Math 185 — UCB, Fall 2016 — William Guss
Problem Set 6, due October 25th

(57.1) .

Note: In the following problem we use that the contour C is homotopic to the circle surrounding the singularity in the analytic domain of each integrand.

(a) $\int_C \frac{e^{-z}}{z-(\pi i/2)};$.

Solution. Using the cauchy integral formula we get that

$$\int_C \frac{e^{-z}}{z-(\pi i/2)} = e^{-(\pi i/2)} 2\pi i = -i 2\pi i = 2\pi.$$

□

(b) $\int_C \frac{\cos(z)/(z^2+8)}{z-0} dz;$

Solution. Using the cauchy integral formula we get that

$$\int_C \frac{\cos(z)/(z^2+8)}{z-0} dz = 2\pi i \cos(0)/8 = \pi i/4$$

□

(c) $\int_C \frac{z/2}{z-(-1/2)} dz;$

Solution. Using the cauchy integral formula we get that

$$\int_C \frac{z/2}{z-(-1/2)} dz = 2\pi i - 1/4 = -\pi i/2$$

□

(d) $\int_C \frac{\cosh z}{(z-0)^{3+1}} dz;$

Solution. Using the cauchy integral formula for the derivative we get that

$$\int_C \frac{\cosh z}{(z-0)^{3+1}} dz = 2\pi i/6 \times \cosh^{(3)}(0) = 2\pi i/6 \times (-\sinh(0)) = 0$$

□

(e) $\int_C \frac{\tan z/2}{(z-x_0)^{1+1}} dz;$

Solution. Using the cauchy integral formula for the derivative we get that

$$\int_C \frac{\tan z/2}{(z-x_0)^{1+1}} dz = 2\pi i \times \tan^{(1)}(x_0/2) = \pi i \sec^2(x_0/2).$$

□

(57.2) .
(a) $g(z) = \frac{1}{z^2+4}$

Solution. First $g(z) = \frac{1/(z+2i)}{(z-2i)}$ and so we can apply the cauchy integral formula and yield

$$\int_C g(z)\ dz = \frac{2\pi i}{4i} = \frac{\pi}{2}$$

$\square$

(b) $g'(z) = \frac{1}{(z^2+4)^2}$.

Solution. First $g'(z) = g(z) \times g(z)$ so $g'(z) = \frac{1/(z+2i)^2}{(z-2i)^2}$ and thus

$$\int_C g'(z)\ dz = \frac{d}{dz}\frac{2\pi i}{(z+2i)^2}\Big|_{z=2i} = \frac{-4\pi i}{(4i)^3} = \pi/16.$$

$\square$

(57.3) Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\ ds \quad (|z| \neq 3)$$

then $g(2) = 8\pi i$.

Proof. If C is the boundary of the ball B then $2 \in int(B)$ implies that the Cauchy integral formula is applicable in the evaluation of $g(2)$. Therefore

$$g(z) = 2\pi(2z^2 - z - 2) = 2\pi i(8 - 2 - 2) = 8\pi i.$$

Additionally if $|z| \geq 3$ then the singularity of the integrand is no longer within the contour and so the integral has value 0 by the anylicity of the elementary functions of which the integrand is composed (Cauchy-Goursat). $\square$

(57.4) Let C be any simple closed contour, described in the postive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\ ds.$$

It follows that $g(z) = 6\pi iz$ when z is inside C and that $g(z) = 0$ when z is outside.

Proof. This exercise is in the same flavor as the previous. Let C be the boundary of some manifold B; that is $C = \partial B$. When z is inside of C, $z \in int(B)$. Therefore C is homotopic to a manifold S^1 containing z and the Cauchy-Integral formula applies; that is,

$$\int_C \frac{s^3 + 2s}{(s - z)^3}\ ds = \frac{2\pi i}{2!}(6z) = 6z\pi i.$$

In the case that z is outside of C then, $z \notin B$ and so ∂B is homotopic to a manifold S^1 not containing any singularities as the analyic integrand does not contain singularities on the interior of B. By the Cauchy-Goursat theorem we have that

$$\int_C \frac{s^3 + 2s}{(s - z)^3}\ ds = 0$$

when z is outside of C. $\square$

(57.5) Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)}{z-z_0}\, dz = \int_C \frac{f(z)}{(z-z_0)^2}\, dz.$$

Proof. If f is analytic on and within the simple contour, we have that f' exists and is analytic on the simple closed contour by Theorem 1 of Sec 57. Again since C is a simple closed contour C is the boundary of a manifold B, such that f and f' are analytic on B. This is the interior manifold as before.

If $z_0 \in B$ then $z_0 \in int(B)$ and $z_0 \notin \partial B = C$. Therefore we can homotop C to some S^1 inside of $\int B$ such that z_0 is inside of S^1 and so the Cauchy-Integral theorem applies and we yield

$$\int_C \frac{f'(z)}{z-z_0}\, dz = 2\pi i f'(z_0) = 2\pi i/1! f'(z_0) = \int_C \frac{f(z)}{(z-z_0)^2}\, dz.$$

If $z_0 \notin B$ then the integrand in both equations is analytic on B and B is fully connected and contains no holes. Thus the Cauchy Goursat theorem gives

$$\int_C \frac{f'(z)}{z-z_0}\, dz = 0 = \int_C \frac{f(z)}{(z-z_0)^2}\, dz.$$

Thus the two contour integrals are equal. □

(57.7) Let C be the unit circle $z = e^{i\theta}$ with $\theta \in [-\pi, \pi]$. For any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Additionally

