Lecture 1

(1 August 2024)

Probability and Random Processes

Module 1 - Basics of Probability

Module 2 - Discrete Random Variables

Module 3 - Continuous Random Variables

Module 4 - Tail Bounds and Limit Theorems

Module 5 - Random Processes

Textbooks:

- 'Probability, Random variables and stochastic processes', Papoulis and Pillai.
- 'Introduction to Probability', Bertsekas and Tsitsiklis.

Grading Plan (Tentative)

Assignments	–	15 %
Quiz 1	–	15 %
Quiz 2	–	15 %
In-class Quizzes	–	5 %
Mid-sem	–	20 %
End-sem	–	30 %

Module 1

- Different approaches to probability
- Probability space
- Conditional probability, Independence
- Total probability theorem, Bayes' theorem
- Continuity of Probability
- Review of counting

We may encounter several things in life that cannot be explained in the language of certainity.

A random experiment is an experiment whose outcomes we cannot predict with certainity.

We need to develop a language to speak about such problems which could not be formulated in the language of certainity.

Probability theory is a mathematical framework that allows us to describe and analyze random experiments

$\boxed{\text{Probability (roughly) means possibility}}$

It helps us to predict how likely or unlikely an event will occur.

Different Approaches to Probability

A. Classical Approach

Probability of an event E, P(E)

$$= \frac{\text{No. of outcomes favourable to event } E}{\text{Total no. of possible outcomes}}$$

- Equitable distribution of ignorance

Example. We roll a pair of unbiased dice. What is the probability that the sum of numbers equals 10?

Ans. $P = 3/36 = 1/12$.

This approach suffers from at least two problems

(1) It cannot deal with outcomes that are not 'equally likely'.

In the example above total no. of possible sum values is 11 (2, 3, ..., 12).

Only one sum 10 is favourable.

Can the probability $= 1/11$?

(2) It cannot handle scenarios when the total no. of possible outcomes is infinite.

B. Relative Frequency Approach

Perform the experiment n times.

Let n_E be the no. of times E occurs.

$$P(E) = \lim_{n \to \infty} \frac{n_E}{n}.$$

The issues with this approach are

(1) we cannot perform the experiment infinite number of times

(2) The ratio $\frac{n_E}{n}$ may not converge as $n \to \infty$.

[Despite the problems with the relative frequency approach of probability the concept of relative frequency is essential in applying probability theory to the real world.]

we should develop an approach that is coherent.

C. Axiomatic Approach

This approach is based on conceptual/thought experiment.

Probability space – $(\Omega, \mathcal{F}, P)$

Ω – Sample space

$\mathcal{F}$ – Event space

P – Probability law

We shall review set theory before we

Set Theory

A set is a well-defined collection of objects, which are called the elements of the set.

A set with no elements is called the empty set, denoted by ϕ or $\{\}$.

A set with a finite number of elements is a finite set.

If a sets contains infinitely many elements which can be enumerated in a list (i.e., a bijective mapping with natural numbers), we write $S = \{x_1, x_2, \ldots\}$ and call S as a countably infinite set.

E.g. Set of even integers $= \{0, 2, -2, 4, -4, \ldots\}$

A set is uncountable if its elements cannot be enumerated in a list.

Exercise: Prove that $\mathbb{Q} \cap [0, 1]$ is a countably infinite set.

Exercise: Prove that $\{0, 1\}^{\infty}$ is an uncountably infinite set.
[Use Cantor's diagonalization argument]

Subset notation: $A \subseteq B \Rightarrow (x \in A \Rightarrow x \in B)$.

Set difference: $A \setminus B = \{x \in \Omega : x \in A \ \& \ x \notin B\}$.

Universal set Ω contains all objects that could be of interest in a particular context.

Set operations:

- Complement of a set S

$$S^c = \{x \in \Omega : x \notin S\}.$$

- Union of two sets A and B

$$A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\}$$

- Intersection of two sets A and B

$$A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\}$$

- Infinite union

If for every $n \in \mathbb{N}$ we are given S_n

$$\bigcup_{n=1}^{\infty} S_n = \{x \in \Omega : x \in S_n \text{ for some } n \in \mathbb{N}\}$$

- Infinite intersection

$$\bigcap_{n=1}^{\infty} S_n = \{x \in \Omega : x \in S_n \text{ for all } n \in \mathbb{N}\}$$

Properties

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Demorgan's laws

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Exercise (i) For $n \in \underline{N}$ show that

$$\left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c .$$

[use mathematical induction]

(ii) Given sets $\underline{S_1}, \underline{S_2}, \ldots$ show that

$$\left(\bigcup_{i=1}^{\infty} S_i \right)^c = \bigcap_{i=1}^{\infty} S_i^c .$$

Note that (i) $\not\Rightarrow$ (ii). That is if we prove a statement T_n for all $n \in \mathbb{N}$ then this may not imply that T_∞ is true.

To see this consider the following.

Let $A_n = \{n, n+1, \ldots\}$, $n \in \mathbb{N}$.

$$T_n = \bigcap_{i=1}^{n} A_i \text{ is non-empty}$$

But T_∞ is an empty set. We prove this via contradiction.

Suppose T_∞ is non-empty.

$$\exists\, m \in \bigcap_{i=1}^{\infty} A_i \Rightarrow m \in A_i \text{ for all } i \in \mathbb{N}.$$

$$\Rightarrow m \in A_{m+1}.$$

However $A_{m+1} = \{m+1, m+2, \ldots\}$

and $m \notin A_{m+1}$.

This is a contradiction,

so we need to prove part (ii) of this exercise independent of (i).

Hint, Use the definitions of infinite union and infinite intersection of sets.