IB Mathematics AA HL

Internal Assessment

Surface Areas and Volumes of n-Balls

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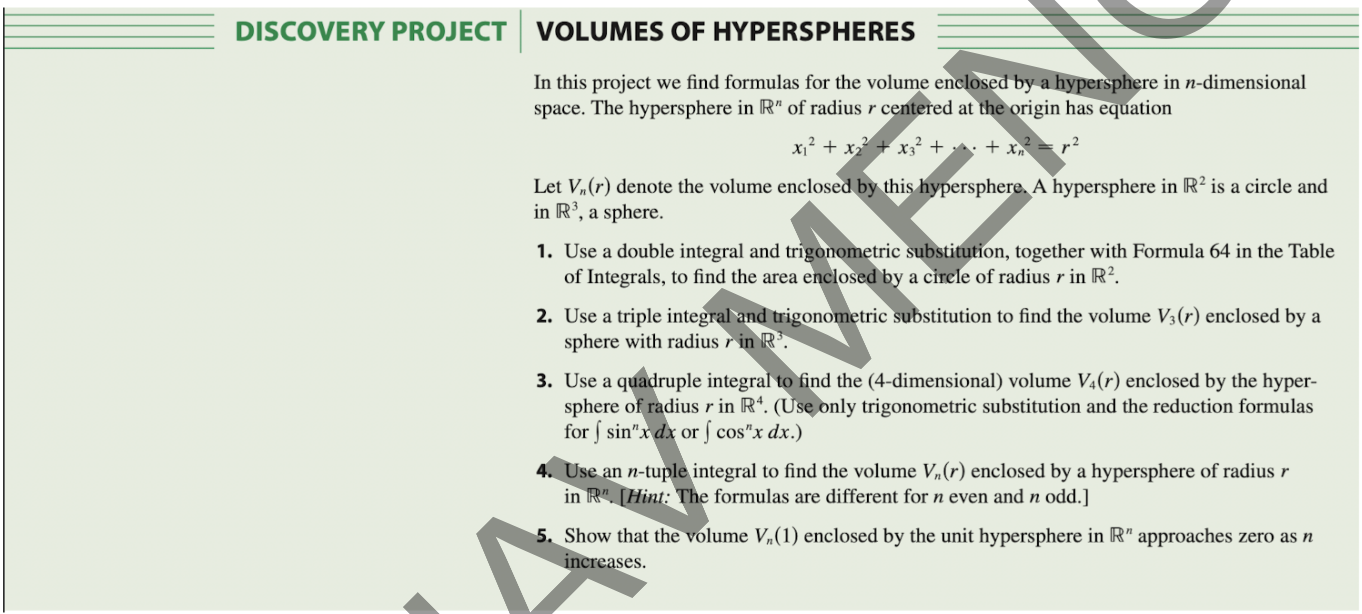
Contents

1 Introduction

As someone passionate about studying physics in university, I decided to work through *Calculus: Early Transcendentals 9E* by James Stewart in order to gain the mathematical maturity required to engage in Physics from a more mathematically rigorous perspective. Thus, I found myself working through the various "discovery projects" present in the textbook to help me hone my mathematical abilities. Through my study, I came across a fascinating project titled "Volumes of Hyperspheres".



DISCOVERY PROJECT | **VOLUMES OF HYPERSPHERES**

In this project we find formulas for the volume enclosed by a hypersphere in n-dimensional space. The hypersphere in $\mathbb{R}^n$ of radius r centered at the origin has equation

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = r^2$$

Let $V_n(r)$ denote the volume enclosed by this hypersphere. A hypersphere in $\mathbb{R}^2$ is a circle and in $\mathbb{R}^3$, a sphere.

1. Use a double integral and trigonometric substitution, together with Formula 64 in the Table of Integrals, to find the area enclosed by a circle of radius r in $\mathbb{R}^2$.
2. Use a triple integral and trigonometric substitution to find the volume $V_3(r)$ enclosed by a sphere with radius r in $\mathbb{R}^3$.
3. Use a quadruple integral to find the (4-dimensional) volume $V_4(r)$ enclosed by the hypersphere of radius r in $\mathbb{R}^4$. (Use only trigonometric substitution and the reduction formulas for $\int \sin^n x\, dx$ or $\int \cos^n x\, dx$.)
4. Use an n-tuple integral to find the volume $V_n(r)$ enclosed by a hypersphere of radius r in $\mathbb{R}^n$. [*Hint:* The formulas are different for n even and n odd.]
5. Show that the volume $V_n(1)$ enclosed by the unit hypersphere in $\mathbb{R}^n$ approaches zero as n increases.

