MATH H104: Homework 8

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30. I'll prove this one in reverse, first defining the general β cantor set.

Definition 1. *The β cantor set, F_β, is defined as an iterative process of which the first iteration is defined for $0 < \beta < 1$ by taking the middle β of $[0, 1]$ and letting F_β^1 be the remaining two intervals of outermeasure $\frac{1-\beta}{2}$. In general, at any iteration of the process n, F_β^n is comprised of 2^n pieces, each of outer measure P_n. The process for generating P_{n+1} is the same as for the first iteration: remove $P_n\beta$ from each piece of outer measure P_n and yield two pieces of outer measure $P_n(1-\beta)/2$. Lastly, $F_\beta \subset F_\beta^n \subset F_\beta^{n-1.}$ for all n.*

Theorem 1. *For any $0 < \beta < 1$, F_β is a zero set.*

Proof. To show this, for every $\epsilon > 0$ we need to find a countable collection of open sets which cover F_β and have outer measure less than ϵ. By the definition of F_n we have that at any iteration n the total outer measure is $P_n 2^n$. Thus we solve the recurrance relation,

$$P_n = P_{n-1}\frac{(1-\beta)}{2},$$

by letting $P_n = \frac{(1-\beta)}{2}$ and solving for the initial conditions that $P_0 = 1$. Thus the total outer measure of F_β^n is defined as

$$outer(F_\beta^n) = \frac{(1-\beta)^n}{2^n}2^n = (1-\beta)^n \to 0$$

by $0 < 1-\beta < 1$. So for every ϵ there is a large enough N such that by extending F_β^N to a very close open interval containing F_β^N and thereby F_β, the outer measure is less than ϵ. So F_β is a zero set. $\square$

It follows simply that the middle fourths cantor set is a zero-set.

31. Again I'll provide a definition for the general fat cantor set and apply it to the specifc definition provided.

Definition 2. *The fat β cantor set, F_β, is defined as an iterative process of which the first iteration is defined for $0 < \beta < 1$ by taking the middle β^n of $[0,1]$ and letting F_β^1 be the remaining two intervals of outermeasure $\frac{1-\beta^n}{2}$. In general, at any iteration of the process n, F_β^n is comprised of 2^n pieces, each of outer measure P_n. The process for generating P_{n+1} is the same as for the first iteration: remove $P_n\beta^n$ from each piece of outer measure P_n and yield two pieces of outer measure $P_n(1-\beta^n)/2$. Lastly, $F_\beta \subset F_\beta^n \subset F_\beta^{n-1.}$ for all n.*

Theorem 2. *The fat β cantor set is not zero set.*

Proof. We essentially need to show that as F_β^n approaches F_β, the outer measure of F_β^n does not tend towards 0. By the definition of F_n we have that at any iteration n the total outer measure is $P_n 2^n$. Thus we solve the recurrance relation,

$$P_n = P_{n-1}\frac{(1-\beta^n)}{2}.$$

Using intuition from the β canot set case, we let P_n be for the form

$$P_n = \frac{(1-\beta^n)^{a_n}}{2^N}$$

for some sequence a_n depending on n. Then we can find a_n by considering the ratio P_n/P_{n-1}. This essentially yields that $a_n - a_{n-1} = 1$. Ommitting the application of variation of parameters to this recurrence relation, we yield $a_n = n$. Thus the total outer measure of F_β^n is defined as

$$outer(F_\beta^n) = \frac{(1-\beta^n)^n}{2^n}2^n = (1-\beta^n)^n \to 1$$

by $0 < 1-\beta < 1$. So there could not possibly be a sequence of countable open coverings of F_β with outer measure approaching 0. This completes the proof. □

In the particular case of the problem, letting $\beta = 1/4$ apply the previous theorem, and yield that the outer measure of the fat cantor set is 1. This is remarkable!

Theorem 3. *The property that S is a zero set is not topological.*

Proof. It suffices to show that for two sets A, B which are homeomorphic, the zero set property does not hold. Clearly $F_\beta \cong C_\beta$ since they are both cantor spaces. Howerver Thoerem 2 states that F_β is not a zero set, whereas C_β is. So by counter example, the zero set property is not topological. □

34. I give a proof of the general case first for functions which map open sets in $\mathbb{R}$.

Theorem 4. *Let $f : U \subset \mathbb{R} \to \mathbb{R}$ be a continuously differentiable map with a set of critical points C. Then, $f(C)$ has outer measure 0.*

Proof. Take some closed interval inside of U, say K. Then let $J : K^2 \to \mathbb{R}$ maps $x, y \in K$ to

$$\frac{-f'(x)(y-x) - f(x) - f(y)}{|y-x|}$$

when $y \neq x$ and 0 otherwise. Since J is composed of a continuously differentiable function f then it is uniformly continuous and can approach 0 simply.

