MATH H104: Homework 4

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2 A Taste of Topology

29. *Show the following.*

Theorem 1. *Let $\mathcal{T}$ be the collection of open subsets of a metric space M, and $\mathcal{K}$ be the collection of closed subsets. Show that there is a bijection from $\mathcal{T}$ onto $\mathcal{K}$.*

Proof. We wish to find a function $f : \mathcal{T} \to \mathcal{K}$ bijective. To do so observe the following fact about compliments in M: $A^c = B$ is the unique compliment of A. Suppose that there were another compliment such that $A^c = C \neq B$ which was the compliment of A. By definition $C = \{x \in M | x \notin A\} = B$, so there cannot exist another set which is also the compliment of A.

As follows from above, the compliment of an open set is closed and the compliment of a closed set is open. Therefore, let $f : A \mapsto A^c$. Then, f is in an injection by the uniqueness of compliments. Furthermore, if $S \in \mathcal{K}$, there exists a set, Q, in $\mathcal{T}$ such that $f(Q) = S$, nameley S^c. This follows by $S^c \in \mathcal{T}$ and $f(S^c) = S^{cc} = S$. Hence f is a bijection.

This completes the proof, and $\mathcal{K} \sim \mathcal{T}$. □

32. *Prove the following and then remark.*

Theorem 2. *Every subset of* $\mathbb{N}$ *is clopen.*

Proof. To show that every subset of $\mathbb{N}$ is clopen the definitions of openness and closedness must hold on every set. Take an arbitrary subset S of the natural numbers. If S is empty or the whole space $\mathbb{N}$ then it is clopen.

Otherwise, for every $q \in S$ there exists an $r > 0$, say 0.5, such that $d(q, p) \implies p \in S$. To see this, consider that the only such p for which the definition of openness holds is q itself. Therefore, S is open.

The subset S must also be closed because S^c is an open subset of the naturals, and $S^{cc} = S$ must be closed by compliments. Hence S is clopen and the proof is complete. □

Remark. Any function mapping the natural numbers to some metric space M must be continuous. Consider some $Q \subset f(\mathbb{N})$. If Q is open then, $f^{pre}(Q)$ is open. Conversely, if Q is closed then, $f^{pre}(Q)$ is closed. Furthermore if M is any discrete space (or one with a discrete metric) then f is an open mapping.

33. *Find a metric space for which the boundary of the r neighboorhood need not always be the r-sphere.*

Example. Let $M = \mathbb{N}$ be a metric space with its inherited metric from $\mathbb{R}$. We show that it is not true that for each $M_r(p)$, the boundary is the r-sphere. Consider that the closure of $M_r(p)$ is $M_r(p)$ as every set in M is clopen. Then the closure of the compliment is just compliment. By definition $\partial M_r(p) = \overline{M_r(p)} \cap \overline{M_r^c(p)} = \emptyset$. However for all $r \in \mathbb{N}$, $S_r(p) = \{x \in M \mid d(x,p) = r\} \neq \emptyset$. So there are cases in which the boundry is not the r-sphere.

Suppose that x were in the boundary of some $M_r(p)$ and not in the unit sphere. Then $d(x,p) \not\leq r \implies d(x,p) > r$. By virtue of x being in the boundary, x must be in every closed subset containing $M_r(p)$. However, $x \notin S_r(p)$ (the r-sphere at p) and $S_r(p) \supset M_r(p)$ is closed; a contradiction! So, the boundary must be contained within the r-sphere at p.

40. *Prove the following.*

Theorem 3. *If M be a metric space with metric d, then the following are equivalent:*

(a) M is homeomorphic to M equipped with the discrete metric.

(b) Every function $f : M \to M$ is continuous.

(c) Every bijection $g : M \to M$ is a homeomorphism.

(d) M has no cluster points.

(e) Every subset of M is clopen.

(f) Every compact subset of M is finite.

Proof. $(a) \implies (e)$. Since $(M, d) \cong (M, d_{discrete})$, then for some function $f : M \to M$ where the domain has the discrete metric, every subset of the domain is clopen, and thereby every image of a subset of the domain is clopen by the homeomorphism.

