University of Pittsburgh

Algorithms and Data Structures 1
CS 0445

Pitt SCI

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Sherif Khattab

ksm73@pitt.edu

(Slides are adapted from Dr. Ramirez's and Dr. Farnan's CS1501 slides.)

Announcements

- Upcoming Deadlines:
  - Homework 6: this Friday @ 11:59 pm
  - Lab 5: next Monday @ 11:59 pm
  - Programming Assignment 1: Late Deadline: Wednesday Oct. 12th
    - Autograder feedback
- Debugging hints
- If you think you lost points in a lab assignment because of the autograder or because of a simple mistake
  - please reach out to Grader TA over Piazza
- **Student Support Hours** of the teaching team are posted on the Syllabus page

Previous Lecture …

- ADT Stack
  - Application: Building a simple parser of Algebraic expressions
  - Application: Runtime stack
- Recursion
  - Definition
  - Basic examples

Today …

- Recursion
  - More examples
  - Runtime analysis using proof by induction
- Using recursion to solve hard problems
  - Towers of Hanoi

Recursively Processing a Linked Chain

- Display data in first node
- Then, (recursively) display data in rest of chain

```
public void display()
{
   displayChain(firstNode);
} // end display
private void displayChain(Node nodeOne)
{
```

Recursively Processing a Linked Chain

- Display data in first node
- Then, (recursively) display data in rest of chain

```
public void display()
{
   displayChain(firstNode);
} // end display
private void displayChain(Node nodeOne)
{
   if (nodeOne != null)
   {
      System.out.println(nodeOne.getData()); // Display first node
```

Recursively Processing a Linked Chain

- Display data in first node
- Then, (recursively) display data in rest of chain

```
public void display()
{
   displayChain(firstNode);
} // end display
private void displayChain(Node nodeOne)
{
   if (nodeOne != null)
   {
      System.out.println(nodeOne.getData()); // Display first node
      displayChain(nodeOne.getNextNode());   // Display rest of chain
   } // end if
} // end displayChain
```

Traversing a linked chain *backwards*

Traversing chain of linked nodes in reverse order easier when done recursively.

```
public void displayBackward()
{
   displayChainBackward(firstNode);
} // end displayBackward

private void displayChainBackward(Node nodeOne)
{
```

Traversing a linked chain *backwards*

- (recursively) display data in rest of chain

```
public void displayBackward()
{
   displayChainBackward(firstNode);
} // end displayBackward

private void displayChainBackward(Node nodeOne)
{
   if (nodeOne != null)
   {
      displayChainBackward(nodeOne.getNextNode());
```

Traversing a linked chain *backwards*

- (recursively) display data in rest of chain
- Then, display data in first node

```
public void displayBackward()
{
   displayChainBackward(firstNode);
} // end displayBackward

private void displayChainBackward(Node nodeOne)
{
   if (nodeOne != null)
   {
      displayChainBackward(nodeOne.getNextNode());
      System.out.println(nodeOne.getData());
   } // end if
} // end displayChainBackward
```

Running Time Analysis of Recursive Algorithms

- Technique #1: Using **proof by induction**
- Assume running time of countDown is a function of *n*: $T(n)$
- $T(n) = 1 + 1 + ??$
  - What is the running time of countdown(n-1)?
  - Can we use the function $T(n)$?
  - Yes! The running time of *countdown(n-1)* is $T(n-1)$
- $T(n) = 2 + T(n-1)$
- $\quad = T(n-1) + O(1)$
- $T(1) = O(1)$

```java
public static void countDown(int n)
{
    System.out.println(n);
    if (n > 1)
        countDown(n - 1);
} // end countDown
```

Running Time Analysis of Recursive Algorithms

- $T(1) = 1$ … (1)
- $T(n) = T(n-1) + 1$ for $n>1$ … (2)
  - The above equation is called a <u>*Recurrence Relation*</u>
- T(2) = T(1) + 1 = 2
- T(3) = T(2) + 1 = 2 + 1 = 3
- T(4) = T(3) + 1 = 3 + 1 = 4
- …
- We have an intuition that the running time is linear
  - $T(n) = n$ … (3)
  - Let’s prove (3) by induction
- <u>Base Case:</u>
  - From (1): T(1) = 1
  - From (3): T(1) = 1
  - (3) applies to the base case

Running Time Analysis of Recursive Algorithms

- <u>Inductive Step:</u>
  - Assume that (3) is true for *all values* < *k* and prove that it is true for *k*
  - Inductive hypothesis: $T(n) = n$ for all $n<k$ … (4)
  - We want to prove that $T(k) = k$
  - From (2), $T(k) = T(k-1) + 1$
  - From (4), T(k) = (k-1) + 1 = k
  - **End of Proof that $T(n) = n$**
  - So, running time of countdown is $O(n)$

$$T(1) = 1 \quad \ldots (1)$$

$$T(n) = T(n-1) + 1 \text{ for } n>1 \quad \ldots (2)$$

```
public static void countDown(int n)
{
    System.out.println(n);
    if (n > 1)
        countDown(n - 1);
} // end countDown
```

Computing x^n

- Using iteration

```java
public static int powerIterative(int x, int n){
    assert(n >= 0);
    int result = 1;
    for(int i=0; i<n; i++){
        result *= x;
    }
    return result;
}
```

