

Math 202A — UCB, Fall 2016 — M. Christ
Problem Set 13, due Wednesday November 30

(13.1) (Folland problem 4.22) Let X be a topological space, and let (Y, d) be a complete metric space. Let (f_n) be a sequence of functions in Y^X that satisfies $\sup_{x \in X} \lim_{m, n \rightarrow \infty} d(f_m(x), f_n(x)) = 0$.
(a) Show that there exists an $f \in Y^X$ so that $\sup_{x \in X} d(f_n(x), f(x)) \rightarrow 0$.

Proof. We will first construct f and show that the uniform norm distance tends to 0. As Y is a complete metric space under d and for every x , $\lim_{m, n \rightarrow \infty} d(f_m(x), f_n(x))$ is bounded by $\sup_{x \in X} \lim_{m, n \rightarrow \infty} d(f_m(x), f_n(x)) = 0$, the sequence $(f_m(x))_m$ is Cauchy and thus has a limit, say $f(x)$. Now define $f : X \rightarrow Y$ so that $x \mapsto f(x)$ as defined previously for every x . We have shown that this function is well defined.

We now claim that $\sup_{x \in X} d(f_n(x), f(x)) \rightarrow 0$. First¹

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{x \in X} d(f_n(x), f(x)) &= \lim_{n \rightarrow \infty} \sup_{x \in X} \lim_{m \rightarrow \infty} d(f_n(x), f_m(x)) \\ &= \lim_{n \rightarrow \infty} \sup_{x \in X} \lim_{m \rightarrow \infty} \sup_{k \geq m} d(f_n(x), f_k(x)) \\ &\leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sup_{x \in X} \sup_{k \geq m} d(f_n(x), f_k(x)) \quad (\text{sup of dec. seq at } x) \\ &\leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sup_{k \geq m} \sup_{x \in X} d(f_n(x), f_k(x)) \quad (\text{sup of maximal distances over } x.) \\ &= \lim_{m, n \rightarrow \infty} \sup_{x \in X} d(f_n(x), f_k(x)) = 0 \quad (\text{existence by hypothesis}) \end{aligned}$$

This completes the proof. □

(b) Show that f is unique.

Proof. Suppose that f is not unique, that is there is a $g \neq f$ so that $\sup_{x \in X} d(f_n(x), g(x)) \rightarrow 0$. Then for every n

$$\sup_x d(f(x), g(x)) \leq \sup_x (d(f(x), f_n(x)) + d(f_n(x), g(x))) \leq \sup_x d(f(x), f_n(x)) + \sup_x d(g(x), f_n(x)).$$

But then $\sup_x d(g(x), f_n(x)) \rightarrow 0$ and $\sup_x d(f(x), f_n(x)) \rightarrow 0$ implies that $\sup_x d(f(x), g(x)) = 0$; thus for every x , $f(x) = g(x)$, and $f = g$; a contradiction to the existence of g . Therefore f is unique. □

(c) Show that if every function f_n is continuous then so is f .

Proof. Let f_n and f be given as above. Then for every $\epsilon > 0$, there is an N so that for all $n \geq N$ $\sup_{x \in X} d(f_n(x), f(x)) < \epsilon/3$. Then by continuity of f_n there is an open neighborhood $U_n \subset X$ of x so that for all $y \in U_n$ $d(f_n(x), f_n(y)) < \epsilon/3$. Finally

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ &\leq d(f_n(x), f_n(y)) + 2 \sup_{x \in X} d(f_n(x), f(x)) < 2/3\epsilon + \epsilon/3 = \epsilon. \end{aligned}$$

Thus for every ϵ , there is an open neighborhood of x so that for all y in that neighborhood, $d(f(x), f(y)) < \epsilon$. □

¹We will abuse notation and write $\lim_n \sup_{x \in X} d(f_n(x), f(x)) = \lim_n \sup_{j \geq n} \sup_{x \in X} d(f_j(x), f(x))$, but we do not assume the limit exists. In fact we will show that the \limsup_n is bounded above by 0 and the sequence is bounded below by 0 so the limit is 0.