$(e) \implies (b)$. If every set of M is clopen then consider any $f : M \to M$. Since $f(A)$ is clopen for any A, and $f^{pre}(f(A))$ is clopen by the assumption, then f is continuous! This completes the proof.

$(b) \implies (c)$. If every function in M^M is continuous, then consider an arbitrary bijection $g : M \to M$. Clearly g is continuous, and it's inverse map $g^-1 : M \to M$ is also continuous.

$(c) \implies (f)$. We will attempt to show that the converse is true. If S is compact and not finite then there exists a bijectyion $g : M \to M$ such that g is not bicontinuous. Clearly S is compact if and only if for all x_n sequences in S there exists a (n_k) such that $x_{n_k} \to X \in S$. Furthermore S is infinite if and only if there exists a sequence (x_n) in S with all of its elements distinct. These two facts inmply that there exists a sequence (x_n) in S distinct which converges to x. Conisder the set $S = \{x_n\} \cup \{x\}$. Then let us examine the following bijection. Take $g : M \to M$ as the bijection which maps the first x_k which is not x to x and then x to such an x_k. Since $x_n \to x$, if g homeomorphism then $g(x_n) \to g(x)$ but this is not true since $g(x_n) \to g(x_k)$ so g does not preserve convergence and therefore we have found satisfying nonhomeomorphic bijective g. This completes the proof.'

$(e) \implies (d)$. For the purpose of contradiction suppose that every subset clopen implies that M has a cluster point p. Every $S \subset M$ is clopen if and only if every set is closed. Let $S = \{p\}$ be the set of the cluster point in M then by the assumption, for all $x \in S$ there exists an $\epsilon > 0$ such that

$$d(x, q) < \epsilon \implies q \in S$$

, which holds namely if $x = q = s$. Since p is a cluster point for all $r > 0$ there exists a q such that $d(q, p) < r$ and $q \neq p$. Take $r = \epsilon$. and we reach a contradiction because $p \neq q$, but $q \in S$. Hence the assumption implies that M has no cluster points.

$(d) \implies (e)$. Suppose that M has no cluster points. Then for all $p \in M$ there exists an $r > 0$ such that for all $q \in M_r(p)$, $q = p$. Let S be an arbitray subset of M. Then M having no cluster points implies that any subset has no cluster points, and therefore for all $p \in S$ there exists an $r > 0$ such that $d(p, q) < r$ implies that $p = q \in S$.

Because every set in M is open, then all compliments are closed. By virtue of membership in M the compliments are also open. Then the compliment of the compliment being the original set is therefore closed. Hence all sets are clopen. This completes the proof.

$(f) \implies (a)$. S is finite if and only if S is compact. Consider a sequence of distinct point which converges to a. Let the set of elements in the sequence be $\{a_n\}$, then the set is compact and non finite which is a contradiction. Hence, all convergent sequences are not distinct, which implies that eventually they are constant. So let $f : M \to M_d$ be the identity map. This map is clearly a bijection, so all that remains to be shown is that f is a bicontinuous function.

If $a_n \to a$ in M, then there exists an n for all $n > N$ $f(x_n) = c$ which implies that $f(x_n) \to c$. Hence f is continuous. On the other hand if $x_n \to x \in M_d$ then x_n must eventually be constant as M_d is endowed with the discrete metric. Thus $f^{-1}(x_n)$ is eventually constant and hence converges. Thus f is a bicontionuous function, thereby implying that f is a homeomorphism. This completes the proof. □

42. *What is wrong with the proof of Theorem 28?*

The misstep in the proof is the statement that there exist subsequences $(a_{n_k}), (b_{n_k})$ which converge. Compactness sureley implies that there exists an index sequence n_k such that $a_{n_k} \to a \in A$ but that exact index set may not be one which allows $b_{n_k} \to b$.

To solve this problem consider the following argument. Since any subsequence of (a_{n_k}) converges to a by the convergence of (a_{n_k}), and B compact, we can take a subsequence, $(b_{n_{k(l)}})$ which converges to b. So the sequence $((a_{n_{k(l)}}), (b_{n_{k(l)}})) \to (a, b)$.

