MATH 105: Homework 8

William Guss
26793499
wguss@berkeley.edu

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29. Upper semicontinuity.

(a) A graph of an upper semicontinuous graph here:

(b) Show the following.

Definition 1. *We say that a function $f : M \to \mathbb{R}$ is (ϵ, δ)-upper semicontinuous if and only if for every $\epsilon > 0$ there is a $\delta > 0$ so that*

$$0 < d(x, y) < \delta \implies f(y) < f(x) + \epsilon \tag{1}$$

Lemma 1. *Upper semicontinuity is equivalent to the (ϵ, δ)-upper semicontinuity.*

Proof. Observe the following fact about $\limsup$.

$$\limsup_{y \to x} g(y) = \alpha = \lim_{\epsilon \to 0} \sup\{g(y) \ : \ y \in M \cap M_\epsilon(x) \setminus \{x\}\}. \tag{2}$$

Therefore f is upper semicontinuous if and only if

$$\limsup_{y \to x} f(y) \leq f(x) \iff \lim_{\epsilon \to 0} \sup\{f(y) : y \in M \cap M_\epsilon(x) \setminus \{x\}\} \leq f(x). \tag{3}$$

We then know for every $\epsilon > 0$ there exists a δ so that

$$\sup\{f(y) \ : \ y \in M \cap M_\delta(x) \setminus \{x\}\} < f(x) + \epsilon. \tag{4}$$

This is true if and only if

$$d(y, x) < \delta \implies f(y) < f(x) + \epsilon. \tag{5}$$

Therefore f is (ϵ, δ)-upper semicontinuous. $\square$

Theorem 1. *The function $f : M \to \mathbb{R}$ if upper semicontinuous if and only if for every $a \in \mathbb{R}$,*

$$U_a = \{x \ : \ f(x) < a\} \tag{6}$$

is an open subset of M.

Proof. Take some $x \in U_a$. Then upper semicontinuity implies that for every $\epsilon > 0$ there is a δ so that

$$0 < d(x, y) < \delta \implies f(y) < f(x) + \epsilon. \tag{7}$$

We know that $f(x) < a$, so take $\epsilon = f(x) - a$. Then for every y with $d(x, y) < \delta$,

$$f(y) < f(x) + a - f(x) = a, \tag{8}$$

and $y \in U_a$. Therefore for all $u \in U_a$ there exists a δ so that $d(u, v) < \delta \implies v \in U_a$, and U_a is open.

In the opposite direction suppose that U_a is open. Then, for every $x \in U_a$ there exists a δ so that $d(y, x) < \delta \implies y \in U_a$. Therefore $f(y) < a$. Since we can do this for any arbitrary a, take any $\gamma \in M$, then consider $U_{f(\gamma)}$. It follows for every $\epsilon > 0$ there is a δ so that

$$0 < d(y, \gamma) < \delta, y \in U_{f(\gamma)} \implies f(y) < f(\gamma) + \epsilon \tag{9}$$

What can be said about $y \notin U_{f(\gamma)}$. Take the arg max of those y subject to $f(y) \leq f(\gamma) + \epsilon, y \neq \gamma$ (this is possible since $U^C_{f(\gamma)}$ is closed and there is an $a > \gamma$ so that every $x \in U_a \supset U_{f(\gamma)}$ is a point of upper semicontinuity) and we get y' Then take a new

$$\delta' = \min\{\delta, d(y', \gamma)\} \tag{10}$$

and get f upper semicontinuous. $\square$

(c) Negative semicontinuity.

Definition 2. *We say that a function $f : M \to \mathbb{R}$ is negative semicontinuous if and only if $-f$ is upper semicontinuous.*

Theorem 2. *A function is negative semicontinuous if and only if*

$$\lim_{y \to x} f(y) \geq f(x). \tag{11}$$

Proof. Suppose that $-f$ is upper semicontinuous, then

$$\limsup_{y \to x} -f(y) \leq -f(x) \iff -\liminf_{y \to x} f(y) \leq -f(x), \tag{12}$$

by the definition of $\liminf$. Then we negate the inequality and get

$$\liminf_{y \to x} f(y) \geq f(x). \tag{13}$$

This completes the proof. $\square$

30. Show the following.

Theorem 3. *Given K compact in the upper half plane. Then we take $g(x) = \max\{y : (x, y) \in K\}$ when $K \cap x \times \mathbb{R} \neq \emptyset$. Then g is upper semicontinuous.*

Proof. We would like to show that $\limsup g(x_n) \leq g(x)$ for every x. Consider x so that $x, g(x) \in K$. Then take a sequence which converges to x and take the subseqyenxe for whixh $x_n, g(x_n)$ are in K.

