Math 202A — UCB, Fall 2016 — William Guss
Problem Set 8, due Wednesday October 19

(8.1) (Folland problem 3.9) Let ν_j be a sequence of positive measures on a measurable space $(X, \mathcal{M})$, and let μ be a positive measure on $(X, \mathcal{M})$. Set $\nu = \sum_j \nu_j$. Show that if $\nu_j \perp \mu$ for all j then $\nu \perp \mu$, and that if $\nu_j \ll \mu$ for all j then $\nu \ll \mu$.

Proof. We first show that if $\nu \perp \mu$ for all j then $\nu \perp \mu$. For every j $\nu_\perp \mu$ if and only if there exist A_j, B_j so that $\nu_(A_j) = 0$ and $\mu(B_j) = 0$. Then since B_j is a countable collection of μ-null sets $\mu(\bigcup B_j) = 0$. Now $X \setminus \bigcup B_j = \bigcap A_j = A$ (follows from $B_j^C = A_j$) has the property that $\nu_j(A) = 0$ for all j; that is, $\nu(A) = \sum \nu(A) = 0$. Finally $A \cup \bigcup B_j = X$.

We now show that if $\nu_j \ll \mu$ for all j then $\nu \ll \mu$. Take $E \in \mathcal{M}$ so that $\mu(E) = 0$. Then $\nu(E) = 0$ for all j and $\nu(E) = \sum \nu_j(E) = \sum 0 = 0$, and so if $\mu(E) = 0$ then $\nu(E) = 0$ and $\nu \ll \mu$. □

(8.2) (Folland problem 3.11(b)) Let μ be a positive measure on $(X, \mathcal{M})$. A family of functions $\{f_\alpha : \alpha \in A\} \subset L^1(\mu)$ is said to be uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta \Rightarrow |\int_E f_\alpha \, d\mu| < \varepsilon$ for every $\alpha \in A$. Let $A = \mathbb{N}$. Show that if $f \in L^1(\mu)$ and if $f_n \to f$ in the L^1 "metric" then $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable.

Proof. To prove the statement we will show that $|\int f_\alpha|$ is bounded by $|\int f|$ and $|\int f_n - f|$ and then reason that there exist small enough mutual domains to both integrals so that the bound is as small as we like.

Let $\epsilon > 0$ be given. First for any E observe that because $f_n, f \in L^1(\mu)$

$$\left|\int_E f_n d\mu\right| = \left|\int_E f_n - f + f d\mu\right| \leq \left|\int_E f_n - f d\mu\right| + \left|\int_E \int f d\mu\right|$$

Since $f \in L^1(\mu)$ by Corollary 3.6 there is a $\delta > 0$ such that $|\int_E f \; d\mu| < \epsilon/2$ whenever $\mu(E) < \delta$.

Now it remains to show that $\{f_n - f\}$ is uniformly integrable. Since $f_n \to f$ in $L^1(\mu)$ there is an N such that for all $m \geq N$ $\int_X |f_m - f| d\mu < \epsilon/2$. Because $f_m - f \in L^1(\mu)$

$$\left|\int_E f_m - f d\mu\right| \leq \int_E |f_m - f| \, d\mu \leq \int_X |f_m - f| \, d\mu < \epsilon/2,$$

and the famlily $\{f_m - f \;\; : \;\; m \geq N\}$ is uniformly integrable. Now consider the finite family $F^* = \{f_m - f \; : \; n < N\}$. For every $\phi_n \in F^*$ there is a $\delta_n > 0$ so that $|\int_E \phi_n d\mu| < \epsilon/2$ as long as $\mu(E) < \delta_n$. There are finitely many postive δ_n so their minimum exists and is positive. Let

$$\delta^* = \min\left(\min_{1 \leq n < N} \delta_n, \delta\right).$$

If $\mu(E) < \delta^*$ then for all n

$$\left|\int_E f_n d\mu\right| \leq \left|\int_E f_n - f d\mu\right| + \left|\int_E \int f d\mu\right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore $\{f_n\}$ is uniformly integrable. □

(8.3) (Folland problem 3.16) Let μ, ν be σ–finite positive measures on a measurable space $(X, \mathcal{M})$. Let $\lambda = \mu + \nu$. Suppose that $\nu \ll \mu$, and let f be the Radon-Nikodym derivative $f = d\nu/d\lambda$. Show that $0 \leq f < 1$ μ–a.e. and show that $d\nu/d\mu = f/(1 - f)$.

