MATH 202A: Notes

Scribe: William Guss

September 9, 2016

Remark. In mathematics, the key discovery is usually a definition!

Definition 1. *Let* $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *be an outer measure.*

Definition 2. $A \subset X$ *is* μ^**-measurable, if for all* $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

Theorem 1. $\mathcal{M}$ *is a* σ*-algebra and* $\mu^*|_{\mathcal{M}}$ *is a measure on* $\mathcal{M}$.

Remark. You could easily make a σ-algebra on X by letting $\mathcal{M} = \{\emptyset, X\}, \mu(\emptyset) = 0$ and $\mu(X) = \mu^*(X)$.

Proof. Last time we prove that $\mathcal{M}$ was an algebra using Venn-Diagrams. We also proved if $A, B \in \mathcal{M}$ and $A \cap B = \emptyset$ then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Now suppose that $A_n \in \mathcal{M}$ is Cartheodory measurable for all $n \in \mathbb{N}$. We wish to show $A = \bigcup_n A_n \in \mathcal{M}$. Define $B_1 = A_1$, and $B_n = A_n \setminus \bigcup_{k=1}^{n-1}$. We have that $B_k \in \mathcal{M}$ and they are pairwise disjoint. So $A = \bigsqcup_k B_k$. Now consider

$$\begin{aligned}\mu^*(E) &\geq \mu^*\left(E \cap \left(\bigcup_{k=1}^{n} B_k\right)\right) + \mu^*\left((E \setminus \bigcup_{k=1}^{n} B_k)\right)\\ &\geq \mu^*\left(E \cap \left(\bigcup_{k=1}^{n} B_k\right)\right) + \mu^*(E \setminus A).\end{aligned}$$

Recall that $E \subset X$ and $A, B \in \mathcal{M}$ and $A \cap B = \emptyset$ then

$$\mu^*(A \cup B) \cap E) = \mu^*(A \cap E) + \mu^*(B \cap E).$$

So for every n

$$\mu^*(E) \geq \sum_{k=1}^{n} \mu^*(E \cap B_k) + \mu^*(E \setminus A).$$

The series is bounded and therefore as $n \to \infty$

$$\mu^*(E) \geq \sum_{k=1}^{\infty} \mu^*(E \cap B_k) + \mu^*(E \setminus A).$$

With $E \cap A \subset \bigcup_k E \cap B_k$ so $\mu^*(E \cap A) \leq \sum_{k=1}^{\infty} \mu^*(E \cap B_k)$ by countable subaddativity. Therefore

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)$$

and so A is measurable; that is $A \in \mathcal{M}$. Therefore $\mathcal{M}$ is closed under countable unions.

Now we want to show that $\mu^*|_{\mathcal{M}}$. However this is implicit, by applying the above argument to $E = A$; that is, if $A = \bigcup_k B_k, B_k \in \mathcal{M}$ and B_k pairwise disjoint

$$\mu^*(A) = \sum_{k=1}^{\infty} \mu^*(A \cap B_k) + \mu^*(E \setminus A) = \sum_{k=1}^{\infty} \mu^*(B_k) + \mu^*(\emptyset).$$

This completes the proof. □

Remark. The measure $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure.

Proof. Suppose $A \in \mathcal{M}$ and $\mu^*(A) = 0$ and $B \subset A$ not necisarrily measurable. Certainly $\mu^*(B) \leq \mu^*(A) = 0$. The set $E \setminus B = E \setminus A \cup A \setminus B$ so $\mu^*(E \setminus B) \leq \mu^*(E \setminus A) + \mu^*(\cup A \setminus B)$ so we conclude $\mu^*(E \setminus B) = \mu^*(E \setminus A)$. For any set E

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A) \geq \mu^*(E \cap B) + \mu^*(E \setminus B)$$

by A measurable! Therefore B is measurable! □

Definition 3. *A pre measure μ_0 on an algebra $\mathcal{A}$ is a $\mu_0 : \mathcal{A} \to [0, \infty]$ such that $\mu_0(\emptyset) = 0$ and $\mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$ if $A, B \in \mathcal{A}$ disjoint. Additionally*

$$\mu_0\left(\bigcup_n^{\infty} A_n\right) = \sum_n^{\infty}$$

iff $A_n \in \mathcal{A}$ and pairwise disjoint.

Theorem 2. *Let μ_0 be a premeasure on an algebra $\mathcal{A}$. Define μ^* from $\mu_0, \mathcal{E} = \mathcal{A}$, and use the same construction as before. Let $\mathcal{M}, \mu = \mu^*|_{\mathcal{M}}$ be the construction from Carthedory's theorem. Then every $\mathcal{A} \subset \mathcal{M}$ and μ and μ_0 agree on $\mathcal{A}$.*

(Thm 3) In addition if μ_0 is σ-finite then the extension of μ_0 to $\mathcal{M}$.

Proof. Consider $\{E_j \in \mathcal{M}\}$ so that $\bigcup E_j \supset E$ and $E_j \in \mathcal{E} = \mathcal{A}$. First

$$\begin{aligned}\sum_j \mu_0(E_j) &= \sum_j \mu_0(E_j \cap A) + \mu_0(E_j \setminus A) \\ &= \sum_j \mu_0(E_j \cap A) + \sum_j \mu_0(E_j \setminus A)\end{aligned}$$

Furthermore $\{E_J \cap A : j \in \mathbb{N}\}$ covers $E \cap A$ and $\{E_j \setminus A\}$ covers $E \cap A$. Therefore

$$\sum_j \mu_0(E_j \cap A) + \sum_j \mu_0(E_j \setminus A) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)$$

Therefore for all covers of E by $E_j \in \mathcal{A}$ it follows that

$$\inf_{(E_J) \subset \mathcal{M}} \mu_0 \left(\sum_j \mu_0(E_j)\right) = \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)$$

Suppose that ν is a measure on $\mathcal{M}$ and $\nu \equiv \mu_0$ on $\mathcal{A}$. We claim that $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$. Consider any countable cover of E by $E_j \in \mathcal{A}$. So

$$\nu(E) \leq \sum) j \nu(E_j) = \sum_j \mu_0(E_j).$$

Take the inf over all such covers and

$$\nu(E) \leq \sum_j \nu(E_j) = \sum_j \mu(E_j).$$

$\square$