Math 202A — UCB, Fall 2016 — William Guss
Problem Set 14, dueeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeeee

(14.1) (Folland problem 4.65) Let U be an open subset of $\mathbb{C}$, and let $\{f_n\}$ be a sequence of holomorphic functions on U. If $\{f_n\}$ is uniformly bounded on compact subset of U, there is a subsequence that converges uniformly to a holomorphic function on compact subset of U.

Proof. If U is an open subset of $\mathbb{C}$ and $\{f_n\}$ is uniformly bounded on compact subset of U, then for any compact of U, say K, $f_n|_K$ is holomorphic. Additionally, since $\mathbb{C}$ is a second countable LCH space with the open ball topology, $f_n|_K$ is holomorphic on countably many fully connected precompacts, and therefore is analytic there. To lastly characterize f_n, uniform boundeness on K is equivalen to the existence of a real number M so that for all n, $|f_n(k)| \leq M$ when $k \in K$. Therefore $f_n|_K \not\to g$ where g is singular on any K.

Now for any compact $K \subset U$ that is without loss of generality fully connected, and for any $x \in K$. The analycity of f_n for all n lets us consider

$$f_n(x) - f_m(x) = \frac{1}{2\pi i}\int_{C_R(x)} \frac{f_n(y) - f_m(y)}{y - x}\, dy$$
$$|f_n(x) - f_m(x)| \leq \frac{1}{2\pi}\int_{C_R(x)} \left|\frac{f_n(y) - f_m(y)}{y - x}\right| dy \leq \frac{M4\pi R^2}{2\pi R}$$

By the unform boundedness of the sequence and the moduli bound of the function $|y - x|$. Therefore given $\epsilon > 0$ There is a $R = \epsilon/6M$ so that for all n, m, $|f_n(x) - f_m(x)| \leq \epsilon/3$. Now by f_1 continous on K take $\delta = \min(R, \delta_1)$ so that δ_1 is the radius of the domain $C_{\delta_1}(x)$ on which f_1 takes values near x of moduli difference less than $\epsilon/3$. Then when $y \in C_\delta(x)$ for any m

$$\begin{aligned}
|f_m(x) - f_m(y)| &\leq |f_m(x) - f_1(x) + f_1(x) - f_1(y) + f_1(y) - f_m(y)| \\
&\leq |f_m(x) - f_1(x)| + |f_1(x) - f_1(y)| + |f_1(y) - f_1(y)| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \quad :)
\end{aligned}$$

Therefore f_n is equicontinuous on any compact. Returning to a more general setting, we can place $x, y \in U$ arbitrarily becasue $\mathbb{C}$ is an LCH space, we can find a compact ball around x anywehre in U and then proceed with the proof. Therefore f_n is econtinuous on U.

Then by the Arzela-Ascoli Theorem II, there is an $f \in C(U)$ so that some subsequence of $\{f_n\}$ converges to f uniformly on compact sets. Then from compelx analysis, uniform convergence of analytic functions preserves analycity and so f is analytic. $\square$

(14.2) (Folland problem 4.63) Let $K \in C([0,1]\times[0,1])$. For $f \in C([0,1]$, let $Tf(x) = \int_0^1 K(x,y)f(y)\,dy$. Then $Tf \in C([0,1])$, and $\{Tf \ : \|f\|_u \leq 1\}$ is precompact on $C([0,1])$.

Proof. We will use A-A Thoerem 1 to show that $\mathcal{F} = \{Tf \ : \|f\|_u \leq 1\}$ is precompact. In order to do, we must first show that $Tf \in C([0,1])$ for any $f \in C([0,1])$ and furthermore that $\mathcal{F}$ is pointwise bounded, and equicontinuous.