(13.2) Give an elementary proof of the Tietze Extension Theorem for the special case in which $X = \mathbb{R}$.

Proof. If A is a closed subset of \mathbb{R} and $f : A \rightarrow [a, b]$ is continuous, we wish to show the existence of a continuous function $F : \mathbb{R} \rightarrow [a, b]$ so that $F|_A = f$. The main idea of our proof is that since f is defined on a closed subset of \mathbb{R} we need only work to define F on the open complement of A which from elementary real analysis is just the countable union of open intervals.

In the trivial case that $A = \mathbb{R}$ or $A = \emptyset$, let $F = f$ or $F = c$ respectively, the first is obviously a continuous extension and the second is a trivial extension since f is undefined for every $x \in \mathbb{R}$.

Otherwise recall from real analysis that $\mathbb{R} \setminus A = B$ is an open subset of \mathbb{R} and so is the countable union of disjoint open intervals O_n . We will essentially define F as the affine interpolation of f on these open intervals. First assume that $O_n = (a_n, b_n)$ where a_n, b_n are finite; we will address the nonfinite cases later. Define

$$F|_{[a_n, b_n]} : x \mapsto \frac{f(b_n) - f(a_n)}{b_n - a_n}(x - a_n) + f(a_n).$$

$F|_{[a_n, b_n]}(a_n) = f(a_n)$ and $F|_{[a_n, b_n]}(b_n) = f(b_n)$. Furthermore from 104, $F|_{[a_n, b_n]}$ is continuous for all $x \in [a_n, b_n]$ as $F|_{[a_n, b_n]}$ is just the affine linear interpolation of $f(a_n)$ and $f(b_n)$. If there are O_n so that $a_n = -\infty$ or $b_n = \infty$ then let $F|_{(a_n, b_n]} : x \mapsto f(b_n)$ or $F|_{[a_n, b_n)} : x \mapsto f(a_n)$ respectively. In this case these functions are just constant and so $F|_{O_n}$ is also continuous. Lastly let $F|_A = f$, and then $F : \mathbb{R} \rightarrow [a, b]$ so that F satisfies all of the previous restrictions²

By the continuity of f on A and the previous arguments F is continuous with respect to the whole topology on \mathbb{R} at least on A° and $B^\circ = B$. As for continuity at $A \setminus A^\circ = C$, we observe that $cl(A) = A$ and so $C = \partial A$. From real analysis the boundary of a closed set cannot contain any open intervals as A° is the countable union of open intervals, and if ∂A contained an open interval, then A° would not be the largest open set containing A , therefore $\partial A = \{y \in A\}$ so that if $v, w \in \partial A$ $v < w$ or $w < v$. Now as $\delta \rightarrow 0$, $\sup f((v - \delta, v + \delta) \cap A) - \inf f((v - \delta, v + \delta) \cap A) \rightarrow 0$. Observe that now $\sup F((v - \delta, v + \delta)) = f((v - \delta, v + \delta) \cap A)$ since on $(v - \delta, v + \delta) \setminus A$ F is defined on some collection of O_n to never achieve values larger than f on the end points on which it must be defined. Likewise for the infimum $\inf F((v - \delta, v + \delta)) = \inf f((v - \delta, v + \delta) \cap A)$. Therefore $\sup F((v - \delta, v + \delta)) - \inf F((v - \delta, v + \delta)) \rightarrow 0$ and F is continuous at $v \in \partial A$.

Therefore F is continuous on A° and ∂A and A^c with respect to the global topology so it is continuous. This completes the proof. \square

(13.3) Show that any product of connected topological spaces is connected.

