Math 185 — UCB, Fall 2016 — William Guss
Problem Set 3, due October 4th

(42.2) Evaluate the following integrals:
(a)

$$\int_0^1 (1+it)^2 \; dt;$$

Solution. We evaluate the integral along each of its components by first seprating it into its real and complex parts. Let $X = [0,1]$

$$\int_X (1+it)^2 \; dt = \int_X 1^2 + 2it - t^2 \; dt = \int_X 1 - t^2 + 2i \int_X t.$$

Thus evaluation on both parts yields

$$\int_X (1+it)^2 \; dt = \left[t - \frac{1}{3}t^3\right]_X + 2i\left[\frac{1}{2}t^2\right]_X = \frac{2}{3} + i$$

□

(b)

$$\int_1^2 \left(\frac{1}{t} - i\right)^2 \; dt$$

Solution. We evaluate the integral along each of its components by first seprating it into its real and complex parts. Let $X = [1,2]$

$$\int_X \left(\frac{1}{t} - i\right)^2 \; dt = \int_X \frac{1}{t^2} - \frac{2i}{t} - 1 \; dt = \int_X \frac{1}{t^2} - 1 \; dt - 2i \int_X \frac{1}{t}.$$

Thus evaluation on both parts yields

$$\int_X \left(\frac{1}{t} - i\right)^2 \; dt = \left[\frac{-1}{t} - t\right]_X - 2i\left[\ln(t)\right]_X = -\frac{1}{2} - 2i\ln(2).$$

□

(c)

$$\int_0^{\pi/6} e^{i2t} \; dt$$

Solution. We evaluate the integral along each of its components by first seprating it into its real and complex parts. Let $X = [0, \pi/6]$

$$\int_X e^{i2t} \; dt = \left[-i\frac{e^{i2t}}{2}\right]_X = \left[-\frac{e^{i(2t+\pi/2)}}{2}\right]_X = \frac{e^{i(\pi/2)} - e^{i5\pi/6}}{2} = \frac{i - e^{i5\pi/6}}{2} = \frac{i+\sqrt{3}}{4}$$

□

(d)

$$\int_0^\infty e^{-zt} \; dt; \qquad (Re \; z > 0)$$

Solution. We evaluate the integral along each of its components by first seprating it into its real and complex parts. Let $X_n = [0, n)$

$$\lim_{n\to\infty} \int_{X_n} e^{-zt}\ dt = \lim_{n\to\infty} \left[-\frac{e^{-zt}}{z} \right]_{X_n} = \lim_{n\to\infty} \left[-\frac{e^{-zn}}{z} \right] + \frac{1}{z}$$

The value of the limit is established so that if $Re(z) > 0$ we have

$$-\frac{e^{-zn}}{z} = -\frac{e^{-nRe(z)}e^{i\cdot Im(-zn)}}{z} \to 0$$

since the radial magnitude of $|e^{-nz}| = e^{-nRe(z)} \to 0$. Thus

$$\int_0^\infty e^{-zt}\ dt = \frac{1}{z}.$$

□

(42.3) Show that if m and n are integers

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\ d\theta = x$$

where $x = 0$ when $m \neq n$ and 2π when $m = n$.

Proof. Consider the case first when $m = n$. Then

$$\int_0^{2\pi} e^{im\theta}e^{in\theta}\ d\theta = \int_0^{2\pi} e^{im\theta}e^{-im\theta}\ d\theta = \int_0^{2\pi} e^{im\theta - im\theta}\ d\theta = \int_0^{2\pi} 1\ d\theta = 2\pi.$$

In the case that $m \neq n$ then

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\ d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta = -i\left[\frac{e^{i(m-n)\theta}}{(m-n)}\right]_0^{2\pi} = -i\frac{e^{ik2\pi}-1}{m-n} = 0$$

since $1 - 1 = 0$. □

(42.4) Show that

$$\int_0^\pi e^x \cos x\ dx = -\frac{1+e^\pi}{2} \qquad \int_0^\pi e^x \sin x\ dx = \frac{1+e^\pi}{2}$$

Proof. Recall that the above integrals are the real and imaginary parts of $\int_0^\pi e^{(1+i)x}\ dx$ respectiveley. Let $X = [0, \pi]$ and then

$$\int_X e^{(1+i)x}\ dx = \left[\frac{e^{(1+i)x}}{1+i}\right]_X = \frac{e^{\pi+i\pi}-1}{1+i} = \frac{e^\pi(\cos\pi + i\sin\pi)}{1+i}$$

