Math 215B — Smooth Manifolds — William Guss
Lecture Notes

Let's start by expressing some familair objects as topological manifolds.

Example 1. The n-dimensional sphere $\boldsymbol{S}^n$ is a topological manifold.

Proof. First off let's deal with the topological assumptions. The space $\boldsymbol{S}^n$ can be embedded as a subset of $\mathbb{R}^n$ as a bounded compact, so clearly it is paracompact, inheritng the subspace topology we also get that $\boldsymbol{S}^n$ is second countable and Hausdorff.

We now need to equip the sphere with differentiable structure, namely we need to give it a series of atlas's and charts. In the spirit of our paper metaphore, we will construct the sphere by overlapping hemispheres in each coordinate direction.

More formally let U_i^+ and U_i^- be the positive and negative hemisphere of $\boldsymbol{S}^n$ in the i-th coordinate; that is,

$$U_i^+ = \{(x_1, \cdots, x_{n+1}) \in \boldsymbol{S}^n \;\big|\; x_i > 0\}$$
$$U_i^- = \{(x_1, \cdots, x_{n+1}) \in \boldsymbol{S}^n \;\big|\; x_i < 0\}$$

Now we'll consider the projection of $U_i^\pm$ onto $\mathbb{R}^n$ without the i-th coordinate. In particular, let $\phi_i^\pm : U_i^\pm \to \mathbb{R}^n$ such that

$$\phi(x_1, \cdots, x_i, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1n}).$$

To show that these maps are charts for our manifold $\mathbb{S}^n$, they must be homeomorphisms. Clearly they give a bijection as in each x_i we can identify the x_i coordinate directly using the cooridinates; that is, we can describe the inverse of ϕ as follows

$$(\phi^\pm)^{-1}(u_1, \cdots, u_n) = \left(u_1, \cdots, u_{i-1}, \pm\sqrt{1 - \|u\|^2}, u_i, \cdots, u_n\right).$$

Both $\phi_i^\pm$ and its inverse are compositions of elementary continuous functions on valid domains, and so $\phi_i^\pm$ is continuous. Therefore $\phi_i^\pm$ is a homeomorphism.

Now for the fun part! Let's check that the change of coordinates between two overlappings charts is actually C^∞. We namely need to compose $\pi_i^\pm$ with $(\phi_j^\mp)^{-1}$ for all i, j such that $U_i^\pm \cap U_j^\mp$. This overlap happens when $i \neq j$ or $\mp = \pm$ and $i = j$.

Calculation yields that the change of coordinates is

$$\begin{aligned}\phi_i^\pm \circ \left(\phi_j^\mp\right)^{-1} &= \phi_i^\pm \circ \left(u_1, \cdots, u_{i-1}, \pm\sqrt{1 - \|u\|^2}, u_i, \cdots, u_n\right), \\ &= \left(u_1, \cdots, u_{i-1}, u_{i+1}, \cdots u_{j-1}, \pm\sqrt{1 - \|u\|^2}, u_j, \cdots, u_n\right)\end{aligned}$$

The above change is clearly smooth recalling basic calculus.

Therefore we have shown that $\boldsymbol{S}^n$ satisifies the technical conditions of a topological manifold and is equipped with a differentiable structure. $\square$

As aforementioned we can also construct interesting manifolds by using this definition and so one might naturally want to consider the product of manifolds. A motivating example is the torus $\mathbb{T} = \boldsymbol{S}^1 \times \boldsymbol{S}^1$

Proposition 1. *If $(\mathcal{X}, A_\mathcal{X}), (\mathcal{Y}, A_\mathcal{Y})$ are smooth manifolds then $(\mathcal{X} \times \mathcal{Y}, A_\mathcal{X} \otimes A_\mathcal{Y})$ is a smooth manifold.*

Proof. First let us consider the product topology of $\mathcal{X} \times \mathcal{Y}$. We know that the product of Hausdorff is hausdorff, the product of second countable is obviously second countable[1], and finally the product of paracompact is paracompact.

