MATH 185: Homework 1($\tau = 2\pi$)

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1. *Show that multiplication of complex numbers satisfies the associative, commutative, and distributive laws.*

Theorem 1. *Given that $\mathbb{C}$ is Abelian under addition, $\mathbb{C}$ is a field.*

Proof. Let $a, b, c \in \mathbb{C}$. Then recall that for any $z \in \mathbb{C}$, $z = |z|e^{i\theta_z}$, where $\theta_z = Argz$. We show that $\mathbb{C}$ satisfies associative, commutative, and distributive laws.

Using that $\mathbb{R}$ is a field, it follows that

$$\begin{aligned}
(ab)c &= (|a|e^{i\theta_a}|b|e^{i\theta_b})|c|e^{i\theta_c} \\
&= |a||b|e^{i(\theta_a+\theta_b)}|c|e^{i\theta_c} \\
&= |a||b||c|e^{i(\theta_a+\theta_b+\theta_c)} \\
&= |a|e^{i\theta_a}|b||c|e^{i(\theta_b+\theta_c)} \\
&= a(bc).
\end{aligned}$$

Without the assumption of eulers identity , we have that

$$\begin{aligned}
(ab)c &= ((a_1 + ia_2)(b_1 + ib_2))(c_1 + ic_2) \\
&= ((a_1b_1 - a_2b_2) + (a_1b_2 + a_2b_1)i)(c_1 + ic_2) \\
&= ((a_1b_1 - a_2b_2)c_1 - (a_1b_2 + a_2b_1)c_2) \\
&\quad + ((a_1b_1 - a_2b_2)c_2 + (a_1b_2 + a_2b_1)c_1)i \\
&= a_1b_1c_1 - a_2b_2c_1 - a_1b_2c_2 + a_2b_1c_2 \\
&\quad + (a_1b_1c_2 - a_2b_2c_2 + a_1b_2c_1 + a_2b_1c_1)i \\
&= a_1(b_1c_1 - b_2c_2) - a_2(b_2c_1 + b_1c_2) \\
&\quad + (a_1(b_1c_2 + b_2c_1) - a_2(b_2c_2 + b_1c_1))i \\
&= (a_1 + a_2i)((b_1c_1 - b_2c_2) + (b_1c_2 + b_2c_1)i) \\
&= a(bc).
\end{aligned}$$

In a similar fashion, consider the following rearrangement which follows by the field properties of $\mathbb{R}$:

$$\begin{aligned}
ab &= (a_1b_1 - a_2b_2) + (a_1b_2 + a_2b_1)i \\
&= (b_1a_1 - b_2a_2) + (b_2a_1 + b_1a_2)i \\
&= ba.
\end{aligned}$$

Lastly we show the distributive property:

$$\begin{aligned} a(b+c) &= a(b_1 + b_2 i + c_1 + c_2 i) \\ &= a((b_1 + c_1) + (b_2 + c_2)i) \\ &= (a_1(b_1 + c_1) - a_2(b_2 + c_2)) + (a_1(b_2 + c_2) + a_2(b_1 + c_1))i \\ &= (a_1 b_1 - a_2 b_2) + (a_1 c_1 - a_2 c_2) + (a_1 b_2 + a_2 b_1)i + (a_1 c_2 + a_2 c_1)i \\ &= ab + ac \end{aligned}$$

Therefore $\mathbb{C}$ is a ring. □

2. Gamelin Exercise I.1.7 (Chapter I, Section 1, Exercise 7)

Theorem 2. *Let $\rho > 1, \rho \neq 1$ and fix $z_0, z_1 \in \mathbb{C}$. Then*

$$S = \{|z - z_0| = \rho |z - z_1| : z \in \mathbb{C}\}$$

is isometric to some $S^1_r \subset \mathbb{R}^2$ for some r.

Proof. Since all $s \in S$ satisfy the above equation*, we have that

$$\sqrt{(s_1 - z_{01})^2 + (s_2 - z_{02})^2} = \rho \sqrt{((s_1 - z_{11})^2 + (s_2 - z_{12})^2}.$$

The form of (5) is identical to a distance meterization in $\mathbb{R}^2$; that is, take the isometry $\phi : \mathbb{C} \to \mathbb{R}^2, ((x + iy) \mapsto (x, y)$ and

$$d(\phi(s), \phi(z_0)) = \rho d(\phi(s), \phi(z_1)) \frac{d(S, Z_0)}{d(S, Z_1)} = \rho,$$

which from high school geometry one might recognize as the equation* of the circle of Appolonius. □

The geometric proof of a equivalency between Appolonius' circle and the Euclidean circle is omitted.

However, if we take the euclidean distance on $\mathbb{R}^2$, we have the following theorem.

Theorem 3. *Suppose that $P, Q \in \mathbb{R}^2$ and S such that*

$$\frac{\overline{PS}}{\overline{QS}} = k \in (0,1)[WLOG],$$

then S is a point on a circle.

