Appendix B.1 Sets - Condensed Notes

Def: Set: A collection of distinct **members** or **elements**.
Notation: If an object x is a member of set S, we write $x \in S$.
Read "x is a member of S" or "x is in S"
If x is not a member of S, we write $x \notin S$.
We can describe a set explicitly using set notation, example: $\{1, 2, 3\}$.
Two sets A and B are are ***equal***, written $A = B$, if they contain the same elements.
Note: Sets are unordered collections.

Special Notation for frequently encountered sets:
$\emptyset$ denotes the empty set, a set with no members.
$\mathbb{Z}$ denotes the set of **integers** $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{R}$ denotes the set of **real numbers**.
$\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \ldots\}$
Note: Some mathematicians start the natural numbers with 0 or 1.

Set Notation Reading

Let $A = \{a \in \mathbb{N} : a < 10 \text{ and } a \geq b \ (b \in \mathbb{N} \ni b = \lceil f(a) \rceil)\}$.
This can be read as the set A contains the natural numbers less than 10 and greater than b with b being the ceiling of f of a.
This notation can be broken down into several components

1. The enclosing curly brackets { } indicate a collection of elements of length of elements.
2. The variable declaration $a \in \mathbb{N}$ is used to declare the type of variable that is used.
3. The such-that operator, the colon, is used to separate the variable declaration from the conditions.
   Note: The separator may appear as a vertical bar | based on preference.
4. The conditions declaration segment can include multiple conditions and new variables.

Subsets

Def: Subset: If $\forall x \in A \implies x \in B$, then we write $A \subseteq B$ (read as "A is a ***subset*** of B").
Def: Proper Subset: If $\forall x \in A \implies (x \in B) \land (A \neq B)$, then we write $A \subset B$.
Note: The Empty Set $\emptyset$ is a subset of all sets.

Set Operations

Given two sets A and B, we can define new sets by applying ***set operations***.

Union: $A \cup B = \{ x : x \in A \text{ or } x \in B \}$, the set of elements in either A or B.
Intersections: $A \cap B = \{ x : x \in A \text{ and } x \in B \}$, the set of elements in both A and B.
Difference: $A - B = \{ x : x \in A \text{ and } x \notin B \}$, the set of elements in A and not in B.

Set Operations must obey the following laws:

Empty Set Laws:
$A \cap \emptyset = \emptyset$,
$A \cup \emptyset = A$.

Identity Laws:
$A \cap A = A$,
$A \cup A = A$.

Associative Laws:
$A \cap (B \cap C) = (A \cap B) \cap C$,
$A \cup (B \cup C) = (A \cup B) \cup C$.

Distributive Laws:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Absorption Laws:
$A \cap (A \cup B) = A$,
$A \cup (A \cap B) = A$.

DeMorgans Laws:
$A - (B \cap C) = (A - B) \cup (A - C)$,
$A - (B \cup C) = (A - B) \cap (A - C)$.

The Universe and Complements

All sets are typically subsets of some larger set U, the ***universe***.
Example: The set $A = \{1, 2, 3\}$ is a subset of the natural numbers or the integers.
It may be crucial to define $U = \mathbb{N}$ or $U = \mathbb{Z}$ for clarity.

Given a universe U, we define the ***complement*** of a set A as $A' = U - A = \{ x : x \in U \text{ and } x \notin A \}$.
For any set $A \subseteq U$, we have the following laws,
$A'' = A$,
$A \cap A' = \emptyset$,
$A \cup A' = U$.

We can rewrite DeMorgan's laws with set complements.
For any two sets $A, B \subseteq U$, we define:
$(A \cap B)' = A' \cup B'$,
$(A \cup B)' = A' \cap B'$.

Disjoint Sets

Def: Disjoint: Two sets A and B are ***disjoint*** if they have no elements in common, $A \cap B = \emptyset$.
A collection $\mathbb{S} = \{S_i\}$ of nonempty sets forms a ***partition*** of a set S if,

1. The sets are ***pairwise disjoint***, that is $(S_i, S_j \in \mathbb{S})(i \neq j) \implies S_i \cap S_j = \emptyset$,
2. The union of all sets of $\mathbb{S}$ is S. This is represented symbolically as,

$$S = \bigcup_{S_i \in \mathbb{S}} S_i.$$

In other words, $\mathbb{S}$ forms a partition of S if each element of S appears in exactly one S_i member of $\mathbb{S}$.
Notation: Due to set theory rules, $\mathbb{S}$ is referred to as a collection and **not** as a set of sets.
Notation: The Big Union operator $\bigcup$ iterates through all set elements and unions them into a single set.

Counting

Def: Cardinality (Size): The number of elements in a set, denoted as $|S|$.
Note: The cardinality of the empty set $\emptyset = 0$.
If the cardinality of a set is a natural number, then set is ***finite***, else it is ***infinite***.
If an infinite set that can be put into a one-to-one correspondence with the natural numbers $\mathbb{N}$ is ***countably infinite***, else it is ***uncountable***.
The integers $\mathbb{Z}$ are countably infinite while the reals R are uncountable.

For any two finite sets A and B, we have the identity $|A \cup B| = |A| + |B| - |A \cup B|$.
We can deduce that $|A \cup B| \leq |A| + |B|$.
If A and B are disjoint, then $|A \cap B| = 0$, thus $ |A \cup B| = |A| + |B|. If A \subseteq B$, then $|A| \leq |B|$.

A finite set of n element is sometimes called an ***n-set***.
A 1-set is called a ***singleton***.
A subset of k elements of a set is sometimes called a ***k-subset***.

Power Sets

Def: Power Set: The set of all subsets of a set S, include the $\emptyset$ and S itself is called the ***power set*** of S, denoted as $\mathcal{P}(S)$.
Example: Let $S = \{a, b\}$, then $\mathcal{P}(S) = \Big\{\emptyset, \{a\}, \{b\}, \{a, b\}\Big\}$.

Cartesian Product

Def: Cartesian Product: Given two sets A and B, the Cartesian Product, denoted $A \times B$, $is the set$ $\{(a, b) : a \in A \text{ and } b \in B\}$.
Example: $\{a, b\} \times \{a, b, c\} =$

$$\begin{Bmatrix} (a,a),(a,b),(a,c), \\ (b,a),(b,b),(b,c) \end{Bmatrix}$$

When A and B are finite sets, the cardinality of the Cartesian product is $|A \times B| = |A| \cdot |B|$.

The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is the set of ***n-tuples***.
The cardinality of this product is $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$ if all sets are finite.

We denote an ***n-fold*** Cartesian product over a single set A as $A^n = A_1 \times A_2 \times A_n$.
The cardinality of this product $|A^n| = |A|^n$ if A is finite.

Ordered Sets

Recall: Sets are unordered collections of elements.
To get around this property, we can define an ***ordered list*** of numbers as nested sets.
Example:

$$\begin{gathered}\operatorname{set}(a, b, c) \neq \operatorname{set}(c, b, a) \\ \Big\{a, \{a, b\}, \{a, b, c\}\Big\} \neq \Big\{c, \{c, b\}, \{c, b, a\}\Big\}.\end{gathered}$$