MATH 105: Homework 8

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43.

Theorem 1. *Let* $g(y) = \int_0^\infty e^{-x} \sin(x+y)\ dx$. *The function is differentiable with respect to* $g(y)$.

Proof. Labesgue Dominated Convergance theorem. Consider the following construction.

$$\begin{aligned}\lim_{h\to 0} \frac{g(y)+g(h+y)}{h} &= \frac{1}{h}\int \gamma(y) + \int \gamma(h+y) \\ &= \int \frac{e^{-x}sin(x+y)\ + e^{-x}sin(x+y+h)}{h}\ dx \\ &= \int f_h\ d\mu(x).\end{aligned} \tag{1}$$

where μ is Lebesgue regular measure. We would like to use the dominated convergence theorem to show that this sequence of integrals converges to a limit. First we must show that f_h converges to f almpost everywhere. We know from Math $1B$ that the limit of f_h is the derivative of γ, and γ is clearly a differentiable function. So it is obviouse that $f_h \to f$.

Furthermore we know that $\gamma(x)$ is dominated by e^-x. Observe that $e^{-x} \geq f_h(x)$ almost everywhere since $\sin(x+y+h) \leq 1$.

Therefore by the dominated convergence thereom $f_h \to f(x)$ is integrable with respect to the measure μ, and $\int f_h \to \int f = L$ is the derivatice of $g(y)$ at y. Since we did this for abitrary y g is differentiable everywhere. :) □

46.

Theorem 2.

Office Hours: We know that f is rieman integrable since it has one point of fiscontinuity. Therefore we can use calculus. Consider the integratipon by parts. We have

$$\int_0^1 \frac{\pi}{x}\sin\frac{\pi}{x}\ dx = x\cos\frac{\pi}{x}|_a^1 - \int_a^1 \cos\frac{\pi}{x}\ dx. \tag{2}$$

Clearly the right hand side converges to 0 since it is enveloped by x. The right hand side can be considered as follows. Look at the intervals $[1/(k+1), 1/k]$. In this case, f we can bound the integral along this interval by the rectangle of area

$$B_k = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{(k+1)k}. \tag{3}$$

Accounting for the negative oscilation of the $\cos(\pi/k)$ we get that for k even

$$\int_{\frac{1}{k+2}}^{\frac{1}{k}} \cos\frac{\pi}{x}\ dx \leq \frac{2}{(k+2)(k+1)k}. \tag{4}$$

This is obvious since

$$\int_{\frac{1}{k+2}}^{\frac{1}{k+1}} \cos\frac{\pi}{x}\ dx \leq \sum_{\frac{1}{k+2}}^{\frac{1}{k+1}} -1 \qquad \int_{\frac{1}{k+1}}^{\frac{1}{k}} \cos\frac{\pi}{x}\ dx \leq \sum_{\frac{1}{k+}}^{\frac{1}{k}} 1 \tag{5}$$

and we take the sum of the bound and get the same inequality. Essentially we are given a very nice bound

$$0 \leq \int_{\frac{1}{k+2}}^{\frac{1}{k}} \cos\frac{\pi}{x}\ dx \leq \frac{2}{(k+2)(k+1)k}. \tag{6}$$

We then know that the difference of $k = m$ and $k = n$ decreases at least cubically and so the series of summing the integrals is cauchy and bounded by the series

$$a = \sum_{k=0}^{\infty} \frac{2}{(k+2)(k+1)k}. \tag{7}$$

So the function itself is Riemann integrable (improperly)!

Now look at the Labesgue integrability condition. It must be that $|f|$ has finite integrable. The absolute value of f however has area lowerbounded by $\sum 1/100k$ which diverges, therefore it could not be that the area under $|f|$ be finite and so the function is not Labesgue integrable.

48. The set S_A is where J' is greatert than a we can cover the S_A with intervals $x, x+h$ using a vitali covering. WE can extract disjoint guys who effectively do the covering and whoe total length is approximateley the total measure of S_A.

50. Recall Theorem 66 from the book.

Theorem 3. *The circle or equivalently* $[0,1)$, *splits into two nonmeasurable disjoint subset, that each has inner measure zero and outer measure one.*

We then set out to prove the following theorem.

Theorem 4. *Every measurable* $E \subset \mathbb{R}$ *with* $mE > 0$ *contains a nonmeasurable set* N *with* $m^*N = mE, m_*N = 0$ *and for each* $E' \subset E$ *we have* $m(E') = m^*(N \cap E')$.

Proof. Since E is measurable it is the union of an open set F and a zeroset Z. Since $F \subset \mathbb{R}$ it is the countable union of disjoint open intervals

$$F = \bigsqcup_{i=1}^{\eta \in \mathbb{Z} \cup \{\infty\}} I_i. \tag{8}$$

Let T_i be the rigid transformation which maps $(0,1) \to I_i$ with absolute determinant $m(I_i)$. Then take doppleganger set of $(0,1), N_{(0,1)}$ and consider the new set

$$N = \bigsqcup_{i=1}^{\eta} T_i(N_{(0,1)}) \tag{9}$$

with η as before. This set has inner measure 0 and outer measure $\sum_i m(I_i) = m(E)$ and so is not measurable!

Now take any measurable subset of E, say E'. In the same sense that we constructed N, $N \cap E'$ is also not measurable. It furthermore follows that $m^*(N \cap E') = m^*(E')$ since we construct E as a disjoint union of open intervals all of which are strict subsets of the respective I_i forming E.

To complete this statement, we must show that the outer measure of these subsets, say I_i' intersect $T_i(N_{(0,1)})$ have measure $m(I_i')$. Outer measure is importantly additive, therefore

$$\begin{aligned} m^*(I_i) = m^*(I_i') + m^*(I_i'^c \cap I_i) &= m^*(T_i(N_{(0,1)})) \\ &= m^*((T(N_{(0,1)}) \cap I_i')^c \cap I_i) + m^*(I_i') \\ \implies m^*(I_i') &= m^*(T(N_{(0,1)}) \cap I_i') \end{aligned} \tag{10}$$

using Lemma 20. This completes the proof. □

52. Show the following theorem.

Theorem 5. *If f is a measurable function then*

$$(dp(\mathcal{U}f) \cap \mathcal{U}f)^y = dp(\mathcal{U}f^y) \cap \mathcal{U}f^y. \tag{11}$$

Proof. W must show that every $x \in (dp(\mathcal{U}f) \cap \mathcal{U}f)^y = dp(\mathcal{U}f^y)$ is also a member of $dp(\mathcal{U}f^y) \cap \mathcal{U}f^y$. If $x \in (dp(\mathcal{U}f) \cap \mathcal{U}f)^y$ then equivalently we have

$$x \in \{p = (\rho, \gamma) \mid \gamma = y \wedge p \in \mathcal{U}f \wedge p \text{ density point of } \mathcal{U}f\} \tag{12}$$

By Theorem 49, we have that ρ is a density point of $\mathcal{U}f^y$. Next we must show that if $\gamma = y$ and $p \in \mathcal{U}f$ then $x \in \mathcal{U}f^y$ This however is the definitions of $\mathcal{U}f^y$. So it must be that $x \in dp(\mathcal{U}f^y) \cap \mathcal{U}f^y$..

In the opposite direction we can again use the same logic. Theorem 49 puts $x \in \mathcal{U}f^y$ in $(\mathcal{U}f)^y$. Applying the same under graph logic as before the proof is complete. □

43, 46,48,50,52