Math 185 — UCB, Fall 2016 — William Guss
Problem Set 3, due October 4th

(30.7) Prove that $|exp(-2z)| < 1$ if and only if $Re(z) > 0$.

Proof. Let $z = x + iy$. It follows that $|exp(-2z)| < 1$ if and only if $|e^{-2x}e^{-i2y}| = |e^{-2x}||e^{-i2y}| < 1$ if and only if $|e^{-i2y}| < \frac{1}{e^{-2x}} = e^{2x}$. The LHS is equivalent to $|\cos(-2y) + i\sin(-2y)|$ and for every $y|e^{-i2y}| = 1$ so $1 < e^{2x}$ if and only if $0 < 2x$. $\square$

(30.11) Describe the behaviour of e^z as $x \to -\infty$ and $y \to \infty$.

Proof. In the first case $e^z = e^x e^{iy}$ gives that $|e^z| \to 0$ since $e^x \to 0$ as $x \to -\infty$. Therefore the function $e^z \to 0$ in the limit w.r.t x.

In the second case, we fix x and observe that $e^z = e^x e^{iy}$ parameterizes a circle of radius e^x by angle w.r.t y. Therefore, increasing y only results in movement along the disk in the counter clockwise direction. Therefore e^z does not converge since $y \mod 2\pi$ is an equivalence class of infiniteley many elements $y + n2\pi, n \in \mathbb{N}$.

However if the limits are achieved simultaneously then $|e^z| \to 0$ implies that $e^z \to 0$ regardless of the angle of approach[1]. $\square$

(30.12) Write $Re(e^{1/z})$ in terms of x and y. Why is this function harmonic on every domain that does not contain the origin.

Proof. Again let $z = x+iy$. Then $e^{1/z} = e^{z^{-1}}$. First $1/z = \overline{z}/|z|^2$. Therefore $e^{z^{-1}} = exp(x/|z|^2)(\cos(-y/|z|^2) + i\sin(-y/|z|^2)$ So the real part is

$$Re(f) = exp\left(\frac{x}{|z|^2}\right)\cos\left(\frac{y}{|z|^2}\right)$$

Then $Re(f)_{xx}$ is given by

$$\frac{\partial^2}{\partial x^2} exp\left(\frac{x}{|z|^2}\right)\cos\left(\frac{y}{|z|^2}\right) = \frac{\partial}{\partial x}\left[\left(\cos\left(\frac{y}{|z|^2}\right) - \sin\left(\frac{y}{|z|^2}\right)\right) exp\left(\frac{x}{|z|^2}\right)\frac{\partial}{\partial x}\frac{x}{|z|^2}\right] \tag{0.1}$$

$$\frac{\partial^2}{\partial y^2} exp\left(\frac{x}{|z|^2}\right)\cos\left(\frac{y}{|z|^2}\right) = \frac{\partial}{\partial y}\left[\left(\cos\left(\frac{y}{|z|^2}\right) - \sin\left(\frac{y}{|z|^2}\right)\right) exp\left(\frac{x}{|z|^2}\right)\frac{\partial}{\partial y}\frac{y}{|z|^2}\right] \tag{0.2}$$

The first equation gives a product rule with a symetric derivative on $(\partial/\partial x)x/|z|^2$ and $(\partial/\partial y)y/|z|^2$ and the antisymmetry on differentiation of trigonometric functions gives that the second derivatives in x and y are equal, so $Re(f)$ is harmonic as long as $x \neq 0$ and $y \neq 0$ since $1/|z|$ is not defined. $\square$

(33.3) Show $Log(i^3) \neq 3Log(i)$.

Proof. Recall that $Log(z) = \log|i^3| + i(\Theta + 2n\pi)$ where $n = 0$ and $\Theta = Arg(i^3)$. Computation gives $i^3 = -i$ so $Arg(i^3) = Arg(-i) = -\pi/2$. Therefore $\log|-i| = \log 1 = 0$ Therefore $Log(i^3) = -i\pi/2$. However, $Log(i) = 0 + i\Theta = 0 + i\pi/2 \neq -3i\pi/2$. $\square$

[1]Imagine a marble spiruling down a funnel.

