Algorithms and Data Structures 1
CS 0445

Fall 2022

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(Slides are adapted from Dr. Ramirez's and Dr. Farnan's CS1501 slides.)

Announcements

- Upcoming Deadlines:
  - Homework 6: this Friday @ 11:59 pm
  - Lab 5: next Monday @ 11:59 pm
  - Programming Assignment 1: Late Deadline: Wednesday Oct. 12th
    - Autograder feedback
- Debugging hints
- If you think you lost points in a lab assignment because of the autograder or because of a simple mistake
  - please reach out to Grader TA over Piazza
- **Student Support Hours** of the teaching team are posted on the Syllabus page

Previous Lecture …

- ADT Stack
  - Application: Building a simple parser of Algebraic expressions
  - Application: Runtime stack
- Recursion
  - Definition
  - Basic examples

Today …

- Recursion
  - More examples
  - Problem solving techniques that use recursion
    - Divide and Conquer
    - Backtracking

Recursively Processing a Linked Chain

- Display data in first node
- Then, (recursively) display data in rest of chain

```
public void display()
{
   displayChain(firstNode);
} // end display
private void displayChain(Node nodeOne)
{
```

Recursively Processing a Linked Chain

- Display data in first node
- Then, (recursively) display data in rest of chain

```
public void display()
{
   displayChain(firstNode);
} // end display
private void displayChain(Node nodeOne)
{
   if (nodeOne != null)
   {
      System.out.println(nodeOne.getData()); // Display first node
```

Recursively Processing a Linked Chain

- Display data in first node
- Then, (recursively) display data in rest of chain

```
public void display()
{
   displayChain(firstNode);
} // end display
private void displayChain(Node nodeOne)
{
   if (nodeOne != null)
   {
      System.out.println(nodeOne.getData()); // Display first node
      displayChain(nodeOne.getNextNode());   // Display rest of chain
   } // end if
} // end displayChain
```

Traversing a linked chain *backwards*

Traversing chain of linked nodes in reverse order easier when done recursively.

```
public void displayBackward()
{
   displayChainBackward(firstNode);
} // end displayBackward

private void displayChainBackward(Node nodeOne)
{
```

Traversing a linked chain *backwards*

- (recursively) display data in rest of chain

```
public void displayBackward()
{
   displayChainBackward(firstNode);
} // end displayBackward

private void displayChainBackward(Node nodeOne)
{
   if (nodeOne != null)
   {
      displayChainBackward(nodeOne.getNextNode());
```

Traversing a linked chain *backwards*

- (recursively) display data in rest of chain
- Then, display data in first node

```
public void displayBackward()
{
   displayChainBackward(firstNode);
} // end displayBackward

private void displayChainBackward(Node nodeOne)
{
   if (nodeOne != null)
   {
      displayChainBackward(nodeOne.getNextNode());
      System.out.println(nodeOne.getData());
   } // end if
} // end displayChainBackward
```

Running Time Analysis of Recursive Algorithms

- Technique #1: Using **proof by induction**
- Assume running time of countDown is a function of *n*: $T(n)$
- $T(n) = 1 + 1 + ??$
  - What is the running time of countdown(n-1)?
  - Can we use the function $T(n)$?
  - Yes! The running time of *countdown(n-1)* is $T(n-1)$
- $T(n) = 2 + T(n-1)$
- $\quad = T(n-1) + O(1)$
- $T(1) = O(1)$

```
public static void countDown(int n)
{
    System.out.println(n);
    if (n > 1)
        countDown(n - 1);
} // end countDown
```

Running Time Analysis of Recursive Algorithms

- $T(1) = 1$ … (1)
- $T(n) = T(n\text{-}1) + 1$ for $n>1$ … (2)
  - The above equation is called a *Recurrence Relation*
- T(2) = T(1) + 1 = 2
- T(3) = T(2) + 1 = 2 + 1 = 3
- T(4) = T(3) + 1 = 3 + 1 = 4
- …
- We have an intuition that the running time is linear
  - $T(n) = n$ … (3)
  - Let's prove (3) by induction
- Base Case:
  - From (1): T(1) = 1
  - From (3): T(1) = 1
  - (3) applies to the base case

Running Time Analysis of Recursive Algorithms

- Inductive Step:
  - Assume that (3) is true for *all values* < k and prove that it is true for k
  - Inductive hypothesis: $T(n) = n$ for all $n<k$ … (4)
  - We want to prove that $T(k) = k$
  - From (2), $T(k) = T(k-1) + 1$
  - From (4), T(k) = (k-1) + 1 = k
  - **End of Proof that $T(n) = n$**
  - So, running time of countdown is $O(n)$

$$\boxed{\begin{array}{ll} T(1) = 1 & \dots (1) \\ T(n) = T(n-1) + 1 \text{ for } n>1 & \dots (2) \end{array}}$$

```java
public static void countDown(int n)
{
    System.out.println(n);
    if (n > 1)
        countDown(n - 1);
} // end countDown
```

Computing x^n

- Using iteration

```
public static int powerIterative(int x, int n){
    assert(n >= 0);
    int result = 1;
    for(int i=0; i<n; i++){
        result *= x;
    }
    return result;
}
```

Computing x^n

- What is the running time of powerIterative?
  - *O(n)*

```
public static int powerIterative(int x, int n){
  assert(n >= 0);
  int result = 1;
  for(int i=0; i<n; i++){
    result *= x;
  }
  return result;
}
```

Computing x^n

- Using recursion
- How?
  - Let's start with recursive mathematical definition
- $x^n = (x^{n/2})^2$
  - when *n* is even and positive
- $x^n = x(x^{n/2})^2$
  - when *n* is odd and positive
  - *n/2* is integer division
- base case or non-recursive case
  - $x^0 = 1$

Computing x^n

- Using recursion

```
public static int power(int x, int n){
  int result = 1;
  if(n > 0){
    int temp = power(x, n/2);
```

Computing x^n

- Using recursion

```
public static int power(int x, int n){
  int result = 1;
  if(n > 0){
    int temp = power(x, n/2);
    result = temp * temp;
```

Computing x^n

- Using recursion

```
public static int power(int x, int n){
    int result = 1;
    if(n > 0){
        int temp = power(x, n/2);
        result = temp * temp;
        if(n%2 == 1){ //is n odd?
            result = x * result;
        }
```

Computing x^n

- Using recursion

```
public static int power(int x, int n){
    int result = 1;
    if(n > 0){
        int temp = power(x, n/2);
        result = temp * temp;
        if(n%2 == 1){ //is n odd?
            result = x * result;
        }
    }
    return result;
}
```

Running Time Analysis of Recursive Algorithms

- Technique #1: Using **proof by induction**
- Assume running time of recursive power is a function of *n*: *T(n)*
- $T(n) = O(1) + ??$
  - What is the running time of power(x, n/2)?
  - Can we use the function *T(n)?*
  - Yes! The running time of *power(x, n/2)* is *T(n/2)*
- $T(n) = T(n/2) + O(1)$
- $T(1) = O(1)$

```
public static int power(int x, int n){
  int result = 1;
  if(n > 0){
    int temp = power(x, n/2);
    result = temp * temp;
    if(n%2 == 1){ //is n odd?
      result = x * result;
    }
  }
  return result;
}
```

