

MATH 105: Homework 6

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1. Show some things.

- (a) *Show that the definition of linear outer measure is unaffected if we demand that the intervals I_k in the coverings be closed instead of open.*

Definition 1. *The linear outer measure of a set $A \subset \mathbb{R}$ is given by*

$$m^*A = \inf \left\{ \sum_k |I_k| : \{I_k\} \text{ is a covering of } A \text{ by open intervals} \right\}. \quad (1)$$

Definition 2. *The closed linear outer measure of a set $A \subset \mathbb{R}$ is given by*

$$\bar{m}^*A = \inf \left\{ \sum_k |\bar{I}_k| : \{\bar{I}_k\} \text{ is a covering of } A \text{ by **closed** intervals} \right\}. \quad (2)$$

Theorem 1. *Definition 1 and definition 2 give equivalent measures.*

Proof. Take some set A and obtain its linear outer measure m^*A . By the definition of infimum, m^*A is the limit of outer measures of finer and finer countable coverings of A . The same argument can be made for \bar{m}^*A , except for \bar{I}_k closed.

Let the two respective sequences of coverings be given by \mathcal{C}_i and $\bar{\mathcal{C}}_i$. Clearly

$$m^*A \leftarrow m_i^*A = \sum_{C \in \mathcal{C}_i} |C| = \bar{m}_i^*A = \sum_{\bar{C} \in \bar{\mathcal{C}}_i} |\bar{C}| \rightarrow \bar{m}^*A \quad (3)$$

And so $m^*A = \bar{m}^*A$. This follows subtly from $m(I) = m(\bar{I}) = b - a$. The proof is complete. \square

- (b) The middle thirds cantor set has a covering by closed intervals C_i whose constituent area is $1/3^i$ and so the infimum has area 0.
(c) How open should I really be?

Theorem 2. *The outer measure of an interval can be taken without conditions on closedness/openess.*

Proof. Consider that any other covering of A besides that depicted in definition 1 and definition 2, has area in between those two coverings by monotonicity of outer measure. Therefore $m^*A \leq \nu A \leq \bar{m}^*A \implies \nu A = m^*A$. \square

- (d) The same thing holds for planar outer measure, since effectively S as a rectangle is the product of n intervals. Furthermore, we can approximate any rectangle (open, closed, clopen, or neither) $\pm\epsilon$ by a bunch of squares.

3.

Theorem 3. *All lines are zero sets.*

Proof. Recall that (from the book) all rigid transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are measure-preserving. Take any rotation and translation ϕ . By the exercise $m(\mathbb{R} \times \{a\}) = 0$ implies that $m(\phi(\mathbb{R} \times \{a\})) = m(\mathbb{R} \times \{a\}) = 0$. \square

Theorem 4. *All $n - 1$ hyperplanes are zero sets in \mathbb{R}^n .*

Proof. Recall proposition 2 (from the book) then without loss of generality apply the measureomorphism in the previous proof. \square

4. Higher dimensional Lemmas!

Lemma 1. *The boundary of an n -dimensional ball is an n -dimensional zero set.*

Proof. If Δ is the closed unit ball in \mathbb{R}^n , then $0 < m\Delta < \infty$ since $[-1/\sqrt{2}, 1/\sqrt{2}]^n \subset [-1, 1]^0 n$. The unit sphere S^{n-1} is the boundary of Δ . It is sandwiched between balls Δ_- of radius $1 - \epsilon$ and Δ_+ of radius $1 + \epsilon$. Corollary 8 implies

$$m(\Delta_-) = (1 - \epsilon)^n m\Delta < m\Delta < (1 + \epsilon)^n m\Delta = m(\Delta_+). \quad (4)$$

Measurability implies that $m(\Delta_+ \setminus \Delta_-) = m(\Delta_+) - m(\Delta_-) = ((1 + \epsilon)^n - (1 - \epsilon)^n) m\Delta$. This gives us

$$m(S^{n-1}) \leq ((1 + \epsilon)^n - (1 - \epsilon)^n) m\Delta = 2 \left(\sum_{i=0}^n \binom{n}{i} \epsilon^{n-i} \right) m\Delta. \quad (5)$$

Since $\epsilon > 0$ is arbitrary, we get $m(S^{n-1}) = 0$. \square

Lemma 2. *Every open cube is a countable disjoint union of open balls plus a zero set.*