Lemma. Product of finitely many connected spaces is connected:

Proof. Take two spaces A, B connected. Then to show that the product $A \times B$ is connected, first let $i_A^b : A \rightarrow A \times \{b\}$, $b \in B$ be the inclusion mapping. The inclusion map is category theoretically a bijection when b is a singleton, so we need only show that the map is continuous and open. If $V \subset A \times \{b\}$ is open then it inherits the product topology so without loss of generality assume it is from the base and so $V = \pi_1^{-1}(C) \cap \pi_1^{-1}(D) \cap A \times \{b\}$ where $C \subset A$ open and $D \subset B$ open. Then $b \in \pi_1^{-1}(D)$ otherwise V would be the empty set, thus $V = C \times \{b\}$ and the inverse of the map is C which is open in A . Since we have shown every open set is of this form, take any $C \subset A$ open and the $i_A^b = C \times \{b\}$ which is open. Therefore the map is a homeomorphism. Since the product is commutative the argument holds for the inclusion map i_B^a for any $a \in A$. Now we will show that

²This is defined since we described the restriction of F on all x in $\mathbb{R} \setminus A \cup A = \mathbb{R}$.

every $(a, b) \in A \times B$ belong to the same maximally connected component by showing they belong to a connected component and there is a common element to all of those connected components (invoking a proof from a previous exercise.) Fix $(c, d) \in A \times B$. Then for all (a, b) , it's clear that $(a, b) \in i_A^d(A) \cup i_B^a(B)$ and symmetrically $(a, b) \in I_A^b \cup i_A^c(B)$ both of which are connected as they are the union of the continuous images of two connected sets. Thus $(a, b) \in i_A^d(A) \cup i_B^a(B) \cup I_A^b \cup i_A^c(B) \ni (c, d)$ and so (c, d) is common to every connected component containing every element of the space and so the space itself is connected.

Then induct on a finite product to yield the result. \square

We now prove the exercise.

Proof. We will use the classic constancy argument. First let $D = \{0, 1\}$ with the topology inherited from \mathbb{R} . Then a continuous closed map $f : C \rightarrow D$ is constant if and only if C is a connected topological space. If $f : C \rightarrow D$ is constant for all f , but for contradiction that C were not connected, then $C = A \sqcup B$ with A, B clopen disjoint non-trivial. Then let $f : A \rightarrow 0, B \rightarrow 1$ and so trivially f is closed since all elements of $P(D)$ are closed. Since every element of $P(D)$ is open then the inverse of subsets of D is either C , A , or B , which are all open by C disconnected. But this contradicts the fact that all closed continuous maps are constant, so C must be connected. In the other direction, if C is connected, then the image of any f must be connected, so must be either 1 or 0 but not D and so every f must be constant.

Now to the problem, let $X = \prod_A X_a$. Then let X_a connected for all a . Take any $f : X \rightarrow D$ to be a continuous function with X endowed with the product topology. Take any point x in X and let M be its maximally connected component; from a previous exercise $M = \bigcup_K K$ where K is a connected subset of X which contains x and M is connected obviously. Then $f(M)$ is connected and so is $cl(f(M))$, since $M \subset X$ and f is a closed continuous map $cl(f(M)) \subset f(X)$. We would like to show that f is constant.