43. *Prove the following.*

Theorem 4. *If the cartesian product of two non-empty sets $A \subset M, B \subset N$ is compact in $M \times N$, A and B are compact.*

Proof. By the compactness of $C = A \times B$, all sequences (a_n, b_n) have subsequences which converge to some $(a, b) \in C$. Take one such particular sequence. Since $a_n \in A$ and $a \in A$. Then the subsequential convergence of the product sequence implies the subsequential convergence of a_n. The same argument holds for b_n. In general, C contains the product of all sequences in A and B. So for any sequence in A, there exists some sequence in the product whose subsequence converges thereby impling the convergence of some subsequence of the original sequence in A. Again, the same argument holds for any given sequence in B.

This completes the proof. □

48. *Prove the following.*

Theorem 5. *There exists an embedding of the line as a closed subset of the plane, and there is an embedding of the line as a bounded subset of the plane, but there is no embedding of the line as a closed and bounded subset of the plane.*

Proof. By the line, we assume that $\mathbb{R}$ is meant. Consider the following function $f : \mathbb{R} \to L_u \subset \mathbb{R}^2$ such that $x \mapsto (x, 0) \in \mathbb{R}^2$. When $L_u = \{(x, y) \in \mathbb{R}^2 \ : \ y = 0\}$ is the codomain, f is clearly surjective and injective. Hence we have that f is bijective. Furthermore, take some open set in L_u, say S. Then $f^{-1}(S) = \{x \in \mathbb{R} | (x, 0) \in S\}$. If for every $s \in S$ there exists an $r > 0$, such that $d(s, q) < r \implies q \in S$, we have that $d((s_x, 0), (q_x, 0)) < r$. Since $f^{-1}s = s_x$ and $f^{-1}q = q_x$ then $d(s_x, q_x) < r$ and thereby $q_x \in \mathbb{R}$. So it must follow that for every s_x in $\mathbb{R}$ there exists an $r > 0$ such that $d(s_x, q_x) < r \implies q_x \in \mathbb{R}$. It suffices to say that f is a homeomorphism when the converse argument is applied.

Knowing that f embeds $\mathbb{R}$ onto $\mathbb{R}^2$, we show that such an embedding is a closed subset. L_u is closed if and only if it contains all of its limit points. Suppose (x_n) is a sequence in L_u such that $x_n \to x$. We wish to show that $x \in L_u$. By the convergence of x_n for every $\epsilon > 0$, there exists an N, such that for all $n \geq N$, $d(x_n, x) < \epsilon$. if x is not in L_u, then $x = (a, b)$ wih $b \neq 0$. So if $d(x_n, x) < \epsilon$ then take $\epsilon = b - 0.1$. In this case, $d(x_n, x) < \epsilon \implies x_n \notin L_u$ which is a contradiction. Ergo, L_u is a closed embedding of the line in the plane.

In a different case, it is clear that $\mathbb{R} \cong (0, 1)$. It suffices to show that $(0, 1)$ has an embedding in $\mathbb{R}^2$ which is bounded. Simple! Take $f : (0, 1) \to \mathbb{R}^2$ such that $x \mapsto (x, 0)$. The function f embeds $(0, 1)$ by the same argument supplied for the first case. Furthermore, $f((0, 1))$ is bounded because the set $[0, 1] \times [0, 1]$ contains the embedding (x is always between 1 and 0 and the y component is always 0.)

In the last case, suppose there existed a closed and bounded subset of the plane such that $\mathbb{R}$ was embedded to that set by some homeomorphism h. Then, by some theorem that embedding is compact as a subset of $\mathbb{R}^2$ and by topological equivalence, $\mathbb{R}$ must also be compact; a contradiction! Therefore, only the first two cases hold. □

53. *Suppose that (K_n) is a nested sequence of compact nonempty sets, $K_1 \subset K_2 \subset \ldots$, and $K = \bigcap K_n$.*

Theorem 6. *If for some $\mu > 0$, diam $K_n \geq \mu$ for all n, then diam $K \geq \mu$.*

Proof. Simple! If for every n, K_n compact, then the countable intersection of K_n, K, must be compact. By a theorem of the book, $K \times K$ is compact.

