Math H104: Homework 2

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1 Real Numbers

36. *Without using the Schroeder-Bernstein Theorem,*

    (a) *Prove the following*

    Theorem 1. *The cardinalities of* $[a, b], (a, b], (a, b)$ *are equivalent. That is,* $[a, b] \sim (a, b] \sim (a, b)$.

    Proof. To show cardinal equivalence, we must find a bijection between the three sets. This argument follows from Hilberts's hotel. Consider the two sets set $A^o = (a, b), A^h = (a, b]$. Because every uncountable set has a countable subset, let

    $$A_C^o = \left\{ a_n \in A^o \mid a_n = \frac{an + b}{n + 1},\ n \in \mathbb{N} \right\}.$$

    In the same light, let

    $$A_C^h = \left\{ a_n \in A^h \mid a_n = \frac{a(n-1) + b}{n},\ n \in \mathbb{N} \right\}.$$

    Then $f : A_C^o \to A_C^h$, such that $f(\frac{an+b}{n+1}) = \frac{a(n-1)+b}{n}$, is clearly a bijection.

    We now make a functiong $g : A^o \to A^h$ such that for $x \in A_C^o$, $g(x) = f(x)$, otherwise $g(x) = x$. Since f is a a bijection, then g is a bijection on A_C^o. Furthermore since $b \notin A^h \setminus A_C^h$, we have that g surjective when $x \notin A_C^o$. Furthermore if $x \neq y$, then clearly $g(x) = x \neq y = g(y)$. So g is injective, and therefore bijective. We have shown that $A^o \sim A^h$.

    Lastly, we will prove that $A^c = [a, b]$ is bijective to A^h. Redefine again the following countable subsets

    $$A_C^h = \left\{ a_n \in A^g \mid a_n = \frac{a + bn}{n + 1},\ n \in \mathbb{N} \right\}.$$

    In the same light, let

    $$A_C^c = \left\{ a_n \in A^c \mid a_n = \frac{a + b(n-1)}{n},\ n \in \mathbb{N} \right\}.$$

Now take the function $f : A_C^h \to A_C^c$ to map $(\frac{a+bn}{n+1}) \mapsto \frac{a+b(n-1)}{n}$. The function f is a clear bijection. Finally let $g : A^h \to A^c$ be such that $x \in A_C^h \implies g(x) = f(x)$, otherwise $g(x) = x$. This new function is clearly a bijection in the same sense that the previous definition of f was, and hence we have shown that $A^c \sim A^h$.

Therefore by bijective composition, $A^c \sim A^h \sim A^o$, and the proof is complete. $\square$

(b) *Prove the following.*

Theorem 2. *If C is countable, then $\mathbb{R} \setminus C \sim \mathbb{R} \sim \mathbb{R} \cup C$.*

Proof. If C is a countable subset of $\mathbb{R}$, then clearly $\mathbb{R} \cup C = \mathbb{R}$ and hence there is equivalence. In the other case, consider the following.

To show this theorem, we first find a bijection between $\mathbb{R}$ and $\mathbb{R} \setminus C$. Because C is a countable, we can take an index set $I = \{1, 2, \dots\}$, which is finite if and only if C is finite and contiguous such that $|I| = |C|$. We then can define $C = \{a_i\}_{i \in I}$ for real numbers $a_i \in C$, such that $i > j \implies a_i > a_j$.

We have that

$$\mathbb{R} \setminus C = (-\infty, a_1) \cup \left[\bigcup_{i \in I \setminus \{1\}} (a_{i-1}, a_i) \right] \cup (\sup C, \infty).$$

So it follows,

$$\mathbb{R} = (-\infty, a_1] \cup \left[\bigcup_{i \in I \setminus \{1\}} [a_{i-1}, a_i] \right] \cup [\sup C, \infty).$$

We will now constuct a bijection $f : \mathbb{R} \setminus C \to \mathbb{R}$. By Theorem 1, there exists a bijective function $h_i : (a_{i-1}, a_i) \to [a_{i-1}, a_i]$. In the edge cases, we propose the following method of producing a bijection.