For contradiction suppose there is a $y \in D$ so that $f(X) \setminus cl(f(M)) \ni y$, that is there is (are) some $z \in X$ so $f(z) = y$ and not $f(z) \in cl(f(M))$. Then take an open neighborhood T of y which is disjoint from $f(M)$; this is possible in D because $f(M)$ is constant. By the continuity of f , $f^{-1}(T) = W$ is open and so we can take a neighborhood of $z \in W$ which is contained by W and is also in the neighborhood base. Then there are open sets so that $z \in \bigcap_1^m \pi_{a_n}^{-1}(A_{a_n})$ and this intersection is a subset of W . Let v so that $v_a = z_a$ for those finite $a \in \{a_n\}$ and otherwise $v_a = x_a$ where $x \in M$ was the element defined in the previous paragraph. Now x and v both agree except for the coordinates in $\{a_n\}$ so they are both contained in an equivalence class Y_x of points in X which agree with y except for on $\{a_n\}$. Now Y_x is bijective to $\prod_{a \in \{a_n\}} X_a$ as the continuous projection map provides an injection since for all $q \in Y_x$, q is uniquely determined on a_n and of course the projection map is surjective. Now for any $q \in Y_x$ take an open neighborhood V of q then every $p \in V$ is uniquely determined by its coordinates $\{a_n\}$ so $V = \bigcap_{b \in Q \subset \{a_n\}} \pi_b^{-1}(U_b) \cap Y_x$. With U_b open in X_b . Therefore $\pi_b(V) = U_b$ for all $b \in Q$ implies that $\pi_{\{a_n\}}(V) = \bigcap_{b \in Q} \pi_b^{-1}(U_b)$ is open in $\prod_{a \in \{a_n\}} X_a$. Therefore Y_x is homeomorphic to this subspace, and this subspace is connected by the Lemma!. This contradicts the fact that $f(v) \in T$ as Y_x is a subset of the maximal component M which contains x and v , and thus $f(Y_x) \subset cl(f(M))$. Therefore there are no elements whose image is not in closure of the image of the maximally connected component M and so f is constant. Therefore X is connected. \square

(13.4) Let X be a topological space equipped with an equivalence relation, \tilde{X} the set of equivalence classes, $\pi : X \rightarrow \tilde{X}$ the map taking each $x \in X$ to its equivalence class, and $\mathcal{T} = \{U \subset \tilde{X} :$

$\pi^{-1}(U) \in \mathcal{T}_X$.

(a) Show that \mathcal{T} is a topology on X .

Proof. First we show that $\tilde{X} \in \mathcal{T}$. Since π is a surjection onto \tilde{X} it follows that $\pi^{-1}(\tilde{X}) = X \in \mathcal{T}_X$ and so $\tilde{X} \in \mathcal{T}$. Additionally $\pi^{-1}(\emptyset) = \emptyset \in \mathcal{T}_X$ so $\emptyset \in \mathcal{T}$. Now take any $U_\alpha \in \mathcal{T}$ with an arbitrary index set $\alpha \in A$. We claim that $\bigcup_\alpha U_\alpha \in \mathcal{T}$. First by the definition of preimage

$$\pi^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} \pi^{-1}(U_\alpha) \in \mathcal{T}_X$$

since $\pi^{-1}(U_\alpha) \in \mathcal{T}_X$ are open in X and the topology \mathcal{T}_X is closed under arbitrary union. Lastly consider the finite intersection of $U_m \in \mathcal{T}$; that is, We claim that $\bigcap_1^n U_m \in \mathcal{T}$. First by the definition of preimage

$$\pi^{-1}\left(\bigcap_{m=1}^n U_m\right) = \bigcap_{m=1}^n \pi^{-1}(U_m) \in \mathcal{T}_X$$

since $\pi^{-1}(U_\alpha) \in \mathcal{T}_X$ are open in X and the topology \mathcal{T}_X is closed under finite intersection.

Thus \mathcal{T} satisfies the definition of a topology. \square

(b) Show that if Y is a topological space, $f : \tilde{X} \rightarrow Y$ is continuous if and only if $f \circ \pi$ is continuous.

Proof. Suppose that $f : \tilde{X} \rightarrow Y$ is continuous and \mathcal{T}_Y denotes the topology of Y . Then for any open $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}$, but then $\mathcal{T}_X \ni \pi^{-1}(f^{-1}(V)) = (f \circ \pi)^{-1}(V)$. Thus the preimage mapping $(f \circ \pi)^{-1}$ preserves the topology \mathcal{T}_Y and so $f \circ \pi : X \rightarrow Y$ is continuous.

