Math 185 — UCB, Fall 2016 – Smirnov
Problem Set 2 due Thursday September 22 - William Guss

(18.5) Show that the function $f(z) = \left(\frac{z}{\overline{z}}\right)^2$ has the value 1 at all nonzero points on the real and imaginary axes, where $z = (x, 0)$ and $z = (0, y)$ but that it has the value -1 at all non zero points on the line $y = z, z = (x, x)$. Thus show that the limit of $f(z)$ as $z \to 0$ does not exist.

Proof. Consider that for all $z \neq 0$ we have that $z/\overline{z} = z^2/|z|^2$. Therefore $f(z) = z^4/|z|^4$. Now take the limit along the imaginary axes and get $f(x_n^1) = (x_n^1 i)^4/(x_n^1)^4$. Using $i^2 = -1$ we get $f(x_n^1) = (x_n^1)^4/(x_n^1)^4 = 1 \to 1$ as $x_n^1 \to 0$. Additionally take the real sequence $f(x) = (x+0i)^4/x^4 = 1$ as $x \to 0$. Finally take $f(x+xi) = (x+xi)^4/(2x^2)^2 = (x^4+4x^4i-6x^4-4x^4i+x^4)/4x^4 = -4x^4/4x^4 = -1 \to -1$ as $x \to 0$. So since all sequences of $z \to 0$ do not converge to the same limit, the function is not ocntinuous at $z = 0$. □

(18.9) Show that

$$\lim_{z \to z_0} f(z)g(z) = 0$$

if $\lim_{z \to z_0} f(z) = 0$.

Proof. If $\lim_{z \to z_0} f(z) = 0$ then for every $\epsilon > 0$ $|f(z)| < \epsilon$ as $d(z, z_0) < \delta$ for some δ and the distance function $d(z, z_0)$. By convention we define $0 \cdot \infty = 0$. Now within a compact δ-ball, $B_\delta(z_0)$ open around z_0 and any z in that ball, $|f(z)g(z)| \leq |f(z)| \sup_{y \in B_\delta(z_0)} |g(z)| \leq |f(z)| \times \infty$ However as $\delta \to 0$, $|f(z)| \to 0$ and $|f(z)| \times \infty \to 0$ by our convention, so $|f(z)g(z)| \to 0$ and so $\lim_{z \to z_0} f(z)g(z) = 0$. □

(18.9) Use theorem in Sec. 17 to show the convergence of the following limits.

(a) $\lim_{z \to \infty} \frac{4z^2}{(z-1)^2}$.

Proof. We let $f(z)$ be the function of the limit, and then show that $f(1/z) \to w_0$ as $z \to 0$ implies $f(z) \to w_0$ as $z \to \infty$. Clearly $f(1/z)$ is given by

$$\frac{4(1/z)^2}{((1/z)-1)^2} = \frac{4}{z^2((1/z)^2 - 2/z + 1)} = \frac{4}{1 - 2z + z^2} = \frac{4}{1 + z(z-2)}$$

And as $z \to 0$ we have $f(1/z) \to 4/1 = 4$ using that z^2 and z are continuous functions and complex multiplication is continuous. □

(b) $\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$.

Proof. We let $f(z)$ be the function of the limit, and then show that $\lim_{z \to 1} \frac{1}{f(z)} = 0$. Clearly $\frac{1}{f(z)} = (z-1)^3$. From the book, all polynomial functions are continuous for all $\mathbb{R}$ so $\lim_{z \to 1}(z-1)^3 = ((1) - 1)^3 = 0^3 = 0$ and by the Sec 17 theorem the limit in (b) converges to ∞. *Yay single point compactifications!* □

(c) $\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$.

Proof. We let $f(z)$ be the function of the limit. First observe that $(z-i)(z+i) = z^2 - z + -iz + iz - i^2 = z^2 + 1$. We must now show that $\lim_{z\to\infty} \frac{z-1}{(z-i)(z+i)} = 0$ which requires that $\lim_{z\to 0} \frac{1/z-1}{(1/z-i)(1/z+i)} = 0$. Clearly $1/(f(1/z)) = \frac{\overline{z}/|z|^2-1}{\overline{z}^2/|z|^4+1}$. Applying the complex conjugate method again we get

