MATH H104: Homework 13

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4 Function Spaces

30. Consider the following example.

Example 1. *The 1-sphere, S^1 is a compact path-connected and nonempty 1-manifold. Now consider the continuous mapping, $\phi : S^1 \to S^1$ which takes a point $p \in S^1$ and adds to it's angle $\sqrt{2}$. The function ϕ has no fixed points.*

Proof. It is fair to represent a point locally by its angle as S^1 is a one manifold and therefore has an atlas of functions bad grammar here dude! $f : S^1 \to E \subset \mathbb{R}$. The assertion that $\exists p \in S^1$ such that $\phi(p) = p$ implies that there exists an angle θ such that $\sqrt{2} + \theta \equiv \theta \pmod{2\pi}$. Suppose such a θ existed. Then $n\sqrt{2} + \theta = \theta + k2\pi$ implies $n2^{1/2} = k2\pi$, and so π is an algebraic number. A contradiction! □

34. Consider the ODE

$$y' = 2\sqrt{|y|}.$$

Theorem 1. *The ODE does not have unique solutions for $x \geqslant 0$, and $y(0) = 0$.*

Proof. Consider the solution $y_1(x) = 0$. Clearly $y_1(0) = 0$, and $y'(x) = 2\sqrt{|0|} = 0$. Then consider likewise4 the solution $y_2(x) = x^2$. Observer that $y(0) = 0^2 = 0$ and $y'(x) = 2x = 2\sqrt{x^2} = 2x$ when $x \geqslant 0$. □

In fact there are even more examples of solutions which are not unique. See figure 1 for those whose domain is infact in $\mathbb{R}^-$.

This does not however contradict Picard's theorem since, the function $f(y')$ defined is not uniformly lipschitz continuous.

Proof. Suppose that $f(t)$ where lipschitz continuous. Then in particular, there is a constant L such that $d(fx, fy) \leqslant Ld(x, y)$ for all $x, y \in M$ the domain of f. So take for the sake of contradiction $x = 0$, and let y approach 0. By f Lipschitz, we have that

$$\sqrt{y} \leqslant Ly$$

which is true if and only if $y/y^2 \leqslant L$. Since $y \to 0$ let us take $y = 1/n$. This asserts that, $n \leqslant L$ for all n which contradicts the archimedian property of $\mathbb{R}$. □

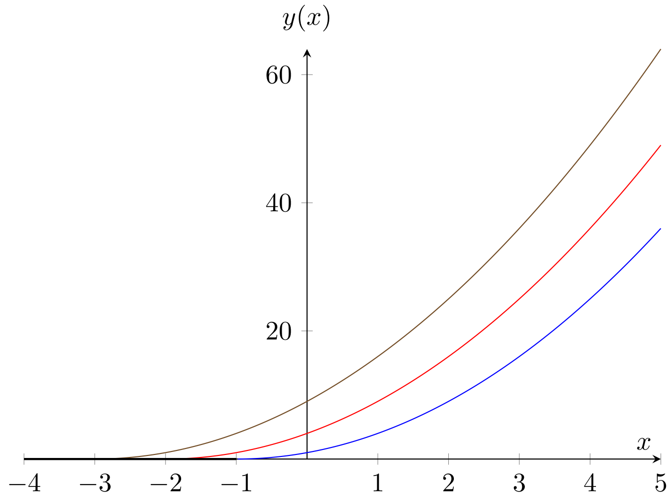

Figure 1: Other solutions to the ODE

35. We conjecture about the following ODE.

$$x' = x^2 \in \mathbb{R}.$$

The solution to the above ODE is obtained through the following calculations.

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ \int \frac{dx}{x^2} &= \int_{t_0}^{t} ds \\ -\frac{1}{2x(t)} + c_1 &= t \end{aligned} \tag{1}$$

and so we have that $x(t) = -\frac{2}{t-c_1}$. Where c_1 shifts the solution to satisfy the initial condition. However consider the solution where $x(-1) = 2$. It's clear that this solution is unbounded as $t \to 0$, and therefore escape to infinity in finite time.

36. We conjecture generally about separable ODE; that is differential equations of the following form.

$$D_s(x) = \mathcal{F}(x) \tag{2}$$

where $x : \mathbb{R} \to \mathbb{R}^m$ is differentiable.

Theorem 2. *Suppose*

$$\mathcal{F} : C^0(\mathbb{R}, \mathbb{R}^m) \to C^0(\mathbb{R}, \mathbb{R}^m).$$

is bounded; that is, there exists an M such that for all $\omega_2, \hat{0} \in C^0(\mathbb{R}, \mathbb{R}^m)$,

$$d_{C^0}\left(\hat{0}, \mathcal{F}[\omega_2]\right) \leqslant M \tag{3}$$

Then if $x \in C^0(\mathbb{R}, \mathbb{R}^m)$ is a solution to (2), it does not escape to infinity in finite time.

