CS 202B — UCB, Spring 2017 — Hammond —- Scribe: William Guss
Lecture Notes

Slices and measure through integration. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. For $E \in \mathcal{A} \times \mathcal{B}$ we set $h(x) = \nu(S_x(E))$ and $k(y) = \mu(t_y(E))$.

Proposition 1. *For every $E \in \mathcal{A} \times \mathcal{B}$, h is $\mathcal{A}$-measurable and k is $\mathcal{B}$-measurable. Furthermore*

$$\int_X h(x)\ d\mu(x) = \int_Y k(y)\ d\nu(y)$$

Proof. Suppose that μ and ν are finite. Let $\mathcal{C}$ be the collection of elements of $\mathcal{A} \times \mathcal{B}$ such that the proposition holds.

Aim. We wish to show two things, that $\mathcal{C}$ contains $\mathcal{C}_0$ (the generating algebra for $\mathcal{A} \times \mathcal{B}$) and that $\mathcal{C}$ is a monotone class. Since $\mathcal{A} \times \mathcal{B} = \sigma(\mathcal{C}_0)$ we will learn from this aim that $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ by the montone class lemma.

Sketch. Let's first approach the first aim, that $\mathcal{C}_0$ satisfies the the proposition.

If $E = A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We need to check that $E \in \mathcal{C}$. In this case $h(x) = \xi_B(x)\nu(B)$ and $k(y) = \xi_A(y)\nu(A)$. The indicator is certainly $\mathcal{A}$ measurable because $h^{-1}([a, \infty)) = A \in \mathcal{A}$ if $a \leq \nu(B)$ and $h^{-1}(\dots) = \emptyset$ otherwise. The same approach might be checked for k as well. Finally $\int_X \chi_A(x)\nu(B)\ d\mu(x) = \mu(A)\nu(B) = \int_Y \xi_B(y)\mu(A)\ d\nu(y)$ by the definition of integration on indicator functions.

Now we should check for disjoint unions. Take $E = \bigcup_{i=1}^n E_i$ where E_i are measurable rectangles and $E_i \cap E_j = \emptyset$ when $i \neq j$. First $S_x(E) = \bigcup S_x(E_i)$ since the E_i are disjoint. Now we can verify that

$$h(x) = \nu(S_x(E)) = \nu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \nu(S_x(E_i))$$

by finite additivity. Then we use that the sum of finitely many measurable functions is measurable to yield that h is measurable for any element $E \in C_0$. To verify the second claim we just use linearity:

$$\int_X h(x)\ d\mu(x) = \int_X \sum_{i=1}^n \nu(S_x(E_i))\ d\mu(x) = \sum_{i=1}^n \mu(A_i)\nu(B_i) = \int_Y \sum_{i=1}^n \mu(T_y(E_i))\ d\nu(y) = \int_Y k(y)\ d\nu(y).$$

Thus $\mathcal{C} \supset \mathcal{C}_0$.

Now we turn to the second aim of the proof, showing that $\mathcal{C}$ is a monotone class.

First consider an increasing sequence $E_n \in \mathcal{C}$ with $E_n \subset E_{n+1}$ and of course $E = \bigcup_{i=1}^\infty E_i$. We want to check that $E \in \mathcal{C}$. Set $h_n(x) = \nu(S_x(E_n))$ and $k_n(y) = \mu(t_y(E_n))$. We know that h_n increases to h and k_n increases to k monotonically. Additionally by $E_n \in \mathcal{C}$ we know h_n, k_n are both measurable in $\mathcal{A}$ and $\mathcal{B}$ respectively. We now apply the monotone convergence theorem and so h, k are measurable and the limit of integration thereof converges to integration of the limit. This covers equality, property two of the proposition.

On the otherhand, a decreaisng sequence $E_n \in \mathcal{C}$ withb $E_n \supset E_{n+1}$ and $E = \bigcap_{i=1}^\infty E_i$. In this case we will apply the dominated convergence theorem with h_1, k_1 clearly dominating their respective sequences. Using the finiteness of of ν, μ the h_1, k_1 are bounded by $\mu(X), \nu(Y)$ respectively. the conclusion of the dominated convergence theorem shows the proposition.

Therefore $\mathcal{C}$ is a monotone class and this completes the proof. $\square$