Now we wish to show that $f(C)$ has measure 0 using the above function. In particular, for every $\epsilon > 0$, divide K into N sub intervals of length $outer(K)/N$ satisfying $|J(x,y)| < \epsilon$ for x, y in a self-similar interval. If $x \in K_l$ (a sub interval) is in this case a critical value, we have that

$$J(x,y) < \epsilon \implies |f(y) - f(x)| < |y - x|\epsilon \leq \frac{outer(K)}{N}\epsilon.$$

Since this holds for arbitrary y, $|f(y) - f(y')| < 2\epsilon\frac{outer(K)}{N}$. Now taking the countable union of these subintervals containing critical value, we yield that $outer(f(C \cap K)) < 2\epsilon outer(f(K))$. This implies directly that the outer measure of cricial values over K is 0.

By the second countability of $\mathbb{R}$ we can build C from these intervals in a countable union, and therefore, the outer measure of $f(C)$ is 0. □

This theorem shows the first case for $[a, b]$ because adjoining the end points does not actually add any outer measure to the set C. Furthermore, the theorem holds in $\mathbb{R}$ as it is an open set of itself.

36. Again, consider the following theorem.

Theorem 5. *Any $f : \mathbb{R} \to \mathbb{R}$ has at most countably many jump discontinuities.*

Proof. Recall that f is jump discontinuous at a if and only if both $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist but are either mutually unequal or unequal to $f(c)$ if it exists. This implies that for some ϵ, f is continuous on both $(c - \epsilon, c)$ and $(c, c + \epsilon)$. Let these intervals be called c_-, c_+. Then if

$$S = \{c \in \mathbb{R} \mid \text{c is a jump discontinuity of } f\},$$

let $\mathcal{B}$ be the collection of those disjoint intervals surrounding the jump discontinuities. The collection can be made to consist of disjoint intervals by taking ϵ small enough for each c.

Since there exists a rational in each of these disjoint intervals, we then can assign to $b \in \mathcal{B}$ a rational. It follows that, $S \sim \mathcal{B} \sim \mathbb{Q} \sim \mathbb{N}$. The proof is complete. □

The example provided in (b) clearly has no jump discontinuities, since the limit from the right does not exist at 0.

The example from (c) has discontinuities at $\mathbb{Q}$ but since $\mathbb{Q}$ as a subset of $\mathbb{R}$ has no isolated points, there are no jump discontinuities. qed.

37. No consider the following counterexample. Let $f(x)$ be defined such that if $x < 0, f(x) = \sin(|\frac{1}{x}|)$, if $x = 0, f(x) = 0$, and if $x > 0, f(x) = \frac{1}{x}$. Since the limit as $x \to 0$ from the right of f does not exist, there is a discontinuity of the second kind there. It is clear however that in the interval $(-1, 1)$ the function does not satisfy the intermediate value property. Take for example $x = -1$ and $x = 1/2$. Does there exist a θ in between those two values such that $f(\theta) = 1.5$? No. There cannot exist such a θ, for if $\theta \leq 0$ tjem $f(\theta) \leq 1$ and if $0 < \theta < 1/2$, $f(\theta) > 2$. So f does not posess the intermediate value property.

50. Observe this simple counter-example. We define $f : [0, 1] \to [0, 1]$ such that x uses the following mapping. If $x \in C_\beta$ then it maps to the corresponding fat cantor set F_β in an order preserving fashion. Otherwise, it maps to $1 - x$, or essentially the identity function in reverse order. Since f has only a set of discontinuities with outer measure 0, it is integrable, but in the inverse case, this is not true as F_β has positive outer measure by Theorem 2.

52. We use this first general theorem to show the properties of rieman integrability.

    Theorem 6. *If $h : [c, d] \supset f([a, b]) \to [e, f]$ is continuous and $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then the composition $h \circ f$ is Riemann integrable.*

