MATH H104: Homework 5

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56. *Prove the following.*

    Theorem 1. *The 2-sphere is not homeomorphic to the plane.*

    Proof. For this proof we make use of the heine-borel theorem and the preservation of topological properties through embedding. Take the two sphere S^2. Take the embedding $h : S^2 \to S \subset \mathbb{R}^3$. This embedding exists as $S = \{r \in \mathbb{R}^3 \mid \|x\| = 1\}$ is a natural image. By the Heine-Borel theorem S is compact because it is closed and bounded. Then, because h is an embedding, S^2 is also compact, a topological invariant. By $\mathbb{R}^2$ not compact, we have that $S^2 \not\cong \mathbb{R}^2$, and the proof is complete. $\square$

57. Prove the following.

    Theorem 2. *If S is connected, its interior may be disconnected.*

    Proof. Consider the following counter example. Denote the closed r-ball $B_r^c(y) = \{x \in \mathbb{R}^2 \mid \|x - y\| \leq r\}$ furthermore let the open$r$-ball B_r^o be the interior of $B_r^c(y)$. If $S = B_1^c(-1, 0) \cup B_1^c(1, 0)$, then the interior of S is clearly $B_1^o(-1, 0) \cup B_1^o(1, 0)$. Since these two sets are disjoint, we have that $int(S) = B_1^o(-1, 0) \sqcup B_1^o(1, 0)$. Lastly since $B_1^o(-1, 0)$ and $B_1^o(1, 0)$ open in $int(S)$, they are also closed since they are compliments. The counter example is complete as $int(S)$ is disconnected in contrast to S connected. $\square$

58. *Theorem 49 states that the closure of a connected set is connected.*

    (a) *The closure of a disconnected set is disconnected.*If M disconnected then, $M = A \sqcup B$ for A, B disjoint clopen subsets of M. The closure of M is the intersection of all; closed sets containing M, which is trivially M. Hence the closure of M is M which is disconnected.

    (b) *What about the interior of a disconnected set?* If M is disconnected, then the interior of M is the union of sets in the topology of M. Since M is clopen and in the topology of M, the interior of M is maximally M. Therefore, the interior of M is disconnected.

59.

60. *Prove the following.*

(a) *Integer domain:*

Theorem 3. *If $f : M \to \mathbb{Z}$ is continuous, then M connected implies that $f(M) = \{c\}$ is a singleton.*

Proof. Suppose for the sake of contradiction that $B = \{a \in M \mid f(a) \neq c\}$ is non=empty. Then $f(M) = \{c\} \sqcup f(B) \subset \mathbb{Z}$. By $\mathbb{Z}$ disconnected, we have that $f(M)$ is discoinnected. This is a contradiction to M connected, implies $f(M)$ connected (by continuity). Hence, $f(M)$ is a singleton. □

(b) *Rational domain:*

Lemma 1. $\mathbb{Q}$ *is totally disconnected.*

Proof. We will show the theorem if for every $x, y \in \mathbb{Q}$ there exist A, B separations of $\mathbb{Q}$ with $x \in A, x \in B$. Without loss of generality, assume $x < y$. Since between two rationals there is an irrational, take the the irrational z to be in between x and y. Let $A' = (-\infty, z)$ and $B' = (z, \infty)$. Then if $A = A'_{\mathbb{Q}} = A' \cap \mathbb{Q}$ and $B = B'_{\mathbb{Q}}$, we have that $\mathbb{Q} = (-\infty, z)_{\mathbb{Q}} \sqcup (z, \infty)_{\mathbb{Q}} = A \sqcup B$. Clearly $x \in A, y \in B$. Therefore $\mathbb{Q}$ is totally disconnected. □

Theorem 4. *If $f : M \to \mathbb{Q}$ continuous, M connected implies that $f(M)$ is trivially the singleton.*

Proof. Suppose $f(M)$ is not trival (not the singleton, nor empty), then $f(M) \subset \mathbb{Q}$ implies that $f(M)$ is totally disconnected by the previous lemma. This is a contradiction to M connected, by M connected implies $f(M)$ connected. Therefore $f(M)$ is the singleton. □

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