Proof. Observe the following algebraic derivation using the parallelagram law inspired by J Wilson at the University of Georgia:

$$\begin{aligned} \frac{|P - S|^2}{|Q - S|^2} &= k^2 \\ |P|^2 + |S|^2 - 2\langle P, S\rangle &= k^2(|Q|^2 + |S|^2 - 2\langle Q, S\rangle) \\ 0 &= |P|^2 + |S|^2 - 2\langle P, S\rangle - k^2(|Q|^2 + |S|^2 - 2\langle Q, S\rangle) \\ &= (1 - k^2)|S|^2 + |P|^2 - k^2|Q|^2 - 2\langle P - Q, k^2 S\rangle \qquad = |S|^2 + \frac{|P|^2}{1 - k^2} - \frac{}{k}^2 |Q| \end{aligned}$$

□

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3. Gamelin Exercise I.2.5

Theorem 4. *For $n \geq 1$ and $z \in \mathbb{C}$ such that $z \neq 1$, we have that*

$$1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1 - z).$$

Proof. Observe that for $z \in \mathbb{C}$ we have that, $z = e^{i\theta}$. Therefore,

$$e^{i0} + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = 1 + z + z^2 + \cdots + z^n$$

Multiplication by $(1 - z)$ gives,

$$\begin{aligned}(1 - e^{i\theta})e^{i0} + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= e^{i0} + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} \\ &\quad - e^{i(0+\theta)} + e^{i(\theta+\theta)} + e^{i(2\theta+\theta)} + \cdots + e^{i(n\theta+\theta)} \\ &= e^{i0} - e^{i(n\theta+\theta)} \\ &= 1 - z^{n+1}.\end{aligned}$$

Reducing using eulers identity it follows that,

$$\begin{aligned}(1 - z)(1 + z + z^2 + \cdots + z^n) &= (1 - z^{n+1}) \\ 1 + z + z^2 + \cdots + z^n &= (1 - z^{n+1})/(1 - z),\end{aligned}$$

when $z \neq 1$. This completes the proof. □

Theorem 5. *For $n \geq 1$ and $z \in \mathbb{C}$ such that $z \neq 1$, we have that*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})}{2\sin\theta/2}$$

Proof. Recall that $z = rcis\theta$. Take in particular all such z whose absolute magnitude is unity. Then Theorem 4 implies that

$$1 + cis\theta + cis2\theta + \cdots + cisn\theta = (1 - z^{n+1})/(1 - z).$$

A little algebra gives us

$$\begin{aligned}\frac{Re(1 - cis(n+1)\theta}{Re(1 - cis\theta)} &= \frac{Re(1 - e^{(n+1)\theta})Re(1 - e^{-i\theta})}{Re(1 - e^{i\theta})Re(1 - e^{-\theta i})} \\ &= \frac{Re(1 - e^{i\theta} - e^{i(n+1)\theta} + e^{in\theta})}{Re(2 - 2cos\theta)} \\ &= \frac{1 - \cos\theta - \cos(n+1)\theta + \cos n\theta}{4\sin^2(\theta/2)} \\ &= \frac{2\sin^2\theta/2 - \sin(n + 1/2)\sin(\theta/2)}{4\sin^2(\theta/2)} \\ &= \frac{1}{2} - \frac{\sin(n + 1/2)}{2\sin(\theta/2)}\end{aligned}$$

Since the above was the real part of $1 + z + z^2 + \cdots + z^n$, the theorem holds. □

4. Gamelin Exercise I.2.6

Theorem 6. *If w_n are the nth roots of unity, then*

(a) $(z - w_0)(z - w_1)\dots(z - w_{n-1}) = z^n - 1$.

(b) $w_0 + \dots + w_{n-1} = 0$.

(c) $w_0 \dots w_n = (-1)^{n-1}$.

(d) $\sum_{j=0}^{n-1} w_j^k = 0, n$.

Proof. (a) Consider that every complex polynomial has roots by the fundamental theorem of algebra. Therefore, every polynomial can be linearized and $z^n - 1$ is no exception. On the left hand side, the expression is zero if and only if $z = w_i$ for some $i \in \{0, \dots, n-1\}$. On the right side, the order n polynomnial is zero if and only if $z^n = 1$, which can only be provided by n distinct roots. By defenition $w_i^n = 1$ and there are n distinct roots of unity. Therefore $z^n - 1 = 0$ if and only if $z = w_i$, which is equivalent to the statement of the left hand side.

(b) Conveniently, let $R = \sum w_i$. Then,

$$e^{i\tau/n}R = e^{i\tau n/n} + (R - 1) = R. \tag{1}$$

So this gives $xR = R$ and since $x \neq R$, we have that $R = 0$.

(c) Let P be the product of the n roots of unity. Then, observe that the product given by eulers formula implies that

$$Arg(P) = \sum_{k=0}^{n-1} Arg(w_k) = \sum_{k=0}^{n-1} k\tau/n = \frac{\tau(n-1)}{2}$$

which is $\tau/2$ if $n-1$ is odd or 0 if $n-1$ is even. Therefore, $P = (-1)^{n-1}$.