(33.4) Show that $log(i^2) \neq 2log(i)$ when the branch

$$\log z = \log r + i\theta \qquad \left(r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right).$$

Proof. In this case $\log i, \log i^2$ has real part 0. Then for the imaginary part. $i^2 = -1$ so $\Theta = -\pi \equiv \pi \in \mathcal{B}_\Theta$ so $\log(i^2) = \pi$ and then $Arg(i) = \pi/2 \equiv 5\pi/2$. And so $5\pi/2 \neq 4\pi/2$ so the logarithims are not equal. □

(33.7) Show that a branch (Sec. 33)

$$\log z = \log r + i\theta \qquad (r?0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function can be written

$$\log z = 1/2 \log(x^2 + y^2) + i \tan^{-1}\left(\frac{x}{y}\right)$$

in rectangular coordinates. Then, using the theorem in Sec 23, show that a the given branch is analytic in its domain of definition and that $\frac{d}{dz}\log z = \frac{1}{z}$.

Proof. For the first assertion let $z = x + iy$ then $\tan^{-1}\left(\frac{x}{y}\right) = arg(z)$ on the partiular branch, by the definition of $\tan\theta = o/a$ for a triangle where o is the height and a is the base/ Therefore $\tan^{-1}(x/y) + n2\pi \in [a, a + 2\pi]$. Next $r = |z|$. So for a real number $|z|$, it follows that $\ln(|z|^2)^{1/2} = 1/2 \ln|z|^2 = \ln(x^2 + y^2)$.

To show analycity, we compute the partial derivatives as follows. First $u = 1/2\ln(x^2 + y^2)$ so $u_x = \frac{x}{(x^2+y^2)}$. Recalling elementary calculus we have that $v_y = \frac{1}{x(1+y^2/x^2)} = \frac{x}{x^2(1+y^2)} = \frac{x}{x^2+y^2}$. Therefore $u_x = u_y$, and the partial derivatives are continuous in the domain (as long as $x = y \neq 0$). Next $v_x = -\frac{y}{x^2(1+y^2/x^2)} = \frac{-y}{x^2+y^2}$, and $u_y = \frac{y^2}{x^2+y^2}$ so the Cauchy riemann equations are sovled and log is analytic on its branch. □

(33.12) Show that

$$Re[\log(z-1)] = \frac{1}{2}\ln[(x-1)^2 + y^2] \qquad (z \neq 1).$$

Proof. If $z = x + iy$ then as long as $z - 1 \neq 0$ then $x = y \neq 0$ and application of the previous formula (33.7) proven gives $Re[\log(z-1)] = \frac{1}{2}\ln[(x-1)^2 + y^2]$. When $z \neq 1$ this function is the real part of the analytic function in (33.7) and so it is a harmonic function which satisfies Laplace's equation as in Theorem 27.

□

(34.1) Show that for any two nonzero complex numbers z_1 and z_2

$$Log(z_1 z_2) = Log(z_1) + Log(z_2) + 2N\pi i$$

where N has one of the values $0, \pm 1$.

Proof. The principle logarithm is defined on the branch of angles by the principle argument. Application of the definition gives $\log(z_1 z_2) = \ln|z_1 z_2| + iarg(z_1 z_2) = \ln|z_1||z_2| + iarg(z_1 z_2) = \ln|z_1| + \ln|z_2| + i(arg(z_1) + arg(z_2)) = \log(z_1) + \log(z_2)$. Now on the principle branch, the real part is stable, but the imaginary part of the log product must be reduced modulo 2π. Since at most $\pi < \theta_1 + \theta_2 3\pi$

the principle modulo reduces the sum by 2π. The reverse holds for the lower bound, increasing by 2π so as to fit the 2π modulo range of the principle argument.

Therefore $N = \pm 1$, and if the sume of the principle angles is in the principle branch $N = 0$. Thus

$$Log(z_1 z_2) = Log(z_1) + Log(z_2) + 2N\pi i$$

and this completes the proof. $\square$

(34.5) Let z denote any nonzero complex number, writeen $z = re^{i\Theta}$, $(-\pi < \Theta < \pi)$, and let n denote any fixed positive integer $(n = 1, 2, \dots)$. Show that all of the values of $\log(z^{1/n})$ are given by the equation

