

# MATH 105: Homework 5

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65. The winding (w)one form.

**Definition 1.** We denote the winding one form,  $d\theta$ , such that

$$d\theta = \frac{-y}{r^2}dx + \frac{x}{r^2}dy. \quad (1)$$

**Theorem 1.** The winding one form is closed but not exact.

*Proof.* We take the exterior derivative of  $d\theta$ . Using  $d(fdx) = df \wedge dx$  we have the following:

$$\begin{aligned} d(d\theta) &= d\left(\frac{-y}{r^2}dx + \frac{x}{r^2}dy\right) \\ &= d\left(\frac{-y}{r^2}dx\right) + d\left(\frac{x}{r^2}dy\right) \\ &= d\left(\frac{-y}{r^2}\right) \wedge dx + d\left(\frac{x}{r^2}\right) \wedge dy \\ &= \left(\frac{\partial}{\partial x} \frac{-y}{x^2+y^2} dx + \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} dy\right) \wedge dx + d\left(\frac{x}{r^2}\right) \wedge dy \\ &= \frac{y^2-x^2}{r^4} dy \wedge dx + d\left(\frac{x}{r^2}\right) \wedge dy \\ &= \frac{y^2-x^2}{r^4} dy \wedge dx + \left(\frac{\partial}{\partial x} \frac{x}{r^2} dx + \frac{\partial}{\partial y} \frac{x}{r^2} dy\right) \wedge dy \\ &= \frac{y^2-x^2}{r^4} dy \wedge dx + \frac{\partial}{\partial x} \frac{x}{r^2} dx \wedge dy \\ &= \frac{y^2-x^2}{r^4} dy \wedge dx + \frac{y^2-x^2}{r^4} dx \wedge dy = 0 \end{aligned} \quad (2)$$

Substitution of  $r \cos \theta$  for  $x$  and similar for  $y$  yields that the form is infact  $d\theta$ , ( $dx = \cos \theta dr - r \sin \theta d\theta$ ). If the form were exact then its anti-exterior derivative should be  $\theta$ , therefore its evaluation along a 1 cell should be its net change along its end points.

Consider the curve which takes the unit circle counter clockwise around the origin.

$$d\theta(c) = \int_0^\tau -\sin t dx + \cos t dy = \int_0^\tau dt = \tau. \quad (3)$$

And  $\tau = 2\pi \neq 0$ , but the net change in  $\theta = 0$  So it could not be that this form is exact.  $\square$

The name  $d\theta$  is totally misleading, it implies that  $d\theta = d(\theta)$  which is false.

68. Closedness of scalar multiplication.

**Theorem 2.** *If  $\omega$  is closed then  $f\omega$  is not necessarily closed.*

*Proof.* Take the exterior derivative of the expression and get

$$d(f\omega) = df \wedge \omega + f \wedge d\omega = df \wedge \omega. \quad (4)$$

So if the differential of  $f$  is 0 then  $f\omega$  is closed. Otherwise, no.  $\square$

**Theorem 3.**

71. Hairy ball theorem! I'll do part (a) separately, but follow my logic for the remaining parts all in part(b).

(a)

**Hairy Ball theorem 1.** *If  $X : S^2 \rightarrow \mathbb{R}^3$  is a continuous vector field so that every point  $p \in S^2$ ,  $X(p)$  is tangent to  $S$ , then there exists a  $q$  so that  $X(q) = 0$ .*

**Hairy Ball theorem 2.** *Other ball theorem.*

**Theorem 4.** *Hairy ball Theorem 1 and Hairy Ball Theorem 2 are logically equivalent.*

*Proof.* Lol.  $\square$

(b) We shall build a set of logical tools to prove the first hairy ball theorem.

**Definition 2.** *If  $X$  is a continuous vector field with no zero points, so that  $X : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we denote **the net winding of  $X$**  along some curve  $\phi \in C_1(\mathbb{R}^n)$ ,  $\omega_X \in \Omega^1(\mathbb{R}^n)$ . We require that the for all  $p \in U$*

$$\langle p, X(p) \rangle = 0, \quad (5)$$

*or intuitively  $X$  is tangential to points in  $U$ , and  $U$  simply connected.*

**Theorem 5.** *If  $\omega_X$  is a continuously tangent to  $S^2$  and has no zeroes, its net winding is 0.*

*Proof.* First consider the point set of a 1-form  $\phi$  on the 2-sphere. If  $\phi$  is a closed loop, then there is a point on the sphere,  $p_0$  where  $p_0$  is not in the point set. Consider the stereographic projection about  $p_0$  on the sphere to the plane. This mapping is a conformal diffeomorphism. Furthermore if we homotop  $\phi$  homeomorphically (with respect to its pointset), we can likewise change  $p_0$

smoothly so that any discussion we make about homotopy of  $T(\phi)$  makes sense on  $S^2$ .

It is therefore true that when  $\phi$  is projected using  $T$  we get a closed curve. Analogously take  $X$  satisfying the winding definition above, and we build an analogous vector field in the image of  $T$  as follows:

- i. For every  $p \in S^2$ , take  $X(p)$  and project it on  $S^2$ .
- ii. The curved pointset produced is  $\overline{p\text{proj}_S^2(X(p))} = L_p$ .
- iii. Define  $Y(T(p)) = T(L_p)$  in the sense that  $Y(T(p))$  is the vector whose head is at the head of the projection of  $L_p$  into the plane.

