Math 202A — UCB, Fall 2016 — M. Christ
Problem Set 13, due Wednesday November 30

(13.1) (Folland problem 4.22) Let X be a topological space, an d let (Y, d) be a complete metric space. Let (f_n) be a seuqnece of functions in Y^X that satisfies $\sup_{x\in X} \lim_{m,n\to\infty} d(f_m(x), f_n(x)) = 0$.
(a) Show that there exists an $f \in Y^X$ so that $\sup_{x\in X} d(f_n(x), f(x)) \to 0$.

Proof. We will first construct f and show that the the uniform norm distance tends to 0. As Y is a complete metric space under d and for every x, $\lim_{m,n\to\infty} d(f_m(x), f_n(x))$ is bounded by $\sup_{x\in X} \lim_{m,n\to\infty} d(f_m(x), f_n(x)) = 0$, the sequence $(f_m(x))_m$ is cauchy and thus has a limit, say $f(x)$. Now define $f : X \to Y$ so that $x \mapsto f(x)$ as defined previously for every x. We have shown that this function is well defined.

We now claim that $\sup_{x\in X} d(f_n(x), f(x)) \to 0$. First[1]

$$\begin{aligned}
\lim_{n\to\infty} \sup_{x\in X} d(f_n(x), f(x)) &= \lim_{n\to\infty} \sup_{x\in X} \lim_{m\to\infty} d(f_n(x), f_m(x)) \\
&= \lim_{n\to\infty} \sup_{x\in X} \lim_{m\to\infty} \sup_{k\geq m} d(f_n(x), f_k(x)) \\
&\leq \lim_{n\to\infty} \lim_{m\to\infty} \sup_{x\in X} \sup_{k\geq m} d(f_n(x), f_k(x)) \quad \text{(sup of dec. seq at x)} \\
&\leq \lim_{n\to\infty} \lim_{m\to\infty} \sup_{k\geq m} \sup_{x\in X} d(f_n(x), f_k(x)) \quad \text{(sup of maximal distances over x.)} \\
&= \lim_{m,n\to\infty} \sup_{x\in X} d(f_n(x), f_k(x)) = 0 \quad \text{(existence by hypothesis)}
\end{aligned}$$

This completes the proof. □

(b) Show that f is unique.

Proof. Suppose that f is not unique, that is there is a $g \neq f$ so that $\sup_{x\in X} d(f_n(x), g(x)) \to 0$. Then for every n

$$\sup_x d(f(x), g(x)) \leq \sup_x (d(f(x), f_n(x)) + d(f_n(x), g(x))) \leq \sup d(f(x), f_n(x)) + \sup d(g(x), f_n(x)).$$

But then $\sup d(g(x), f_n(x)) \to 0$ and $\sup d(f(x), f_n(x)) \to 0$ implies that $\sup_x d(f(x), g(x)) = 0$; thus for every x, $f(x) = g(x)$, and $f = g$; a contradiction to the existence of g. Therefore f is unique. □

(c) Show that if every function f_n is continuous then so is f.

Proof. Let f_n and f be given as above. Then for every $\epsilon > 0$, there is an N so that for all $n \geq N$ $\sup_{x\in X} d(f_n(x), f(x)) < \epsilon/3$. Then by continuity of f_n there is an open neighborhood $U_n \subset X$ of x so that for all $y \in U_n$ $d(f_n(x), f_n(y)) < \epsilon/3$. Finally

$$\begin{aligned}
d(f(x), f(y)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\
&\leq d(f_n(x), f_n(y)) + 2 \sup_{x\in X} d(f_n(x), f(x)) < 2/3\epsilon + \epsilon/3 = \epsilon.
\end{aligned}$$

Thus for every ϵ, there is an open neighborhood of x so that for all y in that neighborhood, $d(f(x), f(y)) < \epsilon$.

