Math 202A— UCB, Fall 2016 — M. Christ
Problem Set 4, due Wednesday September 14 - William Guss

(4.1) Let $f_n : X \to \mathbb{R}^*$ be measurable.
(a) Show that $\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

Proof. First we consider $\{x \in X \mid |\lim_{n\to\infty} f_n(x)| < \infty\} = J$, if $x \in J$ then $g(x) - h(x) = z(x) = 0$ iff $h = \liminf f_n, g = \limsup f_n$, which are measurable so $z^{-1}(0)$ is also measurable. Now for infinite values, $h = g = \infty$ and again by m easurability $h^{-1}(\infty) \cap g^{-1}(\infty)$ measurable, the same aregument ghoes for $-\infty$. $\square$

(4.2) Show that if $f : X \to \mathbb{R}^*$ and if $f^{-1}((q,\infty]) \in \mathcal{M}$ for every $q \in \mathbb{Q}$ then f is measurable.

Proof. We need to show that algebra of half half-infinite rays $\mathcal{E}$ can be generated by $\mathcal{Q} = \{(q,\infty]\}$. Take any $x \in \mathbb{R}^*$, there is a sequence of ascending $q \in \mathbb{Q}$, say $q_n \to x$. T hen consider their manifestation in $\mathcal{Q}$, say $Q_1 = (q_1,\infty], Q_n = (q_n,\infty], \dots$. Then

$$\bigcap_{n=1}^{\infty} Q_n = \{y \in \mathbb{R} \mid y > q_n, n \in \mathbb{N}\} = [x,\infty].$$

Then since $f^{-1}(q,\infty]) \in \mathcal{M}$, it's clear that $f^{-1}(\bigcap Q_n) \in \mathcal{M}$, and so by Proposition 2.3, f is measurable. $\square$

(4.3) Let $(X,\mathcal{M},\mu)$ be a *complete* measure space, then (a) If f is measurable and $f = g$ μ-a.e. then g is measurable.

Proof. Take some set in $\mathcal{N}$, say E, then $f^{-1}(E) \cap f^{-1}(E) \in \mathcal{M}$ and $g^{-1}(E) = f^{-1}(E) \setminus J \cup F$ where $J = \{x \in f^{-1}(E) \mid f(x) \neq f(g)\}$ and $F = \{x \in X \mid g(x) \in E, g(x) \neq f(x)\}$, both of which are subsets of $D = \{x \in X \mid f(x) \neq g(x)\}$. Because $\mu(D) = 0$ and μ complete, then J, F are measurable. Then by measurability of f, $g^{-1}(E) \in \mathcal{M}$. Therefore g measurable and this completes the proof. $\square$

(b) If $f_n \to f$ almost every where and f_n measurable then f measurable.

Proof. Consider $h_n = \sup_{k\geq n} f_n$ and $g_n = \inf_{k\geq n} f_n$. From a proposition of the text h_n and g_n measurable for all n and $\lim h_n = h$ and $\lim g_n = g$ measurable. Furthermore, $h(x) - g(x) = 0 \iff f_n(x) \to f(x)$ for those x. Therefore $D = \{|h(x) - g(x)| > 0\}$ is a zeroset. Furthermore $g(x) = f(x) = h(x)$ on D and by the previous proposition $f = g$ a.e is measurable $\square$

(4.4) If $f \in L^+$ and $\int f < \infty$ then $\{x : f(x) = \infty\}$ is a nullset and $\{x : f(x) > 0\}$ is σ-finite.

Proof. We prove the contrapositive. Suppose that $G = \{x : f(x) = \infty\}$ has measure $m > 0$. Then we have by measure outward continuity (and the results of the section that) $\int_X f > \int_G f$ since $X \supset G$. However, $f|_G$ can be described by a simple function in standard from $f|_G \geq c\chi_G$ where $c = \infty$. Therefore $\int_G f \geq c\mu(\chi) = \infty \times m$, where $m > 0$. Therefore $\int f \geq \infty$. Now suppose that $F = \{x \mid f(x) > 0\}$ is not σ-finite. In such a case, in any countable union forming F has a member set with measure ∞. For example take, $F_n = \{x \mid f(x) > 1/n\}$. Clearly $\bigcup_{n=1}^{\infty} F_n = F$ and there exists an N so that $\mu(F_N) = \infty$. Again $f|_{F_N} > 1/N$ so we know that $\int_{F_N} f \geq \int_{F_N} 1/N\chi_{F_N} = 1/N\mu(F_N) = \infty/N = \infty$. Therefore $\int_X f > \int_{F_N} f = \infty$. So the contrapositive holds.

This completes the proof. $\square$

(4.5) Suppose that $f_n \in L^+$ and f_n decreases pointwise to f and $\int f_1 < \infty$. Show that $\lim n \to \infty \int f_n = \int f$.

