MATH H104: Homework 8

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November 3, 2015

30. I'll prove this one in reverse, first defining the general β cantor set.

    Definition 1. *The β cantor set, F_β, is defined as an iterative process of which the first iteration is defined for $0 < \beta < 1$ by taking the middle β of $[0, 1]$ and letting F_β^1 be the remaining two intervals of outermeasure $\frac{1-\beta}{2}$. In general, at any iteration of the process n, F_β^n is comprised of 2^n pieces, each of outer measure P_n. The process for generating P_{n+1} is the same as for the first iteration: remove $P_n\beta$ from each piece of outer measure P_n and yield two pieces of outer measure $P_n(1-\beta)/2$. Lastly, $F_\beta \subset F_\beta^n \subset F_\beta^{n-1.}$ for all n.*

    Theorem 1. *For any $0 < \beta < 1$, F_β is a zero set.*

    Proof. To show this, for every $\epsilon > 0$ we need to find a countable collection of open sets which cover F_β and have outer measure less than ϵ. By the definition of F_n we have that at any iteration n the total outer measure is $P_n 2^n$. Thus we solve the recurrance relation,

    $$P_n = P_{n-1}\frac{(1-\beta)}{2},$$

    by letting $P_n = \frac{(1-\beta)}{2}$ and solving for the initial conditions that $P_0 = 1$. Thus the total outer measure of F_β^n is defined as

    $$outer(F_\beta^n) = \frac{(1-\beta)^n}{2^n}2^n = (1-\beta)^n \to 0$$

    by $0 < 1-\beta < 1$. So for every ϵ there is a large enough N such that by extending F_β^N to a very close open interval containing F_β^N and thereby F_β, the outer measure is less than ϵ. So F_β is a zero set. $\square$

    It follows simply that the middle fourths cantor set is a zero-set.

31. Again I'll provide a definition for the general fat cantor set and apply it to the specifc definition provided.

Definition 2. *The fat β cantor set, F_β, is defined as an iterative process of which the first iteration is defined for $0 < \beta < 1$ by taking the middle β^n of $[0,1]$ and letting F_β^1 be the remaining two intervals of outermeasure $\frac{1-\beta^n}{2}$. In general, at any iteration of the process n, F_β^n is comprised of 2^n pieces, each of outer measure P_n. The process for generating P_{n+1} is the same as for the first iteration: remove $P_n\beta^n$ from each piece of outer measure P_n and yield two pieces of outer measure $P_n(1-\beta^n)/2$. Lastly, $F_\beta \subset F_\beta^n \subset F_\beta^{n-1\cdot}$ for all n.*

Theorem 2. *The fat β cantor set is not zero set.*

Proof. We essentially need to show that as F_β^n approaches F_β, the outer measure of F_β^n does not tend towards 0. By the definition of F_n we have that at any iteration n the total outer measure is $P_n 2^n$. Thus we solve the recurrance relation,

$$P_n = P_{n-1}\frac{(1-\beta^n)}{2}.$$

Using intuition from the β canot set case, we let P_n be for the form

$$P_n = \frac{(1-\beta^n)}{a_n 2^N}$$

for some sequence a_n depending on n. Then we can find a_n by considering the ratio P_n/P_{n-1}. This essentially yields that $a_n - a_{n-1} = 1$. Ommitting the application of variation of parameters to this recurrence relation, we yield $a_n = n$. Thus the total outer measure of F_β^n is defined as

$$outer(F_\beta^n) = \frac{(1-\beta^n)^n}{2^n}2^n = (1-\beta^n)^n \to 1$$

by $0 < 1-\beta < 1$. So there could not possibly be a sequence of countable open coverings of F_β with outer measure approaching 0. This completes the proof. □

In the particular case of the problem, letting $\beta = 1/4$ apply the previous theorem, and yield that the outer measure of the fat cantor set is 1. This is remarkable!

Theorem 3. *The property that S is a zero set is not topological.*

Proof. It suffices to show that for two sets A, B which are homeomorphic, the zero set property does not hold. Clearly $F_\beta \cong C_\beta$ since they are both cantor spaces. Howerver Thoerem 2 states that F_β is not a zero set, whereas C_β is. So by counter example, the zero set property is not topological. □

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