Let $A_L^o = (-\infty, a_1)$. Then clearly, $(-\infty, b] \subset \mathbb{R}$ for any $b < a_1$. Hence we can use $(-\infty, b] \cup [b, a_1) = A_L^o$. By theorem there exists an $h_L : [b, a_1) \to [b, a_1]$. Again for the upperbound, let $A_U^o = (\sup C, \infty)$. We have that $A_U^o = (\sup C, d] \cup [d, \infty)$. Let $h_U : (\sup C, d] \to [\sup C, d]$ be the bijection producable from Theorem 1. Finally let $g_1 : (-\infty, b] \to (-\infty, b]$ and $g_2 : [d, \infty) \to [d, \infty)$ be the identity functions.

Taking all aforementioned functions and defining a new function f which acts as each function along its corresponding domain, we have a bijection from $\mathbb{R}$ to $\mathbb{R} \setminus C$ when C is bounded.

If C is not bounded, we can use the fact that C is listable to strongly inductively create a mapping. Let $L : \mathbb{N} \to C$ be the listing function for C itself. We propose that $\mathbb{R} \sim \mathbb{R} \setminus \{L(n)\}_{n=0}^{\infty}$.

In the base case we have that $\mathcal{L}_1 = \{L(0)\}$ is a countable and finite set, so by the first half of this proof, $\mathbb{R} \sim \mathbb{R} \setminus \mathcal{L}_1$. Now suppose $\mathbb{R} \sim \mathbb{R} \setminus \bigcup_n^k \mathcal{L}_k$; assume that removing all previous points using this method yields an injection from $\mathbb{R}$ onto $\mathbb{R} \setminus \bigcup_n^k \mathcal{L}_k$ It follows that because $\mathcal{L}_k$ is countable, $\mathcal{L}_{k+1} = \mathcal{L}_k \cup \{L(k+1)\}$

is also countable and bounded. Hence by the first half of the proof, $\mathbb{R} \sim \mathbb{R} \setminus \bigcup_n^k \mathcal{L}_k \sim \mathbb{R} \setminus \left(\bigcup_n^k \mathcal{L}_k \cup \mathcal{L}_{k+1}\right)$. Repeating this process without stopping, we get that $\mathbb{R} \sim \mathbb{R} \setminus \{L(n)\}_{n=0}^{\infty}$ if and only if $\mathbb{R} \sim \mathbb{R} \setminus C$. This completes the proof for C as numbers.

Now if C is not a set of numbers, then clearly $\mathbb{R} \setminus C = \mathbb{R}$ since $\mathbb{R}$ only contains numbers. If we wish to add a countable set of let's say dogs. Let us denote these dogs d_j, simply said "scrappy j", for the jth dog. Now take the following sequence $D_j = \{x \in \mathbb{R} \cup d_j \mid x = n + \frac{1}{j} + j, n = 0 \implies x = d_j, n \in \mathbb{N}\}$. The sequence $D_j^o = \{x \in \mathbb{R} | \ x = n + \frac{1}{j} + j, n = 0 \implies x = \frac{1}{j} + j, n \in \mathbb{N}\}$. Clearly there is a mapping which is bijective between these two sequences as follows. The function f_j maps scrappy j to the first element of D_j^o and then maps the second element of D_j to the third element of D_j^o and so on and so forth. This function is clearly bijective. Then consider a function g_j which will map $x \in \mathbb{R} \setminus D_j$ to $\mathbb{R} \setminus D_j^o$ with the mapping $x \mapsto x$. In the case that $g_j \in D_j$, $x \mapsto f_j(x)$. Since g_j is bijective, we have that $\mathbb{R} \sim \mathbb{R} \cup D_j = \mathbb{R} \cup \{d_j\}$.

Now that we have shown for arbitrary dogs, we can add such dogs to $\mathbb{R}$ and maintain cardinal equivalence, consider the following process. Since all D_j are disjoint, we may repeat this process infinitely. We first take $\mathbb{R}$ and embed d_1 into the reals and find an equivalence. Let us denote the first n scrappies embedded in $\mathbb{R}$, $\mathcal{D}_n$. Then we can put d_2 into $\mathcal{L}_1$ since $D_1 \cap D_2 = \emptyset$. We call this new set $\mathcal{D}_2$, and combining f_1, f_2 and the identity function otherwise to find a bijection g_2 from $\mathbb{R}$ onto $\mathcal{D}_2$. We can repeat this process with out stopping since the set C of dogs is denumerable, and when this process finishes, we will have that $\mathbb{R} \cup C \sim \mathbb{R} \sim \mathbb{R} \sim \mathbb{R} \setminus C$.

