University of Pittsburgh

Algorithms and Data Structures 1
CS 0445

Fall 2022

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(Slides are adapted from Dr. Ramirez's and Dr. Farnan's CS1501 slides.)

Announcements

- Upcoming Deadlines:
  - Lab 5: next Monday @ 11:59 pm
  - No homework due this Friday
- Midterm Exam: Thursday 10/20
  - closed book, paper, in-person
- No recitations tomorrow: Fall Break!

Previous Lecture …

- Recursion
  - More examples
  - Runtime analysis using proof by induction
- Using recursion to solve hard problems
  - Towers of Hanoi

Today …

- Recursion Applications
  - Divide and Conquer
  - Backtracking
- Limitation of Recursion

Recursive Solution to Towers of Hanoi Problem

- Recursive algorithm to solve any number of disks.

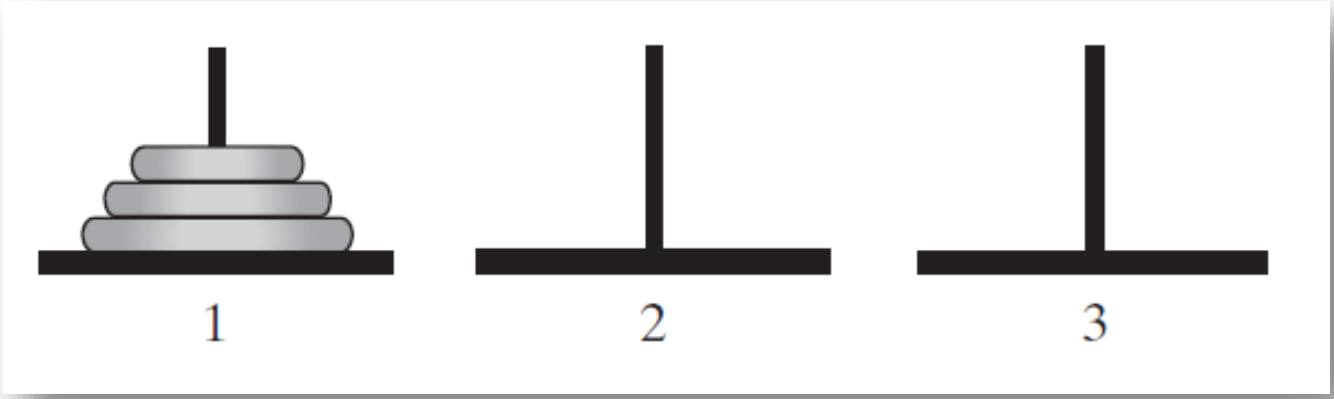
```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
   Move disk from startPole to endPole
else
{
   solveTowers(numberOfDisks - 1, startPole, endPole, tempPole)
   Move disk from startPole to endPole
   solveTowers(numberOfDisks - 1, tempPole, startPole, endPole)
}
```

Runtime analysis of Recursive Algorithms

- Step 1: **Recurrence Relation** for the running time function *T(n)*
  - Don’t forget the base case (e.g., T(1))
- Step 2: Build an intuition to make a guess for a closed form for T(n)
  - By evaluating T(n) for small values of n
- Step 3: Try to prove your guess for T(n) using induction
  - If not successful, go back to Step 2

Runtime analysis of Recursive Algorithms

- Step 1: **Recurrence Relation** for the running time function *T(n)*
- What is n in this problem?
  - number of disks to move
- *T(1) = 1*

