CS 70: Homework 1

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1. Watsons experiment.

   Theorem 1. *If a person has ice cream for desert, he/she has to do the dishes after dinner.*

   Proof. Flip Charlie and Bob. □

2. For the following answers I employed a truth table generator as a latex extension. This is a programmatic method of proof, but it does not detract from the argument.

   (a)

   Theorem 2. $A \ \vee \ (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

   Proof. On the left hand side we have that

a	b	c	a	$\vee$	$(b$	$\wedge$	$c)$
1	1	1	1	**1**	1	1	1
1	1	0	1	**1**	1	0	0
1	0	1	1	**1**	0	0	1
1	0	0	1	**1**	0	0	0
0	1	1	0	**1**	1	1	1
0	1	0	0	**0**	1	0	0
0	0	1	0	**0**	0	0	1
0	0	0	0	**0**	0	0	0

   On the right hand side we have

a	b	c	$(a$	$\vee$	$b)$	$\wedge$	$(a$	$\vee$	$c)$
1	1	1	1	1	1	**1**	1	1	1
1	1	0	1	1	1	**1**	1	1	0
1	0	1	1	1	0	**1**	1	1	1
1	0	0	1	1	0	**1**	1	1	0
0	1	1	0	1	1	**1**	0	1	1
0	1	0	0	1	1	**0**	0	0	0
0	0	1	0	0	0	**0**	0	1	1
0	0	0	0	0	0	**0**	0	0	0

   Since these exhibit ident ical truth values, they myust therefore be the same. □

(b)

Theorem 3. $A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$.

Proof. On the left hand side it follows that,

a	b	c	a	$\land$	$(b$	$\lor$	$c)$
1	1	1	1	**1**	1	1	1
1	1	0	1	**1**	1	1	0
1	0	1	1	**1**	0	1	1
1	0	0	1	**0**	0	0	0
0	1	1	0	**0**	1	1	1
0	1	0	0	**0**	1	1	0
0	0	1	0	**0**	0	1	1
0	0	0	0	**0**	0	0	0

On the right hand side the truth table gives

a	b	c	$(a$	$\land$	$b)$	$\lor$	$(a$	$\land$	$c)$
1	1	1	1	1	1	**1**	1	1	1
1	1	0	1	1	1	**1**	1	0	0
1	0	1	1	0	0	**1**	1	1	1
1	0	0	1	0	0	**0**	1	0	0
0	1	1	0	0	1	**0**	0	0	1
0	1	0	0	0	1	**0**	0	0	0
0	0	1	0	0	0	**0**	0	0	1
0	0	0	0	0	0	**0**	0	0	0

There is logical equivalence and the proof is complete. □

(c)

Theorem 4. $A \implies (B \land C) \equiv (A \implies B) \land (A \implies C)$

Proof. Let $Q = (B \land C)$. Then $A \implies Q$ if and only if $\neg A \lor Q$. And so, $\neg A \lor (B \land C)$ if and only if $(\neg A \lor B) \land (\neg A \lor C)$ by theorem 2 All of that holds if and only if $(A \implies B) \land (A \implies C)$. This completes the proof. □

(d)

Theorem 5. $A \implies (B \lor C) \equiv (A \implies B) \lor (A \implies C)$

Proof. Let $Q = (B \lor C)$. Then $A \implies Q$ if and only if $\neg A \lor Q$. And so, $\neg A \lor (B \lor C)$ if and only if $(\neg A \lor B) \lor (\neg A \lor C)$ by associativity. All of that holds if and only if $(A \implies B) \lor (A \implies C)$. This completes the proof. □

3. Justify equivalence.

(a) There exists an equivalence since the only use of y is for the expressioin involving $Q(x, y)$. In particular the implication is equivalent to $\not P(x) \lor Q(x, \boldsymbol{y})$. So it follows that $\exists$ can be inserted deeper within the statement.