Running Time Analysis of Recursive Algorithms

- $T(1) = 1$ … (1)
- $T(n) = T(n/2) + 1$ for $n>1$ … (2)
  - The above equation is called a *Recurrence Relation*
- T(2) = T(1) + 1 = 2
- T(4) = T(2) + 1 = 2 + 1 = 3
- T(8) = T(4) + 1 = 3 + 1 = 4
- T(16) = T(8) + 1 = 4 + 1 = 5
- When n doubles → T(n) increases by 1
- We have an intuition that the running time is logarithmic
  - $T(n) = log(n) + 1$ … (3)
  - Let's prove (3) by induction
- Base Case:
  - From (1): T(1) = 1
  - From (3): T(1) = log(1) + 1 = 0 + 1 = 1
  - (3) applies to the base case

Running Time Analysis of Recursive Algorithms

- Inductive Step:
  - Assume that (3) is true for all n < k and prove that it is true for k
  - Inductive hypothesis: $T(n) = log(n) + 1$ for all $n<k$ … (4)
  - We want to prove that $T(k) = log(k) + 1$
  - From (2), $T(k) = T(k/2) + 1$
  - From (4), T(k/2) = $log(k/2) + 1$
  - Then, T(k) = $log(k/2) + 1 + 1$
  - $= log(k/2) + 2$
  - $= log(k/2) + log\ 4$
  - $= log(4k/2) = log(2k)$
  - $= log\ k + log\ 2$
  - $= log\ k + 1$
  - **End of Proof that $T(n) = log(n) + 1$**
  - So, running time of recursive power is $O(log\ n)$

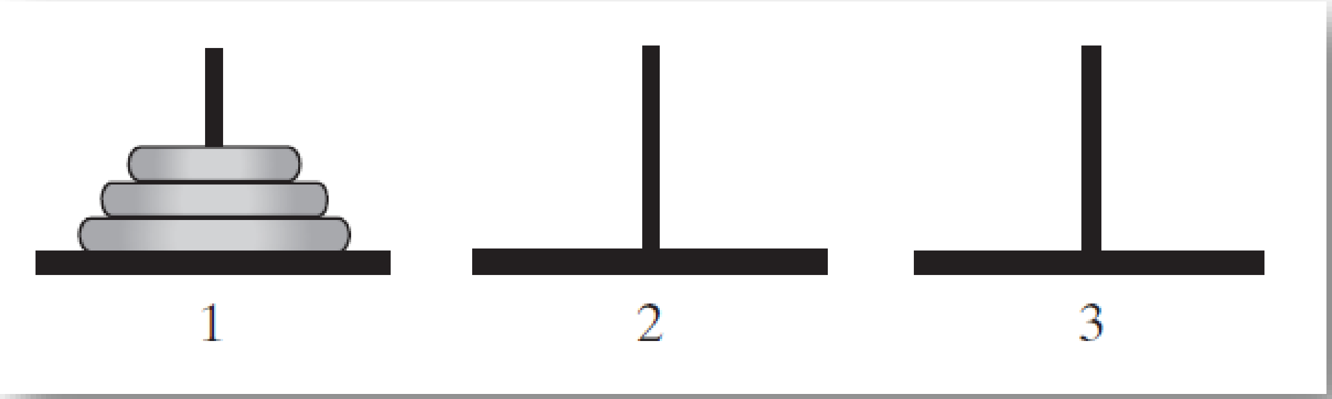
$$\boxed{\begin{array}{ll} T(1) = 1 & \ldots (1) \\ T(n) = T(n/2) + 1 \text{ for } n>1 & \ldots (2) \end{array}}$$

Note on input size

- Our goal is to model running time in terms of input size
- The input size is the number of bits needed to represent the input
- For the power function, the exponent n is represented using how many bits?
  - *log n* bits
  - So, the input size of the exponentiation problem is not *n*, the exponent value
  - The input size is *log n*
- So, the recursive power function has linear running time
  - *O(log n)* is linear in *log n*, the input size

Simple Solution to a Difficult Problem

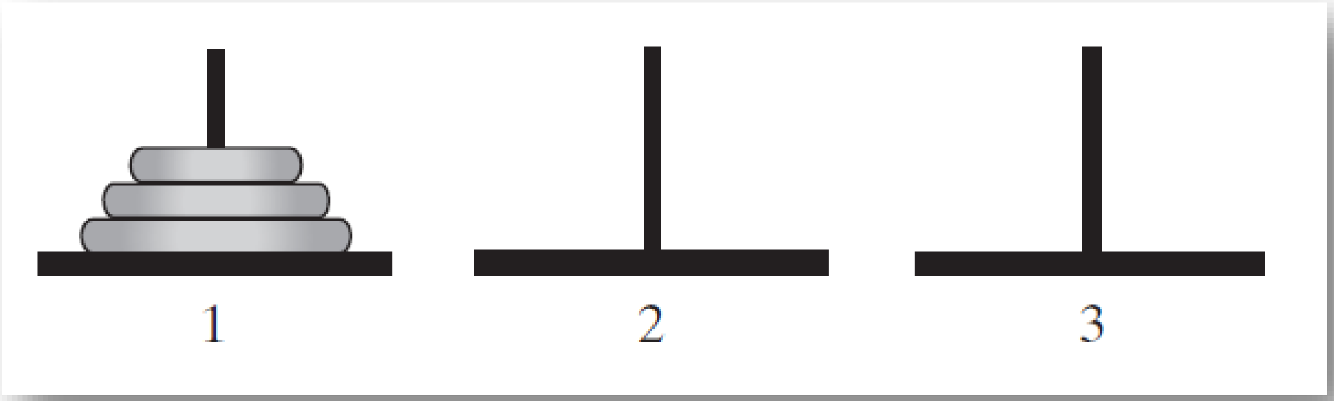
The initial configuration of the
Towers of Hanoi for three disks.



Towers of Hanoi Problem

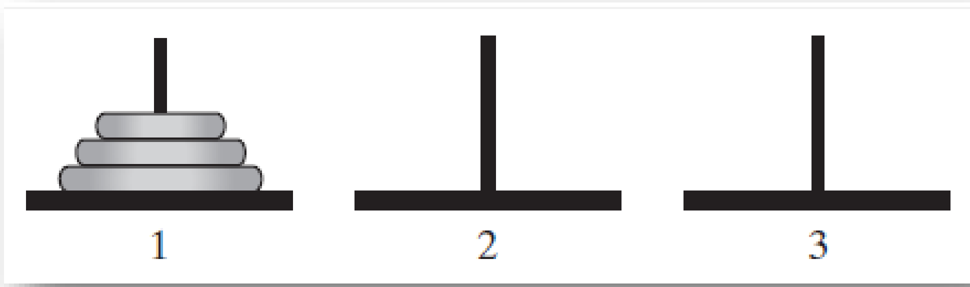
Rules:

1. Move one disk at a time.
2. Disk moved must be topmost disk in its pole
3. No disk may rest on top of a disk smaller than itself
4. You can store disks on the second pole temporarily



