Extended Essay

The Accuracy of Euler's Method in Approximating the Solution to the Differential Equation of the Series RL Circuit

Research Question: To what extent can Euler's method accurately approximate the solution to the differential equation of the series RL circuit?

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1 Introduction

Differential equations are useful tools to model change. From population growth, disease spread, and the demand for commodities in financial markets, they can be used to model several natural phenomena (Zill 1). As a physics enthusiast, I was interested in the ordinary differential equation (ODE) that models the series RL circuit. Series RL circuits are a class of circuits comprised of a resistor, inductor, and a voltage source connected by a single wire. They have a wide range of applications from damping systems to filtering signals ("RL") and are modelled by a first-order ODE.

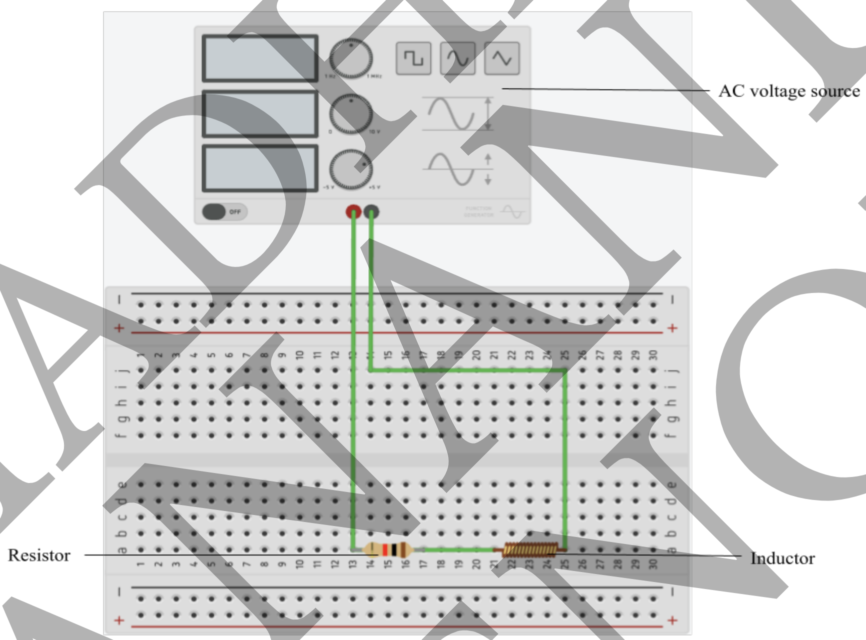

Figure 1: Pictorial Representation of RL Circuit; composed by author on tinkercad.com

In practice, many ODEs cannot be solved analytically (Kaw). That is, the solution to the ODE cannot be expressed as a function of the independent variable. Thus, numerical methods are used to approximate the solution to the ODE. In the movie *Hidden Figures*, NASA engineer Katherine Johnson utilises Euler's method to approximate an astronaut's re-entry into Earth (Luwisha). This surprised me as I always believed Euler's method to be inaccurate given its rudimentary definition as a first-order single-step method to approximate solutions (Tiliksew et al.). However, if it can be used to bring astronauts back to Earth safely, then surely it must be accurate. This prompted me to investigate the accuracy of Euler's method in approximating the solution to the ODE of the RL circuit.

To assess the accuracy of a numerical method, the analytical solution must first be determined. Thus, the first half of this essay will focus on obtaining the analytical solution. The focus will then shift to approximating the solution to the ODE using Euler's method and determining its accuracy by exploring the local and global truncation error of the approximation.

2 Prerequisite Knowledge

2.1 Fundamental Electrical Quantities and Circuit Components

It is important to have an understanding of the electrical quantities and circuit components that comprise the RL circuit to better appreciate the ODE.

2.1.1 Current

Current, $i(t)$, is the rate of change of charge . It is measured in amperes (A) (Young and Freedman 842).

2.1.2 Voltage

Voltage, $v(t)$, is the difference in electric potential energy between the two endpoints of a component in a circuit. Voltage causes current to flow and is measured in volts (V) (Young and Freedman 782).

2.1.3 The Resistor

A resistor opposes the flow of current. The extent to this opposition is quantified by the resistance R of the resistor which is measured in ohms (Ω) (Young and Freedman 848).

Figure 1: Symbol of resistor; taken from www.circuit-diagram.org

The instantaneous voltage, $v_R(t)$, across a resistor is (Young and Freedman 847),

$$v_R(t) = iR \tag{2.1}$$

2.1.4 The Inductor

An inductor generates a voltage due to changes in current through the inductor. The extent to which voltage is generated is quantified by the inductance L of the inductor, which has units of Henries (H) (Young and Freedman 1020).

Figure 2: Symbol of inductor; taken from www.circuit-diagram.org

The magnitude of the instantaneous voltage across an inductor, $v_L(t)$, is (Young and Freedman 1020),

$$v_L(t) = L\frac{di}{dt} \tag{2.2}$$

2.1.5 AC voltage source

An AC voltage source provides voltage that constantly reverses direction. It can be modelled in the form of a sinusoidal function (Young and Freedman 1047).

Figure 4: Symbol of AC voltage source; taken from www.circuit-diagram.org

Depending on initial conditions, the voltage at $v(t = 0)$, the voltage is either a sine or cosine wave. Assuming that $v(0) = 0$,

$$v(t) = V_0 \sin(\omega t) \tag{2.3}$$

where V_0 is the peak voltage and ω is the angular frequency measured in rad s^{-1}.

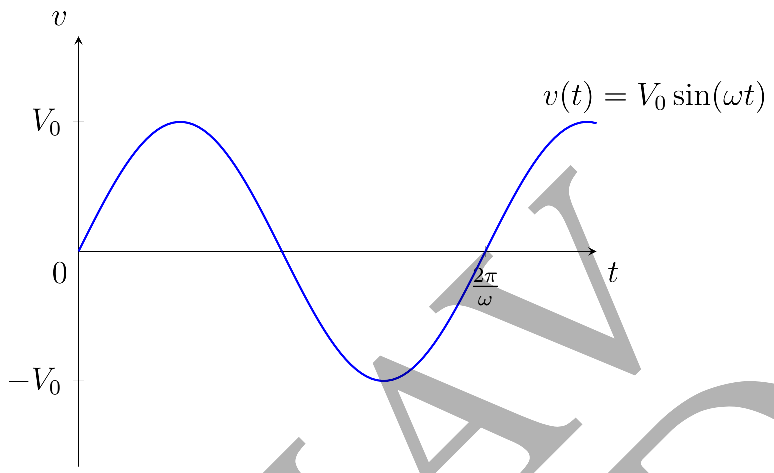

Figure 5: Eq. (2.3) plotted by author using TikZ

2.2 Kirchhoff's Voltage Law

Kirchhoff's Voltage Law (KVL) is crucial to the analysis of circuits and forms the basis for deriving the ODE that will be explored in this essay.

Definition 2.1 (KVL)**.** The algebraic sum of all the voltages in a closed loop circuit must be 0.

$$\sum V = 0 \tag{2.4}$$

KVL is a mathematical representation of the conservation of energy (Alexander and Sadiku 39). It is fascinating how an extremely powerful idea in physics can be represented by a simple mathematical equation.

KVL is applied by starting at an arbitrary point in the circuit , summing up all the voltage gains and losses, and equating the resultant sum to 0. For simplicity, all circuit components except a voltage source cause a voltage loss. These losses are represented by a negative symbol.

3 The RL Circuit Differential Equation

3.1 Deriving the Differential Equation

The circuit that is the focus of this essay is the series RL circuit with a resistor of resistance R, an inductor of inductance L, and an AC voltage source of peak voltage V_0 and angular frequency ω.

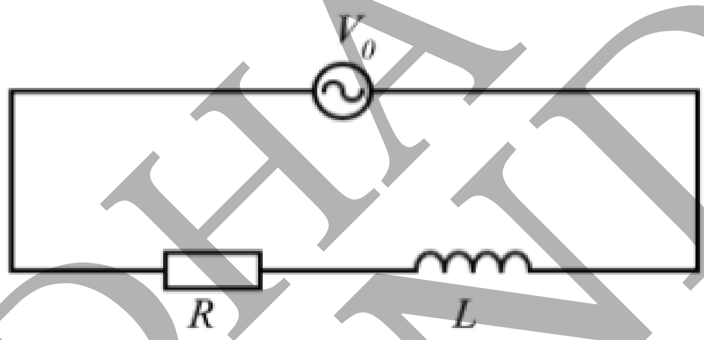

Figure 6: Circuit schematic of RL Circuit; drawn on www.circuit-diagram.org

We obtain the ODE that models this circuit by applying KVL.

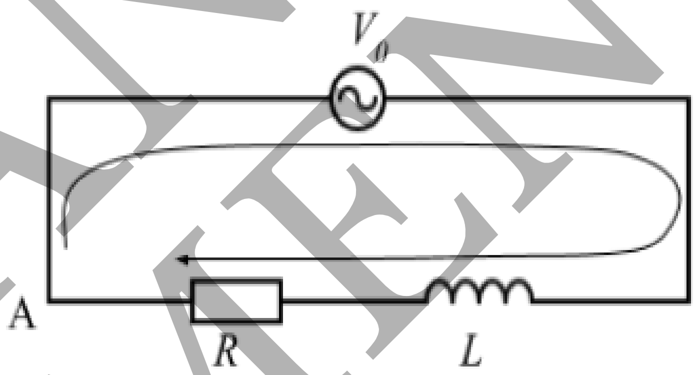

Figure 7: KVL applied to RL Circuit; drawn on www.circuit-diagram.org

Starting at point A and computing the sum of the voltage losses and gains we get,

$$
\begin{aligned}
v(t) - v_L(t) - v_R(t) &= 0 \\
\implies v_L(t) + v_R(t) &= v(t) \qquad (3.1)
\end{aligned}
$$

Substituting in the relevant voltage equations from section 2.1,

$$L\frac{di}{dt} + iR = V_0 \sin(\omega t) \tag{3.2}$$

Dividing both sides of the equation by L,

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_0}{L}\sin(\omega t) \tag{3.3}$$

The obtained ODE is the equation that models the AC RL circuit. We can see that it is a first-order (the highest derivative in the ODE is a first derivative) inhomogeneous (the ODE is equal to some function of the independent variable) ODE with the independent and dependent variables being time and current respectively.

3.2 Initial Conditions and Component Values of the Circuit

Initially, it is assumed that there is no current in the circuit until it is powered on. Thus,

$$i(0) = 0\,\mathrm{A} \tag{3.4}$$

As this essay is based on assessing the accuracy of a numerical method, it is inappropriate to maintain our ODE in variable form. Thus, realistic component values were chosen to model the circuit ("Encyclopedia"):

$$
\begin{aligned}
R &= 1\,\Omega \\
L &= 0.05\,\mathrm{H}
\end{aligned}
$$

$$V_0 = 1\,\mathrm{V}$$
$$\omega = 2\,\mathrm{rad}\ \mathrm{s}^{-1}$$

Substituting these values into eq. (3.3) yields,

$$\frac{di}{dt} + \frac{1}{0.05} i = \frac{1}{0.05}\sin(2t)$$
$$\implies \frac{di}{dt} + 20i = 20\sin(2t) \tag{3.5}$$

Numerous approaches exist for solving this ODE analytically. Initially, the method of integrating factors was employed, but the process became tedious due to the repeated application of integration by parts, as detailed in appendix A. The process of integrating factor is already covered in the IB syllabus and therefore yields no new mathematical insight. Thus, a different method was sought: the Laplace transform. Despite its sophisticated mathematical framework, the transform simplifies the process of solving ODEs, rendering it useful in mathematics, physics, and engineering.

