CS 70: Homework 1

William Guss
26793499
wguss@berkeley.edu

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1. Watsons experiment.

   Theorem 1. *If a person has ice cream for desert, he/she has to do the dishes after dinner.*

   Proof. Flip Charlie and Bob. □

2. For the following answers I employed a truth table generator as a latex extension. This is a programmatic method of proof, but it does not detract from the argument.

   (a)

   Theorem 2. $A \ \vee \ (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

   Proof. On the left hand side we have that

a	b	c	a	$\vee$	$(b$	$\wedge$	$c)$
1	1	1	1	**1**	1	1	1
1	1	0	1	**1**	1	0	0
1	0	1	1	**1**	0	0	1
1	0	0	1	**1**	0	0	0
0	1	1	0	**1**	1	1	1
0	1	0	0	**0**	1	0	0
0	0	1	0	**0**	0	0	1
0	0	0	0	**0**	0	0	0

   On the right hand side we have

a	b	c	$(a$	$\vee$	$b)$	$\wedge$	$(a$	$\vee$	$c)$
1	1	1	1	1	1	**1**	1	1	1
1	1	0	1	1	1	**1**	1	1	0
1	0	1	1	1	0	**1**	1	1	1
1	0	0	1	1	0	**1**	1	1	0
0	1	1	0	1	1	**1**	0	1	1
0	1	0	0	1	1	**0**	0	0	0
0	0	1	0	0	0	**0**	0	1	1
0	0	0	0	0	0	**0**	0	0	0

   Since these exhibit ident ical truth values, they myust therefore be the same. □

(b)

Theorem 3. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

Proof. On the left hand side it follows that,

a	b	c	a	$\wedge$	$(b$	$\vee$	$c)$
1	1	1	1	**1**	1	1	1
1	1	0	1	**1**	1	1	0
1	0	1	1	**1**	0	1	1
1	0	0	1	**0**	0	0	0
0	1	1	0	**0**	1	1	1
0	1	0	0	**0**	1	1	0
0	0	1	0	**0**	0	1	1
0	0	0	0	**0**	0	0	0

On the right hand side the truth table gives

a	b	c	$(a$	$\wedge$	$b)$	$\vee$	$(a$	$\wedge$	$c)$
1	1	1	1	1	1	**1**	1	1	1
1	1	0	1	1	1	**1**	1	0	0
1	0	1	1	0	0	**1**	1	1	1
1	0	0	1	0	0	**0**	1	0	0
0	1	1	0	0	1	**0**	0	0	1
0	1	0	0	0	1	**0**	0	0	0
0	0	1	0	0	0	**0**	0	0	1
0	0	0	0	0	0	**0**	0	0	0

There is logical equivalence and the proof is complete. $\square$

(c)

Theorem 4. $A \implies (B \wedge C) \equiv (A \implies B) \wedge (A \implies C)$

Proof. Let $Q = (B \wedge C)$. Then $A \implies Q$ if and only if $\neg A \vee Q$. And so, $\neg A \vee (B \wedge C)$ if and only if $(\neg A \vee B) \wedge (\neg A \vee C)$ by theorem 2 All of that holds if and only if $(A \implies B) \wedge (A \implies C)$. This completes the proof. $\square$

(d)

Theorem 5. $A \implies (B \vee C) \equiv (A \implies B) \vee (A \implies C)$

Proof. Let $Q = (B \vee C)$. Then $A \implies Q$ if and only if $\neg A \vee Q$. And so, $\neg A \vee (B \vee C)$ if and only if $(\neg A \vee B) \vee (\neg A \vee C)$ by associativity. All of that holds if and only if $(A \implies B) \vee (A \implies C)$. This completes the proof. $\square$

3. Justify equivalence.

   (a) There exists an equivalence since the only use of y is for the expressioin involving $Q(x, y)$. In particular the implication is equivalent to $\not P(x) \vee Q(x, \boldsymbol{y})$. So it follows that $\exists$ can be inserted deeper within the statement.