Solutions

The sequence of moves for solving the Towers of Hanoi problem with three disks



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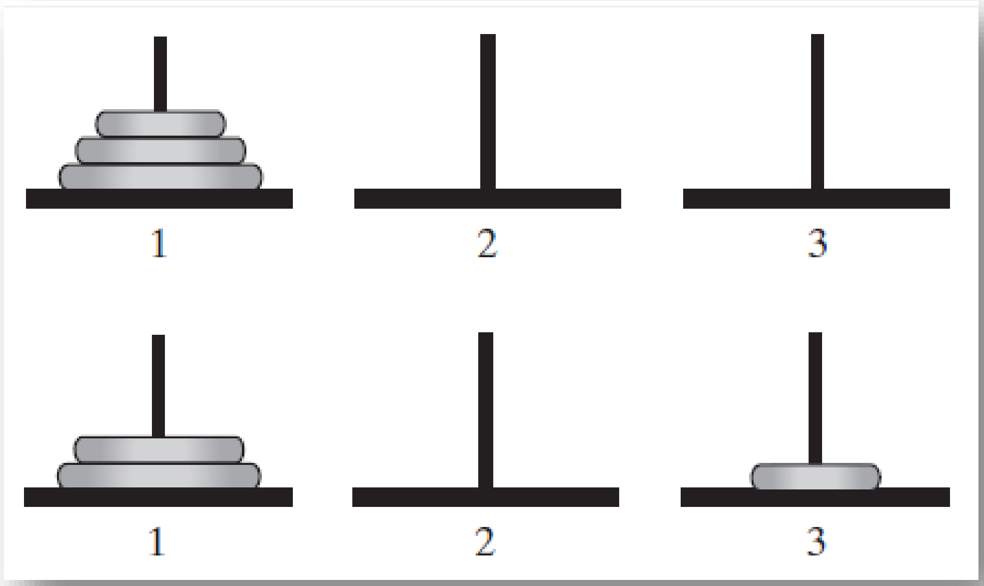

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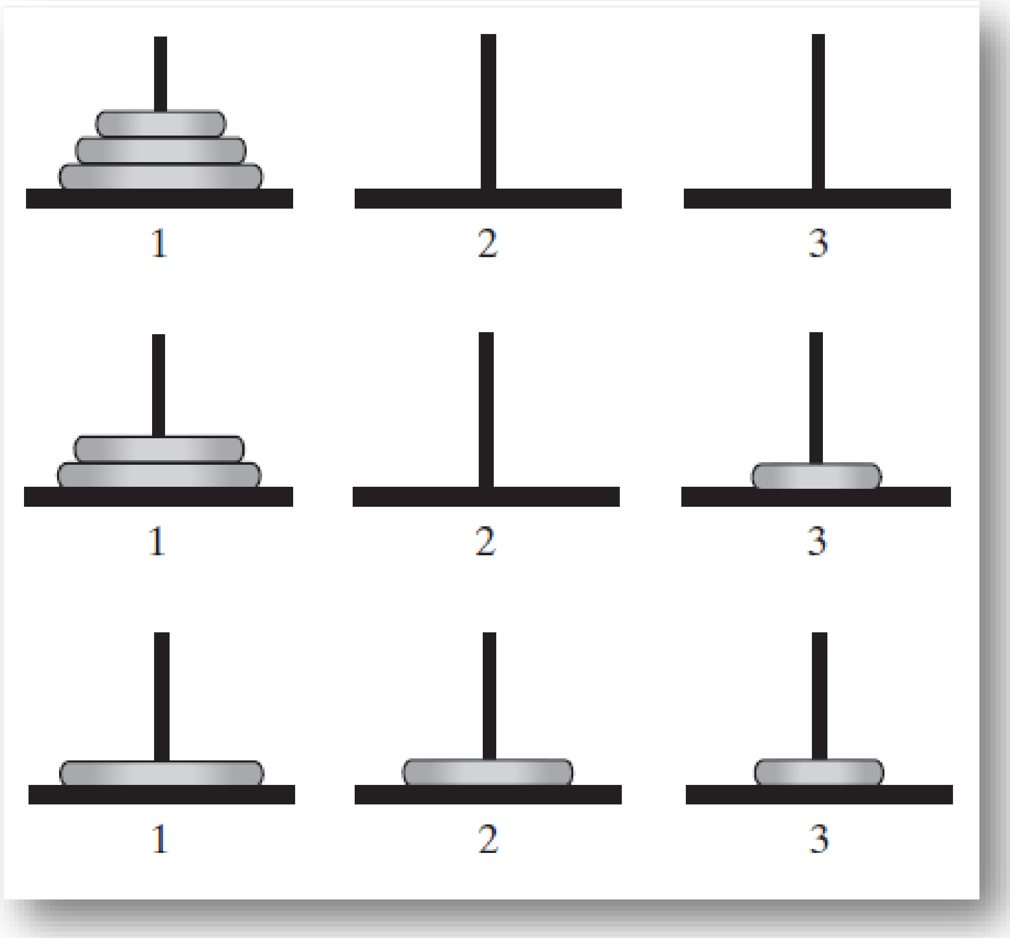

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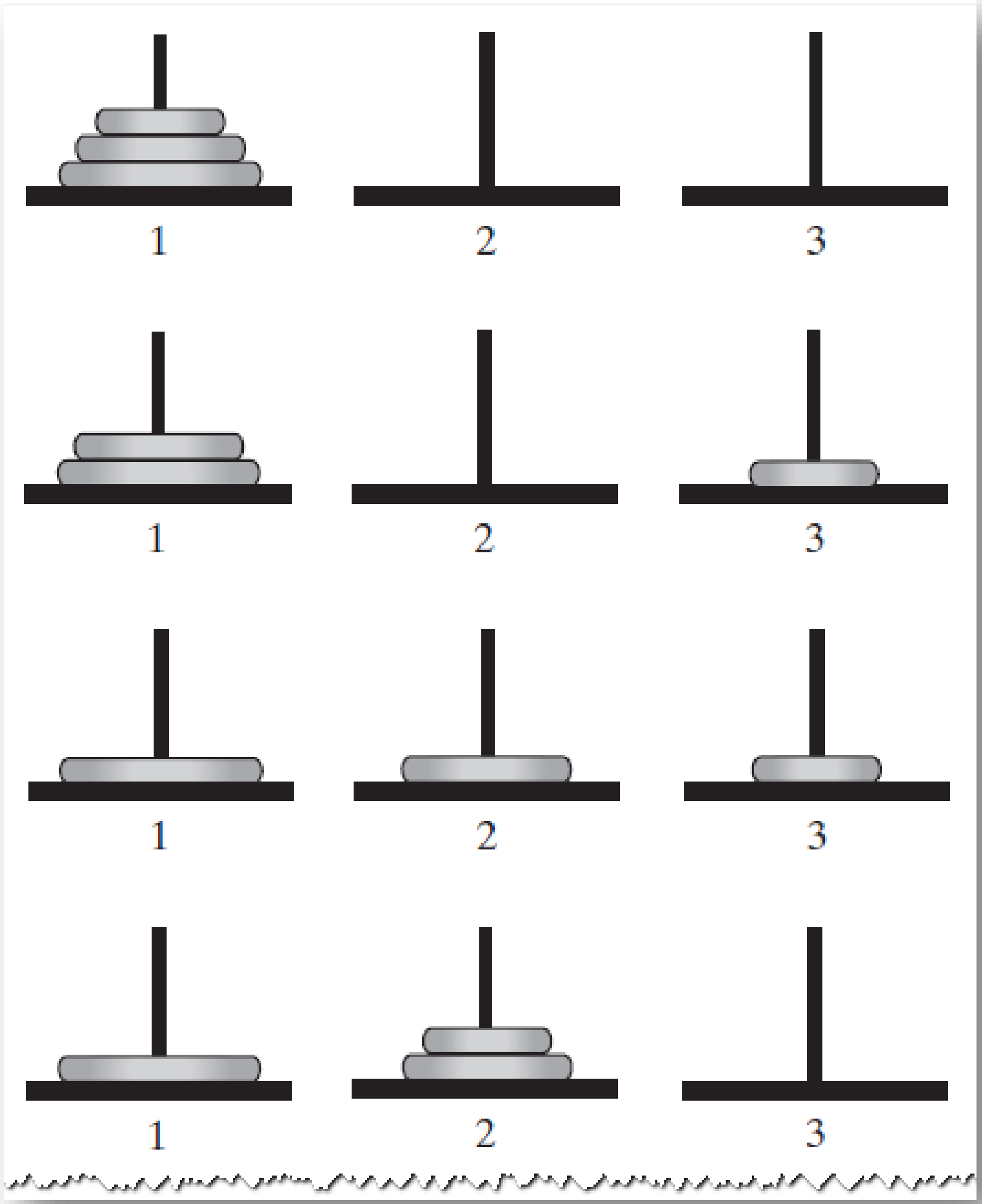

