INTERNATIONAL INSTITUTE OF
INFORMATION TECHNOLOGY

HYDERABAD

Science-1: prep notes

International Institute of Information Technology, Hyderabad

Author: Dr Prabhakar Bhimalapuram

2024/2025

General pointers

1. State of system: 3-dimensional Cartesian coordinate system.
   - scalar quantities (mass, temperature)
   - vector quantities (position vector, velocity and momentum, angular momentum and torque)
2. Vector calculus:
   - vector equalities, addition
   - vector dot product, cross products
   - gradient of a scalar function
   - line integrals: circulation and curl of vector fields
   - surface integrals: flux and divergence of vector fields

Laws of Newton

1. Ist Law: Inertia. Inertial frames. Uniform rectilinear motion of an object in absence of forces
2. IInd Law: Definition of momentum. Rate of change of momentum equals the net external force on the object
3. IIIrd Law: For central forces, the reaction is equal and opposite to action

Great generality of these laws of motion to all kinds of phenomena. Spurred by advances in mathematics.

Work, kinetic energy.

Work done on an object by application of external force $\vec{F}(t)$ is given by:

$$W = \int_{t_1}^{t_2} \vec{F}(t) \cdot d\vec{r}(t)$$

- Using Newton IInd Law, we get $W = \int_1^2 \frac{d\vec{p}}{dt} \cdot \vec{v}(t)dt = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = K_2 - K_1$

Defining $K = \frac{1}{2}mv^2$ as kinetic energy, we see force acting on system changes the kinetic energy of the system.

Conservative forces, potential energy

- If the force is conservative, i.e. $\vec{F}(\vec{r}) = -\nabla U(\vec{r})$, then $W = \int_1^2 (-\nabla U) \cdot d\vec{r} = -(U_2 - U_1)$.
- Thus conservative forces have an associated potential energy function U
- For conservative forces, the above shows that work W is path-independent, depending only on end-point potential energies
- $W = K_2 - K_1 = -(U_2 - U_1)$
- Thus, $K_1 + U_1 = K_2 + U_2$. Law of conservation of mechanical energy

Law of conservation of linear momentum

For a vector $\vec{s}$, if $\vec{F} \cdot \vec{s} = 0$, then

- $\vec{F} \cdot \vec{s} = \frac{d\vec{p}}{dt} \cdot \vec{s} = \frac{d}{dt}\left(\vec{p} \cdot \vec{s}\right)$
- $\vec{F} \cdot \vec{s} = 0$ implies $\left(\vec{p} \cdot \vec{s}\right)$ is a constant
- Thus momentum in the direction of $\vec{s}$ is conserved

Torque & Angular momentum and its conservation

- Angular momentum about origin: $\vec{L} \equiv \vec{r} \times \vec{p}$
- Torque $\vec{N} \equiv \frac{d}{dt}\vec{L} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$

$$\vec{N} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} = 0 + \vec{r} \times \vec{F}$$

$$\vec{N} = \frac{d}{dt}\vec{L} = \vec{r} \times \vec{F}$$

- $\vec{N} \cdot \vec{s} = 0 \implies \frac{d}{dt}\left(\vec{L}.\vec{s}\right) = 0;$
  Conservation of Angular momentum along the direction of zero torque.

For conservative forces, show conservation of energy

- $E = T(\vec{v}(t)) + U(\vec{r}(t), t) \implies \frac{d}{dt}E = \frac{d}{dt}T + \frac{d}{dt}U$
- $\frac{d}{dt}T = m\vec{v} \cdot \frac{d}{dt}\vec{v} = m\vec{v} \cdot \vec{a} = \vec{v} \cdot \vec{F}$
- $\frac{d}{dt}U(\vec{r}, t) = \sum_\alpha \frac{\partial U}{\partial x_\alpha}\frac{dx_\alpha}{dt} + \frac{\partial U}{\partial t} = \vec{v} \cdot \nabla U + \frac{\partial U}{\partial t}$
- $\frac{d}{dt}E = \frac{d}{dt}(T + U) = \vec{v} \cdot (\vec{F} + \nabla U) + \frac{\partial U}{\partial t}$

$\frac{d}{dt}E = \frac{\partial U}{\partial t} \implies E =$constant when $U \equiv U(\vec{r})$

Galilean Relativity

All inertial frames have same form of mechanical law.

Two inertial frames, K and K' having relative velocity v. Then, a particle having $\vec{u}$ in K-frame, $\vec{u}'$ in K'-frame. Then

$$\vec{u}' = \vec{u} - \vec{v}$$

Thus, clearly $\frac{d\vec{p}'}{dt} = \frac{d\vec{p}}{dt}$. This gives

$$\vec{F}' = \vec{F}$$

Hence, all mechanical laws remain in same form in both inertial frames.

Problem solving strategy

- Identify forces, draw free body diagram
- “balance forces” to satisfy constraints
- set up dynamical equations (Use IInd/IIIrd Laws and/or use conservation laws)
- solve math of dynamical equations, with appropriate boundary conditions

Examples: next slide

Single particle, constrained motion

- Need to introduce ‘balancing’ forces. Examples:
  - block moving on a horizontal plane. “normal force”
  - block moving down inclined plane.

For Simple Harmonic Oscillator, find EoM

For Simple Harmonic Oscillator, find EoM

- Hooke's law: $F(x) = -kx$, EoM as $m\ddot{x} = -kx$
- Rewrite as: $\ddot{x}(t) + \omega_0^2 x(t) = 0$, with $\omega_0 = \sqrt{\frac{k}{m}}$
- Differential equation: second order. Linear in x
- One standard practice: convert to two first order differential equation. With $p = m\dot{x}$
  two equations $\frac{d}{dt}p(t) = -kx$ and $\frac{d}{dt}x(t) = p(t)/m$
- $\frac{d}{dt}(x, p) = (p/m, -kx)$, with $(x(0), p(0)) = (x_0, p_0)$
- Conservation of energy, $E = \frac{1}{2m}p^2 + \frac{1}{2}mx^2$
- Note: solution is $x(t) = A_+ e^{i\omega_o t} + A_- e^{-i\omega_o t}$
- Phase diagram: $x(t)$ vs $p(t)$

Show that SHO is applicable for other systems near ‘minima’

Limitations of Newtonian Mechanics

- Handling of constraints (will return to this later)
- Galilean Invariance (Newtonian Relativity) violated by EM: Special Theory of Relativity