Proof. We first show that if λ is a σ–finite positive measure tne $d\lambda/d\lambda = 1$ λ-a.e. First $\lambda = \lambda$ so λ-nullsets are λ-nullsets so $\lambda \ll \lambda$. Since λ is σ-finite and positive then $d\lambda = gd\lambda$, by Radon-Nikodym. Now for every E

$$\int_E d\lambda = gd\lambda \implies \int_E (1-g)d\lambda = 0$$

Therefore, taking absolute values and noting that $1, g \in L^1(\lambda)$ we get that $1 = g\ \lambda - a.e$ since $L^1(\lambda)$ is a metric space and $d(1,g) = 0 \implies 1 \equiv g$.

We now prove the theorem using at a linearity result from the page 91. First $\lambda = \nu + \mu$ and ν, μ positive. Thus if $E \in \mathcal{M}$ such that $\lambda(E) = 0$, then $0 = \lambda(E) = \nu(E) + \mu(E) = 0 + 0$. Hence $\nu + \mu \ll \lambda$ and positivity gives $\nu \ll \lambda$ and $\mu \ll \lambda$. Therefore by the previous paragraph and linearity of the Radon Nikodym deriative

$$\begin{aligned} \frac{d\lambda}{d\lambda} &= \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda} && (\mu - a.e) \\ 1 &= \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} + \frac{d\mu}{d\lambda} && (\text{by } \nu \ll \mu) \\ 1 &= \left(1 + \frac{d\nu}{d\mu}\right)\frac{d\mu}{d\lambda} \end{aligned}$$

Then since $1 + \frac{d\nu}{d\mu} > 0$ μ-a.e by ν, μ positive,

$$\frac{d\mu}{d\lambda} = \frac{1}{1 + \frac{d\nu}{d\mu}} \qquad (\lambda - a.e)$$

We then substitute into the original equation and get

$$0 \le \frac{d\mu}{d\lambda} = 1 - \frac{1}{1 + \frac{d\nu}{d\mu}} < 1 \qquad (\mu - a.e)$$

since $1/(1+x) > 0$ for all $x \in \mathbb{R}_{\ge 0}$. This gives $0 \le f < 1$. Next then by $f < 1$

$$\begin{aligned} \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda} &= 1 \\ \frac{1}{1 - \frac{d\nu}{d\lambda}} &= 1 + \frac{d\nu}{d\mu} \\ \frac{1 - 1 + \frac{d\nu}{d\lambda}}{1 - \frac{d\nu}{d\lambda}} &= \frac{d\nu}{d\mu} \\ \frac{\frac{d\nu}{d\lambda}}{1 - \frac{d\nu}{d\lambda}} &= \frac{d\nu}{d\lambda} \end{aligned}$$

and this completes the proof.

□

(8.4) (Folland problem 3.17) Let $(X, \mathcal{M}, \mu)$ be a measure space with μ σ–finite. Let $\mathcal{N}$ be a σ–algebra that is contained in $\mathcal{M}$, and let $\nu = \mu|_{\mathcal{N}}$. Assume that ν is σ–finite. Let $f \in L^1(\mu)$. Prove that there exists a $\mathcal{N}$–measurable function $g \in L^1(\nu)$ that satisfies $\int_E f\, d\mu = \int_E g\, d\nu$ for every $E \in \mathcal{N}$.

Proof. See attached page.

