Math 202A — UCB, Fall 2016 — William Guss
Problem Set 7, due Wednesday October 14

Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space. $\int f$ is shorthand for $\int_X f\,d\mu$, where μ is a measure which may not be explicitly specified. m denotes Lebesgue measure on either $\mathcal{B}_{\mathbb{R}}$ or $\mathcal{L}_{\mathbb{R}}$. Unless otherwise indicated, "f is measurable" means that $f : X \to \mathbb{C}$ and f is measurable with respect to $\mathcal{M}$. L^1 refers to functions, rather than to equivalence classes of functions, unless otherwise indicated. proof

(7.1) (Folland problem 3.2) If $(X, \mathcal{M}, \nu)$ measure space then if ν is a signed measure then $E \subset X$ is ν-null if and only if $|\nu|(E) = 0$ and $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. If E is ν-null then $A \subset E$ ν-null for every A. Using the Hahn-decomposition $A = P \sqcup N$ such that P, N are ν-null positive and negative sets. By the Jordon decomposition $\nu^+(P) - \nu^-(N) = 0$. Recall that $|\nu|(E) = \nu^+(E) + \nu^-(E)$. Therefore $0 = \nu(N) = \nu^+(N) - \nu^-(N) = -\nu^-(N)$ by $\nu^- \perp \nu^+$. It then follows from the Jordon decomposition that $\nu^+(P) = 0$. Thus giving $(\nu^+ + \nu^-)(E) = 0$.

In the other direction, if $E \in \mathcal{M}$ is $|\nu|$ null then $\nu^+ + \nu^-(E) = 0$. Since both measures are positive $\mu^+(E) = 0 = \nu^-(E)$. Recall that every measurable subset of a zero set of a positive measure is a zero set. Therefoe $\mu^+(A \subset E) - \nu^-(A) = 0$ and E is a ν-null set.

If $\nu \perp \mu$ then $|\nu|(E) = 0$ iff E is ν-null implies that $|\nu|(E) = 0$ iff $\mu(E) \geq 0$. So this gives $\mu \perp |\nu|$. Now suppose the conclusion, then if E is $|\nu|$-null then since ν is the sum of two positive measures it must be that E is ν^+-null and ν^- null. Repeating the same conditions of mutualingularity, it must be that ν^- is mutually singular with μ and ν^+ is mutually singular with μ. Now if the conclusion holds then $\nu^+ - \nu^- = 0 - 0 = 0$ on E and for every subset of E which is measurable so $\nu \perp \mu$. This completes the proof.

□

(7.2) (Folland problem 3.6) Suppose that $\nu(E) = \int f d\mu$ where μ is a positive measure and f is an extended μ-integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variation of ν in terms of f and μ.

Proof. If $(X, \mathcal{M}, \nu)$ is the measure space in reference, the Hahn decomposition is of X into ν-positive and ν-negative sets; that is, $X = P \sqcup N$. From integration theory, if μ is a positive measure then $\int_E f\ d\mu^+ = \int_E f^+ d\mu - \int_E f^-\ d\mu$ where $f^+ > 0 \iff f^- = 0$ and dually $f^- > 0 \iff f^+ = 0$ on $x \in X$. Consider the set $X = f^{+(pre)}((0, \infty]) = X^+$ and dually X^- w.r.t μ . These sets are measurable in μ by the measurability of f. Since integration on subsets ($\int_E f\ d\mu = \int_X \chi_E\ d\mu \iff E \in \mathcal{M}$) is defined only on measurable sets, $\mathcal{M}$ for ν must be equivalent to the σ-algebra of μ. Lastly, X^+ is evidently a ν-positive set and removing the points at which $X^- \ni x$ gives $f(x) = 0$ from X^-, we get X^- is a ν-negative set with $X^+ \cap X^- = \emptyset$ and $X^+ \cup X^- = X$. Additionally the measures $\nu^+ = \int_\cdot f^+\ d\mu$ and $\nu^- = \int_\cdot f^-\ d\mu$ are mutually singular by the above reasoning and give the Jordon decomposition of ν. □

(7.3) (Folland problem 3.7) Suppose that $|nu$ is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$. Show that

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(a) $\nu^+(E) = \sup\{\nu(F) : F \subset E, F \in \mathcal{M}\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$.

Proof. Recall the following logic from a proof othe Hahn Decomposition Theorem. If $X = P \sqcup N$, then it is defined that $m = \sup \nu(J)$ over all positive sets $J \in \mathcal{M}$ which are subsets of X, it is of no loss of generality to apply this logic to our set E itself, but we shall continue down this line of reasoning until we are satisfied that $\nu(P) = m$. In the proof, it is claimed that since m is a supremum of positive sets whose measure tends to m in the limit, that their exists a sequence of positive sets with countable indices so that $P = \bigcup_1^\infty P_j$ and under the conditions of Lemma 3.2 and Proposition 3.1, P is positive and attains measure m in the limit. We can apply the same logic to N with the infimum n.

Now in the logic of the previous exposition replace X with $E \in \mathcal{M}$. Then it follows that $E = P_E \sqcup N_E$ where those two sets attain the same extremum of postiive and negative measures respectiveley. Now under the definition of ν^+ and ν^- given, we claim mutual singularity. If ν^+ is postive on a set $K \subset X$, then ν^+ only "gains" measure by the positive part of the Hahn decomposition of K, say $P_K \subset P$. In fact for the whole space ν^+ only has positive measure on P, and symmetrically ν^- only has measure on N therefore by the disjointness of the Hahn decomposition these measures are mutually signular. $\square$

(b) $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n$ are disjoint, and $\bigcup_1^n E_j = E\}$.

Proof. Using (a) we know that the unique ν Jordon decomposition gives ν^+ and ν^- defined before. We can express E as $E_1 = P_E$ and $E_2 = N_E$ uniqueley, thus $\nu^+(E_1) + \nu^-(E_2) = |\nu(E_1)| + |\nu(E_2)| = \nu^+(E) + \nu^-(E)$ by mutual singularity and the sets E_1 and E_2 achieve the supremum over absolute value sets with the disjointness property. Thus the definition of $|\nu|(E)$ holds as the supremum over the sums, since we have found that supremum. $\square$

(7.4) (Folland problem 3.8) Show that $\nu << \mu$ if and only if $|\nu| << \mu$ if and only if $\nu^+ << \mu$ and $\nu^- << \mu$.

Proof. If $\nu << \mu$ then every E which is a μ-null set is also a ν-null set. From (7.1) we have that E is a ν-null set if and only if it is a $|\nu|$-null set. Therefore $\nu << \mu$ if and only if $|\nu| << \mu$. Next if $|\nu| << \mu$ it follows that every μ-null set is a $|\nu|$-null set, but then by exercise (7.1), $|\nu|$-null if and only if ν^+null and ν^--null. Thus $|\nu| << \mu$ if and only if $\nu^+ << \mu$ and $\nu^- << \mu$. This completes the proof. $\square$