To assert the first claim, we will show that T is a bounded linear operator between $T : C([0,1]) \to C([0,1])$ and then use a basic theorem of functional analysis to assert its continuity. First since $[0,1]^2$ is compact in the box topology (and the product topology by Tychnov) as a subset of a Hausforff space, it is closed and bounded. Additionally K is then bounded by M_K. Additionally every $f \in C([0,1])$ is bounded in absolute value by $M_f \geq \|f\|_u$. Therefore $\|Tf\|_u \leq M_K\|f_u\|$ and so T is a bounded operator. Next for any $f, g \in C([0,1]), T(f+g) = \int K_x(f+g)\, dy = \int K_x f\, dy + \int K_x g\, dy = Tf + Tg$ so the operator is bounded. Lastly we need show continuity of Tf on $[0,1]$.

By uniformy continuity of f on x, given any ϵ, there is a δ so that $|x-y| < \delta$ implies $|T[f](x) - T[f](y)| \leq \|T\| |f(x) - f(y)| \leq \|T\|\epsilon$. Therefore let $\delta' = \min\{\delta, \epsilon/\|T\|\}$ and Tf is continuous. Therefore $T : C([0,1])) \to C([0,1])$ is a bounded linear operator and on the topology of uniform convergence T is continuous.

Next we need show taht $\mathcal{F}$ is equicontinuous. For any $Tf \in \mathcal{F}$ we know that $\|f\|_u \leq 1$ and therefore $\|Tf\|_u = \|T\|\|f\|_u$. Let any ϵ be given and fix $Tf \in \mathcal{F}$, by uniform continuity of Tf for all $\epsilon > 0$ there is a δ with for any $x, y \in [0,1]$ and $|x-y| < \delta$, then $|Tf(x) - Tf(y)| < \epsilon/3$. Furthermore $\|Tf - Tg\|_u \leq \|T\|\|f-g\|_u$. In particular $|Tf(x) - Tg(x)| = |\int K(x,y)(f(y) - g(y))\, dy| \leq \|g-y\|_u \int |K(x,y)|\, dy$.

Now by the continuity of $K(x,y)$ we have that

$$\|g-y\|_u \int_{[0,1]} |K(x,y)|\, dy = \|g-y\|_u F(x) \in C([0,1])$$

Fix a particualr $Tf \in \mathcal{F}$ and let δ be its continuity constant around x for $\epsilon/3$. Then take $\delta' = \min\{\delta, \gamma\}$ where γ is the continuity constant of F around x for $\epsilon/3$; that is the γ so that $|x-y| < \gamma$ implies $|F(x) - F(y)| < \epsilon/3$. Then when $|x-y| < \delta'$ we have that

$$\begin{aligned} |Tg(x) - fg(y)| &\leq |Tg(x) - Tf(x) + Tf(x) - Tf(y) + Tf(y) - Tg(y)| \\ &\leq |Tg(x) - Tf(x)| + |Tf(x) - Tg(y)| + |Tg(y) - Tf(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \quad :) \end{aligned}$$

Then $\mathcal{F}$ is precompact on $C([0,1])$. □

In hindsight, proving contionuity of T was pretty useless, lol.

(14.3) (Folland problem 4.66) Show that $1 - \sum_1^\infty c_n t^n$ is the maclaurin series for $(1-t)^{1/2}$ on compacta of $(-1,1)$.

Proof. First recall that

$$c_n = \frac{1}{n!} \prod_{m=1}^{n} \frac{2m-3}{2}.$$

Then if we compute the series

$$1 + \sum_{n=1}^{\infty} c_n t^n = 1 + \left(-\frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{t^n}{n!} \prod_{m=1}^{n-1} \frac{2m-1}{2}.$$

