Math H104: Homework 2

William Guss
26793499
wguss@berkeley.edu

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1 Real Numbers

36. *Without using the Schroeder-Bernstein Theorem,*

    (a) *Prove the following*

    Theorem 1. *The cardinalities of* $[a,b],(a,b],(a,b)$ *are equivalent. That is,* $[a,b]\sim(a,b]\sim(a,b)$.

    Proof. To show cardinal equivalence, we must find a bijection between the three sets. This argument follows from Hilberts's hotel. Consider the two sets set $A^o=(a,b), A^h=(a,b]$. Because every uncountable set has a countable subset, let

    $$A^o_C=\left\{a_n\in A^o \mid a_n=\frac{an+b}{n+1},\ n\in\mathbb{N}\right\}.$$

    In the same light, let

    $$A^h_C=\left\{a_n\in A^h \mid a_n=\frac{a(n-1)+b}{n},\ n\in\mathbb{N}\right\}.$$

    Then $f:A^o_C\to A^h_C$, such that $f(\frac{an+b}{n+1})=\frac{a(n-1)+b}{n}$, is clearly a bijection.

    We now make a functiong $g:A^o\to A^h$ such that for $x\in A^o_C$, $g(x)=f(x)$, otherwise $g(x)=x$. Since f is a a bijection, then g is a bijection on A^o_C. Furthermore since $b\notin A^h\setminus A^h_C$, we have that g surjective when $x\notin A^o_C$. Furthermore if $x\neq y$, then clearly $g(x)=x\neq y=g(y)$. So g is injective, and therefore bijective. We have shown that $A^o\sim A^h$.

    Lastly, we will prove that $A^c=[a,b]$ is bijective to A^h. Redefine again the following countable subsets

    $$A^h_C=\left\{a_n\in A^g \mid a_n=\frac{a+bn}{n+1},\ n\in\mathbb{N}\right\}.$$

    In the same light, let

    $$A^c_C=\left\{a_n\in A^c \mid a_n=\frac{a+b(n-1)}{n},\ n\in\mathbb{N}\right\}.$$

Now take the function $f : A^h_C \to A^c_C$ to map $(\frac{a+bn}{n+1}) \mapsto \frac{a+b(n-1)}{n}$. The function f is a clear bijection. Finally let $g : A^h \to A^c$ be such that $x \in A^h_C \implies g(x) = f(x)$, otherwise $g(x) = x$. This new function is clearly a bijection in the same sense that the previous definition of f was, and hence we have shown that $A^c \sim A^h$.

Therefore by bijective composition, $A^c \sim A^h \sim A^o$, and the proof is complete.

$\square$

(b) *Prove the following.*

Theorem 2. *If C is countable, then $\mathbb{R} \setminus C \sim \mathbb{R} \sim \mathbb{R} \cup C$.*

Proof. We are going to assume that C is a countable set of real numbers, the union of the real numbers and the set of all dogs, probably is not worth studying, and I presume that's not what this question is about. If I do assume this, it is simple to see that $\mathbb{R} \cup C \cup \mathbb{R} = \mathbb{R}$, which raises the question as to why it has been included in the theorem. However, I shan't digress.

To show this theorem, we first find a bijection between $\mathbb{R}$ and $\mathbb{R} \setminus C$. Because C is a countable, we can take an index set $I = \{1, 2, \dots\}$, which is finite if and only if C is finite and contiguous such that $|I| = |C|$. We then can define $C = \{a_i\}_{i \in I}$ for real numbers $a_i \in C$, such that $i > j \implies a_i > a_j$.

We have that

$$\mathbb{R} \setminus C = (-\infty, a_1) \cup \left[\bigcup_{i \in I \setminus \{1\}} (a_{i-1}, a_i) \right] \cup (\sup C, \infty).$$

So it follows,

$$\mathbb{R} = (-\infty, a_1] \cup \left[\bigcup_{i \in I \setminus \{1\}} [a_{i-1}, a_i] \right] \cup [\sup C, \infty).$$

We will now constuct a bijection $f : \mathbb{R} \setminus C \to \mathbb{R}$. By Theorem 1, there exists a bijective function $h_i : (a_{i-1}, a_i) \to [a_{i-1}, a_i]$. $\square$

2 A Taste of Topology

1. *An ant walks on the floor,ceiling, and walls of a cubical room. What metric is natural for the ant's view of its world? What metric would a spider consider natural? If the ant wants to walk from a point p to a point q, how could it determine the shortest path?* Simple! The ant exists on subspaces of three space. In fact, the ant exists on planes within three space. This means that the ant can only move from one location to another using the inherited metric of a plane subspace in $\mathbb{R}^3$, not the one provided for $\mathbb{R}^3$ itself. To give a precise definition, if the ant wants to travel from the center of the ceiling to the floor, he must take the shortest path to the closest contiguous plane series to that of the floor. Put more simply, the ant needs to walk to the wall and then to the floor to get to his destination.

In this sense we can actually construct a homeomorphism from the ant's 3-space domain, to a simple subset of $\mathbb{R}^2$. Take all of the planes constructing the walls and the floor and knock them down such that they become six planes adjacent in a cross pattern. It is easy to see that the ant must use the standard metric in $\mathbb{R}^2$ between those points, p, q in contiguously adjacent subplanes. If the planes are not adjacent, the ant must find the best possible from one plane to another and then readopt the $\mathbb{R}^2$ metric.

