MATH 185: Homework 1

William Guss
26793499
wguss@berkeley.edu

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2 Algebraic Properties

2.

Theorem 1. *If $z \in \mathbb{C}$ then $Re(iz) = -Im(z)$ and $Im(iz) = Re(z)$.*

Proof. Since $z \in \mathbb{C}$, $z = x + iy$ and $iz = ix - y$ and so $Re(iz) = -y = -Im(z)$. Furthermore $Im(iz) = x = Re(z)$. □

4. *Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 2z + 2 = 0$.*

$$\begin{aligned}(1+i)^2 - 2 - 2i + 2 &= 1 + 2i - 1 - 2 - 2i + 2 = 0\\ (1-i)^2 - 2 + 2i + 2 &= 1 - 2i - 1 - 2 + 2i + 2 = 0.\end{aligned} \tag{1}$$

11. *Solve the equation $z^2 + z + 1 = 0$.*
Using $z \in \mathbb{C}$ we have

$$-\frac{4}{3}(z + 1/2)^2 = 1. \tag{2}$$

So we need solve $w^2 = 3/4e^{i\pi}$. Using eulers formula we get $r^2 e^{i2\theta} = 3/4e^{i\pi}$ and so it must be that $r^2 = 3/4$, so $r = \pm\sqrt{3}/2$ and $\theta = \pi/2$. Therefore $w = \pm\sqrt{3}/2i$. Furthermore, $z = w - 1/2$ so $z = -1/2(1 \mp \sqrt{3}i)$.

3 Further Properties

1. *Reduce the following equations.*

(a)

$$\begin{aligned}\frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \frac{(1+2i)(3+4i)}{25} + \frac{-5i(2-i)}{25}\\ &= \frac{-5 + 10i - 5 - 10i}{25}\\ &= \frac{-10}{25} = -\frac{2}{5}\end{aligned} \tag{3}$$

(b)

$$\frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{-10i} = -\frac{1}{2} \tag{4}$$

(c)

$$\begin{aligned}(1-i)^4 &= 1^4 + 4(-i)^1 + 6(-i)^2 + 4(-i)^3 + (-i)^4 \\ &= 1 - 4i - 6 + 4i + 1 = -4.\end{aligned} \tag{5}$$

2.

Theorem 2. *If $z \in \mathbb{C}$ and $z \neq 0$ then*

$$\frac{1}{1/z} = z. \tag{6}$$

Proof. Recall that $w := 1/z = \overline{z}/|z|^2$. Furthermore $1/w = \overline{w}/|w|^2 = \overline{w}/(1/|z|)^2$ by $\overline{z}/|z|^2 = re^{i-\theta}/r^2 = e^{i-\theta}/r$. Then $\overline{w} = \overline{\overline{z}/|z|^2} = z/|z|^2$ and $\overline{w}/(1/|z|)^2 = z = 1/(1/z)$. This completes the proof. $\square$

5 Vectors and Moduli

4.

Theorem 3. *If $z \in \mathbb{C}$ then*

$$\sqrt{2}|z| \geq |Re(z)| + |Im(z)|. \tag{7}$$

Proof. Let $z = x + iy$ then $|Re(z)| + |Im(z)| = |x| + |y|$ and

$$(|x| + |y|)^2 = |x|^2 + 2|x||y| + |y|^2. \tag{8}$$

Then it remains to prove that $2|x||y| \leq |x|^2 + |y|^2$. Now $0 \leq (|x| - |y|)^2$ implies that $0 \leq |x|^2 - 2|x||y| + |y^2|$ which clearly implies that

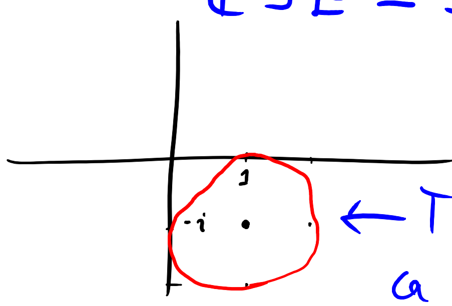
$$2|x||y| \leq |x|^2 + |y|^2. \tag{9}$$

Because $(|x| + |y|)^2 \leq \sqrt{2}|z|^2$, then it follows that $\sqrt{2}|z| \geq |Re(z)| + |Im(z)|$. $\square$

