

MATH H104: Homework 4

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2 A Taste of Topology

29. *Show the following.*

Theorem 1. *Let \mathcal{T} be the collection of open subsets of a metric space M , and \mathcal{K} be the collection of closed subsets. Show that there is a bijection from \mathcal{T} onto \mathcal{K} .*

Proof. We wish to find a function $f : \mathcal{T} \rightarrow \mathcal{K}$ bijective. To do so observe the following fact about compliments in M : $A^c = B$ is the unique compliment of A . Suppose that there were another compliment such that $A^c = C \neq B$ which was the compliment of A . By definition $C = \{x \in M | x \notin A\} = B$, so there cannot exist another set which is also the compliment of A .

As follows from above, the compliment of an open set is closed and the compliment of a closed set is open. Therefore, let $f : A \mapsto A^c$. Then, f is in an injection by the uniqueness of compliments. Furthermore, if $S \in \mathcal{K}$, there exists a set, Q , in \mathcal{T} such that $f(Q) = S$, nameley S^c . This follows by $S^c \in \mathcal{T}$ and $f(S^c) = S^{cc} = S$. Hence f is a bijection.

This completes the proof, and $\mathcal{K} \sim \mathcal{T}$. □

32. *Prove the following and then remark.*

Theorem 2. *Every subset of \mathbb{N} is clopen.*

Proof. To show that every subset of \mathbb{N} is clopen the definitions of openness and closedness must hold on every set. Take an arbitrary subset S of the natural numbers. If S is empty or the whole space \mathbb{N} then it is clopen.

Otherwise, for every $q \in S$ there exists an $r > 0$, say 0.5, such that $d(q, p) \implies p \in S$. To see this, consider that the only such p for which the definition of openness holds is q itself. Therefore, S is open.

The subset S must also be closed because S^c is an open subset of the naturals, and $S^{cc} = S$ must be closed by compliments. Hence S is clopen and the proof is complete. □

Remark. Any function mapping the natural numbers to some metric space M must be continuous. Consider some $Q \subset f(\mathbb{N})$. If Q is open then, $f^{pre}(Q)$ is open. Conversely, if Q is closed then, $f^{pre}(Q)$ is closed. Furthermore if M is any discrete space (or one with a discrete metric) then f is an open mapping.

33. Find a metric space for which the boundary of the r neighborhood need not always be the r -sphere.

Example. Let $M = \mathbb{N}$ be a metric space with its inherited metric from \mathbb{R} . We show that it is not true that for each $M_r(p)$, the boundary is the r -sphere. Consider that the closure of $M_r(p)$ is $M_r(p)$ as every set in M is clopen. Then the closure of the complement is just complement. By definition $\partial M_r(p) = \overline{M_r(p)} \cap \overline{M_r^c(p)} = \emptyset$. However for all $r \in \mathbb{N}$, $S_r(p) = \{x \in M \mid d(x, p) = r\} \neq \emptyset$. So there are cases in which the boundary is not the r -sphere.

Suppose that x were in the boundary of some $M_r(p)$ and not in the unit sphere. Then $d(x, p) \not\leq r \implies d(x, p) > r$. By virtue of x being in the boundary, x must be in every closed subset containing $M_r(p)$. However, $x \notin S_r(p)$ (the r -sphere at p) and $S_r(p) \supset M_r(p)$ is closed; a contradiction! So, the boundary must be contained within the r -sphere at p .

40. Prove the following.

Theorem 3. If M be a metric space with metric d , then the following are equivalent:

- (a) M is homeomorphic to M equipped with the discrete metric.
- (b) Every function $f : M \rightarrow M$ is continuous.
- (c) Every bijection $g : M \rightarrow M$ is a homeomorphism.
- (d) M has no cluster points.
- (e) Every subset of M is clopen.
- (f) Every compact subset of M is finite.

Proof. (a) \implies (e). Since $(M, d) \cong (M, d_{\text{discrete}})$, then for some function $f : M \rightarrow M$ where the domain has the discrete metric, every subset of the domain is clopen, and thereby every image of a subset of the domain is clopen by the homeomorphism.

