Math 110 — Homework 7 — UCB, Summer 2017 — William Guss

(7.1) Let $p, q \in C^\infty([-1,1])$ be real-valued with $p(-1) = 0 = p(1)$, and define $T : C^\infty([-1,1]) \to C^\infty([-1,1])$ by

$$[T(f)](x) := -\frac{d}{dx}\left[p(x)\frac{d}{dx}f(x)\right] - q(x)f(x). \tag{0.1}$$

Show that the eigenvectors of T with distinct eigenvalues are orthogonal.

Proof. For distinct eigenvalues λ_1 and λ_2 take f_1, f_2 to be some respective eigenvectors. Recall that $\langle f_1 | f_2\rangle = 0$ iff $\alpha\langle f_1|f_2\rangle = 0$ for $\alpha \neq 0$. Then

$$\begin{aligned}
\overline{f_1(x)}f_2(x)\lambda_2 - f_2(x)\overline{f_1(x)\lambda_1} &= -\overline{f_1(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}f_2(x)\right] - \overline{f_1(x)}q(x)f_2(x) \\
&\quad + f_2(x)\frac{d}{dx}\left[p(x)\frac{d}{dx}\overline{f_1(x)}\right] + \overline{f_1(x)}q(x)f_2(x) \\
&= -\overline{f_1(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}f_2(x)\right] + f_2(x)\frac{d}{dx}\left[p(x)\frac{d}{dx}\overline{f_1(x)}\right]
\end{aligned}$$

Applying the Liebiniz rule we get

$$\begin{aligned}
(\lambda_2 - \lambda_1)\overline{f_1(x)}f_2(x) &= -\overline{f_1(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}f_2(x)\right] + f_2(x)\frac{d}{dx}\left[p(x)\frac{d}{dx}\overline{f_1(x)}\right] \\
&\quad + \left[p(x)\frac{df_2}{dx}\frac{d\overline{f_1}}{dx} - p(x)\frac{df_2}{dx}\frac{d\overline{f_1}}{dx}\right] \\
&= -\overline{f_1(x)}\frac{d}{dx}\left[p(x)\frac{d}{dx}f_2(x)\right] - p(x)\frac{df_2}{dx}\frac{d\overline{f_1}}{dx} \\
&\quad + f_2(x)\frac{d}{dx}\left[p(x)\frac{d}{dx}\overline{f_1(x)}\right] + p(x)\frac{df_2}{dx}\frac{d\overline{f_1}}{dx} \\
&= -\frac{d}{dx}p(x)\left[\overline{f_1(x)}\frac{df_2}{dx}\right] + \frac{d}{dx}p(x)\left[f_2(x)\frac{d\overline{f_1}}{dx}\right] \\
&= \frac{d}{dx}\left[p(x)\left(f_2(x)\frac{df_1}{dx} - \overline{f_1(x)}\frac{df_2}{dx}\right)\right].
\end{aligned}$$

We now can compute the inner product via integration, and yielf

$$\begin{aligned}
(\lambda_2 - \lambda_1)\langle f_1|f_2\rangle &= (\lambda_2 - \lambda_1)\int_{-1}^{1}\overline{f_1(x)}f_2(x)\,dx \\
&= \left[p(x)\left(f_2(x)\frac{df_1}{dx} - \overline{f_1(x)}\frac{df_2}{dx}\right)\right]_{-1}^{1} \\
&= p(1)\left[f_2(x)\frac{df_1}{dx} - \overline{f_1(x)}\frac{df_2}{dx}\right]_{x=1} - p(0)\left[f_2(x)\frac{df_1}{dx} - \overline{f_1(x)}\frac{df_2}{dx}\right]_{x=0} \\
&= 0 - 0
\end{aligned}$$

Since λ_1 and λ_2 are distinct, we have $\langle f_1|f_2\rangle = 0$ and therefore the eigenvectors of disctinct eigenvalues of T are orthogonal. $\square$

(7.2) Show that $\{\frac{1}{\sqrt{2\pi}e^{inx}}\}$ is orthonormal in $C^\infty((-\pi,\pi))$.

