# Supplements

## A. Vectors in Geometry

### Operations and their properties

The following definition of vectors can be found in elementary geometry textbooks, see for instance [4].

A **directed segment** $\overrightarrow{AB}$ on the plane or in space is specified by an ordered pair of points: the **tail** $A$ and the **head** $B$. Two directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to **represent the same vector** if they are obtained from one another by translation. In other words, the lines $AB$ and $CD$ must be parallel, the lengths $|AB|$ and $|CD|$ must be equal, and the segments must point toward the same of the two possible directions (Figure 10).

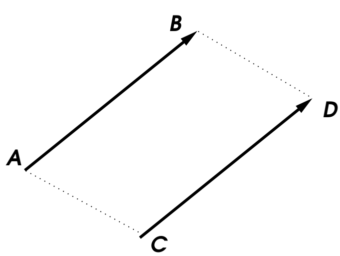



Figure 10

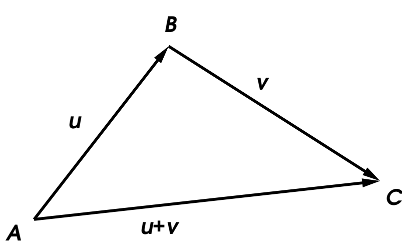



Figure 11

A trip from $A$ to $B$ followed by a trip from $B$ to $C$ results in a trip from $A$ to $C$. This observation motivates the definition of the **vector sum** $\mathbf{w} = \mathbf{v} + \mathbf{u}$ of two vectors $\mathbf{v}$ and $\mathbf{u}$: if $\overrightarrow{AB}$ represents $\mathbf{v}$ and $\overrightarrow{BC}$ represents $\mathbf{u}$ then $\overrightarrow{AC}$ represents their sum $\mathbf{w}$ (Figure 11).

The vector $3\mathbf{v} = \mathbf{v} + \mathbf{v} + \mathbf{v}$ has the same direction as $\mathbf{v}$ but is 3 times longer. Generalizing this example one arrives at the definition of the **multiplication of a vector by a scalar**: given a vector $\mathbf{v}$ and a real number $\alpha$, the result of their multiplication is a vector, denoted $\alpha\mathbf{v}$, which has the same direction as $\mathbf{v}$ but is $\alpha$ times longer. The last phrase calls for comments since it is literally true only for $\alpha > 1$. If $0 < \alpha < 1$, being "$\alpha$ times longer" actually means "shorter." If $\alpha < 0$, the direction of $\alpha\mathbf{v}$ is in fact opposite to the direction of $\mathbf{v}$. Finally, $0\mathbf{v} = \mathbf{0}$ is the **zero vector** represented by directed segments $\overrightarrow{AA}$ of zero length.

Combining the operations of vector addition and multiplication by scalars we can form expressions $\alpha\mathbf{u}+\beta\mathbf{v}+...+\gamma\mathbf{w}$. They are called **linear combinations** of the vectors $\mathbf{u}, \mathbf{v}, ..., \mathbf{w}$ with the coefficients $\alpha, \beta, ..., \gamma$.

The pictures of a parallelogram and parallelepiped (Figures 12 and 13) prove that the addition of vectors is **commutative** and **associative**: for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$,

$$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u} \quad \text{and} \quad (\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}).$$

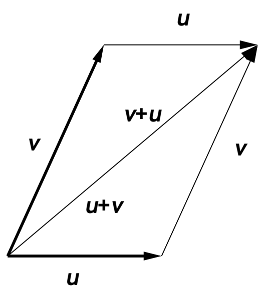



Figure 12

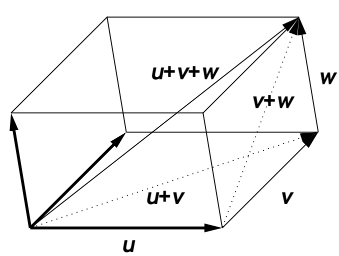



Figure 13

From properties of proportional segments and similar triangles, the reader will easily derive the following two **distributive laws**: for all vectors $\mathbf{u}, \mathbf{v}$ and scalars $\alpha, \beta$,

$$(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u}+\beta\mathbf{u} \quad \text{and} \quad \alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v}.$$

## Coordinates

From a point $O$ in space, draw three directed segments $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ not lying in the same plane and denote by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ the vectors they represent. Then every vector $\mathbf{u} = \overrightarrow{OU}$ can be uniquely written as a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (Figure 14):

$$\mathbf{u} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}.$$

The coefficients form the array $(\alpha, \beta, \gamma)$ of **coordinates** of the vector $\mathbf{u}$ (and of the point $U$) with respect to the **basis** $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (or the **coordinate system** $OABC$).

