MA-202

ASSIGNMENT-4

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Mechanical - Part III.

Q1

Soln: Given, $f_X(x) = \begin{cases} ke^{-b(x-a)}, & \text{if } a \leq x < \infty; \\ 0, & \text{elsewhere.} \end{cases}$

where $k, a, b > 0$.

Clearly, $\int_a^\infty k e^{-b(x-a)} dx = 1.$

$$\frac{k}{-b} e^{-b(x-a)} \Big|_a^\infty = 1.$$

$$\Rightarrow \frac{k}{b} = 1. \quad \Rightarrow \underline{\underline{k = b}}.$$

Now, Mean $= \mu = E(X) = \int x f_X(x) dx$

$$= \int_a^\infty b x e^{-b(x-a)} dx$$

$$= b \int_a^\infty x e^{-b(x-a)} dx$$

Now integrating by parts with $u = x$ and $dv = e^{-bx+ba} dx$.

$$I = uv - \int v\,du.$$

$$\text{So, } \mu = b\left[-\frac{x}{b}e^{-bx+ba} - \int -\frac{1}{b}e^{-bx+ba}\,dx\right].$$

$$= b\left(-\frac{x}{b}e^{-bx+ba} - \frac{1}{b^2}e^{-bx+ba}\right)\Big|_a^\infty.$$

$$= b\left(\frac{a}{b} + \frac{1}{b^2}\right) = \frac{1}{b}(ab+1).$$

$$\text{Now, } E(X^2) = \int_a^\infty b x^2 e^{-bx+ba}\,dx.$$

$$= b\int_a^\infty x^2 e^{-bx+ba}\,dx.$$

Again integrating by parts with $u = x^2$ and $dv = e^{-bx+ba} dx$.

$$\text{So, } I = uv - \int v\,du.$$

$$E(X^2) = b\left(-\frac{x^2}{b}e^{-bx+ba} - \int -\frac{1}{b}e^{-bx+ba}\,2x\,dx\right).$$

$$= b\left(-\frac{x^2}{b}e^{-bx+ba}\Big|_a^\infty - \int_a^\infty -\frac{2}{b}x e^{-bx+ba}\,dx\right).$$

$$= b\left(\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3}\right).$$

$$= a^2 + \frac{2a}{b} + \frac{2}{b^2}.$$

So, $\sigma^2 = E(X^2) - E^2(X)$.

$$= a^2 + \frac{2a}{b} + \frac{2}{b^2} - \frac{1}{b^2}(a^2b^2 + 1 + 2ab).$$

$$= \cancel{a^2} + \cancel{\frac{2a}{b}} + \frac{2}{b^2} - \cancel{a^2} - \frac{1}{b^2} - \cancel{\frac{2a}{b}}.$$

$$\Rightarrow \frac{1}{b^2}.$$

$$\Rightarrow b = \frac{1}{\sigma}. \quad (\because b > 0).$$

$$\because \mu = a + \frac{1}{b}.$$

$$a = \mu - \sigma.$$

So, $\underline{\underline{K = b = \frac{1}{\sigma}}}$ Ans.

$$\underline{\underline{a = \mu - \sigma}}$$

$\because f_X(x) = b e^{-b(x-a)}, \quad x \geq a$

and it is similar to exponential distribution $f_X(x) = \lambda e^{-\lambda x}, \lambda \geq 0$ with shifted origin at $x = a$. Hence the coefficient of skewness and kurtosis of both forms will not change as shape remains same.

It is well known that the coefficient of skewness of exponential distribution is 2 and coefficient of kurtosis of exponential distribution is 9.

So, for given distribution,

$$f_X(x) = be^{-b(x-a)}, \quad x \geq a$$

Coefficient of Skewness = 2.

Coefficient of Kurtosis = 9. Ans

Q2.

Soln: Given, $f_X(x) = \begin{cases} \sin x & , \text{ if } a < x < \pi/2 . \\ 0 & , \text{ elsewhere.} \end{cases}$

Validity of P.d.f:

$$\int_a^{\pi/2} \sin x \, dx = 1.$$

$\Rightarrow -\cos x \Big|_a^{\pi/2} = 1.$

$\Rightarrow \cos a = 1.$ $\Rightarrow a = 0, -2\pi, -4\pi, \ldots$ —— (i)

Distribution function

$$F_X(x) = \int_a^x \sin x \, dx.$$

$$= \cos a - \cos x.$$

for median, $F_X(x) = \frac{1}{2}$.

So, $\cos a - \cos x = \frac{1}{2}$.

$x = \cos^{-1}\left(\cos a - \frac{1}{2}\right)$. —— (ii)

$\because \cos^{-1}(m)$ has range $[0, \pi]$. So from (i) and (ii), a logical choice of a can be 0.

