Classical and semiclassical diamagnetism: A critique of treatment in elementary texts

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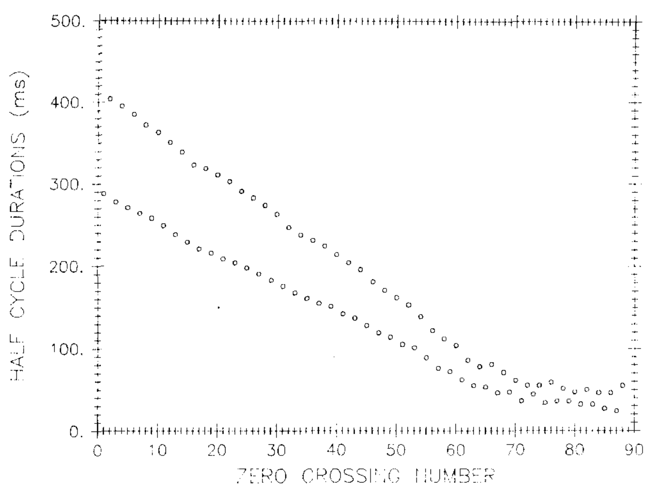

Fig. 5. Measured time intervals between successive zero crossings as a function of zero crossing number for the two-point librator.

in our case, $2b = 9$ mm and $h = 5$ mm. Using the initial value of $a(0)/x_0$, and the data for the durations of the longer half-cycles we were able to calculate the amplitude, and the kinetic energy, as a function of cycle number from the inverse of Eq. (7). A graph of the log of the kinetic energy versus cycle number was fitted rather well by two straight lines. From cycle 1 to 21 (zero crossings 2–42) the slope corresponded to a loss of 8% of the kinetic energy per cycle. For cycles 21 to 37 the slope corresponded to a loss of 16% per cycle. We attribute the increased rate of energy loss after the 21st cycle to the onset of a small rotational motion about the vertical axis, which could not be avoided with our experimental design. The rotational motion extracts additional energy from the librating motion.

D. A system with negative stiffness

For negative stiffness, unlike the cases described in Sec. IV A–C, the expression for the potential energy includes a *negative* term proportional to the absolute value of the displacement. This situation occurs for a parallel plate capacitor (plate separation d) with a dielectric slab (thickness also equal to d and dielectric constant κ) which is free to slide along one direction parallel to the plates. The plates and the slab both have length l and width w, and the plates are connected to a battery with voltage V_B. When the dielectric is centered the potential energy is $V_0 = C_0 V_B^2/2$, where $C_0 = \epsilon_0 \kappa w l/d$. When the dielectric is displaced from center, by x along dimension l, the potential energy decreases to

$$V = V_0[1 - (\kappa - 1)/\kappa]|x/l|. \tag{25}$$

Thus the centered position is one of metastable equilibrium.

The system becomes bistable if springs connected to the dielectric slab tend to restore it to the center. Then the motion of the slab obeys Eq. (3) with a stiffness parameter s equal to $V_0(1-\kappa)/(\kappa l)$.

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[1]N. Minorsky, *Introduction to Nonlinear Mechanics* (Edwards, Ann Arbor, MI, 1947), pp. 424ff.
[2]W. M. Hartmann, J. Acoust. Soc. Am. Suppl. 1 **71**, S83 (1982).
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Classical and semiclassical diamagnetism: A critique of treatment in elementary texts

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Diamagnetism does not exist as a classical phenomenon. Yet many elementary physics texts attempt to give "insight" into this subject using purely classical physics. We believe that these approaches mislead rather than help the students. On the other hand, just as semiclassical methods are employed in the chapters on quantum mechanics, these methods may be used to give an acceptable "explanation" of diamagnetism.

I. INTRODUCTION

When a nonmagnetic material is placed in an external magnetic field **B**, the material typically develops an induced magnetic moment. If this induced moment is opposite to **B**, the system is called "diamagnetic." Being a many-body system generally held at nonzero temperatures, a detailed understanding of diamagnetism must entail statistical mechanics and thermodynamics. It is well known[1] that within this framework, *classical* diamagnetism does not exist. Nevertheless, many introductory physics texts[2–5] attempt to "explain" diamagnetism using purely classical methods. Here we discuss some subtle points which are generally ignored or glossed over in introductory texts. To add these points to present texts would be unnecessarily complex; however, to omit them would be misleading. Consequently, we favor dropping the treatment of diamagnetism within the *classical* context completely.

