Appendix B.4 Graphs

Directed Graphs (Digraphs)

Def: Directed Graphs (Digraphs): Define a graph G as a pair (V, E) such that:
V is a finite set with elements of all vertices of G. Denoted as the Vertex Set.
E is an **ordered** binary relation on V that represent the edges of G. Denoted as the Edge Set.
Note: Self-loops, edges from a vertex to itself, are possible.

Graphical Notation: Directed Graphs are drawn such that vertices are drawn as circles while edges are drawn as arrows.

If $(u, v) \in E$, then (u, v) ***incident from or leaves*** vertex u and ***incidents to or enters*** vertex v.
If $(u, v) \in E$, then vertex u is ***adjacent*** to vertex v, denoted $u \to v$. If $(v, u) \notin E$, then vertex v is not adjacent to vertex u.

Def: Out-Degree of a Vertex: The number of edges leaving a vertex.
Def: In-Degree of a Vertex: The number of edges entering a vertex.

Given a directed graph $G_D = (V_D, E_D)$, the ***undirected version*** $G_E = (V_U, E_U)$ is $(u, v), (v, u) \in E_D \iff (u, v) \in E_U$ and $v \neq u$.
Def: Neighbor: Two vertices that are adjacent in their undirected version.

Undirected Graphs

Def: Directed Graphs (Digraphs): Define a graph G as a pair (V, E) such that:
V is a finite set with elements of all vertices of G. Denoted as the Vertex Set.
E is an **unordered** binary relation on V that represent the edges of G. Denoted as the Edge Set.
Note: Self-loops, edges from a vertex to itself, are possible.
Let $v_1, v_2 \in V$ and $e_1 \in E$. If $e_1 \equiv \{v_1, v_2\}$, then $e_1 \equiv \{v_2, v_1\}$.

Graphical Notation: Undirected Graphs are drawn such that vertices are drawn as circles while edges are drawn as lines.

If $\{u, v\} \in E$, we say that $\{u, v\}$ ***incident on*** on vertices u and v.
If $\{u, v\} \in E$, we say that u and v are ***adjacent***. For Undirected Graphs, this property is symmetric.

Def: Degree of a Vertex: The number of edges incident on a vertex.
Note: A vertex with degree 0 is said to be ***isolated***.

Given a undirected graph $G_U = (V_U, E_U)$, the ***directed version*** $G_D = (V_D, E_D)$ is where $(u, v) \in E_U \iff (u, v), (v, u) \in E_D$.
Def: Neighbor: Two vertices that are adjacent.

Paths

A ***path*** from a vertex u to a vertex vertex u' in a graph G is a sequence $[v_0, \ldots, v_k]$ such that $u = v_0$, $u' = v_k$, and $(v_{i-1}, v_i) \in E$ for $i = 1, \ldots, k$.
The length of the path is value k.
The path contains the vertices $v_0, \ldots, v_k$ and edges $(v_0, v_1), \ldots (v_{k-1}, v_k)$.
Note: For all vertex u, there exists a 0-length path from u to u.
If there exists a path from u to u', we say that u' is ***reachable*** from u via path p.
A path is ***simple*** if all vertices in the path are distinct.

A ***subpath*** is a subsequence of the vertices of path p.
With $p = [v_0, \ldots, v_k]$, if $\exists i, j \ni 0 \leq i \leq j \leq k$, then the subpath $s = [v_i, \ldots, v_k]$.

Cycles

A graph with no cycle is acyclic.

Let G be a directed graph and a path $p = [v_0, \ldots, v_k]$, if $v_0 = v_k$ and the length of p is at least one, then p is a cycle.
  Similar to a simple path, p is a simple cycle if the elements are distinct.
  A directed graph is ***simple*** if there are no self-loops.

Let G be an undirected graph, a path p $[v_0, \ldots, v_k]$ forms a cycle if $k \geq 3$ and $v_0 = v_k$.

Connectivity

> Recall: ***Equivalence Classes*** (Foundations of Mathematics)
> Let S be a nonempty set and let R be a relation. For it to be an equivalence The following properties must hold:
>
> 1. Reflexivity $\forall x \in S,\ (x, x) \in R$
> 2. Symmetric $\forall (x, y) \in R,\ \exists (y, x) \in R$
> 3. Transitivity $\forall (x, y), (y, z) \in R,\ \exists (x, z) \in R$
>
> When referring to the equivalence relation in terms of graphs, $S = V$, the vertex set, and $R = E$, the edge set.

Let G be a directed graph.
G is considered to be ***strongly connected*** if every two vertices are reachable from each other.
The ***strongly connected components*** of a directed graph are the equivalence classes of vertices under the "mutually reachable relation".

Let G be an undirected graph.
G is considered to be ***connected*** if every vertex is reachable from all other vertices.
The ***connected components*** of a graph are the equivalence classes of vertices under the "is reachable" relation.

Isomorphism

> Recall: ***Bijective Functions*** (Foundations of Mathematics)
> A function $f : A \to B$ is said to be bijective (one-to-one) if $(x_1, x_2 \in A),\ f(x_1) \neq f(x_2)$

Two graphs $G = (V, E)$, $G' = (V', E')$ are ***isomorphic*** if there exists a bijection $f : V \to V' \ni (u, v) \in E \iff (f(u), f(v)) \in E'$.

Subgraphs

Given a graph $G = (V, E)$, we define $G' = (V', E')$ as a ***subgraph*** if $V' \subseteq V$ and $E' \subseteq E$.
When given a set $V' \subseteq V$, the subgraph of G ***induced*** by V' is the graph $G' = (V', E') \ni E' = \{(u, v) \in E : u, v \in V'\}$.

Special Graphs

Def: Complete Graph: An undirected graph in which every pair of vertices are adjacent.
Def: Bipartite Graph: An undirected graph $G = (V, E)$ where V is partitioned into
$V_1, V_2 \ni (u, v) \in E \implies u \in V_1, v \in V_2 \text{ or } u \in V_2, v \in V_1$.
Def: Forest: An acyclic, undirected graph.
Def: (Free) Tree: A connected, acyclic, undirected graph.
Def: Dag: A directed acyclic graph.
Def: Multi-graph: An undirected graph that can contain multiple edges going from any two vertices or self-loops.
Def: Hypergraph: Similar to an undirected graph with ***hyperedges***, edges that can connect an arbitrary subset of vertices.
Visually, imagine a hyperedge as a space that envelopes multiple nodes.
Many algorithms written for directed/undirected graphs can be adapted to run on a hypergraph.

Contraction

Def: Contraction of an Undirected Graph:
Given an undirected graph $G = (V, E)$.
The contraction of G by an edge $e = (u, v)$ is a graph $G' = (V', E')$ where,
$V' = V - \{u, v\} \cup \{x\}$ where x is the vertex formed by joining the vertices u and v.
$E' = E - \{e\} \cup E_X$ where E_X is the edges formed by connecting all edges containing u,v to x.
Therefore, the effect is u and v being contracted into a single vertex.