In the other direction if $f \circ \pi : X \rightarrow Y$ is continuous then for every open neighborhood of $(f \circ \pi)(x)$, say V there is an open neighborhood of x , say U , so that $(f \circ \pi)(U) \subset V$. Then $\pi(U) = W \ni \pi(x)$ is a neighborhood of $\pi(x)$ so that $f(W) \subset V$. Thus take W^o and then $f(W^o) \subset V$. Now since π is surjective for every $\tilde{x} \in \tilde{X}$ pick any $x \in \pi^{-1}(\tilde{x})$. Then for any V as above there is an open neighborhood W^o of \tilde{x} so that $f(W^o) \subset V$. Therefore $f : \tilde{X} \rightarrow Y$ is continuous. \square

(c) Show that \tilde{X} is T_1 iff every equivalence class is closed.

Proof. Suppose that (\tilde{X}, \mathcal{T}) is T_1 . Then $X \setminus \{\tilde{x}\} \in \mathcal{T}$ for every $\tilde{x} \in \tilde{X}$ and so $\pi^{-1}(X \setminus \{\tilde{x}\}) \in \mathcal{T}_X$. Now since π maps every x into its exact equivalence class and the fundamental theorem of equivalence relations states that the equivalence classes of x form a partition of the space $\pi^{-1}(\tilde{X} \setminus \{\tilde{x}\}) \cap \pi^{-1}(\{\tilde{x}\}) = \emptyset$ and $\pi^{-1}(\tilde{X} \setminus \{\tilde{x}\}) \cup \pi^{-1}(\{\tilde{x}\}) = X$. Therefore $X \setminus \pi^{-1}(\tilde{X} \setminus \{\tilde{x}\}) = \pi^{-1}(\{\tilde{x}\}) = [x]_\sim$ for some x ; that is $[x]_\sim$ is the complement of an open set and so $[x]_\sim$ is closed. Since we argued this for every \tilde{x} and π is surjective, we get that for every $x \in X$, $[x]_\sim$ is closed.

Suppose that every equivalence class is closed. Then for every $x \in X$, $X \setminus [x]_\sim \in \mathcal{T}_X$. Now take any $\{\tilde{x}\} \subset \tilde{X}$. It follows that $\pi^{-1}(X \setminus \{\tilde{x}\}) = \pi^{-1}(\tilde{X}) \setminus \pi^{-1}(\{\tilde{x}\}) = X \setminus [x]_\sim \in \mathcal{T}_X$ and so $\tilde{X} \setminus \{\tilde{x}\} \in \mathcal{T}$ for every \tilde{x} and so singletons are closed in \tilde{X} and thus (\tilde{X}, \mathcal{T}) is T_1 . \square

(13.5) Let (X, \mathcal{T}) be compact and Hausdorff.

(a) Let \mathcal{T}' be strictly stronger than \mathcal{T} on X . Show that (X, \mathcal{T}') is not compact.

Proof. Suppose that $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{T}' \neq \mathcal{T}$ and \mathcal{T}' compact. Let $id : (X, \mathcal{T}') \rightarrow (X, \mathcal{T})$ so that $id(x) = x$. Then if $V \in \mathcal{T}$ $id^{-1}(V) = V \in \mathcal{T}'$ by our assumption that \mathcal{T}' is a stronger topology. Therefore id is a continuous bijection. Then by Proposition 4.28, (X, \mathcal{T}') compact and (X, \mathcal{T}) Hausdorff implies that id is a homeomorphism. Therefore $\mathcal{T}' = \mathcal{T}$. This is a contradiction and so \mathcal{T} cannot be compact. \square

(b) Let \mathcal{T}' be strictly weaker than \mathcal{T} on X . Show that (X, \mathcal{T}') is not Hausdorff.