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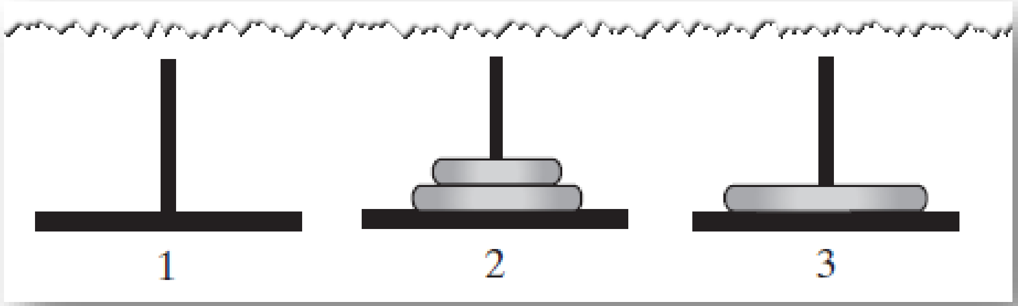

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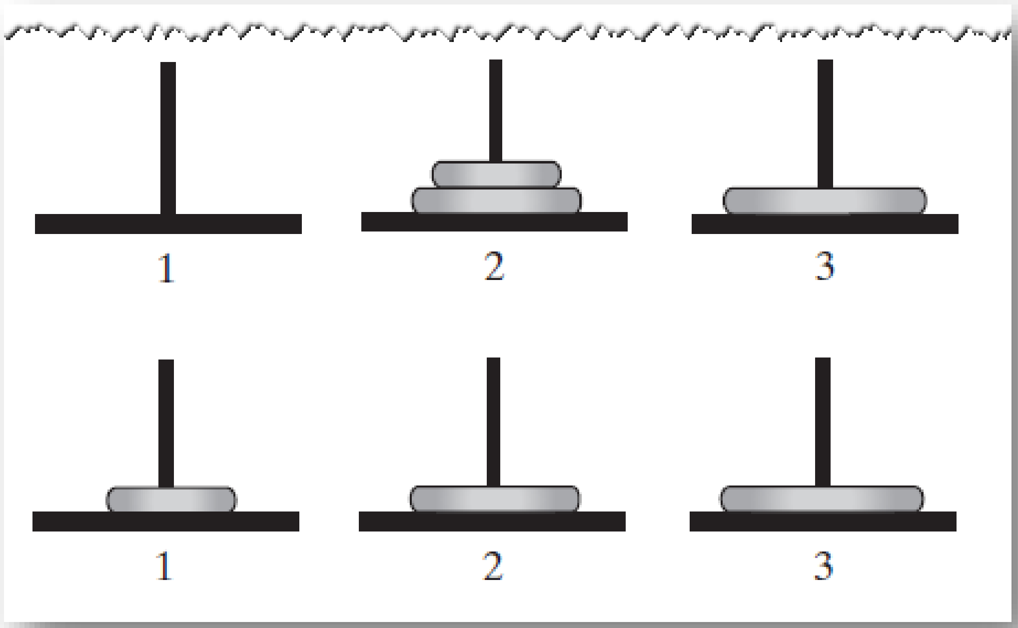

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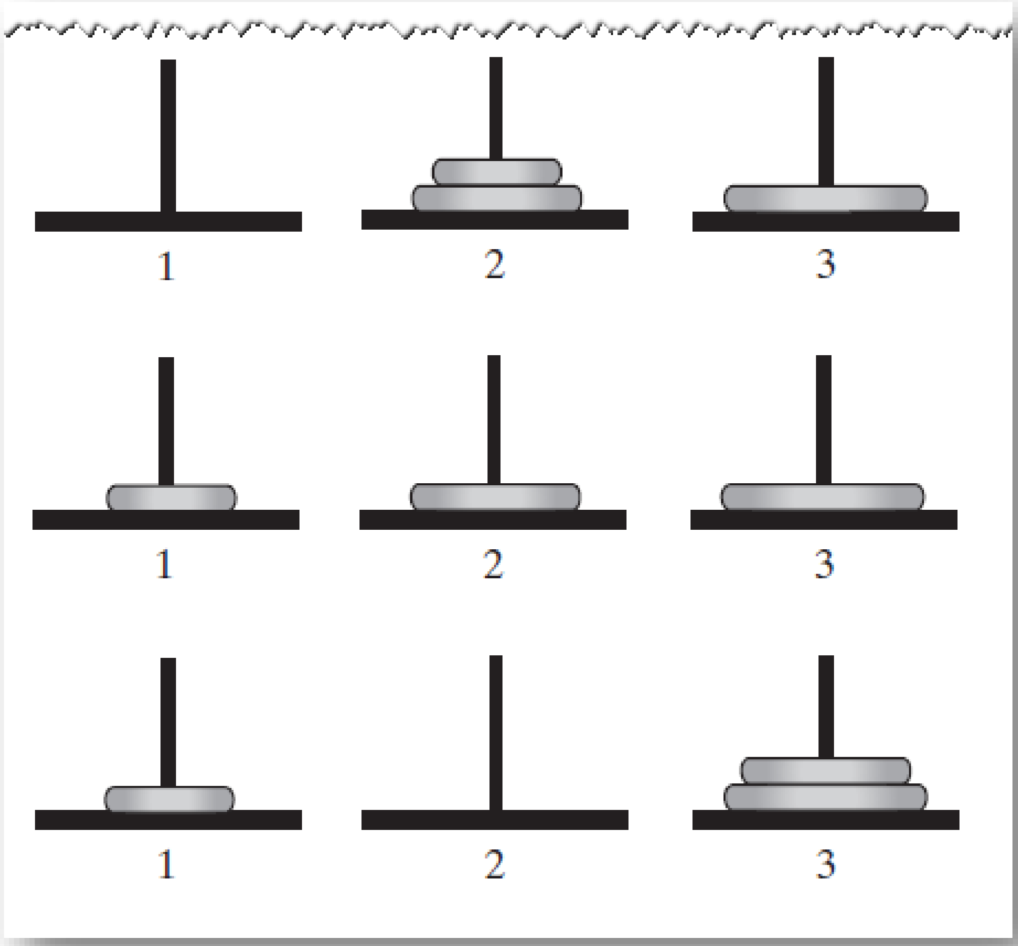