On the other hand, since diamagnetism is a *quantum* mechanical result, it can be included in texts that already contain chapters on quantum mechanics. Needless to say, typical texts cannot, and do not, cover quantum mechanics in its "full glory." Instead, they present semiclassical pictures such as the Bohr model of the atom. In this spirit, we give a semiclassical "explanation" of diamagnetism. Such a treatment is sufficiently simple and instructive to warrant inclusion in introductory texts.

II. A CRITIQUE OF ELEMENTARY TEXTS

Since paramagnetism is typically considerably stronger than diamagnetism, observation of the latter is generally restricted to materials having zero microscopic magnetic moments at the atomic or molecular level. Most texts acknowledge this and, for the sake of simplicity, consider atoms (or molecules) with paired electrons "moving in identical, circular orbits but opposite directions." Needless to say, such a situation is impossible for *classical* particles. Even if such a configuration could be set up initially, magnetic torques would align each (orbital) magnetic moment parallel to $\mathbf{B}$, resulting in paramagnetism. Thus even before undertaking any detailed analysis of diamagnetism, one must introduce quantum mechanical concepts such as states (orbits with opposite angular momenta are different states) and an exclusion principle (two electrons in the same orbit must move in opposite directions). This point is not our main quarrel with elementary texts, however.

Our primary criticism concerns the next step. Starting with two electrons moving in opposite directions with identical speeds v and orbital radii r, the authors typically argue that, upon introduction of a small magnetic field perpendicular to the orbital plane, one electron gains and the other loses a small speed Δv. Since the magnetic moments due to these little "current loops" are proportional to $(v \pm \Delta v)r$, the two are no longer equal but opposite; consequently, a net magnetic moment appears. A "quantitive" analysis leads to a moment which is *opposite* to $\mathbf{B}$—i.e., to diamagnetism. Most authors attribute the speed changes to the additional Lorentz force $q(\mathbf{v}\times\mathbf{B})$, which modifies the centripetal force. This assertion by itself is already quite confusing since the students are taught that magnetic forces cannot do work and cannot change the speed of particles. On closer examination, one finds that the speed change comes from setting the modified centripetal force to mv^2/r and taking r to be constant. Now, the thoughtful student might ask, why should the speed change just so that the orbital radius remains constant?" While a few books do explain this,[2] most provide no clues,[3] assert that "it is an assumption which can be justified,[4] or merely acknowledge the role of Faraday's law in this delicate matter.[5]

There is added confusion for anyone who knows about the purely classical adiabatic invariance of magnetic flux for cyclotron orbits.[6] There, the orbital radius *does in fact change* with B; however, the speed also changes such that the product (vr) and, hence, the angular momentum and magnetic moment of the "current loop" remain *constant*!

What differentiates these cases? Why should the orbital radius change in one case and not in the other? After all, in the atomic (or planetary) problem, the particle will speed up or slow down when going into smaller or larger orbits, respectively. Classically, speed and radius are linked, but neither is "sacred" against changes; nor is any particular value of r or v favored. Indeed, that all values of r (position) and v (velocity) are allowed in a classical statistical mechanical analysis results in the absence of diamagnetism in classical physics. Owing to coupling to the thermal bath, the orbits simply adjust to accommodate a nonzero magnetic field.

Based on these paradoxical questions, we feel that the typical analyses in elementary texts mislead the gullible student while confounding the intelligent. In Sec. III we resolve the puzzles posed here. In a very special sense, to be specified below, there *is classical diamagnetism* for particles in orbit around a central force. In Sec. IV, we propose a semiclassical approach which is valid without "qualifiers."