Proof. Suppose that $\mathcal{T}' \subset \mathcal{T}$ and $\mathcal{T}' \neq \mathcal{T}$ and \mathcal{T}' Hausdorff. Let $id : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$ so that $id(x) = x$. Then if $V \in \mathcal{T}'$ $id^{-1}(V) = V \in \mathcal{T}$ by our assumption that \mathcal{T}' is a weaker topology. Therefore id is a continuous bijection. Then by Proposition 4.28, (X, \mathcal{T}) compact and (X, \mathcal{T}') Hausdorff implies that id is a homeomorphism. Therefore $\mathcal{T}' = \mathcal{T}$. This is a contradiction and so \mathcal{T} cannot be Hausdorff. \square

(13.6) Let X be a topological space. Show that following are equivalent: (i) X is normal; (ii) X satisfies the conclusion of Urysohn's Lemma; (iii) X satisfies the conclusion of the Tietze Extension Theorem.

Proof. Trivially³ (i) implies (ii) and (iii). We will now show that (ii) implies (i) and (iii) implies (ii).

First suppose that X satisfies the conclusions of Urysohn's Lemma, that is for A, B closed disjoint in X then there is a continuous function $f : X \rightarrow [0, 1]$ so that $f(A) = \{1\}$ and $f(B) = \{0\}$. Now when $0 < \epsilon < 1$ consider the open set $V_\epsilon = (\epsilon, 1]$ then $A \subset f^{-1}(V_\epsilon)$. Additionally consider the open set $U_\epsilon = (\epsilon, 1]$ then $B \subset f^{-1}(U_\epsilon)$. We claim that $f^{-1}(U_\epsilon) \cap f^{-1}(V_\epsilon) = \emptyset$. By the set operation homomorphism property of the preimage, we have that

$$\emptyset = f^{-1}(\emptyset) = f^{-1}(U_\epsilon \cap V_\epsilon) = f^{-1}(U_\epsilon) \cap f^{-1}(V_\epsilon).$$

Therefore A, B are contained in open sets which are disjoint. Since this holds for every A, B , the space X is normal.

Now suppose that X satisfies the conclusion of the Tietze Extension Theorem. Then take any two closed disjoint sets A, B in X . Let $f : A \cup B \rightarrow [0, 1]$ so that $f(A) = \{0\}$ and $f(B) = \{1\}$. Since $A, B, A \cup B$ is closed. Now for every $x \in A \cup B$, $x \in A$ or $x \in B$ but not both. Assume that $x \in A$. Then $f(x) = 0$ and for every open neighborhood V of $f(x)$ in $[0, 1]$ we claim that $f^{-1}(V)$ is open. since $f(A \cup B) = \{0, 1\}$ then V must contain 0 and need not contain 1. In the case that $0, 1 \in V$ we have that $f^{-1}(V) = A \cup B$ which is the whole space $A \cup B$ and therefore is open in the relative subspace topology. Otherwise $f^{-1}(V) = A$ which is closed in the relative subspace topology since $A \cap A \cup B = A$ and A is closed globally. Since B is closed in the global topology and $B \cap A \cup B = B$, then B is closed in the subspace topology. By the disjointness of A and B , $A \cup B \setminus B = A$ and so A is open in the subspace topology. Therefore $f^{-1}(V) = A$ which is an open neighborhood of $x \in A$. The same argument can be made by replacing symbols to show that f is continuous for $x \in B$.

Then the Tietze extension theorem gives a continuous extension $F : X \rightarrow [0, 1]$ so that $F|_{A \cup B} = f$; that is $f|_A = 0$ and $f|_B = 1$, therefore the conclusion of Urysohn's Lemma is satisfied.

We have thus shown that (ii) \implies (i) \implies (iii) \implies (i) and thus all of the conditions are equivalent. \square

(13.7) Show that every sequentially compact topological space is countably compact.