Multiplying $\mathbf{u}$ by a scalar $\lambda$ or adding another vector $\mathbf{u}' = \alpha'\mathbf{i} + \beta'\mathbf{j} + \gamma'\mathbf{k}$, and using the above algebraic properties of the operations with vectors, we find:

$$\lambda\mathbf{u} = \lambda\alpha\mathbf{i} + \lambda\beta\mathbf{j} + \lambda\gamma\mathbf{k}, \text{ and } \mathbf{u} + \mathbf{u}' = (\alpha + \alpha')\mathbf{i} + (\beta + \beta')\mathbf{j} + (\gamma + \gamma')\mathbf{k}.$$

Thus, the geometric operations with vectors are expressed by componentwise operations with the arrays of their coordinates:

$$\lambda(\alpha, \beta, \gamma) = (\lambda\alpha, \lambda\beta, \lambda\gamma),$$
$$(\alpha, \beta, \gamma) + (\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma').$$

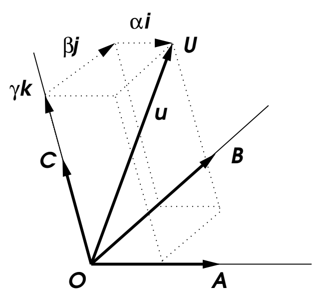



Figure 14

## What is a vector?

No doubt, the idea of vectors is not new to the reader. However, some subtleties of the above introduction do not easily meet the eye, and we would like to say here a few words about them.

As many other mathematical notions, vectors come from physics, where they represent quantities, such as velocities and forces, which are characterized by their magnitude and direction. Yet, the popular slogan "Vectors *are* magnitude and direction" does not qualify for a mathematical definition of vectors, e.g. because it does not tell us how to operate with them.

The computer science definition of vectors as *arrays* of numbers, to be added “apples with apples, oranges with oranges,” will meet the following objection by physicists. When a coordinate system rotates, the coordinates of the *same* force or velocity will change, but the numbers of apples and oranges won’t. Thus forces and velocities *are not* arrays of numbers.

The geometric notion of a directed segment resolves this problem. Note however, that calling directed segments *vectors* would constitute abuse of terminology. Indeed, strictly speaking, directed segments can be added only when the head of one of them coincides with the tail of the other.

So, what is a vector? In our formulations, we actually avoided answering this question directly, and said instead that *two directed segments represent the same vector if . . .* Such wording is due to pedagogical wisdom of the authors of elementary geometry textbooks, because a direct answer sounds quite abstract: *A vector is the class of all directed segments obtained from each other by translation in space.* Such a class is shown in Figure 15.

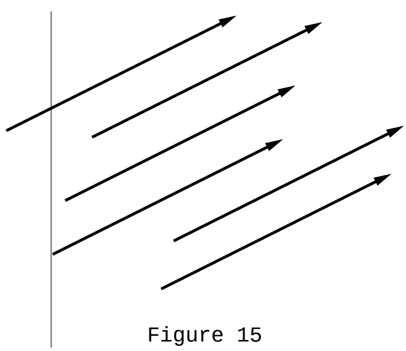


Figure 15

This picture has another interpretation: For every point in space (the tail of an arrow), it indicates a new position (the head). The geometric transformation in space defined this way is translation. This leads to another attractive point of view: a vector *is* a translation. Then the sum of two vectors is the *composition* of the translations.

## The dot product

This operation encodes *metric* concepts of elementary Euclidean geometry, such as lengths and angles. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ of lengths $|\mathbf{u}|$ and $|\mathbf{v}|$ and making the angle $\theta$ to each other, their dot product (also called **inner product** or **scalar product**) is a *number* defined by the formula:

$$\langle \mathbf{u}, \mathbf{v}\rangle = |\mathbf{u}|\ |\mathbf{v}|\ \cos\theta.$$

Of the following properties, the first three are easy (check them!):

(a) $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$ (**symmetricity**);

(b) $\langle \mathbf{u}, \mathbf{u}\rangle = |\mathbf{u}|^2 > 0$ unless $\mathbf{u} = \mathbf{0}$ (**positivity**);

(c) $\langle \lambda\mathbf{u}, \mathbf{v}\rangle = \lambda\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \lambda\mathbf{v}\rangle$ (**homogeneity**);

(d) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$ (**additivity** with respect to the first factor).

To prove the last property, note that due to homogeneity, it suffices to check it assuming that $\mathbf{w}$ is a **unit vector**, i.e. $|\mathbf{w}| = 1$. In this case, consider (Figure 16) a triangle $ABC$ such that $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$, and therefore $\overrightarrow{AC} = \mathbf{u}+\mathbf{v}$, and let $\overrightarrow{OW} = \mathbf{w}$. We can consider the line $OW$ as the number line, with the points $O$ and $W$ representing the numbers 0 and 1 respectively, and denote by $\alpha, \beta, \gamma$ the *numbers* representing perpendicular projections to this line of the vertices $A, B, C$ of the triangle. Then

$$\langle \overrightarrow{AB}, \mathbf{w}\rangle = \beta - \alpha,\ \langle \overrightarrow{BC}, \mathbf{w}\rangle = \gamma - \beta,\ \text{and} \langle \overrightarrow{AC}, \mathbf{w}\rangle = \gamma - \alpha.$$

The required identity follows, because $\gamma - \alpha = (\gamma - \beta) + (\beta - \alpha)$.

