

# MATH H104: Homework 4

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September 20, 2015

## 2 A Taste of Topology

29. *Show the following.*

**Theorem 1.** *Let  $\mathcal{T}$  be the collection of open subsets of a metric space  $M$ , and  $\mathcal{K}$  be the collection of closed subsets. Show that there is a bijection from  $\mathcal{T}$  onto  $\mathcal{K}$ .*

*Proof.* We wish to find a function  $f : \mathcal{T} \rightarrow \mathcal{K}$  bijective. To do so observe the following fact about compliments in  $M$ :  $A^c = B$  is the unique compliment of  $A$ . Suppose that there were another compliment such that  $A^c = C \neq B$  which was the compliment of  $A$ . By definition  $C = \{x \in M | x \notin A\} = B$ , so there cannot exist another set which is also the compliment of  $A$ .

As follows from above, the compliment of an open set is closed and the compliment of a closed set is open. Therefore, let  $f : A \mapsto A^c$ . Then,  $f$  is an injection by the uniqueness of compliments. Furthermore, if  $S \in \mathcal{K}$ , there exists a set,  $Q$ , in  $\mathcal{T}$  such that  $f(Q) = S$ , namely  $S^c$ . This follows by  $S^c \in \mathcal{T}$  and  $f(S^c) = S^{cc} = S$ . Hence  $f$  is a bijection.

This completes the proof, and  $\mathcal{K} \sim \mathcal{T}$ . □

32. *Prove the following and then remark.*

**Theorem 2.** *Every subset of  $\mathbb{N}$  is clopen.*

*Proof.* To show that every subset of  $\mathbb{N}$  is clopen the definitions of openness and closedness must hold on every set. Take an arbitrary subset  $S$  of the natural numbers. If  $S$  is empty or the whole space  $\mathbb{N}$  then it is clopen.

Otherwise, for every  $q \in S$  there exists an  $r > 0$ , say 0.5, such that  $d(q, p) \implies p \in S$ . To see this, consider that the only such  $p$  for which the definition of openness holds is  $q$  itself. Therefore,  $S$  is open.

The subset  $S$  must also be closed because  $S^c$  is an open subset of the naturals, and  $S^{cc} = S$  must be closed by compliments. Hence  $S$  is clopen and the proof is complete. □

**Remark.** Any function mapping the natural numbers to some metric space  $M$  must be continuous. Consider some  $Q \subset f(\mathbb{N})$ . If  $Q$  is open then,  $f^{pre}(Q)$  is open. Conversely, if  $Q$  is closed then,  $f^{pre}(Q)$  is closed. Furthermore if  $M$  is any discrete space (or one with a discrete metric) then  $f$  is an open mapping.

33. Find a metric space for which the boundary of the  $r$  neighborhood need not always be the  $r$ -sphere.

**Example.** Let  $M = \mathbb{N}$  be a metric space with its inherited metric from  $\mathbb{R}$ . We show that it is not true that for each  $M_r(p)$ , the boundary is the  $r$ -sphere. Consider that the closure of  $M_r(p)$  is  $M_r(p)$  as every set in  $M$  is clopen. Then the closure of the complement is just complement. By definition  $\partial M_r(p) = \overline{M_r(p)} \cap \overline{M_r^c(p)} = \emptyset$ . However for all  $r \in \mathbb{N}$ ,  $S_r(p) = \{x \in M \mid d(x, p) = r\} \neq \emptyset$ . So there are cases in which the boundary is not the  $r$ -sphere.

Suppose that  $x$  were in the boundary of some  $M_r(p)$  and not in the unit sphere. Then  $d(x, p) \not\leq r \implies d(x, p) > r$ . By virtue of  $x$  being in the boundary,  $x$  must be in every closed subset containing  $M_r(p)$ . However,  $x \notin S_r(p)$  (the  $r$ -sphere at  $p$ ) and  $S_r(p) \supset M_r(p)$  is closed; a contradiction! So, the boundary must be contained within the  $r$ -sphere at  $p$ .

40. Prove the following.

**Theorem 3.** If  $M$  be a metric space with metric  $d$ , then the following are equivalent:

- (a)  $M$  is homeomorphic to  $M$  equipped with the discrete metric.
- (b) Every function  $f : M \rightarrow M$  is continuous.
- (c) Every bijection  $g : M \rightarrow M$  is a homeomorphism.
- (d)  $M$  has no cluster points.
- (e) Every subset of  $M$  is clopen.
- (f) Every compact subset of  $M$  is finite.