□

[1] We will abuse notation and write $\lim_n \sup_{x\in X} d(f_n(x), f(x)) = \lim_n \sup_{j\geq n} \sup_{x\in X} d(f_j(x), f(x))$, but we do not assume the limit exists. In fact we will show that the the $\limsup_n$ is bounded above by 0 and the sequence is bounded below by 0 so the limit is 0.

(13.2) Give an elementary proof of the Tietze Extension Theorem for the special case in which $X = \mathbb{R}$.

Proof. If A is a closed subset of $\mathbb{R}$ and $f : A \to [a, b]$ is continuous, we wish to show the existence of a continuous function $F : \mathbb{R} \to [a, b]$ so that $F|_A = f$. The main idea of our proof is that since f is defined on a closed subset of $\mathbb{R}$ we need only work to define F on the open compliment of A which from elementary real analysis is just the countable union of open intervals.

In the trivial case that $A = \mathbb{R}$ or $A = \emptyset$, let $F = f$ or $F = c$ respectively, the first is obviously a continuous extension and the second is a trivial extension since f is undefined for every $x \in \mathbb{R}$.

Otherwise recall from real analysis that $\mathbb{R} \setminus A = B$ is an open subset of $\mathbb{R}$ and so is the countable union of disjoint open intervals O_n. We will essentially define F as the affine interpolation of f on these open intervals. First assume that $O_n = (a_n, b_n)$ where a_n, b_n are finite; we will address the nonfinite cases later. Define

$$F|_{[a_n,b_n]} : x \mapsto \frac{f(b_n) - f(a_n)}{b_n - a_n}(x - a_n) + f(a_n).$$

$F|_{[a_n,b_n]}(a_n) = f(a_n)$ and $F|_{[a_n,b_n]}(b_n) = f(b_n)$. Furthermore from 104, $F|_{[a_n,b_n]}$ is continuous for all $x \in [a_n, b_n]$ as $F|_{[a_n,b_n]}$ is just the affine linear interpolation of $f(a_n)$ and $f(b_n)$. If there are O_n so that $a_n = -\infty$ or $b_n = \infty$ then let $F|_{(a_n,b_n]} : x \mapsto f(b_n)$ or $F|_{[a_n,b_n)} : x \mapsto f(a_n)$ respectively. In this case these functions are just constant and so $F|_{O_n}$ is also continuous. Lastly let $F|_A = f$, and then $F : \mathbb{R} \to [a, b]$ so that F satisfies all of the previous restrictions[2]

By the continuity of f on A and the previous arguments F is continuous with respect to the whole topology on $\mathbb{R}$ at least on A^o and $B^o = B$. As for continuity at $A \setminus A^o = C$, we observe that $cl(A) = A$ and so $C = \partial A$. From real analysis the boundary of a closed set cannot contain any open intervals as A^o is the countable union of open intervals, and if ∂A contained an open interval, then A^o would not be the largest open set containing A, therefore $\partial A = \{y \in A\}$ so that if $v, w \in \partial A$ $v < w$ or $w < v$. Now as $\delta \to 0, \sup f((v - \delta, v + \delta) \cap A) - \inf f((v - \delta, v + \delta) \cap A) \to 0$. Observe that now $\sup F((v - \delta, v + \delta)) = f((v - \delta, v + \delta) \cap A)$ since on $(v - \delta, v + \delta) \setminus A$ F is defined on some collection of O_n to never achieve values larger than f on the end points on which it must be defined. Likewise for the infimum $\inf F((v - \delta, v + \delta)) = \inf f((v - \delta, v + \delta) \cap A)$. Therefore $\sup F((v - \delta, v + \delta)) - \inf F((v - \delta, v + \delta)) \to 0$ and F is continuous at $v \in \partial A$.

