Math 215A — UCB, Spring 2017 — William Guss

Partners: Alekos, Chris

Selected Problems: 1

(11.1) *(Chain Homotopy)* Find degree-wise free chain complexes A, B and a chain map $f : A \to B$, not chain homotopic to the zero map, but such that the induced homomorphism $f_* : H_*(A) \to H_*(B)$ is zero.

Solution. We will first describe a suitable general structure for candidate chain complexes and maps, and then provide an example. Let G be any free abelian group (or R-module), then let A, B be chain complexes diagramatically follows

$$\begin{tikzcd} A : 0 \arrow[r] \arrow[d] & G \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"] & G \arrow[r] \arrow[d] & 0 \arrow[d] \\ B : 0 \arrow[r] & G \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

where the chain map f is defined by the downward arrows.

Lemma 0.1. *If $Im(\partial) \subsetneq G$, and for any homomorphism $\gamma : G \to G$, we have that $\gamma \circ \partial = \partial \circ \gamma$ then $f \not\simeq 0$.*

Proof. Suppose that f were chain-nullhomotopic. Then there exists γ and ψ so that $\mathrm{id} - 0 = 0 \circ \psi + \gamma \circ \partial$. By our hypothesis, $\gamma \circ \partial = \partial \circ \gamma$ and therefore $\mathrm{id} = \partial \circ \gamma$. But then this contradicts $Im(\partial \circ \gamma) \subset Im(\partial) \subsetneq G = Im(\mathrm{id})$. Therefore there cannot exist such γ and $f \not\simeq 0$. $\square$

Lemma 0.2. *The induced homomorphism of homologies, f_* is the zero map when $Ker(\partial) = 0$.*

Proof. Application of the homology functor yields automatically the following diagram

$$\begin{tikzcd} A : 0 \arrow[r] \arrow[d] & H_1(A) \arrow[r, "H_*(\partial)"] \arrow[d, "H_*(\mathrm{id})"] & H_0(A) \arrow[r] \arrow[d] & 0 \arrow[d] \\ B : 0 \arrow[r] & H_1(B) \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Then $H_1(B) = G$ since the kernel of thre constant boundary map from G into 0 is G and the image of the constant inclusion map from 0 into G is $\{0\}$. Furthermore, since $Ker(\partial) = 0$ we yield that $H_1(A) = 0$. Therefore f_* is the zero map at every degree. $\square$

With this in mind, finding chain complexes satisfying the statement of the problem is reduced to finding a group G and a homomorphism ∂ with the properties of both lemmas. Take $G = \mathbb{Z}$ and ∂ to be any (non-trivial) multiplicitive operator.