Next we'll verify that the product atlas covers $\mathcal{X} \times \mathcal{Y}$ and maintains differentiable structure. First let us consider set of functions

$$A_{\mathcal{X}} \times A_{\mathcal{Y}} = \{(x, y) \mapsto (f(x), g(y)) \mid f \in A_{\mathcal{X}}, g \in A_{\mathcal{Y}}\}.$$

For any point $p \in \mathcal{X} \times \mathcal{Y}$ we have that $\pi_1(p) = x \in \mathcal{X}$ and $\pi_2(p) = y \in \mathcal{Y}$. Therefore there are neighborhoods U_x^p, U_y^p containing x in $\mathcal{X}$ and y in $\mathcal{Y}$ respectively such that $A_{\mathcal{X}} \ni f : U_x^p \to \mathbb{R}^n$ and $A_{\mathcal{Y}} \ni g : U_y^p \to \mathbb{R}^m$. As the product of finitely many opens is open in the product topology $U_x^p \times U_y^p := U^p \subset \mathcal{X} \times \mathcal{Y}$ is open and there is a map $H : U^p \to \mathbb{R}^n \times \mathbb{R}^m \in A_{\mathcal{X}} \times A_{\mathcal{Y}}$ with $\pi_1 \circ H = f, \pi_2 \circ H = g$. All this is to say, we can cover the manifold $\mathcal{X} \times \mathcal{Y}$ with maps and neiughborhood merely composed from the original charts of both $\mathcal{X}$ and $\mathcal{Y}$.

Next we'll verify properties of H. From real analysis we know that the cartesian product of two bijections is again a bijection. As for continuity we turn to the procut topology. Take an open set $V \in \mathbb{R}^n \times \mathbb{R}^m$ and then

$$H^{-1}(V) = (f^{-1}(\pi_1(V)), g^{-1}(\pi_2(V))) = W \times Z; \quad W \subset \mathcal{X}, Z \subset \mathcal{Y}.$$

Then the finite product of opens is open so H is continuous. Continuity in the opposite direction is easily verified using the Hausdorff hypothesis on $\mathcal{X} \times \mathcal{Y}$.

Putting everything together we have that $\mathcal{X} \times \mathcal{Y}$ is locally homeomorphic to $\mathbb{R}^n \times \mathbb{R}^m$. Next we need to verify that the change of coordinates between to charts is actually smooth. Suppose we have to charts H_1, H_2 that have overlapping domains. Without loss of generality consider $G = H_2 \circ H_1^{-1}$. This is just

$$G = H_2 \circ f_1^{-1} \times g_1^{-1} = f_2 \circ f_1^{-1} \times g_2 \circ g_1^{-1},$$

which is the product of two C^∞ maps by the smoothness of both $\mathcal{X}$ and $\mathcal{Y}$. Since neither coordinate depends on the other, the Frechet derivatives of G occupy only the diagonals of the tensors produced through differentiation and each minor (subcollection of partial-derivatives) is smooth and continuous. Therefore G is smooth.

This verifies that $\mathcal{X} \times \mathcal{Y}$ is a smooth manifold. □

Example 2. The torus is a smooth manifold given by $\boldsymbol{S}^1 \times \boldsymbol{S}^1 = \mathbb{T}$.

We'll now turn to the space of lines in n-dimnensional vector spaces over a field (or division ring) $\mathbb{K}$. Let $V = \mathbb{K}^n \setminus \{0\}$. If $v, w \in V$ then we say that $v \sim w$ if there is a $\lambda \in \mathbb{K}$ so that $v = \lambda w$; that is v and w are *co-linear*[2].

Example 3 (Projective Space)**.** If V and $\sim$ are given as above, we say that $V/\sim$ is the $\mathbb{K}$-projective space for V denoted by $\mathbb{KP}^n$ and if $\mathbb{K}$ admits a calculus, $\mathbb{KP}^n$ is a smooth manifold.

Before we show in generality how $\mathbb{KP}^n$ is a manifold, we will consider what the space is geometrically. Let us start with the real projective line $\mathbb{RP}^1$. We will identify every point in $\mathbb{R}^2$ with the line from the origin on which it lies. In particular $(x, y) \sim (s, t)$ if there is a $\lambda \in \mathbb{R}$ so that $(x, y) = (\lambda s, \lambda t)$. Some authors often use $[x : y] \in \mathbb{KP}^2$ to denote points as the following exposition realizes that the ration between each component of a vector in V colinear to (x, y) maintains the same ratio. We will utilize these ratios to build an atlas for $\mathbb{KP}^n$.

[1] This is true in the same sense that $\mathbb{N} \sim \mathbb{N}^2 \sim \mathbb{Q}$.

