MATH H104: Homework 1

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1 Real Numbers

3. Recast the following English sentences in mathematics, using correct mathematical grammar. Preserve their meaning.

   (a) *2 is the smallest prime number.* Let $P \subset \mathbb{N}$ denote the set of prime numbers. Consider that $t = 2$ is clearly a member of P. Then for all $p \in P$, $t \leqslant P$.

   (b) *The area of any bounded plane region is bisected by some line parallel to x-axis.*

   Before we claim the above we must first rigorize the notion of a bounded plane region. The following uses a notion of open and closed sets in $\mathbb{R}^2$.

   Definition 1. *We say that $B_r(x_0)$ is an open ball of radius $r > 0$ if and only if*

   $$B_r(x_0) = \{x \in \mathbb{R}^2 \mid \|x - x_0\| < r\}.$$

   Furthermore $\bar{B}_r(x_0)$ is a closed ball of radius $r > 0$ if and only if

   $$\bar{B}_r(x_0) = \{x \in \mathbb{R}^2 \mid \|x - x_0\| \leqslant r\}.$$

   Using the above definition we now give our notion of a bounded plane reigon.

   Definition 2. *If A is a subset of $\mathbb{R}^2$ we will say that A is the area of a bounded plane region if and only if for every $x \in A$, there is an open or closed ball centered at x which is a subset of A.*

   Lastly, we give the notion of a parallel line to the x-axis

   Definition 3. *We say that $L_r \subset \mathbb{R}^2$ is a line parallel to the x-axis at radius r if and only if*

   $$L_r = \{(x, y) \in \mathbb{R} \mid y = r\}.$$

   Now it is simple to propose the theorem of symantic equivalence to the question.

   Theorem 1. *Let A be the area of a bounded plane region in $\mathbb{R}^2$. Then, there exists some line parallel to the x-axis of height r, L_r, such that $L_r \cap A \neq \varnothing$ and both*

   $$A_L = \{(x, y) \in A \mid y < r\} \text{ and } A_U = \{(x, y) \in A \mid y \geqslant r\} \tag{1}$$

   are areas of bounded plane regions.

(c) *"All that glitters is mot gold."* Let G be the set of all object which glitter. Then let A be the set of all gold objects. $A \neq G$.

12. *Prove the following.*

Theorem 2. *There exists no smallest positive real number.*

Proof. Suppose that there exists a smallest real number, say $a \in \mathbb{R}$. Clearly $a > 0$ and so is $\frac{a}{2}$. Furthermore $\frac{a}{2} < a$, and hence we reach a contradiciton. Therefore does not exist a smallest postivie real number. □

Theorem 3. *There exist no smallest positive rational number.*

Proof. Suppose that there exists a smallest rational number, say $q \in \mathbb{Q}$. Clearly $q > 0$ and so is $\frac{q}{2}$. Furthermore $\frac{q}{2} < q$, and hence we reach a contradiciton. Therefore does not exist a smallest postivie rational number. □

Theorem 4. *Let $x \in \mathbb{R}$. Then there does not exist a smallest real number y such that $y > x$.*

Proof. Suppose that such a y exists. Now consider $\frac{x+y}{2} = b$. Clearly $b > x$, and remarkably $b < y$. Hence y is not the smallest real number such that $y > x$. This leads to a contradiction, and therefore there is no smallest y satisfying the conditions.

□

22. *Show the following.*

(a) Fixed points:

Theorem 5. *The function $f : A \to A$ has a fixed point if and only if the graph of f interesects the diagonal.*

Proof. We first show the right implication. If f has a fixed point, then there is some $a \in A$ such that $f(a) = a$. Now consider the graph of f,

$$f(A) = \{(a, f(a) \in A\}.$$

Since f has a fixed point, $f(A)$ contains (a, a). Hence the intersection of $f(A)$ with the diagonal of $A \times A$,D, must contain (a, a) at the least and hence is nonempty.