4 Analytical Solution via The Laplace Transform

4.1 The Laplace Transform

The Laplace transform is an integral transform denoted by the operator $\mathscr{L}$. The modern Laplace transform was developed by French polymath Pierre-Simon Laplace ("Laplace Transform", Britannica). It maps a continuous and differentiable function in the time domain t, $f(t)$, to a function in the frequency domain s, $F(s)$.

$$\mathscr{L}\left\{f(t)\right\} : f(t) \longmapsto F(s) \tag{4.1}$$

It should be noted that s is a non-zero complex number. The implication of this is beyond the scope of this essay. The Laplace transform is defined by ("Laplace Transform", Weisstein),

$$F(s) = \mathscr{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}dt \tag{4.2}$$

The Laplace transform is useful as it can transform ODEs into simple algebraic expressions. It is trivial to solve the algebraic expression to obtain solutions in the frequency domain, which can be fed through the inverse Laplace transform to obtain a solution in the time domain.

Before doing this, we must consider the properties of the Laplace transform.

Lemma 4.1 (Linearity)**.** The Laplace transform satisfies the following conditions $\forall \left\{k_1, k_2\right\} \in \mathbb{R}$.

$$\mathscr{L}\left\{k_1 f(t) \pm k_2 g(t)\right\} = k_1 \mathscr{L}\left\{f(t)\right\} \pm k_2 \mathscr{L}\left\{g(t)\right\} \tag{4.3}$$

Proof.

$$
\begin{aligned}
\mathscr{L}\left\{k_1 f(t) \pm k_2 g(t)\right\} &= \int_0^\infty [k_1 f(t) \pm k_2 g(t)] e^{-st} dt \\
&= \int_0^\infty k_1 f(t) e^{-st} \pm k_2 g(t) e^{-st} dt \\
&= k_1 \int_0^\infty f(t) e^{-st} dt \pm k_2 \int_0^\infty g(t) e^{-st} dt \\
&= k_1 \mathscr{L}\left\{f(t)\right\} \pm k_2 \mathscr{L}\left\{g(t)\right\}
\end{aligned}
$$

■

Lemma 4.2 (Laplace Transforms of Derivatives)**.** If the n^{th} derivative of some function $f(t)$ is $f^{(n)}(t)$, then

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - \sum_{r=1}^{n} \left(s^{n-r} f^{(r-1)}(0)\right) \quad \forall \quad n \in \mathbb{Z}^+ \tag{4.4}$$

Proof by mathematical induction. Let P_n be the following proposition,

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - \sum_{r=1}^{n} \left(s^{n-r} f^{(r-1)}(0)\right) \quad \forall \quad n \in \mathbb{Z}^+$$

Testing the base case (P_1):

$$LHS = \mathscr{L}\left\{f'(t)\right\} = \int_0^\infty f'(t) e^{-st} dt$$

Let $u = e^{-st}$ and $dv = f'(t) \implies du = -se^{-st}dt$ and $v = f(t)$. Integrating by parts,

$$
\begin{aligned}
\int_0^\infty f'(t) e^{-st} dt &= [f(t)e^{-st}]_0^\infty - \int_0^\infty -s f(t) e^{-st} dt \\
&= \lim_{t \to \infty} \left(f(t)e^{-st}\right) - [f(0)e^{-s(0)}] + s \int_0^\infty f(t) e^{-st} dt
\end{aligned}
$$

As $s \neq 0$, in the limit $t \to \infty$, $e^{-st} \to 0$

$$\implies \mathscr{L}\left\{f'(t)\right\} = s\int_0^\infty f(t)e^{-st}dt - f(0)$$

$$= s\mathscr{L}\left\{f(t)\right\} - f(0)$$

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

$$RHS = s^1F(s) - \sum_{r=1}^{1}\left(s^{n-r}f^{(r-1)}(0)\right)$$

$$= s^1F(s) - s^{1-1}f^{(1-1)}(0)$$

$$= sF(s) - f(0)$$

Remark. The 0th derivative $f^{(0)}(t)$ is $f(t)$ as the differential operator has not been applied to the function.

As $LHS = RHS$, P_1 holds true.

Assume P_n is true for some $n = k$ where $k \in \mathbb{Z}^+$, $k > 1$. Hence, P_k:

$$\mathscr{L}\left\{f^{(k)}(t)\right\} = s^kF(s) - \sum_{r=1}^{k}\left(s^{k-r}f^{(r-1)}(0)\right)$$

Testing validity for P_{k+1}:

We define a new function $g(t) = f^{(k)}(t)$

$$
\begin{aligned}
\implies \mathscr{L}\left\{f^{(k+1)}(t)\right\} &= \mathscr{L}\left\{g'(t)\right\} \\
&= s\mathscr{L}\left\{g(t)\right\} - g(0) \ \text{(From } P_1\text{)} \\
&= s\mathscr{L}\left\{f^{(k)}(t)\right\} - f^{(k)}(0) \\
&= s\left[s^k F(s) - \sum_{r=1}^{k}\left(s^{k-r}f^{(r-1)}(0)\right)\right] - f^{(k)}(0) \ \text{(From } P_k\text{)} \\
&= s^{k+1}F(s) - \sum_{r=1}^{k}\left(s^{k+1-r}f^{(r-1)}(0)\right) - f^{k}(0) \\
&= s^{k+1}F(s) - \left[s^k f(0) + s^{k-1}f'(0) + s^{k-2}f''(0) + \cdots + sf^{(k-1)}(0)\right] - f^{(k)}(0) \\
&= s^{k+1}F(s) - \left[s^k f(0) + s^{k-1}f'(0) + s^{k-2}f''(0) + \cdots + sf^{(k-1)}(0) + f^{(k)}(0)\right] \\
\mathscr{L}\left\{f^{(k+1)}(t)\right\} &= s^{k+1}F(s) - \sum_{r=1}^{k+1}\left(s^{k+1-r}f^{(r-1)}(0)\right)
\end{aligned}
$$

$\therefore P_{k+1}$ also holds true.

As P_n is true for the base case $n = 1$ and P_{k+1} is true whenever P_k is true, P_n has been proved using the principle of mathematical induction. ■

From P_1,

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0) \tag{4.5}$$

As our ODE includes a sine function, it will be prudent to consider the Laplace transform of this function.

Lemma 4.3 (Laplace transform of the sine function)**.** If $f(t) = \sin(at)$, $a \in \mathbb{R}$ then,

$$\mathscr{L}\left\{\sin(at)\right\} = \frac{a}{s^2 + a^2} \tag{4.6}$$

Proof. Let $I = \mathscr{L}\left\{\sin(at)\right\}$. Thus,

$$I = \int_0^\infty \sin(at)e^{-st}dt$$

Let $u = \sin(at)$ and $dv = e^{-st} \implies du = a\cos(at)$ and $v = -\frac{1}{s}e^{-st}$. Integrating by parts,

$$\begin{aligned}
I &= \left[-\frac{e^{-st}}{s}\sin(at)\right]_0^\infty - \int_0^\infty -\frac{a}{s}\cos(at)e^{-st}dt \\
&= \left[\lim_{t\to\infty}\left(-\frac{e^{-st}}{s}\sin(at)\right) - \left(-\frac{e^0}{s}\sin(0)\right)\right] + \int_0^\infty \frac{a}{s}\cos(at)e^{-st}dt \\
&= (0+0) + \frac{a}{s}\int_0^\infty \cos(at)e^{-st}dt \\
&= \frac{a}{s}\int_0^\infty \cos(at)e^{-st}dt \qquad (4.7)
\end{aligned}$$

Integrating eq. (4.7) by parts with the following substitutions: $\mu = \cos(at)$ and $d\eta = e^{-st} \implies d\mu = -a\sin(at)$ and $\eta = -\frac{1}{s}e^{-st}$

$$\begin{aligned}
\int_0^\infty e^{-st}\cos(at)dt &= \left[-\frac{e^{-st}}{s}\cos(at)\right]_0^\infty - \int_0^\infty \frac{a}{s}e^{-st}\sin(at)dt \\
&= \left[\lim_{t\to\infty}\left(-\frac{e^{-st}}{s}\cos(at)\right) - \left(-\frac{e^0}{s}\cos(0)\right)\right] - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt \\
&= \left(0 + \frac{1}{s}\right) - \frac{a}{s}I \\
&= \frac{1}{s} - \frac{a}{s}I \qquad (4.8)
\end{aligned}$$

Substituting eq. (4.8) into eq. (4.7),

$$\begin{aligned}
I &= \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s}I\right) \\
&= \frac{a}{s^2} - \frac{a^2}{s^2}I \\
\therefore I + \frac{a^2}{s^2}I &= \frac{a}{s^2} \\
I\left(1 + \frac{a^2}{s^2}\right) &= \frac{a}{s^2} \\
I\left(\frac{s^2 + a^2}{s^2}\right) &= \frac{a}{s^2} \\
I &= \frac{a}{s^2} \div \frac{s^2 + a^2}{s^2} \\
I &= \frac{a}{s^2 + a^2} \\
\implies \mathscr{L}\left\{\sin(at)\right\} &= \frac{a}{s^2 + a^2} \qquad (4.9)
\end{aligned}$$

■

A similar process can be applied to show,

$$\mathscr{L}\left\{\cos(at)\right\} = \frac{s}{s^2 + a^2} \qquad (4.10)$$

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s - a} \qquad (4.11)$$

The proofs of these are shown in appendix B.

4.2 Applying the Laplace Transform

Having built up the tools required to use the Laplace Transform, we can apply it to our ODE. Let $I(s)$ be defined as the Laplace transform of $i(t)$. So,

$$I(s) = \mathscr{L}\left\{i(t)\right\} \qquad (4.12)$$

Applying the Laplace Transform to eq. (3.5),

$$\mathscr{L}\left\{\frac{di}{dt}+20i\right\}=\mathscr{L}\left\{20\sin(2t)\right\} \tag{4.13}$$

Applying lemma 4.1, 4.2, and 4.3 to eq. (4.13),

$$\mathscr{L}\left\{\frac{di}{dt}\right\}+20\mathscr{L}\left\{i\right\}=20\mathscr{L}\left\{\sin(2t)\right\}$$

$$\implies sI(s)-i(0)+20I(s)=\frac{20(2)}{s^2+2^2} \tag{4.14}$$

Substituting the initial conditions,

$$sI(s)-0+20I(s)=\frac{40}{s^2+4} \tag{4.15}$$

Factoring and rearranging for $I(s)$,

$$I(s)\left[s+20\right]=\frac{40}{s^2+4}$$

$$I(s)=\frac{40}{(s^2+4)(s+20)} \tag{4.16}$$

Since $\mathscr{L}\left\{i(t)\right\}=I(s)$, having isolated $I(s)$, we can now find $i(t)$

4.3 Inverse Laplace Transform

The inverse Laplace Transform maps a function in the frequency domain s to a function in the time domain t. It returns the original function that was inputted into the Laplace Transform.

If for some continuous and differentiable function $f(t)$, $\mathscr{L}\left\{f(t)\right\}=F(s)$, then the inverse Laplace transform is defined as,

$$\mathscr{L}^{-1}\left\{F(s)\right\}=f(t) \tag{4.17}$$

There is an expression for the inverse Laplace transform known as the Bromwich integral (Arfken et al. 1039). However, it requires an understanding of *contour integrals* which is beyond the scope of this essay. Instead, $I(s)$ will be decomposed via partial fractions such that it can be expressed as a sum of sinusoidal and exponential functions—allowing us to take the inverse Laplace transform easily.

