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Artillery

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ABSTRACT

Starting in the first World War, it was realized that mathematics could be utilized in order to make artillery warfare more efficient. Given a target and a launch speed, the best angle to fire the cannon can be calculated. However, the need to take the time for calculation during battle would become inefficient in itself, so therefore a set of tables with a variety of target distances from the cannon and shooting speeds can be pre-calculated and then memorized in order to optimize cannon firing. These tables can be calculated by using a root-finding algorithm on a non-linear equation and iterating through different starting conditions.

INTRODUCTION

The United States Army has a large need for ballistic data in order to accurately carry out operations. In the first year of WWI, the acquisition of this data was carried out by a small group of people in the Artillery Ammunition section of the Gun Division of the army. MD & Reed (1992a) Quickly, the amount of ballistics data, including firing tables, quickly exceeded the capacity of this small group and the Ballistic Branch was created. This division grew throughout the war, but had some downsizing and reorganizing after WWI. In 1925 the Ballistic Mathematics Unit was created, and it gradually expanded.

In the 1930s, the Bush Differential Analyzer was developed and assembled at the site of the Ballistic Research Laboratory which greatly sped up the calculations and added accuracy. The calculations needed soon outgrew this machine and gradually punch-card machines were acquired from IBM by the lab in 1941. These gave way to the relay calculators made by IBM and after the end of WWII, the Electronic Numerical Integrator and Computer (ENAIC) and the Electronic Discrete Variable Computer (EDVC). The ENAIC was installed in 1947 and was at the start of the widespread use of electronic computers. MD & Reed (1992b) After this the Ballistic Research Laboratories did some of their own electronic development, and utilized those systems until the late 1970s. MD & Reed (1992c)

However, it was realized that in order to increase their accuracy they needed a better theoretical trajectory model. Therefore, a modified point mass model was developed to give a better representation of the trajectory. With modifications, this model was the NATO standard for ballistic control. In the early 2000s, it's documented that the firing table division helps work on improving meteorological algorithms in combination with developing software for computer launched ballistics. ARDEC (2022)

For our experiment, we consider the following problem. Given a target location and initial velocity we can solve for the ideal launch angle of a cannon.

METHODOLOGY

The framework of this ballistics problem is governed by two separate differential equations:

$$v_y^{'}(t) = -g \tag{1}$$

$$v_x^{'}(t) = 0 \tag{2}$$

where g is the acceleration due to gravity. The motion of the cannonball can be separated into two distinct components, x and y. The acceleration in the x-direction is always 0; there is no outside force acting in the x-direction as air resistance is neglected.

The acceleration in the y-direction is solely subject to the gravitational acceleration Earth imparts on the cannonball, which is constant. Integrating these two equations yields:

$$v_y(t) = -gt + v_{y_0} \tag{3}$$

$$v_x(t) = v_{x_0}. \tag{4}$$

As seen in Figure 1, we can use trigonometry to equate $v_{y_0}(t) = v_0 \sin(\theta)$ $v_{x_0}(t) = v_0 \cos(\theta)$.

After substituting those values and integrating yet again to give position as a function of time yields:

$$s_y(t) = \frac{-gt^2}{2} + v_0 \sin\theta + y_0 \tag{5}$$

$$s_x(t) = v_0 \cos\theta + x_0 \tag{6}$$

Taking the cannon as the origin of our coordinate system, we can set $y_0 = 0$ and $x_0 = 0$.

Since we are given the target's coordinates as x_* and y_*, we can set the y-position equal to this final desired coordinate as $y_* = \frac{-gt^2}{2} + v_0 \sin\theta$. Equating this equation to zero yields a quadratic, whose solution is the amount of time it takes the particle to reach the desired height. Because this is a quadratic equation with likely two solutions, the desired solution is the one further away from zero as it represents the time it takes the particle to reach its apogee, then fall back to that height, rather than rise to that height.