Limitations of Newtonian Mechanics

- Handling of constraints (will return to this later)
- Galilean Invariance (Newtonian Relativity) violated by EM: Special Theory of Relativity
- Stability of atom: unexplained by classical physics: Quantum Mechanics

Limitations of Newtonian Mechanics

- Handling of constraints (will return to this later)
- Galilean Invariance (Newtonian Relativity) violated by EM: Special Theory of Relativity
- Stability of atom: unexplained by classical physics: Quantum Mechanics

Reasons & postulates of Special Theory of Relativity

- Electromagnetic phenomena seemingly in violation
  - EM wave equation is not Galilean invariant
  - same phenomena seen differently in inertial frames
    - Moving coil, magnet at rest: magnetic field exerts forces ($qv \times B$) on charges in coil, generating current
    - moving magnet, coil at rest: time varying magnetic flux creates electric field on charges (motive emf, Faraday's Law)
- Speed of light is same in all inertial frames (expts)

Reasons & postulates of Special Theory of Relativity

- Electromagnetic phenomena seemingly in violation
  - EM wave equation is not Galilean invariant
  - same phenomena seen differently in inertial frames
    - Moving coil, magnet at rest: magnetic field exerts forces ($qv \times B$) on charges in coil, generating current
    - moving magnet, coil at rest: time varying magnetic flux creates electric field on charges (motive emf, Faraday's Law)
- Speed of light is same in all inertial frames (expts)

Postulates of Special Theory of Relativity (Einstein 1905)

- Same physical laws in all inertial frames
- Speed of light is same value in all inertial frames

Events simultaneous in all frames? NO!

Concept: Relativity of simultaneity.

Events simultaneous in all frames? NO!

Concept: Relativity of simultaneity.
Observer located at the middle of a train, receives flash from front and back of train; flashes set off at same time.

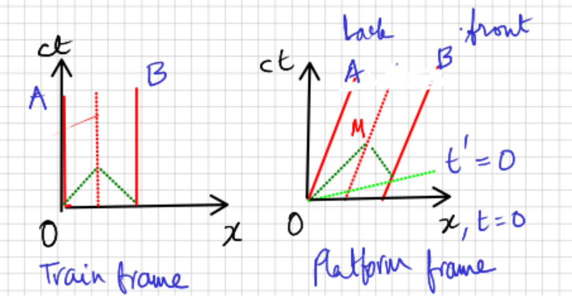


Figure: Space-time diagrams in (left) train-fixed frame and (right) platform-fixed frame, with *c* being same value in both.
Red lines are for front, back and mid-points of train.
Green lines are for light and have same slope in both plots.

How do space-time coordinates transform?

Frame K' with relative velocity v w.r.t. frame K

$$\begin{aligned} x' &= \gamma\,(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\,(t - \frac{xv}{c^2}) \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} x &= \gamma\,(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\,(t' + \frac{x'v}{c^2}) \end{aligned}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$K-$frame the event $e = (x, t)$ has transformed $K'-$frame coordinates $e' = (x', t')$

Space-time invariant has same value in both frames:
$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$

Notes on Lorentz transform

K'-frame is moving with velocity v w.r.t. K-frame

- Given an event $e = (x, y, z, t)$, use forward transform to get $e' = (x', y', z', t')$
- Strategy is to use difference in events as experimental measurable
- Note: interchaning prime with unprimed variables and changing v to $-v$ gives Inverse Lorentz Transform. This is an important symmetry!

What is time difference between ticks of a clock as measured from a moving frame?

Take events, $e_0 = (0, 0)$ and $e_1 = (0, \tau_0)$, $e_2 = (0, 2\tau_0), \cdots$
(i.e. ticking of the clock stationary in K-frame)

What is time difference between ticks of a clock as measured from a moving frame?

Take events, $e_0 = (0, 0)$ and $e_1 = (0, \tau_0)$, $e_2 = (0, 2\tau_0), \cdots$ (i.e. ticking of the clock stationary in K-frame)

In K'-frame, using Lorentz transform, we get $e'_n = (-\gamma v n \tau_0, \gamma n t_0)$.

Hence time difference between two consecutive ticks of the stationary clock will be measured in K'-frame as $\gamma(n - (n-1))t_0 = \gamma t_0$.

This effect is known as Time dilation

What is the length of a rod as measured from moving frame?

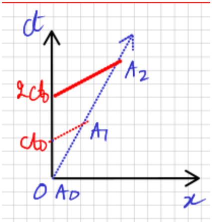
Back end of rod: $e_1 = (0, t)$ and
front-end of rod $e_2 = (L_0, t)$ for any t
To measure rod length in $K'-$frame, we need to find $e'_{1,2}$
with $t'_1 = t'_2$

What is the length of a rod as measured from moving frame?

Back end of rod: $e_1 = (0, t)$ and
front-end of rod $e_2 = (L_0, t)$ for any t
To measure rod length in $K'-$frame, we need to find $e'_{1,2}$ with $t'_1 = t'_2$

$\Delta x' = \gamma(\Delta x - v\Delta t)$
The forward transform involves knowing unprimed quantities , but we do not know which t to use to get x'), we will use inverse Lorentz transform: $x = \gamma(x' + vt')$, giving us $x_{1,2} = \gamma(x'_{1,2} + vt'_{1,2})$ and thus
$\Delta x = x_2 - x_1 = \gamma(\Delta x' + v \cdot 0)$
This gives us
$\Delta x = \gamma \Delta x' \implies L_0 = \gamma L' \implies L' = L_0\sqrt{1 - \frac{v^2}{c^2}}$
This is known as Length Contraction

What is the frequency measured by a moving observer?



$$A_0 = (0,0) \implies A_0' = (0,0)$$

$$A_1 : ct = x + ct_0 \;\&\; x = vt \implies ct_1 = \frac{ct_0}{c - v}$$

$$A_2 : ct = x + 2ct_0 \;\&\; x = vt \implies ct_2 = \frac{2ct_0}{c - v}$$

$$A_1 = \left(\frac{vct_0}{c - v}, \frac{ct_0}{c - v}\right) \implies A_1' = \left(0, \gamma\left[\frac{ct_0}{c - v} - \frac{v}{c^2}\frac{vct_0}{c - v}\right]\right)$$

$$\nu' = \frac{1}{\Delta t'} = \nu\sqrt{\frac{1 - \beta}{1 + \beta}}$$

Frequency shift is velocity dependent; velocity of stars

What about moving rod and a moving clock?