On the otherhand if the graph of f intersects the diagonal, then there exists some $(a, a) \in D$ such that $(a, a) \in f(A)$. Then by definition of the graph of f, $(a, a) = (a, f(a))$, which implies that $f(a) = a$. This completes the proof. □

(b) Intermediate fixed point

Theorem 6. *Every continuous function $f : [0, 1] \to [0, 1]$ has at least one fixed-point.*

Proof. To show this we recall the intermediate value theorem. More specifically we want to find a function whose 0 exists on $[0,1]$ which implies the theorem. Consider that $f(x) = x$ implies that $0 = f(x) - x$, so let's simply let $g(x) = f(x) - x$. By definition of the bound on the codomain, $g(0) \geqslant 0$ and $g(1) \leqslant 0$. Then application of the intermediate value theorem yields that there exists at $c \in [0,1]$ with $g(c) = 0$. Hence, $f(a) = a$. This completes the proof. □

(c) No, consider the case of some function for which $f(x) > x$ on $(0,1)$. Such a function need not attain the value $f(0) = 0, f(1) = 1$ because such values could not possiblt exist on its graph. Hence, $f(x) \neq x$ for all x.

(d) No, consider the function $f(x) = x + 0.5$ when $0 \leqslant x < 0.5$, and $f(x) = x - 0.5$ when $0.5 \leqslant x \leqslant 1$. This function never is equivalent to $g(x) = x$.

23. *Show the following.*

(a) Dyadic squares:

Theorem 7. *If x and y are two dyadic squares, then they are either identical, intersect along a common edge, intersect at a common vertex, or do not intersect at all.*

Proof. Since we must show all cases, let us consider them with respect to the general definition of a planar dyadic cube. In particular, $x, y \in \mathbb{Q}_2^2$. Let us fix x such that

$$x = \left[\frac{p}{2^k}, \frac{p+1}{2^k}\right]^2 \text{ and } y = \left[\frac{q}{2^k}, \frac{q+1}{2^k}\right]^2$$

for some $p, k, q \in \mathbb{Z}$.

If $q = p$, then $y = x$ naturaly. In the case that $q > p+1$ or $q+1 < p$, we have that $x \cap y = \varnothing$. Next consider intersections along different edges. If

$$y = \left[\frac{p}{2^k}, \frac{p+1}{2^k}\right] \times \left[\frac{p+1}{2^k}, \frac{p+2}{2^k}\right],$$

then $y \cap x = [(\frac{p}{2^k} \frac{p+1}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k})]$. In general,

$$y = \left[\frac{p+r}{2^k}, \frac{p+r+1}{2^k}\right] \times \left[\frac{p+s}{2^k}, \frac{p+s+1}{2^k}\right]$$

implies the following intersections.

If $r = 1, s = 0$, then $x \cap y = \left[(\frac{p+1}{2^k}, \frac{p}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k})\right]$. If $r = -1, s = 0$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p}{2^k}), (\frac{p}{2^k}, \frac{p+1}{2^k})\right]$. If $r = 0, s = 1$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p+1}{2^k}), (\frac{p+1}{2^k}, \frac{p+1}{2^k})\right]$. If $r = 0, s = -1$, then $x \cap y = \left[(\frac{p}{2^k}, \frac{p}{2^k}), (\frac{p+1}{2^k}, \frac{p}{2^k})\right]$.

Lastly we need to consider the vertex edge cases. If $r = 1, s = 1$, then $x \cap y = \{(\frac{p+1}{2^k}, \frac{p+1}{2^k})\}$. If $r = -1, s = 1$, then $x \cap y = \{(\frac{p}{2^k}, \frac{p+1}{2^k})\}$. If $r = -1, s = -1$, then $x \cap y = \{(\frac{p}{2^k}, \frac{p}{2^k})\}$. If $r = 1, s = -1$, then $x \cap y = \{(\frac{p+1}{2^k}, \frac{p}{2^k})\}$.

Furthermore if r and s attain other values, we have those cases previously considered. Hence the proof is complete. □

(b) For the following problem we adopt the following notation.

Definition 4. *We say that say that some $X \subset \mathbb{R}^n$ is a dyadic hyper-interval of partition $2^{-\gamma}$ if and only if*

$$X \in \overline{\Delta_n^k} = \left\{ Y \subset \mathbb{R}^n \mid Y = \bigtimes_{i \in \delta_k} 2^{-\gamma} \left[(m_1, \dots, m_n), (m_1, \dots, m_i + 1, \dots, m_n) \right] \right\},$$

where δ_k is the index set of dimensions in which the interval is non-empty and non-singular. Furthermore, $|\delta_k| = k$, and $m_i \in \mathbb{Z}$.