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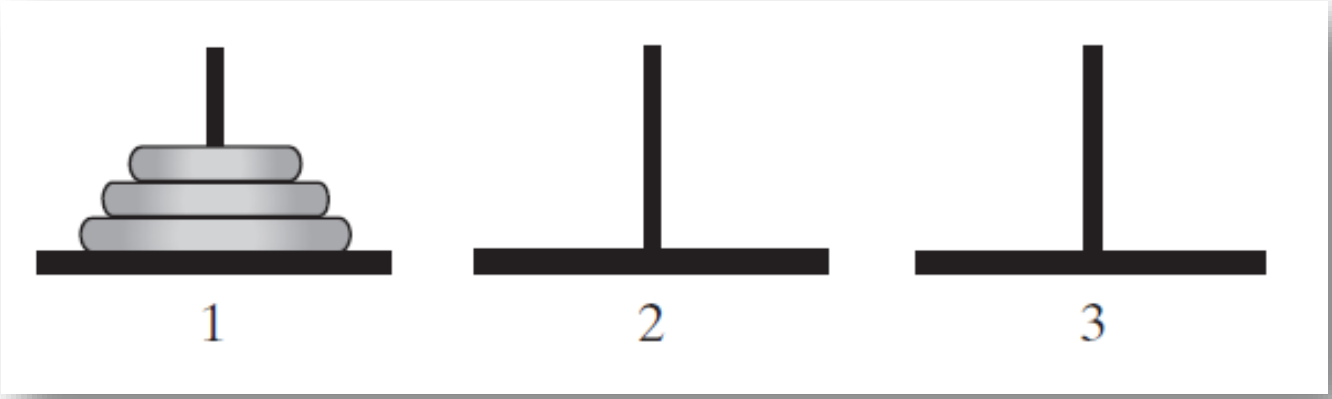
Runtime analysis of Recursive Algorithms

- Step 1: **Recurrence Relation** for the running time function *T(n)*
- What is n in this problem?
  - number of disks to move
- *T(1) = 1*
- *T(n) = ??*
- *T(n) = 2T(n-1) + 1*
  - *why?*

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Runtime analysis of Recursive Algorithms

- Step 1: **Recurrence Relation** for the running time function *T(n)*
  - *T(1) = 1*
  - *T(n) = 2T(n-1) + 1* for *n > 1*
- Step 2: Build an intuition to make a guess for a closed form for T(n)
  - By evaluating T(n) for small values of n
  - T(2) = 2T(1) + 1 = 3
  - T(3) = 2T(2) + 1 = 7
  - T(4) = 2T(3) + 1 = 15
  - T(5) = 2T(4) + 1 = 31
  - …
  - not linear
  - increase by 1 in n → almost doubling of T(n)
  - exponential function
- $T(n) = 2^n$
  - would that work?
  - Nope, why?
- $T(n) = 2^n - 1$

Runtime analysis of Recursive Algorithms

- Step 1: **Recurrence Relation** for the running time function *T(n)*
  - *T(1) = 1 …… (1)*
  - *T(n) = 2T(n-1) + 1* for *n > 1 ……(2)*
- Step 2: Build an intuition to make a guess for a closed form for T(n)
  - $T(n) = 2^n - 1$
- Step 3: Try to prove your guess for T(n) using induction
- Let's try to prove that $T(n) = 2^n-1$ for all n >= 1 …. (3)
- Base Case: when n=1
  - From (1) → T(1) = 1
  - From (3) → $T(1) = 2^1 – 1 = 2 – 1 = 1$
  - (3) holds for n=1

Runtime analysis of Recursive Algorithms

- Step 1: **Recurrence Relation** for the running time function *T(n)*
  - *T(1) = 1 …… (1)*
  - *T(n) = 2T(n-1) + 1* for *n > 1 ……(2)*
- Step 2: Build an intuition to make a guess for a closed form for T(n)
  - $T(n) = 2^n - 1$
- Step 3: Try to prove your guess for T(n) using induction
- Let's try to prove that $T(n) = 2^n-1$ for all n >= 1 …. (3)
- Inductive Step:
  - Assume (3) holds for all values n < k **(Inductive Hypothesis)**
    - $T(n) = 2^n – 1$ for all n< k …….(4)
  - Given that assumption, prove that (3) holds for n=k
    - i.e., prove that $T(k) = 2^k – 1$
    - From (2) → T(k) = 2 T(k-1) + 1
    - $T(k) = 2 (2^{k-1} - 1) + 1$ (Using the inductive hypothesis (4))
    - $T(k) = 2^k – 2 + 1 = 2^k – 1$
    - End of proof!