(b) Since negation flips qualifiers we have the following logic,

$$
\begin{aligned}
&\neg\exists x\forall y(P(x) \implies \neq Q(x,y)) \\
\iff &\forall x\neg\forall y(P(x) \implies \neq Q(x,y)) \\
\iff &\forall x\exists y\neg(P(x) \implies \neq Q(x,y)) \\
\iff &\forall x\exists y\neg(\neg P(x)\vee \neq Q(x,y)) \\
\iff &\forall x\exists y(\neg(\neg P(x)) \wedge \neg(\neq Q(x,y))) \\
\iff &\forall x\exists y(P(x) \wedge Q(x,y)).
\end{aligned} \tag{1}
$$

Therefore, the equivalence holds.

(c) There is not an equivalence by the following argument:

$$
\begin{aligned}
&\forall x\exists y(Q(x,y) \implies P(x)) \\
\iff &\forall x\exists y(\neg Q(x,y) \vee P(x)) \\
\iff &\forall x\exists y\neg Q(x,y) \vee P(x) \\
\iff &\forall x\neg\forall y Q(x,y) \vee P(x) \\
\iff &\forall x(\neg(\forall y Q(x,y)) \vee P(x)) \\
\iff &\forall x(\forall y Q(x,y)) \implies P(x))
\end{aligned} \tag{2}
$$

Whichn is certainly not equal to the right hand side.

4. Prove or disprove!

(a)

Theorem 6. *The following is true. For every x there exists a y such that $xy > 0$ implies $y > 0$.*

Proof. Fix x. Then take any $y > 0$. Clearly, $y > 0$, and so the implication is always true since it is equivalent to $xy \leq 0$ or $y > 0$. This completes the proof. □

(b)

Theorem 7. *The following is false. There exists a x such that for all y, $xy < x^2$.*

Proof. Suppose it were true. Then consider the rectangle of side-length x. The closed and bounded set $S_y = [0,x] \times [0,y]$ must then have outer measure less than that of $X = [0,x]^2$ for all x. Since $x \in \mathbb{R}$, we have that $\forall y, m(S_y) < X$. Then take the sequence $\{a_n\}_{n\in\mathbb{N}}$ where $a_n = n$. The mesure sequence $(m(S_{a_n})$ is bounded and monotone increasing by the initial supposition, so by the monotone convergence theorem, it converges.

Since the measure sequence is bounded and S_y is a closed and bounded compact set for all y, we have that the sequence of diameters is bounded and converges $diam(S_{a_n})$. Furthermore the diameter of such a set is then dominated by a_n by the archimedian property. So we have that $a_n \to a \in \mathbb{R}$. A contradiction to the unboundedness of $\mathbb{N}$!

This completes the proof without loss of generality since negative rectangles make sense from a measure theory prospective. □

(c)

Theorem 8. *There exist a y such that for all x, $xy \geq x^2$.*

Proof. Take the sequence $a_n = n$. Then if there existed y such that $ny \geq n^2$, then $y \geq n$ for all n, a contradiction to the archimedean property of $\mathbb{R}$. QED $\square$

(d) DUCK PROBLEMS DUDE.

i. A. $\forall x D(x) \implies I(x)$.
   B. $\forall x V(x) \implies H_{issues}(x)$
   C. $\forall x C(x) \implies \neg W(x)$
   D. $\forall x H_{issues}(x) \implies W(x)$
   E. $\forall x I(x) \implies C(x)$
   F. $\forall x P(x) \implies V(x)$

ii. A. $\forall x \neg I(x) \implies \neg D(x)$
   B. $\forall x \neg H_{issues}(x) \implies \neg V(x)$
   C. $\forall x W(x) \implies \neg C(x)$
   D. $\forall x \neg W(x) \implies \neg H_{issues}(x)$
   E. $\forall x \neg C(x) \implies \neg I(x)$
   F. $\forall x \neg V(x) \implies \neg P(x)$