So now we need to operationalize this proof. If x and y are two dyadic squares, then they are either identical, intersect along a common edge, intersect at a common vertex, or do not intersect at all.

Theorem 8. *In other words, if $X, Y \in \overline{\Delta_n^n}$ are of the same partition, $2^{-\gamma}$, let*

$$Y = \bigtimes_{i=1}^{k} 2^{-\gamma} \left[(m_1 + r_1, \dots, m_n + r_n), (m_1 + r_1, \dots, m_i + 1 + r_i, \dots, m_n + r_n) \right],$$

where the m_j are those which define X, and $r_j \in \mathbb{Z}$. Then, if $|r_j| \leqslant 1$ for all j, the following two results hold. If $k = n - \sum_i |r_i| > 0$, $X \cap Y \in \overline{\Delta_n^k}$. If $k = 0$, $X \cap Y \subset \mathbb{Q}_2^n$ with $|X \cap Y| = 1$. Otherwise if there exists some j such that $|r_j| > 1$, then $X \cap Y = \varnothing$.

Proof. We denote X_j, Y_j as the j^{th} interval composing X and Y. In the above definition of Y we wish to explore a multitude of different r_j values so as to express the theorem.

In the simplest case, $|r_j| > 1$ for some j then

$y_j = 2^{-k}[(m_1 + r_1, \dots, m_j + r_j \dots, m_1 + r_1), (m_1 + r_1, \dots m_j + r_j + 1, \dots, m_n + r_n)]$.

Clearly $m_j + 1 < m_j + r$ or $m_j > m_j + r_j + 1$, and thus $y_j \cap x_j = \varnothing$, we have that the whole cartesian product,

$$X \cap Y = \varnothing \times \left(\bigtimes_{i \neq j}^{n} x_j \cap y_j \right) = \varnothing,$$

because $\varnothing \times B$ cannot form any pair (a, b) as there is no $a \in \varnothing$.

We claim that when $|r_i| \leqslant 1$, $X \cap Y \in \overline{\Delta_n^k}$ for $k = n - \sum_{i=1}^n |r_i| > 0$. Let (n_p) denote the finite (possibly empty) list of indices for which $|r_j| = 1$. In other words, for all p, $|r_{n_p}| = 1$, else $|r_j| = 0$. The intersection as aforementioned is the cartesian product of all x_j, y_j. Hence for $j \notin \{n_p\}$, $x_j \cap y_j \in \overline{\Delta_n^1}$ with $\delta_1 = j$. Hence, the cartesian product of all such j is $X^* \cap Y^* \in \overline{\Delta_n^c}$ with $\delta_c = \{j \neq n_p \forall p\}$, and $c = n - |\{n_p\}|$. We claim that $X \cap Y$ cannot exist in any higher dimenisonality than $X^* \cap Y^*$.

Suppose $X \cap Y \in \overline{\Delta_n^d}$, with $n \geqslant d > c$. This implies that there exists a $q \in \{n_p\}$ such that $x_q \cap y_q = z_q$ is non-singular and non-empty. We have that

$$\begin{aligned} z_q &= [(m_1, \dots, m_q, \dots, m_n), (m_1, \dots, m_q + 1, \dots, m_n)] \\ &\quad \cap [(m_1, \dots, m_q \pm 1, \dots, m_n), (m_1, \dots, m_q + 1 \pm 1, \dots, m_n)] \\ &= \left\{ \left(m_1, \dots, m_q + \frac{1 \pm 1}{2}, \dots, m_n \right) \right\} \end{aligned}$$

is singular. Hence we reach a contradiction and $X \cap Y \in \overline{\Delta_n^c}$.

□

24. Show the following

(a) Dyadic squares in the unit ball.