□

(8.5) (a) In problem (8.4), let $X = \mathbb{Z}$, let $\mathcal{M} = \mathcal{P}(X)$. Let $\mathcal{N} \subset \mathcal{M}$ be the collection of all $E \subset \mathbb{Z}$ with this property: If n is even then $n \in E \Leftrightarrow n+1 \in E$. Let μ be the natural counting measure on $\mathbb{Z}$. Let f, g be as above, where $f \in L^1(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$ Give a concrete formula for g in terms of f.

To first prove the statement we provide a definition.

Definition 0.1. An $\mathcal{N}$-chain, denoted C_k^m, is a set of contiguous integers such that

$$C_k^m = \{2k, 2k+1, \ldots, 2m, 2m+1\}$$

when $k \leq m$ and $k, m \in \mathbb{Z}$.

We now give a lemma about $\mathcal{N}$-chains.

Lemma 0.1. *If $E \in \mathcal{N}$ then there is an $A \subset \mathbb{Z}^2$ so that*

$$E = \bigcup_{(k,m)\in A} C_k^m.$$

This representation is not unique but can be reduced to a unique representation by merging adjacent $\mathcal{N}$-chains until none can be merged.

Proof. Given some $E \in \mathcal{N}$ as defined above, consider all even $n \in E$. By defintion of $\mathcal{N}$, $n+1 \in E$. We claim that if

$$Q = \bigcup_{n \in E/2} C_{n/2}^{n/2}$$

then $E = Q$, where $E/2$ is the set of even members of E. Let $J = E \setminus Q$. If $n \in J$ then n is such that $n \in E$, n odd, and $n-1$ not in E. It follows in particular that $n-1$ is not in E since if it were then the $\mathcal{N}$-chain $C_{(n-1)/2}^{(n-1)/2}$ would contain n and $n-1$ and therefore by $n-1 \in E$ even $C_{(n-1)/2}^{(n-1)/2}$ is a member of the union in Q. This would all contradict the fact that $n \in J$, so it must be that $n-1$ not in E. We claim that $J = \emptyset$. For all $n \in J$, $n-1 \in \mathbb{Z}$ even implies $n \in E \iff n-1 \in E$ and since $n-1 \notin E$, $n \notin E$ unless $n = \emptyset$. Therefore $J = \emptyset$.

Now for every $E \in \mathcal{N}$, E is just a union of $\mathcal{N}$ chains. A unique reduction exists but will not be proven. $\square$

We now prove (give) that there is a concrete definition for g in terms of f.

Proof. Take $f \in L^1(\mu)$ then take $E \in \mathcal{N}$. ow by the previous exercise there exists a $g \in L^1(\nu)$

$$\int_E f d\mu = \sum_{y \in range(f)} \mu(f^{-1}(\{y\}))y = \sum_{y \in range(g)} \nu(g)(g^{-1}(\{y\})).$$

Since g is $\mathcal{N}$ measurable it must be that $g^{-1}(K) = \bigcup C_j^m$ for a measurable set K by the previous lemma. There does not exist a $n \in g^{-1}(K)$ such that $g(n+1) \neq g(n)$ because is an $\mathcal{N}$-chain $g^{-1}(\{g(n)\})$ containing $\{n, n+1\}$ and is a $\mathcal{N}$-chain $g^{-1}(\{g(n+1)\})$ containing $\{n, n+1\}$ and so

$$g^{-1}(\{g(n)\} \cap \{g(n+1)\}) = g^{-1}(\{g(n)\}) \cap g^{-1}(\{g(n+1)\}) \supset \{n, n+1\}.$$

and thus $\{g(n)\}$ must equal $\{g(n+1)\}$.

