MATH 105: Homework 7

William Guss
26793499
wguss@berkeley.edu

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16. *Write out the proofs of Lemma 23,24,25 in n-dimensions.*

Lemma 1. *If $A, B \subset \mathbb{R}^k$ are boxes then $A \times B$ is measurable and $m(A \times B) - mA \cdot mB$*

Proof. $A \times B$ is a higher dimensional box and the product formula follows from Corollary 15. ☐

Lemma 2. *If A or B is a zero set thern $A \times B$ is measurable and $m(A \times B) = mA \cdot mB = 0$.*

Proof. Without loss of generality let $mA = 0$. For every $\epsilon > 0$, there exists a countable covering of A by open boxes whose volume is ϵ. Crossing those boxes by $(0, 1)$ gives the outer measure $m^*(V_i) = \epsilon$. Then since $\mathbb{R}$ is the countable union of open intervals, take $A_1 = A \times \mathbb{R}$ to be a zero set. Then induct using the above logic recalling that we did not use the dimensionality of V_i. Eventually $0mA_n = m(A \times \mathbb{R}^n) > m(A \times B) = 0$ by $B \subset \mathbb{R}^n$ ☐

Lemma 3. *Every open set in n-space is a countable union of disjoint cubes plus a zeroset.*

Proof. Accept all dyadic cubes that lie in U and reject the rest. n-sect every rejected cube into 2^n subcubes. Accept the interiors of these subcubes which lie in U and reject the rest. Proceed to do this to every single instance of a rejected square infiniteley many times via geometric induction. Eventually every single $x \in U$ will be covered by a cube in this $n - section$ class. ☐

Lemma 4. *If U and V are open then $U \times V$ is measurable and $m(U \times V) = mU \cdot mV$.*

Proof. Since $U \times V$ is open it is measurable. Lemma 24 implies that U is the disjoint union of a bunch of disjoint cubes and a zeroset and V is also the disjoint union of a bunch of cubes and a zeroset. Let J_j, I_i be these two cube sets. Then

$$U \times V = \sqcup_{i,j} I \times J \cup Z \tag{1}$$

where $Z = (Z_U \times V) \cup (U \times Z_V)$ is a zeroset by Lemma 23. Since

$$\left(\sum_i m(I_i)\right)\left(\sum_j m(J_j)\right) = \sum_{i,j} m(I_i)m(J_j) = \sum_{i,j} m(I_i \times J_j) \tag{2}$$

we conclude that $m(U \times V) = mU \cdot mV$. □

17. *Write out the proofs of the measurable product theorem and the zero slice theorem in n dimensional case unbounded.*

Theorem 1. *Measurable Product Theorem.*

Proof. Consider A or B unbounded, then $m^*(A) = \infty$ and it could not possibly be that $m^*(A \times B) \neq \infty$ unless B were a zeroset.

Without loss of generality assume that the sets are subsets of the unit interval. We claim that the hull of a product is the inner product of the hulls and the kernel of a product is the product of the kernels. Since hulls are G_δ sets their product is a G_δ set and is therefore measurable. Similarly the product of kernels is measurable. Clearly,

$$K_A \times K_B \subset A \times B \subset H_A \times H_B \tag{3}$$

and $(H_A \times H_B) \setminus (K_A \times K_B) = (H_A \setminus H_B) \times (H_A \setminus H_B)$. Measurability of A and B implies that $m(H_A \setminus H_B) = m(H_B \setminus K_B) = 0$, so Lemma 23 gives us

$$m(K_A \times K_B) = m(H_A \times H_B). \tag{4}$$

Let U_n and V_n be sequences of open cubes in the unit cube converging down to H_A and H_B. Then $U_n \times V_n$ is a sequence of open sets in I^2 converging down to $H_A \times H_B$. Downward measure continuity implies $m(U_n \times V_n) \to m(H_A \times H_B)$. Lemma 25 imples that $m(U_n \times V_n) = m(U_n)m(V_n)$. Since $m(U_n) \to A$ and the same for V_n to mB we have that $m(A \times B) = mAmB$. □

Theorem 2. *If $E \subset \mathbb{R}^n \times \mathbb{R}^k$ is measurable then E is a zero set if and only if almost every slice of E is a zero set.*

Proof. Without loss of generality assume that E is contained within the unit cube. Suppose that E is measurable and that $m(E)$ is zero.