III. DOES CLASSICAL "DIAMAGNETISM" EXIST?

In this section, we pose a well-defined classical question: How does a (nonrelativistic) charged particle, held in a circular orbit by some central force, respond to an adiabatic turning on of a small magnetic field $\mathbf{B}$? Let q and m be the charge and mass, respectively, of the particle. In analogy with treatments in elementary texts, consider a circular orbit in the x–y plane about the origin of the central force. Thus the magnetic moment $\boldsymbol{\mu}$, which is just $q/(2m)$ times the orbital angular momentum $\mathbf{l}$, is parallel or antiparallel to the z axis.

Now, introduce a *weak* magnetic field in the z direction, of strength $B(t)$ which, initially zero, goes to $\Delta B > 0$. ("Weak" means that the force due to B is small compared to the central one.) The specific form of $B(t)$ is irrelevant provided that it varies *sufficiently slowly*, i.e., on a time scale long compared to the orbital period. The "circular current loop" formed by the charged particle's orbit experiences an induced emf (electromotive force), given by Faraday's law,

$$\text{emf} = -\frac{d(B\pi r^2)}{dt} = E'(2\pi r), \tag{3.1}$$

where E' is the induced field in the reference frame of the "circular current loop" (whose radius r may, in principle, be slowly varying).

Instead of discussing how the fields change the speed and the orbital radius, the route favored by almost all[7] texts, we consider the change of the orbital angular momentum $\mathbf{l}$. This step is well motivated: The magnetic dipole moment $\boldsymbol{\mu}$ due to a particle in orbit is simply $(q/m)\mathbf{l}$; thus we do not need the information on the changes of v and r separately. Thus we consider the torque, which has only a z component, and changes l according to

$$\tau = qrE' = -\frac{1}{2}q\frac{d(Br^2)}{dt} = \frac{dl}{dt}. \tag{3.2}$$

Observe from Eq. (3.2) that we may define a quantity

$$L \equiv l + \tfrac{1}{2}q(Br^2), \tag{3.3}$$

which is a constant of motion. (L is in fact the generalized angular momentum and will be discussed in Sec. IV where it will be quantized.) Consequently, the change in (mechanical) orbital angular momentum is

$$\Delta l = -\tfrac{1}{2}q\Delta(Br^2), \tag{3.4}$$

such that the change in the magnetic dipole moment μ is

$$\Delta\mu = \tfrac{1}{2}(q/m)\Delta l = (q^2/4m)\Delta(Br^2). \tag{3.5}$$

Equation (3.5) demonstrates how unnecessary it is to ask about the detailed dependence of r and v upon B if only a lowest order result is sought. If, as is the case here, the magnetic field B acts merely as a perturbation on the central-force orbit, then to lowest order $\Delta(Br^2) \simeq r_0^2\Delta B$, where r_0 is the unperturbed orbital radius. Any Δr, *whether first order or higher* in ΔB, can only contribute to higher orders in Δl and $\Delta\mu$. Consequently, to lowest order in ΔB,

$$\Delta\mu = (q/2m)\Delta l \simeq -(q^2r_0^2/4m)\,\Delta B, \tag{3.6}$$

where the resulting negative sign indicates "diamagnetism."

Several remarks are in order here.

(a) Strictly speaking, this result is not diamagnetism. There is a nonzero $\boldsymbol{\mu}$ before the field is turned on, contrary to the case of diamagnetic materials. We have found only the *change* in $\boldsymbol{\mu}$, which turns out to oppose $\mathbf{B}$. If $\boldsymbol{\mu}$ is aligned with $\mathbf{B}$, then indeed the moment decreases. If, however, $\boldsymbol{\mu}$ is oppositely aligned, then the magnitude of the moment would increase (so that the change still opposes $\mathbf{B}$) provided it is not disturbed out of its unstable equilibrium. Otherwise, the orbit plane will swing around so that $\boldsymbol{\mu}$ lines up with $\mathbf{B}$ eventually. Taking this swing into account would give us *paramagnetism*! This precarious situation alone warrants the quotation marks we put on the term "diamagnetism."

(b) Accepting all the qualifiers, the result (3.6) is exactly the same as that obtained in a full quantum mechanical calculation, provided one identifies r_0^2 with the quantum expectation value $\langle r_0^2\rangle$.

