MATH 105: Homework 1

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5 Multivariable Calculus

3. Prove the following.

   Theorem 1. *Let $T : V \to W$ be a linear transformation between normed spaces. Then,*

   $$\begin{aligned} \|T\| &= \sup\{|Tv| \ : \ |v| < 1\} \\ &= \sup\{|Tv| \ : \ |v| \leq 1\} \\ &= \sup\{|Tv| \ : \ |v| = 1\} \\ &= \inf\{M \ : \ v \in V \implies |Tv| \leq M|v|\} \end{aligned} \tag{1}$$

   Proof. Let the following defenitions stand,

   $$\begin{aligned} A &= \sup\{|Tv| \ : \ |v| < 1\} \\ B &= \sup\{|Tv| \ : \ |v| \leq 1\} \\ C &= \sup\{|Tv| \ : \ |v| = 1\} \\ D &= \inf\{M \ : \ v \in V \implies |Tv| \leq M|v|\} \end{aligned} \tag{2}$$

   Observe that $A \leq B$ and $C \leq B$ since the family considting of the underlying sets is respectiverley ordered by size. By definition we have that,

   $$\|T\| = \sup\left\{|Tv|/|v|\right\},$$

   and nameley $|Tv|/|v| = |T(v/|v|)|$. Therefore $\|T\| \leq C$. If $|v| \leq 1$ then $|Tv| \leq |Tv|/|v|$ and so $B \leq \|T\|$. We yield that $\|T\| = B = C$.

   By the same logic $A \leq \|T\|$ and therefore is equivalent. Lastly $|Tv| \leq \|T\||v|$ and so by the epsilon property $D = A$. □

4.

6. x

12. Prove the following.

Theorem 2. *If V is a normed finite dimensional vector space, then the unit ball, $B = \{v : |v| = 1\}$ is compact.*

Proof. $\dim V = n \in \mathbb{N} \implies V \cong \mathbb{R}^n \implies B \cong S^{n-1} \implies B$ compact. $\square$

13. Prove the following.

Theorem 3. *The set of invertible $n \times n$ matrices is not dense in $\mathcal{M}$.*

Proof. Consider the set of matrix all of whose entries are the same. They create a linear subspace which is a disjoint open subset of $\mathcal{M}$. Therefore the set of invertible matrices could not possibly be dense in $\mathcal{M}$. $\square$