Proof. Let $S \subset \mathbb{R}^n$ be an open cube. It contains a compact ball Δ whose volume is greater than $1/2^n$ of the volume of the cube. This follows from

$$\frac{m(\Delta)}{m(S)} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} > \frac{1}{2^n}. \quad (6)$$

The difference $U_1 = S \setminus \Delta$ is an open subset of S with $m(U_1) < m(S)((2^n - 1)/2^n)$. It is therefore the disjoint countable union of small open cubes S_i plus a zero set. Each cube contains a ball whose volume is greater than $1/2^n$ of the volume of each cube, and so the total volume of the small balls are more than $1/2^n$ the volume of the

small cubes. So we get that the difference is U_2 whose total volume is less than $m(U_1)((2^n - 1)/2^n) = ((2^n - 1)^2/2^{2n})$.

Repeating this process we get

$$m(U_k) = \frac{(2^n - 1)^k}{2^{kn}} \implies \ln(m(U_k)) = \ln((2^n - 1)^k) - \ln(2^{kn}) = k(\ln(2^n - 1)) - n \ln(2) \rightarrow 0$$

since $\ln(2^n - 1) \rightarrow n \ln(2)$. In other words, repetition gives smaller and smaller compact balls with total measure equal to $m(S)$. Lemma 10 implies that the measure of a closed ball is the same as the measure of its interior, which completes the proof that S consists of countably many disjoint open cubes plus a zero set. \square

Theorem 5. *An affine motion $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a meseomorphism. It multiplies measure by $|\det T|$.*

Proof. Assume that $Tv = Mv$ where M is an invertible matrix. We first claim that if Z is a zero set then so is TZ . Given any $\epsilon > 0$ there is a countable covering of Z by boxes R_k with total volume $< \epsilon$. Each R_k can be covered by cubes with total volume $m(R_k) + \epsilon/2^k$. Hence Z can be covered by countably many cubes S_i with volume 2ϵ . The T image of each S_i is contained in a cube with edge length $\|T\| \text{diam} S_i$. This finally gives, TZ contained by cubes whose total volume is

$$\sum (\|T\| \text{diam} S_i)^n = \sum n^{n/2} \|T\|^n |S_i| \leq 2n^{n/2} \|T\|^2 \epsilon. \quad (7)$$

Since $\epsilon > 0$ is as small as we like, we have $m(TZ) = 0$.

Next we claim that orthogonal transformations are meseometries. Let $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be orthogonal. It carries the ball $B(r, p)$, to the ball $B(r, Op)$, which is a translate of $B(r, p)$. Let S be a cube. The previous lemma implies that $S = \bigsqcup B_i \cup Z$ where B_i are n -balls and Z is a zero set. The O -image of B_i is a ball of equal measure, and the O -image of Z is a zero set. Hence, $m(OS) = mS$. Given $\epsilon > 0$, there is a countable covering of A by cubes S_i with $\sum |S_i| < m^*A + \epsilon$. Thus $\{O(S_i)\}$ covers OA and has total area $< m^*A + \epsilon$. We therefore get

$$m^*(OA) \leq m^*A. \quad (8)$$

Since O^{-1} is also orthogonal, it too does not increase outer measure. Theorem 7 implies that O is a meseometry.

Finally, we use Polar Form to write

$$M = O_1 D O_2 \quad (9)$$

where O_1, O_2 are orthogonal and D is diagonal. Since O_1 and O_2 are meseometries and by Corollary 8 D is a meseomorphism which multiplies measure by $|\det D| = |\det T|$, the proof is complete. \square

5. Interesting general stuff for \mathbb{R}^n !

Theorem 6. *Every closed set in \mathbb{R}^n is a G_δ set, furthermore every open set is a F_σ set.*