Proof. Suppose that X is sequentially compact. Then given any sequence $(x_n)_n$ then there is a subsequence which converges to some $x \in X$; namely $x_{n_k} \rightarrow x$. If \mathcal{V} is a countable open cover of X , then we would like to show that there is a finite subcover. Proposition 4.21 implies that X is countable compact if and only if for every countable family $\{F_n\}$ of closed sets with the finite intersection property ($\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$), it follows that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

To show compactness we will show the dual property with closed sets. Let $\{F_n\}$ be given with the above property and then let choose x_n so that $x_n \in \bigcap_{m=1}^n F_m$. Such an x_n exists for every n by

³ X normal is a hypothesis of both propositions.

the finite intersection property, and there is a subsequence x_{n_k} which converges to x in X . Suppose $x \notin \bigcap_{n=1}^{\infty} F_n$, then there is an m so that $x \notin F_m$. Therefore by F_m closed $x \notin \text{acc}(F_m)$. By definition for all $j > m$, $x_{n_j} \in \bigcap_{p=1}^{x_{n_j}} F_p \subset F_m$. Therefore the sequence $(x_{n_k})_{k=m}^{\infty}$ is entirely contained in F_m . Now since $x \notin \text{acc}(F_m)$, it follows that $x \notin \text{acc}(\{x_{n_k}\}_{k=m}^{\infty})$ and so there is a neighborhood V of x so that $\{x_{n_k}\}_{k=m}^{\infty} \cap (V \setminus \{x\}) = \emptyset$, and therefore exists a neighborhood so that for all $k \geq m$, and for all $j > k$ $x_{n_j} \notin V$ and so $x_{n_k} \not\rightarrow x$, (unless of course $x_{n_k} = x$ becomes constant at some point in the sequence, in which case $x \in F_m$, but this is not possible.) This is a contradiction, and so $x \in \bigcap_{m=1}^{\infty} F_m$ and thus X is countably compact. \square

(13.8) Show that if X is countably compact, then every sequence in X has a cluster point and additionally if X is also first countable then X is sequentially compact.

Proof. Let (x_n) be some sequence in X . Let $S_m = \{x_n\}_{n=m}^{\infty}$ be the value set of the sequence starting at m . Finally let $G = \bigcap_{n=1}^{\infty} \text{acc}(S_n)$. When $n \leq m$, $S_m \subset S_n$ and so $\text{acc}(S_m) \subset \text{acc}(S_n)$. Therefore take any finite $B \subset \mathbb{N}$. There is a maximal element say $k = \max B$, and thus $\text{acc}(S_j) \supset \text{acc}(S_k)$ so the intersection of accumulation points of the partial sequence value sets is non-empty; that is the family $(\text{acc}(S_n))_{n \in \mathbb{N}}$ has the finite intersection property and every member is closed in X . Hence by the countable compactness of X , G is non-empty. We claim that if $x \in G$ then x is a cluster point of the sequence.

Taking $x \in G$ then for all m and for all neighborhoods V of x then $V \cap S_m$ is non-trivial. Hence for all neighborhoods and for all m , there exists an $n > m$ so that $x_n \in S_m$ and $x_n \in V$. Therefore (x_n) is in V infinitely often and so x is a cluster point of the sequence.

Supposing additionally that X is a first countable space, then in Homework 11, exercise 11.4 (or equivalently Folland 4.7) we showed that if x is a cluster point of a sequence in a first countable space, then there is a subsequence which converges to x . Adopting the results of this exercise to the previous proof, $x \in G$ is a cluster point of (x_n) and so there is a subsequence which converges to it. Therefore for any sequence in X there is a convergent subsequence (the existence of a cluster point guaranteed), and so X is sequentially compact.

This completes the proof. \square

(13.9) If X is normal, then X is countably compact iff $C(X) = BC(X)$.

Proof. Suppose that X is countably compact. Now if $f \in C(X)$, then $f : X \rightarrow \mathbb{C}$ has the property that $f(X)$ is countably compact. Since \mathbb{C} is second countable then $f(X)$ is sequentially compact. Additionally since \mathbb{C} is a metric space, from the theory of metric spaces we know that sequential compactness implies compactness which implies total boundedness and closedness (Heine Borel in $\mathbb{R}^2 \simeq \mathbb{C}$)⁴ Therefore $f(X)$ is bounded and so f is bounded; that is $C(X) \subset B(X)$. Then $BC(X) = B(X) \cap C(X) = C(X)$.

