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Lagrange Points of the Sun-Jupiter System

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ABSTRACT

The movement and mechanics of celestial bodies has been studied ever since humans started to gaze upon the stars. Astronomers and mathematicians aimed to describe the dynamics of celestial bodies with mathematical equations and physics. Applying fundamental laws of physics such as Isaac Newton's law of gravitation, the two-body problem of orbiting bodies can be solved. However, in the case of a system with three co-orbiting bodies, the three-body problem contains no closed form solution and produces a chaotic dynamical system. In the 18th century, astronomer Joseph-Louis Lagrange attempted to solve the three-body problem. In his published findings, Lagrange examines the stability of planetary orbits and discovers the Lagrange points. The 5 Lagrange points are points in space where the net force on a third body is zero. The derivation and calculation of the 5 Lagrange points of the Sun-Jupiter system are calculated and their respective stabilities analyzed.

INTRODUCTION

The three-body problem is a problem in classic celestial mechanics which has no closed-form analytical solution. The restricted three-body problem arises in a case where the third body's mass is negligible compared to the first and second mass, meaning it orbits around the center of mass of the system formed by the first and second mass. This allows the system to be analyzed in the scope of two-body motion.

In this case of two-body motion, there exists 5 points where the gravitational forces exerted by the two bodies on the third body is net zero. In the 1750's, Leonhard Euler discovered the collinear Lagrange points, L_1, L_2, and L_3. Euler (1765) Another mathematician, Joseph-Louis Lagrange, examines this problem further in his paper regarding the general three-body problem, *Essai sur le Problème des Trois Corps*, in 1772. In this paper, Lagrange examines the stability of planetary orbits. In the context of the restricted three-body problem Lagrange discovers two further points of gravitational equilibrium, L_4 and L_5. The L_4 and L_5 points are stable, while L_1, L_2, and L_3 are unstable. This is due to the fact that L_4 and L_5 distances to either of the two masses are equal. This implies that L_4 and L_5 lie on points $60°$ ahead of and behind the second body's orbital path, where the distance vectors to the two large masses form the sides of an isosceles triangle. Curtis (2014)

The existence of this points allows for a celestial "parking space". Bodies that find themselves in these areas of stability can remain in pseudo-stable, predictable orbits. For the case of the Sun-Jupiter system, there are certain "Trojan asteroids" that have found themselves in relative stability in the L_4 and L_5 positions. Examinations of these asteroids in a potential future mission can help humans uncover the origins of planetary formation and the solar system's history and origin. Myhaver (2021) Lagrange points also offer other benefits for spacecraft. The James Webb Space Telescope is NASA and the ESA's new premier space telescope launched in December 2021, succeeding the Hubble telescope from 1990. The advanced space telescope is designed to operate in a halo orbit at the Sun-Earth L_2 point. The usefulness of L_2 in this mission is that it allows the telescope to remain out of the shadow of the Earth and Moon for uninterrupted solar power and Earth communication, while simultaneously providing a predictable orbit and allowing the instruments a protected view of outer space.

For our experiment, we consider the circular restricted three-body problem of the Sun-Jupiter system. We will use numerical methods to find solutions to the equations of motion that will yield the locations of the 5 Lagrange points of the Sun-Jupiter system.

METHODOLOGY

The first step to understanding the dynamics of celestial bodies is Newton's Law of Gravitation:

$$F_g = -\frac{GM_1M_2}{r^2}\hat{r} \tag{1}$$

where G is the universal gravitational constant, 6.67 x 10^{-11} $\frac{N\cdot m^2}{kg^2}$, M_1 and M_2 are the masses of the two orbiting bodies, and r is the distance between the center's of gravity of the two bodies.

In the case of our third body, there are two gravitational forces acting on it: one due to Jupiter's gravity and one due to the Sun's. This yields an equation for the net gravitational forces on the third body:

$$F_m = -\frac{GM_{sun}m}{|r-r_1|^3}(r-r_1) - \frac{GM_Jm}{|r-r_2|^3}(r-r_2)$$

where M_{sun}, M_J, and m are the masses of the sun, Jupiter, and third body respectively. Additionally, r_1 is the distance of the sun to the center of mass of the Sun-Jupiter system, r_2 is the distance of Jupiter to the center of mass of the Sun-Jupiter system, and r is the distance of the third body to the center of mass of the Sun-Jupiter system. These are shown below in Figure 1.