Recursion Applications

- So, what is recursion good for?
  - For some problems, a recursive approach is more natural and simpler to understand than an iterative approach
    - e.g., traversing a linked list backwards
    - e.g., reversing a linked list

Reversing a Linked List Recursively

- You have seen in recitations how to reverse a linked list iteratively
  - pretty much involved
- Let's try a recursive solution!
- Imagine you have a friend who can reverse a linked list
  - what should friend return to us?
  - first node? last node? nothing?
  - last node! we can attach the first node after that last node
  - We must return what we expect friend to return
    - That friend is us after all!
- You can use your friend's help
- But …
- You can't ask your friend to reverse the same list given to you
  - has to ask for help on a "smaller" list

Reversing a Linked List Recursively

- Plan of work:
  - Keep track of the first node
  - Ask friend to reverse the rest of the list …
    - i.e., starting from second node
  - … and return the last node in the reversed list
  - Attach the first node after the node returned by friend
  - return the first node, which is now the last node
  - Non-recursive cases
    - reversing an empty list → return null
    - reversing a single node → make it the firstNode
  - Let's code!

Recursion Applications

- So, what is recursion good for?
  - For some problems, a recursive approach is more natural and simpler to understand than an iterative approach
  - For some problems, it is very difficult to even conceive an iterative approach, especially if multiple recursive calls are required in the recursive solution
    - e.g., Towers of Hanoi
    - e.g., Eight Queens

8 Queens Problem

- How can I place 8 queens on a chessboard such that no queen can take any other in the next move?
  - Recall that queens can move horizontally, vertically or diagonally for multiple spaces

Backtracking to solve 8 Queens

- Idea of backtracking:
  - Proceed **forward** to a solution until it becomes apparent that no solution can be achieved along the current path
    - At that point UNDO the solution (backtrack) to a point where we can again proceed forward

8 Queens Problem

- How can we solve this with recursion and backtracking?
  - We note that all queens must be in different rows and different columns, so each row and each column must have exactly one queen when we are finished
    - Complicating it a bit is the fact that queens can move diagonally
  - So, thinking recursively, we see the following
    - To place 8 queens on the board we need to
      - Place a queen in a legal (row, column)
      - Recursively place 7 queens on the rest of the board
  - Where does backtracking come in?
    - Our initial choices may not lead to a solution – we need a way to undo a choice and try another one

8 Queens Problem

- Using this approach, we come up with the solution as shown in BackTracking.java in the code handouts
- Idea of solution:
  - Each recursive call attempts to place a queen in a specific column
    - A loop is used, since there are 8 squares in the column
  - For a given call, the state of the board from previous placements is known (i.e., where are the other queens placed so far?)
    - This is used to determine if a square is legal or not
  - If a placement within the column does not lead to a solution, the queen is removed and moved "down" the column
  - When all rows in a column have been tried, the call terminates and backtracks to the previous call (in the previous column)
  - If a queen cannot be placed into column i, do not even try to place one onto column i+1 – rather, backtrack to column i-1 and move the queen that had been placed there

8 Queens Problem

- Why is this difficult to do iteratively?
  - We need to store a lot of state information as we try (and un-try) many locations on the board
    - For each column so far, where has a queen been placed?
  - The run-time stack does this automatically for us via activation records
    - Without recursion, we would need to store / update this information ourselves
    - This can be done (using our own Stack rather than the run-time stack), but since the mechanism is already built into recursive programming, why not utilize it?
  - There are many other famous backtracking problems
    - http://en.wikipedia.org/wiki/Backtracking
    - PIN Cracking
    - Sudoku