The difference of any two partial of the absolute series when $j \le k$ is just

$$\begin{aligned}
\left|1+\sum_{n=1}^{k}|c_nt^n|-1-\sum_{n=1}^{j}|c_nt^n|\right| &= \left|\left(-\frac{1}{2}\right)\sum_{n=1}^{k}\frac{|t^n|}{n!}\prod_{m=1}^{n-1}\frac{2m-1}{2}-\left(-\frac{1}{2}\right)\sum_{n=1}^{j}\frac{|t^n|}{n!}\prod_{m=1}^{n-1}\frac{2m-1}{2}\right| \\
&= \frac{1}{2}\left|\left(\sum_{n=j}^{k}\frac{|t^n|}{n!}\prod_{m=1}^{n-1}\frac{2m-1}{2}\right)\right| \\
&= \frac{1}{2}\left|\prod_{m=1}^{j-2}\frac{2m-1}{2}\right|\left|\sum_{n=j}^{k}\frac{|t^n|}{n!}\prod_{m=j-1}^{n-1}\frac{2m-1}{2}\right| \\
&\le \frac{1}{2}\left|\prod_{m=1}^{j-2}\frac{2m-1}{2}\right|\sum_{n=j}^{k}\frac{1}{n!}\prod_{m=j-1}^{n-1}\frac{2m-1}{2}; \qquad |t|<1 \\
&= \frac{1}{2^{j-1}}(j-2)!!\sum_{n=j}^{k}\frac{(n-1)!!}{(j-1)!!\cdot n!2^{n-j}} \qquad \cdot !! \text{ is the double factorial} \\
&\le \frac{2^{j/2}(j/2)!}{2^{j-1}}\sum_{n=j}^{k}\frac{2^{\frac{n-1}{2}}((n-1)/2)!}{n!2^{n-j}} \\
&\le \frac{1}{2^{(j-2)/2}\prod_{j/2}^{j}n}\sum_{n=j}^{k}\frac{2^{\frac{2j-n-1}{2}}((n-1)/2)!j!}{n!} \\
&\le \sum_{n=1}^{k-j}\frac{(j/2)!(((j+n)-1)/2)!}{2^{n/2}(j+n)!} \le \sum_{n=j}^{k}\frac{((n-1)/2)!}{2^{n/2}\prod_{j/2}^{n}m} \\
&\le \frac{1}{2^{j/2}}\sum_{n=j}^{k}\frac{((n-1)/2)!}{2^{(n-j)/2}\prod_{j/2}^{n}m} \to 0 \;\; j,k\to\infty.
\end{aligned}$$

Therefore the series is uniformly convergent when $t \le 1$, and so on compact subsets of $(-1,1)$ the uniform convergence of $1+\sum_{n=1}^{\infty}c_nt^n$ yields[1] that if $f(t)=1+\sum_{n=1}^{\infty}c_nt^n$ then $f'(t)=\sum_{n=1}^{\infty}nc_nt^{n-1}$.

[1]Undergraduate real analysis.

Since $f'(t) = \sum_{n=1}^{\infty} nc_n t^{n-1}$ then

$$\begin{aligned}
-2(1-t)f'(t) = 2(t-1)\sum_{n=1}^{\infty} nc_n t^{n-1} &= 2\sum_{n=1}^{\infty} nc_n t^n - 2\sum_{n=1}^{\infty} nc_n t^{n-1} \\
&= 2\sum_{n=1}^{\infty} nc_n (t^n - t^{n-1}) \\
&= 2(-(-1/2))t^0 + 2c_1 t - 2c_2 t + 4c_2 t^2 - 6c_3 t^2 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} 2(c_n - (n+1)c_{n+1})t^n \\
&= 1 + \sum_{n=1}^{\infty} t^n 2\left(\frac{1}{n!}\prod_{m=1}^{n}\frac{2m-3}{2} - \frac{n+1}{(n+1)!}\prod_{m=1}^{n}\frac{2m-3}{2}\right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} 2\prod_{m=1}^{n}\frac{2m-3}{2}\left(1 - \frac{1}{2}\right) \\
&= f(t).
\end{aligned}$$

Therefore $f'(t) = -2(1-t)f'(t)$. Now multiplying by $(1-t)^{-1/2}f(t)$ we take the derivitive thereof and get $((1-t)^{-1/2}f(t))' = (1/2)(1-t)^{-3/2}f(t) + (1-t)^{-1/2}f'(t)$. Therefore $((1-t)^{-1/2}f(t))' = -(1-t)^{-3/2}(1-t)f(t) + (1-t)^{-1/2}f'(t) = 0$. So $(1-t)^{-1/2}f(t)$ is constant. Finally $f(0) = 1$ and thus $f(t) = (1-t)^{1/2}$. This completes the proof. □

(14.4) Let X, Y be compact Hausdorff spaces. Show that the algebra generated by all products of functions $(x, y) \mapsto f(x)g(y)$, where $f \in C(X)$ and $g \in C(Y)$ is dense in $C(X \times Y)$.