Symmetry: both moving and stationary observer will say exactly same thing about other!

- stationary clock is measured to have “longer time difference between consequtive ticks” by moving observer
- a stationary observer will measure a moving clock to have ”longer time difference between consecutive ticks”
- a stationary rod is measured to be shorter than its rest length by moving observer
- a moving rod is measured to be shorter than its rest length by stationary observer

Derive relative velocity formulae

Particle with $x(t) = ut$, what is its velocity in K' frame?

$$x' = \gamma(x - vt) \implies \frac{dx'}{dt} = \gamma(\frac{dx}{dt} - v)$$

$$t' = \gamma\left(t - x\frac{v}{c^2}\right) \implies \frac{dt'}{dt} = \gamma(1 - \frac{dx}{dt}\frac{v}{c^2})$$

$$\frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$\frac{dy'}{dt'} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

$$\frac{dz'}{dt'} = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

What is the relativistic momentum?

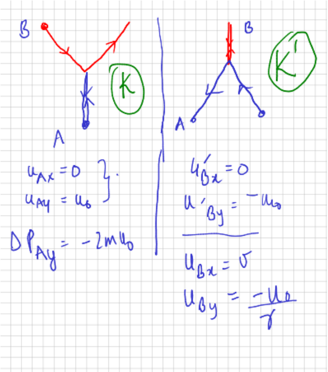


Figure: In $K-$frame (K'-frame), particle A (B) is has u_0 velocity.

Momentum change of A is $-2mu_0$, but NOT for B!
Having $\vec{p} = \gamma(u)\ m\vec{u}$, solves this issue.

From momentum, find energy of particle.

Show $K = (\gamma_u - 1)mc^2$, and hence suggest $E_0 = mc^2$.
Also $E^2 = (pc)^2 + (mc^2)^2$

From momentum, find energy of particle.

Show $K = (\gamma_u - 1)mc^2$, and hence suggest $E_0 = mc^2$.
Also $E^2 = (pc)^2 + (mc^2)^2$

$$W = \int_1^2 F \cdot dr = \int_1^2 \frac{d\vec{p}}{dt} \cdot \vec{u} dt = \int_1^2 \frac{d(\gamma m\vec{u})}{dt} \cdot \vec{u} dt$$

After a bit of math, $W = \int_1^2 \frac{d}{dt}(\gamma mc^2)dt = (\gamma(u) - 1)mc^2$, to get from rest to velocity $\vec{u}$.
Since $W = K_2 - K_1$, we get $K = (\gamma(u) - 1)mc^2$
Hence, total energy $E = \gamma mc^2$, rest mass energy $E_0 = mc^2$, gives $K = E - E_0$

Since $E = \gamma mc^2$, and $p^2 = \gamma^2 m^2 u^2$, we get
$E^2 - p^2c^2 = \gamma^2 m^2 c^4 (1 - u^2/c^2) = m^2 c^4$
Energy formula: $E^2 = (pc)^2 + (mc^2)^2$ for free particle

Space-time invariant

- Events: $e_1 = (x_1, y_1, z_1, t_1)$ and $e_2 = (x_2, y_2, z_2, t_2)$
- Define $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2$
- Lorentz Tranform: $x' = \gamma(x - \beta ct),\ ct' = \gamma(ct - x\beta)$, $y' = y$, $z' = z$ with $\beta = v/c$
- Define $(\Delta s')^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - c^2(\Delta t')^2$
- $(\Delta s')^2 =$ $\gamma^2 \left[(\Delta x - \beta c\Delta t)^2 - (c\Delta t - \beta\Delta x)^2\right] + (\Delta y)^2 + (\Delta z)^2$ $=\gamma^2(1 - \beta^2)\left[(\Delta x)^2 + c^2(\Delta t)^2\right] + (\Delta y)^2 + (\Delta z)^2$ $= (\Delta s)^2$
- Space-time 4-vector $\vec{s} = (x, y, z, ict)$
- If $(\Delta s)^2 < 0$, an inertial frame exists where both events happen at same position but at different times. Time-like interval. Allows for casual relationships between two events.
- If $(\Delta s)^2 > 0$, space-like interval. Both can happen at same time, but at different spatial locations.

Show electric and magnetic effects are equivalent by STR.

Consider a infinite line of charges with density λ on x-axis, and a charge *q* at a distance *d* from this "wire" of charges. All charges at rest. The charge *q* will experience a force $\vec{F} = q\vec{E}$ directly away from the wire ('repelling' force).

Consider now a frame K' where all the charges are moving to right ($+x-$axis) with velocity *v*. Now the charge *q* will experience two forces
(a) 'repelling' electric force
(b) 'attractive' magnetic force

Due to length contraction in K'-frame, density of charges is larger ($\lambda' > \lambda$) and hence the repelling electric force is larger. But the 'new' magnetic force exactly cancels the part that is in excess to case of stationary charges!

Equivalence of electric and magnetic effects (continued)

- Additional reference: https://tinyurl.com/4x2629nv, where force on two parallel line of charges are analysed from two inertial frames
- Maxwell laws of electromagnetism showed that magnetic fields result from time-varying electric fields (due to currents, Biot-Savart Law) and electric fields result from time-varying magnetic fields (Faraday's Law). Maxwell laws demonstrate the connection between electricity and magnetism.
- Special theory of relativity goes one step further: electric forces and magnetic forces are the same phenomena viewed from different inertial frames.

Multi-particle systems: Center of Mass

Consider a system of N particles, with masses $\{m_k\}$, position vectors $\{\vec{r}_k\}$, velocities $\{\vec{v}_k\}$ etc..