Theorem 9. *Given $\epsilon > 0$, show that the unit disc contains finitely many dyadic squares whose total area exceeds $\pi - \epsilon$, and which intersect with eachother only along their boundries.*

Proof. Let B_c^2 be a disk of radius $\sqrt{\frac{\epsilon}{\pi}} \leqslant c < 1$. Then consider the finite set S_k of all dyadic squares of partition $2^{-\gamma} = \frac{1-c}{2}$ such that $B^2 \supset \bigcup S_k \supset B_c^2$. Clearly the area of $\bigcup S_k > \pi - \epsilon$ but less that π. Hence for any $\epsilon > 0$, take S_k as aforementioned, and these satisfying squares do not intersect. The proof is complete. □

(b) Disjoint dyadic squares.

Theorem 10. *Given $\epsilon > 0$, show that the unit disc contains finitely many dyadic squares whose total area exceeds $\pi - \epsilon$, and which are disjoint.*

Proof. For any $\epsilon > 0$, let $r = \frac{1+\sqrt{\frac{\epsilon}{\pi}}}{2}$. Clearly such a point is the average radfius of the unit ball and the unit ball with radius r. Now as before, divide the inside into pices of side length $2^{-n+1} = 1 - \sqrt{\frac{\epsilon}{\pi}}$. If only every second square in every direction is selected, that set, say S_1, is clearly disjoint. Furthermore the total area of this set is at least

$$a_1 = \frac{\alpha_0}{4} = \frac{\pi r^2}{4}.$$

Now for those dyadics not selected, subdivide those sets into 8 pieces in basis direction, and choose every other dyadic which is disjoint from S_1 and dyadics of the same class. Let S_2 be the set of S_1 union with this new set. The area of S_2 is at least

$$a_2 = a_1 + \frac{\alpha_0 - a_1}{4}.$$

Upon repeating this process we yield the following recurrence relation,

$$a_n = a_{n-1} + \frac{\alpha_0 - a_{n-1}}{4}.$$

Hence, we apply the methods of non-homogeneous recurrence relations and find that the general solution is clearly $a_n = c_1 \left(\frac{3}{4}\right)^n$. Then we solve for the particular solution, and yield that $a_n^p = \pi r^2$. So we simply solve $a_1 = \frac{\alpha_0}{4} = c_1 \frac{3}{4} + \alpha_0$ for c_1. Upon yielding $c_1 = -\alpha_0$, we find the total solution to the area upon n repitions of the process is

$$a_n = -\alpha_0 \left(\frac{3}{4}\right)^n + \alpha_0.$$

Now we show that there exists an N such that for some $n > N$, $a_n > \pi - \epsilon$. Observe,

$$\begin{aligned} \alpha_0 - \alpha_0 \left(\frac{3}{4}\right)^n &< \pi - \epsilon \\ \left(\frac{3}{4}\right)^n &> \frac{\epsilon - \pi}{\alpha_0} + 1 \\ n \ln\left(\frac{3}{4}\right) &> \ln\left(\frac{\epsilon - \pi}{\alpha_0} + 1\right). \end{aligned} \tag{2}$$

Hence, let $N(\epsilon) = \frac{\ln\left(\frac{\epsilon-\pi}{\alpha_0}+1\right)}{\ln\left(\frac{3}{4}\right)}$. By the logic of derivation for N, for every $\epsilon > 0$ and for all $n > N(\epsilon)$, $a_n > \pi - \epsilon$.

Take the first such n. Then the set of disjoint dyadics, S_n, which induce the area a_n is finite, and the proof is complete. □

(c) Dyadic hypercubes filling a ball.

Theorem 11. *Given $\epsilon > 0$, show that the unit ball contains finitely many dyadic hypercubes whose total hypervolume exceeds $V_m(1) - \epsilon = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2}+1)} - \epsilon$, and which intersect with eachother only along their boundries.*