(b) Since negation flips qualifiers we have the following logic,

$$\begin{aligned}
&\neg\exists x\forall y(P(x) \implies \neq Q(x,y)) \\
\iff &\forall x\neg\forall y(P(x) \implies \neq Q(x,y)) \\
\iff &\forall x\exists y\neg(P(x) \implies \neq Q(x,y)) \\
\iff &\forall x\exists y\neg(\neg P(x)\vee \neq Q(x,y)) \\
\iff &\forall x\exists y(\neg(\neg P(x)) \wedge \neg(\neq Q(x,y))) \\
\iff &\forall x\exists y(P(x) \wedge Q(x,y)).
\end{aligned} \tag{1}$$

Therefore, the equivalence holds.

(c) There is not an equivalence by the following argument:

$$\begin{aligned}
&\forall x\exists y(Q(x,y) \implies P(x)) \\
\iff &\forall x\exists y(\neg Q(x,y) \vee P(x)) \\
\iff &\forall x\exists y\neg Q(x,y) \vee P(x) \\
\iff &\forall x\neg\forall y Q(x,y) \vee P(x) \\
\iff &\forall x(\neg(\forall y Q(x,y)) \vee P(x)) \\
\iff &\forall x(\forall y Q(x,y)) \implies P(x))
\end{aligned} \tag{2}$$

Whichn is certainly not equal to the right hand side.

4. Prove or disprove!

(a)

Theorem 6. *The following is true. For every x there exists a y such that $xy > 0$ implies $y > 0$.*

Proof. Fix x. Then take any $y > 0$. Clearly, $y > 0$, and so the implication is always true since it is equivalent to $xy \leq 0$ or $y > 0$. This completes the proof. □

(b)

Theorem 7. *The following is false. There exists a x such that for all y, $xy < x^2$.*

Proof. Suppose it were true. Then consider the rectangle of side-length x. The closed and bounded set $S_y = [0,x] \times [0,y]$ must then have outer measure less than that of $X = [0,x]^2$ for all x. Since $x \in \mathbb{R}$, we have that $\forall y, m(S_y) < X$. Then take the sequence $\{a_n\}_{n\in\mathbb{N}}$ where $a_n = n$. The mesure sequence $(m(S_{a_n})$ is bounded and monotone increasing by the initial supposition, so by the monotone convergence theorem, it converges.

Since the measure sequence is bounded and S_y is a closed and bounded compact set for all y, we have that the sequence of diameters is bounded and converges $diam(S_{a_n})$. Furthermore the diameter of such a set is then dominated by a_n by the archimedian property. So we have that $a_n \to a \in \mathbb{R}$. A contradiction to the unboundedness of $\mathbb{N}$!

This completes the proof without loss of generality since negative rectangles make sense from a measure theory prospective. □

(c)

Theorem 8. *There exist a y such that for all x, $xy \geq x^2$.*

Proof. Take the sequence $a_n = n$. Then if there existed y such that $ny \geq n^2$, then $y \geq n$ for all n, a contradiction to the archimedean property of $\mathbb{R}$. QED □

5. Problems concerning ducks.

(a) i. $\forall x D(x) \implies I(x)$.
ii. $\forall x V(x) \implies H_{issues}(x)$
iii. $\forall x C(x) \implies \neg W(x)$
iv. $\forall x H_{issues}(x) \implies W(x)$
v. $\forall x I(x) \implies C(x)$
vi. $\forall x P(x) \implies V(x)$

(b) i. $\forall x \neg I(x) \implies \neg D(x)$
ii. $\forall x \neg H_{issues}(x) \implies \neg V(x)$
iii. $\forall x W(x) \implies \neg C(x)$
iv. $\forall x \neg W(x) \implies \neg H_{issues}(x)$
v. $\forall x \neg C(x) \implies \neg I(x)$
vi. $\forall x \neg V(x) \implies \neg P(x)$

(c) We use the following argument

$$\begin{aligned} P(x) &\implies V(x) \\ &\implies H_{issues}(x) \\ &\implies W(x) \\ &\implies \neg C(x) \\ &\implies \neg I(x) \\ &\implies \neg D(x). \end{aligned}$$

to conclude that those who wear party hats vote; and so they have done their homework on the issues; and so they are well informed; and so they are not confused; and so they have read the candidates positions; and so they are not a Duck dynasty viewer.