- Total mass of system $M = \sum_{k=1}^{N} m_k$
- Center of Mass (CoM) is defined as: $\vec{R} = \frac{1}{M}\sum_{k=1}^{N} m_k \vec{r}_k$
- Momentum of particle k is $\vec{p}_k = m_k \vec{v}_k$
- Total linear momentum of system, $\vec{P} \equiv \sum_k \vec{p}_k$
- Total angular momentum of system, $\vec{L} \equiv \sum_k \vec{r}_k \times \vec{p}_k$
- Net Torque about origin of system, $\vec{N} \equiv \sum_k \vec{r}_k \times \vec{F}_k$
  Check that $\vec{N} \equiv \frac{d}{dt}\vec{L}$ gives above expression for $\vec{N}$

Collection of particles

Forces:

- $F_k = F_k^e + F_k^i$ (external and internal forces, resp.)
- $F_k^i = \sum_j F_{k \leftarrow j}$ (internal interactions)
- For central forces $F_{k \leftarrow j} = -F_{j \leftarrow k}$
  - $\sum_k F_k^i = \sum_{\langle j,k \rangle} F_{k \leftarrow j} = 0!$
  - $\sum_k r_k \times F_k^i = \sum_{\langle j,k \rangle} r_k \times F_{k \leftarrow j} = 0!$

Effects of these forces:

- Net force: $F = \sum_k F_k = \sum_k F_k^e$
- Effective mass M at Center of Mass position $\vec{R}$
- Overall motion: $M \frac{d^2}{dt^2} \vec{R} = \vec{F}$
- Angular momentum: $\vec{L} = \vec{L}_{CoM} + \vec{L}^i_{CoM}$
- Internal forces are central $\implies$ the total internal torque is zero!

Show $\vec{P}$ conserved, when no external forces

Show $\vec{P}$ conserved, when no external forces

- $\frac{d}{dt}\vec{P} = \frac{d}{dt}\left[\sum_k m_k \vec{v}_k\right] = \sum_k m_k \frac{d^2}{dt^2}\vec{r}$
- $\vec{R} = \frac{1}{M}\sum_k m_k \vec{r}_k \implies \frac{d^2}{dt^2}\vec{R} = \frac{1}{M}\frac{d^2}{dt^2}\sum_k m_k \vec{r}_k = \frac{1}{M}\frac{d^2}{dt^2}\vec{P}$
- $\frac{d}{dt}\vec{P} = \frac{d}{dt}\left[\sum_k \vec{p}_k\right] = \sum_k \vec{F}_k$ = net force on system
- If no external forces then
  - $\vec{F}_k = \sum_{j\neq k} \vec{F}_{k\leftarrow j}$, inter-particle interactions only
  - $\sum_k \vec{F}_k = \sum_k \sum_{j\neq k} \vec{F}_{k\leftarrow j} = \sum_{\langle j,k\rangle}\left(\vec{F}_{k\leftarrow j} + \vec{F}_{j\leftarrow k}\right) = 0$!
  - Hence $\frac{d}{dt}\vec{P} = 0$, $\vec{P}$ is a conserved quantity
- When external forces exist, following math similar to above argument, it can be shown that:
  - Define net external force $\vec{F}_{ext} = \sum_k \vec{F}_{k,ext}$
  - $\frac{d}{dt}\vec{P} = \vec{F}_{ext}$
  - And hence $M\frac{d^2}{dt^2}\vec{R} = \vec{F}_{ext}$

Show $\vec{L}$ conserved when no external forces

Show $\vec{L}$ conserved when no external forces

- $r_k = R + r'_k \implies v_k = V + v'_k$
- $L = \sum_k r_k \times p_k = \sum_k m_k(R + r'_k) \times (V + v'_k)$
  which gives $L = MR \times V + \sum_k r'_k \times p'_k$
  Angular momentum= L of CoM + L_{system} about CoM
- $\frac{d}{dt}L = \frac{d}{dt}\sum_k \vec{r}_k \times \vec{p}_k = \sum_k \left[\frac{d}{dt}\vec{r}_k \times \vec{p}_k + \vec{r}_k \times \frac{d}{dt}\vec{p}_k\right]$
- $\frac{d}{dt}\vec{L} = \sum_k [0 + r_k \times F_k] \implies N = \sum_k r_k \times F_k$
- for every particle k, force $F_k = F_{k,ext} + \sum_{j\neq k} F_{k\leftarrow j}$
- $\frac{d}{dt}L = \sum_k r_k \times F^e_k = \sum_k N^e_k$
  Rate of change of L = sum external torque
  If no enxternal torque, angular momentum of system is conserved

Show energy is conserved

Show energy is conserved

- $W = \int_{t_1}^{t_2} \sum_k F_k(t) \cdot dr_k(t) = \sum_k \int_{t_1}^{t_2} m_k \frac{dv_k}{dt} \cdot v_k dt = T_2 - T_1$
  where $T \equiv \sum_k \frac{1}{2} m_k v_k^2$
- Center of Mass: $r_k = R + r'_k, \; v_k = V + v'_k$
- $v_k^2 = V^2 + 2v'_k \cdot V + {v'_k}^2$
- $T = \sum_k \frac{1}{2} m_k v_k^2 = \frac{1}{2} M V^2 + 0 + \sum_k \frac{1}{2} m_k {v'_k}^2$
  KE of system = CoM KE + System KE in CoM frame
- $F_k = F_k^e + \sum_{j \neq k} F_{k \leftarrow j}$
- $W = \sum_k \int_1^2 (F_k^e + F_k^i) \cdot dr_k$
- If $U_{j,k}^i = U(r_k, r_j)$ then $F_{j \leftarrow k} - = -Fk \leftarrow k$
- unfinished

Summary of Multi-particle systems MT:C9.1-9.5

- Center of Mass of collection of particles
- Total momentum of system: $\vec{P} \equiv \sum_k p_k = \vec{P}_{CoM}$
- Net force on system $\vec{F} = M\frac{d}{dt}\vec{P}$
  If there are no external forces, linear momentum of system is conserved and equals the CoM momentum
- Total angular momentum of system about origin:
  $\vec{L} \equiv \sum_k L_k = \sum_k r_k \times p_k = R_{CoM} \times P_{CoM} + \sum_k r'_k \times p'_k$

  = CoM L about origin + sum of particle L'_k about CoM
- Net torque on system $N = \frac{d}{dt}L$ For central forces, net interal torque must vanish
- KE of system equals sum of CoM KE and sum of KE of each partcle in CoM frame
- For a conservative system, the total energy is

One particle Newtonian Mechanics: example

- Identify forces, draw free body diagram
- "balance forces" to satisfy constraints
- set up dynamical equations (Use IInd/IIIrd Laws and/or use conservation laws)
- solve math of dynamical equations, with appropriate boundary conditions

Examples: get Equations of Motion

1. Freely falling body, $\vec{F} = m(0, 0, -g) = m\frac{d^2}{dt^2}(x, y, z)$
2. Charged particle in magnetic field: $\vec{F} = q\vec{v} \times \vec{B}$
3. Block sliding down an incline (no friction)
4. Simple pendulum