Proof. Let B_c^m be a ball of hypervolume $\epsilon < v < V_m(1)$, and therefore radius $\frac{\left(\epsilon\Gamma(\frac{m}{2}+1)\right)^{1/m}}{\sqrt{\pi}} \leqslant c < 1$. Then consider the finite set $S_k \subset \overline{\Delta_m^m}$ of all dyadic hypercubes of partition $2^{-\gamma} = \frac{1-c}{2}$ such that $B^m \supset \bigcup S_\gamma \supset B_c^m$. These cubes will fill the ball of radius $\frac{\left(\epsilon\Gamma(\frac{m}{2}+1)\right)^{1/m}}{\sqrt{\pi}}$ at least. Clearly the hypervolume of $\bigcup S_k > V_m(1) - \epsilon$ but less than $V_m(1)$. Hence for any $\epsilon > 0$, take S_γ as aforementioned, and these satisfying hypercubes do not intersect except along common edges (as proved in 23. The proof is complete. □

(d) *Proof.* Given $\epsilon > 0$. Let $B_{-\gamma/2}^2$ denote the disk inscribed in the dyadic square $\delta \in \overline{\Delta_2^2}$ of partition $2^{-\gamma}$ at some position in $\mathbb{Q}_2^2$. Now consider the unit square and the square at the origin of area ϵ_2 and sidelength $\sqrt{\epsilon} + c$. Define γ to be the rounded solution of $2^{-\gamma} = \frac{1-\sqrt{\epsilon}+c}{2}$. Then let S_1 be the family of every other dyadic square of partition $2^{-\gamma}$ filling the square of area ϵ_2 completeley and then some. The area of such squares is at least $a_1 = \frac{\epsilon_2}{4}$. Then the area of union of the family of ball inscribing all dyadic squares in S_1 is $b_1 = \frac{\epsilon_2}{8}$.

For those squares not selected subdivide them into 16 dyadic squares and choose every other such that these squares are disjoint from one another and their family is disjoint from S_1. Take the union of their family and S_1 to produce S_2 whose area is at least $a_2 = a_1 + \frac{\epsilon_2 - a_1}{8}$. Taking those circles inscribed yields that $b_2 = b_1 + \frac{\epsilon_2 - a_1}{32}$.

Repeating this process yields a geometric series b_n similar to a_n in part (b). By the same logic in part (b), there will exist an n such that $b_n > \epsilon$ and hence a finite disjoint dyadic partitioning of the unit square such that the area of disk inscription of this partitioning has area greater than ϵ which approaches ϵ_2. This completes the proof. □

32. *Suppose that E is a convex region in the plane bounded by a curve C.*

(a) *Show the following*

Theorem 12. *The curve C has a unique tangent line except at a countable number of points.*

Proof. We first show that their exists a tangent line for every point $c \in C$. Let

$$T_c = \left\{ x \in \mathbb{R}^2 \mid x = c + rt, t \in \mathbb{R} \right\},$$

for some slope vector r such that $T_c \cap (E \backslash C) = \varnothing$. We show that $\forall c, T_c \cap E \neq \varnothing$. Take some $c \in C$ and fix it. Then for some sequence of points on the curve, q_n, which start at some other point c' and increase monotonically with respect to angle from the center of E such that $q_n \to c$. Let the secant line to c at some point q be denoted,

$$S_q = \left\{ x \in \mathbb{R}^2 \mid q + \frac{(c-q)}{\|c-q\|} t, t \in \mathbb{R} \right\}.$$

Consider that $[q_n, c] \subset S_{q_n}$, and $S_{q_n} \backslash [q_n, c] \cap E = \varnothing$. For all n, $[q_n, c]$ is clearly non-empty (it contains at least, c), so $\bigcap_n [q_n, c]$ is also non-empty. Therefore, as $q_n \to c$, $S_{q_n} \to S_c \supset \bigcap_n S_{q_n} = c$. S_c could not possibly contain an element of $E \backslash C$. Suppose it contains, $e \in E \backslash C$, for the purpose of reaching a contradiction. Then $e \in [c, c]$ such that $e \neq c$, which leads to a contradiction. Therefore, $S_c = T_c$ for some tangent line satisfying the definition.

Now we show that T_c is unique except at countably many points. Let us define the function $\tau : C \to [0, 2\pi]$ which assigns to every point on the curve C the angle of its tangent line. By the logic above, for every p $\tau(p)$ exists. Let $\phi : \mathbb{R} \to C$ be a bijective paramaterization of C starting at some point q such that one walks counter clockwise with respect to q a distance t and yields $\phi(t)$. We show that $\tau \circ \phi$ is monotonic. $\square$