With this in mind g must take 1/2 the value of $f(n) + f(n+1)$ on all even values in E so as to equate the integral. This function is a measurable combination of constant functions on $\mathcal{N}$-chains of cardinality 1, so it is measurable. Therefore if $E = \bigcup_{m \in A} C_{m/2}^{m/2}$

$$\begin{aligned}
\int_E f d\mu = \sum_{y \in range(f)} \mu(f^{-1}(y) \cap E) y &= \sum_{y \in range(f)} \sum_{n \in f^{-1}(y)} \chi_E(n) \mu(\{n\}) f(n) \\
&= \sum_{y \in range(f)} \sum_{n \in f^{-1}(y)/2} 2 \frac{f(n)\chi_E(n) + f(n+1)\chi_E(n)}{2} \\
&= \sum_{y \in range(g)} \sum_{n \in g^{-1}(y)/2} \nu(C_{n/2}^{n/2}) g(n) \chi_E(n) = \int_E g d\nu
\end{aligned}$$

where $g(n) = g(n+1) = (f(n) + f(n+1))/2$, if n is even and $f^{-1}(x)/2$ is the even subset of the preimage of x. ☐

(b) Now let $X = \mathbb{Z}^2$ and $\mathcal{M} = \mathcal{P}(X)$, with μ equal to counting measure on X. Let $\mathcal{N}$ be the σ–algebra of subsets of X of the form $B \times \mathbb{Z}$, where $B \subset \mathbb{Z}$ is arbitrary. Then $\mu|_{\mathcal{N}}$ is not σ–finite. Modify the conditional expectation construction by defining $\nu(B \times \mathbb{Z})$ to be the number of elements of B, for any $B \subset \mathbb{Z}$. Show that for any $f \in L^1(\mu)$ there exists $g \in L^1(\nu)$ satisfying the required relation, and give a concrete formula for g in terms of f.

Proof. Because $f \in L^1(\mu)$ then $\int_X d\mu < \infty$. Furthermore, $\int_{B \times \mathbb{Z}} f d\mu < \infty$. Thus define $m_b = \int_{\mathbb{Z}} f(b,z) d\mu(z) < \infty$. Now define $g(b,x) = m_b$ for all x. Then if $E \in \mathcal{N}$ and $I(b) = \int_{\mathbb{Z}} f(b,z) d\mu(z) = m_b$, by the FubiniTonelli theorem

$$\begin{aligned}
\infty > \int_E f d\mu = \sum_{y \in range(f)} y\mu(f^{-1}(y) \cap E) &= \sum_{m_b \in range(I)} m_b \mu_1(I^{-1}(m_b) \cap B) \\
\text{(by finiteness of the sum)} \quad &= \sum_{m_b \in range(I)} m_b \sum_{b \in I^{-1}(m_B) \cap B} \nu(\{b\} \times \mathbb{Z}) \\
&= \sum_{m_b \in range(g(\cdot,x))} m_b \sum_{b \in g(x,\cdot)^{-1}(m_b) \cap B} \nu(\{b\} \times \mathbb{Z}) \\
&= \sum_{m_b \in range(g(\cdot,x))} m_b \nu(g(\cdot,x))^{-1}(m_b) \cap B \times \mathbb{Z}) \\
\text{(by } g(b,x) = m_b \text{ for all } x) \quad &= \sum_{m_b \in range(g)} m_b \nu(g^{-1}(m_b) \cap (B \times \mathbb{Z}))
\end{aligned}$$

where μ_1 is the first projection measure (which is nameley equal to ν). ☐

(8.6) (Folland problem 3.18) Prove Proposition 3.13(c) of our text: For any complex measure ν on a measurable space $(X, \mathcal{M})$, $L^1(\nu) = L^1(|\nu|)$, and for any $f \in L^1(\nu)$, $|\int f \, d\nu| \leq \int |f| \, d|\nu|$.

Proof. First let $\mu = |\nu_r| + |\nu_i|$ and then by the book $d\nu = g d\mu$ with $g \in L^1(\mu)$ and $d|\nu| = |g| d\mu$ with $g \in L^1(\mu)$. Finally by Prop 3.13 (a,b), $d\nu = h d|\nu|$ where $h \in L^1(|\nu|)$ and $|h| = 1$ $|\nu|$-a.e. Next

$fd\nu = fgd\mu$ and $f, g \in L^1(\mu)$ by a proposition of the chapter. And we therefore have

$$\begin{aligned} fd\nu &= fgd\mu \\ &= f\frac{d\nu}{d|\nu|}d|\nu| \qquad & f\frac{d\nu}{d|\nu|} \in L^1(|\nu|) \\ &= f\frac{d\nu}{d|\nu|}|g|d\mu \qquad & f\frac{d\nu}{d|\nu|}|g| \in L^1(\mu) \end{aligned}$$