Hence, x(median) $= \cos^{-1}\left(1 - \frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Ans

$\because$ for mode, we need x such that $f_X'(x) = 0$.

or $\cos x = 0$ or $x = \pi/2$ which is not a valid random variable value.

So, mode d.n.e Ans

Q3.

Sol^n:

Given, $X \sim N(\mu_X, \sigma_X^2)$. | X and Y are independent R.V.s.

$Y \sim N(\mu_Y, \sigma_Y^2)$.

To find: Probability density function for $U = X+Y$.

Using characteristic function,

$$\phi_U(t) = \phi_{X+Y}(t) = E\left(e^{it(X+Y)}\right).$$

Also, $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$ ($\because$ X and Y are independent)

The characteristic function of the normal distribution with mean μ and variance σ^2 is

$$\phi(t) = \exp\left(it\mu - \frac{\sigma^2 t^2}{2}\right).$$

So, $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$.

$$= \exp\left(it\mu_X - \frac{\sigma_X^2 t^2}{2}\right) \exp\left(it\mu_Y - \frac{\sigma_Y^2 t^2}{2}\right)$$

$$= \exp\left(it(\mu_X + \mu_Y) - \frac{(\sigma_X^2 + \sigma_Y^2) t^2}{2}\right).$$

So, <u>U also gives</u> a characteristic function of <u>normal distribution</u> with mean as <u>$\mu_X + \mu_Y$</u> and variance as <u>$\sigma_X^2 + \sigma_Y^2$</u>

Also, no two distinct distributions can both have the same characteristic function, so the distribution of $U = X+Y$ must be just this normal distribution.

Q4.
Soln: Given, $f_{XY}(x,y) = \begin{cases} e^{-x-y}, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

(a) <u>Joint distribution function</u>.

$$F_{XY}(a,b) = P(-\infty < x \leq a, -\infty < y \leq b)$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f_{XY}(x,y)\, dy\, dx$$

$$= \int_{0}^{a} \int_{0}^{b} e^{-x-y}\, dy\, dx$$

$$= -\int_{0}^{a} e^{-x-y} \Big|_{0}^{b} dx$$

$$= -\int_{0}^{a} \left(e^{-x-b} - e^{-x-0}\right) dx$$

$$= -\int_{0}^{a} \left(e^{-x-b} - e^{-x}\right) dx$$

$$= \left(e^{-x-b} - e^{-x}\right) \Big|_{0}^{a}$$

$$= \left(e^{-a-b} - e^{-a}\right) - \left(e^{-b} - 1\right)$$

$$= \left(e^{-a} - 1\right)\left(e^{-b} - 1\right)$$

$$= \underline{\underline{\left(1 - e^{-a}\right)\left(1 - e^{-b}\right)}} \text{ Ans}$$

(b) Marginal distribution function

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} \underbrace{\cancel{e^{-x-y}}}_{f_{XY}(x,y)}\, dx\, dy.$$

$$= \int_{0}^{x}\int_{0}^{\infty} e^{-x-y}\, dy\, dx$$

$$= -\int_{0}^{x} e^{-x-y}\Big|_{0}^{\infty} dy.$$

$$= \int_{0}^{x} e^{-y}\, dy.$$

$$= -e^{-y}\Big|_{0}^{x} = \underline{\underline{1-e^{-x}}}. \text{ Ans}$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy.$$

$$= \int_{0}^{y}\int_{0}^{\infty} e^{-x-y}\, dy\, dx.$$

$$= -\int_{0}^{y} e^{-x-y}\Big|_{0}^{\infty} dx.$$

$$= \int_{0}^{y} e^{-x}\, dx$$

$$= -e^{-x}\Big|_{0}^{y} = \underline{\underline{1-e^{-y}}} \text{ Ans}$$

(c) $P(X=y) = F_X(y) = 1-e^{-y}$.

(d) $P(X+Y \le 4)$

$X+Y \le 4$.

$X \le 4-Y$.

$X \in [0,4]$. $\quad \because X \ge 0$

$Y \in [0,4]$. $\quad Y \ge 0$.

$\Rightarrow P(X+Y \le 4) = \int_0^4 \int_0^4 e^{-x-y}\, dx\, dy$.

$= -\int_0^4 e^{-x-y} \Big|_0^4 dy$

$= -\int_0^4 e^{-4-y} - e^{-y}\, dy$.

$= e^{-4-y} - e^{-y} \Big|_0^4$.

$= e^{-4-4} - e^{-4} - e^{-4} + 1$.

