MATH 105: Homework 13

William Guss
26793499
wguss@berkeley.edu

May 6, 2016

81. **Stronger Average Value Theorem.**

Theorem 1. *If f is a measurable function then for all most every p in its domain we have that*

$$\lim_{Q \downarrow p} \frac{1}{mQ} \int_Q |f - fp| \; d\mu(x) = 0 \tag{1}$$

Proof. Get an enumeration of $\mathbb{Q}$, say $\{a_n\}$ there is a sequence $a_n^{fp} \to fp$. Finally consider that for every n the function $|f - a_n|$ is measurable. So we let $f_n^{fp}(x) = |f(x) - a_n^{fp}|$. The limit is measurable. By the average value theorem

$$\lim_{Q \downarrow p} \frac{1}{mQ} \int_Q |f_n - a_n^{fp}| \; d\mu(x) = |f_p - a_n^{fp}|. \tag{2}$$

As $a_n^{fp} \to fp$ the right hand side tends towards to 0 and therefore

$$\begin{aligned} 0 &= \lim_{n\to\infty} \lim_{Q \downarrow p} \frac{1}{mQ} \int_Q |f - a_n^{fp}| \; d\mu(x) \\ &= \lim_{Q \downarrow p} \frac{1}{mQ} \int_Q \lim_{n\to\infty} |f - a_n^{fp}| \; d\mu(x) \\ &= \lim_{Q \downarrow p} \frac{1}{mQ} \int_Q |f - fp| \; d\mu(x). \end{aligned} \tag{3}$$

We can bring the limit inside by the measurability and uniform convergence of the functions. This completes the proof.

□

84. **Almost Absolutely Continuous Functions.**

Lusin's Lemma extends to absolute continuity for the falling reasons. Take an f satisfying the conditions in Lusin's Lemma. Then $f : [a, b] \to \mathbb{R}$ restricted to $E \subset [a, b]$ is continuous and E is a bounded compact set. Since $f_{|E}$ is continuous on a bounded compact subset, then it is absolutely continuous on that subset. So f satisfying Luzin's lemma is almost absolutely contionuous. The lemma used in this reasoning does not require that f be bounded! ‘’

87. **Density Theoretic Boundries**

(a) Measure theoretic boarder.

Theorem 2. *If E is a subset of $\mathbb{R}^n$ and ∂E is its boarder then*

$$\partial_m E \subset \partial E.$$

Proof. If $p \in Ext_m(E)$ then clearly $d(p, E^c) = 1$ and therefore $p \in E^c$. Conversely $\partial_m E \cup Int_m(E) = E$. Suppose for the sake of contradiction that there exists a $p \in \partial_m E$ such that $p \in E^o = Int(E)$. Then there exists an $r > 0$ such that all $x \in B(p, r)$ are in E. Therefore $d(p, E) = 1$. This a contradiction to $p \in \partial_m E$, so $p \in \partial E$. This completes the proof. □

(b) Consider the following construction. Let $f : [-1, 1] \to [0, 2]$ such that $x \mapsto x^{2/3} + 1$. This function has a cusp at $x = 0$ whose walls get sharper and sharper. Imagine the point on the border of the completed undergraph at $x = 0$. As you shrink the ball it encompasses more of the area on the graph. Untill eventually the limit is one. See the picture:

88. **Topological Riemann Integrability**

Theorem 3. *Let X be a compact hypercube in $\mathbb{R}^n$. A function $f : X \to [0, M]$ is Riemann integrable if and only if $m(\partial \mathcal{U} f) = 0$.*

Proof. Recall that Lemma 69 holds for any arbitrary metric space. Therefore,

$$\mathcal{U}\underline{f} = int(\mathcal{U}f) \ \wedge \ \hat{\mathcal{U}}\bar{f} = \overline{\mathcal{U}f} \tag{4}$$

Since open sets and closed set are measurable in $\mathbb{R}^n$, then $\underline{f}$ and $\overline{f}$ are measurable functions. Thus

$$m(\partial(\mathcal{U}f)) = m(\overline{\mathcal{U}f} \setminus int(\mathcal{U}f)) = m(\hat{\mathcal{U}}\bar{f}) - m(\mathcal{U}\underline{f}) = \int_X \bar{f} - \underline{f}. \tag{5}$$

Lebesgue theory tells us that the integral is zero if and only if $\bar{f} = \underline{f}$ almost everywhere, i.e. f is continuous if and only if f is continuous everywhere $(lim_{t \to x} f(t).)$, i.e. by the Multivariate Riemann-Lebesgue Theorem if and only if f is Riemann integrable. $\square$