[2] For non commutative fields, we prefer left multiplication by a scalar in producing the equivalence relation.

Now that we've seen an example of what this real projective line looks like, we'll prove in generality that $\mathbb{KP}^n$ is a smooth manifold.

Proof. Let's first start by verifying the technical conditions on the topology of $\mathbb{KP}^n$. $\square$

Examples Lecture

Smooth Manifold Requirements:

- Second countable
- Hausdorff
- Maximal atlas with C^∞ compatible charts.

Definition 4. Let $\boldsymbol{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ be the n-sphere.

Prop. $\boldsymbol{S}^n$ is a smooth manifold.

- Hausdorff + Second Countable: $\boldsymbol{S}^n \subset \mathbb{R}^{n+1}$
- Open cover:
$$U_i^+ = \{(x_1, \cdots, x_{n+1}) \in \boldsymbol{S}^n \;\big|\; x_i > 0\}$$
$$U_i^- = \{\text{"}\;\;\text{"}\;\;\text{"}\;\;\text{"}\;\; x_i < 0\}$$
- C^∞ charts: $\phi_i^\pm : U_i^\pm \to \mathbb{R}^n$ and
$$\phi(x_1, \cdots, x_i, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}).$$
  - $\phi_i^\pm$ bijective considering $\boldsymbol{S}^n$ as n-surface.
  - C^∞ coordinate change:
$$\begin{aligned} \phi_i^\pm \circ \left(\phi_j^\mp\right)^{-1} &= \phi_i^\pm \circ \left(u_1, \cdots, u_{i-1}, \pm\sqrt{1 - \|u\|^2}, u_i, \cdots, u_n\right), \\ &= \left(u_1, \cdots, u_{i-1}, u_{i+1}, \cdots u_{j-1}, \pm\sqrt{1 - \|u\|^2}, u_j, \cdots, u_n\right) \end{aligned}$$

Definition 5. If $\mathbb{K}$ is a field. Let $x, y \in \mathbb{K}^{n+1} \setminus \{0\}$ and $x \sim y$ iff $x = ty$ for $\mathbb{K} \ni t \neq 0$. Then $\mathbb{KP}^n = \mathbb{K}^{n+1}/\sim$ is called the projective space over $\mathbb{K}$.

Consider $\mathbb{RP}^n$. If $x \in \mathbb{RP}^n$ then $x = [x_1 : x_2 : \cdots : x_{n+1}]$ described by the ratio of the components of each line.

Prop. $\mathbb{RP}^n$ is a manifold.

- X second countable $\implies X/\sim$ is s.c.; ($U \subset X/\sim$ open iff $\pi^{-1}(U)$ open.)
- Chart Idea: $[x_1 : x_2] \sim [x_1/x_2 : 1]$ described by n values.
- Let $\phi_i : \mathbb{KR}^n \to \mathbb{R}^n$ s.t. $x \mapsto [x_1/x_i : x_2/x_i : \cdots : x_{i-1}/x_i : x_{i+1}/x_i : \cdots : x_{n+1}/x_i]$.
- Let $\phi_i^{-1} : \mathbb{R}^n \to \mathbb{KR}^n$ s.t. $u \mapsto [u_1 : \cdots : u_{i-1} : 1 : u_i : \cdots : u_n]$
- Check the rest.

Remark.

- $\mathbb{RP}^1 \simeq S^1$
- $\mathbb{CP}^1 \simeq S^2$
- $\mathbb{HP}^1 \simeq S^4$

Prop. If $(\mathcal{X}, A_\mathcal{X}), (\mathcal{Y}, A_\mathcal{Y})$ are smooth manifolds then $(\mathcal{X} \times \mathcal{Y}, A_\mathcal{X} \otimes A_\mathcal{Y})$ is a smooth manifold.

Proof. Consider the following:

- Product of Hausdorff is Hausdorff.
- Product of topological base $G(\mathcal{B}_X \times \mathcal{B}_Y) \sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$ is second countable.
- Atlas $A_{\mathcal{X}} \otimes A_{\mathcal{Y}}$ ks product of C^∞ diffeomorphisms so is a maximal smooth atlas.

$\square$

Remark. $\mathbb{T} = \boldsymbol{S}^1 \times \boldsymbol{S}^1$ is a smooth manifold!
Remark. Quotients aren't as easy: Quotient Manifold Theorem.