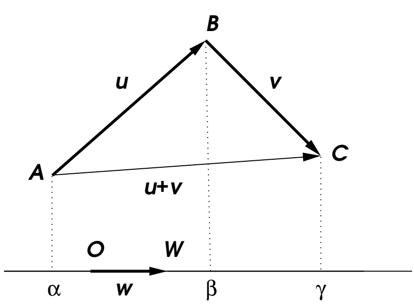



Figure 16

Combining the properties (c) and (d) with (a), we obtain the following identities, expressing **bilinearity** of the dot product (i.e.

linearity with respect to each factor):

$$\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}\rangle = \alpha\langle \mathbf{u}, \mathbf{w}\rangle + \beta\langle \mathbf{v}, \mathbf{w}\rangle$$

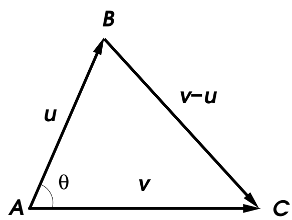
$$\langle \mathbf{w}, \alpha\mathbf{u} + \beta\mathbf{v}\rangle = \alpha\langle \mathbf{w}, \mathbf{u}\rangle + \beta\langle \mathbf{w}, \mathbf{v}\rangle.$$

The following example illustrates the use of nice algebraic properties of dot product in elementary geometry.

`Example.` Given a triangle $ABC$, let us denote by $\mathbf{u}$ and $\mathbf{v}$ the vectors represented by the directed segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and use properties of the inner product in order to compute the length $|BC|$. Notice that the segment $\overrightarrow{BC}$ represents $\mathbf{v} - \mathbf{u}$. We have:

$$\begin{aligned} |BC|^2 = \langle \mathbf{v} - \mathbf{u}, \mathbf{v} - \mathbf{u}\rangle &= \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{u}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle \\ &= |AC|^2 + |AB|^2 - 2|AB|\ |AC|\ \cos\theta. \end{aligned}$$

This is the famous **Law of Cosines** in trigonometry.




`Figure 17`

When the vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal**, i.e. $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, then the formula turns into the **Pythagorean theorem**:

$$|\mathbf{u} \pm \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2.$$

When basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are pairwise orthogonal and *unit*, the coordinate system is called **Cartesian**.[13] We have:

$$\langle \mathbf{i}, \mathbf{i}\rangle = \langle \mathbf{j}, \mathbf{j}\rangle = \langle \mathbf{k}, \mathbf{k}\rangle = 1, \text{ and } \langle \mathbf{i}, \mathbf{j}\rangle = \langle \mathbf{j}, \mathbf{k}\rangle = \langle \mathbf{k}, \mathbf{i}\rangle = 0.$$

Thus, in Cartesian coordinates, the inner squares and the dot products of vectors $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$ are given by the formulas:

$$|\mathbf{r}|^2 = x^2 + y^2 + z^2, \quad \langle \mathbf{r}, \mathbf{r}'\rangle = xx' + yy' + zz'.$$

[13] After René **Descartes** (1596–1650).

***EXERCISES***

***337.*** A mass $m$ rests on an inclined plane making $30°$ with the horizontal plane. Find the forces of friction and reaction acting on the mass. ✓

***338.*** A ferry, capable of making 5 *mph*, shuttles across a river of width 0.6 *mi* with a strong current of 3 *mph*. How long does each round trip take? ✓

***339.*** Prove that for every closed broken line $ABC \dots DE$,

$$\overrightarrow{AB} + \overrightarrow{BC} + \cdots + \overrightarrow{DE} + \overrightarrow{EA} = \mathbf{0}.$$

***340.*** Three medians of a triangle $ABC$ intersect at one point $M$ called the **barycenter** of the triangle. Let $O$ be any point on the plane. Prove that

$$\overrightarrow{OM} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

***341.*** Prove that $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = \mathbf{0}$ if and only if $M$ is the barycenter of the triangle $ABC$.

***342.***$^{\star}$ Along three circles lying in the same plane, the vertices of a triangle are moving clockwise with the equal constant angular velocities. Find how the barycenter of the triangle is moving. ✓

***343.*** Prove that if $AA'$ is a median in a triangle $ABC$, then

$$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$

***344.*** Prove that from medians of a triangle, another triangle can be formed. ϟ

***345.*** Sides of one triangle are parallel to the medians of another. Prove that the medians of the latter triangle are parallel to the sides of the former one.

***346.*** From medians of a given triangle, a new triangle is formed, and from its medians, yet another triangle is formed. Prove that the third triangle is similar to the first one, and find the coefficient of similarity. ✓

***347.*** Midpoints of $AB$ and $CD$, and of $BC$ and $DE$ are connected by two segments, whose midpoints are also connected. Prove that the resulting segment is parallel to $AE$ and congruent to $AE/4$.