Now suppose that X is not countably compact, then by 13.7, X is not sequentially compact, and so there is a sequence with no convergent subsequence. Therefore there is a sequence without a cluster point. Take such a sequence, say (x_n) . Then we claim that the value set $S = \{x_n\}_{n=1}^{\infty}$ is closed. If $x \in \text{acc}(S)$ we wish to show that $x \in S$.

Suppose for the sake of contradiction that $x \notin S$, then because the sequence has no cluster points, there is a neighborhood of x say W so that there is an N so that $x_n \notin W$ when $n > N$. Then if $S_n = S \setminus \{x_n\}_{n=1}^m$ it follows that $S_n \cap W = \emptyset$ for all $n > N$. Since x is an accumulation point it

⁴Closedness comes from \mathbb{C} metric spaces being Hausdorff.

must then be that $S \cap W \ni x_k \neq x$ when $k \leq N$. Now consider any neighborhood $V \subset W$ of x , then $\emptyset \neq V \cap S = V \cap S \setminus S_N$ and so $x \in \text{acc}(S \setminus S_N)$. However, x is T_4 , and therefore $S \setminus S_N$ is closed because it is a finite set of points. Therefore $\text{acc}(S \setminus S_N) \subset S \setminus S_N \subset S$, which contradicts $x \notin S$. Therefore $\text{acc}(S) \subset S$ and so S is closed.

Now let $f : S \rightarrow \mathbb{C}$ so that $f(x) = \max\{n \in \mathbb{N} : x_n = x\} + 0i$. Such a maximum is defined because x_n cannot cluster at any $x \in S$; that is, $\{n \in \mathbb{N} : x_n = x\}$ must be finite since (x_n) has no cluster points and cannot visit x infinitely many times. We claim the function is not bounded. Suppose that there were an N so that for all $x \in S$ $|\max\{n \in \mathbb{N} : x_n = x\}| \leq N$. Then if $m > N$, $x_m \in S$ since N is finite and additionally $N \geq |f(x_m)| = \max\{n \in \mathbb{N} : x_n = x\} = m > N$ which is a contradiction so, f cannot be bounded.

To show f is continuous on S with respect to its subspace topology take any closed subset K of \mathbb{C} , then it suffices to show that $f^{-1}(K)$ is a closed subset of X . Since $S \sim \mathbb{N}$ there are two cases: If $f^{-1}(K)$ is finite then it is closed by X a T_4 topological space. If it is not then it is countable. For each $y \in f^{-1}(K)$ clearly the integer $f(y) = \max\{n : x_n = y\}$ is unique as the natural numbers have a linear order and if $x_f(y) = y$ and $y \neq z$ then $x_f(y) \neq z$. If y_k is an enumeration of $f^{-1}(K)$ then let $x_{f(y_k)}$ be a subsequence of x_n . As $f(y_k)$ does not repeat integers by the aforementioned uniqueness, if $k \neq j$ then $y_k \neq y_j$ and so $f(y_k) \neq f(y_j)$ and so $x_{f(y_k)} \neq x_{f(y_j)}$ and so finally $(x_{f(y_k)})_{k=1}^\infty$ does not repeat elements infinitely many times. Because (x_n) does not have any cluster points the non-repeatability of $x_{f(y_k)}$ implies that $(x_{f(y_k)})$ has no cluster points. Observing that the value set $S(K) = \{x_{f(y_k)}\}_{k=1}^\infty$ is exactly $f^{-1}(K)$, we use the logic showing that (x_n) was closed to conclude that $S(K)$ is closed in X ; therefore $S(K)$ is closed in S . Hence f is continuous.

Now by Corrolary 4.17, it follows that f can be extended to some $F \in C(X)$ so that $F|_S = f$. Therefore F is an unbounded continuous function on X and so $BC(X) \neq C(X)$. \square