MATH H104: Homework 12

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November 24, 2015

4 Function Spaces

25. Prove the following.

Theorem 1. *If $f : M \to M$ be a contraction and M a metric space, then f is uniformly continuous.*

Proof. If f is a contraction, then we have that there exists a $k < 1$ such that for every $x, y \in M$ $d(fx, fy) \leq kd(x, y)$. We wish to show that for every $\epsilon > 0$ there exists a δ such that $d(x, t) < \delta$ implies $d(fx, ft) < \epsilon$. Taking delta to be ϵ/k we have that $d(x, t) < \epsilon/k$ implies that

$$d(fx, ft) \leq kd(x, y) < k\delta = \epsilon.$$

Therefore f is uniformly continuous. □

Theorem 2. *The extension of f to the completion of M, denoted M^*, say $g : M^* \to M^*$ is a unique contraction.*

Proof. Recall that the completion of a metric space $(M, d)$is specifically, a pair consisiting of the completed metric space (M^*, d^*) and an isometry $\phi : M \to M^*$ such that $\phi[M]$ is dense in the completed metric point set M^*. The extension of f on the compelted metric space (M^*, d) must therefore have the following property: $\phi[f(M)]$ is dense in $f^*(M^*)$.

First we show that an extension of a uniformly continuous function f must therefore be uniformly continuous. For every $\epsilon > 0$ there exists a $\delta > 0$ such that $x, y \in M$ and $0 < d(x, y) < \delta$ implies that $d(fx, fy) = d(f^*x, f^*y) < \epsilon$. Take $x^*, y^* \in M$ with $d(x^*, y^*) < \delta(\epsilon)$. Let

$$\theta(\epsilon) = \frac{\delta - d(x, y)}{4}.$$

Observe that $d(x, y) + 2\theta(\epsilon) < \delta$. Then take $x_n, y_n \in M$ such that $d(x_n, x^*) < \theta(\epsilon)/n$ and $d(y_n, y^*) < \theta(\epsilon)/n$. Then for all $n \in \mathbb{N}$

$$d(x_n, y_n) < d(x_n, x^*) + d(x^*, y^*) + d(y^*, y_n) < d(x^*, y^*) + 2\theta(\epsilon) < \delta.$$

So we have that $d(f^*x_n, f^*y_n) < \epsilon$. Finally observe that $x_n \to x^*$ and $y_n \to y^*$ as far as convergence is concerned in M^*. Since for all n we have that the functional composition of the sequence is less than ϵ, sureley we must have that the extension composed with x^*, y^* is less than ϵ.

In particular such satisfying $\delta = \epsilon/k$ and so $d(f^*x^*, f^*y^*) \leq kd(x^*, y^*) < \epsilon$ is a contraction. Uniqueness follows from exercise $54(b)$ of Homework 4. □

26. Consider the following example. Let $M = (-1, 0) \cup (0, 1)$. Then let $f : M \to M$ such that $x \mapsto \frac{1}{2}x$. Then $f^n(M) = (-(\frac{1}{2})^n, 0) \cup (0, (\frac{1}{2})^n)$. Suppose that there were a fixed point in M, say x. Since $x \neq 0$, $|x| > 0$ and there exists an N such that for all $n > N$ $|x| > \frac{1}{2}^n$, so x does not exist past the Nth iterate of the contraction f. There cannot be a fixed point.

27. We now conjecture about weak contractions.

Theorem 3. *Not all weak contractions are contractions.*

Proof. Consider the following example. Let $M = [0, 1]$ be a metric space and $f : M \to M$ such that $x \to \tanh(x - 1) + 1$ be a weak contraction on M. We show that f is not a contraction. Suppose for the sake of contradiction that there were a k such that $|f(x) - f(y)| \leq k|x - y|$ for $k < 1$. Then let $y = 1$, and take $y_n \to y$ such that $y_n = \frac{n-1}{n}$. By f a contraction we have that

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{1 - (\tanh(-\frac{1}{n}) + 1)}{1 - \frac{n-1}{n}} = n \tanh\left(\frac{1}{n}\right) = -n + \frac{2n}{1 + e^{-2/n}} \leq k < 1$$

However, since $n \tanh(1/n) \to 1$ as $n \to \infty$ we have a contradiction since there must be an N for which all $n > N$, $n \tanh(1/n) > k$. □

Furthermore, it follows that even the compactness of M does not give that all weak contractions are contractions by the previous theorem. However, fixed point theorems hold for weak contractions as is demonstrated by the following theorem.

