Maximum A Posteriori Estimation

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August 21, 2023

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Agenda

Revision

Coin Toss Problem

MAP for Logistic Regression

Revision

Bayes Rule

$$P(\theta|D) = \frac{P(D|\theta) \cdot P(\theta)}{P(D)}$$

- $P(\theta|D)$ is called the posterior
- $P(D|\theta)$ is called the likelihood
- $P(\theta)$ is called the prior
- $P(D)$ is called the evidence

Maximum Likelihood Estimation

$$P(\theta|D) = \frac{P(D|\theta) \cdot P(\theta)}{P(D)} = \frac{P(D|\theta) \cdot P(\theta)}{\int_\theta P(D|\theta) \cdot P(\theta) d\theta}$$

Given a dataset D, find the parameters θ that maximize the likelihood of the data.

$$\theta_{\text{MLE}} = \arg\max_\theta P(D|\theta)$$

For example, given a linear regression problem setup, we set the likelihood as normal distribution and find the parameters θ that maximize the likelihood of the data.

Maximum A Posteriori Estimation

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Given a dataset D, find the parameters θ that maximize the posterior of θ considering both the likelihood and the prior.

$$\theta_{\text{MAP}} = \arg\max_\theta P(\theta|D) = \arg\max_\theta P(D|\theta) \cdot P(\theta)$$

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- When do we need prior knowledge?

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- **MLE**: Given N observations, obtain best θ estimate (or θ_{MLE})
- What if we have prior knowledge about θ?
- **MAP**: Given N observations and prior knowledge, obtain best θ estimate (or θ_{MAP})
- When do we need prior knowledge?
  - When the dataset is not a good representation of the true distribution.
  - Can be a data quality and/or quantity issue.

Coin Toss Problem

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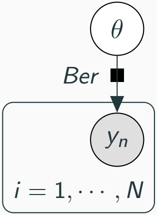
- Assuming $y_i \sim \text{Bernoulli}(\theta)$, $P(y_i|\theta) = \theta^{y_i}(1-\theta)^{1-y_i}$



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- Suppose 10 tosses yield 9 heads and 1 tail. $\theta_{MLE} = 0.9$
- What if we have prior knowledge that the coin is fair?

Incorporating Prior Information

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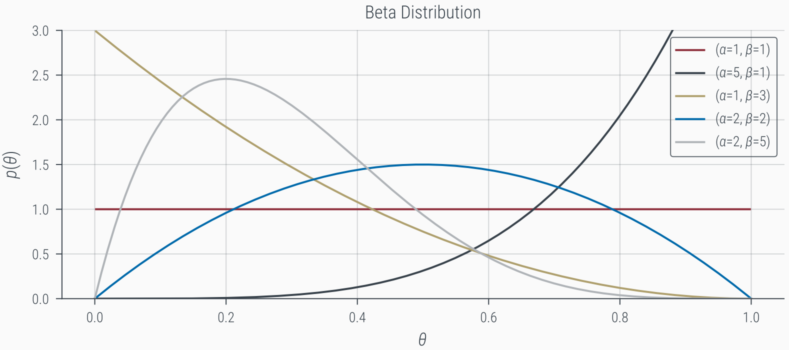
$$\implies P(\theta|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

where,

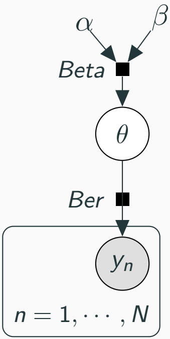
$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \;\; \text{(Gamma Function)}$$

Beta Distribution



Notebook

Coin Toss Problem with Prior



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$$(\alpha - 1) \log\theta + (\beta - 1) \log(1 - \theta)$$

Deriving θ_{MAP}

$$\frac{\partial \log P(\theta|D)}{\partial \theta} = \frac{\sum_{i=1}^{N} y_i}{\theta} - \frac{\sum_{i=1}^{N}(1-y_i)}{1-\theta} + \frac{\alpha-1}{\theta} - \frac{\beta-1}{1-\theta} = 0$$

$$\implies (1-\theta)\sum_{i=1}^{N} y_i + \theta\sum_{i=1}^{N}(1-y_i) + (1-\theta)(\alpha-1) - \theta(\beta-1) = 0$$

$$\implies \sum_{i=1}^{N} y_i - \theta\sum_{i=1}^{N} y_i - N\theta + \theta\sum_{i=1}^{N} y_i + \alpha - 1 - \theta\alpha + \theta - \theta\beta + \theta = 0$$

Deriving θ_{MAP}

$$\implies \sum_{i=1}^{N} y_i + \alpha - 1 - \theta(N + \alpha + \beta - 2) = 0$$

$$\implies \theta_{MAP} = \frac{\sum_{i=1}^{N} y_i + \alpha - 1}{N + \alpha + \beta - 2}$$