Note: In Ex: 4 (5) constraint force is (is not) constant

From EoM to path properties

- Newton's laws: forces change state of system
  $\frac{d}{dt}(x, v) = (v, F/m)$ for each time point t,
- Ex: Harmonic Oscillator (1-dim): $F(x) = -kx = m\ddot{x}$
  Trajectory $x(t) = A\cos(wt),\ p(t) = -Amw\sin(wt)$
  Alternate description $(x/A)^2 + (p/B)^2 = 1$
  Describes whole path, not just at single time
- Ways to get path description:
  - Conservation laws $E = T + U$ along path
  - Straight line path for free particle
  - Also, for free particle minimum average kinetic energy path $T_{avg} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} dt\ T(t)$.
- Fermat minimum time principle for path of light:
  Snell's Law of refraction at interface between medium

Hand-wavy derivation of a path principle

- $F = \dot{p} \implies [F(t) - \dot{p}(t)] \cdot \eta(t) = 0$
- $J \equiv \int_1^2 [F - \dot{p}] \cdot \eta(t)\, dt$.
  For true path $(x^*(t), v^*(t))$, $J = 0$
- $J[\eta] = \int_1^2 [-\frac{\partial U}{\partial x} - m\dot{v}] \cdot \eta\, dt$
- Use $\frac{d}{dt}(v \cdot \eta) = \dot{v} \cdot \eta + v \cdot \dot{\eta}$ to get
  $J = \int_1^2 [-\nabla U \cdot \eta - m\frac{d}{dt}(v \cdot \eta) + mv \cdot \dot{\eta}]dt$
- Set $\eta(t_{1,2}) = 0$. Giving $J = \int_1^2 [-\nabla U \cdot \eta + mv \cdot \dot{\eta}]dt$
- Set $\eta(t) = x(t) - x^*(t)$, i.e. small variation of path from true path. $\delta x(t) = \eta(t), \delta v = \dot{\eta}(t)$
- $L(x, v) = -U(x) + mv^2/2 \implies J = \int_1^2 \delta L(x, v) dt$
- Action $S \equiv \int_1^2 L(x, v)dt$, thus $J = \delta S = 0$ for true path
- Principle of stationary action: $\delta S = 0$

Issues with derivation

- Issue-1: When constraints present, $F = -\nabla U + F_C$.
  But F_C is not included in above derivation
  Note that $F_c(t) \cdot dx^*(t) = 0$ on true path
- Issue-2: Non-simple constraints Ex: Pendulum
- Mathematically rigorous derivation of "Euler-Lagrange equations" MT-C7.6
  - $F_k = m\frac{d^2}{dt^2}x_k = m_k\frac{d^2}{dt^2}x_k$
  - $x_k \equiv x_k(\{q\}, t) \implies$
    $$\dot{x}_k = \sum_m \frac{\partial x_k}{\partial q_m}\dot{q}_m + \frac{\partial x_k}{\partial t} \quad \& \quad \frac{\partial \dot{x}_k}{\partial \dot{q}_n} = \frac{\partial x_k}{\partial q_n}$$
  - After a bit of algebra, using $L = T - U$
    $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}, \quad \forall k$
- Action $S = \int_{t_1}^{t_2} dt\, L(\{q_k\}, \{\dot{q}_k\}, t)$. Then

$$\delta S = 0 \iff \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}, \quad \forall k$$

Calculus of variations: Find stationary soln.

- $J = \int_{x_A}^{x_B} dx\ f\left(y(x), \frac{dy}{dx}(x); x\right)$,
  find $y(x)$ s.t. J is extremum, with fixed $y(x_{A,B}) = y_{A,B}$
- Path space parametrization: $y(\alpha, x) = y(0, x) + \alpha\eta(x)$
- $J(\alpha) = \int_{x_1}^{x_2} f\left(y(\alpha, x), y'(\alpha, x); x\right) dx \implies$
  for true path $y(0, x)$, we have $\frac{\partial J}{\partial \alpha}|_{(\alpha=0)} = 0$
- With $y = y_0 + \alpha\eta$, we have $\frac{\partial y}{\partial \alpha} = \eta$, $\frac{\partial y'}{\partial \alpha} = \frac{d\eta}{dx}$
- $\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} dx\ f(y, y'; x)$, with $y(\alpha), y'(\alpha)$
- $$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx\ \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial \alpha}\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right)$$

Calculus of variations (cntd.)

- $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\eta\right) = \eta \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}$
- Thus for $\eta(t_{1,2}) = 0$, we get

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x)$$

- Setting $\frac{\partial J}{\partial \alpha} = 0$, for every η then gives:

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$$

This is the famous Euler-Lagrange equation, with $f \equiv f\,(y(x), y'(x); x)$

Euler equation of 'second' kind

$f \equiv f(y(x), y'(x))$, hence $\frac{\partial f}{dx} = 0$, then $f - y' \frac{\partial f}{\partial y'}$ =constant

$$df = \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial y'} dy' + \frac{\partial f}{dx} dx \implies \frac{df}{dx} = y' \frac{\partial f}{dy} + y'' \frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}$$

Use E-L equation $\frac{d}{dx} \frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}$ to get

$$\frac{df}{dx} = \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'} \right) + \frac{\partial f}{\partial x}$$

Brachistochrone problem MT-C6-Ex:6.2

What is the shortest time curve for the particle travel while starting from rest at point A to reach point B with $z_A > z_B$?