(d) Applying the same techniques as previously, let $Q = \sum_{j=0}^{n-1} w_j^k$. Then, $e^{i\tau/n}Q = \sum_{j=1}^{n} e^{i\tau kj/n}$. Observe that if $x \equiv j \mod n$, then $kj \equiv x \equiv j \mod n$ when k $\neq mn$ for some $m \in \mathbb{Z}$ Since there is a ring isomorphism between roots of unity and modulo rings, we have that $Q = R = 0$. In the case that k is a multiple of n, we have that $w_j^k = 1$, so the sum must be n.

□

5. Gamelin Exercise I.3.2

Theorem 7. *If P is a point on the sphere which corresponds to $z \in \mathbb{C}$ under stereographic projection, then the antipodal point $-P$ corresponds to $-1/\overline{z}$.*

Proof. As perscribed in the book,

$$P_z = \begin{pmatrix} 2x/(|z|^2 + 1) \\ 2y/(|z|^2 + 1) \\ (|z|^2 - 1)/(|z|^2 + 1) \end{pmatrix}. \tag{2}$$

So it follows that,

$$-P_z = \begin{pmatrix} -2x/(|z|^2+1) \\ -2y/(|z|^2+1) \\ -(|z|^2-1)/(|z|^2+1) \end{pmatrix}. \tag{3}$$

Now let $-1/\overline{z} = w$, so that $w = -z/|z|^2$ and $|w| = 1/|z|$. This gives the following derivation,

$$\begin{aligned} P_w &= \begin{pmatrix} \frac{-2x}{|z|^2(1/|z|^2+1)} \\ \frac{-2y}{|z|^2(1/|z|^2+1)} \\ (1/|z|^2-1)/(1/|z|^2+1) \end{pmatrix} \\ &= \begin{pmatrix} -2x/(|z|^2+1) \\ -2y/(|z|^2+1) \\ \frac{-(1-|z|^2)}{-(1+|z|^2)} \end{pmatrix} \\ &= \begin{pmatrix} -2x/(|z|^2+1) \\ -2y/(|z|^2+1) \\ -(|z|^2-1)/(|z|^2+1) \end{pmatrix} = -P_z. \end{aligned} \tag{4}$$

□

6. Gamelin Exercise I.3.4

Theorem 8. *If S^2 is rotated $\tau/2$ radians about the real axis, show that such a transformation corresponds to the mapping $z \mapsto 1/z$.*

Proof. By theorem 7,

$$P_z = \begin{pmatrix} 2x/(|z|^2+1) \\ 2y/(|z|^2+1) \\ (|z|^2-1)/(|z|^2+1) \end{pmatrix}. \tag{5}$$

. It follows then that a rotation about the x axis, yields

$$R[P_z] = \begin{pmatrix} 2x/(|z|^2+1) \\ -2y/(|z|^2+1) \\ -(|z|^2-1)/(|z|^2+1) \end{pmatrix}. \tag{6}$$

Now let $w = 1/z = \overline{z}/|z|^2$. Then $Re(w) = Re(1/z) = Re(z/|z|^2), Im(w) = Im(1/z) = -Im(z/|z|^2), |w| = |1/z| = 1/|z|$. Projecting w stereoscopically gives the following derivation:

$$\begin{aligned} P_w &= \begin{pmatrix} \frac{2x}{|z|^2(1/|z|^2+1)} \\ \frac{-2y}{|z|^2(1/|z|^2+1)} \\ (1/|z|^2-1)/(1/|z|^2+1) \end{pmatrix} \\ &= \begin{pmatrix} -2x/(|z|^2+1) \\ -2y/(|z|^2+1) \\ \frac{-(1-|z|^2)}{-(1+|z|^2)} \end{pmatrix} \\ &= \begin{pmatrix} -2x/(|z|^2+1) \\ -2y/(|z|^2+1) \\ -(|z|^2-1)/(|z|^2+1) \end{pmatrix} = R[P_z]. \end{aligned} \tag{7}$$

This completes the proof that inversion is a simple rotation of the sphere about the x axis. □

7. Gamelin Exercise I.5.3

Theorem 9. *If $z \in \mathbb{C}$ it follows that $e^{\overline{z}} = \overline{e^z}$.*

Proof. Recall that for $x, y \in \mathbb{R}$, $z = x + iy$. Then

$$\begin{aligned} e^{\overline{z}} &= e^{x-iy} = e^x e^{-iy} \\ &= \frac{e^x}{e^{iy}} = \frac{e^x \overline{e^{iy}}}{|e^{iy}|^2} \\ &= e^x \overline{e^{iy}} = \overline{e^{x+iy}}. \end{aligned} \tag{8}$$

So it follows that complex conjugation distributes through exponentiation. This completes the proof. □