Math 215A — UCB, Spring 2017 — William Guss
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Selected Problems: 1,2 (Depending on that which wasn't submitted by Alekos or Chris.)

(1.1) (Interesting Examples)

- *Write down interesting topological spaces that are not Hausdorff.*

  Recall the definition of a Hausdorff space.

  Definition 0.1. Let X be a topological space endowed with a topology τ. If for every $x, y \in X$ there are open neighborhoods $U \ni x$ and $V \ni y$ so that $U \cap V = \emptyset$, then X is called Hausdorff.

  With this definition in mind we will propose the following examples of a topological space (X, τ) that is not Hausdorff.

  Example 1. Let $X = \{x, y\}$ and $\tau = \{\{x\}, \{x, y\}, \emptyset\}$. First (X, τ) is a topological space as τ contains $X, \emptyset$ and is closed under union and finite intersection. However $x \neq y$ and $\{x\} \ni x$ and $\{x, y\} \ni y$ are not disjoint and there is no $U \in \tau$ so that $U \ni y$ and not $U \ni x$. Therefore (X, τ) is not Hausdorff.

  Example 2. Let $x, y \in \mathbb{R}$ be $\sim$-equivalent $(x \sim y)$ if $x - y \in \mathbb{Q}$. Then let $X = (\mathbb{R}/\sim)$. We claim that the induced quotient topology is not Hausdorff.

  Take $x = [0], y = [e]$. Then let $\tilde{U}, \tilde{V}$ be open neighborhoods of x and y respectively. We have that $\tilde{U} = \pi(U)$ and $\tilde{V} = \pi(V)$ where U, V are open in $\mathbb{R}$ with the standard topology.

  From real analysis we know that every open V in $\mathbb{R}$ must contain a rational number as V is composed of open intervals. Therefore $r \in \mathbb{Q}$ so that $r \in V$. Thus $\pi(r) = [0] \in \tilde{V}$ and thus $\tilde{U} \cap \tilde{V} \neq \emptyset$

  Since this is true for every pair of open neighborhoods $\tilde{U}, \tilde{V}$ the space could not be Hausdorff[1].

- *Write down continuous bijections that aren't homeomorphisms.*
  Remark. If $f : X \to Y$ is continuous and Y is Hausdorff then f is open. Therefore we can only presume to have non-Hausdorff space as an example.

  Example 1. Let (X, τ) be given from Example 1. Then let $Y = X$ and τ_{disc} be the discrete topology generated by $f : Y \to X$ be identity map. Then clearly the preimage of opens, say $\{x, y\} \subset \tau$ is open $\{x, y\} \subset \tau_{disc}$, but the map is not open for $\{y\} \subset \tau_{desc}$ is not a subset of τ.

  I think the easiest way to generate these examples is to take a non-Hausdorff space and build an automorphism.

  Example 2. Let $X = \mathbb{R}$ with the discrete topology, and let $Y = \mathbb{R}$ wkith the standard open-ball topology. Then let $f : X \to Y$ be an identity map, clearly the preimage of opens is open since every set is open in X with the discrete topology, but take the closed interval $[6, 13]$ and its image is not open although under the discrete topology it is open. Therefore f continous bijection and it is not a homeomorphism.

(1.2) (Normality) Prove or disprove the followikng for a topological space (X, τ).

[1] I like this example.

- *X Hausdorff and completely regular implies X normal.*

We will disprove the above statment by proviong the following lemma and providing a counter example.

Lemma 0.1. *If (X, τ) is completely regular than it is Hasudorff.*

Proof. First observe that if X is completely regular then it is Urhysohn; that is, for any points $x, y \in X$ we have that there is a continuous function $f : X \to [0,1]$ where $[0,1]$ is endowed with the subspace topology and $f(x) = 0, f(y) = 1$.

Now let $x, y \in X$ be given and not identical. Take an Urhysohn function f as above and then observe that $W = [0, 0.5)$ and $Z = (0.5, 1]$ are open subsets of $[0,1]$ in the inherited topology. Furthermore $Z \cap W = \emptyset$ implies that $f^{-1}(Z \cap W) = f^{-1}(Z) \cap f^{-1}(W) = \emptyset$. Furthermore $f^{-1}(Z) = V \ni y$, $f^{-1}(W) = U \ni x$.

Therefore X is Hausdorff. □

Counter Example. We will provide a counter example using the topology X from the first example. The space is trivially normal since the only non-empty closed set is $\{y\}$ and so it is normal, but the space is not Hausdorff and so it is not completely regular. Therefore the assertion is false ;).

- *X Hausdorff, completely regular and Lindelof implies X normal.*

Proof. Let $A, B \subset X$ be two disjoint closed sets in X. Then for every $x \in A$ let U_x be an openset containing x disjoint from B and let V_y be the converse for every $y \in B$.

Now $\{U_x\}_{x \in A} = \mathcal{U}$ and $\{V_y\}_{y \in B} = \mathcal{V}$. Then by the Linedelof property there exists a countable subcover indexed by $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ for $\mathcal{U}$ and $\mathcal{V}$ respectively.

Observe the following disjoitness properties:

$$\begin{aligned} U_{x_1} \setminus \overline{V_{y_1}} \cap V_{y_1} \setminus \overline{U_{x_1}} &= \emptyset \\ U_{x_2} \setminus (\overline{V_{y_1}} \cup \overline{V_{y_2}}) \cap V_{y_2} \setminus (\overline{U_{x_1}} \cup \overline{U_{x_2}}) &= \emptyset \\ \vdots \qquad\qquad\qquad \vdots \qquad &= \vdots \\ \bigcup_{i=1}^{\infty} \left[U_{x_i} \setminus \bigcup_{j \leq i} \overline{V_{y_j}} \right] \cap \bigcup_{i=1}^{\infty} \left[V_{y_i} \setminus \bigcup_{j \leq i} \overline{U_{x_j}} \right] &= \emptyset \end{aligned}$$

Furthermore $\bigcup_{i=1}^{\infty} [U_{x_i} \setminus \bigcup_{j \leq i} \overline{V_{y_j}}]$ is a an open set since we subtract the finite union of closed sets from an opensets; that is we subtract a closed set from an open set and thus the resultant is open. Additionally $\bigcup_{i=1}^{\infty} \left[U_{x_i} \setminus \bigcup_{j \leq i} \overline{V_{y_j}} \right] \supset A$ and $\bigcup_{i=1}^{\infty} \left[V_{y_i} \setminus \bigcup_{j \leq i} \overline{U_{x_j}} \right] \supset B$. Therefore there are two disjoint open sets containing both A and B respectively. Since A, B were arbitrary X is normal. This completes the proof. □