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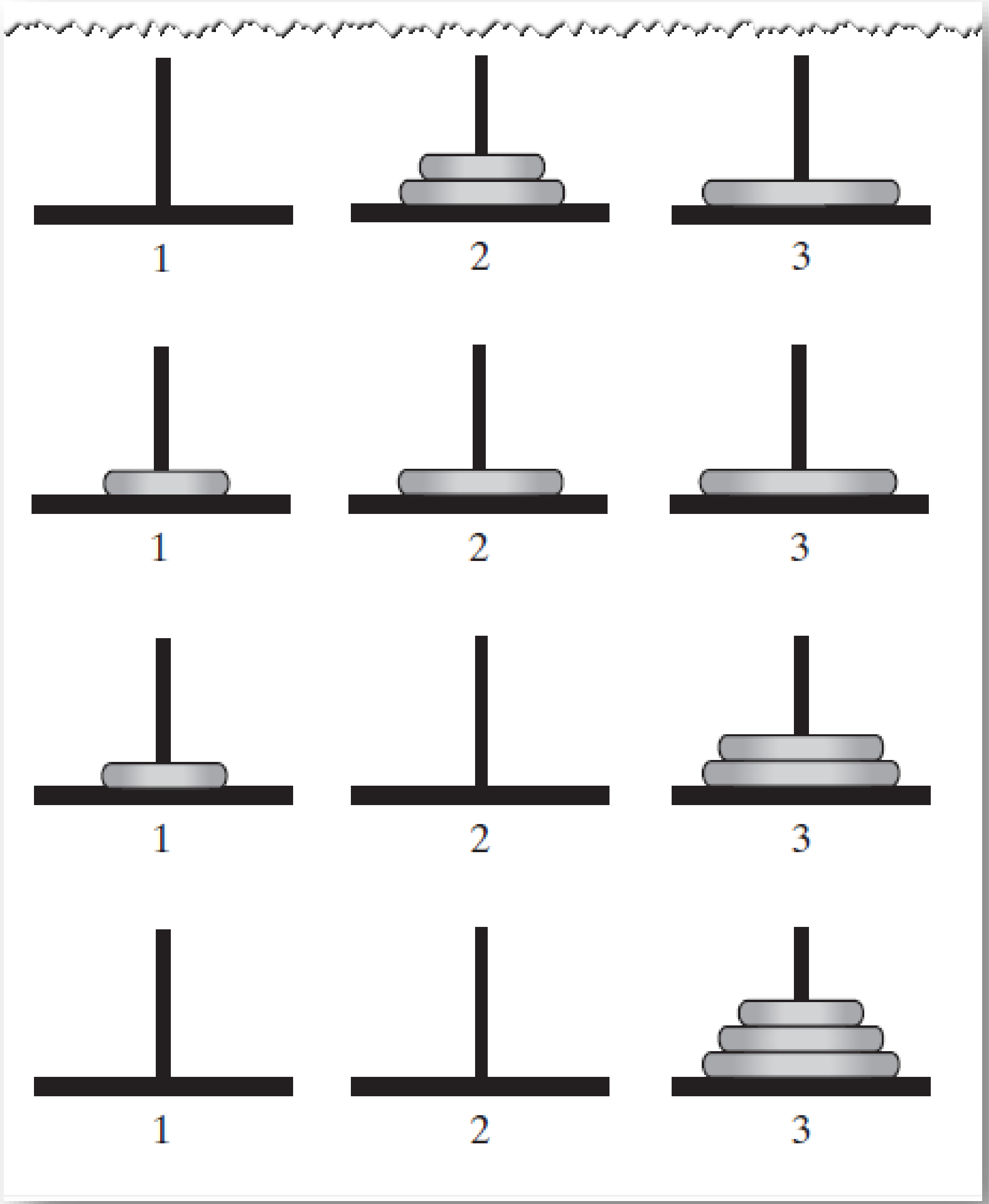
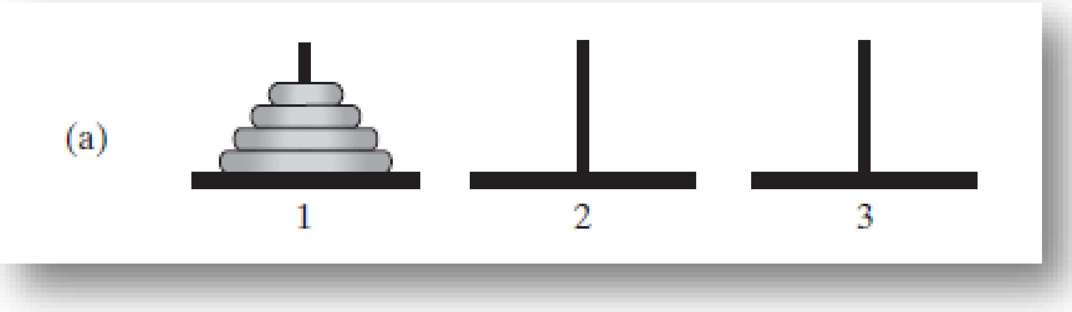


Solutions

The smaller problems in a recursive solution for four disks



Solutions

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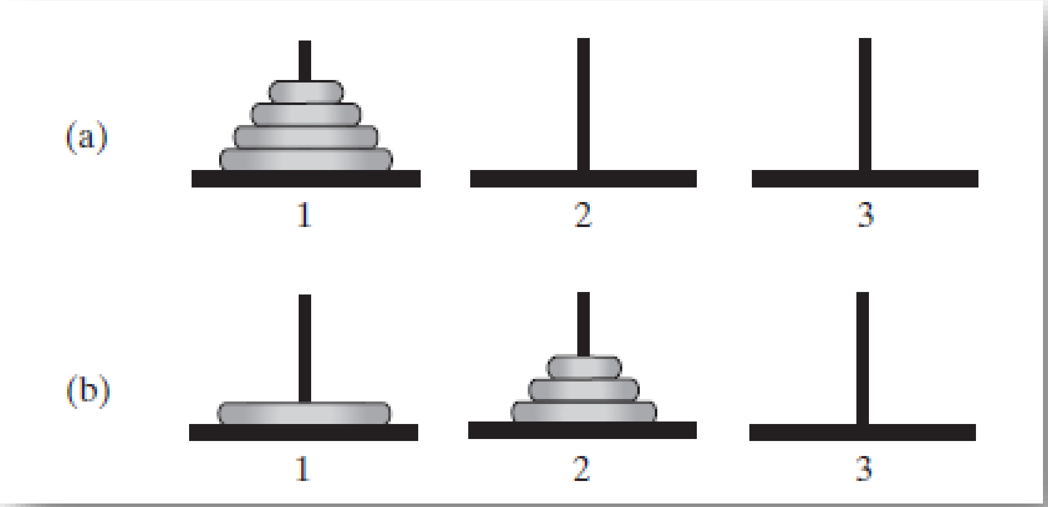