Proof. Take $n \neq m \in \mathbb{Z}$ then

$$\begin{aligned}
\langle f_n, f_m\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{imx}\ dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(m-n)ix}\ dx = \frac{1}{2\pi(m-n)i}e^{(m-n)ix}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi ki}\left[e^{ik\pi} - e^{-ik\pi}\right] = \frac{1}{2\pi ki}\left[e^{ik\pi} - e^{ik\pi}\right] = 0
\end{aligned}$$

since $k := m-n$ is an non-zero integer, and whence the imaginary part of e^{ikx} is zero. Next, fix m and then

$$\|f_m\|^2 = \langle f_m, f_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imx}e^{imx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = \frac{2\pi}{2\pi} = 1.$$

Thus the basis is orthonormal. $\square$

(7.3) Let V be a complex vector space and let $\|\cdot\| : V \to \mathbb{R}_0^+$ be a norm on V. Show that $\|v\|^2 = \langle v|v\rangle$ for some inner-product $\langle\cdot|\cdot\rangle : V\times V \to \mathbb{C}$ if and only if $\|\cdot\|$ satisfies the Parallelogram Law.

Proof. Suppose that $\|v\|^2 = \langle v|v\rangle$ for some inner product $V \times V \to \mathbb{C}$. We calculate as follows

$$\begin{aligned}
\|v+w\|^2 + \|v-w\|^2 &= \langle v+w, v+w\rangle + \langle v-w, v-w\rangle \\
&= \langle v,v\rangle + \langle v,w\rangle + \langle ,v\rangle + \langle w,w\rangle \\
&\quad \langle v,v\rangle - \langle v,w\rangle - \langle w,v\rangle + \langle w,w\rangle \\
&= 2(\|v\|^2 + \|w\|^2).
\end{aligned}$$

where $\|x\|^2 = \langle w,w\rangle$ is applied in the first step. Therefore if V has an inner product, the induced norm satisifes the Parallelogram Law.

In the other direction, suppose that V has a norm which satisfies the parallelogran law. We claim that the following is an inner product whose square is the norm.

$$\langle v|w\rangle := \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 - i\|v+iw\|^2 + i\|v-iw\|^2\right)$$

First, we evaluate the inner product restricted to the diagonal of $V\times V$. Take $v \in V$ then,

$$\begin{aligned}
\langle v|v\rangle &= \frac{1}{4}\left(\|v+v\|^2 - \|v-v\|^2 - i\|v+iv\|^2 + i\|v-iv\|^2\right) \\
&= \frac{1}{4}\left(\|v+v\|^2 - i\|v+iv\|^2 + i\|v-iv\|^2\right) \\
&= \frac{1}{4}\left(\|v+v\|^2 + i(\|v-iv\|^2 - \|v+iv\|^2)\right) \\
&= \frac{1}{4}\left(\|2v\|^2 + i(|1-i|^2\|v\|^2 - |1+i|^2\|v\|^2)\right) \\
&= \|v\|^2.
\end{aligned}$$

Next we check that the inner product structure is satisfied. If $\langle v|v\rangle = 0$ then $\|v\|^2 = 0$ by the foregoing algebra, and by the definition of norms, v must be 0. The other direction follows by plugging 0 into the norm definition. Again by the definition of norm $\langle v,v\rangle = \|v\|^2$ is always non-negative.

For the rest of the proof we will restrict our abnalysis to the real case and then extend to the imaginary case. For symmetry, let $v, w \in V$ and then

$$\begin{aligned}\langle v|w\rangle &= \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)\\ &= \frac{1}{4}\left(\|v+w\|^2 - \|-(w-v)\|^2\right)\\ &= \frac{1}{4}\left(\|v+w\|^2 - \|(w-v)\|^2\right) = \langle w, v\rangle.\end{aligned}$$

Turning our attention to additivity, let $z \in V$ and using the parallelogram law we yield that

$$\begin{aligned}\|v+w+z\|^2 + \|v-w+z\|^2 &= 2\|v+z\| + 2\|w\|\\ \|v+w+z\|^2 &= 2\|v+z\|^2 + 2\|w\|^2 - \|v-w+z\|^2\\ \|v+w+z\|^2 &= 2\|w+z\|^2 + 2\|w\|^2 - \|w-v+z\|^2\end{aligned}$$

