Math 202A— UCB, Fall 2016 — M. Christ
Problem Set 3, due Wednesday September 14[1]

Please read section 1.5 of our text.

(3.1) (Folland problem 1.17) Let μ^* be an outer measure on X and let A_j be pairwise disjoint μ^*–measurable sets. Show that for any $E \subset X$, $\mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$.

Proof. First observe that $E \cap \bigcup A_j = \bigcup (A_j \cap E)$, and the sets $A_j \cap E$ are clearly a pairwise disjoint cut of E. Let $B_j = A_j \cap E$. Then now consider the finite union of B_j. We first show finite addativity by induction. Clearly $\mu^*(B_j) = \mu^*(B_j)$ so the base case holdes. Now suppose that $\mu^*(\bigcup_{j=1}^{n-1} B_j) = \sum_{j=1}^{n-1} \mu^*(B_j)$. Then it follows that

$$\mu^*\left(\bigcup_{j=1}^{n} B_j\right) = \mu^*\left(\bigcup_{j=1}^{n} B_j \cap A_j\right) + \mu^*\left(\bigcup_{j=1}^{n} B_j \cap X \setminus A_j\right)$$

and by the disjointness of A_j we have

$$\mu^*\left(\bigcup_{j=1}^{n} B_j\right) = \mu^*(B_n) + \mu^*\left(\bigcup_{j=1}^{n} B_j\right) = \sum_{j=1}^{n} \mu^*(B_j)$$

for every n. This holds in the limit and $\mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$. ☐

(3.2) (Folland problem 1.18) Let μ_0 be a premeasure on an algebra $\mathcal{A} \subset \mathcal{P}(X)$. Let μ^* be the outer measure induced[2] by μ_0. Let $\mathcal{A}_\sigma$ be the collection of all countable unions of elements of $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ be the collection of all countable intersections of elements of $\mathcal{A}_\sigma$. Prove the following:
(a) For any $E \subset X$ and $\varepsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ such that $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

Proof. If $\mathcal{A}$ is an algebra ☐

(b) If $\mu^*(E) < \infty$ then E is μ^*–measurable if and only if there exists $B \in \mathcal{A}_{\sigma\delta}$ such that $\mu^*(B \setminus E) = 0$.
(c) If μ_0 is σ–finite then the hypothesis $\mu^*(E) < \infty$ in part (b) is superfluous. ☐

(3.3) (Folland problem 1.22(a)) Let $(X, \mathcal{M}, \mu)$ be a measure space, let μ^* be the outer measure induced by μ, let $\mathcal{M}^*$ be the σ–algebra of measurable sets, and let $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ be the associated measure. Assume that μ is σ–finite. Show that $\bar{\mu}$ is the completion[3] of μ. ☐

(3.4) (Folland problem 1.23) Let $\mathcal{A}$ be the collection of all finite unions of sets $(a, b] \cap \mathbb{Q}$ with $-\infty \leq a < b \leq \infty$. Prove the following:
(a) $\mathcal{A}$ is an algebra of subsets of $\mathbb{Q}$.
(b) The σ–algebra generated by $\mathcal{A}$ is equal to $\mathcal{P}(\mathbb{Q})$.
(c) The set function defined on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for all nonempty $A \in \mathcal{A}$ is a premeasure on $\mathcal{A}$. There exists more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is equal to μ_0. ☐

[2]This means that μ^* is defined in terms of μ_0 by (1.12) of our text.
[3]This means that $\bar{\mu}$ coincides with the measure constructed from μ in Theorem 1.9 of our text.

(3.5) Complete the proof of Theorem 1.19 of our text. With the notation of that theorem, show that: (a) $E \in \mathcal{M}_\mu$. (b) $E = A \setminus N_1$ where A is a G_δ set and N_1 is a μ–null set. (c) $E = B \cup N_2$ where B is an F_σ set and N_2 is a μ–null set. $\square$

(3.6) Prove Proposition 1.20 of our text: If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$ then for every $\varepsilon > 0$ there exist finitely many disjoint open intervals I_j such that $\mu(E \,\Delta\, \cup_j I_j) < \varepsilon$. $\square$

(3.7) (Folland problem 1.29) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Let N be the nonmeasurable set constructed in §1.1 of our text and in lecture (where it was denoted by $\mathcal{E}$). (a) Show that $m(E) = 0$ if $E \subset N$. (b) Show that if $E \in \mathcal{L}$ and $\mu(E) > 0$ then E contains a nonmeasurable set. (In part (b) it is not assumed that $E \subset N$.)