Therefore F is continuous on A^o and ∂A and A^c with respect to the global topology so it is continuous. This completes the proof. $\square$

(13.3) Show that any product of connected topological spaces is connected.
Lemma. Product of fininteley many connected spaces is connected:

Proof. Take two spaces A, B connected. Then to show that the produce $A \times B$ is conencted, first let $i_A^b : A \to A \times \{b\}, b \in B$ be the inclusion mapping. The inclusion map is category theoretically a bijection when b is a singleton, so we need only show that the map is continuous and open. If $V \subset A \times \{b\}$ is open then it inherits the product topology so without loss of generality assume it is from the base and so $V = \pi_1^{-1}(C) \cap \pi_1^{-1}(D) \cap A \times \{b\}$ where $C \subset A$ open and $D \subset B$ open. Then $b \in \pi_1^{-1}(D)$ otherwise V would be the emptyset, thus $V = C \times \{b\}$ and the inverse of the map is C which is open in A. Since we have shown every open set is of this form, take any $C \subset A$ open and the $i_A^b = C \times \{b\}$ which is open. Therefore the map is a homemorphism. Since the product is commutative the argument holds for the inclusion map i_B^a for any $a \in A$. Now we will show that

[2]This is defined since we described the restriction of F on all x in $\mathbb{R} \setminus A \cup A = \mathbb{R}$.

every $(a,b) \in A \times B$ belong to the same maximally connected component by showing they belong to a conenected component and there is a common element to all of those connected components (invoking a proof from a previous exercise.) Fix $(c,d) \in A \times B$. Then for all (a,b), its clear that $(a,b) \in i_A^d(A) \cup i_B^a(B)$ and symmetricly $(a,b) \in I_A^b \cup i_A^c(B)$ both of which are connected as they are the union of the continuous images of two connected sets. Thus $(a,b) \in i_A^d(A) \cup i_B^a(B) \cup I_A^b \cup i_A^c(B) \ni (c,d)$ and so (c,d) is common to every connected component contianing every element of the space and so the space itself is connected.

Then induct on a finite product to yield the result. □

We now prove the exercise.

Proof. We will use the classic constancy argument. First let $D = \{0,1\}$ with the topology inherited from $\mathbb{R}$. Then a continuous closed map $f : C \to D$ is constant if and only if C is a connected topological space. If $f : C \to D$ is constant for all f, but for contradiction that C were not connected, then $C = A \sqcup B$ with A, B clopen disjoint non-trival. Then let $f : A \to 0, B \to 1$ and so trivially f is closed since all elements of $P(D)$ are closed. Since every element of $P(D)$ is open then the inverse of subsets of D is either C, A, or B, which are all open by C disconnected. But this contradicts the fact that all closed continuosu maps are constant, so C must be connected. In the other direction, if C is connected, then the image of any f must be connected, so must be either 1 or 0 but not D and so every f must be constant.

Now to the problem, let $X = \prod_A X_a$. Then let X_a connected for all a. Take any $f : X \to D$ to be a continuous function with X endowed with the product topology. Take any point x in X and let M be it maximally connected component; from a previous exercise $M = \bigcup_K K$ where K is a connected subset of X which contains x and M is connected obviously. Then $f(M)$ is connected and so is $cl(f(M)$, since $M \subset X$ and f is a closed continuous map $cl(f(M)) \subset f(X)$. We would like to show that f is constant.