Theorem 4. *If $f : M \to M$ is a weak contraction on a compact metric space M, then f has a unique fixed point.*

Proof. By M compact and f a weak contraction, we have that f continuous. Therefore $f(M)$ is closed up to iteration of f. Observe that since f is a weak contraction we have that $f^{n+1}(M) \subset f^n(M)$. So it follows that

$$\bigcap_{n=1}^{k} f^n(M) = f^k(M).$$

Then since the kth iterate of f on M is closed we have that the infinite intersection is closed and non empty; that is,

$$\bigcap_{n=1}^{\infty} f^n(M) = F \neq \emptyset.$$

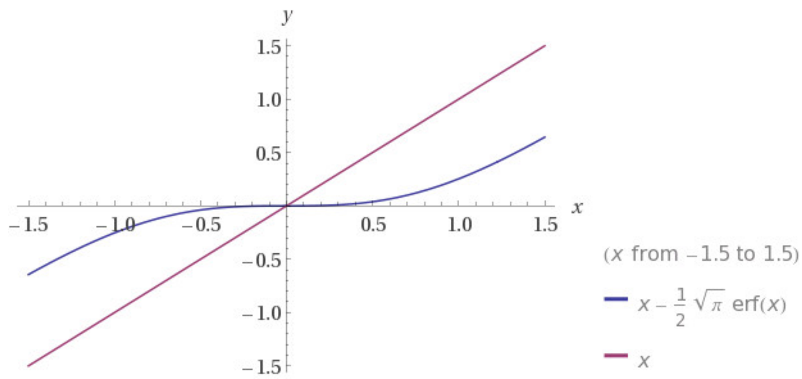

Figure 1: A graph of $f : \mathbb{R} \to \mathbb{R}$ a weak contraction (but not a contraction) with $|f'(x)| < 1$ against the identity function, showing the asymtotic properties of the $f(x)$.

We claim that if $p \in F$, p is a fixed point of f. Suppose for the sake of contradiction that $p \notin f(F)$, then $f(F) \subset F$ which contradicts the fact that $\bigcap_{n=1}^{\infty} f^n(M) = F$.

Lastly we claim that p is unique. Suppose there were another fixed point $q \in F$. Then since $q \neq f$, $d(p,q) = d(fq, fp) < d(p,q)$, which is a contraction, so p is unique. ☐

28. Weak contractions can be brought about by limiting the derivative of real valued functions. We propose the following theorem.

Theorem 5. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and is derivative satisifes $|f'(x)| < 1$ flor all $x \in \mathbb{R}$, then f is a weak contraction.*

Proof. For any $x, y \in \mathbb{R}$ such that $x \neq y$, assume that $x < y$ without loss of generality. Then differentiability of f implies that for some θ between x and y,

$$d(fx, fy) = f(\theta)d(x,y),$$

and since $f(\theta) < 1$ it is clear that $d(fx, fy) < d(x,y)$. So it follows that at least, f is a weak contraction. ☐

Although this property of the derivative guarentees that f is a weak contraciton, it does not necisarrily follow that f is a contraction nor that it has fixed points. Consider the following example.

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ has the property that $|f'(x)| < 1$. In particular let $f'(x) = 1 - e^{-x^2}$. This furthermore gives f the property that $f'(x) \to 1, x \to \infty$.

Clearly f is a weak contraction by the theorem above, but it does not have the property of contraction by the following logic. Suppose that there were a $k < 1$ such that $d(fx, fy) \leq kd(x,y)$. Observe that by f differentiable, we have that for x, y different $d(fx, fy) = f'(\theta)d(x,y)$ where θ is a real number between x and y. Since however as $x, y \to \infty$, $f'(\theta) \to 1$ we may choose x, y large enough that $f'(\theta) > k$, and we have contradicted the relationship given by contraction.

Furthermore since there are infiniteley many such f satisfying the above properties, take

$$f(x) = x - \int_0^x e^{-t^2}\ dt + 420.$$

In this case, there does not exist an x such that $f(x) = x$, since $f(x) - x > 0$ for all x. That is,

$$\inf_{x \in \mathbb{R}} \left(420 - \int_0^x e^{-t^2}\ dt \right) = 419 > 0.$$

So f has no fixed points even though $|f'(x)| < 1$ for all x.

29. We find an interesting counter example to the uniqueness of fixed points in Bruowers Fixed-Point Theorem.

Theorem 6. *Suppose that $f : B^m \to B^m$ is continuous where B^m is the closed unit ball in $\mathbb{R}^m$. Then although f has a fixed point, this point need not be unique.*

Proof. We give a simple proof by example! Take f as above such that $x \mapsto x$. Then for every $x \in B^m$, $f(x) = x$ is a fixed point. Therefore $f^n(x) = x$ for all n and $f^n(x) \to x$ as $n \to \infty$. So there is no unique fixed point. $\square$