Math 110 — Homework 1— UCB, Summer 2017 — William Guss

(1.1) Let $f : X \to Y$ be a function between nonempty sets.

(1) Show that f is injective iff it has a left-inverse iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Proof. The map f is injective if and only if for every $x_1 \neq x_2$, $f(x_1) \neq = f(x_2)$ (by definition). Thus (iff) in the contrapositive $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Furthermore f is surjective to its range and by its injectivity (iff) its preimage on points of its range is always a signleton by the second iff condition. Therefore define g to map points in $f[X]$ to their singleton preimage points. Then $g \circ f(x) = x$. □

(2) Show that f is surjective iff it has a right inverse iff for every $y \in Y$ there is some $x \in X$ such that $f(x) = y$.

Proof. If f is surjective then for every $y \in Y$ there exists an x so that $f(x) = y$. Define a map $g : Y \to X$ so that $g(y) = x$ where $f(x) = y$, this is defined if and only if f is surjective. Then $f(g(y)) = y$ for every y and thus (iff), g is a right inverse of f. The last iff is by definition (Gleezy are u serious dude? I don't even know what is going on with this problem :( ) □

(3) Do your proof work if X or Y is empty? If not, find a counter-example.
Solution. If X is empty then there is a unique function $f : X \to Y$ since $\emptyset$ is initial in the cateogry of sets[1] In the case of (1) is injective since there are no x in the domain and thus the statement of f being injective is a vacuous truth. Futhermore the statement of $f(x_1) = f(x_2)$ implies $x_1 = x_2$ is vacuously true since there are no $x \in \emptyset$. However the proof and construction of g using singleton preimages fails (and generally f has no left-inverse because functions with empty codomain do not exist in the category of sets.)

In the case of (2) f is never surjective since for every y there does not exist an x with $f(x) = y$ since there are no x in the codomain, thus the proof holds. Since f does not exist, it does not make sense to discuss right inverses of f and so this part of the proof fails, as we cannot construct a g if there no f.

Suppose that Y is an emptyset. Then in the case of (1) there are no functions $f : X \to Y$ so every statement is vacuously true. Again the proof is a function on such candidate f, since none exist, then the proof 'works' for every $f : X \to \emptyset$. In the case of (2) there are no funfctions $f : X \to Y$ and using the same logic, the proof is a functioin on such candidate f. Since none exist, the proof 'works'.

(1.2)

(1) Let $\mathbb{F}$ be a division ring and let V be a vector space over $\mathbb{F}$. Show that $\alpha \cdot v = 0$ imples $\alpha = 0$ or $v = 0$.

Proof. If both $\alpha \neq 0$ and $v \neq 0$ then suppose $\alpha v = 0$. Then $0 = \alpha^{-1}0 = \alpha^{-1}\alpha v = 1v = v \neq 0$. So it could not be the case that $\alpha v = 0$ when $\alpha \neq 0$ and $v \neq 0$. Therefore $\alpha = 0$ or $v = 0$ when $\alpha \cdot v = 0$. □

(2) Find an example where this fails when $\mathbb{F}$ is not a division ring.
Solution. Let $\mathbb{F} = \mathbb{Z}_15$ and $V = \mathbb{Z}_15 \oplus \mathbb{Z}_15$. Then $5 \cdot (3, 3) = (15, 15) \mod (15, 15) = 0$ but $5 \neq 0$ and $(3, 3) \neq (0, 0)$.

[1] I am assuming that this category is well defined and has initial and terminal objects.

(1.3) Let $\mathbb{K}$ be a ring and let $T : V \to W$ be a linear transformation between $\mathbb{K}$-modules

(1) Show that $0 \cdot v = 0$ for all $v \in V$.

Proof. First, $0 = (0 \cdot v - 0 \cdot v) = (0 - 0) \cdot v = 0 \cdot v$ since V is abelian and $\mathbb{K}$ is a ring. □

(2) Show that $(-1) \cdot v = -v$ for all $v \in V$.

Proof. First $v + (-1) \cdot v = 1 \cdot v + (-1) \cdot v = (1 - 1) \cdot v = 0 \cdot v = 0$. Therefore by the uniqueness of $-v$ we have $-v = (-1) \cdot v$. □

(3) Show that $T(0) = 0$.