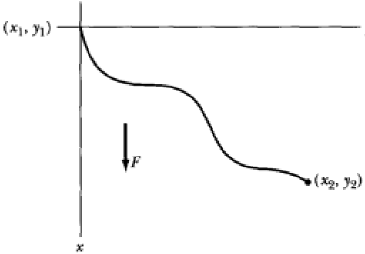


FIGURE 6-3 Example 6.2. The *brachistochrone* problem is to find the path of a particle moving from (x_1, y_1) to (x_2, y_2) that occurs in the least possible time. The force field acting on the particle is F, which is down and constant.

$$mv^2/2 = mgx \implies v = \sqrt{2gx}, ds = vdt = \sqrt{dx^2 + dy^2}$$

$$T = \int_1^2 \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gx}} = \int_{x_1}^{x_2} dx \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}$$

Path Principle: General Pointers

- Generalize to N-particle systems.
  $L = KE - PE = \sum_{k=1}^{N} \frac{1}{2} m_k v_k^2 - U(\{r_k\}_{k=1}^{N})$
- $S = \int_1^2 dt\ L(\{r_k\}, \{v_k\}), \delta S = 0$ over paths.
  $\{r_k\} \mapsto q_j$ (transformation of coordinates) still has
  $\delta S = 0$ with $S = \int_1^2 dt\ L'(\{q_j\}, \{\dot{q}_j\})$
  This is the power of stationary action
- Now,

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} dt\ \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right) \cdot \eta_k(t)$$

  we can derive equation of motion in q_j as

$$L(\{q_k\}, \{\dot{q}_k\}) \text{ has } \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}, \forall k$$

- NOTE: [MT] has direct derivation from $F = ma$

Generalised coordinates

- Simple pendulum MT-E7.2
- Wall of death MT-E7.4
- block sliding down raising incline MT-P7-11

Simple Pendulum MT-E7.2

For the simple pendulum, derive E-L EoM

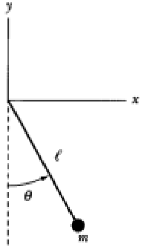


FIGURE 7-1 Example 7.2. A simple pendulum of length ℓ and bob of mass m.

- single generalised coordinates θ
- $L = \frac{1}{2}m(l\dot{\theta})^2 - (-mgl\cos\theta) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$
- $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$
- E-L: $ml^2\ddot{\theta} = -mgl\sin\theta$
- For small osciallatons, we get $\ddot{\theta} = -w^2\theta$, a simple harmonic oscillator

Wall of death MT-E7.4

See: https://www.youtube.com/shorts/HKS31wOunY8
(MT-C9-Example 7.4) Particle travelling on a cone of half-angle α subject to gravitational force. Find EoM.

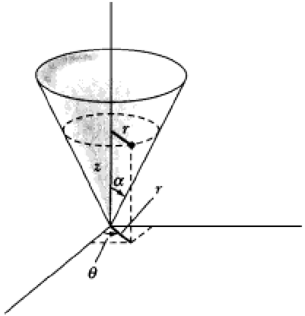


FIGURE 7-2 Example 7.4. A smooth cone of half-angle α. We choose r, θ, and z as the generalized coordinates.

$z = r\cot\alpha$, hence $(x, y, z) = (r\cos\theta, r\sin\theta, r\cot\alpha)$
Clearly, only two generalized coord. required! r and θ

Wall of death (continued)

- $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2 \cot^2\alpha = \dot{r}^2 \csc^2\alpha + r^2\dot{\theta}^2$
- $U = mgz = mgr\cot\alpha$
- $L = \frac{1}{2}m(\dot{r}^2 \csc^2\alpha + r^2\dot{\theta}^2 - mgr\cot\alpha)$
- $\frac{\partial L}{\partial \theta} = 0$ and $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} =$constant [Angular momentum about z-axis is conserved]
- $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cot\alpha$ and $\frac{\partial L}{\partial \dot{r}} = m\dot{r}\csc^2\alpha$
- E-L for r is thus, $m\ddot{r}\csc^2\alpha = mr\dot{\theta}^2 - mg\cot\alpha$ which is $\ddot{r} - r\dot{\theta}^2\sin^2\alpha + \frac{1}{2}\sin(2\alpha) = 0$

Block sliding down raising incline MT-P7.12

A block is located on an incline whose angle is increasing linearly with time. Find EoM

- Angle of incline $\theta = \alpha t$
- Distance of block from point of rotation along incline q
- $L = \left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m(q\dot{\theta})^2\right) - mgq\sin\theta$
  $L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}mq^2\alpha^2 - mgq\sin(\alpha t) = L(q, \dot{q}, t)$
- $\frac{\partial L}{\partial \dot{q}} = m\dot{q} \implies \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ddot{q}$
- $\frac{\partial L}{\partial q} = mq\alpha^2 - mg\sin(\alpha t)$

Thus the equation of motion is:

$$\ddot{q} = \alpha^2 q - g\sin(\alpha t)$$

ML: training as variational problem

Given data set $\{x_k, y_k\}$ find function f s.t. $y = f(x)$
Standard methodology is to minimize $J = \int dx\, [y - f_\theta(x)]^2$

Alain & Bengio, Journal of Machine Learning Research 15 (2014) 3743

Appendix C. Calculus of Variations

Theorem 2 *Let p be a probability density function that is continuously differentiable once and with support $\mathbb{R}^d$ (i.e., $\forall x \in \mathbb{R}^d$ we have $p(x) \neq 0$). Let $\mathcal{L}_{\sigma^2}$ be the loss function defined by*

$$\mathcal{L}_{\sigma^2}(r) = \int_{\mathbb{R}^d} p(x) \left[\|r(x) - x\|_2^2 + \sigma^2 \left\| \frac{\partial r(x)}{\partial x} \right\|_F^2 \right] dx$$

for $r : \mathbb{R}^d \to \mathbb{R}^d$ assumed to be differentiable twice, and $0 \leq \sigma^2 \in \mathbb{R}$ used as factor to the penalty term.

Let $r^*_{\sigma^2}(x)$ denote the optimal function that minimizes $\mathcal{L}_{\sigma^2}$. Then we have that

$$r^*_{\sigma^2}(x) = x + \sigma^2 \frac{\partial \log p(x)}{\partial x} + o(\sigma^2) \quad as \quad \sigma^2 \to 0.$$

Moreover, we also have the following expression for the derivative

$$\frac{\partial r^*_{\sigma^2}(x)}{\partial x} = I + \sigma^2 \frac{\partial^2 \log p(x)}{\partial x^2} + o(\sigma^2) \quad as \quad \sigma^2 \to 0.$$

Both these asymptotic expansions are to be understood in a context where we consider $\{r^*_{\sigma^2}(x)\}_{\sigma^2 \geq 0}$ to be a family of optimal functions minimizing $\mathcal{L}_{\sigma^2}$ for their corresponding value of σ^2. The asymptotic expansions are applicable point-wise in x, that is, with any fixed x we look at the behavior as $\sigma^2 \to 0$.

