MATH H104: Homework 13

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4 Function Spaces

30. Consider the following example.

Example 1. *The 1-sphere, S^1 is a compact path-connected and nonempty 1-manifold. Now consider the continuous mapping, $\phi : S^1 \to S^1$ which takes a point $p \in S^1$ and adds to it's angle $\sqrt{2}$. The function ϕ has no fixed points.*

Proof. It is fair to represent a point locally by its angle as S^1 is a one manifold and therefore has an atlas of functions bad grammar here dude! $f : S^1 \to E \subset \mathbb{R}$. The assertion that $\exists p \in S^1$ such that $\phi(p) = p$ implies that there exists an angle θ such that $\sqrt{2} + \theta \equiv \theta \pmod{2\pi}$. Suppose such a θ existed. Then $n\sqrt{2} + \theta = \theta + k2\pi$ implies $n2^{1/2} = k2\pi$, and so π is an algebraic number. A contradiction! $\square$

34. Consider the ODE

$$y' = 2\sqrt{|y|}.$$

Theorem 1. *The ODE does not have unique solutions for $x \geq 0$, and $y(0) = 0$.*

Proof. Consider the solution $y_1(x) = 0$. Clearly $y_1(0) = 0$, and $y'(x) = 2\sqrt{|0|} = 0$. Then consider likewise4 the solution $y_2(x) = x^2$. Observer that $y(0) = 0^2 = 0$ and $y'(x) = 2x = 2\sqrt{x^2} = 2x$ when $x \geq 0$. $\square$

In fact there are even more examples of solutions which are not unique. See figure 1 for those whose domain is infact in $\mathbb{R}^-$.

This does not however contradict Picard's theorem since, the function $f(y')$ defined is not uniformly lipschitz continuous.

Proof. Suppose that $f(t)$ where lipschitz continuous. Then in particular, there is a constant L such that $d(fx, fy) \leq Ld(x, y)$ for all $x, y \in M$ the domain of f. So take for the sake of contradiction $x = 0$, and let y approach 0. By f Lipschitz, we have that

$$\sqrt{y} \leq Ly$$

which is true if and only if $y/y^2 \leq L$. Since $y \to 0$ let us take $y = 1/n$. This asserts that, $n \leq L$ for all n which contradicts the archimedian property of $\mathbb{R}$. $\square$

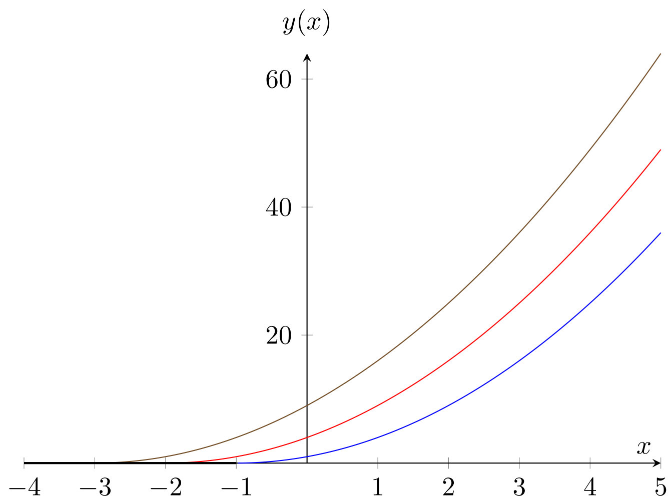

Figure 1: Other solutions to the ODE

35. We conjecture about the following ODE.

$$x' = x^2 \in \mathbb{R}.$$

The solution to the above ODE is obtained through the following calculations.

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ \int \frac{dx}{x^2} &= \int_{t_0}^{t} ds \\ -\frac{1}{2x(t)} + c_1 &= t \end{aligned} \tag{1}$$

and so we have that $x(t) = -\frac{2}{t-c_1}$. Where c_1 shifts the solution to satisfy the initial condition. However consider the solution where $x(-1) = 2$. It's clear that this solution is unbounded as $t \to 0$, and therefore escape to infinity in finite time.

36. We conjecture generally about separable ODE; that is differential equations of the following form.

$$x' = f(x) \in \mathbb{R} \tag{2}$$

Theorem 2. *If $f(x)$ is bounded then no solution of the ODE escape to infinity in finite time.*

Proof. If the ODE is bounded then there exists an M such that for all x, $|f(x)| \leq M$. Furthermore, we have that since the ODE is separable,

$$\begin{aligned} x'(s)\frac{1}{f(x(s))} &= 1 \\ \int_{t_0}^{t} x'(s)\frac{1}{f(x(s))}\, ds &= \int_{t_0}^{t} ds \end{aligned}$$

$\square$