Math 113 — Problem Set 3— William Guss

(P19.24) *Compute* $20.5 +_{25} 19.3$.
Clearly $20.5 + 19.3 = 39.8$ which is equivalent to $39.8 - 25 = 14.8 \mod 25$.

(P19.25) *Compute* $\frac{1}{2} +_1 \frac{7}{8}$.
Clearly $1/2 + 7/8 = 4/8 + 7/8 = 11/8$ which is equivalent to $11/8 - 1 = 3/8 \mod 1$.

(P27.23) *Let* $H \subset M_2(\mathbb{R})$ *consisting all matrices of the form* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *for* $a, b \in \mathbb{R}$. *Is* H *closed under*

a) *matrix addition?* Yes.
b)*matrix multiplication?* Yes.

Lemma 1. *If* H *is defined as above, then* $H \cong \mathbb{C}$.

Proof. Define the mapping $\phi : H \to \mathbb{C}$ such that $\phi(M) = a + ib$. It is obvious that this mapping is a bijection since $a, b \in \mathbb{R}$ implies that for any $x + iy \in \mathbb{C}$ there exist $a, b \in \mathbb{R}$ so that $a + ib = x + iy$.

Next we show that the mapping is a homomorphism under addition and multiplication. Take $Z, W \in H$. Then

$$\begin{aligned}
\phi(ZW) &= \phi\left(\begin{bmatrix} z_1w_1 - z_2w_2 & -(z_2w_1 + z_1w_2) \\ z_2w_1 + z_1w_2 & z_1w_1 - z_2w_2 \end{bmatrix}\right) \\
&= z_1w_1 - z_2w_2 + i(z_2w_1 + z_1w_2) \\
&= (z_1 + iz_2)(w_1 + iw_2) \\
&= \phi(Z)\phi(W) \in \mathbb{C}
\end{aligned}$$

Furthermore we consider the addition operation

$$\begin{aligned}
\phi(Z + W) &= \phi\left(\begin{bmatrix} z_1 + w_1 & -(z_2 + w_2) \\ z_2 + w_2 & z_1 + w_1 \end{bmatrix}\right) \\
&= z_1 + w_1 + i(z_2 + w_2) \\
&= (z_1 + iz_2) + (w_1 + iw_2) \\
&= \phi(Z) + \phi(W) \in \mathbb{C}
\end{aligned}$$

Therefore H is isomorphic to $\mathbb{C}$ under addition and multiplication.

□

Note I implicitly showed that H *was closed in the proof by explicitly calculating* $Z + W, ZW$ *and grouping the terms in the upper right hand corner of the the result. I also take for granted that* $\mathbb{C}$ *is closed under multiplication and addition.*

Corollary 1. *The set* H *is closed under multiplication and addition.*

Proof. If H were not closed under these operations then H would not be isomorphic to $\mathbb{C}$. □

(P27.24)

a) False
b) True
c) False

d) What? lol. **False**
e) True
f) True.

Proof. If $S = a, f : S \times S \to S$ then f must be the mapping $(a, a) \mapsto a$. This is obvious. Then $f(a, a) = f(a, a)$ gives commutativity, and $f(a, f(a, a)) = f(a, a) = f(f(a, a), a)$ gives associativity. $\square$

g) True
h) True
j) False