Computing x^n

- What is the running time of powerIterative?
  - *O(n)*

```java
public static int powerIterative(int x, int n){
  assert(n >= 0);
  int result = 1;
  for(int i=0; i<n; i++){
    result *= x;
  }
  return result;
}
```

Computing x^n

- Using recursion
- How?
  - Let's start with recursive mathematical definition
- $x^n = (x^{n/2})^2$
  - when *n* is even and positive
- $x^n = x(x^{n/2})^2$
  - when *n* is odd and positive
  - *n/2* is integer division
- base case or non-recursive case
  - $x^0 = 1$

Computing x^n

- Using recursion

```
public static int power(int x, int n){
  int result = 1;
  if(n > 0){
    int temp = power(x, n/2);
```

Computing x^n

- Using recursion

```
public static int power(int x, int n){
    int result = 1;
    if(n > 0){
        int temp = power(x, n/2);
        result = temp * temp;
```

Computing x^n

- Using recursion

```
public static int power(int x, int n){
  int result = 1;
  if(n > 0){
    int temp = power(x, n/2);
    result = temp * temp;
    if(n%2 == 1){ //is n odd?
      result = x * result;
    }
```

Computing x^n

- Using recursion

```
public static int power(int x, int n){
    int result = 1;
    if(n > 0){
        int temp = power(x, n/2);
        result = temp * temp;
        if(n%2 == 1){ //is n odd?
            result = x * result;
        }
    }
    return result;
}
```

Running Time Analysis of Recursive Algorithms

- Technique #1: Using **proof by induction**
- Assume running time of recursive power is a function of *n*: $T(n)$
- $T(n) = O(1) + ??$
  - What is the running time of power(x, n/2)?
  - Can we use the function $T(n)$?
  - Yes! The running time of *power(x, n/2)* is $T(n/2)$
- $T(n) = T(n/2) + O(1)$
- $T(1) = O(1)$

```java
public static int power(int x, int n){
  int result = 1;
  if(n > 0){
    int temp = power(x, n/2);
    result = temp * temp;
    if(n%2 == 1){ //is n odd?
      result = x * result;
    }
  }
  return result;
}
```

Running Time Analysis of Recursive Algorithms

- $T(1) = 1$ … (1)
- $T(n) = T(n/2) + 1$ for $n>1$ … (2)
  - The above equation is called a ***<u>Recurrence Relation</u>***
- T(2) = T(1) + 1 = 2
- T(4) = T(2) + 1 = 2 + 1 = 3
- T(8) = T(4) + 1 = 3 + 1 = 4
- T(16) = T(8) + 1 = 4 + 1 = 5
- When n doubles → T(n) increases by 1
- We have an intuition that the running time is logarithmic
  - $T(n) = log(n) + 1$ … (3)
  - Let's prove (3) by induction
- <u>Base Case:</u>
  - From (1): T(1) = 1
  - From (3): T(1) = log(1) + 1 = 0 + 1 = 1
  - (3) applies to the base case

Running Time Analysis of Recursive Algorithms

- <u>Inductive Step:</u>
  - Assume that (3) is true for all n < *k* and prove that it is true for *k*
  - Inductive hypothesis: $T(n) = log(n) + 1$ for all $n<k$ … (4)
  - We want to prove that $T(k) = log(k) + 1$
  - From (2), $T(k) = T(k/2) + 1$
  - From (4), T(k/2) = $log(k/2) + 1$
  - Then, T(k) = $log(k/2) + 1 + 1$
  - $= log(k/2) + 2$
  - $= log(k/2) + log\ 4$
  - $= log(4k/2) = log(2k)$
  - $= log\ k + log\ 2$
  - $= log\ k + 1$
  - **End of Proof that $T(n) = log(n) + 1$**
  - So, running time of recursive power is $O(log\ n)$

$T(1) = 1$ … (1)

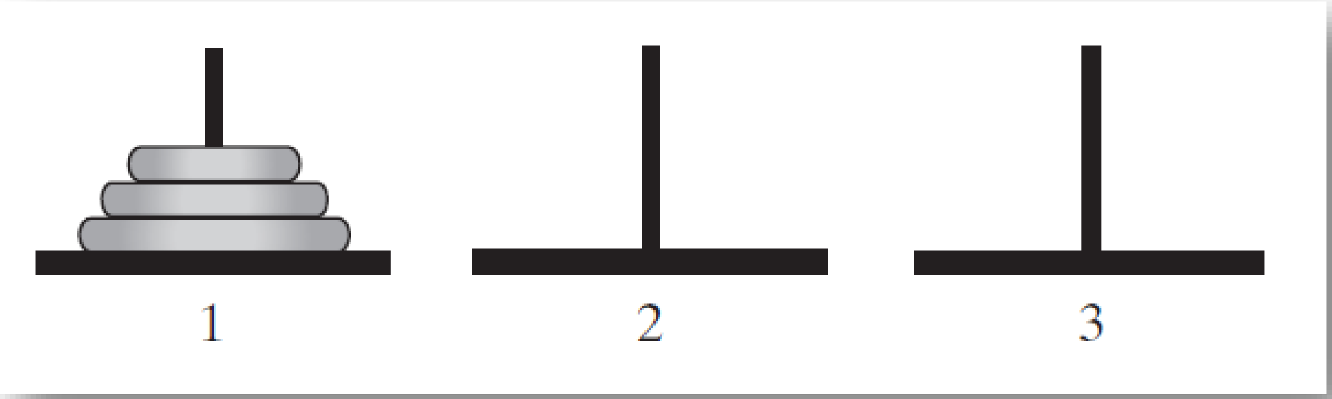
$T(n) = T(n/2) + 1$ for $n>1$ … (2)

