MATH H105: Homework 1

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55. Take a map f which is sufficiently close, that is the max norm of the euclidean distance between. Take the line which takes $f(p)$ to infinity and take a tangent Tp from the point p to a point on the line from $f(p)$. Consider this to be our tangent vector field. If the hairy ball theorem is true then this field must be zero at some p. If this is the case then $f(p) = p$ since the line from the origin to $f(p)$ to infinity need be the line from the origin to p to infinity, otherwise we contradict that the length of this tangent vector is 0.

    Therefore the map f has a fixed point.

    Consider the converse. Suppose all maps which are sufficiently close to the identity with a fixed point. Consider a vector field built in a similar fashion using the vectors $f(p)-p$ and homotopically moving them along the line from $f(p)$ to infinity untill you find a tangent vector to the sphere at p clearly this exists, and since this construction is smooth and $p-f(p)$ is a smooth function, we get that this vector field is continuous and so there is a point where the vector field is 0 since at one point the line from the origin to $f(p)$ is the same as the line from the origin to p.

56. Part b use line integral with differential ewquaiton $w = F_1dx_1 + F_2dx_2 + F_3dx_3$. Then we know that the form applied to a closed loop must be 0 since it is the weighted change of dx from start to finish. Observe that the form evaluates the dot product of F and the tangent of the $1 - cell$ parameterization, so we get that the net change in $\cos\theta$ where θ is the angular difference between the curve and the vector field, is zero and so the total net change in angle is proportional to 2π. The proportion is furthermore only dependent on orientation of the curve.

57. 71**. Hairy Ball Theorem Proof

    Theorem 1. *Any continuous vector field $X : S \to \mathbb{R}^3$ that is every where tangent to S must be zero at some point.*

    Before we give the proof. We first show some lemmas.

    Lemma 1. *If X is a continuous vector field on a closed simply connected domain so that $X \neq 0$, then the winding number of a] closed simple curve in the vector field must be zero.*

Proof. The winding number for a curve in X referrs to the number of times the vector field turns around itself on the curve as $t \to 1$ from 0. It is clear that the winding number of a curve must be an integer since it would violate the continuity of X if at a point infintesmally close the the start of the loop (near the end), the vector in X at that point were a real number different than that of the start point. Therefore, at least the vector field must wind upon itself a whole number of times.

Consider a sufficiently small homotopy of the smooth 1-cell curve we previously referred to. It could not be that H applied to the curve has a winding number much larger than that of the cotarget. Since the winding number itself is a continuous function of the vector field (which is continuous and whose vectors would not change angle much under small homotopy,) it follows that the winding number is invariant to homotopy.

Finally, the winding number of a point is 0 and so by the simple connectedness of the domain, recall that any simple closed curve is homotopic to the point. Therefore, the winding number of any closed curve is 0. □

Lemma 2. *The net winding of a closed curve in a simple closed domain over a continuous vector field X, which is nowhere 0, is invariant to the field. Furthermore if the closed curve is a loop, then it has winding number ± 1 corresponding to its orientation in space.*

Proof. The net winding of a closed simple 1-cell, C in the domain is exactly the net angle between the tangent C and the the vectors X at every point along C. So the net winding of C is the sum of the winding number of C in X and the winding number of C in general. So the net winding of C is ± 1 contingent upon the orientation of C. □

Lemma 3. *Homotopy does not change the net winding of a curve in a continuous vector field X with no 0 points.*

Lemma 4. *If X is a continuous vector field with no zeroes and C is a simple closed curve. Then if $\omega = X_1 dx_1 + X_2 dx_2$,*

$$\oint_C \omega = 0 \implies d\theta(C, X) = z2\pi, z \in \mathbb{Z} \tag{1}$$

Proof. By the definition of the form evaluation of a 1-cell, we have that $f dx_i$ is the f-weighted net change in x from the beginning of the parameterization of the 1-cell to its end. In paricular, $\omega = \langle X, dx_I \rangle$ for $I = \{1, 2\}$. Geometrically we have the following interpretation for

$$\oint_C \omega = 0. \tag{2}$$

Essentially, $\omega = \cos\theta |X||dx_I|(C)$. This gives the intuition that if the contour is closed, then the net angle between the tangents of C and the vector field X is 0.

Suppose we break the C 1-cell into the sum of two (possibly infinite) families of cells, $\{\gamma_t^+)_{t \in J^+}, \{\gamma_s^-\}_{t \in J^-}$ so that

$$\sum_{t \in J^+} \int_{\gamma_t^+} \omega = -\sum_{s \in J^-} \int_{\gamma_s^-} \omega. \tag{3}$$

There exists some sequence $\Gamma = \{\gamma_t^*\}$ such that

$$\oint_C \omega = \sum_{t \in J^- \cup J^+} \int_{\gamma_t^*} \omega = 0. \tag{4}$$

The continuity of X implies that $\square$