*Proof.* (a)  $\implies$  (e). Since  $(M, d) \cong (M, d_{\text{discrete}})$ , then for some function  $f : M \rightarrow M$  where the domain has the discrete metric, every subset of the domain is clopen, and thereby every image of a subset of the domain is clopen by the homeomorphism.

(b)  $\implies$  (c). If every function in  $M^M$  is continuous, then consider an arbitrary bijection  $g : M \rightarrow M$ . Clearly  $g$  is continuous, and its inverse map  $g^{-1} : M \rightarrow M$  is also continuous.

□

42. What is wrong with the proof of Theorem 28?

The misstep in the proof is the statement that there exist subsequences  $(a_{n_k}), (b_{n_k})$  which converge. Compactness surely implies that there exists an index sequence  $n_k$  such that  $a_{n_k} \rightarrow a \in A$  but that exact index set may not be one which allows  $b_{n_k} \rightarrow b$ .

To solve this problem consider the following argument. Since any subsequence of  $(a_{n_k})$  converges to  $a$  by the convergence of  $(a_{n_k})$ , and  $B$  compact, we can take a subsequence,  $(b_{n_{k(l)}})$  which converges to  $b$ . So the sequence  $((a_{n_{k(l)}}), (b_{n_{k(l)}})) \rightarrow (a, b)$ .

43. Prove the following.

**Theorem 4.** *If the cartesian product of two non-empty sets  $A \subset M, B \subset N$  is compact in  $M \times N$ ,  $A$  and  $B$  are compact.*

*Proof.* By the compactness of  $C = A \times B$ , all sequences  $(a_n, b_n)$  have subsequences which converge to some  $(a, b) \in C$ . Take one such particular sequence. Since  $a_n \in A$  and  $a \in A$ . Then the subsequential convergence of the product sequence implies the subsequential convergence of  $a_n$ . The same argument holds for  $b_n$ . In general,  $C$  contains the product of all sequences in  $A$  and  $B$ . So for any sequence in  $A$ , there exists some sequence in the product whose subsequence converges thereby implying the convergence of some subsequence of the original sequence in  $A$ . Again, the same argument holds for any given sequence in  $B$ .

This completes the proof.  $\square$

48. *Prove the following.*

**Theorem 5.** *There exists an embedding of the line as a closed subset of the plane, and there is an embedding of the line as a bounded subset of the plane, but there is no embedding of the line as a closed and bounded subset of the plane.*

*Proof.* By the line, we assume that  $\mathbb{R}$  is meant. Consider the following function  $f : \mathbb{R} \rightarrow L_u \subset \mathbb{R}^2$  such that  $x \mapsto (x, 0) \in \mathbb{R}^2$ . When  $L_u = \{(x, y) \in \mathbb{R}^2 : y = 0\}$  is the codomain,  $f$  is clearly surjective and injective. Hence we have that  $f$  is bijective. Furthermore, take some open set in  $L_u$ , say  $S$ . Then  $f^{-1}(S) = \{x \in \mathbb{R} | (x, 0) \in S\}$ . If for every  $s \in S$  there exists an  $r > 0$ , such that  $d(s, q) < r \implies q \in S$ , we have that  $d((s_x, 0), (q_x, 0)) < r$ . Since  $f^{-1}s = s_x$  and  $f^{-1}q = q_x$  then  $d(s_x, q_x) < r$  and thereby  $q_x \in \mathbb{R}$ . So it must follow that for every  $s_x$  in  $\mathbb{R}$  there exists an  $r > 0$  such that  $d(s_x, q_x) < r \implies q_x \in \mathbb{R}$ . It suffices to say that  $f$  is a homeomorphism when the converse argument is applied.

Knowing that  $f$  embeds  $\mathbb{R}$  onto  $\mathbb{R}^2$ , we show that such an embedding is a closed subset.  $L_u$  is closed if and only if it contains all of its limit points. Suppose  $(x_n)$  is a sequence in  $L_u$  such that  $x_n \rightarrow x$ . We wish to show that  $x \in L_u$ . By the convergence of  $x_n$  for every  $\epsilon > 0$ , there exists an  $N$ , such that for all  $n \geq N$ ,  $d(x_n, x) < \epsilon$ . if  $x$  is not in  $L_u$ , then  $x = (a, b)$  with  $b \neq 0$ . So if  $d(x_n, x) < \epsilon$  then take  $\epsilon = b - 0.1$ . In this case,  $d(x_n, x) < \epsilon \implies x_n \notin L_u$  which is a contradiction. Ergo,  $L_u$  is a closed embedding of the line in the plane.