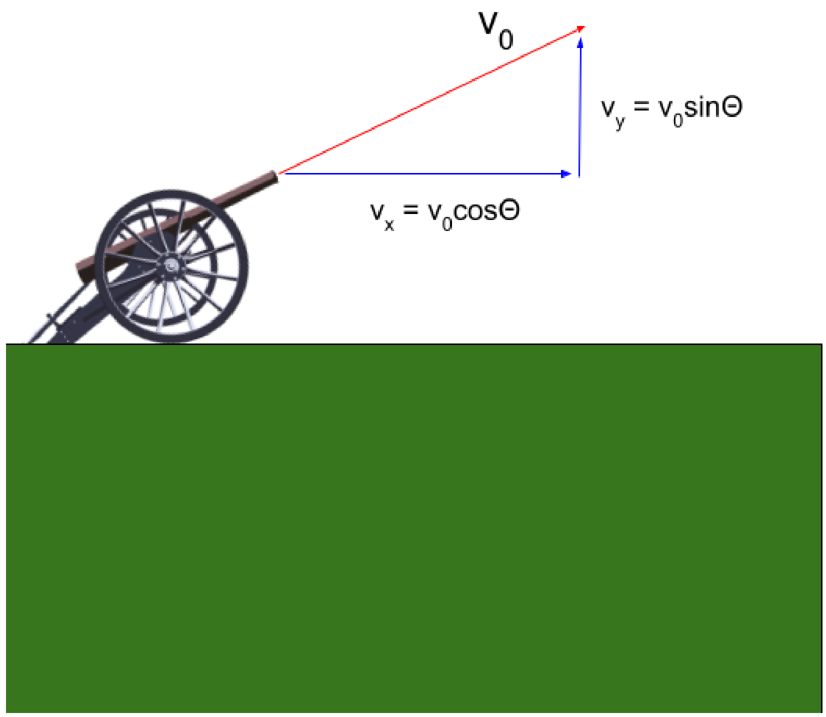

Figure 1. Vector Components

Taking the previous time solution and plugging it into Eq. (6), we calculate the position in the x-direction of the cannonball after t seconds of flight. Using the bisection method, we then adjust the angle of the cannon to hit the target coordinate. If the desired x-coordinate is lesser than current landing spot, raise the angle of the cannon. If the desired x-coordinate is greater than the current landing spot, lower the angle of the cannon. Each instance the cannon's angle is adjusted to correct the x-coordinate landing, the time in flight is recalculated by the quadratic formula according to the new θ. The initial guess for θ is always 45°.

To determine if a target is out of range, the quadratic formula is employed. If the $b^2 - 4ac$ term is less than 0, there are imaginary roots to the quadratic, signaling that the cannon will not reach that height. Additionally, the maximum horizontal range for a projectile is:

$$x_{max} = \frac{v \cos\theta}{g}(v \sin\theta + \sqrt{v^2 \sin^2\theta + 2g(-y_0)}) \tag{7}$$

where θ=45°, as this gives equal horizontal and vertical components creating the furthest trajectory. y_0 is the initial position of the cannon which is 0, but the vertical displacement of y_* can be interpreted as $-y_0$.

DATA

Table 1. $v_0 = 100m/s$

x,y	100.0	400.0	700.0	1000.0	1300.0	1600.0	1900.0	2200.0	2500.0
-100.0	87.314	79.023	69.672	56.585	-1	-1	-1	-1	-1
-75.0	87.285	78.893	69.373	55.613	-1	-1	-1	-1	-1
-50.0	87.254	78.756	69.053	54.441	-1	-1	-1	-1	-1
-25.0	87.222	78.612	68.711	52.94	-1	-1	-1	-1	-1
0.0	87.188	78.46	68.343	50.739	-1	-1	-1	-1	-1
25.0	87.152	78.3	67.944	-1	-1	-1	-1	-1	-1
50.0	87.115	78.13	67.508	-1	-1	-1	-1	-1	-1
75.0	87.076	77.95	67.029	-1	-1	-1	-1	-1	-1
100.0	87.034	77.757	66.496	-1	-1	-1	-1	-1	-1