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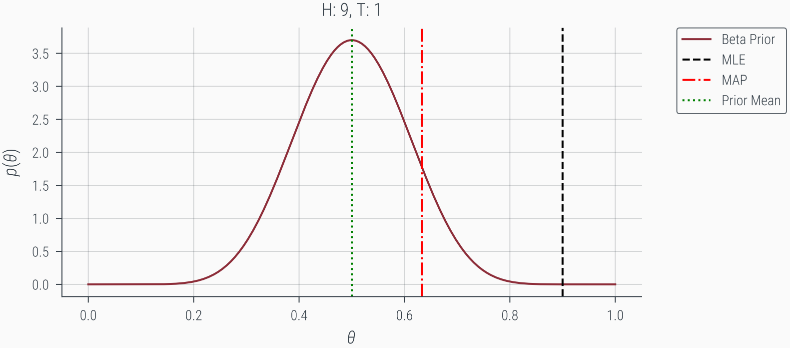
- $\theta_{MAP} = \frac{n_H + \alpha - 1}{N + \alpha + \beta - 2}$
- Prior $= \text{Beta}(\alpha, \beta)$
- Posterior $= \text{Beta}(n_H + \alpha, n_T + \beta)$

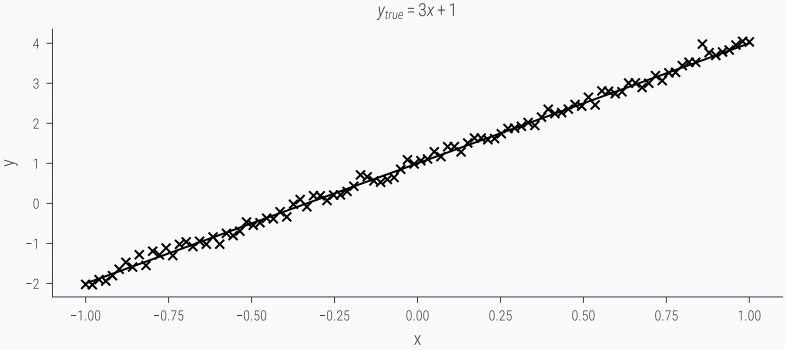
Coin Toss Problem with Prior



Notebook

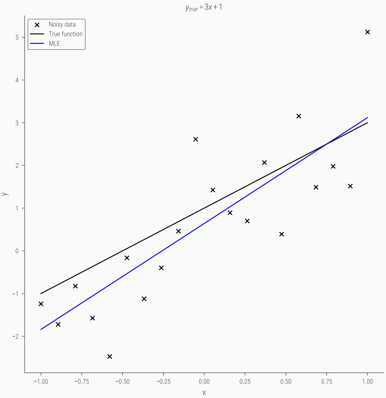
MLE for Linear Regression

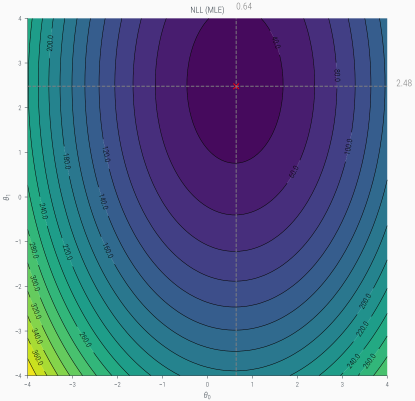
- Consider a dataset $D = \{(x_1, y_1)...(x_N, y_N)\}$ where $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$.
- Suppose the data is generated from a linear model with additive Gaussian noise, i.e., $y_i = \theta^T x_i + \epsilon_i$ where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.



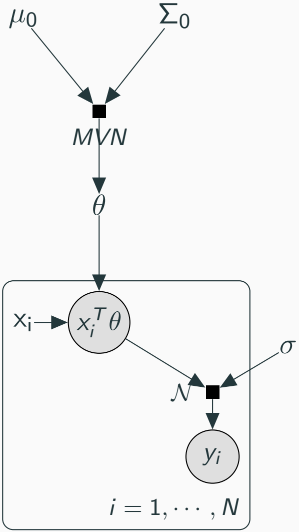
MLE for Linear Regression

- The likelihood is given by, $P(y_i|x_i,\theta) = \mathcal{N}(y_i|\theta^T x_i, \sigma^2)$
- **Recall**: The negative log-likelihood is given by,
  $\mathcal{NLL}(\theta) = \frac{1}{2\sigma^2}(y - X\theta)^T (y - X\theta)$
- **Recall**: The MLE is given by,
  $\theta_{MLE} = \arg\min_\theta \mathcal{NLL}(\theta) = \left(X^T X\right)^{-1} X^T y$





MAP for Linear Regression



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- Log-Likelihood:
  $\mathcal{LL}(\theta) = \log P(D|\theta) = \log \prod_{i=1}^{N} \mathcal{N}(y_i|x_i{}^T\theta, \sigma^2)$
- Prior: $P(\theta) = \mathcal{N}(\theta|\mu_0, \Sigma_0)$
- Log-Prior: $\log P(\theta) = \log \mathcal{N}(\theta|\mu_0, \Sigma_0)$
- Log-Joint: $\log P(\theta|D) = \log P(D|\theta) + \log P(\theta)$

MAP for Linear Regression

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We get

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Answer: Ridge Regression

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Question

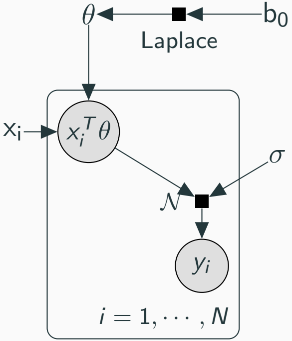
What does this expression remind you of?