For contradiction suppose there is a $y \in D$ so that $f(X) \setminus cl(f(M)) \ni y$, that is there is (are) some $z \in X$ so $f(z) = y$ and not $f(z) \in cl(f(M))$. Then take an openneighboorhood T of y which is disjoint from $f(M)$; this is possible in D because $f(M)$ is constant. By the continuity of f, $f^{-1}(T) = W$ is open and so we can take a neighborhood of $z \in W$ which is contained by W and is also in the neighborhood base. Then there are open sets so that $z \in \bigcap_1^m \pi_{a_n}^{-1}(A_{a_n})$ and this intesection is a subset of W. Let v so that $v_a = z_a$ for those finite $a \in \{a_n\}$ and otherwise $v_a = x_a$ where $x \in M$ was the element defined in the previous paragraph. Now x and v both agree except for the coordinates in $\{a_n\}$ so they are both contained in an equivalence class Y_x of points in X which agree with y except for on $\{a_n\}$. Now Y_x is bijective to $\prod_{a \in \{a_n\}} X_a$ as the continuous projection map provides an injection since for all $q \in Y_x$, q is uniqueley determined on a_n and of course the projection map is surjective. Now for any $q \in Y_x$ take an openneighborhood of V of q then every $p \in V$ is uniqueley determined by its coordinates $\{a_n\}$ so $V = \bigcap_{b \in Q \subset \{a_n\}} \pi_b^{-1}(U_b) \cap Y_x$. With U_b open in X_b. Thereforee $\pi_b(V) = U_b$ for all $b \in Q$ implie that $\pi_{\{a_n\}}(V) = \bigcap_{b \in Q} \pi_b^{-1}(U_b)$ is open in $\prod_{a \in \{a_n\}} X_a$. Therefore Y_x is homemorphic to this subspace, and this subspace is connected by the Lemma!. This contradicts the fact that $f(v) \in T$ as Y_x is a subset of the maximal component M which contains x and v, and thus $f(Y_x) \subset cl(f(M))$. Therefore there are are no elements whose image is not in closure of the image of the maximally connected component M and so so f is constant. Therefore X is connected. □

(13.4) Let X be a topological space equipped with with an equivalence realtion, $\tilde{X}$ the set of equivalence classes, $\pi : X \to \tilde{X}$ the map taking each $x \in X$ to its equivalence class, and $\mathcal{T} = \{U \subset \tilde{X}$:

$\pi^{-1}(U) \in \mathcal{T}_X\}$.
(a) Show that $\mathcal{T}$ is a topology on X.

Proof. First we show that $\tilde{X} \in \mathcal{T}$. Since π is a surjection onto $\tilde{X}$ it follows that $\pi^{-1}(\tilde{X}) = X \in \mathcal{T}_X$ and so $\tilde{X} \in \mathcal{T}$. Additionally $\pi^{-1}(\emptyset) = \emptyset \in \mathcal{T}_X$ so $\emptyset \in \mathcal{T}$. Now take any $U_\alpha \in \mathcal{T}$ with an arbitrary index set $\alpha \in A$. We claim that $\bigcup_\alpha U_\alpha \in \mathcal{T}$. First by the definition of preimage

$$\pi^{-1}\left(\bigcup_{\alpha\in A} U_\alpha\right) = \bigcup_{\alpha\in A} \pi^{-1}\left(U_\alpha\right) \in \mathcal{T}_X$$

since $\pi^{-1}\left(U_\alpha\right) \in \mathcal{T}_X$ are open in X and the topology $\mathcal{T}_X$ is closed under arbitrary union. Lastly consider the finite intersection of $U_m \in \mathcal{T}$; that is, We claim that $\bigcap_1^n U_m \in \mathcal{T}$. First by the definition of preimage

$$\pi^{-1}\left(\bigcap_{m=1}^{n} U_m\right) = \bigcap_{m=1}^{n} \pi^{-1}\left(U_m\right) \in \mathcal{T}_X$$

since $\pi^{-1}\left(U_\alpha\right) \in \mathcal{T}_X$ are open in X and the topology $\mathcal{T}_X$ is closed under finite intersection.

Thus $\mathcal{T}$ satisfies the definition of a topology. $\square$

(b) Show that if Y is a topological space, $f : \tilde{X} \to Y$ is continuous if and only if $f \circ \pi$ is continuous.

Proof. Suppose that $f : \tilde{X} \to Y$ is continuous and $\mathcal{T}_Y$ denotes the topology of Y. Then for any open $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}$, but then $\mathcal{T}_X \ni \pi^{-1}(f^{-1}(V)) = (f \circ \pi)^{-1}(V)$. Thus the preimage mapping $(f \circ \pi)^{-1}$ preserves the topology $\mathcal{T}_Y$ and so $f \circ \pi : X \to Y$ is continuous.