***348.*** Prove that a point $X$ lies on the segment $AB$ if and only if for any origin $O$ and some scalar $0 \le \lambda \le 1$ the radius-vector $\overrightarrow{OX}$ has the form:

$$\overrightarrow{OX} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}.$$

***349.***$^{\star}$ Given a triangle $ABC$, we construct a new triangle $A'B'C'$ in such a way that $A'$ is centrally symmetric to $A$ with respect to the center $B$, $B'$ centrally symmetric to $B$ with respect to $C$, and $C'$ centrally symmetric to $C$ with respect to $A$, and then erase the original triangle. Reconstruct $ABC$ from $A'B'C'$ by straightedge and compass. ✓

***350.*** Prove the **Cauchy – Schwarz inequality**: $\langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle$. In which cases does the inequality turn into equality? Deduce the **triangle inequality**: $|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$. ↯

***351.*** Compute the inner product $\langle \overrightarrow{AB}, \overrightarrow{BC}\rangle$ if $ABC$ is a regular triangle inscribed into a unit circle. ✓

***352.*** Prove that if the sum of three unit vectors is equal to $\mathbf{0}$, then the angle between each pair of these vectors is equal to $120°$.

***353.*** Express the inner product $\langle \mathbf{u}, \mathbf{v}\rangle$ in terms of the lengths $|\mathbf{u}|, |\mathbf{v}|, |\mathbf{u}+\mathbf{v}|$ of the two vectors and of their sum. ✓

***354.*** (a) Prove that if four unit vectors lying in the same plane add up to $\mathbf{0}$, then they form two pairs of opposite vectors. (b) Does this remain true if the vectors do not have to lie in the same plane? ✓

***355.***⋆ Let $AB\dots E$ be a regular polygon with the center $O$. Prove that

$$\overrightarrow{OA} + \overrightarrow{OB} + \cdots + \overrightarrow{OE} = \mathbf{0}. \qquad ↯$$

***356.*** Prove that if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are perpendicular, then $|\mathbf{u}| = |\mathbf{v}|$.

***357.*** For arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$, verify the equality:

$$|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2,$$

and derive the theorem: The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

***358.*** Prove that for every triangle $ABC$ and every point $X$ in space,

$$\overrightarrow{XA}\cdot\overrightarrow{BC} + \overrightarrow{XB}\cdot\overrightarrow{CA} + \overrightarrow{XC}\cdot\overrightarrow{AB} = 0. \qquad ↯$$

***359.***⋆ For four arbitrary points $A$, $B$, $C$, and $D$ in space, prove that if the lines $AC$ and $BD$ are perpendicular, then $AB^2 + CD^2 = BC^2 + DA^2$, and *vice versa*. ↯

***360.***⋆ Given a quadrilateral with perpendicular diagonals. Show that every quadrilateral, whose sides are respectively congruent to the sides of the given one, has perpendicular diagonals. ↯

***361.***⋆ A regular triangle $ABC$ is inscribed into a circle of radius $R$. Prove that for every point $X$ of this circle, $XA^2 + XB^2 + XC^2 = 6R^2$. ↯

***362.***⋆ Let $A_1B_1A_2B_2\dots A_nB_n$ be a $2n$-gon inscribed into a circle. Prove that the length of the vector $\overrightarrow{A_1B_1} + \overrightarrow{A_2B_2} + \cdots + \overrightarrow{A_nB_n}$ does not exceed the diameter. ↯

***363.***⋆ A polyhedron is filled with air under pressure. The pressure force to each face is the vector perpendicular to the face, proportional to the area of the face, and directed to the exterior of the polyhedron. Prove that the sum of these vectors is equal to $\mathbf{0}$. ↯

# B. Complex Numbers

## Law and Order

Life is unfair: The quadratic equation $x^2 - 1 = 0$ has two solutions $x = \pm 1$, but a similar equation $x^2 + 1 = 0$ has no solutions at all. To restore justice one introduces new number $i$, the **imaginary unit**, such that $i^2 = -1$, and thus $x = \pm i$ become two solutions to the equation. This is how complex numbers could have been invented.

More formally, complex numbers are introduced as ordered pairs $(a, b)$ of real numbers, written in the form $z = a + bi$. The real numbers $a$ and $b$ are called respectively the **real part** and **imaginary part** of the complex number $z$, and are denoted $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$.

The sum of $z = a + bi$ and $w = c + di$ is defined as

$$z + w = (a + c) + (b + d)i.$$

The product is defined so as to comply with the relation $i^2 = -1$:

$$zw = ac + bdi^2 + adi + bci = (ac - bd) + (ad + bc)i.$$

The operations of addition and multiplication of complex numbers enjoy the same properties as those of real numbers do. In particular, the product is commutative and associative.