Therefore it follows using $\|y-x\| = \|x-y\|$ and the results from the foregoing application

$$\begin{aligned}\|v+w+z\|^2 &= \|v+z\|^2 + \|w+z\|^2 + \|v\|^2 + \|w\|^2 - \frac{\|v-w+z\|^2 + \|w-v+z\|^2}{2}.\\ \|v+w-z\|^2 &= \|v-z\|^2 + \|w-z\|^2 + \|v\|^2 + \|w\|^2 - \frac{\|w+z-v\|^2 + \|z+v-w\|^2}{2}.\end{aligned}$$

In the additive case,

$$\begin{aligned}4\langle v+w|z\rangle &= \|v+w+z\|^2 - \|v+w-z\|^2\\ &= \|v+z\|^2 + \|w+z\|^2 + \|v\|^2 + \|w\|^2 - \frac{\|v-w+z\|^2 + \|w-v+z\|^2}{2}\\ &\quad - \left(\|v-z\|^2 + \|w-z\|^2 + \|v\|^2 + \|w\|^2 - \frac{\|w+z-v\|^2 + \|z+v-w\|^2}{2}\right)\\ &= \|v+z\|^2 + \|w+z\|^2 - \|v-z\|^2 - \|w-z\|^2\\ &= \|v+z\|^2 - \|v-z\|^2 + \|w+z\|^2 - \|w-z\|^2\\ &= 4\langle v|z\rangle + 4\langle w|z\rangle.\end{aligned}$$

We now will show that this inner product commutes with scalars[1]. In particular we wish to show $\lambda\langle v|w\rangle = \langle \lambda v, w\rangle$ for all $\lambda \in \mathbb{R}$. Take the case of $\lambda = -1$, then

$$\langle -v|w\rangle = \frac{1}{4}\left(\|-v+w\|^2 - \|-v-w\|^2\right) = \frac{1}{4}\left(-\|v+w\|^2 + \|v-w\|^2\right) = -\langle v|w\rangle.$$

Then by induction the result holds for all $\mathbb{Z}$ applying the previous result. Let $p/q = \lambda \in \mathbb{Q}$, then

$$q\langle p/qv|w\rangle = p\langle q/qv|w\rangle = p\langle v|w\rangle \implies \langle p/qv|w\rangle = p/q\langle v|w\rangle$$

Thus homogeneity holds for all $\lambda \in \mathbb{Q}$. We can extend this to all $\mathbb{R}$ by recalling that scalar multiplication, vector addition, and the norm itself are continuous functions on V, in which case, equality on a dense subset ($\mathbb{Q}$) yields equality on the closure ($\mathbb{R}$). This completes the proof.

[1] I've borrowed this argument from an operator theory textbook

The complex case is as follows, we need show that $\langle iv|w\rangle = i\langle v|w\rangle$. It folllows as

$$\begin{aligned}\langle iv|w\rangle &= \frac{1}{4}\left(\|iv+w\|^2 - \|iv-w\|^2 - i\|iv+iw\|^2 + i\|iv-iw\|^2\right)\\ &= \frac{1}{4}\left(-i\|v+w\|^2 + i\|v-w\|^2 + \|iv+w\|^2 - \|iv-w\|^2\right)\\ &= -i\langle v|w\rangle.\end{aligned}$$

This completes the proof. □

(7.4) Let V be a complex vector space, then

$$\langle v|w\rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 + i\|v-iw\|^2\right).$$

Proof. To verify this proof, we will first compute the inner product in terms of $\|v+w\|$.

$$\|v+w\|^2 = \langle v+w|v+w\rangle = \|v\|^2 + \|w\|^2 + Re(\langle v|w\rangle),$$

where the last term comes from the sum of the inner product with its conjugate.