$$\begin{aligned}\frac{1}{f(1/z)} = \frac{(\overline{z}^2/|z|^4+1)(\overline{z}/|z|^2-1)}{(\overline{z}^2/|z|^4+1)\overline{(\overline{z}^2/|z|^4+1)}} &= \frac{(z^2/|z|^4+1)(\overline{z}/|z|^2-1)}{(\overline{z}^2/|z|^4+1)(z^2/|z|^4+1)} \\ &= \frac{(z^2/|z|^4+1)(\overline{z}/|z|^2-1)}{z^2\overline{z}^2/|z|^8+\overline{z}^2/|z|^4+z^2/|z|^4+1} \\ &= \frac{z(1/|z|^4 - z/|z|^4) + \overline{z}/|z|^2 - 1}{\overline{z}^2/|z|^4 + z^2/|z|^4 + 2}\end{aligned}$$

Taking the absolute value of the expression it is immediate that $|1/f(1/z)| \leq \frac{|z-1+1-1|}{|1+1+3|} \to 0$ as $|z| \to 0$ so the infinite limit holds. Another way to see this is that $|(1/z-1)|/|1/z^2+1| \leq C|1/z|/|1/z^2| \leq |z| \to 0$. Then follow application of Sec 17 Theorem twice and get the limit in (c) $\square$

(18.11) With the aid of the theorem in Sec 17. show that when

$$T(z) = \frac{az+b}{cz+d},$$

(a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$

Proof. First we show that $\lim_{z\to\infty} 1/T(z) = 0$ iff $\lim_{z\to 0} 1/T(1/z)) = 0$ iff

$$\frac{d}{az+b} \to 0,\ z \to \infty \iff \frac{d}{a/z+b} \to 0,\ z \to 0$$

Consider the magnitude $|1/(a/z+b)| \leq |d|/|a/z+b|$. Clearly $ab \neq 0$ so $|d|/|a/z+b| \leq d(1/b)/|a/zb+1| \leq |d(1/b)|/|a/zb| \leq |dz/b|/|a| \to 0$ as $z \to 0$, so the first assertion is proved by folling the if ($\Leftarrow$) logic. $\square$

(b) $\lim_{z\to\infty} T(z) = a/c$ and $\lim_{z\to d/c} T(z) = \infty$ if $c \neq 0$

Proof. If $c \neq 0$ we first show that $\lim_{z\to\infty} T(z) = a/c$ iff $\lim_{z\to 0} T(1/z) = a/c$. It follows

$$\frac{(a\overline{z}/|z|^2+B)cz/|z|^2+d}{|c\overline{z}/|z|^2+d|^2} \sim \frac{ac\overline{z}/|z|^4}{c^2|\overline{z}/|z|^2|^2} \sim \frac{a}{c} \to \frac{a}{c}.$$

Now for the second assertion, we will show that $\lim_{z\to d/c} 1/T(z) = 0$ which holds if and only if the second assertion does. Using

$$\lim_{z\to d/c} 1/T(z) = \lim_{z\to d/c} \frac{cz+d}{az+b} = \lim_{z\to d/c} f(z)g(z)$$

where $f(z) = cz+d$ and $1/g(z) = az+b$ and a previous proven theorem in the homework, we need show that $f(z) \to 0$ as $z \to d/c$. This is clear since $c(d/c) - d = d - d = 0$ so $fg \to 0$ so the limit goes to 0 so the inverse of the limit goes to infinity so the assertion is proved. $\square$

(20.4) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, wehre $g'(z_0) \neq 0$ then show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof. Using the definition of the derivative we have that

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim x \to z_0 \frac{f(x-z_0)-f(z_0)}{x-z_0}}{\lim y \to z_0 \frac{g(y-z_0)-g(z_0)}{y-z_0}}$$

Since x, y are any arbitrary sequence (by the existence of f', g') take any sequence $z \to z_0$ then

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim z \to z_0 \frac{f(z-z_0)-f(z_0)}{z-z_0}}{\lim z \to z_0 \frac{g(z-z_0)-g(z_0)}{z-z_0}} = \lim_{z \to z_0} \frac{(f(z-z_0) - f(z_0))(z-z_0)}{(g(z-z_0) - g(z_0))(z-z_0)}$$

it follows that

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{z \to z_0} \frac{f(z-z_0) - f(z_0)}{g(z-z_0) - g(z_0)} = \lim_{z \to z_0} \frac{f(z-z_0) - 0}{g(z-z_0) - 0}$$

by the hypothesis $f(z_0) = g(z_0) = 0$ and so $f'(z_0)/g'(z_0)$ is the limit of the fraction! □

(20.8) Show that $f'(z)$ does not exist at any point z when
(a) $f(z) = Re(z)$

Proof. Observe that $f(z) = \frac{z+\overline{z}}{2}$ and so $D_{\overline{z}} f \neq 0$ clearly and so Cauchy Riemann equations do not hold at any point z and so f is not differentiable. □

(b) $f(z) = Im(z) = \frac{iz+\overline{iz}}{2} = Im(z)$, but this is dependent on $\overline{z}$ so the Cauchy Riemann equations are satisfied nowhere and f is nowhere differentiable.