The complex number $\bar{z} = a - bi$ is called **complex conjugate** to $z = a + bi$. The operation of complex conjugation *respects* sums and products:

$$\overline{z + w} = \bar{z} + \bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

This can be easily checked from definitions, but there is a more profound explanation. The equation $x^2 + 1 = 0$ has two roots, $i$ and $-i$, and the choice of the one to be called $i$ is totally ambiguous. The complex conjugation consists in systematic renaming $i$ by $-i$ and *vice versa*, and such renaming cannot affect properties of complex numbers.

Complex numbers satisfying $\bar{z} = z$ are exactly the real numbers $a + 0i$. We will see that this point of view on real numbers as complex numbers *invariant* under complex conjugation is quite fruitful.

The product $z\bar{z} = a^2 + b^2$ (check this formula!) is real, and is positive unless $z = 0 + 0i = 0$. This shows that

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

Hence the division by $z$ is well-defined for any non-zero complex number $z$. In terminology of Abstract Algebra, complex numbers form therefore a **field**[14] (just as real or rational numbers do).

The field of complex numbers is denoted by $\mathbb{C}$ (while $\mathbb{R}$ stands for reals, and $\mathbb{Q}$ for rationals).

The non-negative real number $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$ is called the **absolute value** of $z$. The absolute value function is **multiplicative**:

$$|zw| = \sqrt{zw\overline{zw}} = \sqrt{z\bar{z}w\bar{w}} = |z| \cdot |w|.$$

It actually coincides with the absolute value of real numbers when applied to complex numbers with zero imaginary part: $|a + 0i| = |a|$.

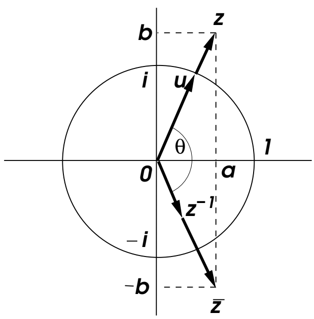



Figure 18

## Geometry

We can identify complex numbers $z = a + bi$ with points $(a, b)$ on the real coordinate plane (Figure 18). This way, the number 0 is identified with the origin, and 1 and $i$ become the unit basis vectors $(1, 0)$ and $(0, 1)$. The coordinate axes are called respectively the real and imaginary axes. Addition of complex numbers coincides with the operation of vector sum (Figure 19).

The absolute value function has the geometrical meaning of the distance from the origin: $|z| = \langle z, z\rangle^{1/2}$. In particular, the triangle inequality $|z + w| \leq |z| + |w|$ holds true. Complex numbers of unit absolute value $|z| = 1$ form the unit circle centered at the origin.

The operation of complex conjugation acts on the radius-vectors $z$ as the reflection about the real axis.

[14] This requires that a set be equipped with commutative and associative operations (called addition and multiplication) satisfying the **distributive law** $z(v + w) = zv + zw$, possessing the zero and unit elements 0 and 1, additive opposites $-z$ for every $z$, and multiplicative inverses $1/z$ *for every* $z \neq 0$.

In order to describe a geometric meaning of complex multiplication, let us study the way multiplication by a given complex number $z$ acts on all complex numbers $w$, i.e. consider the function $w \mapsto zw$. For this, write the vector representing a non-zero complex number $z$ in the **polar** (or trigonometric) form $z = ru$ where $r = |z|$ is a positive real number, and $u = z/|z| = \mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta$ has absolute value 1 (see Figure 20). Here $\theta = \mathbf{arg}\,z$, called the **argument** of the complex number $z$, is the angle that $z$ as a vector makes with the positive direction of the real axis.

Clearly, multiplication by $r$ acts on all vectors $w$ by stretching them $r$ times. Multiplication by $u$ applied to $w = x + yi$ yields a new complex number $uw = X + Yi$ according to the rule:

$$X = \mathrm{Re}\,[(\mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta)(x + yi)] = x\,\mathbf{cos}\,\theta - y\,\mathbf{sin}\,\theta$$
$$Y = \mathrm{Im}\,[(\mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta)(x + yi)] = x\,\mathbf{sin}\,\theta + y\,\mathbf{cos}\,\theta.$$

Comparing with the formula $(*)$ in Section 2, we conclude that the transformation $w \mapsto uw$ is the counter-clockwise rotation through the angle $\theta$.

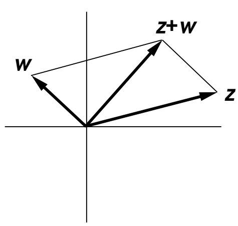



Figure 19

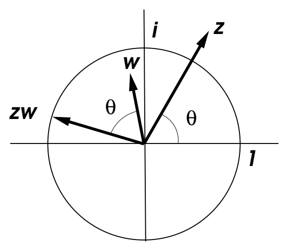



Figure 20

Notice a difference though: In Section 2, we rotated the coordinate system, and the formulas $(*)$ expressed old coordinates of a vector via new coordinates of the *same* vector. This time, we transform vectors, while the coordinate system remains unchanged. The same formulas now express coordinates $(X, Y)$ of a new vector in terms of the coordinates $(x, y)$ of a the old one.