Proof. Let $Q_t = [t_0, t]$ be the finite time in which the differential form is evaluated. Then since the fundamental theorem of calculus yields

$$\int_{Q_t} D_s[x](s)\ ds = x(t) - x(t_0) = \int_{Q_t} \mathcal{F}[x](s)\ ds \tag{4}$$

and so we need only show that the right hand side does not escape to infinity in finite time. It follows that since $\mathcal{F}$ is a bounded mapping,

$$\begin{aligned} \left| \int_{Q_t} \mathcal{F}[x](s)\ ds \right|_2 &\leqslant \sup_{C^0} \left| \int_{Q_t} \mathcal{F}[\omega](s)\ ds \right|_2 \\ &\leqslant \sup_{\omega \in C^0} \sup_{z \in Q_t} |\mathcal{F}[\omega](z)|_2 m(Q_t) \\ &\leqslant \sup_{\omega \in C^0} \sup_{z \in \mathbb{R}} |\mathcal{F}[\omega](z)|_2 m(Q_t) \\ &\leqslant M m(Q_t) \end{aligned} \tag{5}$$

□

Theorem 3. *Suppose that $\mathcal{F}$ from above satisfies the lipschitz condition. Then any solutions do not explode in finite time.*

Proof. Recall that Picard's theorem asserts that $\mathcal{F}$ lipschitz implies that there is a locally unique solution to (2). Such a solution is a continuous mapping from $(a, b) \to \mathbb{R}^m$ which can be made to map the whole domain $\mathbb{R}$. Since this mapping is defined for the whole path connected set $\mathbb{R}$, it does not blow up in finite time. □

Lemma 1. *Suppose that $f : X \to Y$ is a mapping between two metric spaces satisfying the Lipschitz condition. Then it is uniformly continuous.*

Proof. Let $\epsilon > 0$, and $x, y \in X$. Take K to be the lipschitz constant for f. If $\delta = \epsilon/K$ and $d_X(x, y) < \delta$, then clearly $K d_X(x, y) < \epsilon$. Using lipschitz, we know that

$$d_Y(fx, fy) \leqslant K d_X(x, y) < \epsilon,$$

so f is uniformly continuous. □

Theorem 4. *If $\mathcal{F}$ is a uniformly continuous mapping then the solution to (2) does not explode in finite time.*

Proof. By the preceding lemma $\mathcal{F}$ satisfies the lipschitz condition and therefore by a previous theorem does not explode in finite time. □

39. We give an alternative proof for completion.

Theorem 5. *Every metric space can be completed.*

Proof. Let M be a metric space with distance d. Fix a point $p \in M$ and for each $q \in M$ define a function $f_q(x) = d(p, x) - d(p, x)$. Clearly $d(q, x) - d(p, x)$ is bounded since it is never more than $d(q, p)$. It is continuous since arithmetic in $\mathbb{R}$ is continuous.

Now consider the banach space $F = C_b^0(M, \mathbb{R})$. We have that $\phi : x \to f_x$ is an isometry from M into F since,

$$d(f_a, f_b) = \sup_{x \in M} |d(a, x) - d(p, x) - d(b, x) + d(p, x)| = d(a, b). \tag{6}$$

Since the M is dense in its closure and ϕ is an isometry, it follows that $\phi(M)$ is dense in the closure of M in F. Since $cl_F(M)$ is a closed subset of the complete metric space $C_b(M)$ it follows that $cl_F(M)$ is complete and therefore is the completion of the metric space M. □

41. For the purpose of this example, consider the following theorem of Keshner.

Theorem 6. *For any set $D \subset \mathbb{R}^n$ there is a separately continuous function $f : \mathbb{R}^n\mathbb{R}$ with $D(f) = D$ if and only if D is an F_σ set and every orthogonal projection of D onto a coordinate hyperplane has first category image.*

By Keshner's theorem it would seem that there are limits to just how discontinuous a function which is separately continuous can be. Therefore I will describe a function which is only discontinuous at a point (ie. not continuous as per the questions exact wording.)

Example 2. *Let $f : [0, 1] \to [0, 1]$ be defined such that*

$$(x, y) \mapsto \frac{(x - 0.5)(y - 0.5)}{(x - 0.5)^2 + (y - 0.5)^2} \tag{7}$$

when $(x, y) \neq 0$ and $(x, y) \mapsto 0$ at $(x, y) = 0$.

The partial derivatives of f exist at $(0.5, 0.5)$ but the function is clearly discontinuous as per *K-Ciesielski et al.*

43. The joke is that there ODE on a greecian urn. Nice one!