Note on input size

- Our goal is to model running time in terms of input size
- The input size is the number of bits needed to represent the input
- For the power function, the exponent n is represented using how many bits?
  - *log n* bits
  - So, the input size of the exponentiation problem is not *n*, the exponent value
  - The input size is *log n*
- So, the recursive power function has linear running time
  - *O(log n)* is linear in *log n*, the input size

Simple Solution to a Difficult Problem

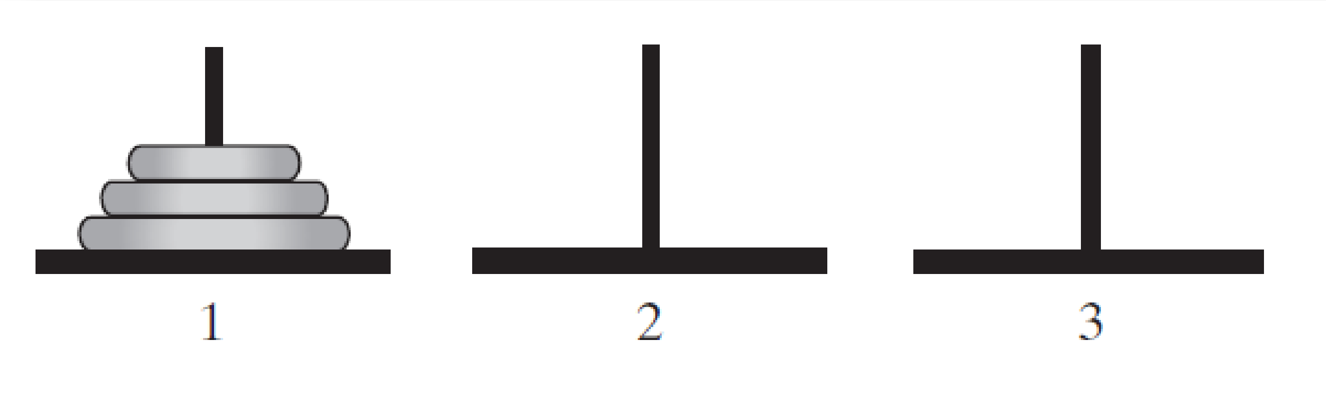
The initial configuration of the
Towers of Hanoi for three disks.



Towers of Hanoi Problem

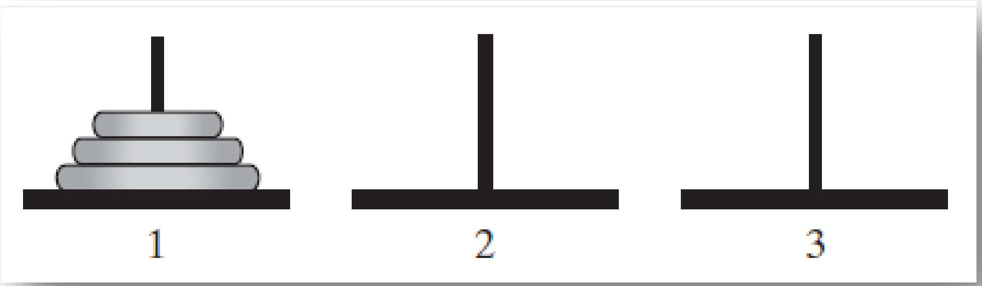
Rules:

1. Move one disk at a time.
2. Disk moved must be topmost disk in its pole
3. No disk may rest on top of a disk smaller than itself
4. You can store disks on the second pole temporarily



Solutions

The sequence of moves for solving the Towers of Hanoi problem with three disks



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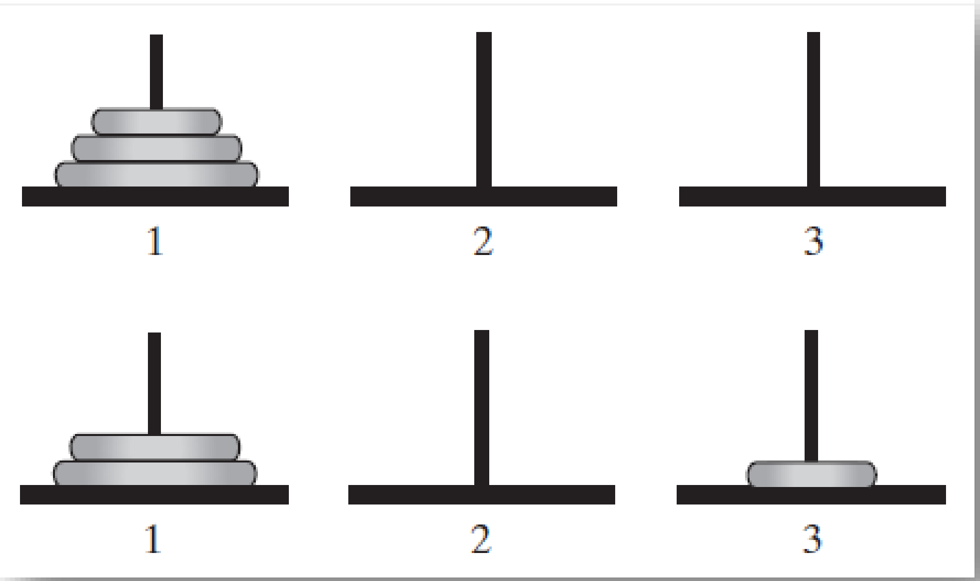

