Introduction to Coding Theory - Spring 2025
Assignment 2
Submission Deadline: 06 feb

1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two binary linear block codes.
   (a) Under what conditions is $\mathcal{C}_1 \cup \mathcal{C}_2$ a linear code?
   (b) Under what conditions is $\mathcal{C}_1 \cap \mathcal{C}_2$ a linear code?

2. Let $\mathcal{C}$ be an $[n = 1023, k, d]$ binary linear block code such that the vector $(1, 1, 1, \ldots, 1) \in \mathcal{C}$. Compute

$$\frac{1}{|\mathcal{C}|} \sum_{\mathbf{c} \in \mathcal{C}} (w_H(\mathbf{c}) \bmod 2)$$

   [Hint: Consider $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^m$. How is the parity of $w_H(\mathbf{u}+\mathbf{v})$ related of parities of $w_H(\mathbf{u})$ and $w_H(\mathbf{v})$? Can you find a relation between the number of even weight codewords and the number of odd weight codewords?]

3. Let $\mathcal{C}$ be an $[n, k, d]$ binary linear block code such that every column of its generator matrix has at least one 1. Let $\mathcal{C}_i = \{\mathbf{c} \in \mathcal{C} : c_i = 0\}$.
   (a) Prove that $\mathcal{C}_i$ is a linear code.
   (b) Find the dimension of $\mathcal{C}_i$.
   (c) Comment on the minimum distance of $\mathcal{C}_i$.

4. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ be a matrix over $\mathbb{F}_3$. Let W denote the space spanned by the columns of $\mathbf{A}$.
   (a) Verify if W is a vector space over $\mathbb{F}_3$.
   (b) Find three different bases of W.
   (c) Define the dimension of a subspace. What is the dimension of W?
   (d) Define the rank of a matrix. Show by calculation that the row rank of $\mathbf{A}$ and the column rank of $\mathbf{A}$ are the same and, therefore, obtain the rank of $\mathbf{A}$.

5. A code $\mathcal{C}$ is called a self-dual code if $\mathcal{C} = \mathcal{C}^\perp$. Prove that if $\mathcal{C}$ is a self-dual code, the code length must be even, and the code must have a rate equal to 1/2. Further, if G is a generator matrix of $\mathcal{C}$, show that $GG^T = \mathbf{0}$.