6. (a) The following truth table is produced

a	b	c	d	$O(a,b,c,d)$
1	1	1	1	**0**
1	1	1	0	**0**
1	1	0	1	**0**
1	1	0	0	**0**
1	0	1	1	**0**
1	0	1	0	**1**
1	0	0	1	**0**
1	0	0	0	**1**
0	1	1	1	**0**
0	1	1	0	**0**
0	1	0	1	**0**
0	1	0	0	**0**
0	0	1	1	**0**
0	0	1	0	**1**
0	0	0	1	**0**
0	0	0	0	**1**

(3)

(b) Thereby giving the following Karneugh table:

	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

(4)

(c) It is equivalent to $\neg B \wedge \neg D$. This follows since we have $(\neg B \wedge \neg D) \vee \neg(A \vee C) \vee \neg(A \vee \neg C) \vee \neg(\neg A \vee C) \vee \neg(\neg A \vee \neg C)$. And so we have cancellation.

7. Proof by contrapositive

(a)

Theorem 9. *If $x, y, a \in \mathbb{Z}$ if a does not divide xy, then a does not divide x and x does not divide y.*

Proof. Suppose that $a|x$ or $a|y$. Then there exists a k so that $ka = x$ or $ma = y$. Then $xy = kay$ or $xy = max$. In either case $a|xy$. Take the contraposition and the theorem holds. $\square$

(b) See the proof of (a).

(c) Consider the case when $a = 9$, $b = 12$, $c = 30$, clearly 9 doesnt divide 12 and 30, but it does divide 360. So the converse is not true.

8. Proof time.

(a) Direct

Theorem 10. *For all natural numbers n, if n is odd then $n^2 + 3n$ is even.*

Proof. if n is odd, then $n = 2k+1$, it follows that $n^2 = 4k^2 + 4k + 1$ and $3n = 6k+3$, so $n^2 + 3n = 4k^2 + 10k + 4$ which is divisible by 2. $\square$

(b) Direct

Theorem 11. *For all natural numbers n, $n^2 + 7n$ is even.*

Proof. If n is even $n^2 + 7n = 4k^2 + 14k$, and the theorem is complete. If n is odd then $n^2 + 7n = n^2 + 3n + 4n$ which is $2m + 4n$ by the previous theorem and so $n^2 + 7n$ is divisible by 2. This completes the proof. $\square$

(c) Contraposition

Theorem 12. *If $a, b \in \mathbb{R}$ and $a + b \geq 10$ then $a \geq 7$ or $b \geq 3$.*

Proof. Consider the contrapositive. If $a < 7$ and $b < 3$ then $a + b < 7 + 3 = 10$. There fore $a + b \geq 10$ implies $a \geq 7$ or $b \geq 3$. $\square$

(d) Contraposition

Theorem 13. *If $r \in \mathbb{Q}^c$ the $r + 1 \in \mathbb{Q}^c$.*

Proof. Consider the contrapositive. If $r + 1 \in \mathbb{Q}$ then $r + 1 = \frac{a}{b}$ and $r = \frac{a}{b} - \frac{b}{b} = \frac{a-b}{b}$. So $r \in \mathbb{Q}$. Therefore the contrapositive holds and the proof is complete. $\square$

(e) Counterexample

Proof. Take $n = 100$, then clearly $1000 < 100 * 10 * 9 < 100!$. $\square$

(f) Contrapositive,

Theorem 14. *For all natural numbers a where a^5 is odd.*

Proof. Consider the contrapositive. If a is even the $a = 2k$, and $a^5 = 2^5 k$ which is divisible by 2. So the contrapositive holds. This completes the proof. $\square$