5. *Sketch the points determed by the given condition.*

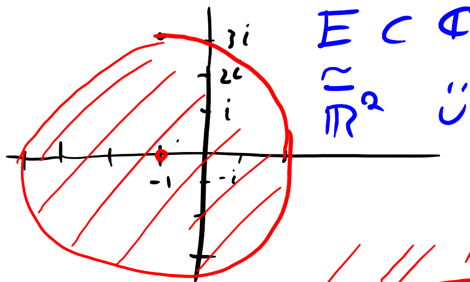
(a) $|z - 1 + i| = 1.$ $= |z - (1-i)|$

$\mathbb{C} \supset E \simeq S^1$

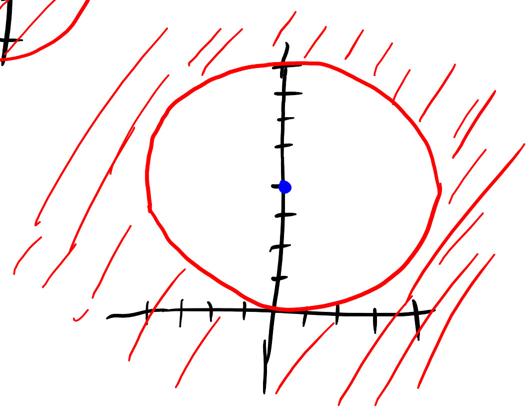


← That is a circle

(b) $|z+1| \leq 3$.



(c) $|z-4i| \geq 4$.



6. *Use geometric arguments!*

(a)

Theorem 4. *The set of points $z \in S \subset \mathbb{C}$ such that $|z-4i|+|z+4i| = 10$ is an elipse*

Proof. Let $z = x+iy$. Then for every point $z \in S$ the points $w = -4i$ and $w = 4i$ are always a summed distance of 10 from z. By definition these points are foci of the set S. Furthermore

$$\begin{aligned} 10 &= \sqrt{x^2+(y-4)^2} + \sqrt{x^2+(y+4)^2} \\ 100 &= (f(x,y)+g(x,y))^2 = f(x,y)^2 + f(x,y)g(x,y) + g(x,y)^2 \qquad (10) \end{aligned}$$

and $f(x,y)^2$ is a quadratic, $f(x,y)g(x,y)$ is a quadratic. and $g(x,y)^2$ is a quadratic where no coefficients on the quadratic mononomial projection are 0. Therefore the levelset must be an elipse. □

9. *Prove the following.*

Theorem 5. *Let $z \in \mathbb{C}$ and n a positive integer. Then $|z^n| = |z|^n$.*

Proof. We induct on n. Let $n = 1$. Then $|z^1| = |z| = |z|^1$. Suppose that $|z^k| = |z|^k$. Then $|z^{k+1}| = |z^k \cdot z| = |z^k||z|$ by (8). Then by our assumption $|z^k||z| = |z|^k|z| = |z|^{k+1}$ and so the theorem holds fo $k+1.s$ By induction the proof is complete. □

6 Complex Conjugates

4. *Prove the following.*

Theorem 6. *If z, z_1, z_2, z_3 are complex numbers then*

$$\overline{z_1 z_2 z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}; \qquad \overline{z^4} = \overline{z}^4 \tag{11}$$

Proof. By associativity of $\mathbb{C}$ and (4) it follows that without loss of generality $\overline{z_1 z_2 z_3} = \overline{z_1(z_2 z_3)} = \overline{z_1}\,\overline{z_2 z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}$. Then $\overline{z^4} = \overline{z(zzz)} = \overline{z}^3\overline{z} = \overline{z}^4$. □

5. *Verify the following.*

Theorem 7. *If z_1, z_2 are complex numbers and $|\cdot| : \mathbb{C} \to \mathbb{R}$ is the complex moduli, then*

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{12}$$

Proof. Recall that $z_1/z_2 = z_1\overline{z_2}/|z_2|^2$. Then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1\overline{z_2}|}{|z_2|^2} = \frac{|z_1||\overline{z_2}|}{|z_2|^2} = \frac{|z_1|}{|z_2|} \tag{13}$$

since $|z_1| = |\overline{z_2}|$ is trivially true, and $|ab| = \left|r_1 r_2 e^{i(\theta_a + \theta_b)}\right| = r_1 r_2 = |a||b|$. □

6. *Prove the following.*

Theorem 8. *Let $z_1, z_2, z_3 \in \mathbb{C}$ with $z_2, z_3 \neq 0$. Then*

$$\overline{\left(\frac{z_1}{z_2 z_3}\right)} = \frac{\overline{z_1}}{\overline{z_1}\,\overline{z_2}}. \tag{14}$$