Table 2. $v_0 = 110m/s$

x,y	100.0	400.0	700.0	1000.0	1300.0	1600.0	1900.0	2200.0	2500.0
-100.0	87.765	80.924	73.523	64.674	49.461	-1	-1	-1	-1
-75.0	87.744	80.836	73.34	64.292	46.605	-1	-1	-1	-1
-50.0	87.723	80.744	73.147	63.882	-1	-1	-1	-1	-1
-25.0	87.7	80.648	72.945	63.439	-1	-1	-1	-1	-1
0.0	87.677	80.549	72.731	62.956	-1	-1	-1	-1	-1
25.0	87.653	80.444	72.506	62.426	-1	-1	-1	-1	-1
50.0	87.628	80.335	72.268	61.838	-1	-1	-1	-1	-1
75.0	87.602	80.22	72.015	61.176	-1	-1	-1	-1	-1
100.0	87.574	80.1	71.744	60.417	-1	-1	-1	-1	-1

Table 3. $v_0 = 120m/s$

x,y	100.0	400.0	700.0	1000.0	1300.0	1600.0	1900.0	2200.0	2500.0
-100.0	88.111	82.364	76.295	69.501	60.975	-1	-1	-1	-1
-75.0	88.096	82.302	76.172	69.282	60.521	-1	-1	-1	-1
-50.0	88.081	82.237	76.045	69.052	60.029	-1	-1	-1	-1
-25.0	88.065	82.171	75.913	68.811	59.489	-1	-1	-1	-1
0.0	88.049	82.102	75.775	68.556	58.891	-1	-1	-1	-1
25.0	88.032	82.03	75.632	68.288	58.221	-1	-1	-1	-1
50.0	88.014	81.956	75.482	68.003	57.454	-1	-1	-1	-1
75.0	87.996	81.879	75.325	67.7	56.554	-1	-1	-1	-1
100.0	87.977	81.798	75.161	67.376	55.45	-1	-1	-1	-1

Table 4. $v_0 = 130m/s$

x,y	100.0	400.0	700.0	1000.0	1300.0	1600.0	1900.0	2200.0	2500.0
-100.0	88.383	83.483	78.388	72.884	66.576	58.349	-1	-1	-1
-75.0	88.373	83.438	78.301	72.741	66.336	57.839	-1	-1	-1
-50.0	88.361	83.391	78.212	72.592	66.083	57.279	-1	-1	-1
-25.0	88.35	83.343	78.121	72.439	65.818	56.656	-1	-1	-1
0.0	88.338	83.294	78.026	72.279	65.538	55.952	-1	-1	-1
25.0	88.326	83.243	77.928	72.113	65.242	55.139	-1	-1	-1
50.0	88.313	83.19	77.826	71.939	64.928	54.168	-1	-1	-1
75.0	88.3	83.136	77.721	71.759	64.594	52.937	-1	-1	-1
100.0	88.287	83.08	77.612	71.57	64.236	51.159	-1	-1	-1

Table 5. $v_0 = 140m/s$

x,y	100.0	400.0	700.0	1000.0	1300.0	1600.0	1900.0	2200.0	2500.0
-100.0	88.601	84.372	80.018	75.418	70.377	64.513	56.667	-1	-1
-75.0	88.593	84.338	79.955	75.318	70.224	64.263	56.13	-1	-1
-50.0	88.585	84.304	79.891	75.215	70.065	64.001	55.535	-1	-1
-25.0	88.576	84.268	79.824	75.109	69.9	63.725	54.866	-1	-1
0.0	88.567	84.232	79.756	75.0	69.729	63.435	54.097	-1	-1
25.0	88.558	84.194	79.686	74.887	69.552	63.128	53.186	-1	-1
50.0	88.549	84.156	79.614	74.771	69.367	62.802	52.046	-1	-1
75.0	88.539	84.116	79.54	74.651	69.175	62.455	50.44	-1	-1
100.0	88.529	84.076	79.464	74.527	68.974	62.084	-1	-1	-1