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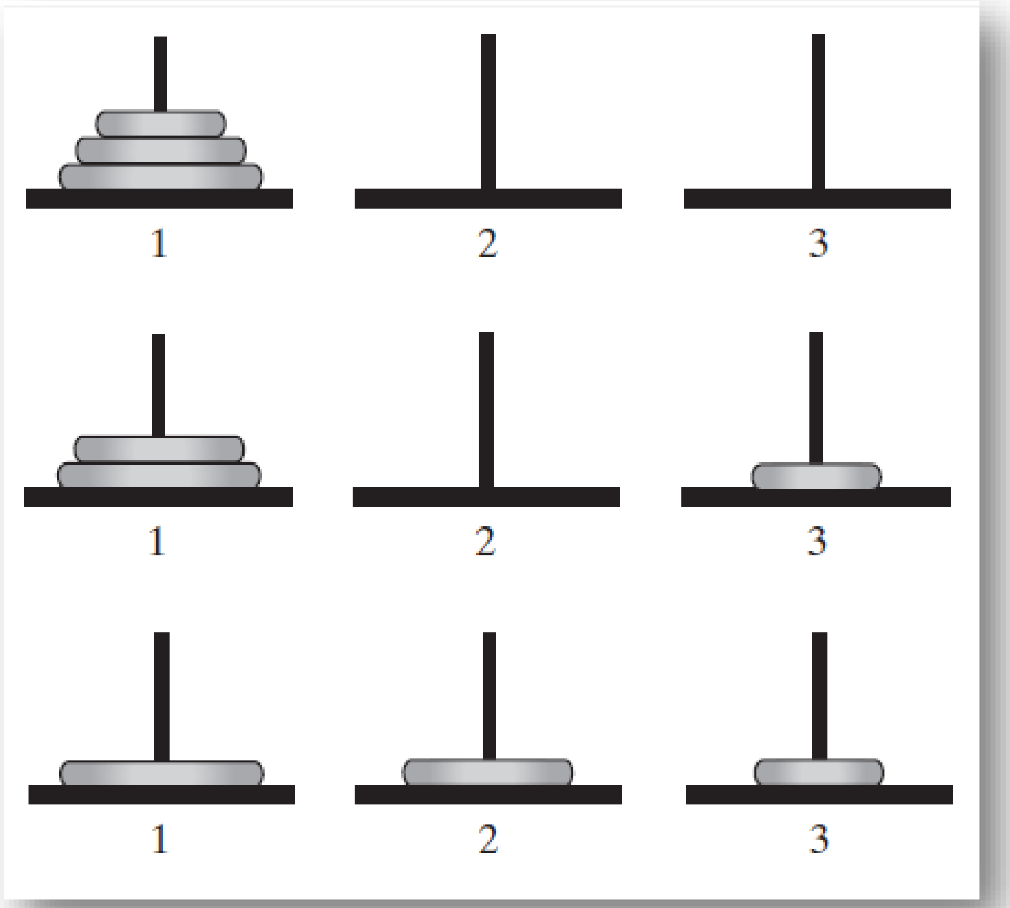

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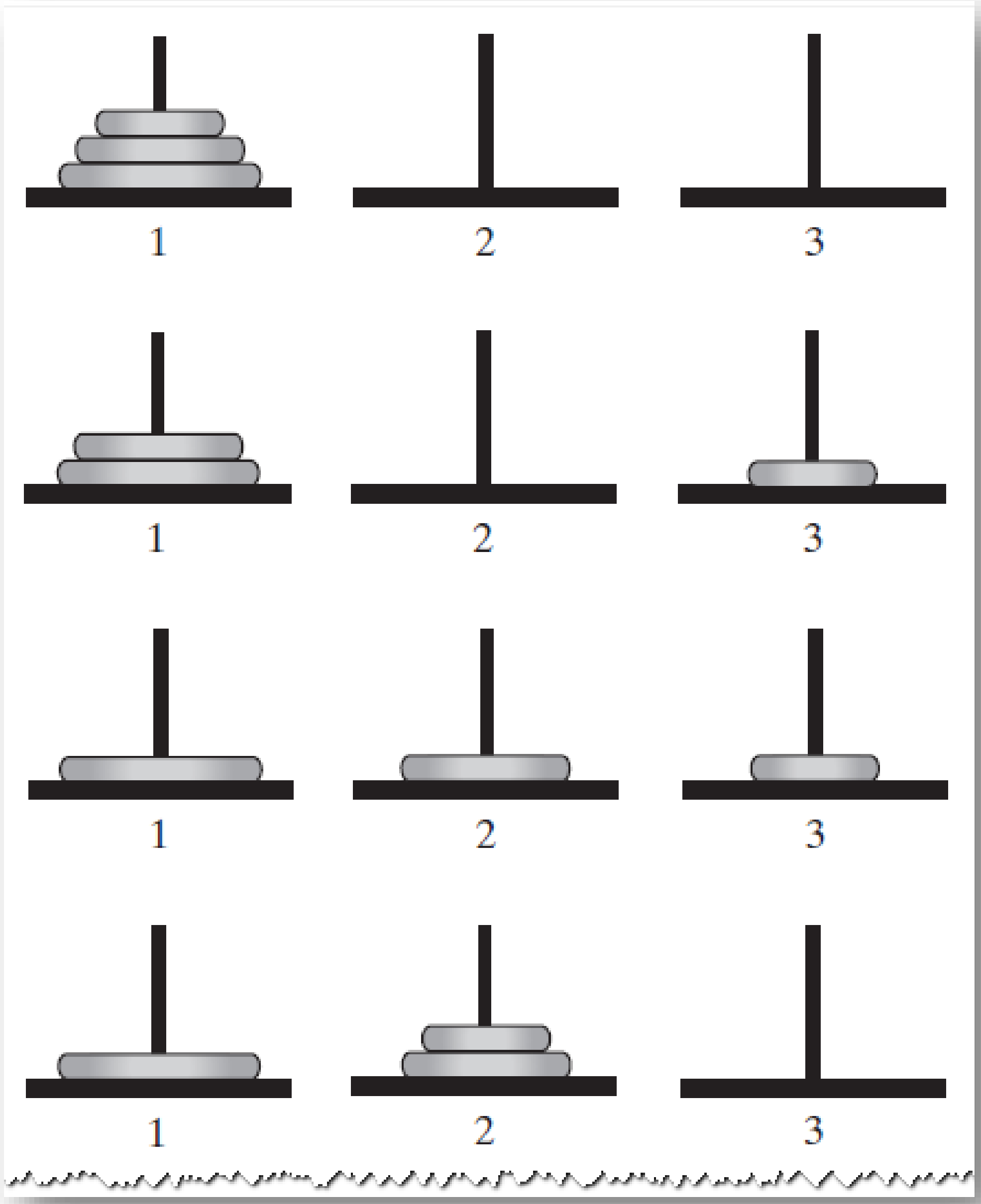