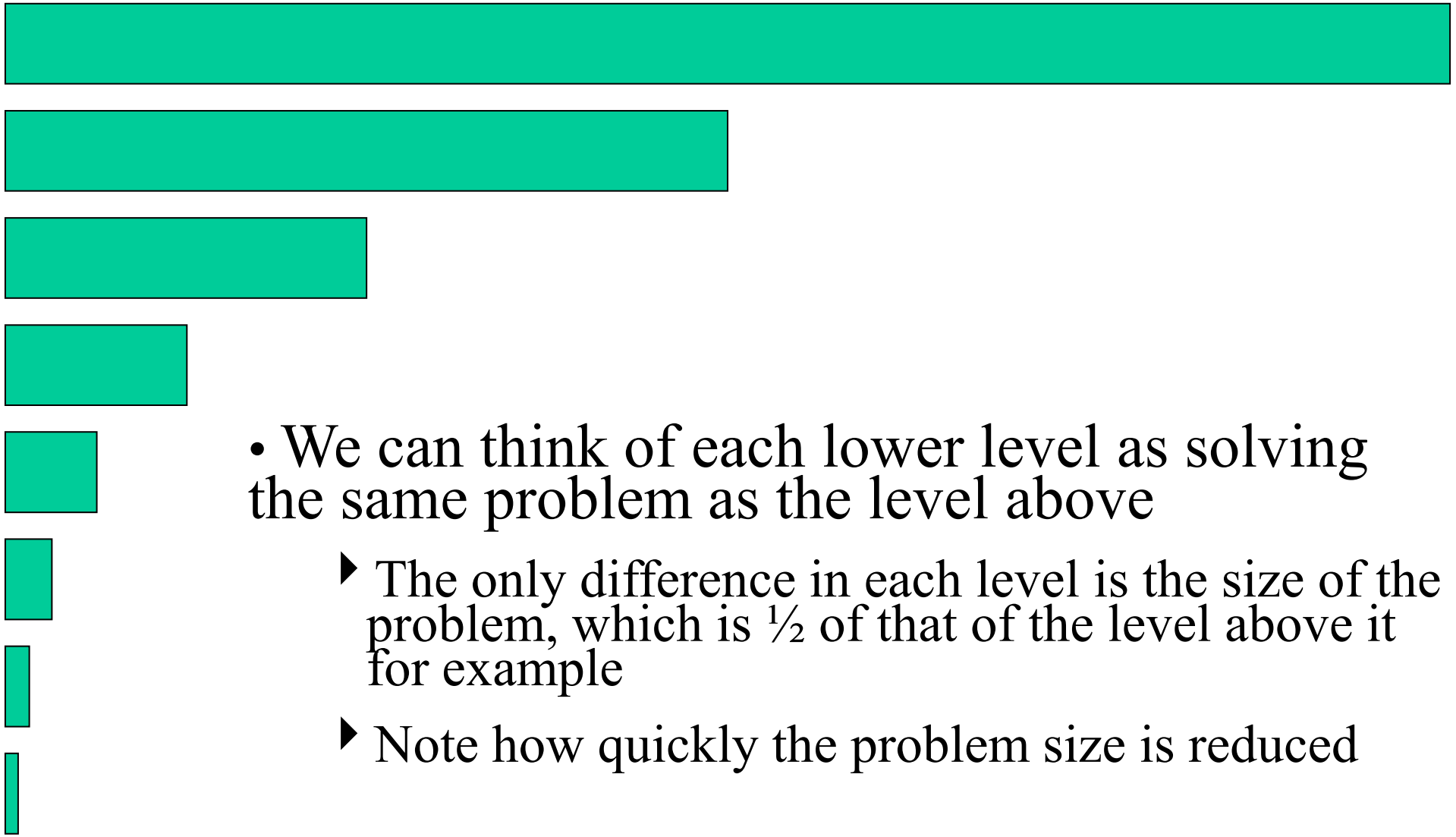
Recursion Applications

- So, what is recursion good for?
  - For some problems, a recursive approach is more natural and simpler to understand than an iterative approach
  - For some problems, it is very difficult to even conceive an iterative approach, especially if multiple recursive calls are required in the recursive solution
  - For some problems, a recursive approach allows for a more efficient solution than an iterative approach
    - e.g., exponentiation x^n
    - general approach is: Divide and Conquer
      - more examples later in the course