Table 6. $v_0 = 150m/s$

x,y	100.0	400.0	700.0	1000.0	1300.0	1600.0	1900.0	2200.0	2500.0
-100.0	88.778	85.09	81.32	77.392	73.201	68.57	63.143	55.819	-1
-75.0	88.772	85.064	81.273	77.319	73.094	68.414	62.895	55.295	-1
-50.0	88.765	85.038	81.224	77.245	72.985	68.252	62.635	54.714	-1
-25.0	88.759	85.011	81.175	77.168	72.872	68.084	62.363	54.061	-1
0.0	88.752	84.983	81.124	77.09	72.756	67.911	62.076	53.31	-1
25.0	88.745	84.955	81.073	77.009	72.637	67.731	61.773	52.416	-1
50.0	88.738	84.926	81.019	76.927	72.514	67.545	61.452	51.289	-1
75.0	88.731	84.897	80.965	76.842	72.388	67.35	61.111	49.662	-1
100.0	88.723	84.867	80.909	76.755	72.257	67.148	60.747	-1	-1

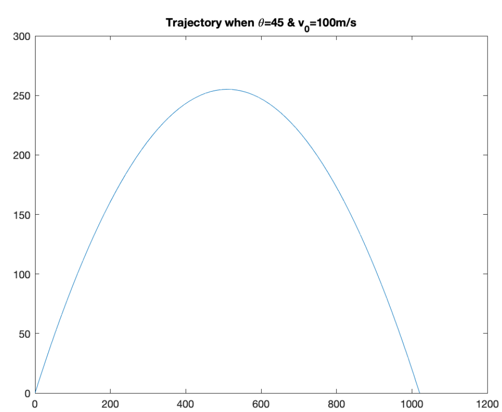

Figure 2. Trajectory when $\theta = 45$ and $v_0 = 100m/s$

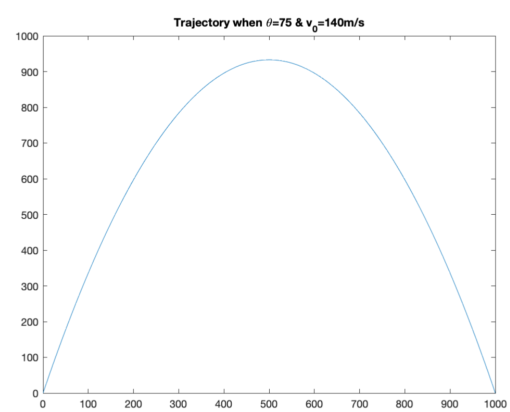

Figure 3. Trajectory when $\theta = 75$ and $v_0 = 140m/s$

RESULTS

The tables above show the theta in degrees that shooting the cannon at would result in a hit at the target location given the initial velocity. Any place where the theta is listed as -1 is an entry where the target location is out of range given the initial velocity. One error of the computations is that through using root finding techniques, there are certain scenarios where two thetas exist that would yield a hit at the target location. The root finding algorithm does not know when it has the ability to sacrifice height and lower the cannon for a given x_* and y_*. The angle given is the most parabolic flight, not the most direct to the target. This would result in a longer time to impact, meaning the target has potentially moved out of their previous location. However, these situations are few and only apply to very close targets, to the point where a firing table may not be required by even a novice artilleryman. Additionally, this provides the firing team the ability to fire behind cover. We can verify the accuracy of the tables with graphs of the trajectories of the projectile. In Figure 2, the projectile's max range can be seen to be slightly greater than 1000, verifying that 1300+ meters is out of range. In Figure 3, we verify the accuracy of Table 5. For the given v_0 and target location of (1000,0), a theta of 75° has a direct impact.

SUMMARY / CONCLUSION

The run-time of the function, even iterating through all the combinations needed to make all the tables, is almost instantaneous. Thus for the problem asked of us, to quickly calculate the ideal launch angle for the given distance range, our algorithm is up to par. The artillery officer would have no great delay in launch using our algorithm.

If we had air resistance we could have computed more accurate tables. The equations for air resistance are

$$\frac{\partial v_x}{\partial t} = -cv_x\sqrt{v_x^2 + v_y^2} \tag{8}$$

$$\frac{\partial v_y}{\partial t} = -g - cv_y\sqrt{v_x^2 + v_y^2} \tag{9}$$

where c_v is a coefficient that is dependent on the shape of the projectile and the density of the air. A blunt body such as a cannonball will experience a high amount of pressure drag. Additionally, density of air is a function of altitude and temperature, requiring the need for more computations.

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