $B(r,p)$. Let S be a cube. The previous lemma implies that $S = \bigsqcup B_i \cup Z$ where B_i are n-balls and Z is an inner measure zero set. The O-image of B_i is a ball of equal inner measure, and the O-image of Z is an inner measure zero set. Hence, $m_*(OS) = m_*S$. Given $\epsilon > 0$, there is a countable covering of A by cubes S_i with $\sum |S_i| < m_*A + \epsilon$. Thus $\{O(S_i)\}$ covers OA and has total inner volume $< m_*A + \epsilon$. We therefore get

$$m_*(OA) \leq m_*A. \tag{14}$$

Since O^{-1} is also orthogonal, it too does not increase inner measure. Theorem 7 implies that O is an inner meseometry.

Finally, we use Polar Form to write

$$M = O_1 D O_2 \tag{15}$$

where O_1, O_2 are orthogonal and D is diagonal. Since O_1 and O_2 are inner meseometries and by Corrolary 8 D is an inner meseomorphism which multiplies inner measure by $|detD| = |detT|$, the proof is complete. □

Therefore translations are mesometries.

9. The if and only if of measure theory. We take the approach of Folland's book. Let X be an open bounded subset of some measure space for which a measure ν is defined which induces the outer and inner measure on the space. For example, in Labesgue measure theory take X to be a massive rectangle. For some algebra in the power set of X denote F_σ to be the collection of countable unions of set in that algebra. Then denote $F_\sigma\delta$ to be the collection of countable intersections of sets in F_σ.

Theorem 9. *For every $E \subset X$ nonempty and for any $\epsilon > 0$, there exists $EA \in F_\sigma$ so that $m^*(A) \leq m^*(E) + \epsilon$.*

Proof. The covering which induces the outer measure of E is an G_δ set induced by sets for which ν is defined and contain E. Therefore by our previous theorem, its complement is an F_σ set, say $A' \subset E$ not proper. Since A' is an F_σ, set it is the countable union of closed sets in X. Using the axiom of choice, repeat the following process: Remove sets from the collection defining A' in an enumerable fashion until the resulting F_σ union of that collection has outer measure less than A'. This resulting set, A is F_σ and has the following property:

$$m^*(A) \leq E + \epsilon. \tag{16}$$

□

Theorem 10. *For some measure space $(M, \mathfrak{M}, \mu)$, we have that $A \subset X$ gives $\mu^*(A) = \mu_*(A)$, then $A \in \mathfrak{M}$.*

Proof. Let μ be some measure on $\mathfrak{M}$. For any $X \subset M$, we define the outer measure of X induced by μ

$$\mu^* X = \inf \{\mu(S) : S \in \mathfrak{M} \wedge S \supset X\}. \tag{17}$$

Dually we define the inner measure induced by μ as

$$\mu_* X = \sup \{\mu(S) : S \in \mathfrak{M} \wedge S \subset X\}. \tag{18}$$

Now suppose that $\mu_* X = \mu^* X$. We claim that $X \in \mathfrak{M}$. The set A is measurable if and only if for every test set $X \subset M$, we get

$$\mu^*(A) = \mu^*(A \cap X) + \mu^*(A^c \cap X). \tag{19}$$

Take some open subset of M, say H which contains A with finite measure ν which induces m^*. Then clearly, $m^*(A) = m_*(A) = \nu(H) - m^*(A)$. Since $\nu(H)$ is finite we are allowed to use the following

$$m^*(H) = \nu(H) = m^*(A) + m^*(A^c). \tag{20}$$

By the previous theorem for every ϵ, in particular every n such that $1/n = \epsilon$, there is a $B_n \in M_\sigma$ so that $B_n \subset A$, and $\mu^*(B_n) = \mu^*(A) + 1/n$. We get that the infinite intersection of the family $\{B_n\}$ has the property that

$$m^*(A) \leq m^*(\bigcap B_n) \leq m^*(B_n) \leq m^*(A) + 1/n. \tag{21}$$

We know that this infinite intersection forms an algebra on the sigma algebra of H so it is measurable. Letting n be as large as we like, $m^*(A) + m^(A^c) = m^*(A) + m^*(B \cap A^c) + m^*(A^c)$, we therefore get that A is measurable by the previous theorem and the first logic of the proof. $\square$