Proof. Define a sequence of simple functions $\mathfrak{F}_n$ so that for every n, $\mathfrak{F}_n \leq f_n$ and $|\int \mathfrak{F}_n - \int f_n| < 2^{-n}$. Such a sequence exist since $\int f_n < \infty$ so the subtraction does not violate any rules of the extended reals. Furthermore there are such simple functions since $\int f_n < \infty$ gives $\{x : f(x) > 0\}$ σ-finite, so partitioning the domain into constants and indicator functions gives a finite integral for such a simple function $\leq f_n$. First we know that $\int \mathfrak{F}_n \geq \int f$ for every n since $\mathfrak{F}_n \geq f$. Furthermore $a_n = \int \mathfrak{F}_n$ is Cauchy since $|\int \mathfrak{F}_n - \int \mathfrak{F}_m| \leq |\int \mathfrak{F}_n - \int f_n| + |\int \mathfrak{F}_m - \int f_m| + -|\int f_m - \int f_n| < \epsilon + 2^{-n} + 2^{-m} \to 0$ as $m, n \to \infty$.

Next we use the sandwhich theorem and

$$\begin{tikzcd}
f \arrow[r, dash, "\leq"] \arrow[d, "n"] & \mathfrak{F}_n \arrow[r, dash, "\leq"] \arrow[d, dashed, "n"] & f_n \arrow[d, "n"] \\
f \arrow[r, equal] & f \arrow[r, equal] & f
\end{tikzcd}$$

Let $\epsilon > 0$. Consider the sequence $d_n = 2\sup_x \mathfrak{F}_n - f$. By the above limit diagram, $d_n \to 0$ and $\mathfrak{F}_n - d_n = \mathfrak{G}_n(x) = \sum_{y_n \in range(\mathfrak{F})}(y_n - d_n)\chi_{y_n}(x)$ is an element of a family of siple functions so that $\mathfrak{G}_n \uparrow f$ pointwise. By the *upward* monotone convergence theorem of measure theory $\int \mathfrak{G}_n \uparrow \int f$. Now observe that

$$\left| \int \mathfrak{G}_n - \int \mathfrak{F}_n \right| = \sum (y_n - (y_n - d_n))\, \mu(\{x : \mathfrak{F}(x) = y_n\}) = \sum (d_n)\, \mu(\{x : \mathfrak{F}(x) = y_n\}) \to 0$$

So $\int \mathfrak{G}_n \to \int f$ implies that $\int \mathfrak{F}_n \to \int f$. Finally we can now take n large enough that $2^{-n} < \epsilon/2$ and $\left|\int \mathfrak{F}_n - \int f\right| < \epsilon/2$.

$$\left| \int f_n - \int f \right| \leq \left| \int f_n - \int \mathfrak{F}_n \right| + \left| \int \mathfrak{F}_n - \int f \right| < \epsilon/2 + \epsilon/2 < \epsilon.$$

□

This completes the proof.

(4.6) Let C be the Cantor ternary set, and let $f : [0,1] \to [0,1]$ be the Devil's staircase, as defined in 1.5 of our text. Define $g(x) = f(x) + x$. Prove the following: (a) g is homeomorphism from $[0,1] \to [0,2]$.

Lemma 0.1. *If $f : E \subset \mathbb{R} \to F \subset \mathbb{R}$ non decreasing and E, compact then $f + id$ is bijective on its range.*

Proof. Take any $x \neq y$, then without loss of generality $x > y$, so $f(x) \geq f(y)$ and $id(x) > id(y)$ so $f(x) + id(x) > id(y) + f(y)$ and $(f + id)(x) \neq (f + id)(y)$ and $f + id$ is injective, and so $f + id$ bijective on its range from E. □

Proof. (Of 4.6.a) The function is bijective by the previous lemma. The sum of to continuous functions is continuous. Every closed subset of $[0,1]$ is compact. Since the function is continuous the image of a compact set is a compact and closed subset of $[0,2]$ so the map is closed so the map is open and bijective continuous and so g is a homeo. □

(b) $m(g(C)) = 1$ even though C is a null set.

Proof. We know that $[0,1] \setminus C = U = \bigsqcup_j B_j$ is the union of open intervals. Then take

$$g(U) = \bigsqcup_{n=1}^{j} c_j + B_j$$

and $m(B_j) = m(B_j + c_j)$ because $B_j \in \mathcal{B}_{\mathbb{R}}$ open and $B_j + c_j \in \mathcal{B}_{\mathbb{R}}$ so

$$m(U) = \sum_{j=1}^{\infty} m(B_j) = \sum_{j=1}^{\infty} m(B_j + c_j) = m(g(U)).$$

Therefore $m([0,2]) = m(g(U)) + m(g(C)) \implies m(g(C)) = 1$ and $m(C) = 0$. $\square$

(c) Let A be any subset of $g(C)$ that is not Lebesgue measurable. Show that $B = g^{-1}(A)$ is Lebesgue measurable, but is not Borel measurable.

Proof. Let $A \subset g(C)$ non measurable. $B = g^{-1}(A)$ is a subset of C which is a Lebesgue null-set. Since labesgue μ is complete then any subset of C is measurable with measure 0. Since g^{-1} is a homeomorphism it preserves the Borel σ-algebra, and so since A is not Lebesgue measurable it is not Borel measurable and so if B were Borel measurable it would conteradict the topological invariance of g. $\square$

(d) There exist a Lebesgue measurable function $F : \mathbb{R} \to \mathbb{R}$ and a continuous function $G : \mathbb{R} \to \mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Proof. Let $F = \chi_B$ then $G = g^{-1}$ when $x \in [0,2]$ other wise if $x < 0$, $G(x) = 0$, or if $x > 2$, $G(x) = 1$. Then $(F \circ G)^{-1} = g \circ F^{-1}(1) = g(B) = A$. $\square$