(c) Equation (3.6) would not follow from (3.5) *if B were not small but only ΔB, its changes, were small* over a period in question. Such would be the case for "free" particles held in circular orbits by an already existing magnetic field. Then the dependence of r on B could play an important role since we now have an added term

$$-qBr_0\Delta r,$$

where Δr could be first order in ΔB. Under these conditions, further analysis is needed to show that this added term cancels the one in (3.6), leading to no change in l or μ. From these remarks, the subtle role of the central force is revealed. Although it does not appear explicitly in any of the Δl equations, it is crucial in providing an r_0 which does not vanish with $B \to 0$.

(d) Finally, we re-emphasize that, in the case where a central force is present, "classical diamagnetism" can be derived without invoking details about the speed and radius changes. For the sake of completeness, we simply state the results[8]: Due to the E field, Δv is first order in ΔB while Δr is second order. It is in this sense that "the orbital radius does not change."

We see how "classical diamagnetism" may exist, if we consider very special cases and keep track of detailed dynamical variations while the B field is turned on. The phenomenon has nothing to do with (stable thermodynamic) *equilibrium* of particles moving under the influence of combined central and magnetic forces. Thus we do not feel that the tract presented in this section is suitable for inclusion in typical introductory texts. Even putting it in an Appendix seems unjustifiable, considering how voluminous these texts are already. Instead, we favor the semiclassical approach, presented in Sec. IV.

IV. A SEMICLASSICAL APPROACH TO DIAMAGNETISM

In elementary texts, quantum mechanics and its application to models of the atom are usually set in the semiclassical language. "Special" orbits are introduced through the quantization of l to integral multiples of h (Planck's constant) divided by 2π. Similarly, we can speak of "special" electronic orbits of atoms placed in an external magnetic field. One can completely ignore the delicate question of dynamical changes while the field is being turned on adiabatically. In fact, this is *the* appropriate viewpoint to consider true diamagnetism. How fast or slowly the B field is applied to a sample is completely irrelevant. Diamagnetism is an *equilibrium phenomenon*, after the atom has "settled down."

Motivated to look along similar lines, we next ask what is the quantity that should be quantized in the case of atomic orbitals in a uniform, time-independent magnetic field $\mathbf{B}$. (The uniformity assumption is well satisfied for typical static $\mathbf{B}$ fields when one looks at distances of the order of angstroms!) The (mechanical) orbital angular momentum l is no longer a conserved quantity, varying according to Eq. (3.4). Therefore, it makes no sense to set it equal to constant multiples of $h/2\pi$. Instead, we seek a generalized angular momentum which is a constant of motion and which reduces to l when $B \to 0$. Equations (3.2) and (3.3) yield

$$\frac{dL}{dt} = 0, \tag{4.1}$$

where

$$L \equiv l + \tfrac{1}{2}q(Br^2). \tag{4.2}$$

This generalized angular momentum, because we are considering orbits in the x–y plane alone, is in the z (i.e., $\mathbf{B}$) direction only. Its magnitude will be quantized in exactly the same manner as the $B \equiv 0$ case. That is to say,

$$L \equiv l(B) + \tfrac{1}{2}q(Br^2) = l(0), \tag{4.3}$$

where B is the magnitude of $\mathbf{B}$. The difference between the two magnetic moments is, therefore,

$$\Delta\mu = (\tfrac{1}{2}q/m)\langle[l(B) - l(0)]\rangle = (-q^2\langle r^2\rangle/4m)B, \tag{4.4}$$

where the brackets denote an average over the orbit (which is not necessarily circular in this approach). This is (Larmor) diamagnetism in the semiclassical spirit.

To be a little more precise, we should note that (4.4) is the change in μ of a particular orbit. True diamagnetism is a phenomenon associated with atoms or ions with only $l = 0$ electrons or completely filled $l \neq 0$ shells. Thus the next moment for $B = 0$ is zero even though individual electronic orbits contribute a nonzero μ. Considering these atoms in a $B \neq 0$ field and taking into account the Pauli exclusion principle, (4.4) implies a net $\boldsymbol{\mu}$ which is opposed to $\mathbf{B}$.

