MATH H104: Homework 7

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3 Functions of a Real Variable

1.

Theorem 1. *If $f : \mathbb{R} \to \mathbb{R}$ and $|f(t) - f(x)| \leq |t - x|^2 \forall t, x$, then $f'(x) = 0 \forall x$.*

Proof. For all t, x we have the above relation. In particular when $x \neq t$ we also have that

$$\frac{|f(t) - f(x)|}{|t - x|^2} \leq |t - x|^2.$$

So we have essentially that $\left|\frac{\Delta f}{\Delta x}\right| \leq |\Delta x|$. Using the limit definition of the derivative we have that $f'(x) \leq \lim_{\Delta x \to 0} |\Delta x| = 0$ for every x. Hence, $\forall x$, $f'(x) = 0$. This completes the proof. $\square$

4.

Theorem 2. *The following is true, $f(n) = \sqrt{n+1} - \sqrt{n} \to 0$.*

Proof. Consider that the above expression is equivalent in the following fassion to,

$$f(n) = \sqrt{n+1} - \sqrt{n} = \frac{n + 1 - n}{\sqrt{n+1} + \sqrt{n}}.$$

So it follows that $f(n) = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. It is clear that the above expression tends directly to 0. This completes the proof. $\square$

5.

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Furthermore, for all $x \neq 0$ have $f'(x)$ exist. If $\lim_{x \to 0} f'(x) = L$, $f'(0)$ exists and is equal to L.*

Proof. Simple. By definition, $f'(0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{t \to 0} \frac{f(0) - f(t)}{t}$ which by L'hopital's rule gives that $f'(0) = \lim_{t \to 0} f'(t) =:$ which completes the proof. $\square$

11. (a)

Theorem 4. *Let f be a real valued function with doamin (a, b). Given that $f''(x)$ exists, then*

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x).$$

Proof. Since $f''(x)$ exists it must follow that $f'(x)$ and $f(x)$ must exist. Furthermore, these functions are both continuous. Consider the following derivation.

Recall that

$$f''(x) = \lim_{h\to 0} \frac{f'(x+\frac{h}{2}) - f'(x-\frac{h}{2})}{h}$$

. Since $f'(z)$ is also itself a limit, we yield the following definition for $f''(x)$.

$$f''(x) = \lim_{h\to 0}\lim_{g\to 0} \frac{f'(x+\frac{h+g}{2}) - f'(x+\frac{h-g}{2}) - f'(x-\frac{h-g}{2}) + f'(x-\frac{-h-g}{2})}{gh}.$$

Since the there is no dependence of g on h and their combination is at most linear, we may redenote the limit as a single limit of h by the change of variables $h = g$. Upon reordering, this gives exactly definition,

$$f''(x) = \lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}.$$

$\square$

(b) An example of a function where the limit exists is $f : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x|x|$. In this case, taking the limit yields

$$\lim_{h\to 0} \frac{(0-h)|0-h| - 2(0) + (0+h)|0+h|}{h^2}$$

or equivalently

$$\lim_{h\to 0} \frac{-h|h| + h|h|}{h^2} = \frac{0}{h^2} = 0$$

by L'hopitals theorem. However $f'(x) = 1/2|x|$ whose derivative clearly does not exist at $x = 0$, and thus the theorem does not hold if we are not known to $f''(x)$ existing.

The better question now is, what is the geometric interpretation of this limit? Clearly in some capacity, $x = 0$, is an inflection point of the function. Concavity directly changes at this point, and yet the second derivative does not exist. Taking such a limit and yielding a result may be akin to calculating something called the subgradient of a function. In fact, this approach may yield the most natural subgradient (for the SG method states that any value in $[-1, 1]$ may be taken as the subgradient at this point, but 0 reflects the most information about the given function, does it not?)

15. *Is the following argument 'bogus'? "If f is a real valued function such that $f(x) = x^2$ when $x < 0$ and $f(x) = x + x^2$ when $x \geq 0$, we have that $f''(x) = 2$ for **every** x."*

The argument is completeley bogus. The claim may very well be true for $x \in (0, \infty) \cup (-\infty, 0)$ but consider the derivative of the left and right sides of this function.

On the left hand we have $2x \to 0$ as $x \to 0$, and on the left, we have $1 + 2x \to 1$ as $x \to 0$. This means that the limits do not agree at 0 from the left and right. Hence the second derivative could not possibly exist at this point. BOGUS!

16. (a) Before we prove the statement, consider the following lemma.

Lemma 1. *If $f : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x^k$, then the derivative of f at all x is kx^{k-1}.*

Proof. To show the theorem we must equivalently show that for all x, $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = kx^{k-1}$. This is simply done by expansion of the binomial coefficients of $(x+h)^k$ as follows,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^k - x^k}{h} \\ &= \lim_{h\to 0} \frac{\sum_{m=0}^{k} \binom{k}{m} x^m h^{k-m} - x^k}{h} \\ &= \lim_{h\to 0} \binom{k}{k-1} \sum_{m=0}^{k-1} \binom{k-1}{m} x^m h^{k-m} \\ &= \lim_{h\to 0} \binom{k}{k-1} (x+h)^{k-1} \\ &= kx^{k-1}, \end{aligned}$$

which is clearly the claim. Hence the proof is complete. □

Now we pose the following theorem.