    Proof. Recall that f satisfies the Riemann integrability criterion if and only if for every $\epsilon > 0$ there is a partition such that

    $$(M_i - m_i) < \frac{\epsilon}{b-a},$$

    where $M_i = \sup_{x \in I_i} f(x)$, and $m_i = \inf_{x \in I_i} f(x)$. By $[c, d]$ compact we have that for every $\gamma > 0$ there exists a δ such that

    $$|x, y| < \delta \implies |\phi(x) - \phi(y)| < \frac{\gamma}{f - e}.$$

    Let α satisfy $h(f(\alpha)) = \sup_w h(f(w))$. Likewise, let β satisfy $h(f(\beta)) = \inf_w h(f(w))$. Then clearly $|f(\alpha) - f(\beta)| < \epsilon/(b - a)$. Pick ϵ such that $\epsilon/(b - a) < \delta$, and we have that

    $$|h(f(\alpha)) - h(f(\beta))| = \sup_w h(f(x)) - \inf_w h(f(x)) < \frac{\gamma}{f - e}.$$

    Therefore for any $\gamma > 0$ there is a partition P such that $U(h \circ f, P) - L(h \circ f, P) < \gamma$. This completes the proof. ∎

    In the specific problem, (a) holds by the above theorem, by supplying $h = |x|$. The statement of (b) does not by counter example: Take $f : x \mapsto x$ if $x \notin F_\beta$ otherwise, $x \mapsto -x$. In this case $|f| = id \in \mathcal{R}$, but f cannot be Riemann integrable since it is discontinuous on a non-zero set. Furthermore, (c) holds since $h : x \mapsto 1/x$ is continuous on $[0 < c, d > c]$. The converse direction could not hold since if $f^2 < 0$ for all x, f has no real values and cannot be Riemann integrable. For (f) take $h : x \mapsto x^{1/3}$, continuous, and (f) is tautological. For (g) take $h : x \mapsto x^{1/2}$, for $x \geq 0$, and the result holds by the above theorem.

53. We propose a similar theorem to Theorem 6, except for functions of two variables.

    Theorem 7. *Provided $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, let $K \subset \mathbb{R}^2$ with $K \supset f([a, b]) \times g([a, b])$ and $[e, f] \subset \mathbb{R}$ be two compact sets. If $h : K \to [e, f]$ is continuous then the composition $h \circ (f, g)$ is Riemann integrable.*

    Proof. Recall that f, g satisfies the Riemann integrability criterion if and only if for every $\epsilon > 0$ there are paritions P_1, P_2 and a refinement $P = P_1 \cup P_2$ such that

    $$(M_i - m_i) < \frac{\epsilon}{2(b-a)} \wedge (M_i' - m_i') < \frac{\epsilon}{2(b-a)}$$

where $M_i = \sup_{x \in I_i} f(x)$, $m_i = \inf_{x \in I_i} f(x)$, $M_i' = \sup_{x \in I_i} g(x)$, and $m_i' = \inf_{x \in I_i} g(x)$ for $I_i \subset P$.

By K compact, h uniformly continuous, and we have that for every $\gamma > 0$ there exists a δ such that

$$d(w_1, w_2) < \delta \implies |\phi(w_1) - \phi(w_2)| < \frac{\gamma}{f - e}.$$

Let α satisfy $h(f(\alpha), g(\alpha)) = \sup_x h(f(w), g(w))$. Likewise, let β satisfy $h(f(\beta), g(\beta)) = \inf_x h(f(w), g(w))$. Then clearly $|f(\alpha) - f(\beta)| < \epsilon/(2(b-a))$ and $|g(\alpha) - g(\beta)| < \epsilon/(2(b-a))$. Pick ϵ such that $\epsilon/(b-a) < \delta$, and we have that by the triangle inequality $d((f(\alpha), g(\alpha))(f(\beta), g(\beta)) < \delta$. Simply, this realtion yields,

$$|h(f(\alpha), g(\alpha)) - h(f(\beta), g(\beta))| = \sup_w h(f(x)) - \inf_w h(f(x)) < \frac{\gamma}{f - e}.$$

Therefore for any $\gamma > 0$ there is a partition P such that $U(h \circ (f, g), P) - L(h \circ (f, g), P) < \gamma$. This completes the proof. $\square$

Observe that $\min(x, y), \max(x, y)$ are continuous on K compact a subset of $\mathbb{R}^2$. Apply Theorem 7, and $\max(f, g), \min(f, g) \in \mathcal{R}$.

55. Let $f : \mathbb{R} \to \mathbb{R}$ be definedd as follows. Given some β, the denote C as the countable union of the cantor set C_β offset onto the interval $[k, k+1]$ for $k \in \mathbb{Z}$. Imagine putting cantor dust on every interval of length 1 starting from $[0, 1]$. The let $f : x \mapsto x$ iff $x \in C$, otherwise, $x \mapsto 0$. The improper integral over $\mathbb{R}$ is $\int_0^\infty f\, dx + \int_0^{-\infty} f\, dx = 0 + 0$ since C has outer measure 0. Furthermore, f is unbounded.