Appendix B.2 Relations - Condensed Notes

A ***Binary Relation*** R on two sets A and B is a subset of the Cartesian product $A \times B$, symbolically as $R \subseteq A \times B$.
If $(a, b) \in R$, we sometimes write $a\, R\, b$.
When we say R is a binary relation on a set A, we mean that $R \subseteq A \times A$.
Ex: let R be the "$\leq$" relation on $\mathbb{N}$, so $R = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a < b\}$.

A binary relation $R \subseteq A \times A$, is ***reflexive*** if $(a, a) \in R$.
Ex: The "$=$" and "$\leq$" are reflexive relations on $\mathbb{N}$ but "$<$" is not.
A binary relation R is ***symmetric*** if $(a, b) \in R \implies \exists (b, a) \in R$.
Ex: The "$=$" is symmetric relation, but "$\leq$" and "$<$" are not.
A binary relation R is ***transitive*** if $(a, b), (b, c) \in R \implies (a, c) \in R$.
Ex: The relations "$=$", "$\leq$", and "$<$" are transitive.
The relation $R = \{(a, b) : a, b \in \mathbb{N} \text{ and } a = b - 1\}$ is not transitive as $3\, R\, 4$ and $4\, R\, 5$ do not imply $3\, R\, 5$.

A relation R that is reflexive, symmetric, and transitive is an ***equivalence relation***.
If R is an equivalence relation on a set A, then for $a \in A$, the ***equivalence class*** of a is $[a] = \{b \in A : a\, R\, b\}$, or the set of all elements equivalent to a.

Theorem: An Equivalence Relation is the Same as a Partition

(1) Let the equivalence classes of any equivalence relation R on a set A form a partition of A.
(2) Any partition of A determines an equivalence on A for which the sets in the partition are the equivalence classes.

Proof:

> (1) Equivalence classes of any relation R on a set A form a partition of A
> We must show that the equivalence classes are nonempty, pairwise disjoint sets whose union is A.
> As R is reflexive, $a \in [a]$ so the equivalence classes are nonempty.
> Further, as $\forall a \in A$ belongs to the equivalence class $[a]$, the union of all equivalence classes is A.
> To show that the equivalence classes are pairwise disjoint, if $[a], [b]$ have an element c in common, they are the same set.
> Suppose that aRc and bRc.
> By the symmetry and transitivity property of equivalence relations, aRb.
> Thus, $\forall x \in [a]$, we have that xRa.
> By transitivity, xRb.
> Thus $[a] \subseteq [b]$ and that $[b] \subseteq [a]$.
> It follows that $[a] = [b]$.
> $\therefore$ An equivalence class of a set A is also a partition of A.

> (2) Any partition of A determines an equivalence on A for which the sets in the partition are the equivalence classes.
> Let $\mathcal{A}$ = {A_i} be a partition of A and define $R = \{(a, b) : \exists i \ni a \in A_i \text{ and } b \in B_i\}$.
> Suppose that R is an equivalence relation A, so we must prove the three qualitites of equivalence relations.
> Reflexivity: Since $a \in A_i \implies aRa$.
> Symmetry: If $aRb, then$a$ and b are in the same set A_i, so bRa.
> Transitivity: If aRb and bRc, then all three elements must be in the same set A_i.
> As the three properties of equivalence relations are satisfied,
> $\therefore$ A partition of A is also an equivalence class of a set A.

Partial Order

A binary relation R on a set A is ***antisymmetric*** if aRb and bRa implies $a = b$.
Ex: The $\leq$ relation on $\mathbb{N}$ is antisymmetric, since $a \leq b$ and $b \leq a$ implies $a = b$.
A relation that is reflexive, antisymmetric, and transitive is a ***partial order***.
The set that a partial order is defined on is a ***partially ordered set***.

In a partial ordered set A, there may be no single ***maximal*** element from the relation R.
Instead, there may be several maximal elements.
Ex: A collection of different-sized boxes may have multiple maximal boxes that don't fit inside any other box.

Total Relation

A relation R on a set A is a ***total relation*** if $\forall a, b \in A$, we have aRb, bRa, or both, with every pairing of elements of A is related by R.
Ex: The relation "$\leq$" is a total order on the natural numbers.
The relation "is a descendant of" is not a total order on the set of all people as two individuals may not be descendants from one or the other.
A partial order that is also a total relation is a ***total order*** or ***linear order***.
A total relation that is transitive, but not reflexive and antisymmetric, is a ***total preorder***.