Proof. Observe the following

$$\overline{\left(\frac{z_1}{z_2 z_3}\right)} = \overline{z_1}\,\overline{\frac{1}{z_2 z_3}} = \overline{z_1}\,\overline{\frac{\overline{z_2 z_3}}{|z_2 z_3|^2}} = \frac{\overline{z_1} z_2 z_3}{\overline{z_2 z_3} z_2 z_3} = \frac{\overline{z_1}}{\overline{z_1 z_2}} \tag{15}$$

using the identities of the section. □

Theorem 9. *Let $z_1, z_2, z_3 \in \mathbb{C}$ with $z_2, z_3 \neq 0$. Then*

$$\left|\frac{z_1}{z_2 z_3}\right| = \frac{|z_1|}{|z_1||z_2|}. \tag{16}$$

Proof. Let $a = z_1, b = z_2 z_3$ then by the previous exercise

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} = \frac{|z_1|}{|z_2 z_3|} = \frac{|z_1|}{|z_2||z_3|}. \tag{17}$$

This completes the proof. □

9. *Prove the following.*

Theorem 10. *If z lies on the circle $|z| = 2$ then*

$$\frac{1}{|z^4 - 4z^2 + 3|} \leq \frac{1}{3} \tag{18}$$

Proof. Consider the factorization, $|z^4 - 4z^2 + 3| = |z^2 - 3||z^2 - 1|$. It follows that $||z^2| - 3|||z^2| - 1| = |4 - 3||4 - 1| = 3 \leq |z^4 - 4z^2 + 3|$ from (9) section 4. So the reciprocal inequality holds. □

14. *Prove the following.*

Theorem 11. *Let $z \in \mathbb{C}$. Show that the hyperbola $x^2 - y^2 = 1$ can be written*

$$z^2 + \overline{z}^2 = 2. \tag{19}$$

Proof. Let $z = x + iy$, then algebra gives

$$z^2 + \overline{z}^2 = x^2 + 2ixy - y^2 + x^2 - 2ixy - y^2 = 2x^2 - 2y^2 = 2. \tag{20}$$

Dividing by two gives that all z satisfying the complex equation describe the hyperbola. □

9 Arguments of Products and Quotients

1. *Find the principle argument $Argz$.*
   (a) Let $z = \frac{-2}{1+\sqrt{3}i}$. Then $Argz = Arg(-2) - Arg(1 + \sqrt{3}i) = \pi - Arg(1 + \sqrt{3}i) = \pi - tan^{-1}(\sqrt{3}). = 2\pi/3$
   (b) Let $z = \left(\sqrt{3} - i\right)^6$. Then $Argz = 6Arg(\sqrt{3} - i) = -6\pi/6 = \pi$ in principle.
2. *Prove the following theorem.*

Theorem 12. *If $\theta \in \mathbb{R}$ then $|e^{i\theta}| = 1$ and $\overline{e^{i\theta}} = e^{-i\theta}$.*

Proof. Observe that $e^{i\theta} = \cos\theta + i\sin\theta$, so $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. Now $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta}$. Then by cos even and sin odd $e^{-i\theta} = \cos(-\theta) + i\sin -\theta = \cos\theta - i\sin\theta = \overline{e^{-i\theta}}$. □

9. *Establish the Lagrange's trigonometric identity.*
Using the following trick,

$$(1 + z + z^2 + \cdots + z^n)(1 - z) = 1 + (z - z) + \cdots + (z^n - z^n) + z^{n+1}, \tag{21}$$

we get that

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}. \tag{22}$$

Using $z = e^{i\theta}$ we get

$$\begin{aligned} 1 + \sum_k^n \cos(n\theta) + i\sum_k^n \sin k\theta &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \\ &= \frac{(1 - e^{i(n+1)\theta})(1 - e^{-i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})} \\ &= \frac{1 - e^{-i\theta} - e^{i(n+1)\theta} + e^{in\theta}}{1 - e^{i\theta} - e^{-i\theta} + 1} \\ &= \frac{1 - i2\sin(\theta) + e^{i(n+1)\theta}}{2 + 2\cos(\theta)} \end{aligned} \tag{23}$$

10. *Use de Moivre's formula to derive the following.*

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta. \end{aligned} \tag{24}$$

Proof. Let $\theta \in \mathbb{R}$, then

$$\cos 3\theta + i\sin\theta = (\cos\theta + i\sin\theta)^3. \tag{25}$$

By binomial expansion then

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\cos^3. \tag{26}$$

Separating the imaginary and real parts gives the formulas exactly. □

11 Roots of Complex Numbers

4. *Identify the Principle Root.*
We find the roots of $(-2-1)^{1/3}$ by taking $z_0 = e^{i\pi}$. Then $z_0^{1/3} = e^{i\pi/3 + i2k\pi/3}$. This gives a triangle and principle root $e^{i\pi/3}$.

