This map is clearly smooth as $\binom{0}{1}$ is a constant vector.

2. Let $U = (-1,1) \times [(\frac{1}{2}, 1] \cup [-1, \frac{1}{2})] / \sim$ and φ ~~bee~~ a local coordinate chart s.t.

$$\varphi : [(x,y)] \longmapsto \begin{cases} (x, y+1) & \text{if } y \in [-1, -\frac{1}{2}) \\ (-x, y-1) & \text{if } y \in (\frac{1}{2}, 1] \end{cases}$$

We know φ is a diffeomorphism from Hw3. We now consider the coordinate charts on $F(U)$, say ψ so that

$\psi : F(U) \to \mathbb{R}^2 \times \mathbb{R}^2$ w/

$$\psi = (\varphi_1 \circ \pi, \varphi_2 \circ \pi, d\varphi_1, d\varphi_2).$$

Then $\psi \circ F \circ \varphi^{-1} = \psi \circ F((x,y))$

$$= \psi \circ (x, y, 0, 1)$$

$$= \psi(\varphi(x), \varphi(y), 0, d\varphi_2 \tbinom{0}{1}))$$

which is clearly a smooth map of (x,y) and using smoothness of φ we get that F is a smooth vector field $\dot{U}$.

Problem 2 Show that the geom Möbiusband $\mathcal{M}$ admits a non-vanishing vector field but is not parallelizable.

Prf Recall that a vector field on $\mathcal{M}$ is a (smooth) map $F : \mathcal{M} \to T\mathcal{M}$ st. $\pi \circ F = id_{\mathcal{M}}$.

We say that F is not vanishing if $\forall x \in \mathcal{M}$, $\cancel{F(x) \neq 0(x,0)}$
$F(x) \neq (x, \vec{0})$.

Define the following vector field.
$F : \mathcal{M} \to T\mathcal{M}$ st $[x] \mapsto (x, e_2)$ where e_2 is the 2^{nd} elementary vector in $\mathbb{R}^2$.
F is trivially a vector field and it vanishes nowhere. To show that F is smooth, we'll consider two charts on $\mathcal{M}$:

1. Let $U = [(-1,1)\times(-1,1)/\sim]$, then $U \overset{\varphi}{\cong} \mathbb{R}^2$ via the identity map. Then take the charts $\psi = (x_1 \circ \pi, x_2 \circ \pi, dx_1, dx_2)$ on $F(U)$ and given that

$$\psi \circ F \circ \varphi^{-1} = \psi \circ (x_1^{-1}, x_2^{-1}, 0, 1)$$
$$= (x_1, x_2, 0, dx_2\begin{pmatrix}0\\1\end{pmatrix})$$