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta) d\theta = \pi.$$

Proof. By the analyicity of $\exp(a\cdot)$ we can apply directly the Cauchy integral formula and get

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i e^{0\times a} = 2\pi i.$$

Next expanding the 1-form we get that

$$
\begin{aligned}
2\pi i = \int_C \frac{e^{az}}{z}\ dz &= i\int_0^{\pi} \frac{e^{a(\cos(\theta)+i\sin(\theta))}}{e^{i\theta}} e^{i\theta}\ d\theta - i\int_0^{-\pi} \frac{e^{a(\cos(\theta)+i\sin(\theta))}}{e^{i\theta}} e^{i\theta}\ d\theta \\
&= i\left(\int_0^{\pi} e^{a(\cos(\theta)+i\sin(\theta))}\ d\theta - \int_0^{-\pi} e^{a(\cos(\theta)+i\sin(\theta))}\ d\theta\right) \\
&= i\int_0^{\pi} e^{a(\cos(\theta)+i\sin(\theta))}\ d\theta + \dots \\
&= i\int_0^{\pi} e^{a(\cos(\theta))}e^{ai\sin(\theta)}\ d\theta + \dots \\
&= i\int_0^{\pi} e^{a(\cos(\theta))}\cos(a\sin(\theta)) + i\sin(a\sin(\theta))d\theta + \dots \\
&= i\int_0^{\pi} e^{a(\cos(\theta))}\cos(a\sin(\theta)) + i\sin(a\sin(\theta))d\theta \\
&+ i\int_0^{\pi} e^{a(\cos(\theta))}\cos(a\sin(\theta)) - i\sin(a\sin(\theta))d\theta \\
&= 2i\int_0^{\pi} e^{a(\cos(\theta))}\cos(a\sin(\theta))\ d\theta
\end{aligned}
$$

The last two equations were given by sin an odd function and cos and even function (on the reparameterization of the integrals the cos keeps its sign and sin switches). Therefore, dividing by $2i$ on each side we get

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi.$$

□

(59.1) Suppose $f(z)$ is entire and the harmonic function $u(x,y) = Re[f(z)]$ has an upperbound u_0; that is $u(x,y) \leq u_0$ for all points in (z,y) in the xy plane. Show that $u(x,y)$ must be constant throughout the plane.

Proof. If $f(z)$ is entire and its real part $u(x,y)$ has an upperbound, then the function $g(z) = e^{f(z)} = e^{u(x,y)}e^{iIm(f(z))} \leq e^{u_0}e^{iIm(f(z))}$ and since $|e^{iq}| \leq 1$ for all $q \in \mathbb{R}$ it follow that $|g(z)| \leq e^{u_0}$. By the entirety of g and Louiville's theorem, it follows that $g(z)$ is constant on its entire domain. Lastly every g in the family of functions $log(g) = log(e^{z_0}) \equiv z_0 + i2\pi n$, $n \in \mathbb{N}$ and z_0 constant is therefore constant. Lastly $f \in log(g)$ since $g = e^f$, thus f is constant throughout its domain, ie. the xy plane. □

(59.2) Let a function f be continuous on a closed bounded region R and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, then $|f(z)|$ has a minimum value $m \in R$ which occurs on the boundary of R and never in the interior.

Proof. We examine the function $g(z) = 1/f(z)$ which is analytic in $int(R)$ since $f(z) \neq 0$ everywhere in R and f analytic in $int(R)$. Since g is not constant by the Maximum modulus principle, $|g|$ has no maximum value in $int(R)$. By $|g|$ continuous on R R compact we have from undergraduate real analysis that (105) that $|g|$ attains a maximum on R, however this maximum cannot be in $int(R)$ so it must be in ∂R since compactness gives $R = \int(R) \sqcup \partial R$. Lastly if $|g|$ is a maximum then $1/|g|$ is a minimum, thus $|f|$ attains a minimum on ∂R. This completes the proof. $\square$

(59.4) Let R be the region depicted in the book. that taht the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = (\pi/2) + i$.

Proof. By the previous theorem $|f|$ can only attain a maximum on ∂R by the compactness of R (see the book, it is a product of closed intervals and in the product topology R must be closed.) Now we claim that the maximum point is $z = (\pi/2) + i$.

Recall that $|f(z)|^2 = \sin^2(x) + \sinh(y)$ by section 37. Then in ∂R we must find x such that $\sin^2(x)$ is maximized. Since $x \in [0, \pi]$, we know that sin achieves a maximum at $\pi/2$ from Math 1A. Next since $\sinh^2$ is montone increasing the largest possible value of $Im(z), z \in \partial z$ achieves the maximum. Thus $y = 1$ and so $|f(z)|^2$ is maximal at $z = (\pi/2) + i$. Therefore $|f(z)|$ is maximal there too. $\square$

(59.7) Let the function $f = u + iv$ be continous on a closed bounded region R suppose that it is analytic and not constant in the interior R. Show that $u(x, y)$ has maximum and minimum values in R which are reached on the boundary of R and never the interior, where it is harmonic.

Proof. First $|f|$ and thus $|f|^2$ has no extremal values for $z \in int(R)$. Thus $|f|^2 = |u|^2 + |v|^2$ is maximal for $(x, y) \in \partial R$ since R is compact and f analytic implies $|f|$ and therefore $|f|^2$ continuous. Additionally Exercise 5 implies that u attains a minimum in ∂R and by the compactness of ∂R and the continuity of u in ∂R, u also attains a maximum. Letting $g = -if$ and applying the prvious logic to $u' = v$ we get that v attains a minimum and a maximum in ∂R.

Thus the theorem holds. $\square$