Probability Refresher

Univariate

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Introduction

References

- MIT OCW MIT RES.6-012 Introduction to Probability, Spring 2018. (https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-041sc-probabilistic-systems-analysis-and-applied-probability-fall-2013/unit-i/lecture-1-introduction-to-probability/)

Sample Space

- A sample space is a set of all possible outcomes of an experiment.
- Typically denoted by Ω.
- For example, if we toss a coin, the sample space is $\Omega = \{H, T\}$.
- If we toss two coins, the sample space is $\{HH, HT, TH, TT\}$.
- If we toss a coin and roll a die, the sample space is $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.
- Continuous sample spaces are also possible. For example, if we measure the height of a person, the sample space is $\mathbb{R}$.

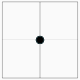
Sample Space

Consider two rolls of a die. What is the sample space?

X/Y	1	2	3	4	5	6
1	11	12	13	14	15	16
2	21	22	23	24	25	26
3	31	32	33	34	35	36
4	41	42	43	44	45	46
5	51	52	53	54	55	56
6	61	62	63	64	65	66

Sample Space

Consider you are throwing a dart at a dartboard (square from (0, 0) to (1, 1)). What is the sample space?



The sample space is $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Events

- An event is a subset of the sample space.
- For example, if we toss a coin, the sample space is $\Omega = \{H, T\}$. The event that the coin lands heads is $A = \{H\}$.
- If we toss two coins, the sample space is $\{HH, HT, TH, TT\}$. The event that the first coin lands heads is $A = \{HH, HT\}$.
- If we toss a coin and roll a die, the sample space is $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$. The event that the coin lands heads is $A = \{H1, H2, H3, H4, H5, H6\}$.
- If we measure the height of a person, the sample space is $\mathbb{R}$. The event that the height is greater than 6 feet is $A = \{x \in \mathbb{R} : x > 6\}$.

Axioms of Probability

1. $P(A) \geq 0$ for all events A.
2. $P(\Omega) = 1$.
3. If $A_1, A_2, \ldots$ are disjoint events, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Consequences of Axioms

1. $P(\emptyset) = 0$.
2. $P(A^c) = 1 - P(A)$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
4. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

Conditional Probability

- The conditional probability of A given B is defined as $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
- If A and B are independent, then $P(A|B) = P(A)$.
- If A and B are independent, then $P(A \cap B) = P(A)P(B)$.

Random Variables

- A random variable is a function from the sample space to the real numbers, i.e., $X : \Omega \to \mathbb{R}$.
- For example, if we toss a coin, the sample space is $\Omega = \{H, T\}$.
    - We can define a random variable X as $X(H) = 1$ and $X(T) = 0$.
    - We could also have a random variable Y to denote our gain from the coin toss, i.e., $Y(H) = 1$ and $Y(T) = -1$.
- If we toss two coins, the sample space is $\{HH, HT, TH, TT\}$.
    - We can define a random variable X as $X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$, and $X(TT) = 0$, where X denotes the number of heads.

Random Variables Notation

X denotes a random variable. x denotes a particular value of the random variable.

Probability Mass Function

- The probability mass function (PMF) of a discrete random variable X is defined as $p_X(x) = P(X = x)$.
- For example, if we toss a coin, the sample space is $\Omega = \{H, T\}$.
    - We can define a random variable X as $X(H) = 1$ and $X(T) = 0$.
    - The PMF of X is $p_X(1) = P(X = 1) = P(H) = 0.5$ and $p_X(0) = P(X = 0) = P(T) = 0.5$.
- If we toss two coins, the sample space is $\{HH, HT, TH, TT\}$.
    - We can define a random variable X as $X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$, and $X(TT) = 0$, where X denotes the number of heads.
    - The PMF of X is $p_X(2) = P(X = 2) = P(HH) = 0.25$, $p_X(1) = P(X = 1) = P(HT) + P(TH) = 0.5$, and $p_X(0) = P(X = 0) = P(TT) = 0.25$.

Probability Mass Function

Consider two rolls of die. The following is the sample space.

X/Y	1	2	3	4	5	6
1	11	12	13	14	15	16
2	21	22	23	24	25	26
3	31	32	33	34	35	36
4	41	42	43	44	45	46
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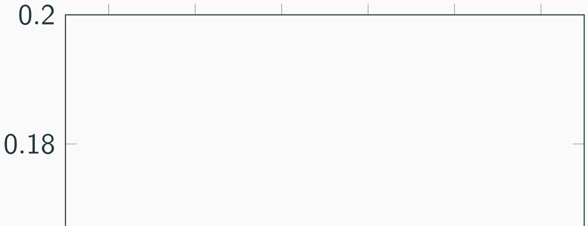
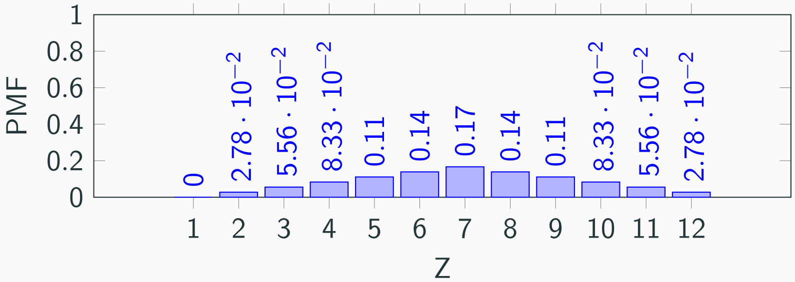
- X and Y are random variables (value of the 1st and 2nd die)
- $p_X(1) = P(X = 1) =$
  $P(11) + P(11) + P(13) + P(14) + P(15) + P(16) = \frac{1}{6}$



Probability Mass Function

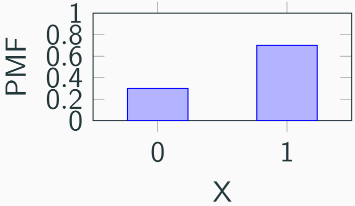
- Let us create a new random variable $Z = X + Y$.
  - $p_Z(1) = 0$ as there is no way to get a sum of 1 from two die rolls.
  - $p_Z(2) = P(Z = 2) = P(11) = \frac{1}{36}$



Discrete Random Variables

Bernoulli Distribution

- A random variable X is said to follow a Bernoulli distribution with parameter p if X can take only two values, 0 and 1, and $P(X = 1) = p$.
- The PMF of X is $p_X(x) = p^x(1-p)^{1-x}$.
- For example, if we toss a coin, the sample space is $\Omega = \{H, T\}$.
  - We can define a random variable X as $X(H) = 1$ and $X(T) = 0$.
  - The PMF of X is $p_X(1) = P(X = 1) = P(H) = p$ and $p_X(0) = P(X = 0) = P(T) = 1 - p$.



Expected Value

- The expected value of a random variable X is defined as $E[X] = \sum_{x \in \mathcal{X}} x.p_X(x)$.
- For example, consider a random variable X that follows a Bernoulli distribution with parameter p.
  - The expected value of X is $E[X] = 0 \times (1-p) + 1 \times p = p$.
- Consider a random variable X that follows a uniform distribution over the set $\{1, 2, 3, 4, 5, 6\}$.
- The expected value of X is $E[X] = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 3.5$.