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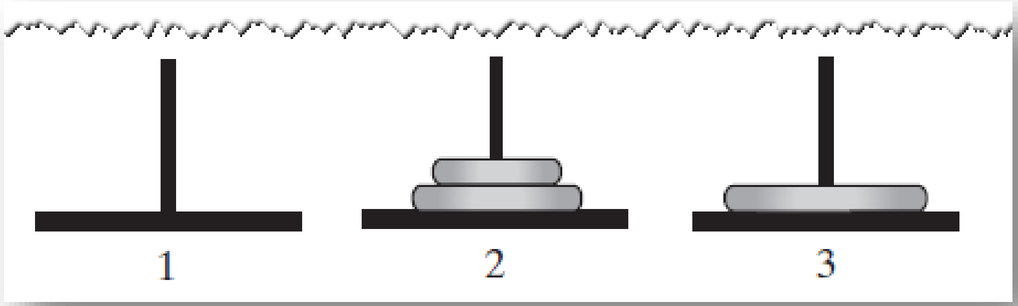

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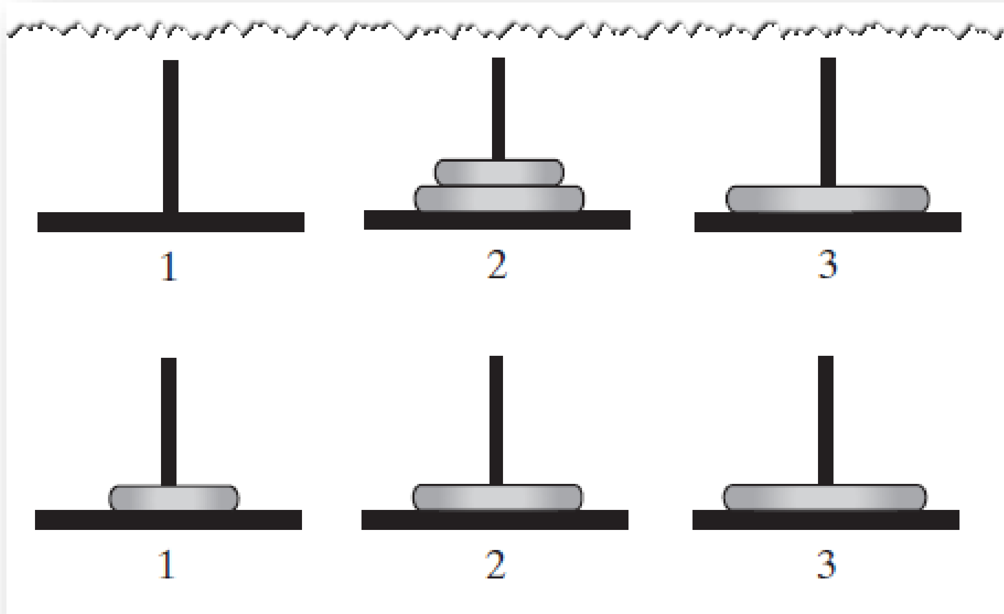