This completes the proof. □

(c) *Infer the thing about $\mathbb{Q}$*

Theorem 3. *The set of irrational numbers has the same cardinality as $\mathbb{R}$.*

Proof. Simple! We establish that $\mathbb{Q}$ is countable. Then, it follows since $\mathbb{Q}$ is not bounded, that $\mathbb{R} \sim \mathbb{R} \setminus \mathbb{Q}$ by Theorem 2. □

37. *Prove that the plane is bijective to the line.*

Theorem 4. *The real plane $\mathbb{R}^2$ has the same cardinality as the real line $\mathbb{R}$.*

Proof. To show that there is equivalent cardinality between the plane and the line, we must find a bijection. It follows from the Schroeder-Bernstein theorem, that if two injections are found, there exists a hyper-composite bijection. Let $f : \mathbb{R} \to \mathbb{R}^2$ such that $x \mapsto (0, x) \in \mathbb{R}$. If $z \neq w$, then $f(z) = (0, z) \neq (0, w) = f(w)$. Hence, f is injective. Let $g : \mathbb{R}^2 \to \mathbb{R}$, such that for every decimal expansion (not nine-non terminating) of $(a, b) = (a_1.a_2a_3\ldots, b_1.b_2b_3\ldots)$, $g((a, b)) \mapsto a_1.b_1a_2b_2\ldots$. Clearly g is an injection for if $z \neq w$, then the decimal expansion is non-equal at some index. This implies that $f(z) \neq f(w)$.

Now by the the Shroeder-Bernstein theorem, we have that there exists a bijection from $\mathbb{R}^2 \to \mathbb{R}$. Hence $\mathbb{R} \sim \mathbb{R}^2$. □

2 A Taste of Topology

1. *An ant walks on the floor,ceiling, and walls of a cubical room. What metric is natural for the ant's view of its world? What metric would a spider consider natural? If the ant wants to walk from a point p to a point q, how could it determine the shortest path?* Simple! The ant exists on subspaces of three space. In fact, the ant exists on planes within three space. This means that the ant can only move from one location to another using the inherited metric of a plane subspace in $\mathbb{R}^3$, not the one provided for $\mathbb{R}^3$ itself. To give a precise definition, if the ant wants to travel from the center of the ceiling to the floor, he must take the shortest path to the closest contiguous plane series to that of the floor. Put more simply, the ant needs to walk to the wall and then to the floor to get to his destination.

   In this sense we can actually construct a homeomorphism from the ant's 3-space domain, to a simple subset of $\mathbb{R}^2$. Take all of the planes constructing the walls and the floor and knock them down such that they become six planes adjacent in a cross pattern. It is easy to see that the ant must use the standard metric in $\mathbb{R}^2$ between those points, p, q in contiguously adjacent subplanes. If the planes are not adjacent, the ant must find the best possible from one plane to another and then readopt the $\mathbb{R}^2$ metric.

   The spider's space would be endowed with the ant metric except for when the spider wishes to go from points on the ceiling, C, to the floor F, in that case, the spider would use the taxicab metric for $\mathbb{R}^3$ to reach a point directly below the point on the ceiling, on the floor, and then again assume the ant metric. This process would not work in reverse however, considering that the ant cannot ascend, unless a web has already been created (a can of worms into which we will not venture!).

2. If $M = \mathbb{R}^2$ is a metric space, we say that it is endowed with the taxicab metric if and only if
$$d(x, y) = \|x - y\|_1 = |x_1 - y_1| + |x_2 - y_2|.$$
   This name arises if one considers the natural metric for a taxicab driver. Clearly the driver must stay aligned to the grid system which is a subset of $\mathbb{R}^2$. Hence if such a driver wishes to travel from point q to point p. He must first go a long the grid in the x direction until his x position coincides with p, and then head in the y direction untill his x and y positions are at p. The process requires that he travel the distances in the x and y directions independently. Essentially $d(p, q) =$ distance_x_direction + distance_y_direction $= \|x - y\|_1 = |p_1 - q_1| + |p_2 - q_2|$. Therefore it is natural to arive at our definition of the taxicab metric.