4.3.1 Partial Fraction Decomposition

The general form of the decomposed partial fraction is,

$$I(s) = \frac{40}{(s^2+4)(s+20)} = \frac{A}{s+20} + \frac{Bs+C}{s^2+4}, \text{ where } \{A, B, C\} \in \mathbb{R} \tag{4.18}$$

Cross multiplying and expanding,

$$A(s^2+4) + (Bs+C)(s+20) = 40$$

$$\implies As^2 + 4A + Bs^2 + 20Bs + Cs + 20C = 40$$

$$\implies (A+B)s^2 + (20B+C)s + (4A+20C) = 40 \tag{4.19}$$

Comparing coefficients,

$$4A + 20C = 40 \tag{4.20}$$

$$A + B = 0 \tag{4.21}$$

$$20B + C = 0 \tag{4.22}$$

As $B = -A$,

$$-20A + C = 0 \tag{4.23}$$

$$20A + 100C = 200 \tag{4.24}$$

Therefore,

$$101C = 200$$
$$\implies C = \frac{200}{101} \tag{4.25}$$

From eq. (4.23),

$$\begin{aligned} A &= \frac{C}{20} \\ &= \frac{200}{101} \times \frac{1}{20} \\ A &= \frac{10}{101} \end{aligned} \tag{4.26}$$

From eq. (4.21),

$$B = -\frac{10}{101} \tag{4.27}$$

Thus,

$$A = \frac{10}{101} \tag{4.28}$$

$$B = -\frac{10}{101} \tag{4.29}$$

$$C = \frac{200}{101} \tag{4.30}$$

Substituting A, B and C into eq. (4.18)

$$\begin{aligned} I(s) &= \frac{\frac{10}{101}}{s+20} + \frac{\frac{-10s+200}{101}}{s^2+4} \\ &= \frac{\frac{10}{101}}{s+20} + \frac{-\frac{10}{101}s}{s^2+4} + \frac{\frac{200}{101}}{s^2+4} \\ I(s) &= \frac{10}{101}\left(\frac{1}{s+20}\right) - \frac{10}{101}\left(\frac{s}{s^2+4}\right) + \frac{\frac{200}{101}}{2}\left(\frac{2}{s^2+4}\right) \end{aligned} \tag{4.31}$$

Taking the inverse Laplace transform of both sides and recognising that we have an exponential, cosine and sine function respectively on the RHS (from lemma 4.3, eq. (4.10) and eq. (4.11)),

$$\begin{aligned} i(t) &= \mathscr{L}^{-1}\left\{I(s)\right\} \qquad (4.32)\\ &= \mathscr{L}^{-1}\left\{\frac{10}{101}\left(\frac{1}{s+20}\right)-\frac{10}{101}\left(\frac{s}{s^2+4}\right)+\frac{\frac{200}{101}}{2}\left(\frac{2}{s^2+4}\right)\right\}\\ &= \frac{10}{101}e^{-20t}-\frac{10}{101}\cos(2t)+\frac{100}{101}\sin(2t)\\ \implies i(t) &= \frac{10}{101}\left(e^{-20t}-\cos(2t)+10\sin(2t)\right) \qquad (4.33)\end{aligned}$$

4.4 The Analytical Solution

In this essay, we will be approximating the solution to the differential equation at an arbitrarily chosen timestep $t = 0.5$ via Euler's method and assessing the accuracy of the approximation. To do so, the analytical solution must be determined,

$$i(0.5) = \frac{10}{101}\left(e^{-10}-\cos(1)+10\sin(1)\right) \approx 0.7796488061525335 \text{ A} \qquad (4.34)$$

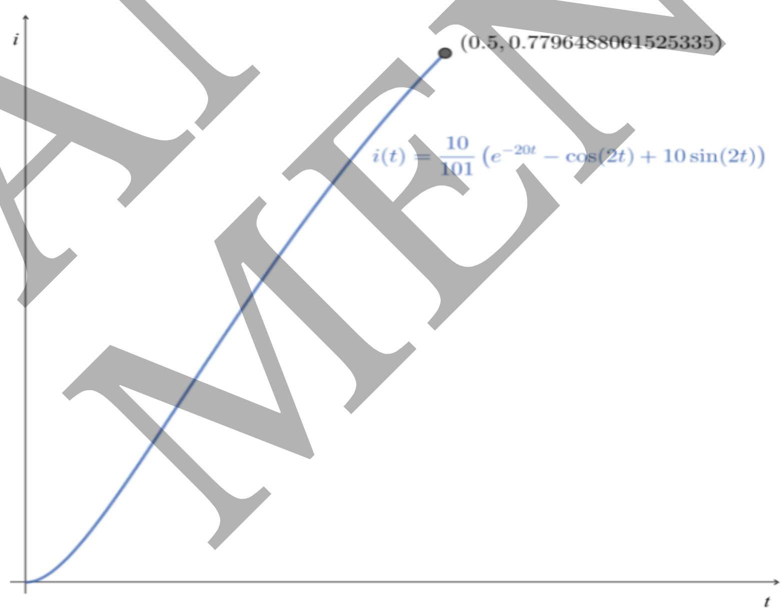

Figure 8: Plot of eq. (4.33) ($0 \leq t \leq 0.5$); created on Geogebra

This number was obtained with a high degree of precision due to the implementation of a Python algorithm (see appendix C). Given our focus on evaluating accuracy, attaining a high number of decimal places is essential. This precision allows for an accurate estimation of the error, thereby providing a clear insight into the method's accuracy.

5 Euler's Method

5.1 Deriving Euler's Method

Consider a general ODE expressed as an initial value problem (IVP),

$$\frac{dy}{dt} = f(t, y), \text{ where } y(t_0) = y_0 \tag{5.1}$$

The solution to this ODE can be expressed as $y = f(t)$. Consider two arbitrary points on the solution curve $(t_n, f(t_n) = y_n)$ and $(t_{n+1}, f(t_{n+1}) = y_{n+1})$. The instantaneous slope at t_n is then given by,

$$\frac{dy}{dt}|_{t=t_n} = f(t_n, y_n) = \frac{y_{n+1} - y_n}{t_{n+1} - t_n} \tag{5.2}$$

This assumes that $t_{n+1} - t_n$ is infinitesimally small. We can represent this difference by h, which is referred to as the step size. Thus,

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n) \tag{5.3}$$

Rearranging gives us Euler's method,

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{5.4}$$

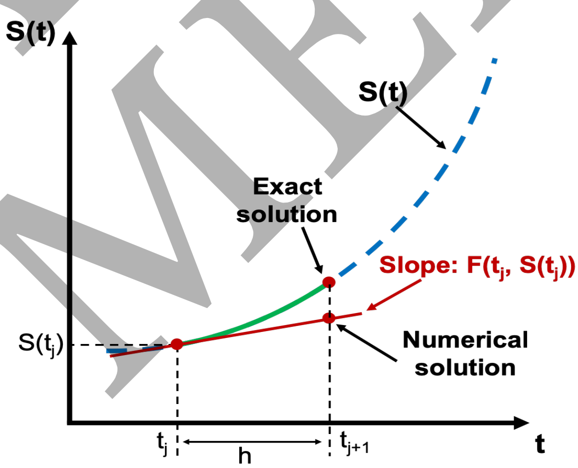

Figure 9: Visualisation of Euler's Method (Kong et al.)

Fig. 9 illustrates that Euler's method approximates the solution curve at a particular timestep by utilising tangent lines from the preceding timestep. Writing eq. (3.5) in the form eq. (5.1),

$$\frac{di}{dt} = 20\sin(2t) - 20i, \text{ where } i(0) = 0 \text{ (A)} \tag{5.5}$$

5.2 Approximations via Euler's Method

We start by applying Euler's method to approximate $i(0.5)$. An arbitrary step-size of $h = 0.01$ was chosen. It should be noted that the values listed below were truncated so they fit on the page.

$$\begin{aligned}
i_0 &= 0 \text{ (A)} \\
i_1 &= 0 + 0.01\,[20\sin(2(0)) - 20(0)] = 0 \text{ (A)} \\
i_2 &\approx 0 + 0.01\,[20\sin(2(0.01)) - 20(0)] \approx 0.00399973 \text{ (A)} \\
i_3 &\approx 0.00399973 + 0.01\,[20\sin(2(0.02)) - 20(0.00399973)] \approx 0.01119765 \text{ (A)} \\
&\vdots \\
i_{48} &\approx 0.74184393 + 0.01\,[20\sin(2(0.47)) - 20(0.74184393)] \approx 0.75498677 \text{ (A)} \\
i_{49} &\approx 0.75498677 + 0.01\,[20\sin(2(0.48)) - 20(0.75498677)] \approx 0.76782772 \text{ (A)} \\
i_{50} &\approx 0.76782772 + 0.01\,[20\sin(2(0.49)) - 20(0.76782772)] \approx 0.78036166 \text{ (A)}
\end{aligned} \tag{5.6}$$

Thus, according to Euler's method,

$$i(0.50) \approx 0.7803616552326172 \text{ A} \quad \text{(Untruncated)} \tag{5.7}$$

The full untruncated output for each timestep can be found in appendix D.

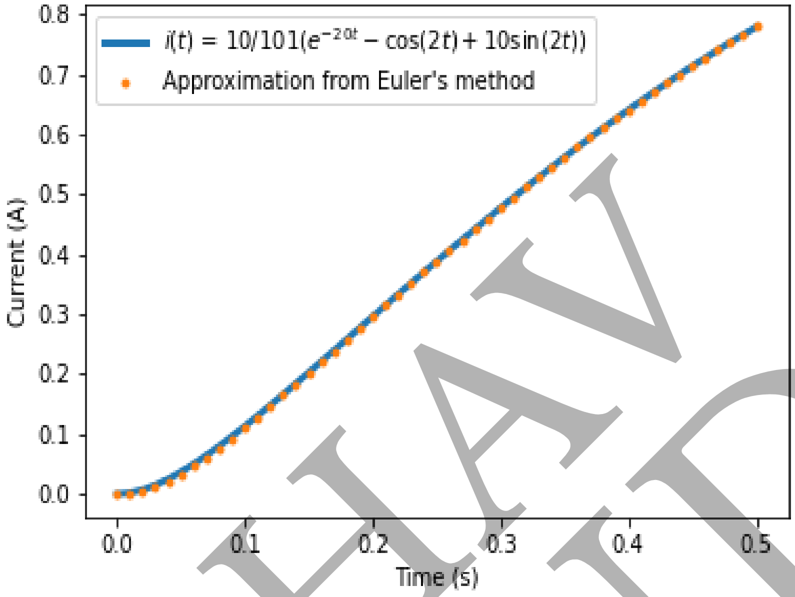

Figure 10: Output from Euler's method against solution curve; plotted on Pyplot

Fig. 10 demonstrates that Euler's method yields a high level of accuracy as the approximated points closely align with the solution curve. However, there will inevitably be inaccuracies in our approximations; we now focus on quantifying this by considering the error in our approximation.

5.3 Local and Global Truncation Errors

5.3.1 Interlude I: Multi-variable Functions and Partial Derivatives

A single-variable function maps one real number to another. A multi-variable function maps multiple real numbers to a single real number. In the case of a two-variable function, they can be denoted by $f(t, y)$ where the inputs are t and y.

$$f(t) : \mathbb{R} \longmapsto \mathbb{R}$$

$$f(t, y) : \mathbb{R}^2 \longmapsto \mathbb{R}$$

Remark. The RHS of our IVP is a multi-variable function,

$$\frac{di}{dt} = f(t, i) = 20\sin(2t) - 20i \tag{5.8}$$

A partial derivative is defined as the derivative of a multi-variable function with respect to one of its inputs; the other input is treated as a constant. The partial derivative of f with respect to t and y, for infinitesimally small Δt and Δy, is (Strang),

$$\frac{\partial f}{\partial t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t, y) - f(t, y)}{\Delta t} \tag{5.9}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(t, y+\Delta y) - f(t, y)}{\Delta y} \tag{5.10}$$

Example:

$$f(t, y) = t^2 y + 2t^3 y^2 + t + y$$

$$\implies \frac{\partial f}{\partial t} = 2ty + 6t^2 y^2 + 1 + 0$$

and

$$\frac{\partial f}{\partial y} = t^2 + 4t^3 y + 0 + 1$$

5.3.2 Interlude II: The Mean Value Theorem (MVT)

Theorem 5.1 (MVT)**.** If f is a function that is continuous in the closed interval $[a,\ b]$ and differentiable in the open interval $(a,\ b)$ where $\{a, b\} \in \mathbb{R}$, then $\exists\ \tau \in [a, b]$ where,

$$f'(\tau) = \frac{f(b) - f(a)}{b - a} \tag{5.11}$$

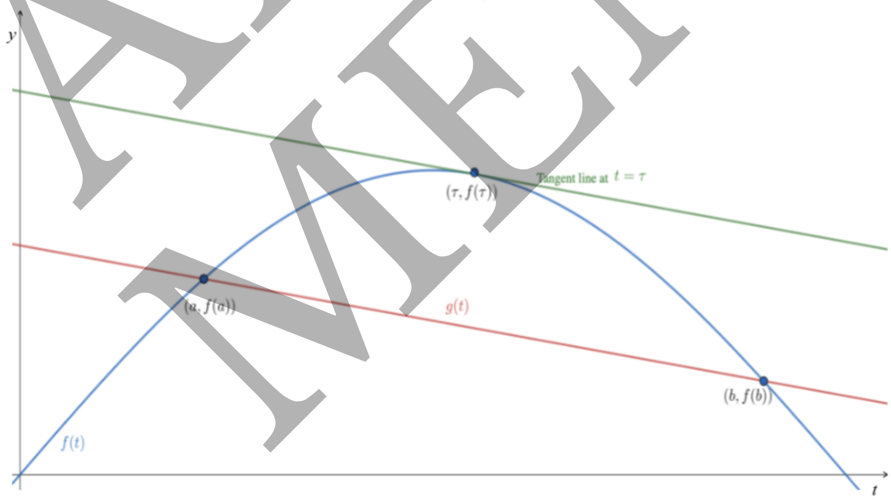

Figure 11: Visualisation of mean value theorem; created on Geogebra

As shown in fig. 11, the MVT can be thought of as the existence of some τ where the slope of the tangent at $(\tau, f(\tau))$ is equal to the slope of the secant line that passes through a and b.

