MATH 105: Homework 8

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29. Upper semicontinuity.

(a) A graph of an upper semicontinuous graph here:

(b) Show the following.

Definition 1. *We say that a function $f : M \to \mathbb{R}$ is (ϵ, δ)-upper semicontinuous if and only if for every $\epsilon > 0$ there is a $\delta > 0$ so that*

$$0 < d(x, y) < \delta \implies f(y) < f(x) + \epsilon \tag{1}$$

Lemma 1. *Upper semicontinuity is equivalent to the (ϵ, δ)-upper semicontinuity.*

Proof. Observe the following fact about $\limsup$.

$$\limsup_{y \to x} g(y) = \alpha = \lim_{\epsilon \to 0} \sup\{g(y) \ : \ y \in M \cap M_\epsilon(x) \setminus \{x\}\}. \tag{2}$$

Therefore f is upper semicontinuous if and only if

$$\limsup_{y \to x} f(y) \leq f(x) \iff \lim_{\epsilon \to 0} \sup\{f(y) : y \in M \cap M_\epsilon(x) \setminus \{x\}\} \leq f(x). \tag{3}$$

We then know for every $\epsilon > 0$ there exists a δ so that

$$\sup\{f(y) \ : \ y \in M \cap M_\delta(x) \setminus \{x\}\} < f(x) + \epsilon. \tag{4}$$

This is true if and only if

$$d(y, x) < \delta \implies f(y) < f(x) + \epsilon. \tag{5}$$

Therefore f is (ϵ, δ)-upper semicontinuous. $\square$

Theorem 1. *The function $f : M \to \mathbb{R}$ if upper semicontinuous if and only if for every $a \in \mathbb{R}$,*

$$U_a = \{x \ : \ f(x) < a\} \tag{6}$$

is an open subset of M.

Proof. Take some $x \in U_a$. Then upper semicontinuity implies that for every $\epsilon > 0$ there is a δ so that

$$0 < d(x,y) < \delta \implies f(y) < f(x) + \epsilon. \tag{7}$$

We know that $f(x) < a$, so take $\epsilon = f(x) - a$. Then for every y with $d(x,y) < \delta$,

$$f(y) < f(x) + a - f(x) = a, \tag{8}$$

and $y \in U_a$. Therefore for all $u \in U_a$ there exists a δ so that $d(u,v) < \delta \implies v \in U_a$, and U_a is open.

In the opposite direction suppose that U_a is open. Then, for every $x \in U_a$ there exists a δ so that $d(y,x) < \delta \implies y \in U_a$. Therefore $f(y) < a$. Since we can do this for any arbitrary a, take any $\gamma \in M$, then consider $U_{f(\gamma)}$. It follows for every $\epsilon > 0$ there is a δ so that

$$0 < d(y,\gamma) < \delta, y \in U_{f(\gamma)} \implies f(y) < f(\gamma) + \epsilon \tag{9}$$

What can be said about $y \notin U_{f(\gamma)}$. Take the arg max of those y subject to $f(y) \leq f(\gamma) + \epsilon, y \neq \gamma$ (this is possible since $U^C_{f(\gamma)}$ is closed and there is an $a > \gamma$ so that every $x \in U_a \supset U_{f(\gamma)}$ is a point of upper semicontinuity) and we get y' Then take a new

$$\delta' = \min\{\delta, d(y', \gamma)\} \tag{10}$$

and get f upper semicontinuous. □

(c) Negative semicontinuity.

Definition 2. *We say that a function $f : M \to \mathbb{R}$ is negative semicontinuous if and only if $-f$ is upper semicontinuous.*

Theorem 2. *A function is negative semicontinuous if and only if*

$$\lim_{y \to x} f(y) \geq f(x). \tag{11}$$

Proof. Suppose that $-f$ is upper semicontinuous, then

$$\limsup_{y \to x} -f(y) \leq -f(x) \iff -\liminf_{y \to x} f(y) \leq -f(x), \tag{12}$$

by the definition of $\liminf$. Then we negate the inequality and get

$$\liminf_{y \to x} f(y) \geq f(x). \tag{13}$$

This completes the proof. □

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34. Prove the following

Theorem 3. *Suppose that $f_n : \mathbb{R} \to [0, \infty)$ is a sequence of integrable functions, $f_n \downarrow f$ a.e. as $n \to \infty$ and $\int f_n \downarrow 0$, then $f = 0$ almost everywhere.*

Proof. Suppose that that $f \neq 0$ almost everywhere. Then the undergraph of f would have nonzero measure. If this is the case then it is not true that $\int f_n \downarrow 0$ since if it were the case then not $f_n \downarrow f$ since the undergraph of f is not a zeroset. Therefore $f = 0$. This completes the proof. $\square$