Proof. Fix some $v \in V$. Then $T(0_V) = T(v + (-v)) = T(v + (-1)(v))T(v) + (-1)T(v) = T(v) - T(v) = 0_W$. □

(4) Show that $T(-v) = -T(v)$.

Proof. For any $v \in V$, $T(-v) = T((-1) \cdot v) = (-1) \cdot T(v) = -T(v)$. This completes the proof. □

(1.4) Let $\mathbb{K}$ be a ring, and let $T : V \to W$ be a linear transformation between $\mathbb{K}$-modules.

(1) Show that $Ker(T)$ is a subspace of V.

Proof. We shall check the subspace axioms on $Ker(T)$. Let $v, w \in Ker(T)$ then $T(v + w) = T(v) + T(w) = 0 + 0$ and so $v + w \in Ker(T)$. Furthermore $T(0) = 0$ implies $0 \in Ker(T)$. Additionally let $a \in \mathbb{K}$ then for $v \in Ker(T)$, $T(av) = a \cdot T(v) = 0$ by a previous exercise. □

(2) Show that $Im(T)$ is a subspace of W.

Proof. We shall check the subspace axioms on $Im(T)$. Let $f(v), f(w) \in Im(T)$. Then $f(v) + f(w) = f(v+w) \in Im(T)$. Furthemore let $a \in \mathbb{K}$, then $a \cdot f(v) = f(a \cdot v) \in Im(T)$ as $a \cdot v \in V$. Finally $T(0_V) = 0_W$ and so $0_W \in Im(T)$ by a previous exercise. Therefore $Im(T)$ satisfies the subspace axioms of W and is a subspace. □

(1.5) Let V and W be vector spaces over $\mathbb{R}$ and let $T : V \to W$ be a function so that $T(v_1 + v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in V$.

(1) Show that $T(\alpha \cdot v_= \alpha \cdot T(v)$ for all $\alpha \in \mathbb{Q}$.

Proof. Since $\alpha \in \mathbb{Q}$ there are $a, b \in \mathbb{Z}$ so that $\alpha = a/b$. Without loss of generality assume that $\alpha \geq 0$. Then

$$T(\alpha v) = T\left(\sum_{i=1}^{a} \frac{v}{b}\right) = \sum_{i=1}^{a} T\left(v/b\right) = aT(v/b).$$

Multiplying we get

$$b(a \cdot T\left(v/b\right)) = a \sum_{j=1}^{b} T\left(v/b\right) = a \cdot T\left(\sum_{j=1}^{b} \frac{v}{b}\right) = a \cdot T\left(v\right).$$

Therefore

$$\frac{a}{b} \cdot T\left(v\right) = \frac{b}{b} a \cdot T(v/b) = T\left(\frac{a}{b} v\right).$$

This completes the proof. □

(2) Show that T need not be linear.

(1.6) Let V and W be a vector space over $\mathbb{R}$ and let $T : V \to W$ be a function such that $T(\alpha v) = \alpha T(v)$ for all $v \in V$ and $\alpha \in \mathbb{R}$.

(1.7) Let $\mathbb{K}$ be a ring, let V be a $\mathbb{K}$-module, and let $W_1, W_2 \subset V$ be subspaces of V.

(1) Show that $W_1 \cap W_2$ is a subspace of V.

Proof. We show that $W_1 \cap W_2$ satisfies the subspace axioms. If $v, w \in W_1 \cap W_2$ then $v + w \in W_1$ and $v + w \in W_2$ since $v, w \in W_1$ and $v, w \in W_2$. Furthermore for $\alpha \in \mathbb{K}$ it follows that for any v $\alpha v \in W_1$ and $\alpha v \in W_2$ by W_1, W_2 subspaces thus $\alpha v \in W_1 \cap W_2$. Finally $0 \in W_1$ and $0 \in W_2$, thus $0 \in W_1 \cap W_2$. Therefore $W_1 \cap W_2$ is a subspace. □

(2) Show that $W_1 \cup W_2$ is a subspace of V, iff $W_2 \subset W_1$ or $W_1 \subset W_2$.

