Information Theory for Machine Learning

Nipun Batra

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IIT Gandhinagar

The Data Compression Problem

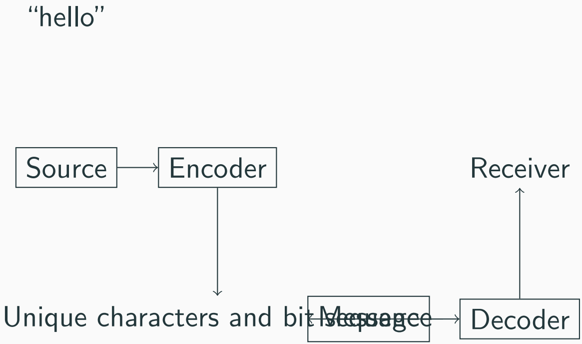


Figure 1: Data Compression Problem

Self Information

- What is more surprising: Snowing in Kashmir or Snowing in Gandhinagar?
- To formalize, let us assume that the probability of snowing in Kashmir is p_1 and that in Gandhinagar is p_2, and that $p_1 >> p_2$.
- How can we quantify the surprise?

Self Information

- Events that are less likely to occur are more surprising.
- Also, if an event is 100% likely to occur, it is not surprising at all.
- Also, if two events are independent, then the surprise of both of them occurring together is the sum of the surprise of each of them occurring individually.
- So, we need a function that maps probability to a number. Function should be: monotonic, and additive, and is 0 when the probability is 1.
- The function is $I(x) = -\log_2(x)$ also called the self information or surprisal.

Self Information

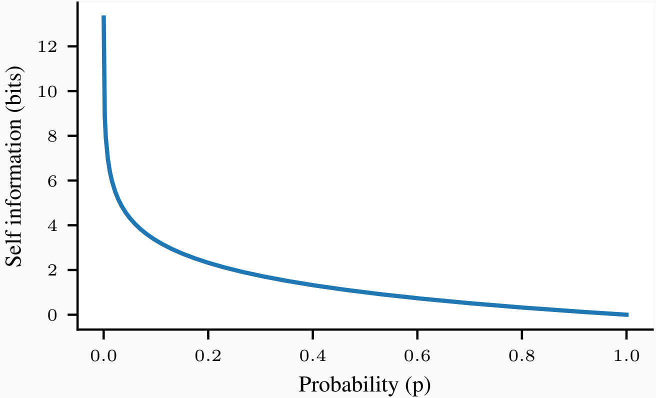
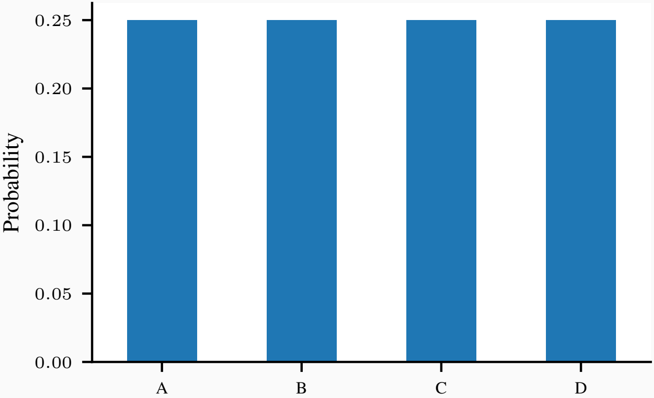


Figure 2: Self Information

Self Information

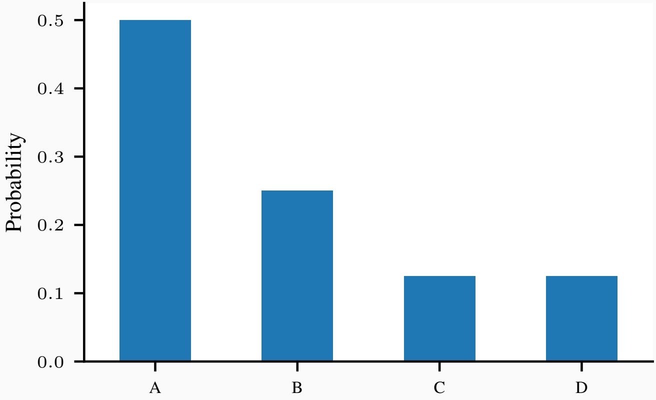
Consider a categorical random variable X with 4 possible outcomes: A, B, C, D. The probability of each of these outcomes is 0.25. What is the self information of each of these outcomes?



$I(A) = I(B) = I(C) = I(D) = 2$ bits.

Self Information

Consider a categorical random variable X with 4 possible outcomes: A, B, C, D. The probability these outcomes is 0.5, 0.25, 0.125, and 0.125. What is the self information of each of these outcomes?



$I(A) = 1$ bit, $I(B) = 2$ bits, $I(C) = I(D) = 3$ bits.