Anyway, the conclusion is that multiplication by $z$ is the composition of two operations: stretching $|z|$ times, and rotating through the angle $\mathbf{arg}\,z$.

In other words, the product operation of two complex numbers sums their arguments and multiplies absolute values:

$$|zw| = |z| \cdot |w|, \quad \mathbf{arg}\,zw = \mathbf{arg}\,z + \mathbf{arg}\,w \text{ modulo } 2\pi.$$

For example, if $z = r(\mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta)$, then $z^n = r^n(\mathbf{cos}\,n\theta + i\,\mathbf{sin}\,n\theta)$.

## The Fundamental Theorem of Algebra

A degree 2 polynomial $z^2 + pz + q$ has two roots

$$z_{\pm} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

This **quadratic formula** works regardless of the sign of the **discriminant** $p^2 - 4q$, provided that we allow the roots to be complex, and take in account **multiplicity**. Namely, if $p^2 - 4q = 0$, $z^2 + pz + q = (z + p/2)^2$ and therefore the single root $z = -p/2$ has multiplicity two. If $p^2 - 4q < 0$ the roots are complex conjugate with $\operatorname{Re} z_{\pm} = -p/2$, $\operatorname{Im} z_{\pm} = \pm\sqrt{|p^2 - 4q|}/2$. The Fundamental Theorem of Algebra shows that not only justice has been restored, but that any degree $n$ polynomial has $n$ complex roots, possibly — multiple.

`Theorem.` ***A degree*** $n$ ***polynomial***

$$P(z) = z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n$$

***with complex coefficients*** $a_1, ..., a_n$ ***factors as***

$$P(z) = (z - z_1)^{m_1} ... (z - z_r)^{m_r}.$$

***Here*** $z_1, ..., z_r$ ***are complex roots of*** $P$***, and*** $m_1, ..., m_r$ ***their multiplicities,*** $m_1 + \cdots + m_r = n$***.***

A proof of this theorem deserves a separate chapter (if not a book). Many proofs are known, based on various ideas of Algebra, Analysis or Topology. We refer to [6] for an exposition of the classical proof due to Euler, Lagrange and de Foncenex, which is almost entirely algebraic. Here we merely illustrate the theorem with several examples.

`Examples.` (a) To solve the quadratic equation $z^2 = w$, equate the absolute value $r$ and argument $\theta$ of the given complex number $w$ with those of $z^2$:

$$|z|^2 = \rho, \quad 2\,\mathbf{arg}\, z = \phi + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

We find: $|z| = \sqrt{\rho}$, and $\mathbf{arg}\, z = \phi/2 + \pi k$. Increasing $\mathbf{arg}\, z$ by even multiples $\pi$ does not change $z$, and by odd changes $z$ to $-z$. Thus the equation has two solutions:

$$z = \pm\sqrt{\rho}\left(\mathbf{cos}\,\frac{\phi}{2} + i\,\mathbf{sin}\,\frac{\phi}{2}\right).$$

(b) The equation $z^2 + pz + q = 0$ with coefficients $p, q \in \mathbb{C}$ has two complex solutions given by the quadratic formula (see above), because according to Example (a), the **square root** of a complex number takes on two opposite values (distinct, unless both are equal to 0).

(c) The complex numbers $1, i, -1, -i$ are the roots of the polynomial $z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z - i)(z + i)$.

(d) There are $n$ complex $n$**th roots of unity** (see Figure 21, where $n = 5$). Namely, if $z = r(\mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta)$ satisfies $z^n = 1$, then $r^n = 1$ (and hence $r = 1$), and $n\theta = 2\pi k,\ k = 0, \pm 1, \pm 2, ....$ Therefore $\theta = 2\pi k/n$, where only the remainder of $k$ modulo $n$ is relevant. Thus the $n$ roots are:

$$z = \mathbf{cos}\,\frac{2\pi k}{n} + i\,\mathbf{sin}\,\frac{2\pi k}{n},\ \ k = 0, 1, 2, ..., n - 1.$$

For instance, if $n = 3$, the roots are 1 and

$$\mathbf{cos}\,\frac{2\pi}{3} \pm i\,\mathbf{sin}\,\frac{2\pi}{3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