Proof. If $E \subset N$ is a measurable set then $\mu^*(E \cap N) + \mu^*(E^c \cap N) = \mu^*(N) = \mu(E) + \mu^*(N \setminus E)$. If $\mu^*(N) = 0$ the proof is complete, but this cannot be the case since $\mu^*(N)$ gives N measurabe; that is for any set $\mu^*(X \cap N) + \mu^*(X^c \cap N) \leq 2\mu^*(N) \leq 0$, therefore $\mu^*(N) > 0$. If $\mu(E) > 0$, then $\mu^*(N) - \mu^*(N \setminus E) = \mu^*(E)$ which gives N measurable. Contradiction. $\square$

(3.8) (Folland problem 1.30) Let $E \in \mathcal{L}$ and $0 < \alpha < 1$. Show that if $m(E) > 0$ then there exists an open interval I satisfying $m(E \cap I) > \alpha m(I)$. Since E is Labesgue measurable, there is an openset such that E is strictly contained in that openset say U. Furthermore, since $m(\mathbb{R}) = \infty$ then by a previous exercise/theorem we can find opensets which are larger than U (contain it) and have larger measure than U. Now take any $\alpha \in (0, 1)$ since $m(E)$ is the infimum of the measures of open U containing it, we can take U small enough that $m(E \cap U) = m(E)$ and $m(U) - m(E) < \epsilon$. In particular there are U such that $m(E)/m(U) = 1 - \epsilon$, so take $\epsilon \in (0, 1)$ and we have α so that $m(E)\alpha = m(E \cap U)\alpha = m(U)$.

(3.9) (Folland problem 1.33) Show that there exists a Borel set $A \subset [0, 1]$ such that for every subinterval $I \subset [0, 1]$ of positive length, $0 < m(A \cap I) < m(I)$.

Proof. Consider the following construction, using knowledge from Pugh. Denote $E_1 = [0, 1]$. Then let $E_2 = E_1 \setminus (1/3, 2/3)$. Then remove the middle open set of length $1/9$ from each connected segment of E_2. Then remove the middle open set of length $1/3^n$ from each segment of E_{n-1}. The limiting process is a countable intersection process so thhe set $E^{n \to \infty} = F_C$ is Borel measurable. Now take any interval $I \subset [0, 1]$ and intersect F_C. We claim that there are no intervals in F_C and so F_C mudst be strictly smaller than I. Suppose that there were an I in F_C. This means that for every n, I must lie in some segment of E_n. If it did not for every n then the symmetric difference of E_n and I would be an interval of finite length. Now observe that as $n \to \infty$ the difference endpoints of segments in $E_n \to 0$, so I must be a point, a contradiction to I an interval (in this problem we are sure that intervals are not points). Therefore $m(F_C \cap I) \leq m(I)$.

Now we must modify our construction so that $m(F_C \cap I) > 0$ for every I. For every n, construct an F_C^n such that every gap $I \setminus E_n$ which has not already been filled in a previous iteration is filled by a set F_c by the same construction as before except where the initial set $E_1^n = [c, g]$ where c, g are the end points of a gap in this construction. This looks like a fractal. Repeat this process forever, and since it is countable we still have a Boral set, A.

Again take any interval in $[a, b]$ we claim that the intersection has area $p(b - a)$. First recall that the F_C, fat cantor set has area $m([0, 1]) - 1/3 - 2/27 - 4/27 * 27 - \cdots - 2^n/3^{2n+2}$ which gives the set a postive proportion of $[0, 1]$ say p. Then for any F_C^n we can scale the original set to fit in that interval giving us (by the same argument) a set of area $p(c - g)$ Then any interval $[a, b] \subset [0, 1]$ intersects a number of these F_C^n whose area could never be $[a, b]$ since $0 < p < 1$.

This completes the proof $\square$