Proof. Let the secant line that passes through a and b be denoted by $g(t)$. As $g(t)$ intersects $f(t)$ at a and b, it follows that,

$$f(a) = g(a)$$
$$f(b) = g(b)$$

The equation of $g(t)$ is,

$$g(t) - g(a) = \frac{g(b) - g(a)}{b - a}(t - a)$$
$$\implies g(t) = \frac{f(b) - f(a)}{b - a}(t - a) + f(a) \tag{5.12}$$

Let us define a new function $h(t)$ where,

$$h(t) = f(t) - g(t)$$
$$\implies h(t) = f(t) - \left[\frac{f(b) - f(a)}{b - a}(t - a) + f(a)\right]$$
$$\implies h'(t) = f'(t) - \frac{f(b) - f(a)}{b - a} \cdot \frac{d}{dt}(t - a)$$
$$\implies h'(t) = f'(t) - \frac{f(b) - f(a)}{b - a} \tag{5.13}$$

Rolle's theorem states that if a function f satisfies the conditions in theorem 5.1 and $f(a) = f(b)$, then there exists some τ where $f'(\tau) = 0$. This theorem essentially identifies the existence of a stationary point, and for brevity, the proof of this theorem was left in appendix E.

As $h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$, $h(a) = h(b)$. Thus, from

Rolle's theorem,

$$h'(\tau) = f'(\tau) - \frac{f(b) - f(a)}{b - a} = 0 \tag{5.14}$$

$$\implies f'(\tau) = \frac{f(b) - f(a)}{b - a} \tag{5.15}$$

■

It is important to note that the MVT does not tell us what τ is, instead it only tells us that there must exist some τ.

The MVT can be further extended to an MVT for integrals.

Theorem 5.2 (MVT for Integrals)**.** If f is a function that is continuous in $[a, b]$ and differentiable in (a, b), where $\{a, b\} \in \mathbb{R}$ then $\exists\ \tau \in [a, b]$ where,

$$f(\tau)(b - a) = \int_a^b f(t)dt \tag{5.16}$$

Proof. From the fundamental theorem of calculus let $F(t)$ be defined as,

$$F(t) = \int_0^t f(x)dx \text{ (x is a dummy variable)}$$

Differentiating $F(t)$ yields,

$$F'(t) = f(t)$$

If f satisfies the conditions stated in the theorem, then so does $F'(t)$. Thus, by the MVT

$$F'(\tau) = \frac{F(b) - F(a)}{b - a}$$

$$\implies f(\tau) = \frac{F(b) - F(a)}{b - a}$$

$$\implies f(\tau)(b - a) = F(b) - F(a)$$

Invoking the fundamental theorem of calculus,

$$f(\tau)(b-a) = \int_a^b f(t)dt$$

■

5.3.3 Interlude III: Taylor Series

The Taylor Series is a polynomial approximation for a function around a point a. It can be thought of as a "shift" of the Maclaurin series by a units, and is given by (Hartman),

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(t-a)^n \tag{5.17}$$

As we approximate our function by more terms in the Taylor series, the polynomial approximation becomes more accurate as seen below

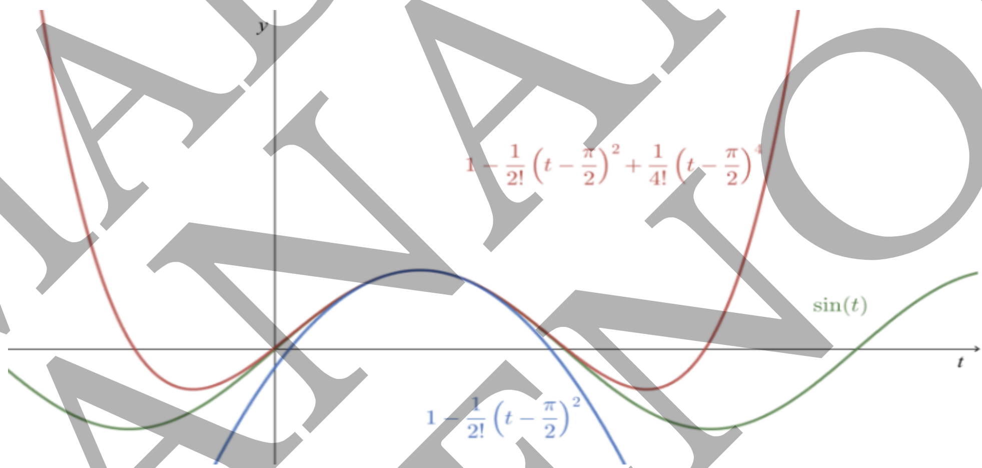

Figure 12: Taylor expansion of $\sin(t)$ around $\frac{\pi}{2}$; created on Geogebra

5.3.4 Interlude IV: The Lagrange Form of the Remainder

When we consider a Taylor series expansion, the resultant polynomial is an approximation as it is impossible expand up to infinite terms. We define the remainder term as the difference between the actual function and its polynomial approximation.

Let the Taylor expansion of a function up to n terms be represented by $p_n(t)$. Then,

$$p_n(t) = f(a)+f'(a)(t-a)+\frac{f''(a)}{2!}(t-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(t-a)^n = \sum_{r=0}^{n}\frac{f^{(r)}(a)}{r!}(t-a)^r \quad (5.18)$$

The remainder term is then,

$$R_n(t) = f(t) - p_n(t) \quad (5.19)$$

Theorem 5.3 (Taylor's Remainder Theorem)**.** Consider a function f that is continuous in the interval $[a,t]$ and $n+1$ times differentiable in the interval (a,t). For a Taylor expansion of f centred at $t = a$, the remainder term can be expressed as (Rosenberg),

$$R_n(t) = f(t) - \sum_{r=0}^{n}\frac{f^{(r)}(a)}{r!}(t-a)^r = \int_a^t \frac{(t-x)^n}{n!}f^{(n+1)}(x)dx \quad (5.20)$$

Remark. x is a dummy variable.

Proof by mathematical induction. Let P_n be the following proposition

$$R_n(t) = f(t) - \sum_{r=0}^{n}\frac{f^{(r)}(a)}{r!}(t-a)^r = \int_a^t \frac{(t-x)^n}{n!}f^{(n+1)}(x)dx \ \forall\, n \geq 0, n \in \mathbb{Z}$$

Testing the base case (P_0):

$$\begin{aligned}
LHS &= f(t) - \frac{f(a)}{0!}(t-a)^0 \\
&= f(t) - f(a) \\
RHS &= \int_a^t \frac{(t-x)^0}{0!}f'(x)dx \\
&= \int_a^t f'(x)dx \\
&= [f(x)]_a^t \\
&= f(t) - f(a)
\end{aligned}$$

As $LHS = RHS, P_0$ holds true. Assume P_n holds true for some $n = k, k \in \mathbb{Z}$

$$f(t) - \sum_{r=0}^{k} \frac{f^{(r)}(a)}{r!}(t-a)^r = \int_a^t \frac{(t-x)^k}{k!} f^{(k+1)}(t)dx$$

Testing validity for P_{k+1}, we start with RHS,

$$\int_a^t \frac{(t-x)^k}{k!} f^{(k+1)}(x)dx$$

Let $u = f^{(k+1)}(x)$ and $dv = \frac{(t-x)^k}{k!} \implies du = f^{(k+2)}(x)dx$ and $v = -\frac{(t-x)^{k+1}}{(k+1)!}$. Integrating by parts,

$$\begin{aligned}
\int_a^t \frac{(t-x)^k}{k!} f^{(k+1)}(x)dx &= \left[-\frac{(t-x)^{k+1}}{(k+1)!} f^{(k+1)}(x)\right]_a^t - \int_a^t -\frac{(t-x)^{k+1}}{(k+1)!} f^{(k+2)}(x)dx \\
&= 0 - \left[-\frac{(t-a)^{k+1}}{(k+1)!} f^{(k+1)}(a)\right] + \int_a^t \frac{(t-x)^{k+1}}{(k+1)!} f^{(k+2)}(x)dx \\
&= \frac{(t-a)^{k+1}}{(k+1)!} f^{(k+1)}(a) + \int_a^t \frac{(t-x)^{k+1}}{(k+1)!} f^{(k+2)}(x)dx
\end{aligned}$$

Thus from P_k,

$$f(t) - \sum_{r=0}^{k} \frac{f^{(r)}(a)}{r!}(t-a)^r - \frac{(t-a)^{k+1}}{(k+1)!} f^{(k+1)}(a) = \int_a^t \frac{(t-x)^{k+1}}{(k+1)!} f^{(k+2)}(x)dx$$

$$\begin{aligned}
\implies f(t) - \left[\sum_{r=0}^{k} \frac{f^{(r)}(a)}{r!}(t-a)^r + \frac{(t-a)^{k+1}}{(k+1)!} f^{(k+1)}(a)\right] &= \int_a^t \frac{(t-x)^{k+1}}{(k+1)!} f^{(k+2)}(x)dx \\
\implies f(t) - \sum_{r=0}^{k+1} \frac{f^{(r)}(a)}{r!}(t-a)^r &= \int_a^t \frac{(t-x)^{k+1}}{(k+1)!} f^{((k+1)+1)}(x)dx \\
\implies R_{k+1}(t) &= \int_a^t \frac{(t-x)^{k+1}}{(k+1)!} f^{((k+1)+1)}(x)dx
\end{aligned} \tag{5.21}$$

$\therefore P_{k+1}$ also holds true.

As P_0 holds true and P_{k+1} holds true whenever P_k holds true, P_n has been proved using the principle of mathematical induction. ■

By applying the MVT for integrals to the integral form of our remainder,

$$R_n(t) = \frac{f^{(n+1)}(\tau)}{(n+1)!}(t-a)^{n+1} \tag{5.22}$$

This is known as the Lagrange form of the remainder (Weisstein). Alternatively, this is called the *error* of our Taylor approximation (Hartman).