Now we consider the imaginary piece and yield

$$\|v+iw\|^2 = \langle v+iw|v+iw\rangle = \|v\|^2 + \|w\|^2 - 2Im(\langle v|w\rangle).$$

Repeating the steps above with $v-w$, we get that

$$\begin{aligned}\|v-w\|^2 &= \|v\|^2 + \|w\|^2 - 2Re(\langle v|w\rangle)\\ \|v-iw\|^2 &= \|v\|^2 + \|w\|^2 + 2Im(\langle v|w\rangle)\end{aligned}$$

By combinign the two relations in both the real and imaginary case we get that

$$\langle v|w\rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 + i\|v-iw\|^2\right).$$

This completes the proof. □

(7.5) Let V be an inner product space, let $\{e_1,\ldots,e_d\} \subset V$ be an orthonormal basis of V, and let $\{v_1,\ldots,v_d\} \subset V$ be such that $\|v_k - e_k\| < \frac{1}{\sqrt{d}}$ for $1 \le k \le d$. Show that $\{v_1,\ldots,v_d\}$ is a basis of $V/$

Proof. We need show that the set is linearly indepeendent to show that it is a basis (given its cardinality). Thus for the sake of contradiction suppose it is not.

Then there exist non-trivial c_k so that

$$0 = c_1v_1 + \cdots + c_dv_d.$$

We add zero to the sum by subtracting e_k for each k and readding those terms,

$$\begin{aligned}
0 &= c_1(v_1 - e_1) + \cdots + c_d(v_d - e_d) + c_1e_1 + \cdots + c_de_d. \\
-(c_1e_1 + \cdots + c_de_d) &= c_1(v_1 - e_1) + \cdots + c_d(v_d - e_d) + c_1e_1 + \cdots + c_de_d. \\
\|c_1e_1 + \cdots + c_de_d\| &= \|c_1(v_1 - e_1) + \cdots + c_d(v_d - e_d) + c_1e_1 + \cdots + c_de_d\| \\
\sqrt{|c_1|^2\|e_1\|^2 + \cdots + |c_d|^2\|e_d\|^2} &= \|c_1(v_1 - e_1) + \cdots + c_d(v_d - e_d) + c_1e_1 + \cdots + c_de_d\| \\
\sqrt{\sum_{k=1}^{d} |c_k|^2} &= \|c_1(v_1 - e_1) + \cdots + c_d(v_d - e_d) + c_1e_1 + \cdots + c_de_d\| \\
\sqrt{\sum_{k=1}^{d} |c_k|^2} &\leq \sum_{k=1}^{d} |c_k|\|v_k - e_k\| < \sum_{k=1}^{d} \frac{|c_k|}{\sqrt{d}}. \\
\sqrt{d}\sqrt{\sum_{k=1}^{d} |c_k|^2} &< \sum_{k=1}^{d} |c_k|
\end{aligned}$$

Now consider the following product

$$\sum_{k=1}^{n} |c_k| = |\langle c|1\rangle| \leq \|c\|\|1\| = \sqrt{d}\sqrt{\sum_{k=1}^{n} |c_k|^2}$$

which follows from the Cauchy Schwartz equality. This clearly contradicts the foregoing inequality, and therefore it must be the case that the set is linearly independent. Therefore it spans V. THis completes the proof.

□

(7.6) Let V be an inner-product space and let $T : V \to V$. Show that if T is orthogonally diagonalizable, then T is normal.

Proof. Recall that T is normal iff $T^*T = TT^*$, thus we wish to show the resultant property. First if T is orthogonally diagonalizable, then we will first show that $[T]^*_{\mathcal{B}\to\mathcal{B}} = [T^*]_{\mathcal{B}\to\mathcal{B}}$. Observe that $T(u_j) = t_{1j}b_1 + \cdots a_{nj}b_n$ and since the basis is orthonormal, $a_{kj} = \langle T(b_j)|b_k\rangle$. Likewise the kjth element of T^* is given by $\langle T^*(b_j)|b_k\rangle = \overline{\langle b_k|T^*(b_j)\rangle} = \overline{\langle T(b_k)|b_j\rangle} = \overline{a_{jk}}$.

