MATH 105: Homework 12

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65. **Critical Values!** Critical values of sin are $\{-1, 1\}$. Critical points are the multiples of π.

Theorem 1. *If $f : [a,b] \to \mathbb{R}, f \in C^1$ then $cp(f), cv(f)$ are compact.*

Proof. We need to show that $cp(f) = \{x : f'(x) = 0\}$ is closed. Since f' is continuous, we consider the f' 0-locus. It is true that $cp(f) = f'^{pre}(0)$ is closed since $\{0\}$ is closed. Therefore $cp(f)$ is compact. Finally by the continuity of f, $f(cp(f)) = cv(f)$ is compact. □

Theorem 2. *If $f \in C^1(\mathbb{R})$ then $cv(f)$ is a zeroset.*

Proof. It must be shown that $cv(f)$ can be covered with intervals of small length. Take any $\epsilon > 0$. Then for every $\theta \in cp(f)$ take $|a - \theta| < \sqrt{\epsilon}/2$ and a close enough that $f'(a) - f'(\theta) < \sqrt{\epsilon}$. It follows that $m((f(\theta), f(a)) = 1/2m((\theta - a, \theta + a))$. Furthermore by the mean value theorem there is a γ such that

$$f(\theta) - f(a) = f'(\gamma)\frac{\sqrt{\epsilon}}{2} \le \frac{\epsilon}{2}. \tag{1}$$

Therefore it follows that $m((f(a), 2(f(\theta) - f(a)) + f(a))) = \epsilon$ where $I_\theta = (f(a), 2(f(\theta) - f(a)) + f(a))$,

For every $\vartheta \in cp(v)$ if $f(\theta) = f(\vartheta)$ then we have an equivalence realtion $\vartheta \sim \theta$. There fore we consider the disjoint union

$$S = \bigsqcup_{\theta \in [cp(f)]_\sim} I_\theta \tag{2}$$

as a covering of $cv(f)$. Since in each interval there is a $q_\theta \in \mathbb{Q}$ the union is countable. Then as $\epsilon \to 0$, $m(I_\theta) \to 0$ implies that for all $\delta > 0$ and for all θ there are intervals I_θ such that

$$\bigsqcup_{\theta \in [cp(f)]_\sim} I_\theta < \delta \implies cv(f) \text{ a zeroset.} \tag{3}$$

We can then generalize to all of $\mathbb{R}$ by restricting f to $(z, z+1)$ for every $z \in \mathbb{Z}$, call that f_z. The total set of critical values is

$$cv(f) = \bigcup_{z \in \mathbb{Z}} cv(f_z) \tag{4}$$

which is the countable union of zerosets, ie, a zeroset. $\square$

66. **An interesting function!**

Theorem 3. *There exists a monotone function $f : [0,1] \to \mathbb{R}$ whose discotninuity set is exactly the set $\mathbb{Q} \cap [0,1]$.*

Proof. Take any enumeration of $\mathbb{Q} \cap [0,1]$, say $\{a_k\} \subset \mathbb{Q}$. Then let $f_0 : x \mapsto x$. We define f_n as follows. If $x < a_n$ then we simply have $f_n : x \mapsto f_{n-1}(x)$. For $x = a_n$,

$$f_n : x \mapsto \sup_{y \leq a_n} f_{n-1}(y) + \frac{1}{2n^2}. \tag{5}$$

At $x > a_n$, then

$$f_n : x \mapsto f_{n-1}(x) + \frac{1}{n^2}. \tag{6}$$

We know that $\limsup f_n \leq \sum_{n=1}^{\infty} 1/n^2 \in \mathbb{R}$. So we can bound the function.

We show that f_n is uniformly cauchy; that is, for every $\epsilon > 0$ we claim that there exists an N such that for all $n, m > N$ $\|f_n - f_m\| < \epsilon$. To see this consider that the main difference of these functions is exacerbated at the end of the intervals, at $f_n(1)$ and $f_m(1)$.

The difference $f_n(1) - f_m(1) = \sum_k^n 1/k^2 - \sum_k^m 1/k^2$ gives without loss of generality

$$\|f_n - f_m\| \leq \sum_{k=m}^{n} \frac{1}{k^2}. \tag{7}$$

Since there series $\sum 1/k^2$ converges take N so large that the partial sums of that series differ by no more than ϵ. Therefore f_n converges uniformly to some f.

Now, every f_n is riemann integrable since its set of discontinuities is a zeroset. Therefore f is riemann integrable and therefore its set of discontinuities is a zeroset. This completes the proof. $\square$

70. **Kernel's Hull's and other pretty cool stuff!**

(a) Uniqueness.