(part 1 of the proof)

We make use of the Euler-Lagrange equation from the Calculus of Variations. We would refer the reader to either (Dacorogna, 2004) or Wikipedia for more on the topic. Let

$$f(x_1, \ldots, x_n, r, r_{x_1}, \ldots, r_{x_n}) = p(x) \left[\|r(x) - x\|_2^2 + \sigma^2 \left\| \frac{\partial r(x)}{\partial x} \right\|_F^2 \right]$$

Two body problem: m_1, m_2 & gravity[MT:C8]

- $L = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - U(|r_1 - r_2|)$
  Two particle system: 6 coordinates (and 6 velocities)
- Choose appropriate generalised coordinates
  - Center of Mass: $\vec{R} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2$
  - Diff vector: $\vec{r} = \vec{r}_1 - \vec{r}_2$
- With $\vec{R}$ and $\vec{r}$, we find: $L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$
- Set $\dot{\vec{R}} = 0$ (Why?). And $\vec{r}$ as 2d vector (Why?):
  $L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) = L(r, \dot{r}, \theta, \dot{\theta})$
  generalised coordinates r and θ: 1 particle in 2D
- $\frac{\partial L}{\partial \theta} = 0 \implies \frac{\partial L}{\partial \dot{\theta}} = \ell$, a constant. Thus
  $\mu r^2 \dot{\theta} \implies \ell = r \cdot r\dot{\theta} = \ell$, Kepler's second law
- $\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$ and $\frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - U'(r)$. Thus,
  $\mu\ddot{r} = \mu r\dot{\theta}^2 - U'(r) \implies \mu\ddot{r} = \ell^2/(\mu r^3) - U'(r)$
  (a) Circular orbit (b) non-circulur orbit

E-L with undetermined multipliers

Constraints $f_k(\{x\}, t) = 0$
Example of disk rolling down the incline in next slide.

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0$$

$$g(y, \theta) = y - R\theta = 0 \qquad (6.73)$$

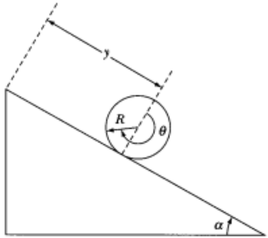


FIGURE 6-7 Example 6.5. A disk rolls down an inclined plane without slipping.

Figure: Disk rolling without slipping

Example: Disk rolling down incline MT-E7.9

- Generalized coordinates y and θ
- Constraint $g(y,\theta) = y - R\theta = 0$; $\quad \frac{\partial g}{\partial y} = 1, \ \frac{\partial g}{\partial \theta} = -R$
- $L = \left(\frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2\right) - Mg(-y\sin\alpha) = L(y, \dot{y}, \dot{\theta})$
- $$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda\frac{\partial f}{\partial y} = 0$$
  $$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0$$
- $Mg\sin\alpha - M\ddot{y} + \lambda 1 = 0$ and
  $0 - I\ddot{\theta} - \lambda R = 0$
  $y - R\theta = 0$
- Eliminate λ: $\lambda = -I\ddot{\theta}/R = -I\ddot{y}/R^2$, which gives
  $Mg\sin\alpha - M\ddot{y} - I\ddot{y}/R^2 = 0$
  $$\ddot{y} = \left(\frac{M}{M + I/R^2}\right) g\sin\alpha$$

Comparison: Newtonian and Lagrangian formulations

	Newtonian	Lagrangian
Equation of Motion	$m_k \frac{d^2}{dt^2} x_k = F_k$	$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}$
Order of diff. equation	Second order	Second order
Coordinates	Cartesian only	generalised coordinates
Interactions via	using forces	using energies
Constraints	'balancing' forces	naturally or by equation of constraints

Time symmetry leads to Conservation of Energy

Time symmetry: A path is 'translated' time, and has no change in value of Lagrangian,i.e. when $\frac{\partial L}{\partial t} = 0$

$$\frac{dL}{dt} = \nabla_q L \cdot dq + \nabla_{\dot{q}} L \cdot d\dot{q} + \frac{\partial L}{\partial t}$$

which gives

$$\frac{d}{dt}\left(L - \dot{q} \cdot \nabla_{\dot{q}} L\right) = \frac{\partial L}{\partial t} = 0$$

This shows that $H = \sum_k q_k p_k - L =$ constant when $\frac{\partial L}{\partial t} = 0$
Show that $H = E$ when $L = mv^2/2 + U(x)$ proving energy is conserved

Translation symmetry leads to Conservation of Momentum

Translation symmetry: A path 'translated' in space, has no change in value of Lagrangian i.e. $\delta L = 0$ for $\delta \vec{r}$ in path

$$\delta L = \nabla_x L \cdot \delta x + \nabla_{\dot{x}} L \cdot \delta \dot{x} = 0$$

Now noting that translation means that$\delta \dot{x} = 0$, we get

$$\delta L = \nabla_x L \cdot \delta x = 0 \implies \frac{\partial L}{\partial x_i} = 0$$

which from EL equations, give $\frac{d}{dt}\frac{\partial L}{\partial v_i} = 0 \implies mv_i =$ constant!

Rotational symmetry

Rotational Symmetry: A path ”rotated” in space, has no change in value of Lagrangian $\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$ and hence $\delta\vec{v} = \delta\vec{\theta} \times \vec{v}$

HOMEWORK Show $\delta L = 0 \implies \vec{r} \times \vec{p} =$ constant

See MT-7.9

Hamiltonian Dynamics

- $p_k = \frac{\partial L}{\partial \dot{q}_k}$, whose solution gives $\dot{q}_k = \dot{q}_k(q_k, p_k, t)$
- $H = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$ can be seen as Legrende transform of L to replace variable $\dot{q}_k$
- $H \equiv H(\{q_k\}, \{\dot{q}_k\}, t)$ i.e. natural variables: q_k and p_k
- dH with $H(q_k, p_k, t)$ and $H = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$
- Hamilton equation of motion

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \forall k$$

- p_k and q_k are treated equivalently.
- for each k, two first-order differential equations

Statistical Mechanics

- Atoms / molecules approximated as interacting particles (Lennard-Jones, electrostatic, bonds (as springs) etc)
- Classical mechanics reasonably model dynamics
- ISSUE: $N \sim 10^{23}$ atoms. Too large for tracking numerically
- Statistics is the way to go
- Statistical Mechanics: child of Boltzmann, based on strong belief on reality of atoms
- Statistical Mechanics gives a probabilistic description of system, moving away from deterministic Classical Mechanics. This was major sticking point/contention of its acceptability among Boltzmann's contemporary scientists (thermodynamics, mechanics)