Proof. Take $S \subset N$ to be some closed set. Then for every $n \in \mathbb{N}$ let

$$O_n = \bigcup_{x \in S} B\left(x, \frac{1}{n}\right), \quad (10)$$

where $B(p, r)$, is the open ball of radius r at p . Then clearly

$$\bigcap_{n=1}^{\infty} O_n = S, \quad (11)$$

and S is a G_δ set. Let Y be some open set in N . Then Y^c is closed and therefore is an G_δ set. That is, there exist some open family $\{O_n\}$ so that

$$Y^c = \bigcap_{n=1}^{\infty} O_n \implies Y^{cc} = \bigcup_{n=1}^{\infty} O_n^c \quad (12)$$

and Y is an F_σ set. □

7. *Prove that inner measure is translation invariant.* Observe that translation, $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine motion with $|\det T| = 1$. This can be seen from linear algebra class using an augmented matrix! Furthermore all dilations are affine motions so we propose the following theorems warranting that

Theorem 7. *A set E is measurable if and only if $m^*E = m_*E$.*

which will be shown in Question 9.

Lemma 3. *The boundary of an n -dimensional ball is an n -dimensional zero inner measure set.*

Proof. If the outer measure of a set is 0, then the inner measure must be 0. □

Lemma 4. *Every open cube is a countable disjoint union of open balls plus a zero inner measure set.*

Proof. If it is true for a zero outer measure set, then it must be that the inner measure of such a set is a zero set. Furthermore, a ball is measurable so inner measure is outer measure. □

Theorem 8. *An affine motion $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a inner mesomorphism. It multiplies inner measure by $|\det T|$.*

Proof. Assume that $Tv = Mv$ where M is an invertible matrix. We first claim that if Z is a zero inner measure set then so is TZ . Given any $\epsilon > 0$ there is a countable covering of Z by boxes R_k with total inner volume $< \epsilon$. Each R_k can be covered by cubes with total volume $m_*(R_k) + \epsilon/2^k$. Hence Z can be covered by countably many cubes S_i with inner volume 2ϵ . The T image of each S_i is contained in a cube with inner edge length $\|T\| \text{diam} S_i$. This finally gives, TZ contained by cubes whose total inner volume is

$$\sum (\|T\| \text{diam} S_i)^n = \sum n^{n/2} \|T\|^n |S_i| \leq 2n^{n/2} \|T\|^2 \epsilon. \quad (13)$$

Since $\epsilon > 0$ is as small as we like, we have $m_*(TZ) = 0$.

Next we claim that orthogonal transformations are inner meseometries. Let $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be orthogonal. It carries the ball $B(r, p)$, to the ball $B(r, Op)$, which is a translate of $B(r, p)$. Let S be a cube. The previous lemma implies that $S = \bigsqcup B_i \cup Z$ where B_i are n -balls and Z is an inner measure zero set. The O -image of B_i is a ball of equal inner measure, and the O -image of Z is an inner measure zero set. Hence, $m_*(OS) = m_*S$. Given $\epsilon > 0$, there is a countable covering of A by cubes S_i with $\sum |S_i| < m_*A + \epsilon$. Thus $\{O(S_i)\}$ covers OA and has total inner volume $< m_*A + \epsilon$. We therefore get

$$m_*(OA) \leq m_*A. \quad (14)$$

Since O^{-1} is also orthogonal, it too does not increase inner measure. Theorem 7 implies that O is an inner meseometry.

Finally, we use Polar Form to write

$$M = O_1 D O_2 \quad (15)$$

where O_1, O_2 are orthogonal and D is diagonal. Since O_1 and O_2 are inner meseometries and by Corrolary 8 D is an inner meseomorphism which multiplies inner measure by $|\det D| = |\det T|$, the proof is complete. \square

Therefore translations are meseometries.

9. The if and only if of measure theory.

Theorem 9. *For some measure space (M, \mathfrak{M}, μ) , we have that $A \subset X$ gives $\mu^*(A) = \mu_*(A)$, then $A \in \mathfrak{M}$.*

Proof. Let μ be some measure on \mathfrak{M} . For any $X \subset M$, we define the outer measure of X induced by μ

$$\mu^*X = \inf \{ \mu(S) : S \in \mathfrak{M} \wedge S \supset X \}. \quad (16)$$

Dually we define the inner measure induced by μ as

$$\mu_*X = \sup \{ \mu(S) : S \in \mathfrak{M} \wedge S \subset X \}. \quad (17)$$

Now suppose that $\mu_*X = \mu^*X$. We claim that $X \in \mathfrak{M}$. The set X is measurable if and only if for every test set $A \subset M$, we get

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap A^c). \quad (18)$$

\square