Theorem 4. *Let $A \subset \mathbb{R}^n$ be a bounded set. Then K_A and H_A are unique up to a zeroset.*

Proof. Take two kernels of A say K, K'. These sets are F_σ and there measure is the supremum of all of the closeds, $\kappa \subset A$. We claim that these two sets have a mutual set of nonzero measure assuming that K, K' are not zerosets (if they were then they would be uniqueley empty up to zerosets!)

Suppose they did not. This would mean that there are two families of closeds within A say $\mathcal{K}_1$ and $\mathcal{K}_2$ such every element in the first is disjoint $\mathcal{K}_1$ and $\mathcal{K}_2$from every element of the second. To see this, imagine that K and like wise K' are the unions of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectiveley and kernels are F_σ sets. If all of this were true, then the kernel $K'' = \bigcup \mathcal{K}_1 \cup \bigcup \mathcal{K}_2$ mus have more measure than both kernels and thus a larger measure, which contradicts the kernels being maximal with respect to the supremum of closeds within A. Therefore they have a common set, call it B.

Now $K_1 = B \cup Z_1$ and $K_2 = B \cup Z_2$. Since they differ by more than a zeroset, it must be that Z_1 or Z_2 is not a zero set. In fact one or both of these sets has positive measure. Since sets of positive measure contain a closed with some measure, it follows that the measure of either K or K' is greater than the other. This again contradicts the fact that K and K' are maximal. Therefore Z_1 and Z_2 must be zero sets. This is a contradition to our hypothesis that K and K' differ by more than a zeroset.

Take two hulls of A, say H and H'. These sets are G_δ and their measures are the infimum of opens containing A. Supose that H and H' differed by more than a zeroset. Bound this A by a box of at least twice the diameter of A. Then let this box be our universe, such that compliments are taken in $\mathbb{R}^n$ and then intersected with the box.

If $H = O_1 \cup Z_1, H' = O_2 \cup Z_2$are the minimal sets then H^c, H'^c are F_σ sets which are maximal with respect to the kernel on A^c. If H and H' differ by more than a zero set then H^c, H'^c certainly differ by more than a zero set and yet they are hulls of A^c, which is a contradiction. Therefore H and H' must not differ by more than a zero set. ☐

(b) Jam on bread.

Theorem 5. *For every $A \subset \mathbb{R}^n$ and every E measurable,*

$$m^*(A \cap E) = m(H_A \cap E). \tag{8}$$

Proof. This result is natural! Recall that $m(H_A) = m^*(A)$. We then need to show that $H_A \cap E$ is a hull for $A \cap E$. Take the hull of E, this set is a closed and a zeroset. Therefore the intersction, $H_A \cap E$ is a closed an a zeroset and an G_δ. We need to show that this set is minimal with respect to $A \cap E$. This easy since $x \in H_A \cap H_E$ implies that x is in the minimal most covering of A and the minimal most covering of E, measure theoretically. The measurabiliy of E gives the same result of $x \in H_A \cap E$; that is $H_A \cap E$ is the hull for $A \cap E$. Therefore $m^*(A \cap E) = m(H_A \cap E)$. ☐

(c) Density in hulls.

Theorem 6. *For almost every $p \in H_A$ we have*

$$\lim_{Q \downarrow p} \frac{m^*(A \cap Q)}{mQ} = 1. \tag{9}$$

Proof. Since H_A is a closed and a zeroset, ie. G_δ, we take every point within the closed, ie. almost every point. Call the closed set $\mathfrak{H}$. Since $\mathfrak{H}$ is closed the

Labesgue Density Theorem along with (b) gives

$$1 = \lim_{Q \downarrow p} \frac{m(\mathfrak{H} \cap Q)}{mQ} = \lim_{Q \downarrow p} \frac{m(H_A \cap Q)}{mQ} = \lim_{Q \downarrow p} \frac{m^*(A \cap Q)}{mQ}. \tag{10}$$

This completes the proof. □

71. It is true that $H_A \setminus A$ is a zeroset, this follows from H_A a G_δ.

76. **Closed locus diffeomorphisms.**

Theorem 7. *Given a closed set $L \subset \mathbb{R}$ there exists a C^∞ function $\beta : \mathbb{R} \to [0, \infty)$ whose zero locus $\{x \ : \ \beta(x) = 0\} = L$.*

Proof. L is closed and therefore its compliment L^c is open. By the compactness of L we have that there exist a large ball $Q \supset L$ which is finite. Take $R = L^c \cap Q$. Furthermore there exists a countable disjoint efficient Vitali covering of Q by a family of supper effective balls $\mathcal{B} = \{B_i\}$.