5.3.5 Local Truncation Error (LTE)

Definition 5.1 (LTE)**.** The LTE at an arbitrary timestep t_{n+1} is the magnitude of the difference between the analytical output $y(t_{n+1})$ and the approximation from Euler's method which employs the analytical solution of the preceding timestep as its initial condition (Wong).

$$\varepsilon_l = |y(t_{n+1}) - [y(t_n) + hf(t_n, y(t_n))]| \tag{5.23}$$

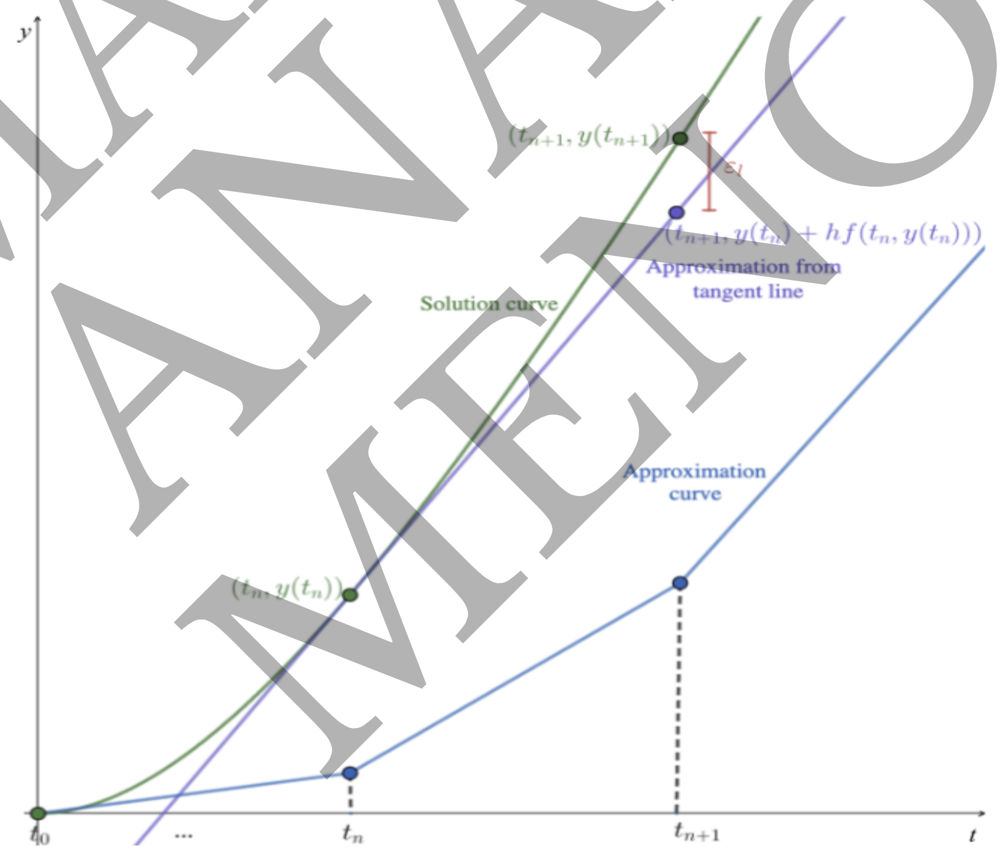

Figure 13: Visualisation of LTE; created on Geogebra

From fig. 13 we can see that we approximate some point at t_{n+1} using the tangent line of the analytical solution at t_n. The distance between this approximated point and the analytical solution at t_{n+1} is the LTE. It is assumed that there is no error due to rounding in obtaining $y(t_n)+hf(t_n, y(t_n))$. This is impossible as modern computers have a limit on the number of floating points they can store ("Unit 6"). As a result of this assumption, the LTE is another approximation.

Eq. (5.23) isn't useful to work with as we do not know $y(t_{n+1})$ (the point of a numerical method is to approximate $y(t_{n+1})$). However, it is assumed we know the analytical solution at the preceding timestep $y(t_n)$. Thus, we can use the Taylor expansion of our analytical solution $y(t)$ to get an approximation at t_{n+1}. The Taylor expansion of $y(t+h)$ centred at t is

$$\begin{aligned} y(t+h) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(t)}{n!}(t+h-t)^n \\ &= \sum_{n=0}^{\infty} \frac{y^{(n)}(t)}{n!}h^n \end{aligned} \tag{5.24}$$

Thus,

$$\begin{aligned} y(t_{n+1}) &= y(t_n+h) \\ &= \sum_{n=0}^{\infty} \frac{h^n y^{(n)}(t_n)}{n!} \\ &= y(t_n) + hy'(t_n) + \sum_{n=2}^{\infty} \frac{h^n y^{(n)}(t_n)}{n!} \end{aligned} \tag{5.25}$$

From eq. (5.1), $y'(t_n) = \frac{dy}{dt}|_{t=t_n} = f(t_n, y_n)$. Substituting this into eq. (5.25),

$$y(t_{n+1}) = y(t_n) + hf(t_n, y_n) + \sum_{n=2}^{\infty} \frac{h^n y^{(n)}(t_n)}{n!} \tag{5.26}$$

The first two terms in the expansion are the terms in Euler's method, highlighting the connection between it and the Taylor series expansion. This is sensible as the Taylor series expansion provides information about a function's derivatives (which is needed to complete the expansion). Euler's method uses these derivatives to build its approximations. Given this link, the tools we have built to determine the Taylor series error can help determine the error in Euler's method. Substituting eq. (5.26) into eq. (5.23),

$$\begin{aligned}\varepsilon_l &= \left| y(t_n) + hf(t_n, y(t_n)) + \sum_{n=2}^{\infty} \frac{h^n y^{(n)}(t_n)}{n!} - [y(t_n) + hf(t_n, y(t_n))] \right| \\ &= \left| \sum_{n=2}^{\infty} \frac{h^n y^{(n)}(t_n)}{n!} \right| \end{aligned} \tag{5.27}$$

We see that our LTE is the remainder term of our Taylor series expansion. Expanding the first few terms of our LTE,

$$\varepsilon_l = \left| \frac{h^2}{2!} y''(t_n) + \frac{h^3}{3!} y^{(3)}(t_n) + \frac{h^4}{4!} y^{(4)}(t_n) + \sum_{n=5}^{\infty} \frac{h^n y^{(n)}(t_n)}{n!} \right| \tag{5.28}$$

As $h \to 0$, the terms larger than h^2 term will tend to 0 faster. Thus, we consider our LTE to be of *order* h^2, represented by $\mathcal{O}(h^2)$, as this gives us the largest value of our LTE.

From theorem 5.3,

$$\varepsilon_l \approx \left| \frac{h^2}{2} y''(\tau) \right| \tag{5.29}$$

We can re-write Euler's method to include the LTE as,

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2} y''(\tau) \tag{5.30}$$

$$y_{n+1} = y_n + hf(t_n, y_n) + \mathcal{O}\left(h^2\right) \tag{5.31}$$

This notation implies that a decrease in the step size h by a factor of n will decrease the LTE by a factor of n^2.

The LTE for each timestep can be found in appendix D.

5.3.6 The Global Truncation Error (GTE) bound

Definition 5.2 (GTE)**.** The GTE is the difference between the analytical solution at the final timestep t_{n+1} and the approximation from Euler's Method. It is the propagation of all the LTEs through our approximation (Wong).

$$\varepsilon_g = |y(t_{n+1}) - y_{n+1}| \tag{5.32}$$

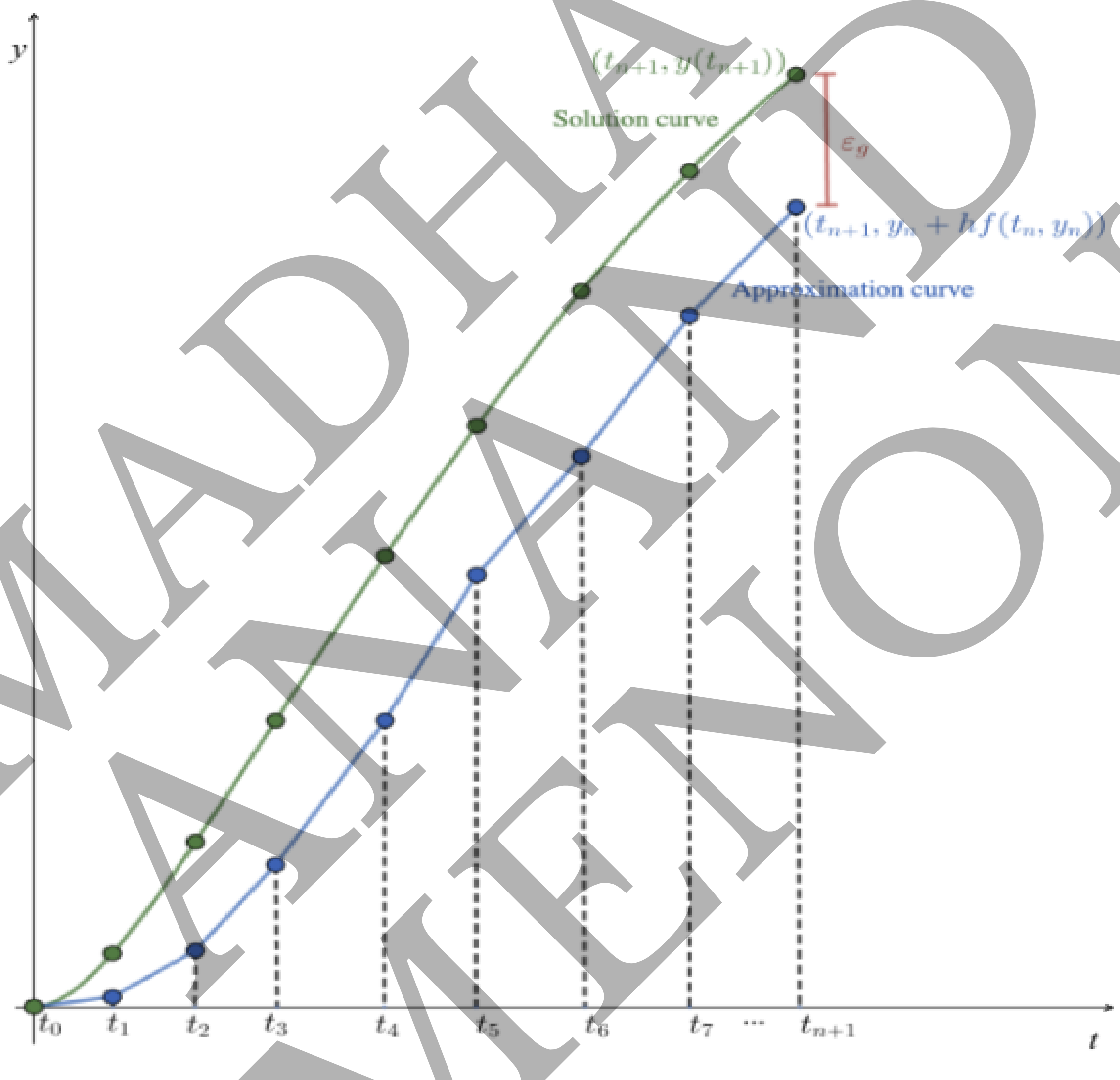

Figure 14: Visualisation of GTE; created by author on Geogebra

From fig. 14 we see that the GTE is the distance between the analytical solution and the approximation from Euler's method at the final timestep. Unlike the LTE which only determines the error in approximation at a single step, the GTE considers the propagation of the LTE across all approximations.

As this essay focuses on evaluating the accuracy of Euler's method, we now attempt to bound the maximum value of our GTE.

For the GTE bound to exist, the following conditions for the general IVP $\frac{dy}{dt} = f(t, y)$ defined in the interval $[a, b]$ must be satisfied (Burden and Faires 271).

1. (Lipschitz Condition). $f(t, y)$ satisfies the following condition:

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \; \forall \, t \in [a, b] \tag{5.33}$$

   L is termed the Lipschitz constant and is defined as,

$$L = \max_{t \in [a,b]} \left| \frac{\partial}{\partial y} f(t, y) \right| \tag{5.34}$$

   This notation represents the maximum value of $\frac{\partial f}{\partial y}$ in the interval $[a, b]$. The Lipschitz constant constrains how fast the function changes due to perturbations in the initial conditions of the ODE; the solution is said to be *stable* if it satisfies the Lipschitz condition (Wong).

2. A constant M exists such that the analytical solution to the IVP denoted by $y(t)$ satisfies the following,

$$\left| \frac{d^2 y}{dt^2} \right| \leq M \; \forall \, t \in [a, b] \tag{5.35}$$

From eq. (5.30),

$$y(t_{n+1}) = y(t_n) + hf(t_n) + \frac{h^2}{2} y''(\tau) \tag{5.36}$$

As the GTE is defined as the propagation of the LTE, and the LTE is of order h^2, it is sufficient to expand until the h^2 term as further terms contribute very little to the error.