Let $Z = \{x : E_x notazeroset\}$. Z is a zeroset. The slices E_x for which E_X is not zeroset are contained in $Z \times \mathbb{R}$ which as proved above is a zero set in $\mathbb{R}^n$. Then $E \setminus (Z \times \mathbb{R}^m)$ is measurable and has the same measure as E, and so it is no loss of generality to assume that every slice E_x is a zeroset.

It is sufficient to show that the inner measure of E is zero. Let K be any compact subset of E and let $\epsilon > 0$ be given. The slice K_x is comapct and it has slice measure 0. Therefore it has an open neighboorhood $V(x)$ so that $m(V(x)) < \epsilon$. Compactness of K implies that for all x' near x we have $y \notin K_x$. Closedness of K implies that $(x, y) \in K$ so $y \in K_x$ a contradiction. Hence if $U(x)$ is small then for all $x' \in U(x)$ we have $x' \times K_{x'} \subset W(x) = U(x) \times V(X)$. It makes sense!

We can choose these small open sets $U(x)$ from a countable base of the topology of $\mathbb{R}^n$, for instance the open cubes with rational vertices. This gives a countable covering of K by thin product set $W_i = U_i \times V_i$ such that $m(V_i) < \epsilon$ for every single i. We disjointify the covering by setting

$$U_i' = U_i \setminus (U_1 \cup \cdots \cup U_{i-1}). \tag{5}$$

The sets U_i' are measurable, disjoint, and since E is contained in the unit $m+1$ cube they all line in the unit $k$cube. Hence their total n dimensional meausre is less than 1. The sets $W_i' = U_i' \times V_i$ are dsjoin, are measurable, and coverm K. Theorem 21 implies that $m(W_i') = m(U_i')m(V_i)$ so their total $m+1$ dimewsional meaure is $< \sum m(U_i') \cdot \epsilon \leq \epsilon$.

Converseley, suppose that E is a zero set. Regularity implies there is a G_δ set $G \subset E$ with $mG = 0$ and it suffices to show that almost every slice of G is a zero set. The slices of a G_δ set are G_δ sets and in particular each slice G_x is measurable. Let $X(\alpha) = \{x : m(G_x > \alpha\}$. We claim that $m^*(X(\alpha)) = 0$. Each G_x contains a cpokmpact set $K(x)$ with $m(K(x)) = m(G_x)$.

Let U be any open subset of I^n that contains G. If $x \in X(\alpha)$ then $x \times K(x)$ is a compac subset of U and there is a product neighboorhood $W(x) = U(x) \times V(x)$ of $x \times K(x)$ with $W(x) \subset U$. Since $K(x) \subset V(x)$ we have that $m(V(x)) > \alpha$. Again we can assume neighboorhoods $U(x)$ belong to some countable base for the topology of $\mathbb{R}^n$. This gives a countable family U_i which covers $X(\alpha)$. Ads above, set $Ui' = U_i \setminus (U_1 \cup dots \cup U_{i-1})$. Disjointness and theorem 21 imply that

$$\begin{aligned} mU \geq \sum m(U_i' \times V_i') = \sum m(U_i')m(V_i) \\ \geq \sum m(U_i')\alpha \geq \alpha m^*(X(\alpha)) \end{aligned} \tag{6}$$

Since $mG = 0$ there are open sets $U \supset G \supset E$ with arbitrarily small measure. Thus $X(\alpha)$ is a zero set and so is $\bigcup_{\ell \in \mathbb{N}} X(1/\ell)$. That is, $m(E_x) = 0$ for almost every x. $\square$