Proof. Without loss of generality assume that $W_1 \subset W_2$. Then $W_1 \cup W_2 = W_2$ and by W_2 a subspace, $W_1 \cup W_2$ is a subspace.

In the other direction suppose that $W_1 \cup W_2$ is a subspace of V. Let $w, v \in W_1 \cup W_2$, then if $w \in W_1$ and $v \in W_2$, $v + w \in W_1 \cup W_2$ so $v + w \in W_1$ or $v + w \in W_2$. If $v + w \in W_1$ then $v + w - v = w \in W_1$ and so $W_2 \subset W_1$. Otherwise $v + w \in W_2$ implies $v + w - w = v \in W_2$ and so $W_1 \subset W_2$. □

(1.8) Let V be a vector space and let $\mathcal{B} \subset V$. Show TFAE.

(1) $\mathcal{B}$ is a basis.
(2) $\mathcal{B}$ is a maximal linearly-independent set.
(3) $\mathcal{B}$ is a minimal spanning set.

Proof. Suppose that $\mathcal{B}$ is a basis. Then by definition $\mathcal{B}$ is linearly independent and spans V. Then suppose that $\mathcal{B}$ is not maximal, then there is $w \in V$ so that $\mathcal{B} \cup \{w\}$ is linearly independent. But since $w \in V$ and $span(\mathcal{B}) = V$, $w = \sum_{v \in \mathcal{B}} \alpha_v v$ and so $\mathcal{B} \cup \{w\}$ is not linearly independent. Thus $\mathcal{B}$ is a maximal linearly-independent set.

Supose that $\mathcal{B}$ is a minimal spanning set. Then there is no $w \in \mathcal{B}$ so that $\mathcal{B} \setminus \{w\}$ also spans V. Suppose that there exists a $w \in \mathcal{B}$ so that w is a linear combination $w = \sum_{v \in \mathcal{B} \setminus \{w\}} \alpha_v v$, then any $z \in V$ can be expressed as

$$z = \alpha_w w + \sum_{v \in \mathcal{B} \setminus \{w\}} \alpha'_v v = \sum_{v \in \mathcal{B} \setminus \{w\}} (\alpha_v + \alpha'_v) v$$

and so $\mathcal{B} \setminus \{w\}$ spans V which contradicts the minimality of $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis.

Finally suppose that $\mathcal{B}$ is a maximally linearly independent set. Then for any $v \in V$ suppose that v is not a linear combination of $\mathcal{B}$ then $w \perp \mathcal{B}$, but this would contradict the maximality of $\mathcal{B}$. If $w \in \mathcal{B}$ so that $\mathcal{B} \setminus \{w\}$ spans V then $w \in V$ implies that w is a linear combination of $\mathcal{B} \setminus \{w\}$ and so $w \not\perp \mathcal{B} \setminus \{w\}$. This contradicts the linear indepdendence of $\mathcal{B}$. Therefore $\mathcal{B}$ must also be a minimal spanning set of V.

Since (1) $\implies$ (2) and (3) $\implies$ (1) and (2) $\implies$ (3), the statements are equivalent. □

(1.9) Define $T : \mathbb{R}[x] \to \mathbb{R}[x]$ by

$$[T(p)](x) := \int_0^x dt\ p(t)$$

(1) What is $Ker(T)$?

Solution. We claim that $Ker(T) = \{0\}$. To see this let $0 \neq p \in \mathbb{R}[x]$. Then

$$\int_0^x p(t)\ dt = \sum_{k=1}^{n} a_k/(k+1)x^{n+1} \neq 0 + 0x + 0x^2 + \cdots$$

. However $T(0) = 0$ since $\int_0^x 0\ dt = 0$.

(2) What is $Im(T)$?

Solution. $Im(T) = \mathbb{R}[x]$. To see this let $p \in \mathbb{R}[x]$ with n coefficients a_i. Then $q[x] = \sum_{i=1}^{n} i \times a_i x^i$ has $T[q](x) = p[x]$ using the integration formula as above. Therefore T is surjective and $Im[T] = \mathbb{R}[x]$.