In the other direction if $f \circ \pi : X \to Y$ is continuous then for every open neighborhood of $(f \circ \pi)(x)$, say V there is an open neighborhood of x, say U, so that $(f \circ \pi)(U) \subset V$. Then $\pi(U) = W \ni \pi(x)$ is a neighboorhood of $\pi(x)$ so that $f(W) \subset V$. Thus take W^o and then $f(W^o) \subset V$. Now since π is surjective for every $\tilde{x} \in \tilde{X}$ pick any $x \in \pi^{-1}(x)$. Then for any V as above there is an open neighboorhood W^o of $\tilde{x}$ so that $f(W^o) \subset C$. Therefore $f : \tilde{X} \to Y$ is continuous. $\square$

(c) Show that $\tilde{X}$ is T_1 iff every equivalence class is closed.

Proof. Suppose that $(\tilde{X}, \mathcal{T})$ is T_1. Then $X \setminus \{\tilde{x}\} \in \mathcal{T}$ for every $\tilde{x} \in \tilde{X}$ and so $\pi^{-1}(\tilde{X} \setminus \{\tilde{x}\}) \in \mathcal{T}_X$. Now since π maps every x into its exact equivalence class and the fundamental theorem of equivalence relations states that the equivalence classes of x form a partiiton of the space $\pi^{-1}(\tilde{X}\setminus\{\tilde{x}\}) \cap \pi^{-1}(\{\tilde{x}\}) = \emptyset$ and $\pi^{-1}(\tilde{X} \setminus \{\tilde{x}\}) \cup \pi^{-1}(\{\tilde{x}\}) = X$. Therefore $X \setminus \pi^{-1}(\tilde{X} \setminus \{\tilde{x}\}) = \pi^{-1}(\{\tilde{x}\}) = [x]_\sim$ for some x; that is $[x]_\sim$ is the compliment of an open set and so $[x]_\sim$ is closed. Since we argued this for every $\tilde{x}$ and π is surjective, we get that for every $x \in X$, $[x]_\sim$ is closed.

Suppose that every equivalence class is closed. Then for every $x \in X$, $X \setminus [x]_\sim \in \mathcal{T}_X$. Now take any $\{\tilde{x}\} \subset \tilde{X}$. It follows that $\pi^{-1}(X \setminus \{\tilde{x}\}) = \pi^{-1}(\tilde{X}) \setminus \pi^{-1}(\tilde{x}) = X \setminus [x]_\sim \in \mathcal{T}_X$ and so $\tilde{X} \setminus \{\tilde{x}\} \in \mathcal{T}$ for every $\tilde{x}$ and so singletons are closed in $\tilde{X}$ and thus $(\tilde{X}, \mathcal{T})$ is T_1. $\square$

(13.5) Let $(X, \mathcal{T})$ be compact and Hausdorff.
(a) Let $\mathcal{T}'$ be strictly stronger than $\mathcal{T}$ on X. Show that $(X, \mathcal{T}')$ is not compact.

Proof. Suppose that $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{T}' \neq \mathcal{T}$ and $\mathcal{T}'$ compact. Let $id : (X, \mathcal{T}') \to (X, \mathcal{T})$ so that $id(x) = x$. Then if $V \in \mathcal{T}$ $id^{-1}(V) = V \in \mathcal{T}'$ by our assumption that $\mathcal{T}'$ is a stronger topology. Therefore id is a continuous bijection. Then by Proposition 4.28, $(X, \mathcal{T}')$ compact and $(X, \mathcal{T})$ Hausdorff implies that id is a homeomorphism. Therefore $\mathcal{T}' = \mathcal{T}$. This is a contradiction ahd so $\mathcal{T}$ cannot be compact. $\square$

(b) Let $\mathcal{T}'$ be strictly weaker than $\mathcal{T}$ on X. Show that $(X, \mathcal{T}')$ is not Hausdorff.

