Math 110 — Homework 3 — UCB, Summer 2017 — William Guss

(3.1)

- Show that $V = W_1 \oplus \cdots \oplus W_m$ if and only if $V = W_1 + \cdots + W_m$ and $0 = w_1 + \cdots + w_m$, $w_k \in W_k$ implies that $w_1, \cdots, w_m = 0$.

  Proof. Suppose that V is the given direct sum. Then if $0 = w_1 + \cdots + w_m$ each w_k is unique. Then $0 \in W_k$ for all k and $0 + \cdots + 0 = 0 \in V$ gives that this is the only such linear combination yielding 0, by uniqueness. Therefore $w_k = 0$ for all w_k.

  Now suppose that there is a vector $v \in V$ with non unique decomposition; that is, let $w_k, z_k \in W_k$ for all k, then v is such that

  $$v = w_1 + \cdots + w_m = z_1 + \cdots + z_m$$

  where there exists a j so that $W_j \ni w_j \neq z_j \in W_j$.

  Then it follows that

  $$\begin{aligned} 0 = 0 + \cdots + 0 &= v + (-v) \\ &= w_1 + \cdots + w_m + -(w_1 + \cdots + w_m) \\ &= w_1 - z_1 + \cdots + w_m - z_m. \end{aligned}$$

  Since $\mathbb{K}$ is a ring, and thus addition is commutative. Since $w_j \neq v_j$, then $w_j - v_j \neq 0$ and thus there are non-zero terms in the decomposition of 0. Therefore, $0 = d_1 + \cdot + d_m$ does not imply $d_k = 0$ for all k. Thus by contraposition we have shown the other direction. This completes the proof. $\square$

- Show that $V = U \oplus W$ iff ($U + W = V$ and $U \cap W = 0$).

  Proof. Suppose that $V = U \oplus W$. If $v \in U \cap W$, by definition $v = u + w$ uniquely. Futhermore there are $u' \in U$ and $w' \in W$ so that $u' = u + w = w'$. Thus $0 = v + (-v) = (u' - u) - w = -u + (w' - w)$ with $(w' - w) \in W$ and $(u' - u) \in U$. Using the uniqueness of 0 in its direct sum decomposition $(u' - u) = u'$ and $(w' - w) = w'$. Therefore $-w = 0$ and $-u = 0$, so $v = 0 + 0$ implies $v = 0$, and $U \cap V = 0$. Next $V = U \oplus W$ implies that every $v \in V$ has a unique decomposition in $U \oplus V$ and so $V \subset U + W$. On the other hand $U + W \subset V$ as $W \subset V$ and $U \subset V$ as subspaces, and therefore any combination of vectors in both need be in V as to not violate V being a $\mathbb{K}$-module.

  In the other direction suppose that $U + W = V$ and $U \cap W = 0$. Now take $v \in V$ so that $v = u + w$. We will show that $u, w \in U, W$ are unique. Suppose there were $u', w' \in U, W$ with $v = u' + w'$ and $(u, w) \neq (u', w')$ for the sake of contradiction. Then $0 = v + (-v) \in U \cap V$ with $0 = (u - u') + (w - w')$, but one of thes terms must be non-zero by our supposition, and therefore the other must be its inverse. That is, $(u - u') = -(w - w')$ and so $(u - u') \in W$ and $(w - w') \in U$ by U, W subspaces. So $w - w' = 0$ and $u - u' = 0$ by the hypothesis, which contracits $(u, w) \neq (u', w')$. Therefore the decompositon $v = u + w$ is unique and $V = U + W$ is direct. $\square$

(3.2)

- Show that $Ker(T)$ is T-invariant.

Proof. We use the alternative definition for T-invariant given in 4.4.85. Then if $w \in Ker(T)$, $T(w) = 0$, and since $T(T(w)) = T(0) = 0$ we have $T(w) \in Ker(T)$. Therefore $Ker(T)$ is T-invariant. □

- Show that $Im(T)$ is T-invariant.