V. SUMMARY AND CONCLUSION

Strictly speaking, classical diamagnetism does not exist. We have clarified the restricted sense within which the response of a classical particle, bound by a central force, to an external **B** field may be thought of as "diamagnetic." Most elementary texts "explain" diamagnetism based on this restricted notion. Unfortunately, the typical analysis is incomplete. We feel that this approach is, at best, misleading and adds little insight. To present a complete analysis of such a response is far too complex to be worthwhile.

By contrast, a semiclassical approach to this phenomenon requires no more than four equations and two paragraphs. In addition to presenting a complete and simple analysis, this approach introduces the useful concept of generalized angular momentum. Since semiclassical pictures are often portrayed in texts at this level, this analysis does not put extra demands on the mathematical ability of the student. If avoiding the full machinery of quantum mechanics is prerequisite to an elementary text, and if the author insists on including diamagnetism, then we hope that this route will be taken in the future.

[1]See, e.g., F. Mohling, *Statistical Mechanics Methods and Applications* (Publishers Creative Services, Wiley, New York, 1982).

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[5]G. B. Arfken, D. F. Griffing, D. C. Kelly, and J. Priest, *University Physics* (Academic, New York, 1984); M. C. Martin and C. A. Hewett, *Elements of Classical Physics* (Pergamon, New York, 1975); J. W. Kane and M. M. Sternheim, *Life Science Physics* (Wiley, New York, 1978); E. M. Rogers, *Physics for the Inquiring Mind* (Princeton U. P., Princeton, NJ, 1960); R. T. Weidner and R. L. Sells, *Elementary Classical Physics*, (Allyn & Bacon, Boston, 1973), 2nd ed.; O. Oldenberg and N. C. Rasmussen, *Modern Physics for Engineers* (McGraw-Hill, New York, 1966); J. B. Marion and W. F. Hornyak, *Physics for Science and Engineering* (Saunders, Philadelphia, 1982); D. E. Tilley, *University Physics for Science and Engineering* (Cummings, Menlo Park, CA, 1976).

[6]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed.

[7]Only Feynman (Ref. 2) avoids this cumbersome route. Our analysis here is identical to his.

[8]Purcell (Ref. 2) sets up a situation similar to the one posed here, with the central force provided with a cord. His demonstration that, to first order in dB, there is no change in the tension on the cord, is tantamount to saying that there is no change in radius.

Unveiling the solitons mystery: The Jacobi elliptic functions

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Several examples are given in which the soliton solutions belonging to well-known nonlinear differential equations arise as simple limits of the Jacobian elliptic functions. The knowledge of the solutions of a single ordinary nonlinear differential equation is sufficient for the subsequent derivation of soliton solutions. Various explicit examples are worked out.

I. INTRODUCTION

In the last decade we have witnessed a notorious revival of interest in nonlinear equations of several kinds in many branches of physics, as well as a growing use of their solutions in solving long-standing problems of nonlinear physics: turbulence, induced self-transparency, nonlinear optics and, indeed, two-dimensional models in mathematical physics and elementary particle physics.

There exists nowadays several up to date reviews on this subject which provide an excellent account, even at the elementary level, of the physical properties of these "solitary waves." Also, they usually contain accurate discussions on related topics such as topological quantum numbers, conserved topological currents, and many other applications.[1-3] However, even in the more advanced reviews of this kind, the soliton solutions seem to emerge from the darkness; they are usually written down with no reference to the methods by which they can easily (or not so easily) be obtained. This situation gives rise to the wrong assumption that those important nonlinear solutions have been always obtained by extremely clever tricks, dwelling on the physicist's skills more than on well-established and accurately defined principles in the study of nonlinear differential equations. More important, this situation leads the nonspecialist to believe that this is a subject in which there is no room for people lacking in advanced mathematical education. Since solitons will likely play a crucial role in our understanding of nature, I find this situation somewhat uncomfortable. This paper is an attempt to show that solitons can be found with no more knowledge than is required for the teacher to develop the general solutions of the assymmetric top, as is done, for instance, in the textbook Ref. 4. All that is needed is to spend about half an