We find the roots of $8^1/6$. Let $z_0 = 8e^{i0+i2k\pi}$. Then we get $8^{1/6} = \sqrt{2}e^{i2k\pi/6}$ which forms a hexagon with principle root $\sqrt{2}$.

6. *Find the four zeros of $z^4 + 4$.*
This problem is equivalent to finding the 4th root of $-4 = 4e^{i\pi}$. This gives

$$z = \sqrt{2}e^{i\pi/4 + ik\pi/2}. \tag{27}$$

7. *Prove the following.*

Theorem 13. *If c is an n^{th} root of unity then*

$$1 + c + \cdots + c^{n-1} = 0. \tag{28}$$

Proof. Recall the formula

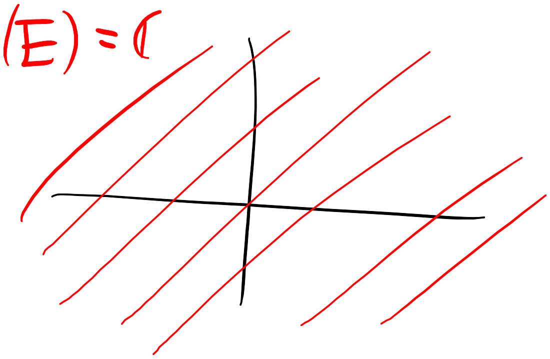
$$1 + c + c^2 + \cdots + c^{n-1} = \frac{1-c^n}{1-c} = \frac{0}{1-c}. \tag{29}$$

This completes the proof. □

12 Regions in the Complex Plane

4. . a) $\{-\pi < \arg z < \pi\} = E.$

$cl(E) = \mathbb{C}$



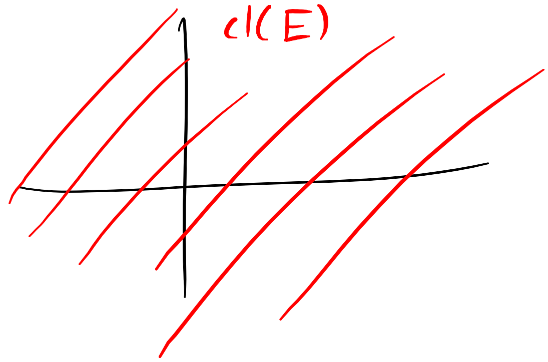
b) $\{|\mathrm{Re}\, z| < |z|\} = E$.

$x \in E \Rightarrow \mathrm{Im}\, z \neq 0$

$\Rightarrow E = \mathbb{C} \setminus \mathbb{R}$.

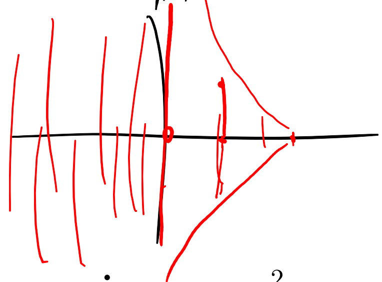
$\Rightarrow cl(E) = \mathbb{C}$, by $\mathbb{C}\setminus\mathbb{R}$ dense in $\mathbb{R}$

cl(E)



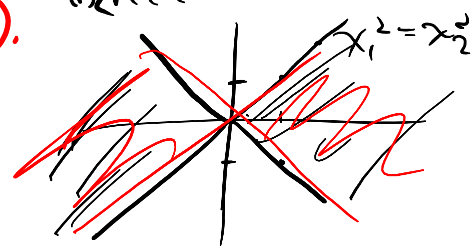
c) $\{\mathrm{Re}(\frac{1}{z}) \le \frac{1}{2}\} = E$. $x \in E \Rightarrow x \neq 0$,

$\Rightarrow \mathrm{Re}\left(\frac{\bar{x}}{|x|^2}\right) \le \frac{1}{2} \to \frac{x_1}{x_1^2 + x_2^2} \le \frac{1}{2}$



cl(E).

d) $\{\mathrm{Re}(z^2) > 0\}$. $\Rightarrow x_1^2 - x_2^2 > 0$



14 The mapping $w = z^2$.

4. *Write $f(z) = z + 1/z$ in parametric form.* Observe that

$$f(z) = z + \frac{1}{z} = re^{i\theta} + \frac{e^{-i\theta}}{r} = r(\cos(\theta) + i\sin(\theta)) + \frac{1}{r}(\cos(\theta) - i\sin(\theta)) \tag{30}$$

Clearly by separating the brackets and parameterizing the function we get

$$u(r,\theta) = (r + 1/r)\cos(\theta), v(r,\theta) = (r - 1/r)\sin(\theta). \tag{31}$$