8. *Prove the following*

   (a) *Absolute convegence*

   Theorem 5. *Let (x_n) be a convegent sequence in $\mathbb{R}$. Then, the sequence of absolute values $(|x_n|)$ converges in $\mathbb{R}$.*

   Proof. If (x_n) converges, then there exists a limit x such that for every $\epsilon > 0$, there exists an N such that for all $n > N$, $d(x_n, x) < \epsilon$. Since the natural metric on $\mathbb{R}$ is the absolute value, we have that $|x_n - x| < \epsilon$. This holds for any sign of x_n and x, so $||x_n| - |x|| \leq |x_n - x| < \epsilon$ implies that $|x_n| \to |x|$. $\square$

(b) *State the converse.*

Theorem 6. *If for some sequence (x_n) in $\mathbb{R}$, $|x_n| \to |L|$, then $*(x_n)$ converges to some limit.*

(c) *Disprove the the previous statement.* We show by counter example, the theorem cannot be true. Take $(a_n) = (-1)^n$. Clearly $|a_n| \to 1$, but there exists and $\epsilon > 0$, say $\epsilon = 0.5$, such that for all N there exists an n, (take the next odd n), such that $|a_n - 1| = |-1-1| = 2 \geq 0.5 = \epsilon$. Hence (a_n) does not converge to the limit of its absolute value sequence. The theorem cannot be true.

12. *Prove the following*

(a) *Bijective:*

Theorem 7. *If (p_n) is a sequence, and $f : \mathbb{N} \to \mathbb{N}$ is a bijection, then the sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is a rearrangement. If $p_n \to L$, then $q_n \to L$ for all such f.*

Proof. If $p_n \to L$, then for all $\epsilon > 0$ p_n is a distance less than ϵ from L for all but finiteley many n. Let N be the number of those first elements of the sequence which are more than ϵ to L. Since f is bijective, the set $f^{-1}(\{0, ..., N\})$ is also finite, and therefore there exists and element $M \in \mathbb{N}$ such that for all $n \in f^{-1}(\{0, \dots, N\}), M > n$. Therefore, for all $m > M$, we have that $|q_m - L| < \epsilon$ if and only if $q_m \to L$. □

(b) *Injective*

Theorem 8. *If (p_n) is a sequence, and $f : \mathbb{N} \to \mathbb{N}$ is an injection, then the sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is a rearrangement. If $p_n \to L$, then $q_n \to L$ for all such f.*

Proof. If f is an injection the theorem will still hold. Recall that an injection implies that each element in the range has a singleton pre-image. Furthermore, each element in the co-domain has a singleton or empty pre-image. Thus if p_n is convergent, then for all $\epsilon > 0$, there exists an N such that for all $n > N$, $|p_n - L| < \epsilon$. Hence $\bigcup_{j=1}^{N} f^{pre}(j) = P$ is finite and contains the n for which q_n is a distance greater than or equal to ϵ away from L. Thus take the maximal element of, P, say M. Then for all $n > M$ (such that there exists a x with $f(x) = n$. Note: there must be infinitely many such n), $|q_n - L| < \epsilon$, and hence $q_n \to L$. □

(c) *Surjective:* In the case that f is only a surjection, then we show that not all rearrangements converge by counterexample. Take, for example, a sequence $p_n = \frac{n}{n+1}$. This sequence clearly converges to 1, but consider the surjective rearrangement, $f(2n) = 1, f(2n-1) = n$. Such a map takes even elements and maps them to 1, and otherwise takes odd elements and maps them to half their double. It's easy to see that this function is surjective, $f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 1, \dots$. However, there exists an $\epsilon > 0$, say $\epsilon = 0.5$, such that for all N, $n > N$, and $n = 2N$ implies that $|p_{f(n)} - 1| \geq \epsilon$. Hence, the rearrangement does not converge. By counterexample, not all surjective rearrangements converge if the normal sequence does.