Theorem 5. *If $\ln(x) = \int_1^x \frac{1}{t} dt$, for $x > 0$ then $\ln$ is a smooth function.*

Proof. To show that *ln* is a smooth function, we need only show that it has infinite derivatives defined for all values of $x > 0$. This implicitly states that all derivatives, including ln itself, are continuous, because if the next derivative exists for all $x > 0$ then the derivative or function itself must be continuous.

We claim that for all n and for all $x > 0$ $\ln^{(n)}(x)$ exists and

$$\ln^{(n)}(x) = \frac{(n-1)!(-1)^{n-1}}{x^n}.$$

Consider the base case. By the fundamental theorem of calculus $\ln^{(1)}(x) = \frac{1}{x}$, which clearly exists for all $x > 0$ and satisfies the definition posed for $\ln^{(1)}(x)$ Now suppose that $\ln^{(k)}(x)$ exists for all $x > 0$, then we wish to show that $\ln^{(k+1)}(x)$ satisfies the above the form. In the following differentiation of $\ln^{(k)}(x)$ we use the quotient rule and Lemma 1.

$$\begin{aligned}
\ln^{(n)\,\prime}(x) &= (k-1)!(-1)^{k-1}\left(\left(x^k\right)^{-1}\right)' \\
&= (k-1)!(-1)^{k-1}\frac{0-(x^k)'}{(x^k)^2} \\
&= (k-1)!(-1)^{k-1}\frac{-kx^{k-1}}{(x^k)^2} \\
&= (k-1)!(-1)^{k-1}\left(-kx^{k-1}x^{-2k}\right) \\
&= ((k+1)-1)!(-1)^{(k+1)-1}\left(x^{-(k+1)}\right).
\end{aligned}$$

This illustrates that $\ln^{(k)}(x)$ of the form aforementioned implies $\ln^{(k+1)}(x)$ is also of the same form.

By induction all derivatives must be of the form. Furthermore the form exists in the reals for $x > 0$ and by the logic from the beginning of the proof, we have that all derivatives are therefore continuous. Hence ln is a smooth function! $\square$

(b)

Theorem 6. *For every $x, y > 0$, $\ln(xy) = \ln(x) + \ln(y)$.*

Proof. Fix y and let $f(x) = \ln(xy) - \ln(x) - \ln(y)$. Equivalently we have that

$$\begin{aligned}
f(x) &= \int_1^{xy} 1/t\ dt - \int_1^{y} 1/t\ dt - \int_1^{y} 1/t\ dt \\
&= \int_x^{xy} 1/t\ dt - \int_1^{y} 1/t\ dt
\end{aligned}$$

Performing the change of variables, $u = t/x$ yields that,

$$f(x) = \int_1^{y} 1/u\ du - \int_1^{y} 1/t\ dt \equiv 0$$

for all x. Thus $\ln(xy) - \ln(x) - \ln(y) = 0$ for all x, y since y was fixed. This completes the proof. $\square$

(c)

Theorem 7. *The function* ln *is monotone increasing and* ln *is surjective.*

Proof. We show that ln is monotone increasing. By the fundamental theorem of calculus, we have that $\ln(x)$ has derivative $1/x$ which is positive for all $x > 0$ if and only if the $\ln(x)$ is monotone increasing by definition.

Now we wish to show that the range of ln is all of $\mathbb{R}$. We knlow that $\ln \pi = \int_1^\pi 1/t\ dt > 0$ and by $\ln(xy) = \ln x + \ln y, \ln \pi^2 = 2 \ln \pi$. In fact for all n $\ln \pi^n = n \ln \pi$. By the intermediate value theorem, every value between $\ln \pi^n$ and $\ln \pi^{n+1}$ is achieved as ln is smooth. Furthermore, since $n \to \infty$ implies $n \ln \pi \to \infty$ we can expand the intermediate value theorem to all positive n and yield that ln achives all values between $\ln \pi$ and ∞. The same process can be applied for negative n. Thus ln achieves all values of $\mathbb{R}$. $\square$

20.

Theorem 8. *Show that $\mathbb{Q}^c$ cannot be the coutnable untion of closed intervals in $\mathbb{R}$.*

Proof. Since $cl(\mathbb{Q}^c) = \mathbb{R} = cl(\mathbb{Q})$, it is dense in $\mathbb{R}$. Furthermore we have that the $cl(\mathbb{Q}) = int(\mathbb{Q}^c)^c$ and that $int(\mathbb{Q})^c = cl(\mathbb{Q}^c)$. Finally $\mathbb{R} = int(\mathbb{Q}^c)^c \implies \mathbb{R}^c = int(\mathbb{Q}^c) = \emptyset$. So the interior of $\mathbb{Q}^c$ is empty.

Now suppose that $\mathbb{Q}^c$ is the countable union of closed sets in $\mathbb{R}$. Baire's Theorem states that at least one of these sets must have a non-empty interior, which implies that $\mathbb{Q}^c$ has a non-empty interior. This contradicts the fact that $\mathbb{Q}^c$ has a empty interior, and hence $\mathbb{Q}^c$ cannot be the countable union of closed intervals in $\mathbb{R}$. $\square$