Proof. Suppose that $\mathcal{T}' \subset \mathcal{T}$ and $\mathcal{T}' \neq \mathcal{T}$ and $\mathcal{T}'$ Hausdorff. Let $id : (X, \mathcal{T}) \to (X, \mathcal{T}')$ so that $id(x) = x$. Then if $V \in \mathcal{T}'$ $id^{-1}(V) = V \in \mathcal{T}$ by our assumption that $\mathcal{T}'$ is a weaker topology. Therefore id is a continuous bijection. Then by Proposition 4.28, $(X, \mathcal{T})$ compact and $(X, \mathcal{T}')$ Hausdorff implies that id is a homeomorphism. Therefore $\mathcal{T}' = \mathcal{T}$. This is a contradiction ahd so $\mathcal{T}$ cannot be Hausdorff. $\square$

(13.6) Let X be a topological space. Show that following are equivalent: (i) X is normal; (ii) X satisfies the conclusion of Urysohn's Lemma; (iii) X satisfies the conclusion of the Tietze Extension Theorem.

Proof. Trivially[3] (i) implies (ii) and (iii). We will now show that (ii) implies (i) and (iii) implies (ii).

First suppose that X satisfies the conclusions of Urysohn's Lemma, that is for A, B closed disjoint in X then there is a continuous function $f : X \to [0, 1]$ so that $f(A) = \{1\}$ and $f(B) = \{0\}$. Now when $0 < \epsilon < 1$ consider the open set $V_\epsilon = (\epsilon, 1]$ then $A \subset f^{-1}(V_\epsilon)$. Additionaly consider the open set $U_\epsilon = (\epsilon, 1]$ then $B \subset f^{-1}(U_\epsilon)$. We claim that $f^{-1}(U_\epsilon) \cap f^{-1}(V_\epsilon) = \emptyset$. By the set operation homomorphism property of the preimage, we have that

$$\emptyset = f^{-1}(\emptyset) = f^{-1}(U_\epsilon \cap V_\epsilon) = f^{-1}(U_\epsilon) \cap f^{-1}(V_\epsilon).$$

Therefore A, B are contained in open sets which are disjoint. Since this holds for every A, B, the space X is normal.

Now suppose that X satisfies the conclusion of the Tietze Extension Theorem. Then take any two closed disjoint sets A, B in X. Let $f : A \cup B \to [0, 1]$ so that $f(A) = \{0\}$ and $f(B) = \{1\}$. Since A, B, $A \cup B$ is closed. Now for every $x \in A \cup B$, $x \in A$ or $x \in B$ but not both. Assume that $x \in A$. Then $f(x) = 0$ and for every open neighborhood V of $f(x)$ in $[0, 1]$ we claim that $f^{-1}(V)$ is open. since $f(A \cup B) = \{0, 1\}$ then V must contain 0 and need not contain 1. In the case that $0, 1 \in V$ we have that $f^{-1}(V) = A \cup B$ which is the whole space $A \cup B$ and therefore is open in the relative subspace topology. Otherwise $f^{-1}(V) = A$ which is closed in the relative subspace topology since $A \cap A \cup B = A$ and A is closed globally. Since B is closed in the global topology and $B \cap A \cup B = B$, then B is closed in the subspace topology. By the disjointness of A and B, $A \cup B \setminus B = A$ and so A is open in the subspace topology. Therefore $f^{-1}(V) = A$ which is an open neighborhood of $x \in A$. The same argument can be made by replacing symbols to show that f is continuous for $x \in B$.

Then the Tietze extension theorem gives a continuous extension $F : X \to [0, 1]$ so that $F|_{A \cup B} = f$; that is $f|_A = 0$ and $f|_B = 1$, therefore the conclusion of Urysohn's Lemma is satisfied.

We have thus shown that $(ii) \implies (i) \implies (iii) \implies (i)$ and thus all of the conditions are equivalent. $\square$

(13.7) Show that every sequentially compact topological space is countably compact.

