Maximum Likelihood Estimation

Univariate

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Introduction

Univariate Normal Distribution

The probability density function of a univariate normal distribution is given by:

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

Let us assume we have a dataset $D = \{x_1, x_2, \ldots, x_n\}$, where each x_i is an independent sample from the above distribution. We want to estimate the parameters $\theta = \{\mu, \sigma\}$ from the data.

Our likelihood function is given by:

$$P(D|\theta) = \mathcal{L}(\mu,\sigma^2) = \prod_{i=1}^{n} f(x_i|\mu,\sigma^2) \tag{2}$$

Log Likelihood Function

Log-likelihood function:

$$\log \mathcal{L}(\mu, \sigma^2) = \sum_{i=1}^{n} \log f(x_i|\mu, \sigma^2) \tag{3}$$

Simplifying the above equation, we get:

$$\begin{aligned}
\log \mathcal{L}(\mu, \sigma^2) &= \sum_{i=1}^{n} \log f(x_i|\mu, \sigma^2) \\
&= \sum_{i=1}^{n} \log \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \right) \\
&= \sum_{i=1}^{n} \left(\log \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right) + \log \left(\exp \left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \right) \right)
\end{aligned}$$

$$
\begin{aligned}
\log \mathcal{L}(\mu, \sigma^2) &= \sum_{i=1}^{n} \left(\log \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right) - \frac{(x_i - \mu)^2}{2\sigma^2} \right) \\
&= \sum_{i=1}^{n} \left(-\frac{1}{2} \log(2\pi\sigma^2) - \frac{(x_i - \mu)^2}{2\sigma^2} \right) \\
&= -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2
\end{aligned}
$$

Log Likelihood Function for Univariate Normal Distribution

Log-likelihood function for normally distributed data is given by:

$$
\log \mathcal{L}(\mu, \sigma^2) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2
$$

Maximum Likelihood Estimate for μ

To find the MLE for μ, we differentiate the log-likelihood function with respect to μ and set it to zero:

$$\frac{\partial \log \mathcal{L}(\mu, \sigma^2)}{\partial \mu} = \frac{\partial}{\partial \mu} \left(-\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right) = 0$$

$$\frac{\partial}{\partial \mu} \left(\sum_{i=1}^{n} (x_i - \mu)^2 \right) = 0$$

Maximum Likelihood Estimate for μ

MLE of μ, denoted as $\hat{\mu}_{\text{MLE}}$, is given by:

$$\hat{\mu}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

MLE for σ for normally distributed data

Recall that the log-likelihood function is given by:

$$\log \mathcal{L}(\mu, \sigma^2) = \sum_{i=1}^{n} \log f(x_i | \mu, \sigma^2) \tag{4}$$

Let us find the maximum likelihood estimate of σ^2 now. We can do this by taking the derivative of the log-likelihood function with respect to σ^2 and equating it to zero.

$$\frac{\partial \log \mathcal{L}(\mu, \sigma^2)}{\partial \sigma^2} = \sum_{i=1}^{n} \frac{\partial \log f(x_i | \mu, \sigma^2)}{\partial \sigma^2} = 0 \tag{5}$$

MLE for σ for normally distributed data

Continuing from the previous equation, we can simplify the expression by substituting $t = \sigma^2$:

$$
\begin{aligned}
\frac{\partial \log \mathcal{L}(\mu, \sigma^2)}{\partial \sigma^2} &= \sum_{i=1}^{n} \frac{\partial \log f(x_i|\mu, \sigma^2)}{\partial \sigma^2} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \sigma^2} \left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \\
&= \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2(\sigma^2)^2} - \frac{1}{2\sigma^2}
\end{aligned}
$$

Substituting $t = \sigma^2$, we have:

$$
\frac{\partial \log \mathcal{L}(\mu, t)}{\partial t} = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2t^2} - \frac{1}{2t}
$$