Simple Harmonic oscillator

- $m\ddot{x} = -kx \implies x = A\cos(wt), w = \sqrt{k/m}$
- $x = A\cos(wt) \implies p = m\dot{x} = -Amw\sin(wt)$
- $E = \frac{1}{2}kx^2 + \frac{1}{2m}p^2 = \frac{1}{2}kA^2$
- phase plot (x vs p): point on ‘circle’ with angle wt. Phase-point $\mathbf{x} = (x, p)$ visits all points equally
- Make a probabilistic statement: ”Equal probability for system to be in any of the *possible* phase points”

A mathematically rigorous proof for Hamiltonian systems: Louivlle theorem for phase space density (in an ensemble) $\frac{d}{dt}\rho(q_k, p_k, t) = 0$

Postulates of statistical mechanics

- State of system $\mathbf{x} = \left(\{\vec{r}_k\}_{k=1}^N, \{\vec{p}_k\})_{k=1}^N\right)$, $\mathbf{x} \in \mathcal{R}^{6N}$. Phase-space point
- POSTULATE: "Equal a priori probability of states with equal energy"
- EQUIVALENTLY: "probability of state **x** is proportional to $e^{-E(x)/kT}$". This is commonly known as Boltzmann Law.
- Ensemble: a collection of states consistent with thermodynamic constraint
  - Micro-canonical: particles, volume and energy
  - Canonical: particles, volume and temperature
  - Constant pressure, temperature and particles
  - Grand canonical: chemical potential, volume and temperature
- POSTULATE: (Ergodicity) Time average equals Ensemble average

Canonical ensemble: general setup

- phase point $\mathbf{x} \in \mathcal{R}^{3N} \times \mathcal{R}^{3N}$ (generalized coordinate, generalised momentum space)
- $p(\mathbf{x}) \propto e^{-E(\mathbf{x})/kT} = \frac{1}{Z} e^{-E(\mathbf{x})/kT}$
- Partition function $Z = \int d\mathbf{x}\ e^{-E(\mathbf{x})/kT} = \frac{1}{Z} e^{-\beta E}$, $\beta = 1/kT$
- $\langle E \rangle = \int d\mathbf{x}\ p(\mathbf{x}) E(\mathbf{x}) = -\frac{\partial}{\partial \beta} \ln Z$
- Reminder Gibbs-Helmholtz relation in Thermodynamics:

$$U = \left(\frac{\partial (A/T)}{\partial (1/T)} \right)_{V,N}$$

We identify $A/kT = -\ln Z + \phi(V, N)$, setting $\phi = 0$

- $A = -kT \ln Z$, all thermodynamics from this relation!

Degeneracy of energy level, Entropy

- $A = -kT \ln Z = -kT \int d\mathbf{x} \exp - \frac{E(\mathbf{x})}{kT}$
- $Z = \int_{E_0}^{\infty} dE \; W(E) \exp(-E/kT)$
- Integrand is gaussian with peak $\bar{E}$
  $Z = e^{-\bar{E}/kT} W(\bar{E}) \Delta E$ where ΔE is spread
- $S = (U - A)/T = (\bar{E} + kT \ln Z)/T = k \ln \left[W(\bar{E}) \Delta E \right]$
- Entropy S is thus a measure of number of states assailable to system given by $S(T) = k \ln W(\bar{E}) \Delta E$ where $\bar{E}$ and spread ΔE both are T dependent
- Microcanonaical (const E,V,N): has
  $S(E, V, N) = k \ln W(E)$
- uncertainity principle $\Delta x \Delta p_x \geq h$. So gives volume of each state is given by $\Delta x \Delta p_x = h$

System of non-interacting particles

Ideal system i.e. identical non-interacting (independent) particles at temperature T

- Total energy $E = \sum_{E} k$ where E_k is KE of particle k
- Probability density $p(\mathbf{x}) = \Pi_{k=1}^{N} p_1(\mathbf{x}_k)$,
- $p_1(\mathbf{x}_k) = \frac{1}{Z_1} e^{-E_1(\mathbf{x}_k)/kT}$ with $Z_1 = \int d\mathbf{x}_k \; e^{-E_1(\mathbf{x}_k)/kT}$
- Barometric Law: $E_1 = mgh \implies \rho(h) \propto exp(\frac{mgh}{kT})$
- Maxwell-Boltzmann distribution:
  $E_1 = \frac{1}{2} m\vec{v} \cdot \vec{v} \implies p_1(\vec{v}) \propto exp(-\frac{mv^2}{2kT})$

System of non-interacting particles (cntd.

Equi-partition theorem:

- $Z_1 = \int d\mathbf{x_1} e^{-p_1^2/2mkT}$
- $d\mathbf{x} = d\vec{r} d\vec{p} = \int dx \int dy \int dz \int dp_x \int dp_y \int dp_z$
- $Z_1 = \left[\int d\vec{r} 1\right] \left[\int d\vec{p}\, e^{-p^2/2mkT}\right] =$
  $V \left[\int_{-\infty}^{+\infty} dp_x e^{-p_x^2/2mkT}\right]^3 = V\left(\sqrt{2\pi mkT}\right)^3$
  $Z_1 \propto T^{3/2}$
- $\langle E_1 \rangle = -\frac{\partial ln Z_1}{\partial \beta} = \frac{3}{2} kT$
- for N Particle system
  - $\langle E \rangle = \frac{3}{2} Nk\ T = \frac{3}{2} RT$
  - $Z_N = (Z_1)^N \implies p = -\left(\frac{\partial A}{\partial V}\right)_{T,N} = kT \frac{N}{V}$
- Every quadratic term in energy contributes $\frac{1}{2} kT$

Grand-canonical ensemble statistics: Ideal systems i.e. non-interacting particles

- Constant chemical potential μ, volume V and Temperature T.
  Varying number of particles N
- each state can have occupancy of:
  - $n_s = 0, 1 \implies$ Fermi-Dirac statistics
  - $n_s = 0, 1, 2, \cdots, N \implies$ Bose-Einstein statistics
- Occupation probability: $prob(n_s) = \frac{1}{\exp\left(\frac{E_s - \mu}{kT}\right) \pm 1}$, with
  - $+$ sign for Fermi-Dirac statistics
  - $-$ sign for Bose-Einstein statistics
  - When 1 is removed in denominator, we get Boltzmann statistics

Black-body radiation: Boson photon gas

Specific heat of solids