(b) \implies (c). If every function in M^M is continuous, then consider an arbitrary bijection $g : M \rightarrow M$. Clearly g is continuous, and its inverse map $g^{-1} : M \rightarrow M$ is also continuous.

TODO: PROVE

□

42. What is wrong with the proof of Theorem 28?

The misstep in the proof is the statement that there exist subsequences $(a_{n_k}), (b_{n_k})$ which converge. Compactness surely implies that there exists an index sequence n_k such that $a_{n_k} \rightarrow a \in A$ but that exact index set may not be one which allows $b_{n_k} \rightarrow b$.

To solve this problem consider the following argument. Since any subsequence of (a_{n_k}) converges to a by the convergence of (a_{n_k}) , and B compact, we can take a subsequence, $(b_{n_{k(l)}})$ which converges to b . So the sequence $((a_{n_{k(l)}}), (b_{n_{k(l)}})) \rightarrow (a, b)$.

43. Prove the following.

Theorem 4. *If the cartesian product of two non-empty sets $A \subset M, B \subset N$ is compact in $M \times N$, A and B are compact.*

Proof. By the compactness of $C = A \times B$, all sequences (a_n, b_n) have subsequences which converge to some $(a, b) \in C$. Take one such particular sequence. Since $a_n \in A$ and $a \in A$. Then the subsequential convergence of the product sequence implies the subsequential convergence of a_n . The same argument holds for b_n . In general, C contains the product of all sequences in A and B . So for any sequence in A , there exists some sequence in the product whose subsequence converges thereby implying the convergence of some subsequence of the original sequence in A . Again, the same argument holds for any given sequence in B .

This completes the proof. \square

48. *Prove the following.*

Theorem 5. *There exists an embedding of the line as a closed subset of the plane, and there is an embedding of the line as a bounded subset of the plane, but there is no embedding of the line as a closed and bounded subset of the plane.*

Proof. By the line, we assume that \mathbb{R} is meant. Consider the following function $f : \mathbb{R} \rightarrow L_u \subset \mathbb{R}^2$ such that $x \mapsto (x, 0) \in \mathbb{R}^2$. When $L_u = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ is the codomain, f is clearly surjective and injective. Hence we have that f is bijective. Furthermore, take some open set in L_u , say S . Then $f^{-1}(S) = \{x \in \mathbb{R} | (x, 0) \in S\}$. If for every $s \in S$ there exists an $r > 0$, such that $d(s, q) < r \implies q \in S$, we have that $d((s_x, 0), (q_x, 0)) < r$. Since $f^{-1}s = s_x$ and $f^{-1}q = q_x$ then $d(s_x, q_x) < r$ and thereby $q_x \in \mathbb{R}$. So it must follow that for every s_x in \mathbb{R} there exists an $r > 0$ such that $d(s_x, q_x) < r \implies q_x \in \mathbb{R}$. It suffices to say that f is a homeomorphism when the converse argument is applied.

Knowing that f embeds \mathbb{R} onto \mathbb{R}^2 , we show that such an embedding is a closed subset. L_u is closed if and only if it contains all of its limit points. Suppose (x_n) is a sequence in L_u such that $x_n \rightarrow x$. We wish to show that $x \in L_u$. By the convergence of x_n for every $\epsilon > 0$, there exists an N , such that for all $n \geq N$, $d(x_n, x) < \epsilon$. if x is not in L_u , then $x = (a, b)$ with $b \neq 0$. So if $d(x_n, x) < \epsilon$ then take $\epsilon = b - 0.1$. In this case, $d(x_n, x) < \epsilon \implies x_n \notin L_u$ which is a contradiction. Ergo, L_u is a closed embedding of the line in the plane.