Answer: Ridge Regression

Using Laplace prior

We can also use a Laplace prior on the weights, i.e.,

$$P(\theta) = \frac{1}{2b_0} \exp\left(-\frac{|x-\mu|}{b_0}\right)$$



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The MAP takes the form,

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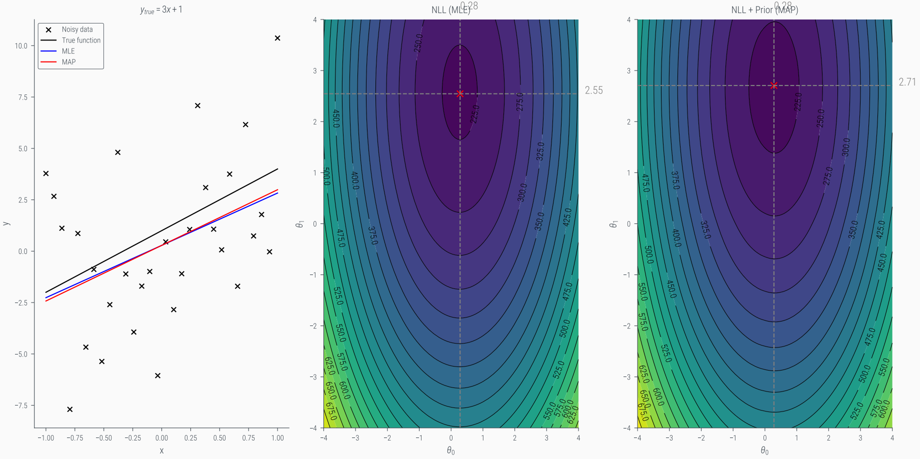
Answer: Lasso Regression

Using Laplace prior



Notebook

MAP for Logistic Regression

MLE for Logistic Regression

Consider a dataset $D = \{(x_1, y_1)...(x_N, y_N)\}$, where $x_i \in \mathbb{R}^d$ and $y_i \in \{0, 1\}$ such that

$$P(y = 1|x) = \hat{y} = \frac{1}{1 + \exp(-X^T\theta)} = \sigma(X^T\theta)$$

Take $y \sim \text{Bernoulli}\left(\sigma(X^T\theta)\right)$

MLE for Logistic Regression

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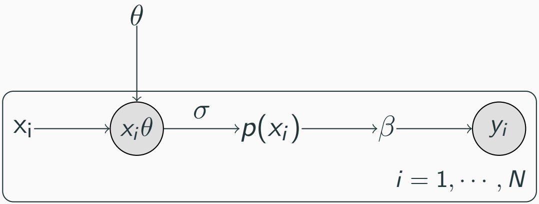
The likelihood is given by

$$\mathcal{L}(\theta) = \prod_{i=1}^{N} \hat{y}_i^{y_i}(i - \hat{y}_i)^{1-y_i}$$

$$\implies \mathcal{LL}(\theta) = \sum_{i=1}^{N} y_i \log \hat{y}_i + (1 - y_i) \log(1 - \hat{y}_i)$$

MLE for Logistic Regression



Binary Classification:

$$P(Y = 1|X) = \frac{1}{1 + e^{-(\theta_0 + \theta_1 X)}}$$

$$\therefore \mathcal{LL}(\theta) = \sum_{i=1}^{N} y_i \log p(x_i) + (1 - y_i) \log(1 - p(x_i))$$

Using zero-mean Gaussian prior

Considering a zero-mean Gaussian prior on the weights, i.e., $P(\theta) = \mathcal{N}(\theta|0, \sigma_0^2)$, the MAP is given by,

$$\theta_{MAP} = \arg\min \log(1 + \exp(-\theta^T X)) + \frac{1}{\sigma_0^2}\theta^T\theta$$

Using Laplace prior

Considering a Laplace prior on the weights, i.e., $P(\theta) = \prod_D \text{Laplace}(\theta_i|0, b_0) \propto \prod_D \exp(-\frac{1}{b_0}|\theta_i|)$, the MAP is given by,

$$\theta_{MAP} = \arg\min \log(1 + \exp(-\theta^T X)) + \frac{1}{b_0}|\theta|$$

MAP for Logistic Regression

Self-Study: Modify the code for Linear Regression to implement MAP for Logistic Regression.