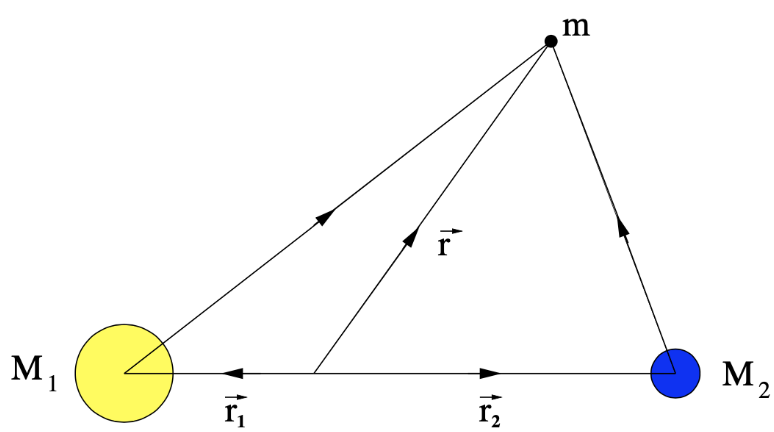

Figure 1. Radius Vectors to Center of Mass of System

However, these are not the only forces we have to take into account before finding the location of the Lagrange points. We have defined a coordinate system here with the origin of the center of mass of the Sun-Jupiter system. Due to this definition, the coordinate system is rotating and non-inertial. There are two forces that must be taken into account in this rotating reference frame, the Coriolis force and the centrifugal force. The equations for these forces is shown below:

$$F_{Coriolis} = 2m\omega\frac{dr}{dt} \tag{2}$$

$$F_{centrifugal} = m\omega^2 r \tag{3}$$

where $\frac{dr}{dt}$ is the rate of change with respect to time of the r vector, and ω is the angular velocity of the rotating coordinate system. The angular velocity can be found by taking the total radians swept out by Jupiter's orbit divided by the period of the orbit, the time it takes Jupiter to complete one revolution around the sun.

$$\omega = \frac{2\pi}{T}$$

Combining these four forces on the third body yields the equation for net force on the third body:

$$F_m = -\frac{GM_{sun}m}{|r-r_1|^3}(r-r_1) - \frac{GM_Jm}{|r-r_2|^3}(r-r_2) + m\omega^2 r - 2m\omega\frac{dr}{dt} \tag{4}$$

For circular orbits, it is known that the mean motion, n, or the angular velocity ω of the orbit is related to the masses of the bodies and the radius of the orbit, center of Sun to center of Jupiter, by the equation:

$$\omega^2 = \frac{\mu}{a^3} = \frac{G(M_{sun} + M_J)}{|r_1 - r_2|^3} \tag{5}$$

where $\mu = G(M_{sun} + M_J)$ and r_1 and r_2 are the distances of the Sun and Jupiter to the center of mass of the Sun-Jupiter system. From simple vector math, a, Jupiter's orbit radius, can be shown to be equal to $|r_1 - r_2|$.

Now to find the Lagrange points, let's consider some facts about the system. We can make an assumption that the $\frac{dr}{dt}$ term goes to zero as we do not want relative motion of the r vector at the Lagrange point due to its nature. Secondly, we can sub in Eq.5 into Eq.4. Using Newton's second law, ΣF=ma, we can divide both sides of the equation by m to yield an equation for the net acceleration of the third body.

$$a = -\frac{GM_{sun}}{|r - r_1|^3}(r - r_1) - \frac{GM_J}{|r - r_2|^3}(r - r_2) + \frac{G(M_{sun} + M_J)}{|r_1 - r_2|^3} r \tag{6}$$

Simplifying the problem into 1 dimension allows us to derive the equations to find the first 3 collinear Lagrange points. Using x as the distance of the Lagrange point to the center of mass of the Sun-Jupiter system and setting acceleration to 0 yields 3 equations:

For L1:

$$0 = \frac{GM_{sun}}{x^2} - \frac{GM_J}{(r_2 - x)^2} - \frac{G(M_{sun} + M_J)}{r_2^3} x \tag{7}$$

For L2:

$$0 = \frac{GM_{sun}}{x^2} + \frac{GM_J}{(x - r_2)^2} - \frac{G(M_{sun} + M_J)}{r_2^3} x \tag{8}$$

For L3:

$$0 = \frac{GM_{sun}}{x^2} + \frac{GM_J}{(x + r_2)^2} + \frac{G(M_{sun} + M_J)}{r_2^3} x \tag{9}$$

We can now use numerical root finding methods on each of these 3 equations to find x, the distance of the Lagrange points which are in line with the Sun and Jupiter. The results for these calculations are shown in Table **??**. The solver used also requires an initial guess for the points of L_1, L_2, and L_3. The initial guess is determined by graphing the right hand side of the equations of motion seen in Figures 2, 3, and 4.

