MATH 185: Homework 2

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1.

Definition 1. *A set $S \subset \mathbb{C}$ is bounded if and only if there exists $z \in \mathbb{C}$ such that for every $s \in S$, $|s| \leq |z|$*

Definition 2. *Alternativeley, a set $S \subset \mathbb{C}$ is bounded if and only if there is an r such that $S \subset B_r(0)$, where $B_s(z)$ is the ball of radius s with center z.*

Theorem 1. *If $(z_n)_{n=1}^{\infty}$ is a convergent sequence of compex numbers, then the sequence is bounded.*

Proof. Take the value set $S = \{z_n\}$. Then suppose there were no r such that $S \subset B_r(0)$. If this is the case, the countability of S implies that for every n, $S \cap B_n(0)$ is finite. Since $z_n \to z$, take $N \in \mathbb{N}$ such that $N > |z|$. Such an n exists by the archimedian principle of $\mathbb{R}$. Then $S \cap N$ must be finite.

Take $\epsilon = N - |z|$, then there is an M such that for all $m > M$, $d(z_n, z) < \epsilon$. That is there are infinite elements within ϵ of z, and thereby there are infinite elements in $S \cap B_N(0)$. This is a conradiction to its finiteness.

Therefore it must be that the value set is contained within the N ball, and therefore, (z_n) is bounded. □

2. Exercise II.1.11

Theorem 2. *The function $Arg : \mathbb{C} \to \mathbb{R}$ is continuous except for along the line $L = \{z : Im(z) = 0 \wedge Re(z) < 0\}$.*

Proof. A function is continuous if and only if it preserves limits. Specifically, if $\lim_{h \to x} f(h) = f(x)$ implies that f is continuous at h. Consider the restricted Arg function, say $A : \mathbb{C} \backslash L \to \mathbb{R}$. Then it is clear that $\lim_{\mathbb{C} \backslash L} A(h) = (-\pi, \pi)$, since if a point is within an ϵ neighborhood of another point, its gradial distance is proportionate to $\sin^{-1}$ of its ϵ distance, (a continuous function).

However consider any $z \in L$ Such that $h \to z$ approaches from the upper half plane and $g \to z$ from the lower. Clearly $Arg(h) \to \pi$ and $Arg(g) \to -\pi$, so no limit exists and the function is not continuous at z. This completes the proof. □

3. Exercise II.1.16

Theorem 3. *The punctured plane $\mathbb{C} \setminus L = \mathbb{C}_P$ is star shaped but not convex.*

Proof. Take any $z \in \mathbb{C}_P$. Then for any $r \geq 1$, z/r is clearly in $\mathbb{C}_P$ since r is always positive and the imaginary part of z is always non-zero or its real part is non-negative. In the first case z/r is never in L for all finite r, and when $r \to \infty$, then $r = 0 \in \mathbb{C}_P$. In the second case, its real part is always positive or 0 until it reaches 0 by the same logic. In the case that both are true, we consider again the same logic. If $z = 0$, we are done.

Clearly, $\mathbb{C}_P$ is not convex when considering the line, $B = \{x + iy : x = -1\}$ which contains $-1 \in L$. □

Definition 3. *A space X is contractible if the identity map is homotopic to some constant map.*

Definition 4. *A homotopy between two continuous functions f, g from a topological space X to a topological space Y is a continuous function $H : X \times [0,1] \to Y$, such that if $x \in X$ then, $H(x,0) = f(x), H(x,1) = g(x)$. [Wikipedia]*

Theorem 4. *Every homeomorphism is a homotopy equivalence.*

Theorem 5. *A star-shaped space X is homotopic to a point.*

Proof. Let $H(x,t) = x(1-t) + z_0 t$, then $H(x,0) = id_X$, and $H(x,1)$ is the constant identity. H is continuous by the definition of H as a star shaped space. Therefore, the star-shaped space is homotopic to a point. □

Theorem 6. *The space $\gamma = \mathbb{C} \setminus [-1,1]$ is not star shaped.*

Proof. The set γ is not homeomorphic to the unit ball B^2 and since any homemorphism would contain a radial unit submap around the unit interval which is disconnected. Therefore, γ is not homotopic to B^2 which is homotopic to a point since B^2 is star shaped. The space γ could not be star shaped since if it were it would be homotopic to a point which it is not. Therefore, γ is not star shaped. □

Theorem 7. *The punctured disk is not star shaped.*

Proof. The punctured disk is not homeomorphic to B^2 since the diameter is disconnected and any submap there would violate the homeomorphism. Therefore it is not homeotopic, and by the logic of the above proof, it is not homeotopic to a point, and so it could not possibly be star shaped as that would lead to a contradiction. This completes the proof. This completes the proof. □