The net winding of  $X$  along some curve is the 1-form,

$$\omega_Y = \frac{Y_1}{\sqrt{Y_1^2 + Y^2}} dy_1 + \frac{Y_2}{\sqrt{Y_1^2 + Y^2}} dy_2 := \left\langle Y, \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \right\rangle. \quad (6)$$

This definition makes sense, since it accumulates the sum of the net change in  $\cos \theta$  over the vector field against the vector  $(1, \dots, 1)^T$ . So we are concerned with the field not the curve over which we accumulate the winding.

We show the net winding over  $Y$  is 0. First off, continuity of  $Y$  implies that winding must be *integer proportional* to  $2\pi$ . If it were not then at a point infinitesimally close to the initial parameter (start) of a loop  $\gamma$  on  $Y$ , there would be largeley different (not infinitesimally different) vector  $Y$  than that of the initial parameter. If it is not at this inial point then it must be somewhere else. However the mear existence of this 'jump' violates the continuity of  $X$ .

Now by the simply connected nature of the projection,  $T(S^2)$  imagine a sufficiently small homotopy of  $\gamma$ . It could not be the case that  $\omega_Y$  changes much, that is for this homotopy  $H$ . Then  $|\omega_Y(H(0, \phi)) - \omega_Y(H(1, \gamma))| < \epsilon$ , by continuity of  $Y$ . However this difference can only be *integer proportional* to  $2\pi i$ , so it must be zero. Finally consider the homotopy of  $\gamma$  to a constant map (a point)  $\mu \in T(S^2)$ . The net winding of  $Y$  in this map is nameley 0, and since  $\omega_Y$  is invariant to homotopy, we have that  $\omega_Y(\gamma) = 0$  for all  $\gamma$  homotopic to a point (closed curves). *Note: the homotopy of  $\gamma$  to a point makes sense in terms of winding if and only if  $T(S^2)$  is simply connected, which it is. Furthermore, we will be using this an homolygous homotopy on the sphere but recall the argument of the smooth map  $T$  changing smoothly with homotopy.*

Now we wish to pull back the winding of curves in  $Y$  to the tangential vector field on  $S^2$ . In other terms,  $T^*\omega_Y = \omega_X$ . Consider the followuing derivation,

$$\begin{aligned} T^*\omega_Y &= T^*(\langle Y, (dy_1, dy_2) \rangle) \\ &= T^*(\cos \theta_Y dy_\Theta), \end{aligned} \quad (7)$$

where  $dx_\Theta$  is the reparameterization of  $(dy_1, dy_2)$  in polar coordinates. We use the conformality of  $T$  and get

$$\omega_X = \cos \theta_X dx_\Theta \implies T^*(\omega_Y) = (T^* \cos \theta_Y) dT_\Theta = \omega_X. \quad (8)$$

Finally by the definition of pullback we get  $T^*(\omega_Y) : \phi \mapsto \omega(T \circ \phi) = 0$  for all  $\phi$  that are simple closed 1-cell loops. So  $\omega_X(\phi) = 0$  and the proof is complete.  $\square$

**Definition 3.** The winding of  $\phi \in C_1(U)$  against a vector field  $X : U \rightarrow \mathbb{R}^n$  is the net angle between the tangent vectors of the curve and the  $X$  denoted  $w(X) \in \omega^1(X(U))$ , so that

$$w(X) = \omega_X + d\theta, \quad (9)$$

where  $d\theta$  calculates the angle accumulated on  $\phi$ .

**Corollary 1.** If  $\phi \in C_1(S^2)$  is a simple closed loop, then its net winding against a continuous vector field  $X$  on  $S^2$  with no holes is  $\pm 2\pi$  according to its orientation.

*Proof.* Using the previous theorem, by definition we have

$$w(X)(\phi) = \omega_X(\phi) + d\theta(\phi) = 0 + d\theta(\phi). \quad (10)$$

Lastly,  $d\theta(\phi) = \pm 2\pi$  from class. This completes the proof.  $\square$

**Corollary 2.** Homeomorphic homotopy of a simple closed curve,  $\phi$  in  $U$  does not change  $\omega(\phi, X)$ .

*Proof.* Homeomorphic homotopy is meant in the sense that  $\phi$  must have a loop point set after deformation. By the same argument in theorem 1, continuity of  $X$  implies that  $w(X)(\phi)$  is a continuous functional of small homotopic changes of  $\phi$  whose value set has a discrete topology. The only continuous mappings from some set  $D \rightarrow C$  where  $C$  has the discrete topology are constant maps. So the theorem holds.  $\square$

We now prove the hairy ball theorem!

*Proof of Hairy Ball Theorem 1.* Suppose that there does exist such a beautiful wonderful vector field  $X$  with no zeroes!

Let  $\phi$  be a nice artic circle closed loop 1-form. We call the homotopy from Figure 135

$$H_r : C_1(U) \times [0, 1] \rightarrow C_1(U), \gamma, 0 \mapsto \phi \wedge \gamma, 1 \mapsto \phi_o. \quad (11)$$

Where  $\phi_o$  is the same curve but with its orientation reversed. Let  $w(X)(\phi) = a$ . Then clearly  $w(X)(\phi_o) = -a$ . But by the previous corollary, the homeomorphic homotopy equivalence of  $\phi_o$  and  $\phi$  implies that  $a = -a = 0$ , which contradicts the corollary before that. Therefore it could not possibly be that  $X$  has no zeroes, and the proof is complete!