$$\log(z^{1/n}) = \frac{1}{n} \ln r + i\frac{\Theta + 2(pn + k)\pi}{n}$$

Proof. First recall from a previous exercise if $z = x + y$

$$\log z = 1/2 \ln(x^2 + y^2) + i(\Theta \mod \mathcal{B}_m)$$

If $w = z^{1/n}$ then $w = |z|^{1/n} e^{\frac{i\Theta}{n}}$. Then $|w|^2 = (|z|^{1/n})^2 = \sqrt[n]{|z|^2}$ so $Re(\log w) = \frac{1}{2n} \ln(x^2 + y^2) = \frac{1}{n} \ln r$. Since there are n solutions to w we get that the principle argument of each forms a set $Argw = \{\Theta/n + 2k\pi/n \mid k \in \mathbb{Z}_n\}$ using eulers formula on the polar form of z. Clearly $-\pi < \Theta/n + 2k\pi/n < \pi$ so now each branch of the logarithm will be identified by moving the principle by 2π. Thus $argw = \{\Theta/n + 2k\pi/n + \rho 2\pi \mid k \in \mathbb{Z}_n, \rho \in \mathbb{Z}\}$ gives

$$\log(z^{1/n}) = \frac{1}{n} \ln r + i\frac{\Theta + 2(pn + k)\pi}{n}, \qquad ; \quad p \in \mathbb{Z}, k \in \mathbb{Z}_n$$

$\square$

(38.2) (a) With the aid of expression (4), Sec 37. show that

$$exp(iz_1)exp(iz_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$$

Proof. Recall that $e^{iz} = \cos z + i \sin z$. Then $e^{iz_1} e^{iz_2} = (\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2)$. Expanding this relation we get $e^{iz_1} e^{iz_2} = \cos z_1 \cos z_2 + i^2 \sin z_1 \sin z_2 + i(\cos z_1 \sin z_2 + \cos z_2 \sin z_1)$ which gives the statement of the exercise. $\square$

(b) Using the results in part (a) and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

Proof. Observe the following algebra.

$$
\begin{aligned}
\sin(z+z_2) &= \frac{1}{2i}\left(e^{iz}e^{iz_2} - e^{-iz}e^{-iz_2}\right) \\
&= \frac{1}{2i}(\cos z \cos z_2 - \sin z \sin z_2 + i(\sin z \cos z_2 + \cos z \sin z_2) \\
&- (\cos(-z)\cos(-z_2) - \sin(-z)\sin(-z_2) + i(\sin(-z)\cos(-z_2) + \cos(-z)\sin(-z_2)))) \\
&= \frac{1}{2i}\cos z \cos z_2 - \sin z \sin z_2 + i(\sin z \cos z_2 + \cos z \sin z_2 \\
&- (\cos(z)\cos(z_2) - \sin(z)\sin(z_2) - i(\sin(z)\cos(z_2) + \cos(z)\sin(z_2)))) \\
&= \frac{2i}{2i}(\sin(z)\cos(z_2) + \cos(z)\sin(z_2)) \\
&= \sin(z)\cos(z_2) + \cos(z)\sin(z_2)
\end{aligned}
$$

□

(38.3) Show that $\cos(z_1 + z_2) = \cos z_2 \cos z_2 - \sin z_1 \sin z_2$.

Proof. From the previous exercise we differentiate the expression

$$d/dz \sin(z+z_2) = \cos(z+z_2) = d/dz \sin(z)\cos(z_2) + \cos(z) + \sin(z_2)$$

So it follows immediately that $\cos(z_1+z_2) = \cos z_2 \cos z_2 - \sin z_1 \sin z_2$. □

(39.2) Prove that $\sinh 2z = 2 \sinh z \cosh z$.

Proof. We do the following algebra

$$\sinh 2z = \frac{(e^z - e^{-z})}{2} = \frac{(e^z e^z + e^z e^{-z} - e^z e^{-z} e^{-z} e^{-z})}{2} = 2\frac{(e^z - e^{-z})}{2}\frac{(e^z + e^{-z})}{2}$$

Observe the right side is $\sinh 2z = 2\cosh z \sinh z$ and the algebra is if and only if. □

(40.3) Solve $\cos z = \sqrt{2}$ for z.

Proof. Using the formula, we have that

$$\cos^{-1}(z) = -i \log\left[z + i(1-z^2)^{1/2}\right].$$

Then $\cos^{-1}(\sqrt{2})$, it follows that $-i\log\left[\sqrt{2} + i(1-2)^{1/2}\right]$. Then we have all solutions are $-i\log\left[\sqrt{2} \pm i\right]$.
Therefore $\cos^{-1} = -i\left[\frac{1}{2}\ln 3 \pm i(\tan\left(\frac{1}{\sqrt{2}}\right) + 2k\pi)\right] = \pm\tan\left(\frac{1}{\sqrt{2}}\right) + 2k\pi - i\frac{\ln 3}{2}$. □