Math 110 — Homework 8 — UCB, Summer 2017 — William Guss

(8.1) Find an example of an inner-product space V and a linear transformation $T : V \to V$ with eigen value λ such that $\lambda^* \notin Eig(T^*)$.

Proof. Let $V = \ell_2(\mathbb{C})$ and let $T := L$ where L is the left-shift operator. Then λ with $|\lambda| < 1$ is an eigenvalue of the T; that is, when $W_\lambda = span\left((\lambda^i)_{i=1}^\infty\right)$ we have that $L(w) = \lambda w$, $w_\lambda \in W$. As in the notes $T^* = R$ where R is the right shift operator, however, the right shift operator R has no eigenvalues as proven on a previous homework. □

(8.2) Let V be a finite-dimensional inner-product space, and let $T : V \to V$ be linear. Show that $\langle v|T(v)\rangle \geq 0$ for all $v \in V$ if and only if there is some $S : V \to V$ such that $T = S^*S$.

Proof. If $\langle v|T(v)\rangle \geq 0$ then $\langle v|T(v)\rangle = \langle T^*(v)|v\rangle = \overline{\langle v|T^*(v)\rangle} = \langle v|T^*(v)\rangle = \langle T(v)|v\rangle$, and therefore T is self-adjoint.

Then T has a diagonalizable in some basis, that is there exist a $\mathcal{B}$ so that $[T]_{\mathcal{B}\to\mathcal{B}}$ is diagonal. Since it is semi-positive definite for all v,

$$0 \leq \langle v|T(v)\rangle = [v]_{\mathcal{B}}^*[T]_{\mathcal{B}\to\mathcal{B}}[v]_{\mathcal{B}} = \sum_{k=1}^{d} |v_{\mathcal{B}}^k|^2\lambda_k$$

which can only be the case if $\lambda_k \geq 0$. Then define S such that $[S]_{\mathcal{B}\to\mathcal{B}} = \sqrt{[T]_{\mathcal{B}\to\mathcal{B}}}$ where the square root is applied pointwise. Then $T = S^*S$ as the square roots are real and on the diagonal. This completes the implication.

In the other direciton if $T = S^*S$ then

$$\langle v|T(v)\rangle = \langle v|S^*Sv\rangle = \langle S(v)|S(v)\rangle = \|S(v)\|^2 \geq 0.$$

This completes the proof. □

(8.3) Define $T : C^\infty[-1,1] \to C^\infty[-1,1]$ by $T(f) = f''$. Show that $-T$ is nonnegative.

Proof. Recall from the text that the derivative D is anti-self adjoint. Furthermore, $T = D^2 = -D^*D$ by the anti-self-adjoint property. Thus $-T$ is nonnegative. □

(8.4) Let $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the right-shift operator. Compute $|R|$.

Proof. By definition $|R| = \sqrt{R^*R}$. Recall that $R^* = L$ and therefore $R^*R = LR : (v_1, \dots) \mapsto L(0, v_1, \dots) \mapsto (v_1, \dots)$. Therefore $LR = Id_{\ell_2(\mathbb{N})}$. Since the identity is idempotent, it is also its own square root. Thus $|R| = Id_{\ell_2(\mathbb{N})}$. □

(8.5) Let V and W be inner-product spaces and let $T : V \to W$ be linear. Show that T is unitary iff $\|T(v)\| = \|v\|$ for all v.

Proof. If $\|T(v)\| = \|v\|$ for all v, then $\langle T(v)|T(v)\rangle = \|T(v)\|^2 = \|v\|^2\langle v|v\rangle$ for all v. Therefore T is unitary. The foregoing steps algebraically are logically bidirectional and thus the proof is complete. □

(8.6) Find an exampole of a finite-dimensional inner-product space V, a linear-transformation $T : V \to V$, and a basis $\mathcal{B}$ of V such that $[T^*]_{\mathcal{B}} \neq [T]_{\mathcal{B}}^*$.

Proof. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have matrix representation in the standard basis $\mathcal{E}$ given by

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now, let $\mathcal{B} = \{(1,0),(1,1)\} \subset \mathbb{R}^2$ be the desired basis. We have that

$$[T^*]_{\mathcal{B}} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Applying computation to the basic operator we get

$$[T]_{\mathcal{B}} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

By observation of the foregoing $[T^*]_{\mathcal{B}} \neq [T]_{\mathcal{B}}^*$. This completes the proof. $\square$