Consider the sequence $\{(x_n, y_n)\} \subset K \times K$. For this sequence in particular, take (x_1, y_1) as the pair of vectors such that $d(x_1, y_1) = diam\ K_1 > \mu$. Repeating this process for any n, take (x_n, y_n) as the pair of vectors such that $d(x_n, y_n) = diam\ K_n > \mu$. Since there exists a $\mu > 0$ such that for all N and for every $n \geq N$, $d(x_n, y_n) > \mu$ we might say that $d(x_n, y_n) \nrightarrow c$ where $c < \mu$.

Now by the compactness of K there exists a subsequence of $((x_n, y_n))$ which converges, and furthermore by the monotonicity and boundedness of $d(x_n, y_n)$ we have that it converges to a distance say $d(x, y) \geq \mu$. Recall however that $x, y \in K_n$ for every n and so $diam\ K \geq d(x, y) \geq \mu$. This completes the proof. □

54. *If $f : A \to B$ and $g : C \to B$ such that $A \subset C$ and for each $a \in A$ we have that $f(a) = g(a)$ then g* ***extends*** *f. We also say that g* ***extends to*** *g. Assume that $f : S \to \mathbb{R}$ is a uniformly continuous function defined on a subset S of a metric space M. Prove the following:*

   (a) *Extension to closure.*

   Theorem 7. *The function f extends to a uniformly continuous function $\bar{f} : \bar{S} \to \mathbb{R}$.*

   Proof. If f is uniformly continuous, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in S$, $d(p, q) < \delta \implies d(fp, dq) < \epsilon$. Since f is continuous it preserves convergence of sequences. So adding the closure of S to S through union lets all sequences in this new set $\bar{S}$ converge to elements in $\bar{S}$. Adding these elements we construct a function based on the convergence of limits. $g : \bar{S} \to \mathbb{R}$ such that if $x \in S$, then $x \mapsto fx$ and otherwise if $x \notin S$ and $x \in \bar{S}$ we know the following. The element x is a limit of a sequence in s, say x_n. Then for every $r > 0$ there exists an N such that for all $n > N$, $d(x_n, x) < r$. Using the function, $f(x_n) \to y \in \mathbb{R}$. Let $g(x) = y$. Then for all $\epsilon < 0$, there exists such an N that $n > N$ implies $d(gx_n, gx) < \epsilon$. In this case let $\delta = r = \epsilon$ from before. Then the limit is perserved and g is uniformly continuous at x. Hence f extends to a uniformly continuous function $\bar{f} = g$. □

   (b) *Uniqueness*

   Theorem 8. *The function $\bar{f}$ is the unique extension of f.*

   Proof. Suppose that there exists another extension of $\bar{f}$ to the closure of S, say g. Then for every $a \in S$, $f(a) = \bar{f}(a) = g(a)$, by extension, and if $x \in \bar{S}$ then $\bar{f}(x) \neq g(x)$. Consider a sequence which converges to x as a subset of S. Then for all $\epsilon > 0$ there exists an N_1 such that for all $n > N_1$,

$$d(\bar{f}x_n, fx) < \epsilon/2.$$

   Since g is also continuous we have that for some N_2 and all $n > N_2$

$$d(gx_n, gx) < \epsilon/2.$$

Remember that our assumption implies that $\bar{f}(x) \neq g(x)$. Take $N = \max N_1, N_2$ then for all $n > N$ we have that

$$d(\bar{f}x, gx) \leq d(\bar{f}x, \bar{f}x_n) + d(\bar{f}x_n, gx_n) + d(gx_n, gx) < \epsilon/2 + 0 + \epsilon/2,$$

by extension of f. So it is clear, $\bar{f}(x) = g(x)$; a contradiction!

Therefore $\bar{f}$ is unique and the proof is complete. $\square$