Now since T is orthogonally diagonalizeable, $[T]_{\mathcal{B}\to\mathcal{B}}$ diagonal implies $[T^*]_{\mathcal{B}\to\mathcal{B}}$ diagonal by the above proof. Thus, $[T^*T]_{\mathcal{B}\to\mathcal{B}} = [T^*]_{\mathcal{B}\to\mathcal{B}}[T]_{\mathcal{B}\to\mathcal{B}} = [T]_{\mathcal{B}\to\mathcal{B}}[T^*]_{\mathcal{B}\to\mathcal{B}} = [TT^*]_{\mathcal{B}\to\mathcal{B}}$ since multiplication by diagonal matrices commutes. In conclusion, $T^*T = TT^*$. □

(7.7) Let V be a finite-dimensional inner-product space, let $T : V \to V$ be self-adjoint, and let $W \subset V$ be a subspace. Show that W is T-invariant iff $W^\perp$ is T-invariant.

Proof. Suppose that W is T-invariant. Then $T[W] \subset W$, and in particular for any $w \in W, w' \in W^\perp$, $\langle Tw|w'\rangle = 0$. Since T is self-adjoint, we have that $\langle Tw|w'\rangle = \langle w|Tw'\rangle = 0$. Therefore $Tw' \perp w$ and thus $Tw' \in W^\perp$; that is, $T[W^\perp] \subset W^\perp$.

In the other direction if $W^\perp$ is T-invariant then for any $w' \in W^\perp$ and any $w \in W$ we have that $0 = \langle Tw'|w\rangle = \langle w'|Tw\rangle = 0$. So, $Tw \in (W^\perp)^\perp = W$; that is W is T-invariant. □

(7.8) Let V be a finite-dimensional inner-product space and let $P : V \to V$ be such that $P^2 = P$ and $P^* = P$. Show that there is a subspace $W \subset V$ such that $P = proj_W$.

Proof. Let $W := P[V]$, we want to show that $P(v) \in W$ is the unique element of W such that $v - P(v) \in W^\perp$. First,

$$\langle P(v)|v - P(v)\rangle = \langle P^2(v)|v - P(v)\rangle = \langle P(v)|P(v) - P^2(v)\rangle.$$

But then as $P^2(v) = P(v)$, we yield that $\langle P(v)|v - P(v)\rangle = \langle P(v)|0\rangle = 0$. Therefore $v - P(v) \in W^\perp$, and in fact $v - P(v) \in Ker(P)$.

Now suppose that there were another element, y, in W so that $v - y \in W^\perp$. Then $0 = \langle y|v - y\rangle$. It follows that $(v - y) + (P(v) - v) = P(v) - y \in W^\perp$, and thus

$$0 = \langle y|P(v) - y\rangle.$$

Furthermore $P(v) - y \in W$ since $P(v), y \in W$. Therefore[2] $P(v) - y \in W \cap W^\perp$ so $P(v) - y = 0$ and so $P(v) = y$, which is a contradiction. Therefore $P(v)$ is unique, and $P = proj_W$. □

(7.9) Give an example of an inner-product space V and a subspace $W \subset V$ such that it is not the case that $W \oplus W^\perp = V$.

Soluion. In the case that the whole space is $C[a, b]$ take W to be all elements $f \in V$ so that $f(a) = 0$. Then if g such that for all $f \in W$, $\langle g, f\rangle = 0$ we must also have that $\langle g, id_{[a,b]} - a\rangle = 0$, but then for arbitrary $q \in V$ we have $\langle h, (id_{[a,b]} - a)q\rangle = 0$. Thus

$$0 = \int_a^b (x - a)q(x)g(x)\ dx \implies \int_a^b xq(x)g(x)\ dx = a\int_a^b q(x)g(x)\ dx$$

In particular take $q = g$ and then then $\int_a^b xg^2\ d = 0x$. Since xg^2 is a positive, continuous map, we have that $xg^2 = 0$ for all x and thus $g = 0$ on $[a, b]$. Therefore $W^\perp = \{0\}$, but it is not the case that $W \oplus \{0\} = V$.

[2] We use that finite dimensional inner-product spaces are endowed with a natural topology giving them qa Hilbert space structure.