B.3 Functions

Given two sets A and B, a function f is a binary relation on A and B such that $(\forall a \in A)(\exists b \in B \ni (a,b) \in f \text{ and } (a,b) \text{ is distinct})$.
Def: Domain: The set of all inputs that f operates.
Def: Codomain: The set of all outputs that f can return.
Note: While the ***range*** of f is the actual output values of f, the codomain is the **set** of all possible outputs.
Notation: We write $f : A \to B$.
Notation: If $(a,b) \in f$, we write $b = f(a)$ as b is uniquely determined by the choice of a.

The function f assigns an element of B to each element of A.
As no element of A is assigned two different elements of B, the same element of B can be assigned to different elements of A.

Given a function $f : A \to B$, if $f(a) = b$, then we says that a is the ***argument*** of f and b is the ***value*** of f at a.
Two functions f and g are ***equal*** if they have the same domain and codomain, if for all a in the domain, $f(a) = g(a)$.

Sequences

A ***finite sequence*** of length n is a function f whose domain is the set of n integers $\{0, 1, \ldots, n-1\}$.
A finite sequence is denoted by listing its values: $\langle f(0), f(1), \ldots, f(n-1) \rangle$.

An infinite sequence is a function whose domain is the set of natural numbers $\mathbb{N}$.
Ex: The Fibonacci sequence defined by recursion, is the infinite sequence $\langle 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \rangle$.

Functions of Cartesian Products

Notation: When a domain of a function f is a Cartesian product, the extra parentheses of the argument are omitted.
Ex: Let $f : A_1 \times \cdots \times A_n \to B$, we write $b = f(a_1, \ldots, a_n)$ and **not** $b = f((a_1, \ldots, a_n))$.
Note: Each argument a_i is called an argument even though the only single argument of f is the n-tuple $(a_1, a_2, \ldots, a_n)$.

Images

If $f : A \to B$ is function and $b = f(a)$, then we can say that b is the ***image*** of a under f.
The image of a set $A' \subseteq A$ under f is defined by $f(A') = \{b \in B : b = f(a) \text{ for some } a \in A'\}$.
The ***range*** of f is the image of its domain, $f(A)$.
Ex: If $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$,
then the range is $f(\mathbb{N}) = \{m : m = 2n \ni \exists n \in \mathbb{N}\}$, the set of nonnegative even integers.

Types of Function

A function is a ***surjection*** if its range is its codomain.

Ex: $f(n) = \lfloor n/2 \rfloor$ is a surjective function $\mathbb{N} \to \mathbb{N}$ as every element in $\mathbb{N}$ appears as the value of f for some argument.

Counter: $f(n) = 2n$ is not surjective from $\mathbb{N} \to \mathbb{N}$ as $(a \in \mathbb{N}) f(a) \neq 3$.

However, $f(n) = 2n$ is surjective from $\mathbb{N} \to \{\text{Evens Naturals}\}$.

Def: Onto: A function $f : A \to A$ that is surjective is also referred to as ***onto***, read as f maps A onto B.

A function $f : A \to B$ is an **injective_** if the distinct arguments to f produce distinct values.

Symbolically, If $a \neq a'$, then $f(a) \neq f(a')$.

Ex: $f(n) = 2n$ is an injective function from $\mathbb{N} \to \mathbb{N}$ as no two arguments result in the same value.

Counter: $f(n) = \lfloor n/2 \rfloor$ as $f(2) = f(3) = 1$.

Def: One-to-One: A function $f : A \to B$ that is injective is also referred to as ***one-to-one***.

A function $f : A \to B$ is a ***bijection*** if it is injective and surjective (one-to-one and onto).

$f = (n) = (-1)^n \lceil n/2 \rceil$ is a bijection from $\mathbb{N} \to \mathbb{Z}$.

The function injective (one-to-one) as no element of $\mathbb{Z}$ is the result of more than one element of $\mathbb{N}$.

The function subjective (onto) as every element of $\mathbb{Z}$ appears as the result of some element $\mathbb{N}$.

A bijective function is sometimes called a ***one-to-one correspondence*** as it pairs elements in the domain with the codomain.
A bijection from a set A to itself is sometimes called a ***permutation***.

When a function f is bijective, the inverse f^{-1} is defined as $f^{-1}(b) = a$ if and only if $f(a) = b$.