Then since $|\frac{d\nu}{d|\nu|}| = |h| = 1$ $|\nu|$-a.e we get (using the first and last equation from above)

$$\begin{aligned} |f|d\nu = |fg|d\mu \qquad &= |fg|d\mu \\ = |f\frac{d\nu}{d\mu}|d\mu \quad &= |f\frac{d|\nu|}{d\mu}|d\mu \\ = |f|d\nu \qquad &= |f|d|\nu| \end{aligned}$$

Now the equality gives $\int fd\nu < \infty$ if and only if $\int fd|\nu| < \infty$. So $f \in L^1(|\nu|)$ if and only if $f \in L^1(\nu)$ and so the function spaces are equal.

Finally

$$\begin{aligned} \left|\int f \ d\nu\right| &\le \int |f| \ d\nu = \int |f|\frac{d\nu}{d|\nu|}d|\nu| \\ &\le \int \left|f\frac{d\nu}{d|\nu|}\right| d|\nu| = \int |f|d|\nu| \end{aligned}$$

by $|\frac{d\nu}{d|\nu|}| = 1$ a.e. □

(8.7) (Folland problem 3.19) Let ν, μ be complex measures, and let λ be a positive measure, on a common measurable space. Show that $\nu \perp \mu$ if and only if $|\nu| \perp |\mu|$. Show that $\nu \ll \lambda$ if and only if $|\nu| \ll |\lambda$.

Proof. If $\nu \perp \mu$ then there exist A, B such that $A \bigcup B = X$ and $|\nu_i|(A) + |\nu_r|(A) = 0$ and $|\mu_r|(B) + |\mu_i|(B) = 0$. Let f, g such that $d\nu = fd(|\nu_i|(A) + |\nu_r|)$ and $d\mu = gd(|\mu_r| + |\mu_i|)$. Without loss of generality we will treat ν and then assume the same treatment for μ on B It follows that

$$|\nu|(A) = \int_A |f|d(|\nu_i|(A) + |\nu_r|) = 0$$

because $(|\nu_i|(A) + |\nu_r|)(A) = 0$. Thus $|\nu| \perp |\mu|$.

In the other direction If $|\nu|(A) = 0$ and $|\mu|(B) = 0$ then $\nu \ll |\nu|$ implies that A is a ν-nullset. Likewise $\mu \ll |\mu|$ implies that B is a μ-nullset.

Now if $\nu \ll \lambda$ then $E\lambda$ null implies that E ν null. Again

$$|\nu|(A) = \int_A |f|d(|\nu_i|(A) + |\nu_r|) = 0$$

by $d(|\nu_i|(A) + |\nu_r|) = 0$.

In the other directon if $|\nu| \ll \lambda$ then if $E\lambda$ null then E is $|\nu|$ null. Since $\nu \ll |\nu|$ then E is ν null. Therefore $\nu \ll \lambda$

□

(8.8) (Folland problem 3.21) Let ν be a complex measure on $(X, \mathcal{M})$. For any $E \in \mathcal{M}$ define measures μ_j, for $j \in \{1, 2, 3\}$, as in the problem statement in our text. Show that these measures are equal to one another, and are all equal to $|\nu|$.

Proof. Let $\mu_j(E)$ be given from the text and the sets over whuich they are the resultant supremums be $S_j(E)$.