Self Information

Proof on additivity of self information: Consider two independent random variables X and Y with PMFs $p_X(x)$ and $p_Y(y)$ respectively. The joint PMF is $p_{X,Y}(x, y) = p_X(x)p_Y(y)$. The self information of the joint PMF is:

$$\begin{aligned} I(X = x, Y = y) &= -\log_2(p_X(x)p_Y(y)) \\ &= -\log_2(p_X(x)) - \log_2(p_Y(y)) \\ &= I(X = x) + I(Y = y) \end{aligned}$$

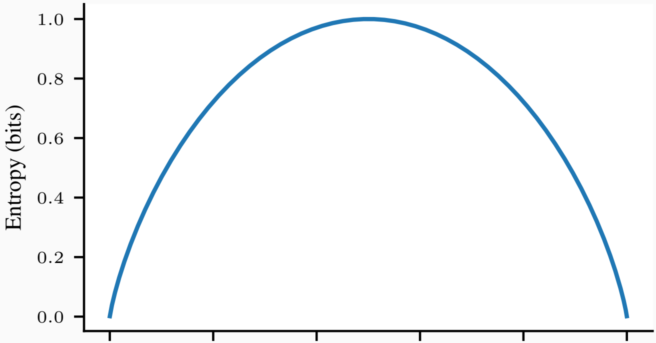
Entropy

- The entropy of a random variable is the expected value of the self information.
- $H(X) = \mathbb{E}_{X \sim p(x)}[I(X)] = \mathbb{E}_{X \sim p(x)}[-\log_2(p(x))]$
- The entropy of a random variable is the expected number of bits required to encode the random variable.
- The entropy of a random variable is the minimum number of bits required to encode the random variable.

Entropy

For a Bernoulli random variable X with probability p of success, the entropy is:

$$\begin{aligned} H(X) &= \mathbb{E}_{X\sim p(x)}[-\log_2(p(x))] \\ &= -\log_2(p) \times p - \log_2(1-p) \times (1-p) \\ &= -p\log_2(p) - (1-p)\log_2(1-p) \end{aligned}$$



Entropy

For a k class categorical random variable X with probability p_i of class i, the entropy is:

$$H(X) = \mathbb{E}_{X\sim p(x)}[-\log_2(p(x))]$$
$$= -\sum_{i=1}^{k} p_i \log_2(p_i)$$

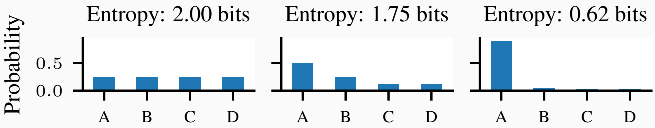


Figure 4: Entropy

Code Length

Let us assume our symbols are: A, B, C, D. Let us assume that the probability of each of these symbols is 0.25. Let us assume we use the following code to encode these symbols:

$$A \to 00$$
$$B \to 01$$
$$C \to 10$$
$$D \to 11$$

What is the expected code length?

Expected code length $= \sum_{i=1}^{4} p_i \times l_i = 2$ bits.

Code Length

Let us assume our symbols are: A, B, C, D. Let us assume that the probability of these symbols is 0.5, 0.25, 0.125, and 0.125. Let us assume we use the following code to encode these symbols:

$$A \rightarrow 00$$
$$B \rightarrow 01$$
$$C \rightarrow 10$$
$$D \rightarrow 11$$

What is the expected code length?

Expected code length $= \sum_{i=1}^{4} p_i \times l_i = 2$ bits. But, is this the most efficient code? No! What is the entropy of this random variable? $H(X) = 1.75$ bits.

Code Length

Let us assume our symbols are: A, B, C, D. Let us assume that the probability of these symbols is 0.5, 0.25, 0.125, and 0.125. Using fixed length codes, we need 2 bits to encode each symbol.

Key idea: Use shorter codes for more frequent symbols and longer codes for less frequent symbols.

How about the following code?

$$A \to 0$$

$$B \to 10$$

$$C \to 110$$

$$D \to 111$$

Expected code length $= \sum_{i=1}^{4} p_i \times l_i = 1.75$ bits.

Huffman Encoding

- Huffman encoding is a method to construct a variable length code for a random variable.
- The code is constructed such that the expected code length is equal to the entropy of the random variable.
- The code is constructed such that the code is a prefix code.

KL divergence

Suppose we have four symbols A, B, C, D with probabilities 0.5, 0.25, 0.125, and 0.125 respectively. Let us call this distribution $p(x)$. We want to transmit some data using these symbols. The optimum encoding scheme is:

$$A \to 0$$
$$B \to 10$$
$$C \to 110$$
$$D \to 111$$

But, for some reason, we believe that the four symbols are distributed as per $q(x)$: 0.25, 0.25, 0.25, and 0.25. For this distribution, the optimum encoding scheme is:

$$A \to 00$$
$$B \to 01$$