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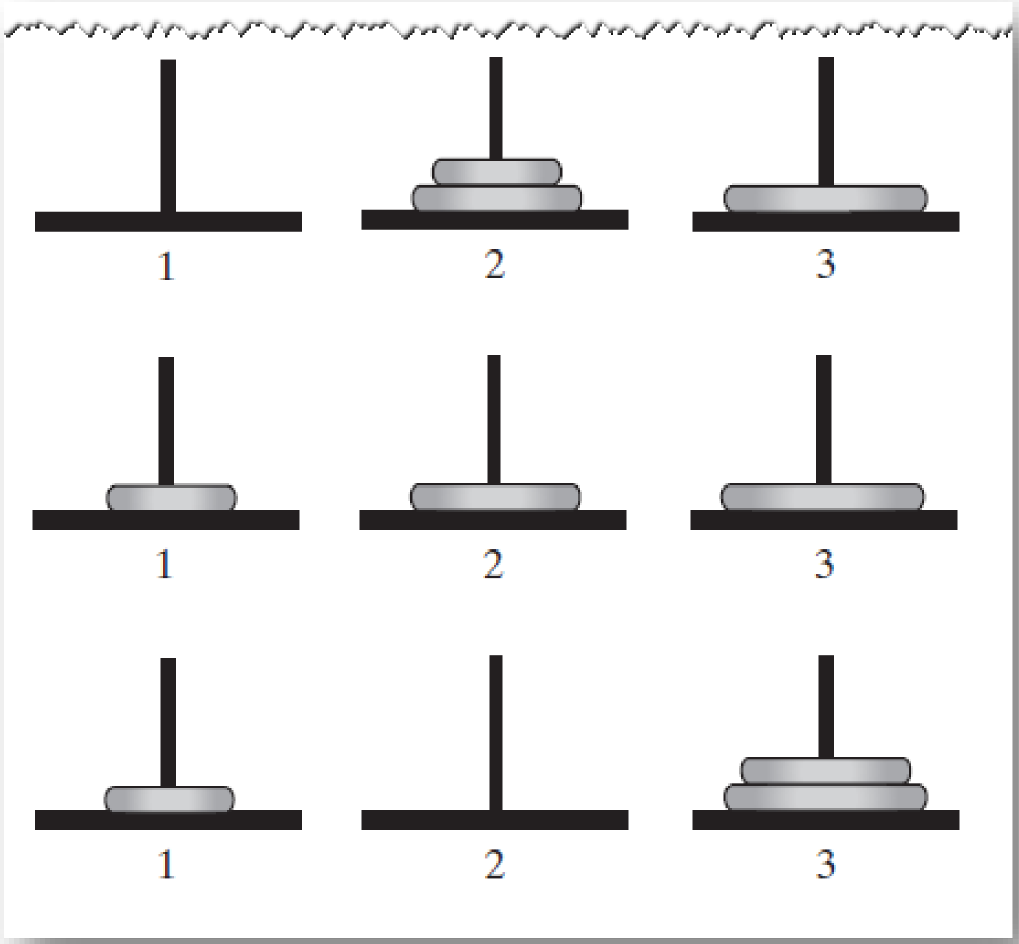

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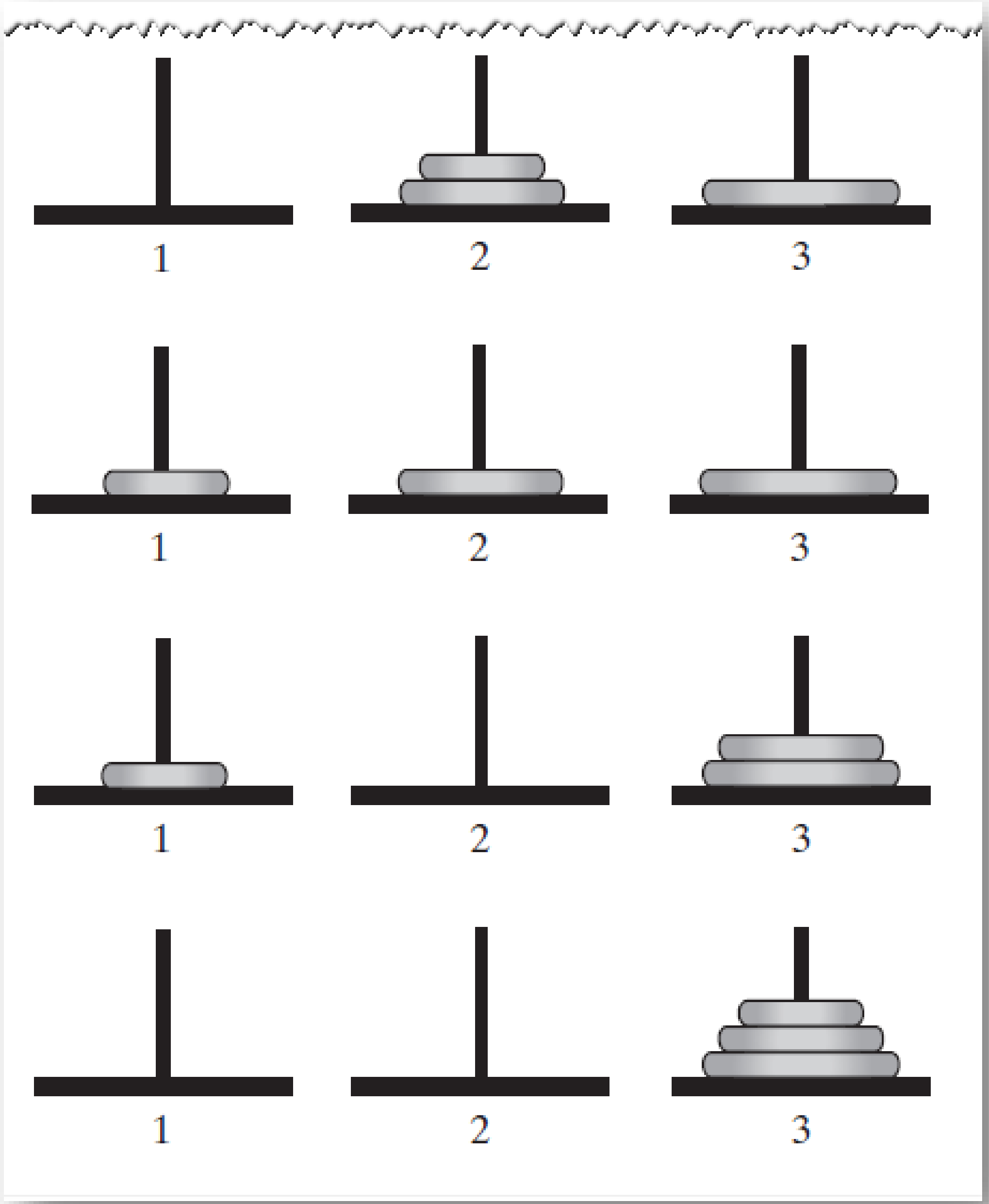
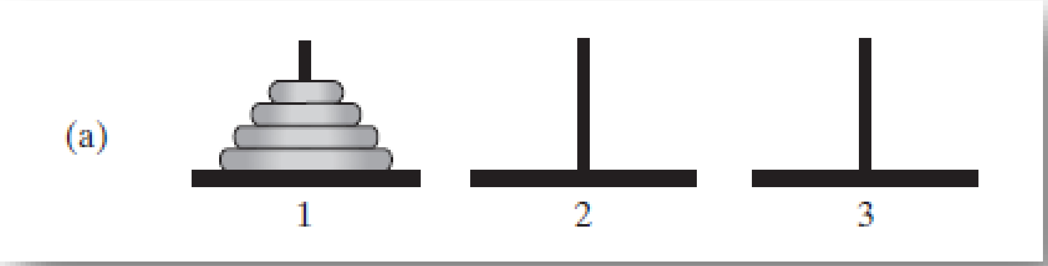