Suppose that $\limsup g(x_n) > g(x)$. In which case $g(x_n)$ has a convergent subsequence. Suppose that $g(x_{n_k}) \to a > g(x)$. Then $x_{n_k}, g(x_{n_k}) \to x, \alpha$ not in K which contradicts K closed since $x_{n_k}, g(x_{n_k}) \in K$. Therefore g is upper semicontinuous along K. Outside, it is $f(x) = 0$ which is upper semicontinuous. □

31. This problem has been made optional.

33. Show some interesting examples breaking things.

    (a) Consider the following counterexample (lol). The steeple function defined as

$$s_m(x) = \begin{cases} 2m(1 - m(1/2 - x)) \text{ if } x \in (1/2 - 1/m, 1/2], \\ 2m(1 + m(1/2 - x)) \text{ if } x \in (1/2, 1/2 + 1/m) \\ 0 \text{ otherwise.} \end{cases} \tag{14}$$

    Clearly this sequence of functions has limit 0 almost everywhere, but the area of the undergraph is 1 for all m. So, the conclusion of the dominated convergence theorem is not true ion this context.

    (b) Consider the sequence of functions $f_m(x)$ so that if m is odd, $f_m(x) = s_8(x-0.25)$ and if m is even, $f_m(x) = s_8(x + 0.25)$. Clearly $\liminf f_m = 0$ but the lim inf of the integrals is always 1. Therefore

$$\int \liminf f_m < \liminf \int f_m. \tag{15}$$

34. Prove the following

Theorem 4. *Suppose that $f_n : \mathbb{R} \to [0, \infty)$ is a sequence of integrable functions, $f_n \downarrow f$ a.e. as $n \to \infty$ and $\int f_n \downarrow 0$, then $f = 0$ almost everywhere.*

Proof. Because $f_n \downarrow f$ and $\int f_n \downarrow f$, measure continuity implies $m_2(U(f)) = 0$. By the slice theorem almost every slice of a zeroset implies that slice measure zero must be zero. Since the undergraph of a function is not disconnected with respect to its slices, the only connected set in $\mathbb{R}$ with measure 0 is a point. Therefore, the completed undergraph must be a point, must be 0 almost everywhere. □

35. Consider the sequence of intervals,

$$R_{m,n} = [m/n, m + 1/n] \tag{16}$$

. Then let f_k be a sequence of indicator functions defined so that

$$f_1 = \chi_{R_{0,1}}, f_2 = \chi_{R_{0,2}}, f_3 = \chi_{R_{1,2}}, \ldots \tag{17}$$

It is clear that this sequence does not converge to 0 pointwise since at every irrational point and for every n there is an N more than n so that a smaller compact support R_n covers the point.

However, the undergraph of the sequence is always decreasing and has measure proportional to $1/\sqrt{n}$ which tends towards 0. This completes the counter example.

To visualzie this example, imagine a scanner of compact supports moving across the real line smoothly but shrinking as $n \to \infty$, never stopping.

36. Show the converse to the dominated convergence theorem fails.

Theorem 5. *There is a sequence of functions $f_k : [0, 2] \to [0, \infty)$ such that $f_k \to 0$ almost everywhere $\int f_k \to 0$ but there is no dominator g.*

Proof. Consider the following sequence of sets, $R_k = [1/k, 1/k + 1/k^2] \times [0, k]$. Then let $f_k = \chi_{R_k}$. The dominator must have an undergraph at least as large as the union of all $U(f_k)$. Since the undergraph of each f_k has volume $1/k$, the total volume of the union by measure additivity is $\sum 1/k = \infty$ which implies that $\int g = \infty$. Therefore there cannot exist a dominating dude. □

37. Show the absolute value dominated convergence theorem kind of.

Theorem 6. *Suppose $f_k \to f$ and f_k takes on both positive and negative values. If there exists and integrable function g such that for almost every x we have $|f_k(x)| \leq g(x)$, then $\int f_k \to \int f$.*

Proof. We can write $f_k = f_{+,k} - f_{-,k}$ so that $f_{+,k} = \max\{0, f\}$, $f_{-,k} = \min\{0, f\}$. For f we can write $f_+ = \max\{0, f\}$, $f_- = \min\{0, f\}$.

It is obvious that $f_k \to f$ implies $f_{k,+} \to f_+$ and $f_{k,-} \to f_-$. Lastly, $\int f = \int f_+ + \int -f_-$. Furthermore $\int f_k = \int f_{k,+} + \int -f_{k,-}$. By the dominated convergence theorem, $\int f_{k,+} \to \int f_+$ and $\int -f_{k,-} \to \int -f_-$. Therefore $\int f_k \to \int f$. □

38. Min max integrability.

Theorem 7. *If f, g are integrable, then $\max\{f, g\}$ and $\min\{f, g\}$ are integrable.*

Proof. We start with minimum and illustrate a point which can be generalized to the maximum case. Observe that

$$\hat{U}(f) \cap \hat{U}(g) = \{(x, y) : y \leq f(x) \wedge y \leq g(x) \iff y \leq \min\{g(x), f(x)\}\}. \tag{18}$$

Therefore $\hat{U}(f) \cap \hat{U}(g) = \hat{U}(\min\{f, g\})$. And the intersections of closeds is closed. Therefore $U(\min\{f, g\})$ measurable and $\min\{f, g\}$ integrable.

Applying the same methodology to the max function except using the undergraph and not the completed the completed undergraph, we get that $\max\{f, g\}$ is integrable (taking unions not intersections). □