Define the interior of R to be R^0 and the interior of $\bigcup \mathcal{B}$ to be B^0. Then, $B^0 \cup Z = R^0$ where Z is a zero set. We wish to complete B^0 in a finite way that gives us all of R^0. In other words we'd like to cover Z with finitely many balls.

Since $Z \subset R^0$ we have that for every $z \in Z$ there is an $r_z > 0$ so that $\mathbb{B}^0_{r_z}(z) \subset R^0$, where $\mathbb{B}^0_\rho$ denotes the open ball centered at z with radius ρ. We also denote the center of the ball $o(\mathbb{B}^0_r(z)) = z$.

Let $\mathcal{E} = \{B_{r_z}(z)\}_{z \in Z}$ be the family of all such balls. This family covers $cl(Z) \subset R$ almost everywhere and reduces to a finite subcovering of Z, $\mathcal{E}_F$.

Recall the bump function ϕ such that $\int \phi = 1$ and $\phi \in C^\infty$. We define $\gamma : Q \to [0, \infty)$ as the disjoint vitali map such that if $x \in B_o \in \mathcal{B}$,

$$\gamma : x \mapsto \phi\left(\frac{x - o(B_i)}{\mathrm{diam}(B_i)}\right). \tag{11}$$

Otherwise $\gamma : x \mapsto 0$. Clearly γ is C^∞ since $B_i \cap B_j = \emptyset$ for all $i \neq j$.

Then we let $\beta_Q : Q \to [0, \infty)$ be defined as the full covering map

$$\beta_Q : x \mapsto \gamma(x) + \sum_{E \in \mathcal{E}_F} \phi\left(\frac{x - o(E)}{\mathrm{diam}(E)}\right). \tag{12}$$

This map is C^∞ since it is the finite sum of C^∞functions. Furthermore β_Q is only 0 if $x \notin R^0$ or equivelently $x \in L$.

Finally we can extend β_Q to the whole space smoothly using $\mathbf{e}$ from Chapter 3. That is if $x \in Q$ then define $\beta : x \mapsto \beta_Q(x)$. Otherwise we let

$$\beta : x \mapsto \mathbf{e}\left(\left\| x - \frac{\mathrm{diam}(Q)}{\|x - o(Q)\|}(x - o(Q))\right\|\right). \tag{13}$$

This map essentially takes x from the boundary of Q and brings it towards infinity smoothly so that no value outside of Q is acutally 0.

Therefore $\beta \in C^\infty(\mathbb{R}^n, [0, \infty))$ with zero locus only at L this completes the proof. □

77. **Smooth cantor shrinkification.**

Theorem 8. *Suppose that $F \subset [0,2]$ is a fat cantor set of measure 1. There exists a C^∞ homeomorphism of $h : \mathbb{R} \to \mathbb{R}$ which carries $[0,2]$ to $[0,1]$ and F to a cantor set hF with measure 0.*

Proof. Take the mapping $\beta \in C^\infty$ with zero locus F using Theorem 7. Then let $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = c \int_0^x \beta(t)\ dt, \quad c = \frac{1}{\int_0^2 \beta(t)\ dt}. \tag{14}$$

It follows that $h'(x) = c\beta(x) = 0$ if $x \in F$. Therefore by Theorem 2 we have that hF is the set of critical values for h and therefore a zeroset. Since h is homeomorphic we have that hF is a cantor set with measure zero. Finally c is such that $h : [0,2] \mapsto [0,1]$. This completes the proof. □

78. **Compositional measurability.**

Theorem 9. *Suppose $f : \mathbb{R} \to [0,\infty)$ is Lebesgue measureable and $g : [0,\infty) \to [0,\infty)$ is monotone or continuous. The composition $g \circ f$ is Lebesgue measurable.*

Proof. We wish to show that for every open set E F_σ, $(g \circ f)^{pre}(E)$ measurable.

For any open E take $F = f^{pre}(E)$. This set is measurable by the measurabiltiy of f. It is enough to show that $g^{pre}(F)$ is measurable. If g is continuous this is immediate if we take Borel σ-algebra (Lebesgue is not so kind). If g is montone its set of discontinuities is a zeroset. In this case take the continuous restriction of g to be g_C and the discontinuous to be g_D. Then $g^{pre}(F) = g_C^{pre}(F) \cup g_D pre(F)$ which is a measurable union and a zeroset. Therefore in any case g^{pre} is measurable!

In the Lebesgue case I'm not convinced this theorem is true! Actually you can take g to be the identity map between $(\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{L})$ and it is certain that there are sets in $\mathcal{L}$ which are not in $\mathcal{B}$. So let's stick to Borel measurable sets. □