Real Analysis

End-Sem 2023

Time - 3.00 hours

Full marks 100

1. a) Prove that a sequence can have atmost one limit
b) Consider $\{u_n\}$ and $\{v_n\}$ are two converging sequences which converges to u and v respectively. Then prove the following identities.

$i)$ $\lim_{n\infty}(u_n + v_n) = u + v$
$ii)$ if $c \in \mathbb{R}$, $\lim_{n\infty}(cu_n) = cu$
$iii)$ $\lim_{n\infty}(u_n v_n) = uv$
$iv)$ $\lim_{n\infty}(u_n/v_n) = u/v$ providing $\{v_n\}$ is a sequence of non zero elements and it does not converge to 0.

(5+15)

2. Test the convergences of the following two series:

$$S_1 = \sum_{n=1}^{\infty}\left(\frac{1}{2^n} + \frac{1}{3^n}\right)$$

$$S_2 = 1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4}\dots\dots\dots$$

(5+5)

3. a) State and prove the Sandwitch theorem of limits.
b) State and prove the Cauchy principle of limit.

(10+15)

4. Use the definition of continuity at a point to prove that
i) $f(x) = 3x - 5$ is continuous at $x = 2$.
ii) $f(x) = x^2$ is continuous at $x = 3$.
iii) $f(x) = 1/x$ is continuous at $x = 1/2$.

(5+5+5)

5. a) From the definition of differentiation prove that $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$, where $f(x)$ and $g(x)$ are differentiable functions in the interval I.
b) Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ is a real valued function differentiable at $c \in I$. Then prove that if $f'(x) > 0$ $(or\ f'(x) < 0)$ at c, then the function is increasing (or decreasing) at c.
c) State and prove Taylor's theorem.

(5+10+15)

20
10
15
15
60

$\frac{fg - f(h)g(h)}{h}$

$f'(c) > 0$

$\frac{f(c+h) - f(c)}{}$

$c+h > c$

$f'(c) = \frac{f(c+h) - f(c)}{h}$

$\frac{f - f(c)}{h}$

$f'(x)(x-c) + f''(x)\frac{(x-c)^2}{2!}$