Continuing the algebra

$$\int_X e^{(1+i)x}\ dx = \frac{-e^\pi - 1}{1+i} = \frac{-(e^\pi+1)(1-i)}{2} = \frac{-(e^\pi+1)}{2} + i\frac{(e^\pi+1)}{2}.$$

Observing the statement of the proposition and the components of the complex number to which the integral evaluated completes the proof. □

(46.1) Let $f = (z+2)/z$. Then for the following contours, C evaluate

$$\int_C f(z)\ dz.$$

(a) Let $C = \theta \mapsto 2e^{i\theta}$ such that $\theta \in X = [0, \pi]$.

Solution. Then we evaluate the contour integral parametrically using

$$\int_C f(z)\ dz = \int_X \frac{2e^{i\theta}+2}{2e^{i\theta}} \left(2ie^{i\theta}\right)\ d\theta = 2i \int_X \left(e^{i\theta}+1\right) d\theta$$

Computing the volumetric integral over the 1-cell gives

$$\int_C f(z)\ dz = 2i\left[-ie^{i\theta}+\theta\right]_X = 2i\left[-ie^{i\theta}\right]_X + i2\pi = 2\left[-2\right] + i2\pi = -4 + i2\pi$$

□

(b) Let $C = \theta \mapsto 2e^{i\theta}$ such that $\theta \in X = [\pi, 2\pi]$.

Solution. Then we evaluate the contour integral parametrically using

$$\int_C f(z)\ dz = \int_X \frac{2e^{i\theta}+2}{2e^{i\theta}} \left(2ie^{i\theta}\right)\ d\theta = 2i \int_X \left(e^{i\theta}+1\right) d\theta$$

Computing the volumetric integral over the 1-cell gives

$$\int_C f(z)\ dz = 2i\left[-ie^{i\theta}+\theta\right]_X = 2i\left[-ie^{i\theta}\right]_X + i2\pi = 2\left[e^{i\theta}\right] + i2\pi = (1-(-1)) + i2\pi = 4 + 2\pi$$

□

(c) Let $C = \theta \mapsto 2e^{i\theta}$ such that $\theta \in X = [0, 2\pi]$.

Solution. Then we evaluate the contour integral parametrically using

$$\int_C f(z)\ dz = \int_X \frac{2e^{i\theta}+2}{2e^{i\theta}} \left(2ie^{i\theta}\right)\ d\theta = 2i \int_X \left(e^{i\theta}+1\right) d\theta$$

Computing the volumetric integral over the 1-cell gives

$$\int_C f(z)\ dz = 2i\left[-ie^{i\theta}+\theta\right]_X = 2i\left[-ie^{i\theta}\right]_X + i4\pi = 2\left[e^{i\theta}\right] + i4\pi = 2(1-1) + i4\pi = i4\pi.$$

One will observe that the 1-form can be evaluated by summing the results of the two evaluations on C_a, C_b since differential forms are linear up to reparameterizations. □

(46.4) Define $f : \mathbb{C} \to \mathbb{C}$ so that $z \mapsto 1$ when $Im(z) < 0$ and $z \mapsto 4y$ when $Im(z) > 0$. Then let $C : E \subset \mathbb{C}$ be a 1-cell such that $C(0) = -z - i$ and $C(1) = 1 + i$ along the curve $y = x^3$. Evaluate the 1-form

$$f\ dz(C) = \int_C f\ dz.$$

Solution. If $x = -1$ then $y = x^3 = -1$, additionally if $x = 1$ then $y = x^3 = 1$. Therefore let $\xi : X = [-1, 1] \to \mathbb{C}$ such that $t \mapsto t^3$ be a diffeomorphism, and therefore, a 1-cell which reparameterizes C. By the theory of differential forms $f\ dz(\xi) = f\ dz(C)$. Now we compute the form on ξ.

$$f\ dz(\xi) = \int_\xi f\ dz = \int_X f(\xi(t))det\left|\frac{\partial \xi(t)}{\partial t}\right|\ dt = \int_X f(\xi(t))\xi'(t)\ dt.$$