  Proof. Recall that as a subspace $Im(T) \subset V$. Then $T(Im(V)) \subset \{v \in V \ : \ v = T(w), w \in V\}$; since $w \in Im(V)$ gives us $T(Im(V))$. Therefore $T(Im(V)) \subset Im(v)$ and $Im(V)$ is T-invariant. □

(3.3) Let V be a vector space over $\mathbb{F}$, let $X, H : V \to V$ be linear, let $\alpha, \lambda \in \mathbb{F}$ be central, let $v \in V$ be an eigenvector of H with eigenvalue λ. Show that if $H \circ X - X \circ H = \alpha X$ then $X(v)$ is an eigen vector of H with eigenvalue $\alpha + \lambda$.

Proof. Using theorem 4.2.12 we yield that λ need be central and $H(v) = \lambda v$. Then $\alpha X(v) = H \circ X(v) - X \circ H(v) = H(X(v)) - X(\lambda v)$. Then using centrality of λ we get $H(X(v)) = (\alpha + \lambda)X(v)$; and thus $\alpha + \lambda$ is an eigenvalue of H, with $X(v) \in Eig_{\lambda,H}$. □

(3.4)

(3.5)

- Find the eigenvalues and eigenspaces of both L and R. [1]

  Solution. For the right shift operator, every $r \in \mathbb{R}$ is an eigenvalue. To see this take $c \in \mathbb{C}^{\mathbb{N}}$ so that $c = \langle z, rz, r^2z, r^3z, \cdots \rangle$. with $z \in \mathbb{C}$. Then $R(c) = \langle rz, r^2z, r^3z, \cdots \rangle = rc$. The cooresponding eigen spaces are $Eig_{r,R} = \{\langle z, rz, r^2z, \cdots \rangle \ : \ z \in \mathbb{C}\}$.

  For the left shift operator, we claim that there are no eigenvalues. To see this suppose that there were some non-zero eigenvalue, say $r = \lambda$. Then if $c \in Eig_{\lambda,L}$ non-trivially we have $L(c) = rc$. Take the first non-zero element in c, say c_j, and then $L(c)_j = 0$ and so it could not be that $rc_j = 0$ unless $r = 0$. Now suppose that $r = 0$ were an eigenvalue and take a non-zero eigenvector $c \in Eig_{0,L}$. Then there is a j so that $c_j \neq 0$ and then $L(c)_j = 0$, but $L(c)_{j+1} \neq 0$ which contradicts that $L(c) = rc = 0$. Therefore $r = 0$ is not an eigenvalue of L.

- Find the eigenvalues and eigenspaces of both $L|_{\ell^2(\mathbb{C}))}, R|_{\ell^2(\mathbb{C})} : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$.

  Solution. First let $T : V \to V$ as vector spaces over a division ring $\mathbb{K}$. Then T has no eigenvalues if and only if there are no $W \subset V$ with $T|_W = rid_W$ for some $r \in \mathbb{K}$. Therefore if E is a subspace V then $T|_E$ has no eigenvalues as any subspace of E, say J, is mereley a subspace of V expressed $J = E \cap J$. Therefore $L|_{\ell^2(\mathbb{C})}$ has no eigenvalues using this spectral subspace principle.

  We apply similar reasoning as above to R. We consider all those eigenspaces which form a strict subspace of $\ell^2(\mathbb{C})$, and thus all those eigen spaces for which the series $\sum_{k=1}^{\infty} r^n z \in \mathbb{C}$ converges. Therefore r so that $|r| < 1$ yields convergence and therefore defines the sert of acceptable eigen values. This completes the solution.

[1]Gleezy actually got the operators wrong in his notes, but I'll just keep consistent with them; that is, $L(0, 1, 2, \cdots) = (0, 0, 1, 2, 3, 4, \cdots)$.

(3.6) Let V be a vector space over a division ring $\mathbb{F}$, let $T : V \to V$ be linear, and $p \in \mathbb{F}[x]$ be a polynomial.