Thus,

$$
\begin{aligned}
\varepsilon_g &= |y(t_{n+1}) - y_{n+1}| \\
&= \left| y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} y''(\tau) - y_n - hf(t_n, y_n) \right| \\
&= \left| y(t_n) - y_n + h[f(t_n, y(t_n)) - f(t_n, y_n)] + \frac{h^2}{2} y''(\tau) \right| \qquad (5.37)
\end{aligned}
$$

Theorem 5.4 (Triangle Inequality). $\forall \{a, b\} \in \mathbb{R}$,

$$|a + b| \leq |a| + |b| \qquad (5.38)$$

Proof. Note,

$$|a|^2 = a^2$$

Thus,

$$|a + b|^2 = (a + b)^2 = a^2 + 2ab + b^2$$

As $|a| \geq a$ and $|b| \geq b$,

$$
\begin{aligned}
a^2 + 2ab + b^2 &\leq |a|^2 + 2|a||b| + |b|^2 \\
\implies (a + b)^2 &\leq (|a| + |b|)^2 \\
\implies |a + b|^2 &\leq (|a| + |b|)^2 \\
\implies |a + b| &\leq |a| + |b| \qquad (5.39)
\end{aligned}
$$

∎

From theorem 5.4,

$$|y(t_{n+1}) - y_{n+1}| \leq |y(t_n) - y_n| + h|f(t_n, y(t_n)) - f(t_n, y_n)| + \frac{h^2}{2} |y''(\tau)| \qquad (5.40)$$

From the Lipschitz condition,

$$|f(t_n, y(t_n)) - f(t_n, y_n)| \leq L|y(t_n) - y_n|$$

$$\implies h|f(t_n, y(t_n)) - f(t_n, y_n)| \leq hL|y(t_n) - y_n| \tag{5.41}$$

From the second condition,

$$\frac{h^2}{2}|y''(\tau)| \leq \frac{h^2 M}{2} \tag{5.42}$$

Thus,

$$\begin{aligned} |y(t_{n+1}) - y_{n+1}| &\leq |y(t_n) - y_n| + hL|y(t_n) - y_n| + \frac{h^2 M}{2} \\ &\leq (1 + hL)|y(t_n) - y_n| + \frac{h^2 M}{2} \end{aligned} \tag{5.43}$$

Lemma 5.5. $\forall\, t \geq$ -1 and some $\alpha \in \mathbb{R}^+$, the following holds true

$$0 \leq (1 + t)^\alpha \leq e^{\alpha t} \tag{5.44}$$

Proof. Consider the Taylor series expansion of e^t centred at the origin.

$$\begin{aligned} e^t &\approx e^0 + \frac{(t-0)^1}{1!} e^0 + R_2(t) \\ &\approx 1 + t + \frac{t^2}{2} e^\tau \end{aligned}$$

Because of the remainder term,

$$1 + t \leq 1 + t + \frac{t^2}{2} e^\tau$$

As $t \geq -1$,

$$0 \leq 1 + t \leq 1 + t + \frac{t^2}{2} e^\tau$$

As our Taylor series expansion is an approximation, we truncate all the higher powers of

t. Thus,

$$1 + t + \frac{t^2}{2}e^{\tau} \leq \sum_{r=0}^{\infty} \frac{t^r}{r!}e^0 = e^t$$

$$\implies 0 \leq 1 + t \leq e^t$$

Raising each side of the inequality to the power α,

$$0 \leq (1+t)^{\alpha} \leq e^{\alpha t} \tag{5.45}$$

■

Lemma 5.6. If $\{\lambda, \mu, k\} \in \mathbb{R}$ and β_n forms the terms of a sequence for $n = 0, 1, 2 \ldots, k-1$ such that the following is true,

$$\beta_0 \geq -\frac{\lambda}{\mu}$$

$$\beta_{n+1} \leq (1+\lambda)\beta_k + \mu$$

then the following is true,

$$\beta_{n+1} \leq e^{(n+1)\lambda}\left(\beta_0 + \frac{\mu}{\lambda}\right) - \frac{\mu}{\lambda} \tag{5.46}$$

Proof. Repeatedly substituting in the second inequality condition in descending order of k,

$$\begin{aligned}
\beta_{n+1} &\leq (1+\lambda)\beta_k + \mu \\
&\leq (1+\lambda)\left[(1+\lambda)\beta_{k-1} + \mu\right] + \mu = (1+\lambda)^2\beta_{k-1} + [1 + (1+\lambda)]\mu \\
&\leq (1+\lambda)^2[(1+\lambda)\beta_{k-2} + \mu] + [1 + (1+\lambda)]\mu = (1+\lambda)^3\beta_{k-2} + [1 + (1+\lambda) + (1+\lambda)^2]\mu \\
&\vdots \\
&\leq (1+\lambda)^{n+1}\beta_0 + [1 + (1+\lambda) + (1+\lambda)^2 + \cdots + (1+\lambda)^n]\mu
\end{aligned}$$

The second sum involving λ is the sum to n terms of a geometric sequence with first term 1 and common ratio $1+\lambda$. Thus,

$$\begin{aligned}
\beta_{n+1} &\leq (1+\lambda)^{n+1}\beta_0 + \left(\frac{(1+\lambda)^n - 1}{1+\lambda-1}\right)\mu \\
&\leq (1+\lambda)^{n+1}\beta_0 + \left(\frac{(1+\lambda)^n - 1}{\lambda}\right)\mu \\
&\leq (1+\lambda)^{n+1}\beta_0 + \frac{\mu(1+\lambda)^n}{\lambda} - \frac{\mu}{\lambda} \\
&\leq (1+\lambda)^{n+1}\beta_0 + \frac{\mu(1+\lambda)^{n+1}}{\lambda} - \frac{\mu}{\lambda} \\
\beta_{n+1} &\leq (1+\lambda)^{n+1}\left[\beta_0 + \frac{\mu}{\lambda}\right] - \frac{\mu}{\lambda}
\end{aligned}$$

Invoking lemma 5.5 where $t = \lambda$,

$$\begin{aligned}
(1+\lambda)^{n+1} &\leq e^{(n+1)\lambda} \\
\implies \beta_{n+1} &\leq \left(\beta_0 + \frac{\mu}{\lambda}\right)e^{(n+1)\lambda} - \frac{\mu}{\lambda}
\end{aligned} \tag{5.47}$$

■

We now invoke lemma 5.6 to simplify eq. (5.43). Defining the parameters of the lemma as,

$$\lambda = hL \tag{5.48}$$

$$\mu = \frac{h^2 M}{2} \tag{5.49}$$

$$\beta_n = |y(t_n) - y_n| \tag{5.50}$$

Substituting into lemma 5.6,

$$\begin{aligned}
|y(t_{n+1}) - y_{n+1}| &\leq (1+hL)|y(t_n) - y_n| + \frac{h^2 M}{2hL} \\
\implies |y(t_{n+1}) - y_{n+1}| &\leq e^{(n+1)hL}\left(|y(t_0) - y_0| + \frac{h^2 M}{2hL}\right) - \frac{h^2 M}{2hL}
\end{aligned} \tag{5.51}$$

For an IVP $y(t_0) = y_0 \implies |y(t_0) - y_0| = 0$. Thus,

$$|y(t_{n+1}) - y_{n+1}| \leq \frac{hM}{2L} e^{(n+1)hL} - \frac{hM}{2L} \tag{5.52}$$

The number of steps in our approximation is given by,

$$\begin{aligned} n + 1 &= \frac{t_{n+1} - t_0}{h} \\ \implies (n+1)h &= t_{n+1} - t_0 \end{aligned} \tag{5.53}$$

Substituting this into our inequality,

$$|y(t_{n+1}) - y_{n+1}| \leq \frac{hM}{2L} e^{(t_{n+1}-t_0)L} - \frac{hM}{2L} \tag{5.54}$$

Thus, the bound for our GTE is given by,

$$\varepsilon_g \leq \frac{hM}{2L} \left(e^{(t_{n+1}-t_0)L} - 1 \right) \tag{5.55}$$

Remark. The GTE bound for Euler's method includes Euler's number!

As the GTE bound is proportional to h, it is said to be of order h, $\mathcal{O}(h)$. Since the power of h is 1, Euler's method is defined as a first-order method. We can show that Euler's method converges,

$$\begin{aligned} \lim_{h\to 0} \varepsilon_g &= \lim_{h\to 0} \frac{hM}{2L} \left(e^{(t_{n+1}-t_0)L} - 1 \right) \\ &= \left(e^{(t_{n+1}-t_0)L} - 1 \right) \lim_{h\to 0} \frac{hM}{2L} \\ \implies \lim_{h\to 0} \varepsilon_g &= 0 \end{aligned} \tag{5.56}$$

As our step-size approaches 0, the error also approaches 0 which indicates that our approximation will converge to the analytical solution. This is proof that a smaller step-size implies greater accuracy.

By removing the modulus sign on our GTE bound, we see that our analytical solution lies in the following range,

$$\begin{aligned}
|y(t_{n+1}) - y_{n+1}| &\leq \frac{hM}{2L}\left(e^{(t_{n+1}-t_0)L} - 1\right) \\
\implies -\frac{hM}{2L}\left(e^{(t_{n+1}-t_0)L} - 1\right) \leq y(t_{n+1}) - y_{n+1} &\leq \frac{hM}{2L}\left(e^{(t_{n+1}-t_0)L} - 1\right) \\
\implies y_{n+1} - \frac{hM}{2L}\left(e^{(t_{n+1}-t_0)L} - 1\right) \leq y(t_{n+1}) &\leq y_{n+1} + \frac{hM}{2L}\left(e^{(t_{n+1}-t_0)L} - 1\right) \qquad (5.57)
\end{aligned}$$

5.4 Bounding the Error of the Approximation

Now that we have derived the expression for the bound of our GTE, we can apply it to our ODE. Remembering our IVP,

$$\frac{di}{dt} = f(t,i) = 20\sin(2t) - 20i \tag{5.58}$$

We must compute our Lipschitz constant L using eq. (5.34). As we approximate our solutions from $t = 0$ to $t = 0.5$, this becomes a and b respectively.

$$\begin{aligned}
L &= \max_{t\in[a,b]}\left|\frac{\partial}{\partial i} f(t,i)\right| \\
&= \max_{t\in[0,\ 0.5]}\left|\frac{\partial}{\partial i}\left[20\sin(2t) - 20i\right]\right| \\
&= \max_{t\in[0,\ 0.5]} |-20| \\
\implies L &= 20 \qquad (5.59)
\end{aligned}$$

Remark: The maximum value of a constant is the constant itself.