(24.1) Use the theorem in Section 21 to show that $f'(z)$ does not exist at any point if
(c) $f(z) = 2x + ixy^2$.

Proof. If f' exists the cauchy riemann equations are satisfied; that is $2 = 2yx$ and $0 = y^2$, so $2 = 0$ if the cauchy riemann equations hold, this is a contradiction. Therefore the derivative lives no where. □

(d) $f(z) = e^x e^{-iy}$.

Proof. Equivalently we have that $f(z) = e^{x-iy} = e^{\overline{z}}$. Therefore $\partial_{\overline{z}} f(z) = e^{\overline{z}} \neq 0$! So the Cauchy-Riemann equations could not hold at any z and the function is nowhere differentiable. □

(24.3) From results obtained in 21 and 23 determine where $f'(z)$ exists and find its value when
(a) $f(z) = 1/z$.

Proof. Using the power rules for differentiation we have that $f'(z) = -z^{-2}$ iff f is diferentiable. To show differentiability we recall that $f(z) = \frac{\overline{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$. So the real component of the derivative is consistent iff $\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2}$ wgucg follows since $2y^2 - y^2 - x^2 = x^2 + y^2 - 2x^2$. For the second component of the derivative we have Cauchy riemman conistency since $\frac{-2yx}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$. So the function is differentiable every where except for $z = 0$. □

(b) $f(z) = x^2 + iy^2$.

Proof. We can actually calculate the derivative using the Cauchy-Riemann equations; that is by the isomorphism between $Df \in E \subset \mathbb{R}^2 \otimes \mathbb{R}^2$ and $f' \in \mathbb{C}$, we use the following derivation to calculaute f'. First $2x = 2y \implies x = y$ and $0 = -0$ so it must be that $x = y$, lest the derivative not exist. Therefore we have $f'(z) = 2x + 0i = 2y - 0i$ $\square$

(24.7) (a) With the aid of the polar form (6), derive the alternative form $f'(z_0) = -\frac{i}{z_0}(u_\theta + iv_\theta)$.

Proof. From the section we know that $v_\theta = ru_r$ and $u_\theta = -rv_r$. Therefore $f'(z_0) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(v_\theta/r - iu_\theta/r) = e^{-i\theta}/(ri)(u_\theta + iv_\theta) = \frac{-i}{r}e^{-i\theta}(u_\theta + iv_\theta)$. Next $z_0 = re^{i\theta}$ so $1/z_0 = 1/re^{-i\theta}$ and we have the theorem

$$f'(z_0) = -\frac{i}{z_0}(u_\theta + iv_\theta).$$

This completes the proof. $\square$

(b) Derive the derivative of $f(z) = 1/z$ using the above formula.

Proof. We use the expression and find that $f(z) = 1/z = 1/re^{-i\theta} = 1/r(\cos\theta - i\sin\theta)$. Then $f'(z) = -i/z(-\sin\theta + i\cos\theta) = -1/z(\cos\theta - i\sin\theta) = -1/z^2$. $\square$

(26.1) Apply the main theorem of Section 23 to verify that each of these functions is entire.
(a) $f(z) = e^{-y}\sin x - ie^{-y}\cos x$.

Proof. C.R gives $(LHS)\ \ e^{-y}\cos x = e^{-y}\cos(x)\ \ (RHS)$ and $-e^{-y}\sin x = -(-\sin xe^{-y})$ and so the functions are analytic since the partial derivatives are continuous on $\mathbb{C}$. $\square$

(d) $f(z) = (z^2 - 2)/z$

Proof. We show that the partial derivative of $f(z)$ w.r.t the conjugate of z is always 0; that is since $f(z) = (z^2 - 2) \times 1/z$, $f(z)$ is the product of two analytic functions, again analytic on the largest open covering contained in the intersections of their domains. $\square$

For 27.4, 27.5, 27.6, the sketches are attached!