Solutions

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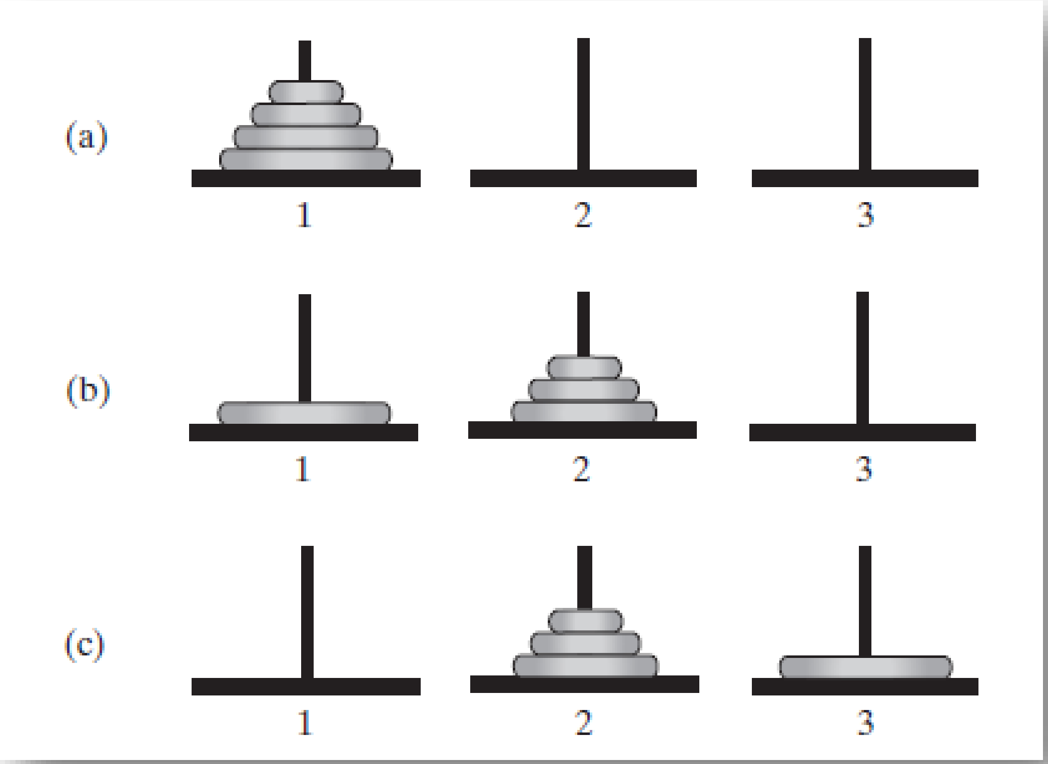
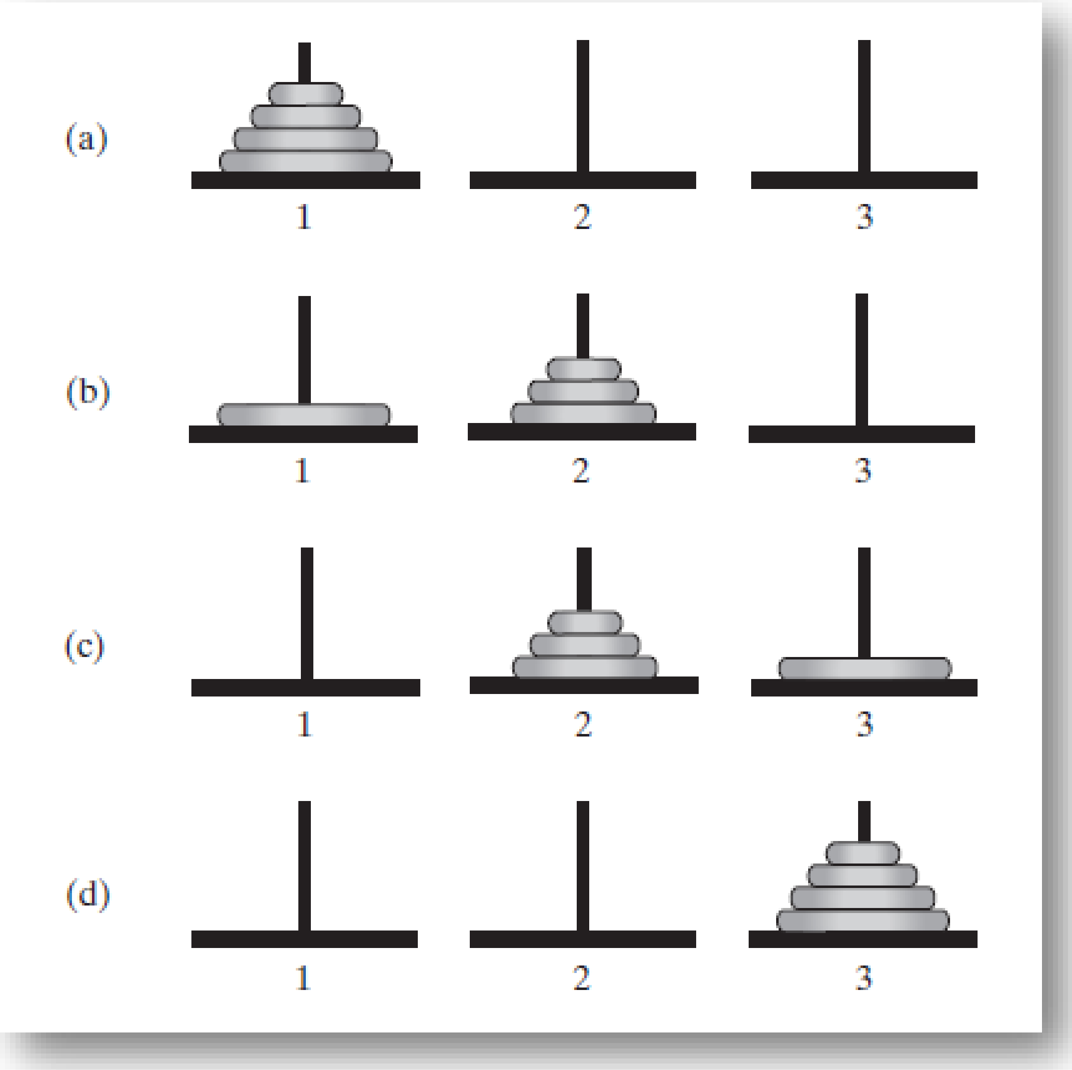