We now determine M by computing the second derivative of our analytical solution,

$$\begin{aligned}
i(t) &= \frac{10}{101}\left(e^{-20t} - \cos(2t) + 10\sin(2t)\right) \\
\implies \frac{di}{dt} &= \frac{20}{101}\left(\sin(2t) + 10\cos(2t) - 10e^{-20t}\right)
\end{aligned}$$

$$\implies \frac{d^2i}{dt^2} = \frac{40}{101}\left(\cos(2t) - 10\sin(2t) + 100e^{-20t}\right)$$

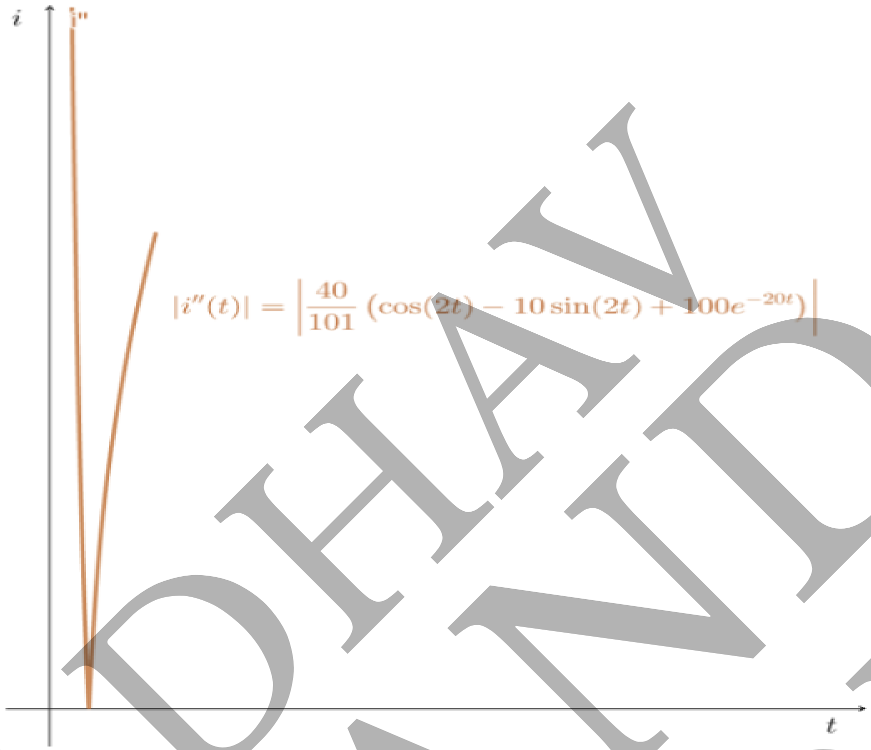

Figure 15: Plot of $|i''(t)|$; created on Geogebra

The maximum value M in $[0, 0.5]$ occurs at $t = 0$.

$$\implies M = \frac{d^2i}{dt^2}|_{t=0}| = 40 \tag{5.60}$$

We can now bound our GTE with the following,

$$h = 0.01$$

$$M = 40$$

$$L = 20$$

$$t_{n+1} = 0.5$$

$$t_0 = 0$$

Substituting these into our bound gives us,

$$
\begin{aligned}
\varepsilon_g &\leq \frac{0.01 \times 40}{2 \times 20}\left(e^{(0.5-0)\times 20} - 1\right) \\
&\leq \frac{0.4}{40}\left(e^{10} - 1\right) \\
\implies \varepsilon_g &\leq 220.254657948
\end{aligned} \tag{5.61}
$$

The actual GTE is,

$$\varepsilon_g = |0.7796488061525335 - 0.7803616552326172| \approx 0.0007128490800837 \tag{5.62}$$

Calculating the percentage error,

$$\frac{0.0007128490800837}{0.7796488061525335} \times 100 \approx 0.0914320748596433\% \tag{5.63}$$

5.5 Analysis of the Error Bound

The computed error bound should be surprising as we get a large error. According to eq. (5.57), our analytical solution lies in the following range,

$$0.78036165523261 - 220.254657948 \leq i(0.5) \leq 0.78036165523261 + 220.254657948 \tag{5.64}$$

At first glance, Euler's method can seem inaccurate. However, this assessment is misleading as the obtained value is only an upper-bound and therefore a "worst-case" scenario. Thus, the bound only tells us that the maximum value the GTE can take is 220.254657948; it is not the actual GTE. The large error bound can mainly be attributed to the large Lipschitz constant which shows that the ODE is sensitive to small perturbations. A different ODE may have a smaller Lipschitz constant implying greater stability in its solution. The large value could be due to the convexity of the solution curve. From fig. 8 we see that the curvature near $t = 0$ would imply higher inaccuracy when approximating with tangent lines in a single-step—as is the case with Euler's method.

The accuracy of Euler's method is emphasised by the fact that it was demonstrated that the approximated solution converges to the analytical solution as h tends to 0. This holds true on a local scale as its LTE is of order h^2 which is reasonable for most use-cases. The percentage error being significantly smaller than 1% shows that Euler's method is highly accurate in approximating the solution to the ODE of the RL circuit as seen in fig. 10 as well. However, in cases where the analytical solution cannot be determined, the GTE cannot be computed. Thus, a more accurate numerical method must be sought to reduce the GTE bound and thus, the maximum possible error.

6 Conclusion and Extension

6.1 Scope for Future Work

In practice, The GTE bound is used as a measure of accuracy over the actual GTE. Firstly, the Picard-Lindelöf theorem guarantees that an ODE will have an analytical solution if it satisfies the Lipschitz condition (Passias and Wegner). Thus, ODEs which do not satisfy this condition do not have an analytical solution, making it impossible to compute the GTE. Furthermore, the GTE bound provides a more conservative measure of the error. Although Euler's method shows a low GTE, this cannot be confirmed for ODEs with no analytical solution. This results in the need for us to pursue more accurate numerical methods such as the Runge-Kutta methods. The most famous example being the Runge-Kutta 4 method which is defined as follows: For a general IVP (Burden and Faires 288),

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + 2k_4)$$

where,

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right)$$

$$k_4 = f\left(t_n + h, y_n + hk_3\right)$$

The LTE and GTE are of the order $\mathcal{O}\left(h^5\right)$ and $\mathcal{O}\left(h^4\right)$ respectively (Burden and Faires 290). It would be interesting to see how the expression for the GTE bound changes as the order of the error term becomes much larger (ie. the magnitude of the error decreases).

In this essay, it was assumed that there were no round-off errors in our approximation. Thus, it would be interesting to see how round-off errors are accounted for in the real world. The round-off error is quantified by the machine-epsilon, ϵ_{mach} of the number system a computer uses in calculations. The machine epsilon is defined as the distance between 1 and the next largest floating point number in a given number system (Silva). It would be fascinating to see how the expression for the GTE bound could be modified to account for round-off errors using the machine-epsilon.

6.2 Conclusion

"The grandest discoveries of science have been but the rewards of accurate measurement and patient long-continued labour in the minute sifting of numerical results."
- Lord Kelvin (Bryant).

The importance of numerical methods extends beyond the realm of approximations—they are foundational to the processes of discovery and innovation in science and engineering. These methods have opened areas for exploration by enabling the pursuit of complex systems and phenomena that are intractable using purely algebraic methods.

This essay demonstrated that Euler's method, despite its simplicity, is a remarkably useful method to have in one's toolkit. By first obtaining the analytic solution via the Laplace transform and deriving the error bounds of the method and applying it within context of the RL circuit, the accuracy of Euler's method was evaluated successfully. It was shown that Euler's method can be said to be accurate when the analytical solution can be determined. However, in the event that no analytical solution exists, a more accurate numerical method must be sought to reduce the GTE bound. It is very rare for differential equations, even those that can be solved analytically, to be solved by hand these days. Computer packages such as SciPy and MATLAB, which are extensively used, use numerical methods to solve these equations. This reliance brings about the need to understand the accuracy of these numerical methods.

Although more advanced numerical methods are in use today, at their core, all of them attempt to trace out the solution curve using tangent lines. Thus, it is always worth returning to the most basic method to foster an intuitive understanding and deep appreciation of the iterative process of numerical approximations as this can often be lost in the derivations of higher-order methods.

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Appendix A Solving the RL DE via integrating factor

The ODE and initial condition is given as,

$$\frac{di}{dt} + 20i = 20\sin(2t),\ i(0) = 0 \tag{A.1}$$

The integrating factor is given by,

$$e^{\int 20dt} = e^{20t} \tag{A.2}$$

Multiplying both sides of the differential equation by the integrating factor,

$$\begin{aligned}
e^{20t}\frac{di}{dt} + 20e^{20t}i &= 20e^{20t}\sin(2t) \\
\implies \frac{d}{dt}(ie^{20t}) &= 20e^{20t}\sin(2t) \\
\implies ie^{20t} &= 20\int e^{20t}\sin(2t)\,dt \\
\implies \frac{e^{20t}}{20}i &= \int e^{20t}\sin(2t)\,dt
\end{aligned} \tag{A.3}$$

Let us define the following,

$$I = \int e^{20t}\sin(2t)\,dt \tag{A.4}$$

Integrating by parts with the following substitutions, $u = \sin(2t)$ and $dv = e^{20t} \implies du = 2\cos(2t)$ and $v = \frac{e^{20t}}{20}$,

$$\implies I = \frac{e^{20t}}{20}\sin(2t) - \frac{1}{10}\int e^{20t}\cos(2t)\,dt \tag{A.5}$$

Integrating the integral with the following substitutions, $\lambda = \cos(2t)$ and $d\eta = e^{20t} \implies d\lambda = -2\sin(2t)$ and $\eta = \frac{e^{20t}}{20}$

$$\begin{aligned}\int e^{20t}\cos(2t)\,dt &= \frac{e^{20t}}{20}\cos(2t) + \frac{1}{10}\int e^{20t}\sin(2t)\,dt \\ &= \frac{e^{20t}}{20}\cos(2t) + \frac{I}{10} + C,\;\; C \in \mathbb{R} \end{aligned} \tag{A.6}$$

Substituting eq. (A.6) into eq. (A.5),

$$\begin{aligned} I &= \frac{e^{20t}}{20}\sin(2t) - \frac{1}{10}\left[\frac{e^{20t}}{20}\cos(2t) + \frac{I}{10} + C\right] \\ &= \frac{e^{20t}}{20}\sin(2t) - \frac{e^{20t}}{200}\cos(2t) - \frac{I}{100} + \frac{C}{10} \\ \implies \frac{101}{100}I &= \frac{e^{20t}}{20}\sin(2t) - \frac{e^{20t}}{200}\cos(2t) + \frac{C}{10} \end{aligned} \tag{A.7}$$

From the equality of eq. (A.3) and eq. (A.4),

$$\begin{aligned} \frac{101}{100}\left(\frac{ie^{20t}}{20}\right) &= \frac{e^{20t}}{20}\sin(2t) - \frac{e^{20t}}{200}\cos(2t) + \frac{C}{10} \\ \implies \frac{101}{100}i &= \sin(2t) - \frac{\cos(2t)}{10} + 2Ce^{-20t} \\ \implies i &= \frac{100}{101}\sin(2t) - \frac{10}{101}\cos(2t) + \frac{200}{101}Ce^{-20t} \end{aligned} \tag{A.8}$$

From the initial conditions,

$$\begin{aligned} 0 &= 0 - \frac{10}{101} + \frac{200}{101}C \\ \implies C &= \frac{1}{20} \end{aligned} \tag{A.9}$$

Thus,

$$i(t) = \frac{10}{101}\left(e^{-20t} - \cos(2t) + 10\sin(2t)\right) \tag{A.10}$$

Thus, we can see that the process of repeated integration can be trivialised with the Laplace transform.

Appendix B Laplace Transforms of $\cos(at)$ and e^{at}

,

Lemma B.1 (Laplace Transform of the cosine function)**.** If $f(t) = \cos(at)$, $a \in \mathbb{R}$ then,

$$\mathscr{L}\left\{\cos(at)\right\} = \frac{s}{s^2 + a^2}$$

Proof. Let $I = \mathscr{L}\left\{\cos(at)\right\}$. So,

$$I = \int_0^\infty \cos(at)e^{-st}dt$$

Let $u = \cos(at)$ and $dv = e^{-st} \implies du = -a\sin(at)$ and $v = -\frac{1}{s}e^{-st}$. Integrating by parts,

$$\begin{aligned}
I &= \left[-\frac{e^{-st}}{s}\cos(at)\right]_0^\infty - \int_0^\infty \frac{a}{s}\sin(at)e^{-st}dt \\
&= \left[\lim_{t\to\infty}\left(-\frac{e^{-st}}{s}\cos(at)\right) - \left(-\frac{e^0}{s}\cos(0)\right)\right] - \int_0^\infty \frac{a}{s}\sin(at)e^{-st}dt \\
&= \left(0 - \left(-\frac{1}{s}\right)\right) - \frac{a}{s}\int_0^\infty \sin(at)e^{-st}dt \\
&= \frac{1}{s} - \frac{a}{s}\int_0^\infty \sin(at)e^{-st}dt \qquad \text{(B.1)}
\end{aligned}$$