For the Lagrange points L_4 and L_5, we must now take into account a 2-D perspective of the orbital plane as opposed to 1-D due to the locations of L_4 and L_5 being non-collinear with the Sun and Jupiter. Separating the acceleration vector into an x and y component will yield a system of equations that when solved, will provide two solutions are the coordinates of the L_4 and L_5 points. The system of equations is given below:

$$a_x(x, y) = -\frac{GM_{sun}(x + r_1)}{((x + r_1)^2 + y^2)^{\frac{3}{2}}} - \frac{GM_J(x - r_2)}{((x - r_2)^2 + y^2)^{\frac{3}{2}}} + \frac{G(M_{sun} + M_J)}{|r_1 - r_2|^3} x \tag{10}$$

$$a_y(x, y) = -\frac{GM_{sun} y}{((x + r_1)^2 + y^2)^{\frac{3}{2}}} - \frac{GM_J y}{((x - r_2)^2 + y^2)^{\frac{3}{2}}} + \frac{G(M_{sun} + M_J)}{|r_1 - r_2|^3} y \tag{11}$$

where x is the x-component of the distance vector from the Lagrange point to the center of mass of the system and y is the y-component of the distance vector from the Lagrange point to the center of mass of the system. r_1 and r_2 are the distance of the Sun and Jupiter to the center of mass of the system respectively. Setting the acceleration to 0 and solving the system of equations using a generalized solver gives the coordinates of L_4 and L_5. Noting that there is two solutions to this system of equations we need to condition our guesses for the iterative solver correctly. However, we can use the knowledge that L_4 and L_5 will lie approximately 60° ahead of and behind Jupiter's orbit to condition the guesses. Thus, the coordinates for the L_4 guess are going to be ($r_2cos(60)$,$r_2sin(60)$). Since L_5 is behind Jupiter's orbit, the initial guess will be ($r_2cos(60)$,$-r_2sin(60)$).

DATA

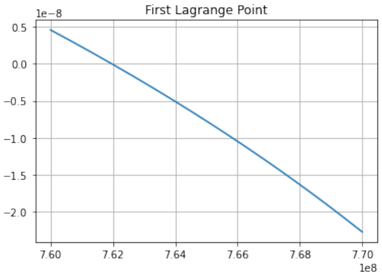

Figure 2. First Lagrange Point

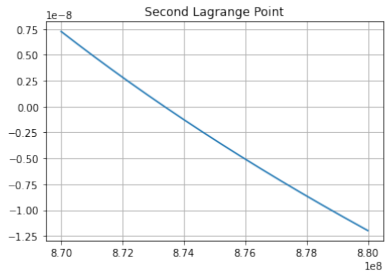

Figure 3. Second Lagrange Point

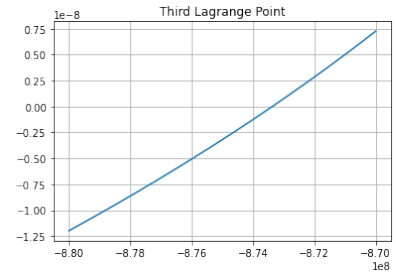

Figure 4. Third Lagrange Point

Table 1. Lagrange Points

Point	x (km)	y (km)	Time (s)
L_1	761961313.7868444	0	0.0009992122650146484
L_2	873367501.5651376	0	0.0009999275207519531
L_3	-873367501.5651377	0	0.0010004043579101562
L_4	240845674.556546	781000443.395860	0.0007410049438476562
L_5	240844605.350067	-781000772.702288	0.0006608963012695312