Proof. Let $\mathcal{A}$ be the set of functions mentioned in the problem. Clearly $\mathcal{A}$ is a subalgebra since it is by definition the minimal family closed under multiplication and addition containing all $p \in C(X \times Y)$ such that $p = (x, y) \mapsto f(x)g(y)$ for some $g \in C(Y), f \in C(X)$. We need to show that this algebra sepeartes points. Take any distinct $(x, y), (w, z) \in X \times Y$. Since X, Y are compact Hausdorff spaces, they are normal and so by Urhysohn's lemma the following functions exist. Let $f \in C(X)$ so that $f(x) = 1, f(w) = 0$. Let $g \in C(Y)$ so that $g(y) = 1$ and $g(z) = 0$. Additionally let $h \in C(X)$ so that $h(x) = 0, h(w) = 1$ and $k \in C(Y)$ so that $k(y) = 0$ and $k(z) = 1$. Then there are clearly $p : (a, b) \mapsto f(a)g(b)$ and $q : (a, b) \mapsto h(a)k(b)$ so that $p, q \in \mathcal{A}$ and aitionally $p(x, y) = 1 \neq q(x, y) = 0$ and $p(w, z) = 0 \neq q(w, z) = 1$. Therefore $\mathcal{A}$ seperates points.

Since multiplication is a continuous operator on $\mathbb{C}^2$ it follows that all $p \in G$ where G is the generating family of $\mathcal{A}$ are continuous, and continuous functions on $X \times Y$ are closed under algebraic operations (pointwise multiplication and addition). Additionally $\mathcal{A}$ is closed under conjugation since the generating family is closed under c.c. $\overline{fg} = \overline{f}\overline{g}$ where $\overline{f} \in C(X), \overline{g} \in C(Y)$. Lastly the non-zero constant map is in both $C(X)$ and $C(Y)$ so it is also in $\mathcal{A}$[2]so by the Complex Stone-Weierstrass Theorem we have $cl(\mathcal{A}) = C(X \times Y)$. □

(14.5) let $X = [0, 1]^A$ where $A \neq \emptyset$. Show that algebra generated by all coordinate maps $\pi_\alpha : X \to [0, 1]$ together with the constant function 1 is dense in $C(X)$.

[2]This is a proof that $\mathcal{A}$ saitisifies the second condition of the theorem.

Proof. Let $\mathcal{A}$ be the subalgebra described in the statement of the problem. The generating set of coordinate maps are continuous. We need show that $\mathcal{A}$ seperates points. Take any $x, y \in X$, distinct. Then $x = \prod_{\alpha \in A} x_\alpha \neq y = \prod_{\alpha \in A} y_\alpha$. Therefore there must be a $\alpha \in A$ so that $\pi_\alpha(x) = x_\alpha \neq y_\alpha = \pi_\alpha(y)$. So for every pair of distinct points $x, y \in X$ there is an α such that π_α seperates x, y in $[0,1] \subset \mathbb{C}$. Therefore $\mathcal{A}$ seperates points. Next if $\overline{\pi_\alpha} = Re(\pi_\alpha) - iIm(\pi_\alpha) = Re(\pi_\alpha) = \pi_\alpha$. Therefore $\mathcal{A}$ is generated by a family which is closed under complex conjugation and so $\mathcal{A}$ is closed under complex conjugation. Lastly since the constant map 1 is in $\mathcal{A}$ it cannnot be a subset of $\{f \in C(X) \ : f(x_0) = 0\}$. Therefore by the Stone-Weierstrass theorem $cl(\mathcal{A}) = C(X)$. This completes the proof. $\square$