Now integrating the integral in eq. (B.1) via integration by parts with the following substitutions: Let $\mu = \sin(at)$ and $d\eta = e^{-st} \implies d\mu = a\cos(at)$ and $\eta = -\frac{1}{s}e^{-st}$

$$\begin{aligned}
\implies \int_0^\infty e^{-st}\sin(at) &= \left[-\frac{e^{-st}}{s}\sin(at)\right]_0^\infty - \int_0^\infty -\frac{a}{s}e^{-st}\cos(at) \\
&= \left[\lim_{t\to\infty}\left(-\frac{e^{-st}}{s}\sin(at)\right) - \left(-\frac{e^0}{s}\sin(0)\right)\right] + \frac{a}{s}\int_0^\infty e^{-st}\cos(at) \\
&= (0-0) + \frac{a}{s}I \\
&= \frac{a}{s}I \qquad \text{(B.2)}
\end{aligned}$$

Substituting eq. (B.2) into eq. (B.1),

$$\begin{aligned}
I &= \frac{1}{s} - \frac{a}{s}\left(\frac{a}{s}I\right) \\
&= \frac{1}{s} - \frac{a^2}{s^2}I \\
\implies I\left(1+\frac{a^2}{s^2}\right) &= \frac{1}{s} \\
\implies I\left(\frac{s^2+a^2}{s^2}\right) &= \frac{1}{s} \qquad \text{(B.3)} \\
\implies I &= \frac{1}{s} \div \frac{s^2+a^2}{s^2} \\
\implies I &= \frac{s}{s^2+a^2} \qquad \text{(B.4)}
\end{aligned}$$

As $I = \mathscr{L}\{\cos(at)\}$, it has been shown that

$$\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \qquad \text{(B.5)}$$

■

Lemma B.2 (Laplace Transform of the exponential function)**.** If $f(t) = e^{at}$, $a \in \mathbb{R}$ then,

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

Proof. Let $I = \mathscr{L}\left\{e^{at}\right\}$. So,

$$\begin{aligned}
I &= \int_0^\infty e^{at}e^{-st}dt \\
&= \int_0^\infty e^{-(s-a)t}dt \\
&= \left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^\infty \\
&= -\frac{1}{s-a}\left[\lim_{t\to\infty}\left(e^{-(s-a)t}\right) - e^0\right] \\
&= -\frac{1}{s-a}(-1) \\
I &= \frac{1}{s-a}
\end{aligned}$$

As $I = \mathscr{L}\left\{e^{at}\right\}$, it has been shown that

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a} \tag{B.6}$$

∎

Appendix C Implementation of Euler's Method in Python

```
import numpy as np

def f(t):
    return (10/101)*(np.exp(-20*t) - np.cos(2*t) + 10*np.sin(2*t))
        # Defining the solution function

def didt(i, t):
    return 20*np.sin(2*t)-20*i # Defining the differential equation

h = 0.01 # step size

i = 0 # initial condition

timesteps = np.arange(0,0.51,h)

i_values = [] # stores actual value of function at each timestep

for t in timesteps:
    i_values.append(f(t))

approximated_values = [] # stores result from Euler's method at
    each timestep

approximated_values.append(i)

for t in timesteps: # Loop to iterate over all t in the timesteps
    array
```

```
        # Adds each output from Euler's method into the array, starting
            with the initial condition
        i = i + h*didt(i, t) # Euler's method
        approximated_values.append(i)

    approximated_values.pop() # The counter in the for-loop adds the
        approximation for t=0.51, this is removed here

    local_error = [] # Initialises a list to store the local
        truncation error

    local_error.append(0)

    timesteps_null = np.arange(0.01,0.51,h)

    for t in timesteps_null:
        local_error.append(abs(f(t) - f(t-0.01) - h*didt(f(t-0.01),
            t-0.01))) # Calculates the local truncation error and adds
            it to the list
```

Appendix D Output from Euler's Method

Table 1: Output from Euler's Method

Step (n)	t_n	$i(t_n)$	i_n	ε_l
0	0	0	0	0
1	0.01	0.0018730112222509778	0.0	0.0018730112222509778
2	0.02	0.007031017944653766	0.0039997333386666166	0.0015328756281863679
2	0.03	0.014876352569630524	0.011197653508260127	0.0012536713765806898
4	0.04	0.024918237376864272	0.02095092410249702	0.0010243540252709328
5	0.05	0.036753410134821606	0.03274367807583216	0.0008358814394956483
6	0.06	0.05005026227203262	0.04616162579003136	0.0006808508348097216
7	0.07	0.06453585288825092	0.06087174208980896	0.0005532016128409183
8	0.08	0.07998527730311765	0.07660601660069447	0.0004479720636696126
9	0.09	0.09621296333581256	0.09314845460340476	0.0003611001704692425
10	0.10	0.11306554587583995	0.11032467836788865	0.0002892605220250234
11	0.11	0.1304160336475665	0.12799360885332317	0.0002297307878822881
12	0.12	0.14815903393155733	0.1460408116988324	0.00018028239733027668
13	0.13	0.16620684346548237	0.16437317464449283	0.0001390910348095832
14	0.14	0.18448624851045392	0.1829146500940253	0.00010466335963700557
15	0.15	0.20293590553026272	0.201602849788043	0.00007577700907689111
16	0.16	0.2215041972333765	0.22038632116270232	0.00005143147689841074
17	0.17	0.24014747780595289	0.2392223690533854	0.000030807896028137854
18	0.18	0.2588286367841839	0.2580753136708712	0.000013236111258748545
19	0.19	0.27751592380291906	0.27691509759171495	0.000001.8322794460325087

Continued on next page

Table 1: Output from Euler's Method (Continued)

20	0.20	0.2961819869279559	0.2957161719559685	0.000014845996975900583
21	0.21	0.31480308585186845	0.31445660602650494	0.000026172152226371787
22	0.22	0.3333584482517922	0.333117375433118	0.000036111041616562756
23	0.23	0.3518297433538917	0.3516817933596943	0.00004490826074199025
24	0.24	0.37020065145391734	0.3701350560808594	0.00005276462229999654
25	0.25	0.3884565119951231	0.3884638799729841	0.00005984427630735187
26	0.26	0.40658403595846326	0.4066562116992279	0.00006628135847581637
27	0.27	0.42457107090197227	0.4247009969281297	0.00007218543354569132
28	0.28	0.4424064091001882	0.44258799587312636	0.00007764595201225147
29	0.29	0.46007963096524007	0.4603076362826778	0.00008273589908709342
30	0.30	0.47758097734825167	0.47785089638451694	0.0000875147823151054
31	0.31	0.49490124547988246	0.49520921178662064	0.0000920310777259431
32	0.32	0.5120317042586933	0.5123744015367575	0.0000963242326737293
33	0.33	0.528964025373715	0.5293386095018844	0.00010042630571801756
34	0.34	0.545690227384317	0.5460942579961943	0.00010436330934171859
35	0.35	0.56220263040178	0.5626340112006492	0.00010815630936737014
36	0.36	0.5784938194437835	0.5789507464080575	0.00011182232517857257
37	0.37	0.5945566148824745	0.5950375315207407	0.00011537506684699411
38	0.38	0.6103840486928761	0.6108876075422216	0.00011882553873247825
39	0.39	0.6259693454426538	0.6264943750558876	0.00012218253375732152
40	0.40	0.6413059071560362	0.6418513838847921	0.00012545303816889793
41	0.41	0.6563873013417146	0.6569523252877383	0.00012864256301887804
42	0.42	0.6712072516031152	0.6717910261755697	0.0001317554156355389

Continued on next page

Table 1: Output from Euler's Method (Continued)

43	0.43	0.6857596303546917	0.6863614449346277	0.00013479492197237923
44	0.44	0.70003845325408	0.7006576685267576	0.00013776360872876092
45	0.45	0.7140378750305203	0.7146739106011999	0.00014066335253756085
46	0.46	0.7277521864477262	0.7284045104064566	0.00014349550218668676
47	0.47	0.741175812186693	0.7418439323324385	0.00014626097876112298
48	0.48	0.754303309472675	0.7549867659469737	0.00014896035770226544
49	0.49	0.7671293673022693	0.7678277264177786	0.00015159393607030655
50	0.50	0.7796488061525335	0.7803616552326172	0.00015416178767605794

Appendix E Rolle's Theorem

Theorem E.1 (Rolle's Theorem). If f is a function that is continuous on the interval $[a, b]$, $\{a, b\} \in \mathbb{R}$, differentiable on the interval (a, b) and $f(a) = f(b)$, then $\exists\, \tau \in (a, b)$ where,

$$f'(\tau) = 0 \tag{E.1}$$

Proof. Proving Rolle's theorem algebraically involves invoking several other theorems that will further dilute this essay. As a result, a graphical approach was considered. If the function has the same value at its endpoints, then it follows that the function is either a constant value (a horizontal line) or has a turning point so that the function returns to the original point. This can be shown below:

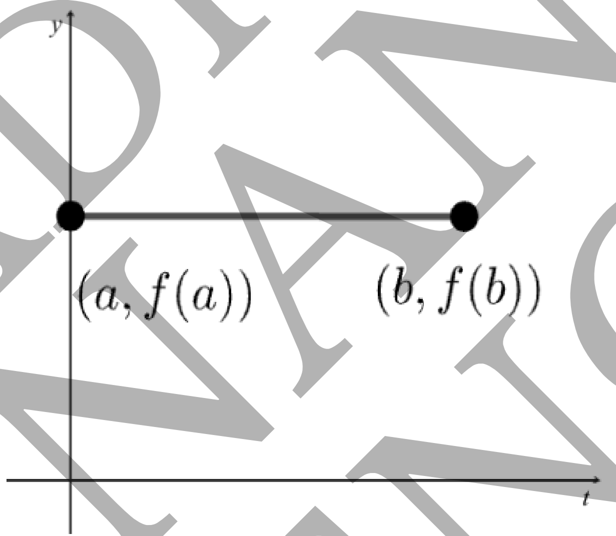

Figure 16: Case 1: $f(t)$ is a constant

In this case, $f(t) = k \implies f'(t) = 0\ \forall\, t \in [a, b]$. Thus, there is a τ where $f'(\tau) = 0$.

When $f(t)$ is not a constant there has to be a turning point so that $f(a) = f(b)$. The turning point is either a minimum or maximum as shown below,

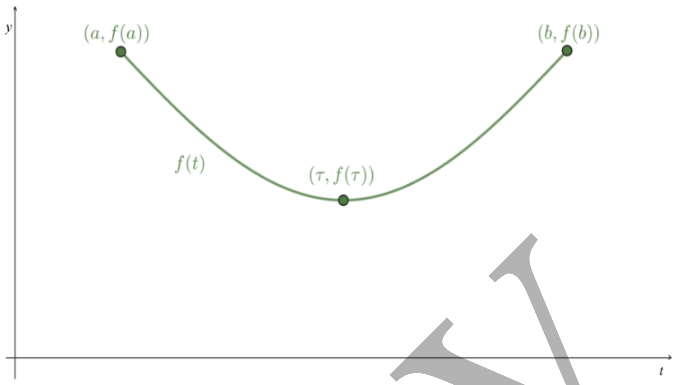

Figure 17: Case 2: $f(t)$ has a minimum

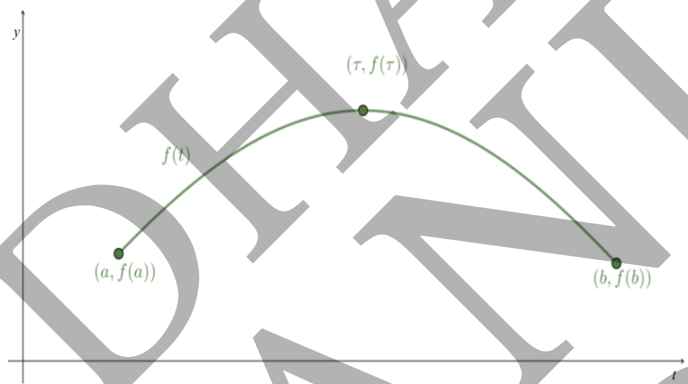

Figure 18: Case 3: $f(t)$ has a maximum

In both cases, the existence of a turning point implies that there is some value of τ where $f'(\tau) = 0$ as the slope of the tangent line at a turning point is 0. ■

It should be noted that Rolle's theorem does not tell us how many such τ values there are, or where τ is. Instead, it only tells us that there exists a τ.