In a different case, it is clear that $\mathbb{R} \cong (0, 1)$. It suffices to show that $(0, 1)$ has an embedding in \mathbb{R}^2 which is bounded. Simple! Take $f : (0, 1) \rightarrow \mathbb{R}^2$ such that $x \mapsto (x, 0)$. The function f embeds $(0, 1)$ by the same argument supplied for the first case. Furthermore, $f((0, 1))$ is bounded because the set $[0, 1] \times [0, 1]$ contains the embedding (x is always between 1 and 0 and the y component is always 0.)

In the last case, suppose there existed a closed and bounded subset of the plane such that \mathbb{R} was embedded to that set by some homeomorphism h . Then, by some theorem that embedding is compact as a subset of \mathbb{R}^2 and by topological equivalence, \mathbb{R} must also be compact; a contradiction! Therefore, only the first two cases hold. \square

54. *If $f : A \rightarrow B$ and $g : C \rightarrow B$ such that $A \subset C$ and for each $a \in A$ we have that $f(a) = g(a)$ then g **extends** f . We also say that g **extends to** g . Assume that*

$f : S \rightarrow \mathbb{R}$ is a uniformly continuous function defined on a subset S of a metric space M . Prove the following:

(a) *Extension to closure.*

Theorem 6. *The function f extends to a uniformly continuous function $\bar{f} : \bar{S} \rightarrow \mathbb{R}$.*

Proof. If f is uniformly continuous, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in S$, $d(p, q) < \delta \implies d(fp, dq) < \epsilon$. Since f is continuous it preserves convergence of sequences. So adding the closure of S to S through union lets all sequences in this new set \bar{S} converge to elements in \bar{S} . Adding these elements we construct a function based on the convergence of limits. $g : \bar{S} \rightarrow \mathbb{R}$ such that if $x \in S$, then $x \mapsto fx$ and otherwise if $x \notin S$ and $x \in \bar{S}$ we know the following. The element x is a limit of a sequence in S , say x_n . Then for every $r > 0$ there exists an N such that for all $n > N$, $d(x_n, x) < r$. Using the function, $f(x_n) \rightarrow y \in \mathbb{R}$. Let $g(x) = y$. Then for all $\epsilon > 0$, there exists such an N that $n > N$ implies $d(gx_n, gx) < \epsilon$. In this case let $\delta = r = \epsilon$ from before. Then the limit is preserved and g is uniformly continuous at x . Hence f extends to a uniformly continuous function $\bar{f} = g$. \square

(b) *Uniqueness*

Theorem 7. *The function \bar{f} is the unique extension of f .*

Proof. Suppose that there exists another extension of \bar{f} to the closure of S , say g . Then for every $a \in S$, $f(a) = \bar{f}(a) = g(a)$, by extension, and if $x \in \bar{S}$ then $\bar{f}(x) \neq g(x)$. Consider a sequence which converges to x as a subset of S . Then for all $\epsilon > 0$ there exists an N_1 such that for all $n > N_1$,

$$d(\bar{f}x_n, fx) < \epsilon/2.$$

Since g is also continuous we have that for some N_2 and all $n > N_2$

$$d(gx_n, gx) < \epsilon/2.$$

Remember that our assumption implies that $\bar{f}(x) \neq g(x)$. Take $N = \max N_1, N_2$ then for all $n > N$ we have that

$$d(\bar{f}x, gx) \leq d(\bar{f}x, \bar{f}x_n) + d(\bar{f}x_n, gx_n) + d(gx_n, gx) < \epsilon/2 + 0 + \epsilon/2,$$

by extension of f . So it is clear, $\bar{f}(x) = g(x)$; a contradiction!

Therefore \bar{f} is unique and the proof is complete. \square

(c) *General complete metric space.*

Theorem 8. *Let $f : S \rightarrow N$ be a uniformly continuous function defined on $S \subset M$ with M a metric space. Then $\bar{f} : \bar{S} \rightarrow N$ is a unique uniformly continuous extension of f .*

Proof. TODO: PROVE The uniqueness of \bar{f} is proved by (b) without loss of generality. Hence take the previous theorem with N as the domain. \square