Solutions

The smaller problems in a recursive solution for four disks



Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
```

Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
    Move disk from startPole to endPole
```

Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
    Move disk from startPole to endPole
else
{
    solveTowers(numberOfDisks - 1, startPole, endPole, tempPole)
```

Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
    Move disk from startPole to endPole
else
{
    solveTowers(numberOfDisks - 1, startPole, endPole, tempPole)
    Move disk from startPole to endPole
```

Solutions

- Recursive algorithm to solve any number of disks.

```
Algorithm solveTowers(numberOfDisks, startPole, tempPole, endPole)
if (numberOfDisks == 1)
    Move disk from startPole to endPole
else
{
    solveTowers(numberOfDisks - 1, startPole, endPole, tempPole)
    Move disk from startPole to endPole
    solveTowers(numberOfDisks - 1, tempPole, startPole, endPole)
}
```

Exercise

- Prove that the running time of solveTowers is $2^n - 1$
- *Hint: use proof by induction*

Poor Solution to a Simple Problem

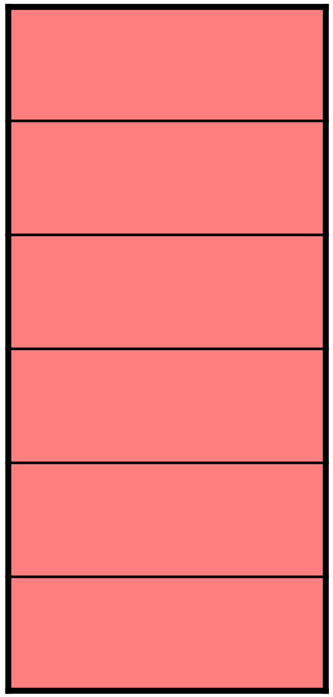
- Algorithm to generate Fibonacci numbers.
- Why is this inefficient?

Algorithm Fibonacci(n)

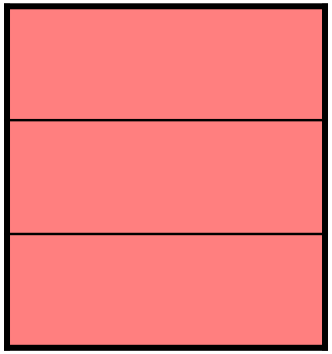
```
if (n <= 1)
    return 1
else
    return Fibonacci(n - 1) + Fibonacci(n - 2)
```

Single recursion

- A recursive algorithm with a single recursive call still provides a linear chain of calls



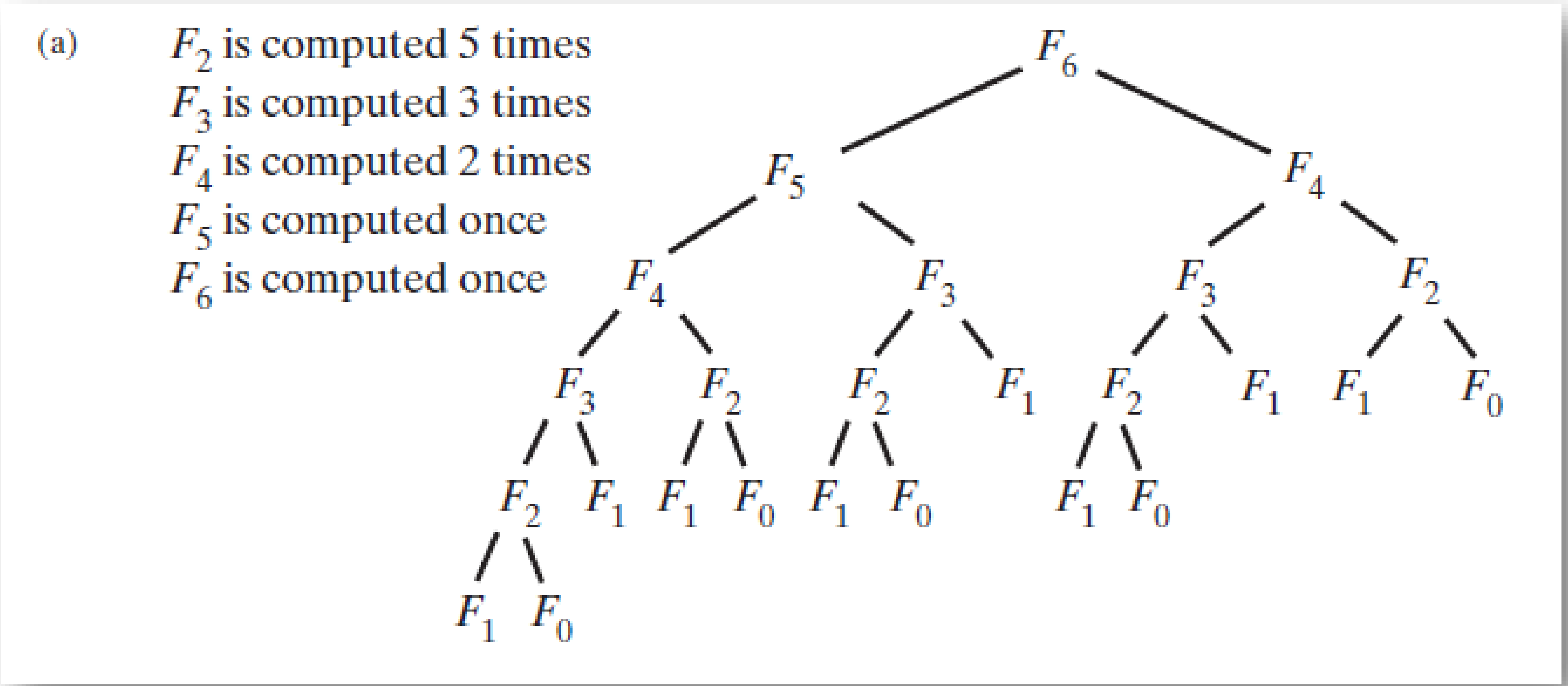
Calls build run-time stack



Stack shrinks as calls finish

Poor Solution to a Simple Problem

- The computation of the Fibonacci number F_6 using recursion



Double recursion

- When a recursive algorithm has 2 calls, the execution trace is now a ***binary tree***, as we saw with the trace on the board of Fibonacci
  - This execution is more difficult to do without recursion
    - To do it, programmer must create and maintain his/her own stack to keep all of the various data values
    - This increases the likelihood of errors / bugs in the code
- Later we will see some other classic recursive algorithms with multiple calls
  - Ex: MergeSort, QuickSort

Converting Recursion into Iteration

- Can we tell if a recursive algorithm can be easily done in an iterative way?
  - Yes – any recursive algorithm that is exclusively tail recursive can be done simply using iteration without recursion
  - Some algorithms we have seen so far are tail recursive

Tail Recursion

- So, what is tail recursion?
  - Recursive algorithm in which the recursive call is the LAST statement of the method
- What are the implications of tail recursion?
  - Any tail recursive algorithm can be converted into an iterative algorithm in a methodical way
    - some compilers do this automatically

Tail Recursion

- When the last action performed by a recursive method is a recursive call.

