MATH 202A: Notes

Scribe: William Guss

September 8, 2016

We will learn how to make a measure out of an outer measure.

Definition 1. *We call μ^* an outer measure if, $\mu^* : P(X) \to [0, \infty]$ such that $\mu^*(\emptyset) = 0$, $\mu^*(A) \le \mu^*(B)$ if $A \subset B$, and for all E_n,$\mu^*(\bigcup_{n=1}^{\infty} E_n) \le \sum_{j=1}^{\infty} \mu^*(E_j)$*

Definition 2. *Define some volume set function, $\rho : \mathcal{E} \subset P(X) \to [0, \infty]$*

Definition 3. *For any $A \subset X, X \in \mathcal{E}$, then the outer measure of A is*

$$\mu^*(A) = \inf_{(E_i) \in \mathcal{E}, A \subset \bigcup E_i} \sum_{n=1}^{\infty} \rho(E_n).$$

Lemma 1. *Assume that $\mathcal{E} \subset P(x)$ and that $\emptyset, X \in \mathcal{E}$. Then let $\rho : \mathcal{E} \to [0, \infty]$, $\rho(\emptyset) = 0$ and define μ^* as above. Then μ^* is an outer measure.*

Proof. Clearly $\mu^*(\emptyset) = 0$ since the empty set covers itself. If $A \subset B$ then we get subaddativity. Let $A_n \subset X$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Let $\epsilon > 0$. *Naive*: For each n there is $E_{n,i} \in \mathcal{E}$ so that $A_n \subset \bigcup_{i=1}^{\infty} E_{n,i}$ and

$$\sum_{i=1}^{\infty} \rho^*(E_{n,i}) \le \mu^*(A_n) + 2^{-n}\epsilon.$$

Now the family $E_{n,i}$ covers A. This is ac ountable family so $\mu^*(A) \le \sum \sum \rho(E_{n,i}) \le \sum_n \mu^*(A_n) + \sum_{n=1}^{\infty} 2^{-n}\epsilon$. The other direction was proven in the homeworks. $\square$

Remark. The $2^{-n}\epsilon$ trick is effectively a divide and conquer.

Fact 1. *The outer measure of $\mathbb{Q}$ is 0.*

Fact 2. *The out measure of $[a, b]$ is $|b - a|$.*

Definition 4. *Let μ^* be an outer measure on $P(X)$. A set $A \subset X$ is called μ^*-measurable and $A \in \mathcal{M}$ iff for every $E \subset X$*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

Remark. $E = (E \cap A) \sqcup (E \setminus A)$ and so $\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \setminus A)$ for any A. **Remark**. A measurable set is like a knife which cuts any E so that every part of E which has mass is conserve; "there is no blood loss, or unintententional injuries on the edges."

Theorem 1. *If μ^* is an outer measure on $P(X)$ then $\mathcal{M}$ is a σ-algebra and $\mu^*|_{\mathcal{M}} = \mu$ is a complete measure.*

Proof. Sketchy proof sketch.
1. A set $A \in \mathcal{M}$ if and only if $X \setminus A \in \mathcal{M}$. Therefore $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A))$. Therefore $\mathcal{M}$ is closed under compliments.

2. Suppose $A, B \in \mathcal{M}$. Let $E \subset X$. By A measurable

$$\begin{aligned}
\mu^*(E) &\geq \mu^*(E \cap A) + \mu^*(E \setminus A) \\
&\geq \mu^*(E \cap A \cap B) + \mu^*((E \cap A) \setminus B) + \mu^*((E \setminus A) \cap B) + \mu^*((E \setminus A) \setminus B) \\
&= \mu^*(E \cap A \cap B) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A)) + \mu^*(E \setminus (A \cup B))
\end{aligned}$$

Now we want to show that $\mu^* E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B))$. We can use that $\mu^*(E \cap (A \cup B) \leq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + ...$ So $\mathcal{M}$ is closed under unions.

3. Assume A_n measurable for $n \in \mathbb{N}$ if $A = \bigcup A_n$ then want to show A measurable. Without loss of generality, assume that $A_1 \subset A_2 \subset A_3 \subset \cdots$. Set $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$, and $\bigcup B_n = A$. So each $B_n \in \mathcal{M}$ and they are pairwise disjoint. Let $E \subset X$.... $\square$