Solutions

The smaller problems in a recursive solution for four disks



Solutions

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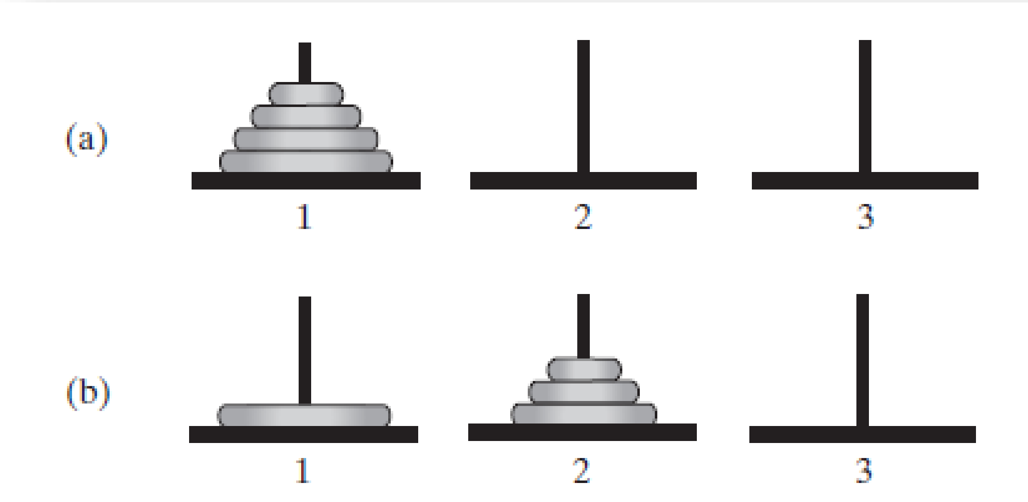