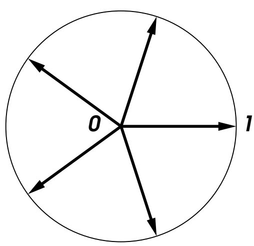



Figure 21

As illustrated by the previous two examples, even if all coefficients $a_1, \ldots, a_n$ of a polynomial $P$ are real, its roots don't have to be real. But then the non-real roots come in pairs of complex conjugate ones. To verify this, we can use the fact that being real means stay *invariant* (i.e. unchanged) under complex conjugation. Namely, $\bar{a}_i = a_i$ for all $i$ means that

$$\overline{P(\bar{z})} = z^n + \bar{a}_1 z^{n-1} + ... + \bar{a}_n = P(z).$$

Therefore we have two factorizations of the same polynomial:

$$\bar{P}(\bar{z}) = (z - \bar{z}_1)^{m_1} ... (z - \bar{z}_r)^{m_r} = (z - z_1)^{m_1} ... (z - z_r)^{m_r} = P(z).$$

They can differ only by orders of the factors. Thus, for each non-real root $z_i$ of $P$, the complex conjugate $\bar{z}_i$ must be also a root, and of the same multiplicity.

Expanding the product

$$(z - z_1)...(z - z_n) = z^n - (z_1 + ... + z_n)z^{n-1} + ... + (-1)^n z_1...z_n$$

we can express coefficients $a_1, ..., a_n$ of the polynomial in terms of the roots $z_1, ..., z_n$ (here multiple roots are repeated according to their multiplicities). In particular, the sum and product of the roots are

$$z_1 + ... + z_n = -a_1, \quad z_1...z_n = (-1)^n a_n.$$

These relations generalize **Vieta's theorem** $z_+ + z_- = -p$, $z_+ z_- = q$ about roots $z_\pm$ of quadratic equations $z^2 + pz + q = 0$.

## The Exponential Function

Consider the series

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ... + \frac{z^n}{n!} + ...$$

Applying the ratio test for convergence of infinite series,

$$\left| \frac{z^n (n-1)!}{n! z^{n-1}} \right| = \frac{|z|}{n} \to 0 < 1 \text{ as } n \to \infty,$$

we conclude that the series converges absolutely for any complex number $z$. The sum of the series is a complex number denoted $\exp z$ and the rule $z \mapsto \exp z$ defines the **exponential function** of the complex variable $z$.

The exponential function transforms sums to products:

$$\exp(z + w) = (\exp z)(\exp w) \text{ for any complex } z \text{ and } w.$$

Indeed, due to the binomial formula, we have

$$(z + w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = n! \sum_{k+l=n} \frac{z^k}{k!} \frac{w^l}{l!}.$$

Rearranging the sum over all $n$ as a double sum over $k$ and $l$ we get

$$\sum_{n=0}^{\infty} \frac{(z + w)^n}{n!} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{z^k}{k!} \frac{w^l}{l!} = (\sum_{k=0}^{\infty} \frac{z^k}{k!})(\sum_{l=0}^{\infty} \frac{w^l}{l!}).$$

The exponentials of complex conjugated numbers are conjugated:

$$\exp \bar{z} = \sum \frac{\bar{z}^n}{n!} = \overline{\sum \frac{z^n}{n!}} = \overline{\exp z}.$$

In particular, on the real axis the exponential function is real and coincides with the usual real exponential function $\exp x = e^x$ where $e = 1+1/2+1/6+...+1/n!+... = \exp(1)$. Extending this notation to complex numbers we can rewrite the above properties of $e^z = \exp z$ as $e^{z+w} = e^z e^w$, $e^{\bar{z}} = \overline{e^z}$.

On the imaginary axis, $w = e^{iy}$ satisfies $w\bar{w} = e^0 = 1$ and hence $|e^{iy}| = 1$. The way the imaginary axis is mapped by the exponential function to the unit circle is described by the following **Euler's formula**:

$$e^{i\theta} = \mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta.$$

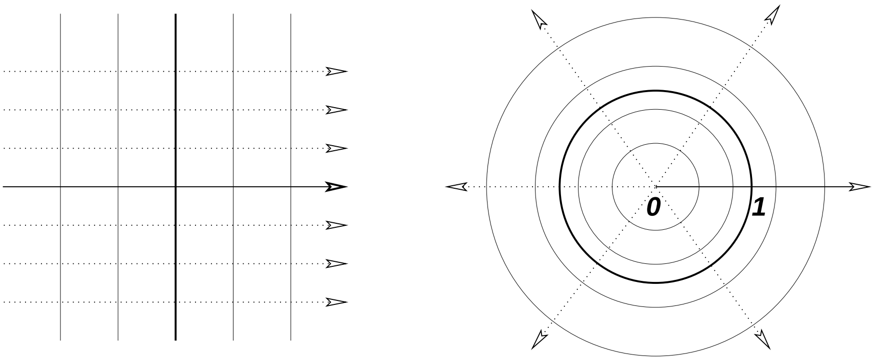



Figure S2–5 $z \longmapsto w = \exp z$

It is proved by comparison of $e^{i\theta} = \sum (i\theta)^n/n!$ with Taylor series for $\mathbf{cos}\,\theta$ and $\mathbf{sin}\,\theta$:

$$\begin{array}{l} \mathrm{Re}\, e^{i\theta} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - ... = \sum (-1)^k \frac{\theta^{2k}}{(2k)!} = \mathbf{cos}\,\theta \\ \mathrm{Im}\, e^{i\theta} = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - ... = \sum (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} = \mathbf{sin}\,\theta \end{array}.$$