First. We will show that $\mu_3 \leq \mu_1$. Since μ_3 is the supremum there exists a sequence of $L^1(\nu)$ measurable functions f_n with $|f_n| \leq 1$ so that $a_n = |\int_E f_n d\nu| \to \mu_3(E)$. If we show that a sequence in $S_1(E)$ tends to $\mu_1(E)$ above a_n then $\mu_3 \leq \mu_1$. For each n there is a sequence of simple functions $phi_{n,m}$ approximating f_n so that $\int |\phi_{n,m} - f_n| d\nu \to 0$ as $m \to \infty$. Then consider the embodiment of $\phi_{n,n}$ in S_3; that is

$$\begin{aligned}
\left|\left|\int_E \phi_{n,n} d\nu\right| - \mu_3(E)\right| &= \left|\left|\int_E \phi_{n,n} d\nu\right| - \left|\int_E f_n d\nu\right| - \mu_3(E) + \left|\int_E f_n d\nu\right|\right| \\
&\leq \left|\left|\int_E \phi_{n,n} d\nu\right| - \left|\int_E f_n d\nu\right|\right| + \left|\left|\int_E f_n d\nu\right| - \mu_3(E)\right| \\
&< \left|\left|\int_E \phi_{n,n} d\nu\right| - \left|\int_E f_n d\nu\right|\right| + \epsilon/2 \\
&\leq \int_E ||\phi_{n,n}| - |f_n||\, d\nu + \epsilon/2 \\
&\leq \int_E |\phi_{n,n} - f_n|\, d\nu + \epsilon/2 < \epsilon
\end{aligned}$$

Knowing that there is a sequence of simple function $\psi_n = \phi_{n,n}$ whose embodiment in S_3 tends towards the supremum with $|\psi_n| \leq 1$ and $|\psi_E \psi_n| \to \mu_3(E)$. Next we expand the definition of a simple function using the standard representation on E and get

$$\left|\int_E \psi_n d\nu\right| \leq \int_E |\psi_n| d\nu = \sum_{j=1}^{m} |y_n^j| \nu(E_j) \leq \sum_{j=1}^{m} |y_n^j||\nu(E_j)| \in S_1(E)^{\mathbb{N}}$$

where $S_1(E)^{\mathbb{N}}$ denotes the set of sequences in $S_1(E)$. Since the embodiment of ψ_n in $S_3(E)$ tends towards the supremum and is dominated by a sequence in $S_1(E)$ we have that the supremum of $S_(E)1$ dominates $S_3(E)$ and $\mu_3(E) \leq \mu)1(E)$.

Second. We would like to show $\mu_1 \leq \mu_2$. We will use the same technique as before. Let $(a_n) \in S_1(E)^{\mathbb{N}}$ so that $a_n \to \mu_1(E)$. Then

$$a_n = \sum_{j=1}^{m_n} \left|\nu(E_j^n)\right| \qquad \bigsqcup_{j=1}^{m} E_j^n = E.$$

Let $b_n = \sum_{j=1}^{\infty} \left|\nu(K_j^n)\right| \qquad \bigsqcup_{j=1}^{m} K_j^n = E$ where $K_j^n = \emptyset$ when $j > m_n$ and $K_j^n = E_j^n$ otherwise. Therefore $K_p^n \cap K_q^n = \emptyset$ where $p, q \geq m_n, q > p$ so by disjointness of the E_j^n then K_j^n are disjoint. Now $a_n \leq b_n$ for all n and since $a_n \to m_1(E)$ we use the same domination argument as before. Therefore $\mu_1(E) \leq \mu_2(E)$.

Third. We would like to show that $\mu_2 \le |\nu|$. This is immediate from one of the corrolaries (3.13). Consider $(a_n) \in S_2(E)^{\mathbb{N}}$ such that $a_N \to \mu_2$. Then

$$\begin{aligned} a_n &= \sum_{j=1}^{\infty} \left|\nu(E_j^n)\right| && \bigsqcup_{j=1}^{\infty} E_j^n = E. \\ &\le \sum_{j=1}^{\infty} |\nu|\left(E_j^n\right) \\ &= |\nu|(E) && \text{(by disjointness)} \end{aligned}$$

Fourth. We lastly would like to show that $\mu_3 = |\nu|$. We will use the Radon-Nikodym theorem. Let $f = \frac{d\nu}{d|\nu|}$, this exists by $\nu \ll |\nu|$. Then $\left|\int_E f d\nu\right| \le \int_E |f| d|\nu|$. But $\int_E |f| d|\nu|$ is just $\int_E d|\nu|$ because $|f| = |\frac{d\nu}{d|\nu|}|$ is 1 $|\nu|$-a.e. Finally by $L^1(|\nu|) = L^1(\nu)$ and $1 = |f| \le 1$, $|\nu|$-a.e. we have that $|f| \in S_3(E)$. Now suppose that there were a $g \in S_3(E)$ so that $g > |f|$ then $|g| > 1$ $|\nu|$-a.e. and this is not possible. Therefore $|f|$ is the supremum.

