# MATH: H104: Homework 11

William Guss
26793499
wguss@berkeley.edu

November 18, 2015

8.

9.

10.

13.

18.

22. Consider the following example.

**Theorem 1.** *Let $f : [0,1] \to \mathbb{R}$ such that if $x \neq 0, x \mapsto x \sin\left(\frac{1}{x}\right)$ and $x \mapsto 0$ otherwise. Furthermore, let $(g_n)$ be a family of functions such that $g_n : [0,1] \to \mathbb{R}$ such that*

$$g_n(x) = \begin{cases} 0, \ \forall x \in [0, 1/n] \\ e^{\frac{1}{(x-1/n)^2}}, \ \forall x \in (1/n, 2/n) \\ 1, \ \forall x \in [2/n, 1] \end{cases}$$

*If $(f_n)$ is defined such that $f_n(x) = f(x)g_n(x)$, then the family $(f_n)$ is smooth, equicontinuous, with unbounded derivatives.*

*Proof.* Let $x \in (0,1], \gamma > 0$. Then there exists an $N$ such that $2/n < x - \gamma$. In this case for all $n > N$ $f_n(y) = f(y)$ for each and every $y \in (x-\gamma, x+\gamma)$. Then for every $\epsilon > 0$, the continuity of $f$ gives that there is a $\delta < \gamma$ with $|f_n(y) - f_n(x)| = f(y) - f(x) < \epsilon$. Take the smallest delta for which all $f_1, \dots, f_N$ are satisfied and yield that this $\delta'$ gives equicontinuity. At $x = 0$, $|f_n(y)| \leq y \sin(1/y) \leq y$ for all $y \in [0,1]$ and for all $n > N$. So $f_n(0)$ is equicontinuous. Then the compactness of $[0,1]$ implies uniform equicontinuity by the Arzela Ascoli theorem. Clearly $f'(x)$ is unbounded as it approaches 0 so in every case the derivatives of $f_n$ are unbounded by the product rule. Smoothness comes from the fact that $f_n = 0$ in $[0, 1/n]$ and the derivatives at 0 are 0. $\square$