Thus $\theta \mapsto e^{i\theta}$ is the usual parameterization of the unit circle by the angular coordinate $\theta$. In particular, $e^{2\pi i} = 1$ and therefore the exponential function is $2\pi i$-periodic: $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$. Using Euler's formula we can rewrite the polar form of a non-zero complex number $w$ as

$$w = |w| e^{i\,\mathbf{arg}\,w}.$$

### *EXERCISES*

***364.*** Can complex numbers be: real? real *and* imaginary? neither? ✓

***365.*** Compute: (a) $(1+i)/(3-2i)$; (b) $(\mathbf{cos}\,\pi/3 + i\,\mathbf{sin}\,\pi/3)^{-1}$. ✓

***366.*** Verify the commutative and distributive laws for multiplication of complex numbers.

***367.*** Show that $z^{-1}$ is real proportional to $\bar{z}$ and find the proportionality coefficient. ✓

***368.*** Find all $z$ satisfying the equations: $|z-1| = |z+1| = 2$. ✓

***369.*** Sketch the solution set to the following system of inequalities:

$$|z-1| \leq 1, \quad |z| \leq 1, \quad \mathrm{Re}(iz) \leq 0.$$

***370.*** Compute absolute values and arguments of (a) $1-i$, (b) $1-i\sqrt{3}$. ✓

***371.*** Compute $\left(\frac{\sqrt{3}+i}{2}\right)^{100}$. ✓

***372.*** Express $\mathbf{cos}\,3\theta$ and $\mathbf{sin}\,3\theta$ in terms of $\mathbf{cos}\,\theta$ and $\mathbf{sin}\,\theta$. ϟ

***373.*** Express $\mathbf{cos}(\theta_1+\theta_2)$ and $\mathbf{sin}(\theta_1+\theta_2)$ in terms of $\mathbf{cos}\,\theta_i$ and $\mathbf{sin}\,\theta_i$.

***374.*** Prove **Bézout's theorem**[15]: A number $z_0$ is a root of a polynomial $P$ in one variable $z$ if and only if $P$ is divisible by $z - z_0$. ϟ

***375.*** Find roots of degree 2 polynomials:

$$z^2-4z+5, \quad z^2-iz+1, \quad z^2-2(1+i)z+2i, \quad z^2-2z+i\sqrt{3}. \qquad \checkmark$$

***376.*** Find all roots of polynomials:

$$z^3+8, \quad z^3+i, \quad z^4+4z^2+4, \quad z^4-2z^2+4, \quad z^6+1. \qquad \checkmark$$

***377.*** Prove that every polynomial with real coefficients factors into the product of polynomials of degree 1 and 2 with real coefficients. ϟ

***378.*** Prove that the sum of all 5th roots of unity is equal to 0. ϟ

***379.***$^\star$ Find general **Vieta's formulas** [16] expressing all coefficients of a polynomial in terms of its roots. ✓

***380.*** Prove the **Fundamental Formula of Mathematics** $e^{\pi i} + 1 = 0$ (which is so nicknamed, because it unifies the equality relation, and the operations of addition, multiplication, exponentiation with the famous numbers 0, 1, $e$, $\pi$, and $i$).

***381.*** Represent $1-i$ and $1-i\sqrt{3}$ in the polar form $re^{i\theta}$.

***382.*** Express $\mathbf{cos}\,\theta$ and $\mathbf{sin}\,\theta$ in terms of the complex exponential function. ϟ

[15] Named after Étienne **Bézout** (1730–1783).
[16] Named after François Viéte (1540–1603) also known as Franciscus **Vieta**.

***383.***$^\star$ Show that the complex exponential function $w = \exp z$ maps the $z$-plane onto the entire $w$-plane except the origin $w = 0$, and describe geometrically the images on the $w$-plane of the lines of the coordinate grid of the $z$-plane. ↯

***384.*** Describe the image of the region $0 < \operatorname{Im} z < \pi$ under the exponential function $w = \exp z$. ✓

***385.*** Compute the real and imaginary parts of the product $e^{i\phi}e^{i\psi}$, using Euler's formula, and deduce (once again) the addition formulas for $\mathbf{cos}(\phi + \psi)$ and $\mathbf{sin}(\phi + \psi)$.

***386.*** Express the real and imaginary parts of $e^{3i\theta}$ in terms of $\mathbf{cos}\,\theta$ and $\mathbf{sin}\,\theta$, and deduce the triple argument formula for $\mathbf{cos}\,3\theta$ and $\mathbf{sin}\,3\theta$. ✓

***387.***$^\star$ Prove the binomial formula $(z + w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$, where $\binom{n}{k} = n!/k!(n-k)!$. ↯