We then use the partwise decomposition property of integration and let $X_- = [-1, 0)$ and $X_+ = (0, 1]$ and since $\{0\}$ is a zeroset w.r.t standard Labesgue measure on $\mathbb{R}$ without loss of generality the we redefine f such that $z \mapsto 1$ when $y = 0$ and thus

$$f\ dz(\xi) = \int_0^1 4t^3 \cdot (1 + i3t^2)\ dt + \int_{-1}^0 1 \cdot (1 + i3t^2)\ dt.$$

Evaluation of the right-most quantity side gives $\int_0^1 3t^2 i + 1\ dt = 1 + i$. Evaluation of left leg gives

$$\int_1^0 4t^3 + 12t^5 i\ dt = \left[t^4 + 2t^6 i\right]_{-1}^0 = ((1)^4 + 2i) - 0 - 0i = 1 + 2i$$

Therefore $f\ dz(\xi) = 2 + 3i$

□

(46.9) Let $C : [0, 2\pi) \to S^1 \subset \mathbb{C}$ be a diffeomorphic 1-cell.
(a) Show that if $f(z)$ is the principle branch

$$z^{-3/4} = exp\left(-\frac{3}{4}Log(z)\right) \qquad (|z| > 0,\ \ -\pi < Arg(z) < \pi)$$

then $f\ dz(C) = 4\sqrt{2}i$.

Proof. Seperating the evaluation of the differential for m$f\ dz$ on C into two 1-cells C_1 from $0 \to \pi$ and C_2 from $0 \to -\pi$. Then $C_1 : \theta \mapsto e^{i\theta}$ and the jacobian of C_1 is C_1' w.r.t θ yielding $C_1' = ie^{i\theta}$. The same can be done for C_2 when letting $\theta_2 = -\theta$ parameterize the 1-cell. We therefore reduce the calculation to

$$f\ dz(C_1) = i\int_0^\pi exp\left(-\frac{3}{4}i\theta\right)e^{i\theta}\ d\theta = i\int_0^\pi exp\left(-\frac{3}{4}i\theta + i\theta\right)\ d\theta.$$

Then we perform normal integration giving

$$f\ dz(C_1) = i\int_0^\pi exp\left(\frac{i\theta}{4}\right)\ d\theta = 4i\left[exp\left(\frac{i\theta}{4}\right)\right]_0^\pi = -4i(\sqrt{2}/2 + \sqrt{2}/2i - 1)$$

A similar calculation for C_2 is performed yielding

$$f\ dz(C_2) = i\int_0^\pi exp\left(\frac{(3-4)i\theta}{4}\right)\ d\theta = 4i\left[exp\left(\frac{-i\theta}{4}\right)\right]_0^\pi = 4i(\sqrt{2}/2 - \sqrt{2}/2i - 1)$$

Finally $f\ dz(C) = f\ dz(-C_2 + C_1) = f\ dz(C_1) - f\ dz(C_2)$. Therefore

$$f\ dz(C) = -4i((\sqrt{2}/2 + \sqrt{2}/2i + 1) + (-\sqrt{2}/2 + \sqrt{2}/2i - 1)) = 4i\sqrt{2}$$

□

(b) Show that if $g(z)$ is the following branch

$$z^{-3/4} = exp\left(-\frac{3}{4}log(z)\right) \quad (|z| > 0,\ \ 0 < arg(z) < 2\pi)$$

then $g\ dz(C) = -4 + 4i$.

Proof. Recall that $C : [0, 2\pi) \to S^1 \subset \mathbb{C}$ is a diffeomorphic 1-cell. Therefore we evaluate $g\ dz(C)$ as follows

$$g\ dz(C_1) = i\int_0^{\pi} exp\left(-\frac{3}{4}i\theta\right)e^{i\theta}\ d\theta = i\int_0^{\pi} exp\left(-\frac{3}{4}i\theta + i\theta\right)\ d\theta.$$

Then we perform normal integration giving

$$g\ dz(C_1) = i\int_0^{\pi} exp\left(\frac{i\theta}{4}\right)\ d\theta = 4\left[exp\left(\frac{i\theta}{4}\right)\right]_0^{\pi} = 4(\sqrt[4]{-1} - 1)$$

A similar calculation for C_2 is performed yielding $g\ dz(C_2) = 4i(1 + (-1)^{3/4})$. Therefore $g\ dz(C_1 + C_2) = g\ dz(C_1) + g\ dz(C_2) = -4 + 4i$ $\square$

(46.13) (47.1)