In a different case, it is clear that  $\mathbb{R} \cong (0, 1)$ . It suffices to show that  $(0, 1)$  has an embedding in  $\mathbb{R}^2$  which is bounded. Simple! Take  $f : (0, 1) \rightarrow \mathbb{R}^2$  such that  $x \mapsto (x, 0)$ . The function  $f$  embeds  $(0, 1)$  by the same argument supplied for the first case. Furthermore,  $f((0, 1))$  is bounded because the set  $[0, 1] \times [0, 1]$  contains the embedding ( $x$  is always between 1 and 0 and the  $y$  component is always 0.)

In the last case, suppose there existed a closed and bounded subset of the plane such that  $\mathbb{R}$  was embedded to that set by some homeomorphism  $h$ . Then, by some theorem that embedding is compact as a subset of  $\mathbb{R}^2$  and by topological equivalence,  $\mathbb{R}$  must also be compact; a contradiction! Therefore, only the first two cases hold.  $\square$

54. *If  $f : A \rightarrow B$  and  $g : C \rightarrow B$  such that  $A \subset C$  and for each  $a \in A$  we have that  $f(a) = g(a)$  then  $g$  **extends**  $f$ . We also say that  $g$  **extends to**  $g$ . Assume that*

$f : S \rightarrow \mathbb{R}$  is a uniformly continuous function defined on a subset  $S$  of a metric space  $M$ . Prove the following:

(a) *Extension to closure.*

**Theorem 6.** *The function  $f$  extends to a uniformly continuous function  $\bar{f} : \bar{S} \rightarrow \mathbb{R}$ .*

*Proof.* If  $f$  is uniformly continuous, then for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that for all  $p, q \in S$ ,  $d(p, q) < \delta \implies d(fp, dq) < \epsilon$ . Since  $f$  is continuous it preserves convergence of sequences. So adding the closure of  $S$  to  $S$  through union lets all sequences in this new set  $\bar{S}$  converge to elements in  $\bar{S}$ . Adding these elements we construct a function based on the convergence of limits.  $g : \bar{S} \rightarrow \mathbb{R}$  such that if  $x \in S$ , then  $x \mapsto fx$  and otherwise if  $x \notin S$  and  $x \in \bar{S}$  we know the following. The element  $x$  is a limit of a sequence in  $S$ , say  $x_n$ . Then for every  $r > 0$  there exists an  $N$  such that for all  $n > N$ ,  $d(x_n, x) < r$ . Using the function,  $f(x_n) \rightarrow y \in \mathbb{R}$ . Let  $g(x) = y$ . Then for all  $\epsilon > 0$ , there exists such an  $N$  that  $n > N$  implies  $d(gx_n, gx) < \epsilon$ . In this case let  $\delta = r = \epsilon$  from before. Then the limit is preserved and  $g$  is uniformly continuous at  $x$ . Hence  $f$  extends to a uniformly continuous function  $\bar{f} = g$ .  $\square$

(b) *Uniqueness*

**Theorem 7.** *The function  $\bar{f}$  is the unique extension of  $f$ .*

*Proof.* Suppose that there exists another extension of  $\bar{f}$  to the closure of  $S$ , say  $g$ . Then for every  $a \in S$ ,  $f(a) = \bar{f}(a) = g(a)$ , by extension, and if  $x \in \bar{S}$  then  $\bar{f}(x) \neq g(x)$ . Consider a sequence which converges to  $x$  as a subset of  $S$ . Then for all  $\epsilon > 0$  there exists an  $N_1$  such that for all  $n > N_1$ ,

$$d(\bar{f}x_n, fx) < \epsilon/2.$$

Since  $g$  is also continuous we have that for some  $N_2$  and all  $n > N_2$

$$d(gx_n, gx) < \epsilon/2.$$

Remember that our assumption implies that  $\bar{f}(x) \neq g(x)$ . Take  $N = \max N_1, N_2$  then for all  $n > N$  we have that

$$d(\bar{f}x, gx) \leq d(\bar{f}x, \bar{f}x_n) + d(\bar{f}x_n, gx_n) + d(gx_n, gx) < \epsilon/2 + 0 + \epsilon/2,$$

by extension of  $f$ . So it is clear,  $\bar{f}(x) = g(x)$ ; a contradiction!

Therefore  $\bar{f}$  is unique and the proof is complete.  $\square$

(c) *General complete metric space.*

**Theorem 8.** *Let  $f : S \rightarrow N$  be a uniformly continuous function defined on  $S \subset M$  with  $M$  a metric space. Then  $\bar{f} : \bar{S} \rightarrow N$  is a unique uniformly continuous extension of  $f$ .*

*Proof.* TODO: PROVE The uniqueness of  $\bar{f}$  is proved by (b) without loss of generality. Hence take the previous theorem with  $N$  as the domain.  $\square$