

MATH H104: Homework 1

William Guss
26793499
wguss@berkeley.edu

August 31, 2015

1 Real Numbers

3. Recast the following English sentences in mathematics, using correct mathematical grammar. Preserve their meaning.

- (a) *2 is the smallest prime number.* Let $P \subset \mathbb{N}$ denote the set of prime numbers. Consider that $t = 2$ is clearly a member of P . Then for all $p \in P$, $t \leq p$.
- (b) *The area of any bounded plane region is bisected by some line parallel to x -axis.*

Before we claim the above we must first rigorize the notion of a bounded plane region. The following uses a notion of open and closed sets in \mathbb{R}^2 .

Definition 1. We say that $B_r(x_0)$ is an open ball of radius $r > 0$ if and only if

$$B_r(x_0) = \{x \in \mathbb{R}^2 \mid \|x - x_0\| < r\}.$$

Furthermore $\bar{B}_r(x_0)$ is a closed ball of radius $r > 0$ if and only if

$$\bar{B}_r(x_0) = \{x \in \mathbb{R}^2 \mid \|x - x_0\| \leq r\}.$$

Using the above definition we now give our notion of a bounded plane region.

Definition 2. If A is a subset of \mathbb{R}^2 we will say that A is the area of a bounded plane region if and only if for every $x \in A$, there is an open or closed ball centered at x which is a subset of A .

Lastly, we give the notion of a parallel line to the x -axis

Definition 3. We say that $L_r \subset \mathbb{R}^2$ is a line parallel to the x -axis at radius r if and only if

$$L_r = \{(x, y) \in \mathbb{R}^2 \mid y = r\}.$$

Now it is simple to propose the theorem of symmetric equivalence to the question.

Theorem 1. Let A be the area of a bounded plane region in \mathbb{R}^2 . Then, there exists some line parallel to the x -axis of height r , L_r , such that $L_r \cap A \neq \emptyset$ and both

$$A_L = \{(x, y) \in A \mid y < r\} \text{ and } A_U = \{(x, y) \in A \mid y \geq r\} \quad (1)$$

are areas of bounded plane regions.

- (c) "All that glitters is not gold." Let G be the set of all object which glitter. Then let A be the set of all gold objects. $A \neq G$.

12. Prove the following.

Theorem 2. *There exists no smallest positive real number.*

Proof. Suppose that there exists a smallest real number, say $a \in \mathbb{R}$. Clearly $a > 0$ and so is $\frac{a}{2}$. Furthermore $\frac{a}{2} < a$, and hence we reach a contradiction. Therefore does not exist a smallest positive real number. \square

Theorem 3. *There exist no smallest positive rational number.*

Proof. Suppose that there exists a smallest rational number, say $q \in \mathbb{Q}$. Clearly $q > 0$ and so is $\frac{q}{2}$. Furthermore $\frac{q}{2} < q$, and hence we reach a contradiction. Therefore does not exist a smallest positive rational number. \square

Theorem 4. *Let $x \in \mathbb{R}$. Then there does not exist a smallest real number y such that $y > x$.*

Proof. Suppose that such a y exists. Now consider $\frac{x+y}{2} = b$. Clearly $b > x$, and remarkably $b < y$. Hence y is not the smallest real number such that $y > x$. This leads to a contradiction, and therefore there is no smallest y satisfying the conditions. \square

22. Show the following.

- (a) Fixed points:

Theorem 5. *The function $f : A \rightarrow A$ has a fixed point if and only if the graph of f intersects the diagonal.*

Proof. We first show the right implication. If f has a fixed point, then there is some $a \in A$ such that $f(a) = a$. Now consider the graph of f ,

$$f(A) = \{(a, f(a)) \in A\}.$$

Since f has a fixed point, $f(A)$ contains (a, a) . Hence the intersection of $f(A)$ with the diagonal of $A \times A$, must contain (a, a) at the least and hence is nonempty.

On the otherhand if the graph of f intersects the diagonal, then there exists some $(a, a) \in D$ such that $(a, a) \in f(A)$. Then by definition of the graph of f , $(a, a) = (a, f(a))$, which implies that $f(a) = a$. This completes the proof. \square

- (b) Intermediate fixed point

Theorem 6. *Every continuous function $f : [0, 1] \rightarrow [0, 1]$ has at least one fixed-point.*

Proof. To show this we recall the intermediate value theorem. More specifically we want to find a function whose 0 exists on $[0, 1]$ which implies the theorem. Consider that $f(x) = x$ implies that $0 = f(x) - x$, so let's simply let $g(x) = f(x) - x$. By definition of the bound on the codomain, $g(0) \geq 0$ and $g(1) \leq 0$. Then application of the intermediate value theorem yields that there exists at $c \in [0, 1]$ with $g(c) = 0$. Hence, $f(a) = a$. This completes the proof. \square

- (c) No, consider the case of some function for which $f(x) > x$ on $(0, 1)$. Such a function need not attain the value $f(0) = 0, f(1) = 1$ because such values could not possibly exist on its graph. Hence, $f(x) \neq x$ for all x .
- (d) No, consider the function $f(x) = x + 0.5$ when $0 \leq x < 0.5$, and $f(x) = x - 0.5$ when $0.5 \leq x \leq 1$. This function never is equivalent to $g(x) = x$.