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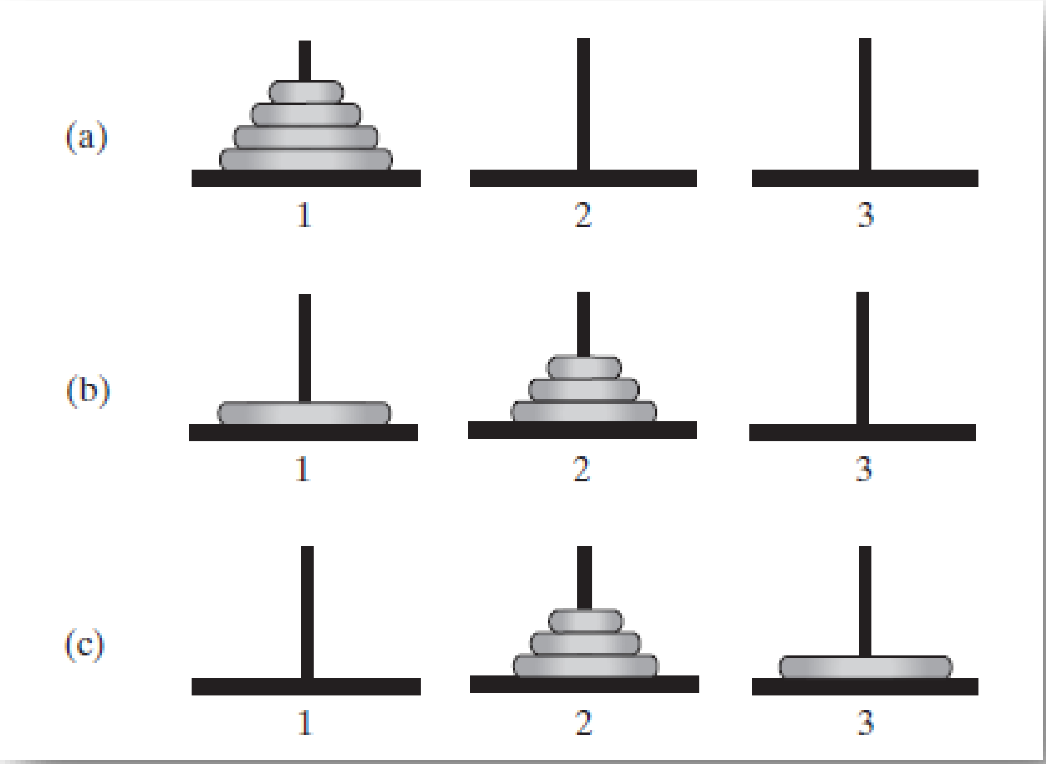
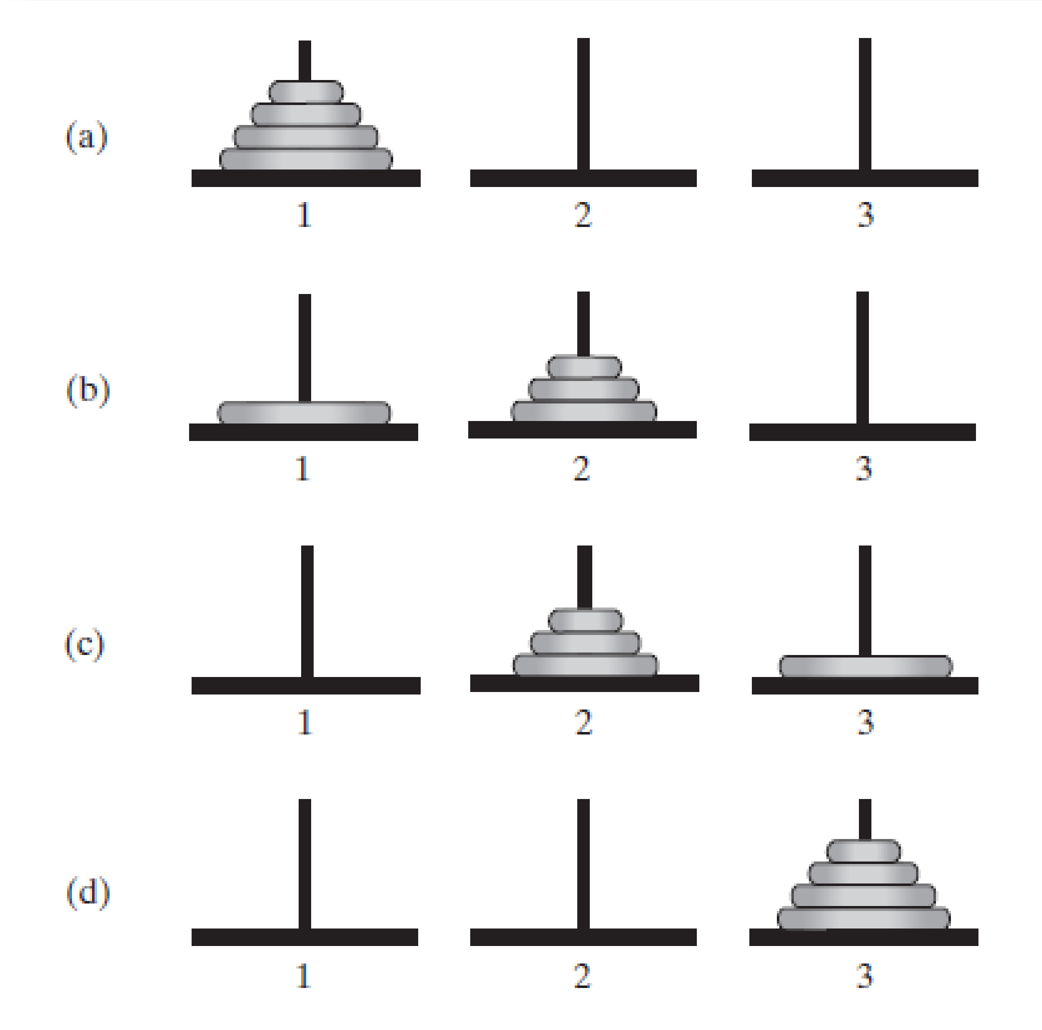


Solutions

The smaller problems in a recursive solution for four disks



Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
```

Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
    Move disk from startPole to endPole
```

Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
    Move disk from startPole to endPole
else
{
    solveTowers(numberOfDisks - 1, startPole, endPole, tempPole)
```

Solutions

- Recursive algorithm to solve any number of disks.

Algorithm `solveTowers(numberOfDisks, startPole, tempPole, endPole)`
```
if (numberOfDisks == 1)
    Move disk from startPole to endPole
else
{
    solveTowers(numberOfDisks - 1, startPole, endPole, tempPole)
    Move disk from startPole to endPole
```

Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
    Move disk from startPole to endPole
else
{
    solveTowers(numberOfDisks - 1, startPole, endPole, tempPole)
    Move disk from startPole to endPole
    solveTowers(numberOfDisks - 1, tempPole, startPole, endPole)
}
```

Exercise

- Prove that the running time of solveTowers is $2^n - 1$
- *Hint: use proof by induction*