The spider's space would be endowed with the ant metric except for when the spider wishes to go from points on the ceiling, C, to the floor F, in that case, the spider would use the taxicab metric for $\mathbb{R}^3$ to reach a point directly below the point on the ceiling, on the floor, and then again assume the ant metric. This process would not work in reverse however, considering that the ant cannot ascend, unless a web has already been created (a can of worms into which we will not venture!).

2. If $M = \mathbb{R}^2$ is a metric space, we say that it is endowed with the taxicab metric if and only if

$$d(x, y) = \|x - y\|_1 = |x_1 - y_1| + |x_2 - y_2|.$$

This name arises if one considers the natural metric for a taxicab driver. Clearly the driver must stay aligned to the grid system which is a subset of $\mathbb{R}^2$. Hence if such a driver wishes to travel from point q to point p. He must first go a long the grid in the x direction until his x position coincides with p, and then head in the y direction untill his x and y positions are at p. The process requires that he travel the distances in the x and y directions independently. Essentially $d(p, q) =$ distance_x_direction + distance_y_direction $= \|x - y\|_1 = |p_1 - q_1| + |p_2 - q_2|$. Therefore it is natural to arive at our definition of the taxicab metric.

8. *Prove the following*

   (a) *Absolute convegence*

   Theorem 3. *Let (x_n) be a convegent sequence in $\mathbb{R}$. Then, the sequence of absolute values $(|x_n|)$ converges in $\mathbb{R}$.*

   Proof. If (x_n) converges, then there exists a limit x such that for every $\epsilon > 0$, there exists an N such that for all $n > N$, $d(x_n, x) < \epsilon$. Since the natural metric on $\mathbb{R}$ is the absolute value, we have that $|x_n - x| < \epsilon$. This holds for any sign of x_n and x, so $||x_n| - |x|| \leq |x_n - x| < \epsilon$ implies that $|x_n| \to |x|$. □

   (b) *State the converse.*

   Theorem 4. *If for some sequence (x_n) in $\mathbb{R}$, $|x_n| \to |L|$, then $*(x_n)$ converges to some limit.*

   (c) *Disprove the the previous statement.* We show by counter example, the theorem cannot be true. Take $(a_n) = (-1)^n$. Clearly $|a_n| \to 1$, but there exists and $\epsilon > 0$, say $\epsilon = 0.5$, such that for all N there exists an n, (take the next odd n), such that $|a_n - 1| = |-1 - 1| = 2 \geq 0.5 = \epsilon$. Hence (a_n) does not converge to the limit of its absolute value sequence. The theorem cannot be true.

12. *Prove the following*

(a) *Bijective:*

Theorem 5. *If (p_n) is a sequence, and $f : \mathbb{N} \to \mathbb{N}$ is a bijection, then the sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is a rearrangement. If $p_n \to L$, then $q_n \to L$ for all such f.*

Proof. If $p_n \to L$, then for all $\epsilon > 0$ p_n is a distance less than ϵ from L for all but finiteley many n. Let N be the number of those first elements of the sequence which are more than ϵ to L. Since f is bijective, the set $f^{-1}(\{0, ..., N\})$ is also finite, and therefore there exists and element $M \in \mathbb{N}$ such that for all $n \in f^{-1}(\{0, \ldots, N\}), M > n$. Therefore, for all $m > M$, we have that $|q_m - L| < \epsilon$ if and only if $q_m \to L$. □

(b) *Injective*

Theorem 6. *If (p_n) is a sequence, and $f : \mathbb{N} \to \mathbb{N}$ is an injection, then the sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is a rearrangement. If $p_n \to L$, then $q_n \to L$ for all such f.*

Proof. If f is an injection the theorem will still hold. Recall that an injection implies that each element in the range has a singleton pre-image. Furthermore, each element in the co-domain has a singleton or empty pre-image. Thus if p_n is convergent, then for all $\epsilon > 0$, there exists an N such that for all $n > N$, $|p_n - L| < \epsilon$. Hence $\bigcup_{j=1}^{N} f^{pre}(j) = P$ is finite and contains the n for which q_n is a distance greater than or equal to ϵ away from L. Thus take the maximal element of, P, say M. Then for all $n > M$ (such that there exists a x with $f(x) = n$. Note: there must be infinitely many such n), $|q_n - L| < \epsilon$, and hence $q_n \to L$. □

(c) *Surjective:* In the case that f is only a surjection, then we show that not all rearrangements converge by counterexample. Take, for example, a sequence $p_n = \frac{n}{n+1}$. This sequence clearly converges to 1, but consider the surjective rearrangement, $f(2n) = 1, f(2n-1) = n$. Such a map takes even elements and maps them to 1, and otherwise takes odd elements and maps them to half their double. It's easy to see that this function is surjective, $f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 1, \ldots$. However, there exists an $\epsilon > 0$, say $\epsilon = 0.5$, such that for all N, $n > N$, and $n = 2N$ implies that $|p_{f(n)} - 1| \geq \epsilon$. Hence, the rearrangement does not converge. By counterexample, not all surjective rearrangements converge if the normal sequence does.