Math 215A — Sards and Whitney and Tangent Bundles — William Guss
Lecture Notes

1. Lecture: Sards

Consider the following projection on to $\mathbb{R}$.

$$\begin{tikzcd} \mathbb{R} \supset T^2 \arrow[r, "\pi"] & \mathbb{R}^3 \arrow[d, "\phi(x)=(0,0,x)"] \\ \mathbb{R}^2 \arrow[u, "\phi"] \arrow[r] & \mathbb{R} \end{tikzcd}$$

In general if $f : X \to Y$ is a smooth map on manifolds X, Y then the preimage of certain points in Y enable us to yield submanifolds of X. These certain points are regular values.

Definition 1. For $f : X \to Y$ a point $y \in Y$ is a regular value of Y if for each $x \in f^{-1}(y)$, the differential

$$df_x : T_xX \to T_yY$$

is surjective. If y is not regular then it is a critical value.

Theorem 1. *Let $f : X \to Y$ be given as above. If y is a regular value of Y then $f^{-1}(y)$ is a submanifold of X.*

Theorem 2 (Sard's)**.** *If $f : X \to Y$ is smooth, the set of critical values has measure 0 in Y.*

Definition 2. For $f : X \to Y$ such that if f is an immersion of/submersion at $x \in X$ if $df_x : T_xX \to T_{f(x)}Y$ is injective or surjective.

Theorem 3. *If $f : X \to Y$ is an immersion then for every $x \in X$ and $y \in Y$, there exists charts ϕ at x and ψ at y such that*

$$f(x_1, \cdots, x_k) = (x_1, \cdots, x_k, 0, \cdots 0)$$

that is the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathbb{R}^k \supset U \arrow[u, "\phi"] \arrow[r] & V \subset \mathbb{R}^n \arrow[u, "\psi"] \end{tikzcd}$$

Theorem 4. *If $f : X^m \to Y^n$ is a submersion at $x \in X$ and the charts are given above, then*

$$f(x_1, \cdots, x_m) = (x_1, \cdots, x_k)$$

Transversality essentially describes when two manifolds cross eachother and have area in eachother when there is a crossing; that is there is no part of either manifold where the manifolds mearely touch. (*This is weird.*)

Definition 3. If X, Y are two submanifolds of some manifold M. Then if we say that $X \pitchfork Y$ if at each $p \in X \cap Y$, $T_pX + T_pY = T_pM$.

2. Lecture: Flows

Let us give our definition of the tangent space.

$$T_pM = \{\text{curves through } p\}/\sim .$$

Definition 4. We say that ξ is a vector field if

$$\xi : M \to \bigsqcup_{p\in M} T_pM = TM = \{(p,v) \mid p \in M, v \in T_pM\}.$$

where TM is called the tangent bundle.

Proposition 5. *The tangent bundle for a manifold M, TM is a topological manifold.*

Lemma 6 (Lee 1.35)**.** *Let M be a set $(U_i)_{i\in I}$ be a cover of M and $\phi : U \to \mathbb{R}^n$ with U open such that*

- *For all $i, j \in I$ $\phi_i(U_i \cap U_j)$ open;*
- *$\phi_i \circ \phi_j^{-1}$ such that $\phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a diffeomorphism;*
- *For all U_i can be reduced to a countable cover of M;*
- *The space M is hausdorff using $(U_i)_{i\in I}$ as a base.*

Then $\{\phi^{-1}(U) \mid i \in I, U \subset \mathbb{R}^n \text{ open}\}$ form a topology on M and the charts $\{(\phi_i, U_i)\}_{i\in I}$ are a smooth atlas.

Let $\phi : M \to U \subset \mathbb{R}^n$ be a diffeomorphism (chart). Then we claim $(TM, T\phi)$ is a chart. First off $T\phi(p, v) = T\phi(p, \sum_i v_i \frac{\partial}{\partial x_i}\big|_p) \mapsto (\phi p, (v_1, \cdots, v_n))$. Let M be any smooth manifold. (U_i, ϕ_i) be the associated charts to M. Then check that $\{(TU_i, T\phi_i)\}$ is an atlas. Furthermore $T(U_i) \cap T(U_j) = T(U_i \cap U_j)$.

Definition 5. Let $\theta : \mathbb{R} \times M \to M$ so that $\theta(0, p) = p$, $\theta(s, \theta(t, p)) = \theta(s + t, p)$. Then θ is a group action and if θ is a smooth map of manifolds, θ is called a flow.

3. Lecture: Whitney

Definition 6. An embedding $\theta : X \to Y$ is an embedding of topological space whose differential $d\theta$ is one-to-one; θ is an immaersion.

Theorem 7 (Whitney Embedding)**.** *Any smooth manifold M^n can be embedded into $\mathbb{R}^{2nn}$ and immersed into $\mathbb{R}^{2n-1}$.*

Theorem 8 (Weak Whitney Embedding)**.** *Any smooth manifold M^n can be embedded into $\mathbb{R}^N$ for sufficiently large N.*

We'll be proving the following weaker version of the standar Whitney Embedding Theorem.

Theorem 9 (Whitney Embedding)**.** *Any smooth manifold M^n can be embedded into $\mathbb{R}^{2n+1}$ and immersed into $\mathbb{R}^{2n}$.*

This theorem uses much measure theory so we need ot define measure on a manifold.

Definition 7. Let (X, Σ_X, μ) be a measurable sapce with a measure μ and let (Y, σ_Y, ν) be a measurable sapce. IF $f : X \to Y$ is a measurable map then the pushforward measure of μ by f onto y is a measure $\mu^* : \Sigma_Y \to [0, \infty]$ defined by $\mu^*(U) = \mu(f^{-1}(U))$ for all $U \in \Sigma_Y$.

Example 8. Take $X = cl(B^n)$ where $Y = \mathbb{RP}^n$ then we can define am easure easily on Y using the push forward of lebesgue measure and the cannonical projection on to the quotient space ofB^n.

Proof of Whitney Embedding in Compact. Let θ be the embedding given by the weak whitney embedding theorem with $N > 2n+1$. Let $u \in \mathbb{R}^N$ then $[u] \in \mathbb{RP}^{N-1}$ be the cannonical projection of u, then let $u^\perp$ be the orthogonal compliment to u.

Consider all elements $[u] \in \mathbb{RP}^{N-1}$ such that $\theta_u = \pi_u \circ \theta$ is not an embedding, π_u is the orthogonal projection of $\mathbb{R}^n \to u^\perp$. Denote this set A. We claim that $\mu^*(A) = 0$ using the pushforward measure in $\mathbb{KP}^{N-1}$.

Case 1. Suppose that θ_u is not injective. There exist $x_1 \neq x_2$ suvh that $\theta_u(x_1) = \theta_u(x_2)$. Then $[\theta_u(x_1) - \theta_u(x_2)]. = [u]$. Observe that $[u]$ lies in the image of $\tau : (M \times M) \setminus \Delta \to \mathbb{RP}^{N-1}$ given by $(x, y) \mapsto [\theta(x) - \theta(y)]$. By sards theorem $2n < N-1$ implies that $|\lim \tau| = 0$.

See more notes online.