$= e^{-8} - 2e^{-4} + 1$

$= (e^{-4} - 1)^2$

(e) $P(X \geq 1) = F_X(x \to \infty) - F_X(x=1)$.

$= 1 - 0 - (1 - e^{-1})$

$= \frac{1}{e}$ Ans

(7) $P(X \leq Y) = \int_0^\infty \int_0^y e^{-x-y} \, dx \, dy$.

$= -\int_0^\infty e^{-x-y} \Big|_0^y dy$

$= -\int_0^\infty (e^{-y-y} - e^{-y}) dy$.

$= \frac{e^{-2y}}{2} - \frac{e^{-y}}{+1} \Big|_0^\infty$.

$= (0 - 0) - (\frac{1}{2} - 1) = \frac{1}{2}$. Ans

(8) $P(a < X+Y < b)$, $0 < a < b$

$\Rightarrow P(X+Y < b) - P(X+Y > a)$.

$\Rightarrow \int_0^a \int_{a-x}^{b-x} e^{-x-y} dx \, dy + \int_a^b \int_0^{b-x} e^{-x-y} dx \, dy$.

(I) (II)

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$$= -\int_0^a e^{-x-y}\Big|_{a-x}^{b-x} dx \quad \cancel{= -e^{-b}x\Big|_a^b} \quad (I)$$

$$= -\int_0^a e^{-x-(b-x)} - e^{-x-(a-x)}\, dx$$

$$= -\int_0^a e^{-b} - e^{-a}\, dx$$

$$= a\left(e^{-a} - e^{-b}\right).$$

So, $(I) = a\left(e^{-a} - e^{-b}\right)$.

Now $(II) = -\int_a^b e^{-x-y}\Big|_0^{b-x} dx.$

$$= -\int_a^b e^{-x-b+x} - e^{-x}\, dx$$

$$= -\int_a^b \left(e^{-b} - e^{-x}\right) dx$$

$$= e^{-b}x\Big|_a^b + \frac{e^{-x}}{-1}\Big|_a^b$$

$$= -e^{-b}(b-a) - \left(e^{-b} - e^{-a}\right).$$

$$(II) = -e^{-b}(b-a) + e^{-a} - e^{-b}.$$

Now $I + II = \left(e^{-a} - e^{-b}\right)(a+1) - e^{-b}(b-a)$

$$= e^{-a}(a+1) - e^{-b}(b+1)$$

h:

Given, Marginal distribution function.

$$F_X(a) = 1 - e^{-a}.$$

$$F_Y(b) = 1 - e^{-b}.$$

So, $f_X(a) = \frac{d}{da}(1-e^{-a}) = e^{-a}.$

$f_Y(b) = \frac{d}{db}(1-e^{-b}) = e^{-b}.$

Clearly $f_{XY}(a,b) = e^{-a-b}.$

Also, $f_X(a) \cdot f_Y(b) = e^{-a} \times e^{-b} = e^{-a-b}.$

$\because f_{XY}(a,b) = f_X(a) \cdot f_Y(b)$

So, X and Y are independent random variables.

Q5.

Soln:

Given. $f_X(x) = \frac{1}{2} e^{-|x|} \quad -\infty < x < \infty.$

$y = x^2$.

$$F_Y(y) = P(Y \leq y).$$
$$= P(X^2 \leq y).$$
$$= P(-\sqrt{y} \leq X \leq \sqrt{y}).$$
$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|} dx.$$
$$= \int_0^{\sqrt{y}} e^{-x} dx$$
$$= 1 - e^{-\sqrt{y}}.$$

Thus, $F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & y \geq 0 \\ 0 & \text{otherwise}. \end{cases}$

Q6.

Solⁿ:

Given, $f_X(x) = \begin{cases} cx(1-x) & , 0<x<1; \\ 0 & , \text{elsewhere}. \end{cases}$

$\because \int_{-\infty}^{\infty} f_X(x)\,dx = 1.$

$\Rightarrow \int_0^1 cx(1-x)\,dx = 1.$

$\Rightarrow c\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 1.$

$\Rightarrow c \times \frac{1}{6} = 1 \Rightarrow c = 6.$

Now, Corresponding distribution function

$F_X(a) = \int_0^a f_X(x)\,dx.$

$= 6\int_0^a (x - x^2)\,dx$

$= \boxed{6\left(\frac{a^2}{2} - \frac{a^3}{3}\right) \text{ for } a \le 1.}$ Ans

Now, $P(X > \frac{1}{2}) = 1 - P(X \le \frac{1}{2}).$

$= 1 - 6\left(\frac{a^2}{2} - \frac{a^3}{3}\right)_{a=\frac{1}{2}}.$

$= 1 - \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$ Ans