- If $v \in V$ is an eigenvector with eigenvalue λ (so that $T(v) = \lambda v$) show that $[p(T)](v) = p(\lambda)v$.

  Proof. As $p \in \mathbb{F}[x]$, then $p[x] = a_0 + a_1 x + \cdots + a_n x^n$. So we evaluate

  $$\begin{aligned} p[T](v) &= \left[\sum_{k=0}^{n} a_k T^n\right](v) \\ &= \sum_{k=0}^{n} a_k T^n(v) = \sum_{k=0}^{n} a_k \lambda^n v \\ &= \sum_{k=0}^{n} a_k [\lambda^n] v = \left(\sum_{k=0}^{n} a_k [\lambda]^n\right) v = p(\lambda)v. \end{aligned}$$

  The above follows using associativeity of $\mathbb{F}$. This completes the proof. □

- Let $\lambda \in \mathbb{F}$ be an eigenvalue of T and suppose that $p(T) = 0$. Show that $p(\lambda) = 0$

  Proof. If λ is an eigenvalue of T then $p(T)(v) = 0$ for all v, and thus $p(\lambda)v = 0$ for all v. Since $\mathbb{F}$ is a division ring, $p(\lambda)v = 0$ for $v \neq 0$ implies that at least $p(\lambda) = 0$. □

(3.7) Let V be a finite-dimensional vector space and let $T : V \to V$ be linear with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Show that T is diagonalizable iff $V = \bigoplus_{i=1}^m Eig_{\lambda_i}$.

Proof. Suppose T is diagonalizable then by the fundamental theorem of diagonalizability (4.3.2), this is if and only if there is a basis $\mathcal{B}$ of V consisting of eigenvectors of T. By proposition (4.4.34) we have that

$$V = \bigoplus_{b \in \mathcal{B}} Span(b) = \bigoplus_{i=1}^{m} Eig_{\lambda_i, T}.$$

In the other direction suppose that $V = \bigoplus_{i=1}^m Eig_{\lambda_i}$. Then let $\mathcal{B}_i$ be a basis for Eig_{λ_i}. Since V is a direct sum of the eigen spaces, then $0 = e_1 + \cdots + e_m$ implies every $e_k = 0$. Then writing e_l interms of its basis in $\mathcal{B}_k$, we yield that the set $\bigcup \mathcal{B}_i =: \mathcal{B}$ is linearly independent. The set also spans V since $V = \sum_{i=1}^m Eig_{\lambda_i}$. This gives $\mathcal{B}$ a basis of V and therefore by the fundaemental theorem of diagonalizability, T is diagonalizable. □

(3.8) Let V be a finite-dimensional vector space over a field and let $S, T : V \to V$ be diagonalizable linear-maps. Show that S, T are simultaneously diagonalizable if and only if S and T commute.

Proof. Suppose that S, T are simultaneously diagonalizable with common basis $\mathcal{B}$. Then for every $v \in V$ it follow that

$$\begin{aligned}
[S \circ T[v]]_{\mathcal{B}} &= [S \circ T]_{\mathcal{B}}[v]_{\mathcal{B}} \\
&= \left[\sum_{j=1}^{n} S_j^p \sum_{k=1}^{n} T_j^k [v]_{\mathcal{B}}^k \right]_{p=1}^{n} \\
&= \left[T_p^p S_p^p [v]_{\mathcal{B}}^p \right]_{p=1}^{n} = \left[S_p^p T_p^p [v]_{\mathcal{B}}^p \right]_{p=1}^{n} \\
&= \left[\sum_{j=1}^{n} T_j^p \sum_{k=1}^{n} S_j^k [v]_{\mathcal{B}}^k \right]_{p=1}^{n} \\
&= [T \circ S]_{\mathcal{B}}[v]_{\mathcal{B}}
\end{aligned}$$

by the commutativity of the base field. $\square$