Proof. Suppose that X is sequentially compact. Then given any sequence $(x_n)_n$ then there is a subsequence which converges to some $x \in X$; namely $x_{n_k} \to x$. If $\mathcal{V}$ is a countable open cover of X, then we would like to show that there is a finite subcover. Proposition 4.21 implies that X is countable compact if and only if for every countable family $\{F_n\}$ of closed sets with the finite intersection property ($\bigcap_{n \in B \subset \mathbb{N}} F_n \neq \emptyset$), it follows that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

To show compactness we will show the dual property with closed sets. Let $\{F_n\}$ be given with the above property and then let choose x_n so that $x_n \in \bigcap_{m=1}^{n} F_m$. Such an x_n exists for every n by

[3] X normal is a hypothesis of both propositions.

the finite intersection property, and there is a subsequence x_{n_k} which converges to x in X. Suppose $x \notin \bigcap_{n=1}^{\infty} F_n$, then there is an m so that $x \notin F_m$. Therefore by F_m closed $x \notin acc(F_m)$. By definition for all $j > m$, $x_{n_j} \in \bigcap_{p=1}^{x_{n_j}} F_p \subset F_m$. Therefore the sequence $(x_{n_k})_{k=m}^{\infty}$ is entirely contained in F_m. Now since $x \notin acc(F_m)$, it follows that $x \notin acc\left(\{x_{n_k}\}_{k=m}^{\infty}\right)$ and so there is a neighborhood V of x so that $\{x_{n_k}\}_{k=m}^{\infty} \cap (V \setminus \{x\}) = \emptyset$, and therefore exists a neighborhood so that for all $k \geq m$, and for all $j > k$ $x_{n_j} \notin V$ and so $x_{n_k} \not\to x$, (unless of course $x_{n_k} = x$ becomes constant at somepoint in the sequence, in which case $x \in F_m$, but this is not possible.) This is a contradiction, and so $x \in \bigcap_{m=1}^{\infty} F_m$ and thus X is countably compact. □

(13.8) Show that if X is countably compact, then every sequence in X has a cluster point and additionally if X is also first countable then X is sequentially compact.

Proof. Let (x_n) be some sequence in X. Let $S_m = \{x_n\}_{n=m}^{\infty}$ be the value set of the sequence starting at m. Finally let $G = \bigcap_{n=1}^{\infty} acc(S_n)$. When $n \leq m$, $S_m \subset S_n$ and so $acc(S_m) \subset acc(S_n)$. Therefore take any finite $B \subset \mathbb{N}$. There is a maximal element say $k = \max B$, and thus $acc(S_j) \supset acc(S_k)$ so the intersection of accumulation points of the partial sequence value sets is non-empty; that is the family $(acc(S_n))_{n \in \mathbb{N}}$ has the finite intersection property and every member is closed in X. Hence by the countable compactness of X, G is non-empty. We claim that if $x \in G$ then x is a cluster point of the sequence.

Taking $x \in G$ then forall m and for all neighborhoods V of x then $V \cap S_m$ is non-tivial. Hence forall neighborhoods and forall m, there exists an $n > m$ so that $x_n \in S_m$ and $x_n \in V$. Therefore (x_n) is in V infinitely often and so x is a cluster point of the sequence.

Supposing additionally that X is a first countable space, then in Homework 11, exercize 11.4 (or equivalently Folland 4.7) we showed that if x is a cluster point of a sequence in a first countable sequence, then there is a subsequence which converges to x. Adopting the results of this exercise to the previous proof, $x \in G$ is a cluster point of (x_n) and so there is a subsequence which converges to it. Therefore for any sequence in X there is a convergent subsequence (the existence of a cluster point guarenteed), and so X is sequentially compact.

This completes the proof. □

(13.9) If X is normal, then X is countably compact iff $C(X) = BC(X)$.

Proof. Suppose that X is countably compact. Now if $f \in C(X)$, then $f : X \to \mathbb{C}$ has the property that $f(X)$ is countably compact. Since $\mathbb{C}$ is second countable then $f(X)$ is sequentially compact. Additionally since $\mathbb{C}$ is a metric space, from the theory of metric spaces we know that sequentiall compactness implies compactness which implies total boundedness and closedness (Heine Borel in $\mathbb{R}^2 \simeq \mathbb{C}$)[4] Therefore $f(X)$ is bounded and so f is bounded; that is $C(X) \subset B(X)$. Then $BC(X) = B(X) \cap C(X) = C(X)$.