23. Show the following.

- (a) Dyadic squares:

Theorem 7. *If x and y are two dyadic squares, then they are either identical, intersect along a common edge, intersect at a common vertex, or do not intersect at all.*

Proof. Since we must show all cases, let us consider them with respect to the general definition of a planar dyadic cube. In particular, $x, y \in \mathbb{Q}_2^2$. Let us fix x such that

$$x = \left[\frac{p}{2^k}, \frac{p+1}{2^k} \right]^2 \text{ and } y = \left[\frac{q}{2^k}, \frac{q+1}{2^k} \right]^2$$

for some $p, k, q \in \mathbb{Z}$.

If $q = p$, then $y = x$ naturally. In the case that $q > p + 1$ or $q + 1 < p$, we have that $x \cap y = \emptyset$. Next consider intersections along different edges. If

$$y = \left[\frac{p}{2^k}, \frac{p+1}{2^k} \right] \times \left[\frac{p+1}{2^k}, \frac{p+2}{2^k} \right],$$

then $y \cap x = \left[(\frac{p}{2^k}, \frac{p+1}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k}) \right]$. In general,

$$y = \left[\frac{p+r}{2^k}, \frac{p+r+1}{2^k} \right] \times \left[\frac{p+s}{2^k}, \frac{p+s+1}{2^k} \right]$$

implies the following intersections.

If $r = 1, s = 0$, then $x \cap y = \left[(\frac{p+1}{2^k}, \frac{p}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k}) \right]$. If $r = -1, s = 0$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p}{2^k}), (\frac{p}{2^k}, \frac{p+1}{2^k}) \right]$. If $r = 0, s = 1$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p+1}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k}) \right]$. If $r = 0, s = -1$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p}{2^k}), (\frac{p+1}{2^k}, \frac{p}{2^k}) \right]$.

Lastly we need to consider the vertex edge cases. If $r = 1, s = 1$, then $x \cap y = \{(\frac{p+1}{2^k}, \frac{p+1}{2^k})\}$. If $r = -1, s = 1$, then $x \cap y = \{(\frac{p}{2^k}, \frac{p+1}{2^k})\}$. If $r = -1, s = -1$, then $x \cap y = \{(\frac{p}{2^k}, \frac{p}{2^k})\}$. If $r = 1, s = -1$, then $x \cap y = \{(\frac{p+1}{2^k}, \frac{p}{2^k})\}$.

Furthermore if r and s attain other values, we have those cases previously considered. Hence the proof is complete. \square

- (b) For the following problem we adopt the following notation.

Definition 4. We say that some $X \subset \mathbb{R}^n$ is a dyadic hyper-interval of partition $2^{-\gamma}$ if and only if

$$X \in \overline{\Delta_n^k} = \left\{ Y \subset \mathbb{R}^n \mid Y = \bigtimes_{i=1}^k 2^{-\gamma} [(m_1, \dots, m_n), (m_1, \dots, m_i + 1, \dots, m_n)], m_j \in \mathbb{Z} \right\}.$$

So now we need to operationalize this proof. If x and y are two dyadic squares, then they are either identical, intersect along a common edge, intersect at a common vertex, or do not intersect at all.

Theorem 8. In other words, if $X, Y \in \overline{\Delta_n^k}$ are of the same partition, $2^{-\gamma}$, let

$$Y = \bigtimes_{i=1}^k 2^{-\gamma} [(m_1 + r_1, \dots, m_n + r_n), (m_1 + r_1, \dots, m_i + 1 + r_i, \dots, m_n + r_n)],$$

where the m_j are those which define X , and $r_j \in \mathbb{Z}$. Then, if $|r_j| \leq 1$ for all j , the following two results hold. If $k = n - \sum_i |r_i| > 0$, $X \cap Y \in \overline{\Delta_n^k}$. If $k = 0$, $X \cap Y \subset \mathbb{Q}_2^n$. Otherwise if there exists some j such that $|r_j| > 1$ or $k < 0$, then $X \cap Y = \emptyset$.

Proof. We denote X_j, Y_j as the j^{th} interval composing X and Y . In the above definition of Y we wish to explore a multitude of different r_j values so as to express the theorem.

In the simplest case, $|r_j| > 1$ for some j then $y_j = 2^{-k}[m_j + r_j, m_j + r_j + 1]..$ Because $m_j + 1 < m_j + r_j$, then $y_j x_j = \emptyset$ \square

32. Suppose that E is a convex region in the plane bounded by a curve C .

- (a) Show the following

Theorem 9. The curve C has a tangent line except at a countable number of points.

Proof. By definition if E is a convex region, then for any two points $x, y \in E$, all points on the line $L(x, y) = \{z \in E : tx + sy = z, t + s = 1, 0 \leq t, s \leq 1\}$.

Let a, b_t be two points on the curve C . \square