Recursion and Divide and Conquer

- Divide and Conquer
  - The idea is that a problem can be solved by breaking it down to one or more "smaller" problems in a systematic way
    - Usually the subproblem(s) are a fraction of the size of the original problem
    - Usually the subproblems(s) are identical in nature to the original problem
    - It is fairly clear why these algorithms can typically be solved quite nicely using recursion

Recursion and Divide and Conquer



- We can think of each lower level as solving the same problem as the level above
  - The only difference in each level is the size of the problem, which is ½ of that of the level above it for example
  - Note how quickly the problem size is reduced

Limitations of Recursion

- Recursion Overhead
- Recursion may lead a poor solution that an iterative approach

Overhead of Recursion

- Why do we care?
  - Recursive algorithms have overhead associated with them
    - Space: each activation record (AR) takes up memory in the run-time stack (RTS)
      - If too many calls "stack up" memory can be a problem
    - Time: generating ARs and manipulating the RTS takes time
      - A recursive algorithm will always run more slowly than an equivalent iterative version
  - Once a recursive algorithm is developed, in some cases we can convert it into a faster iterative version

Tail Recursion

- Tail recursion
  - Recursive algorithm in which the recursive call is the LAST statement of the method
  - No further processing after the return of the recursive call
- What are the implications of tail recursion?
  - Any tail recursive algorithm can be converted into an iterative algorithm in a methodical way
    - some compilers do this automatically
  - Some algorithms we have seen so far are tail recursive
    - make sure that the recursive call is inside an if statement
    - change the if statement to a while loop
    - replace the recursive call by a set of assignment statements in the form
      - parameter = argument

Example

- The recursive call is inside the if statement

```
public static void countDown(int integer)
{
    if (integer >= 1)
    {
        System.out.println(integer);
        countDown(integer - 1);
    } // end if
} // end countDown
```

Example

- Convert if to while

```
public static void countDown(int integer)
{
    if (integer >= 1)
    {
        System.out.println(integer);
        countDown(integer - 1);
    } // end if
} // end countDown
```

Example

- replace recursive call by assignment statement
  - parameter = argument
  - integer = integer -1

```
public static void countDown(int integer)
{
    if (integer >= 1)
    {
        System.out.println(integer);
        countDown(integer - 1);
    } // end if
} // end countDown
```

while

integer = integer - 1

Parameter

argument

Limitations of Recursion

- Recursion Overhead
- Recursion may lead a poor solution that an iterative approach

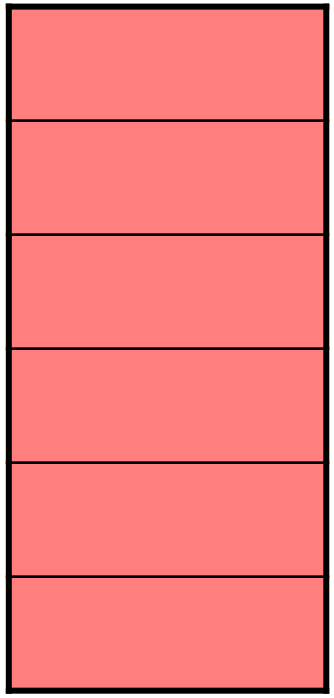
Poor Solution to a Simple Problem

- Algorithm to generate Fibonacci numbers.
- Why is this inefficient?

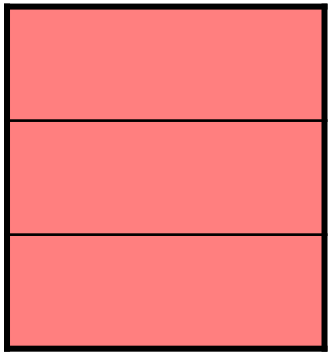
```
Algorithm Fibonacci(n)
if (n <= 1)
    return 1
else
    return Fibonacci(n - 1) + Fibonacci(n - 2)
```

Single recursion

- A recursive algorithm with a single recursive call still provides a linear chain of calls



Calls build run-time stack



Stack shrinks as calls finish

Poor Solution to a Simple Problem

- The computation of the Fibonacci number F_6 using recursion