MATH 105: Homework 8

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29. Upper semicontinuity.

    (a) A graph of an upper semicontinuous graph here:

    (b) Show the following.

    Definition 1. *We say that a function $f : M \to \mathbb{R}$ is (ϵ, δ)-upper semicontinuous if and only if for every $\epsilon > 0$ there is a $\delta > 0$ so that*

    $$0 < d(x, y) < \delta \implies f(y) < f(x) + \epsilon \tag{1}$$

    Lemma 1. *Upper semicontinuity is equivalent to the (ϵ, δ)-upper semicontinuity.*

    Proof. Observe the following fact about $\limsup$.

    $$\limsup_{y \to x} g(y) = \alpha = \lim_{\epsilon \to 0} \sup\{g(y) \ : \ y \in M \cap M_\epsilon(x) \setminus \{x\}\}. \tag{2}$$

    Therefore f is upper semicontinuous if and only if

    $$\limsup_{y \to x} f(y) \le f(x) \iff \lim_{\epsilon \to 0} \sup\{f(y) : y \in M \cap M_\epsilon(x) \setminus \{x\}\} \le f(x). \tag{3}$$

    We then know for every $\epsilon > 0$ there exists a δ so that

    $$\sup\{f(y) \ : \ y \in M \cap M_\delta(x)\{x\}\} < f(x) + \epsilon. \tag{4}$$

    This is true if and only if

    $$d(y, x) < \delta \implies f(y) < f(x) + \epsilon. \tag{5}$$

    Therefore f is (ϵ, δ)-upper semicontinuous. □

Theorem 1. *The function $f : M \to \mathbb{R}$ if upper semicontinuous if and only if for every $a \in \mathbb{R}$,*

$$U_a = \{x \ : \ f(x) < a\} \tag{6}$$

is an open subset of M.

Proof. Take some $x \in U_a$. Then upper semicontinuity implies that for every $\epsilon > 0$ there is a δ so that

$$0 < d(x, y) < \delta \implies f(y) < f(x) + \epsilon. \tag{7}$$

We know that $f(x) < a$, so take $\epsilon = f(x) - a$. Then for every y with $d(x, y) < \delta$,

$$f(y) < f(x) + a - f(x) = a, \tag{8}$$

and $y \in U_a$. Therefore for all $u \in U_a$ there exists a δ so that $d(u, v) < \delta \implies v \in U_a$, and U_a is open.

In the opposite direction suppose that U_a is open. Then, for every $x \in U_a$ there exists a δ so that $d(y, x) < \delta \implies$ $\square$