MATH H104: Homework 9

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59.

Theorem 1. *If $\sum a_n$ converges and $a_n \geq 0$, then show $\sum \sqrt{a_n}/n$ converges.*

Proof. Let $x = (\sqrt{a_n})_n$, $y = \left(\frac{1}{n}\right)_n$. Clearly $y \in \ell_1$, and since $\sum a_n \to c$, $a_n \to 0$ implies that $\sqrt{a_n} \to 0$. Therefore, $x \in \ell_1$. Since ℓ_1 is an inner product space, the cauchy schwartz inequality gives,

$$0 \leq \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} = \langle x, y \rangle \leq |x||y| = \sqrt{\sum_{n=1}^{\infty} a_n}\sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \sqrt{\frac{c}{6}}\pi$$

and so the series is bounded and therefore converges. □

61. Consider the following $\{a_n\} \in \ell_1$. We say that $a_n = 1/4^n$ if n odd and $a_n = 1/2^n$ otherwise. Clearly

$$0 < \sum_{n \in \mathbb{N}} a_n = \sum_{n \text{ odd}} \frac{1}{4^n} + \sum_{n \text{ even}} \frac{1}{2^n} < \sum \frac{1}{2^n} < \sum \frac{1}{n^2} = \frac{\pi^2}{6}.$$

So the series converges. Let $\rho_N = \sup_{n>N} |a_{n+1}|/|a_n| = \sup_{n>N} 2^n = \infty$. So clearly $\rho = \lim \rho_N = \infty$, and yet the series converges. If we were to suppose that $\lambda = \rho$ then the test would be wrong since $\lambda > 1$ implies divergence. So it must be the case that the test is inconclusive when $\rho \geq 1$.

62.

Theorem 2. *Let $\{a_n\} \in \ell_1$ be a monotonically non-increasing sequence. Then, the series $\sum a_n$ converges if and only if the associated dyadic series*

$$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots = \sum 2^k a_{2^k}$$

converges.

Proof. Suppose that $\sum 2^n a_{2^n}$ converges. Since the sequence is monotone decreasing, we have that $a_1 \geq a_1$, $2a_2 > a_2 + a_3, \ldots, 2^k a_{2^k} > a_{2^k} + \cdots + a_{2^{k+1}}$. In other words the series is dominated by the dyagic series. Therefore by the comparison test, the series must converge.

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63.

Theorem 3. *The series $\sum 1/(k(\log k)^p)$ diverges for $p \leq 1$ and converges for $p > 1$.*

Proof. We will show the theorem using the integral test. In particular, we need only show that the function

$$f(x) = \frac{1}{x \log^p x}$$

has convergent/divergent improper integrals for the corresponding p-cases. In the special case of $p = 1$ simple v-substituion yields that, $\int_2^\infty f(x)\ dx = \log(\log(x))|_2^n \to \infty$. When $p \neq 1$, we have that the indefinite integral is

$$F(n) = \int_2^n \frac{1}{x \log^p x}\ dx = \left.\frac{\log^{1-p}(x)}{1-p}\right|_2^n.$$

It is clear in the case that $p > 1$, the log function is in the denominator and therefore $F(n) \to 0 + \log^p(2)/(1-p) \in \mathbb{R}$. Otherwise, for $p < 1, F(n) \to \infty$. Therefore, by the integral test we have that the series converges for $p > 1$ and diverges otherwise. $\square$

67.

Theorem 4. *Let $\{a_k\} \in \ell_1$ such that a_k has the same sign for all k. The series $\sum a_k$ converges if and only if $\prod(a_k + 1)$ converges.*

Proof. The infinite product $\prod(a_k + 1)$ converges if and only if

$$\log\left(\prod_{k=1}^\infty a_k + 1\right) = \sum_{k=1}^\infty \log(a_k + 1)$$

converges. If $\sum a_n$ converges then $a_n \to 0$, and if $\sum a_n$ does not converge then we do not grant this. By the comparison test $\lim ln(1 + a_n)/a_n = 1$ if and only if $\sum a_n$ converges if and only if the infinite product converges. This completes the proof. $\square$

68.

69.

70.