Now suppose that X is not countably countably compact, then by 13.7, X is not sequentially compact, and so there is a sequence with no convergent subsequence. Therefore there is a sequence without a cluster point. Take such a sequence, say (x_n). Then we claim that the value set $S = \{x_n\}_{n=1}^{\infty}$ is closed. If $x \in acc(S)$ we wish to show that $x \in S$.

Suppose for the sake of contradiciton that $x \notin S$, then because the sequence has no cluster points, there is a neighborhood of x say W so that there is an N so that $x_n \notin W$ when $n > N$. Then if $S_n = S \setminus \{x_n\}_{n=1}^{m}$ it follows that $S_n \cap W = \emptyset$ for all $n > N$. Since x is an accumulation point it

[4] Closedness comes from $\mathbb{C}$ metric spaces being Hausfordd.

must then be that $S \cap W \ni x_k \neq x$ when $k \leq N$. Now consider any neighborhood $V \subset W$ of x, then $\emptyset \neq V \cap S = V \cap S \setminus S_N$ and so $x \in acc(S \setminus S_N)$. However, x is T_4, and therefore $S \setminus S_N$ is closed because it is a finite set of points. Therefore $acc(S \setminus S_N) \subset S \setminus S_N \subset S$, which contradicts $x \notin S$. Therefore $acc(S) \subset S$ and so S is closed.

Now let $f : S \to \mathbb{C}$ so that $f(x) = \max\{n \in \mathbb{N} \ : \ x_n = x\} + 0i$. Such a maximum is defined because x_n cannot cluster at any $x \in S$; that is, $\{n \in \mathbb{N} \ : \ x_n = x\}$ must be finite since (x_n) has no cluster points and cannot visit x infiniteley many times. We claim the function is not bounded. Suppose that there were an N so that for all $x \in S$ $|\max\{n \in \mathbb{N} \ : \ x_n = x\}| \leq N$. Then if $m > N$, $x_m \in S$ since N is finite and additionally $N \geq |f(x_m)| = \max\{n \in \mathbb{N} \ : \ x_n = x\} = m > N$ which is a contradiction so, f cannot be bounded.

To show f is continuous on S with respect to its subspace topology take any closed subset K of $\mathbb{C}$, then it suffices to show that $f^{-1}(K)$ is a closed subset of X. Since $S \sim \mathbb{N}$ there are two cases: If $f^{-1}(K)$ is finite then it is closed by X a T_4 topological space. If it is not then it is countable. For each $y \in f^{-1}(K)$ clearly the integer $f(y) = \max\{n : x_n = y\}$ is unique as the natural numbers have a linear order and if $x_f(y) = y$ and $y \neq z$ then $x_f(y) \neq z$. If y_k is an enumeration of $f^{-1}(K)$ then let $x_{f(y_k)}$ be a subsequence of x_n. As $f(y_k)$ does not repeat integers by the aforementioned uniqeuness, if $k \neq j$ then $y_k \neq y_j$ and so $f(y_k) \neq f(y_j)$ and so $x_{f(y_k)} \neq x_{f(y_j)}$ and so finally $(x_{f(y_k)})_{k=1}^{\infty}$ does not repeat elements infinitely many times. Because (x_n) does not have any cluster points the non-repeatability of $x_{f(y_k)}$ implies that $(x_{f(y_k)})$ has no cluster points. Observing that the value set $S(K) = \{x_{f(y_k)}\}_{k=1}^{\infty}$ is exactly $f^{-1}(K)$, we use the logic showing that (x_n) was closed to conclude that $S(K)$ is closed in X; therefore $S(K)$ is closed in S. Hence f is continuous.

Now by Corrolary 4.17, it follows that f can be extended to some $F \in C(X)$ so that $F|_S = f$. Therefore F is an unbounded continuous function on X and so $BC(X) \neq C(X)$. $\square$