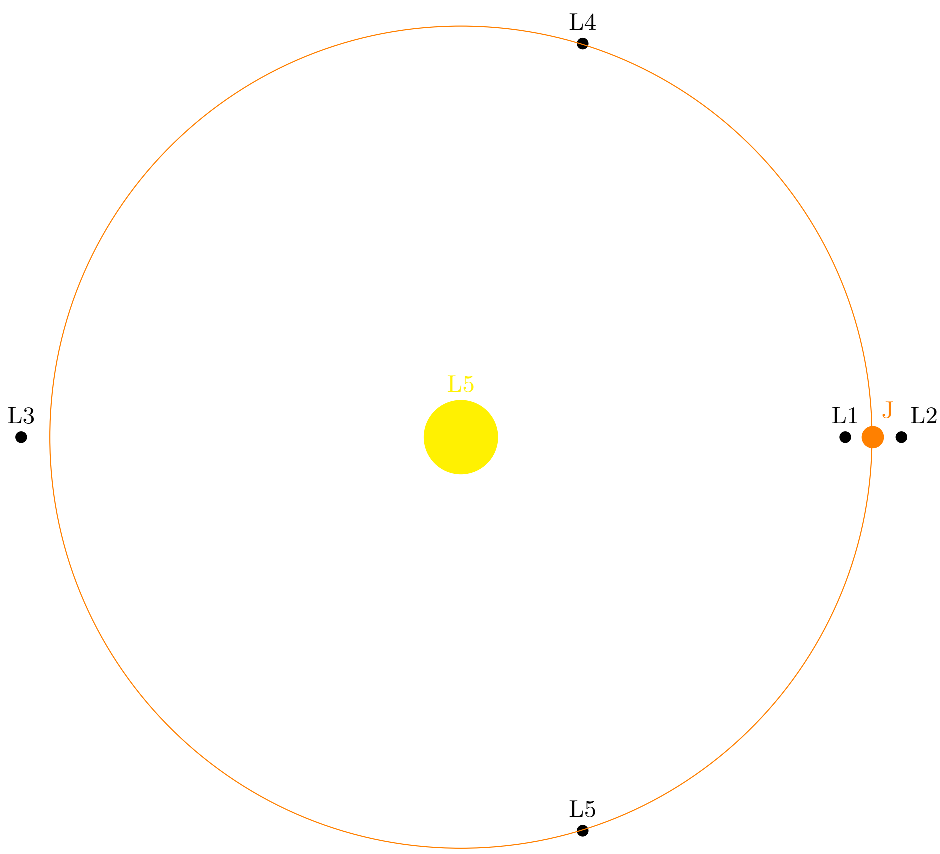

Figure 5. Model of Lagrange Points

RESULTS

For the most accurate results, we found the constants describing MG_{sun} and MG_J directly along with the position describing the Sun and Jupiter Folkner et al. (2014) and the ratio of mass between the Sun and Jupiter Burša (1998).

Using these we used the distance formula to find the total distance between the Sun and Jupiter and then the center of mass equation to find r_{sun} and r_J. Using these values, we plotted the Lagrange point equations (see Figues 2, 3, and 4) and, getting a basic guess, used scipy fsolve to find the Lagrange points (see Table 1). For stability of the Lagrange points, we need to evaluate the eigenvalues of the Jacobian matrix. To obtain the Jacobian matrix, we used the full output of scipy's fsolve function to give the orthogonal Q matrix and upper triangular matrix, R, that results from the QR factorization of the Jacobian. Multiplying the Q and R matrix together will return the Jacobian. Finding the eigenvalues of the Jacobian will gives us the clues necessary to determine stablility of the points. This relationship is known from linear systems where a system with input, u, and matrix, A, in the system $\dot{x}$ = Ax + Bu has stability if and only if the real parts of the eigenvalues of matrix A are negative. Using this knowledge, we can calculate the eigenvalues of the Jacobian for each L_1, L_2, L_3, L_4, and L_5. We conclude from the results that only L_4 and L_5 are stable Lagrange points.

SUMMARY / CONCLUSION

The goal of this experiment was to numerically solve for the coordinates of the 5 Lagrange points of the Sun-Jupiter system. The design of this problem requires numerical root-finding methods to solve 1 dimensional equations and 2 dimensional systems of equations. The method we used to solve these systems is efficient, the slowest time to calculate one of the Lagrange points was roughly .001 seconds. To improve upon this experiment and create higher accuracy, other gravitational effects from Saturn or Jupiter's moons can be added simply by including the gravitational force when summing the forces on the third body. However, this would still be slightly inaccurate due to the restricted n-body problem where we are only considering other bodies' effects on the spacecraft, and not vise-versa. Additionally, another assumption is that Jupiter's orbit is circular, in fact it is not. This would vary the position of the Lagrange points as Jupiter completes a revolution around the sun. Overall, the goal of the experiment was completed.

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