Conclusion. We've shown for aribtraty E $|\nu| = \mu_3 \le \mu_1 \le \mu_2 \le |\nu|$ and so $\mu_1 = \mu_2 = \mu_3 = |\nu|$. This completes the proof.

$\square$

Let $X, \mathcal{M}, \mu$ be a finite measure space, $N \subset \mathcal{M}$, $\nu = \nu|N$.
If $f \in L^1(\mu)$ then $\exists g \in L^1(\nu)$ so $\int_E f\,d\mu = \int_E g\,d\nu$.

<u>Proof</u> Let $\lambda^f = \int_E f\,d\mu$. λ^F is a complex measure since $\lambda^F = \lambda_r^F + i\lambda_i^f = \int Re(f)\,d\mu + i\int Im(f)\,d\mu$. Without loss of generality assume μ is positive (will address la

then $\lambda^F << \mu$, ~~Therefore~~ ~~This holds since~~
$\Uparrow$
$\lambda_i^f << \mu$ and $\lambda_r^F << \mu$, ~~a.c. reres~~

Now consider $\lambda_i^f|_N$. Claim that $\lambda_i^F|_N << \mu|_N$.

If $\mu|_N(E \in N) = 0$ then $\mu(E) = 0 \Rightarrow |\lambda_i^f|(E) = 0$ and $|\lambda_r^f|(E) = 0$
$\Rightarrow |\lambda_i^f|_N|(E) = 0 \wedge |\lambda_r^f|_N|(E) = 0 \Rightarrow \lambda_i^f|_N(E) = 0$.

Therefore $\exists h \in L^1(\mu|_N)$, so $h = \frac{d\lambda^F|_N}{d\mu|_N}$ and

$$\int_E d\lambda^F|_N = \int_E \chi_E \frac{d\lambda^f|_N}{d\mu|_N}\,d\mu|_N = \lambda^F|_N(E) = \lambda^F(E)$$

$$\Rightarrow \int_E \frac{d\lambda^f|_N}{d\mu|_N}\,d\nu = \int_E f\,d\mu.$$

~~see back for μ a complex measure.~~
~~signed~~
~~complex~~

~~Finally if $f \in L^1$, $\in$ μ is a signed measure~~

~~$f\,d\mu = f\frac{d\mu}{d|\mu|}\,d|\mu|$ where $\left|\frac{d\mu}{d|\mu|}\right| = 1$ $|\mu|$-a.e. consider~~

~~Redefine f to be $\frac{f}{\frac{d\mu}{d|\mu|}}$ since $f\frac{d\mu}{d|\mu|} \in L^1(\mu) = L^1(|\mu|)$~~

~~Then $f\,d\mu = f\,d(\mu^+ - \mu^-)$ and so~~

~~$\frac{f}{\frac{d\mu}{d|\mu|}}\,d\mu = f\,d|\mu|$~~ ~~redefine f st $f \neq f$~~

If μ complex valued then $\exists\, r \in L^1(|\mu|)$ s.t.

$\cancel{d\lambda =}\ \int d\mu = \int r \, d\lambda$, take $\lambda = |\mu_r| + |\mu_i|$.

Then $\frac{d\mu}{d\lambda} d\lambda = \overset{r}{\cancel{R}}\, d\lambda = d\mu$.

We then can let $d\mu|_N = \cancel{d\lambda}\ \frac{d\mu|_N}{d\lambda|_N} d\lambda|_N$

and then proceed ~~through~~ the the theorem normally. $\square$.