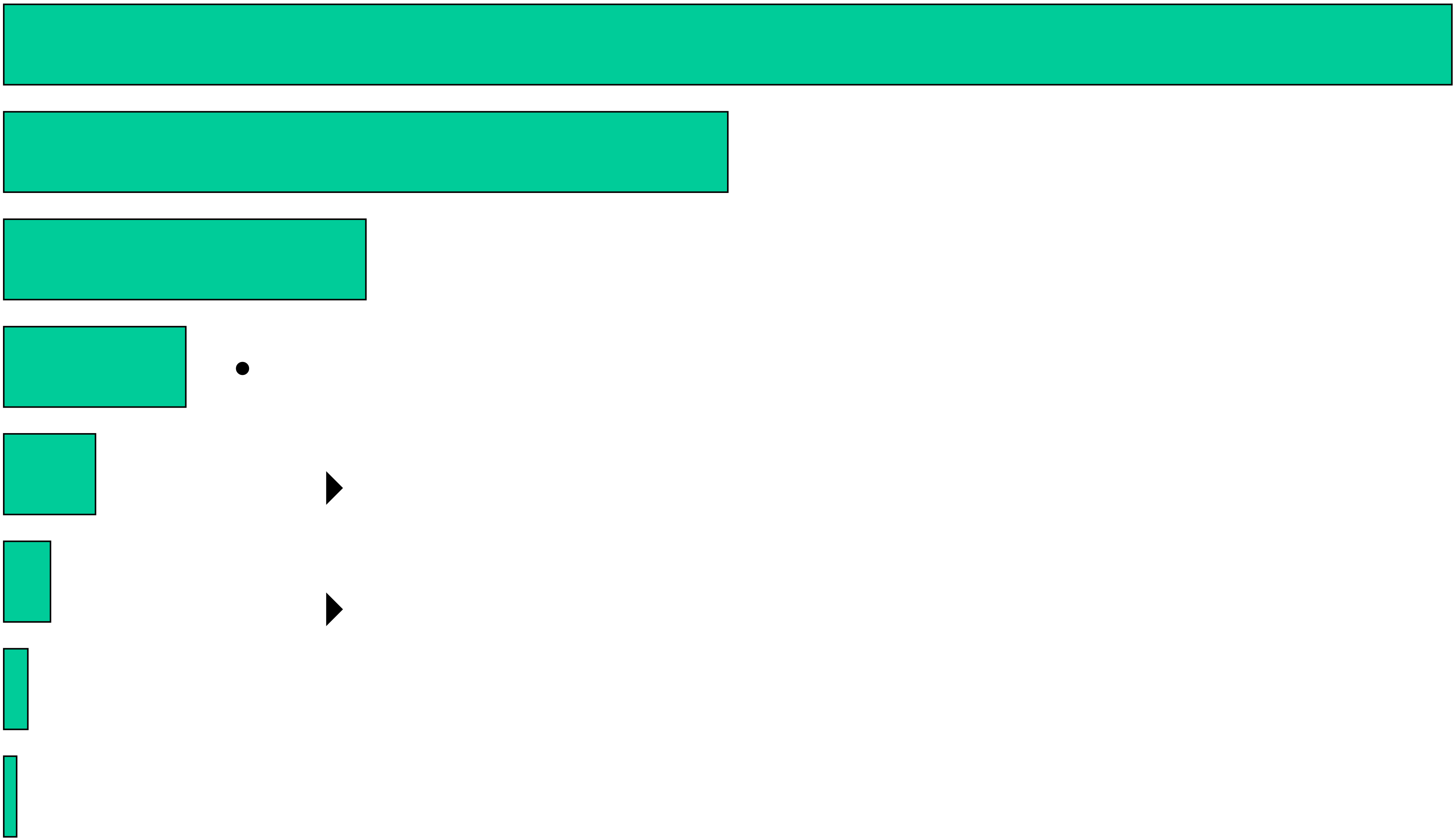
```java
public static void countDown(int integer)
{
    if (integer >= 1)
    {
        System.out.println(integer);
        countDown(integer - 1);
    } // end if
} // end countDown
```

Overhead of Recursion

- Why do we care?
  - Recursive algorithms have overhead associated with them
    - Space: each activation record (AR) takes up memory in the run-time stack (RTS)
      - If too many calls "stack up" memory can be a problem
    - Time: generating ARs and manipulating the RTS takes time
      - A recursive algorithm will always run more slowly than an equivalent iterative version

- Divide and Conquer
  - The idea is that a problem can be solved by breaking it down to one or more "smaller" problems in a systematic way
    - Usually the subproblem(s) are a fraction of the size of the original problem
    - Usually the subproblems(s) are identical in nature to the original problem
    - It is fairly clear why these algorithms can typically be solved quite nicely using recursion

Recursion and Divide and Conquer



Recursion Applications

- So what else is recursion good for?

1) For some problems, a recursive approach is more natural and simpler to understand than an iterative approach
   - Once the algorithm is developed, if it is tail recursive, we can always convert it into a faster iterative version
2) For some problems, it is very difficult to even conceive an iterative approach, especially if multiple recursive calls are required in the recursive solution

- Example: Backtracking problems

Recursion and Backtracking

- Idea of backtracking:
  - Proceed forward to a solution until it becomes apparent that no solution can be achieved along the current path
    - At that point UNDO the solution (backtrack) to a point where we can again proceed forward
  - Example: 8 Queens Problem
    - How can I place 8 queens on a chessboard such that no queen can take any other in the next move?
      - Recall that queens can move horizontally, vertically or diagonally for multiple spaces

8 Queens Problem

- How can we solve this with recursion and backtracking?
  - We note that all queens must be in different rows and different columns, so each row and each column must have exactly one queen when we are finished
    - Complicating it a bit is the fact that queens can move diagonally
  - So, thinking recursively, we see the following
    - To place 8 queens on the board we need to
      - Place a queen in a legal (row, column)
      - Recursively place 7 queens on the rest of the board
  - Where does backtracking come in?
    - Our initial choices may not lead to a solution – we need a way to undo a choice and try another one

8 Queens Problem

- Using this approach we come up with the solution as shown in 8-Queens handout
  - 8Queens.java
- Idea of solution:
  - Each recursive call attempts to place a queen in a specific column
    - A loop is used, since there are 8 squares in the column
  - For a given call, the state of the board from previous placements is known (i.e. where are the other queens?)
    - This is used to determine if a square is legal or not
  - If a placement within the column does not lead to a solution, the queen is removed and moved "down" the column

8 Queens Problem

  - When all rows in a column have been tried, the call terminates and backtracks to the previous call (in the previous column)
  - If a queen cannot be placed into column i, do not even try to place one onto column i+1 – rather, backtrack to column i-1 and move the queen that had been placed there
  - See handout for code details

- Why is this difficult to do iteratively?
  - We need to store a lot of state information as we try (and un-try) many locations on the board
    - For each column so far, where has a queen been placed?

8 Queens Problem

- The run-time stack does this automatically for us via activation records
  - Without recursion, we would need to store / update this information ourselves
  - This can be done (using our own Stack rather than the run-time stack), but since the mechanism is already built into recursive programming, why not utilize it?
- There are many other famous backtracking problems
  - http://en.wikipedia.org/wiki/Backtracking