Figure 1: Discovery Project (Stewart 1095)

This project was intriguing as it prompted me to ask myself the question: *what does it mean for a "hypersphere" to be in n dimensions?*. Being unable to think of an intuitive answer, I decided to research this topic further which led me to discover the concept of n-balls. Fascinated by this concept, I decided to investigate n-balls in this mathematical investigation by exploring the general expression for the volume, V_n, and surface area, S_n, of an n-ball as a function of radius R.

It would be prudent to define key terms that will be adopted throughout the course of this investigation. A dimension refers to the number of real numbers needed to define a point in space (Kuttler). For example, we only need two numbers x and y to define a point in two dimensional space as

(x, y) whereas in three dimensions we need three numbers x, y and z to define a point (x, y, z). Thus, it follows that we need n number of real numbers to define a point in n-dimensional space. According to Weisstein, there are two definitions for a hypersphere, also known as an n-sphere. We will be using the "geometer's definition" which states that an n-sphere refers to the generalisation of a circle in n-dimensions. Hence, a 2-sphere is a circle while a 3-sphere is a traditional sphere ("Hypersphere"). It is important to note that the n-sphere only refers to the set of all points that are a common distance r away from a central point (Watkins). Hence, a 2-sphere refers to the set of all paired real numbers (x, y) that are a constant distance r—the radius of the circle—from a central point. Similarly, a 3-sphere is a traditional sphere which refers to the set of all triple real numbers (x, y, z) that are a constant distance r—the radius of the sphere—from a central point. Thus, it follows that an n-sphere refers to the set of all real numbers $(x_1, x_2, x_3 \ldots x_n)$ that are a constant distance r away from a central point. For simplicity, this investigation will assume that the central point is the origin in all dimensions. The n-ball, on the other hand, refers to the points that are also enclosed within the n-sphere (Watkins). Thus a 2-ball is a disc (the points inside the circle are included) while a 2-sphere is the circle that bounds the disk. Similarly, a 3-ball is a traditional solid ball (the points inside the sphere are included) while a 3-sphere is the sphere that encloses the ball. Hence, we can see that the term "volume" strictly refers to the space within the n-ball while "surface area" refers to the area bounded by the n-sphere. In other words, "surface area" refers to the area that bounds the n-ball.

2 Surface Areas and Volumes in Higher Dimensions

2.1 2-ball

2.1.1 Deriving $S_2(R)$

A 2-sphere is simply a circle and thus a 2-ball refers to all the points within the circle. It should be intuitive that the volume $V_2(R)$, which is a measure of "space" enclosed, of the 2-ball must simply then be the area of the circle while the surface area $S_2(R)$ is simply the circumference of the circle. The circumference of a circle is well known to be $C = 2\pi R$ where C is the circumference and R is the radius of the circle. However, invoking this equation would assume that we already know the surface area of the 2-ball. Hence, we must start by first proving this. The equation of a circle in a Cartesian coordinate system with radius R, centred at the origin is:

$$x^2 + y^2 = R^2 \tag{2.1}$$

By rearranging this equation for y and only considering the positive root, we get:

$$y = \sqrt{R^2 - x^2} \tag{2.2}$$

Plotting this function gives us the following semicircle, $y = f(x)$:

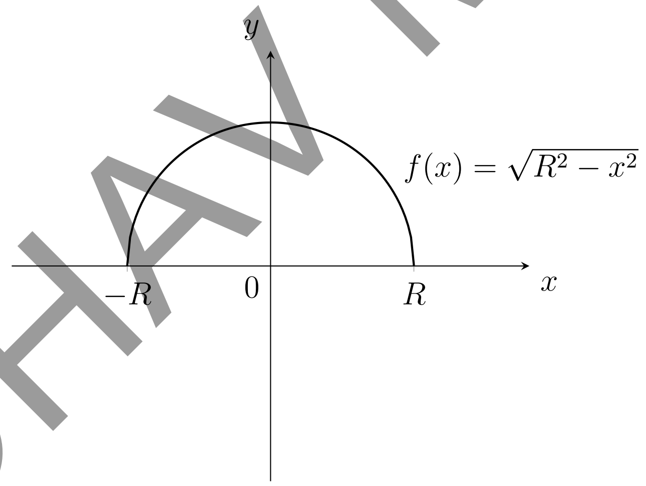


Figure 2: eq. (2.2) plotted by author using TikZ

If we were to double the length of this curve, we would be able to get $S_2(R)$. To find the length of this curve, we can start by plotting a few points and finding the distance between each point and summing them up.

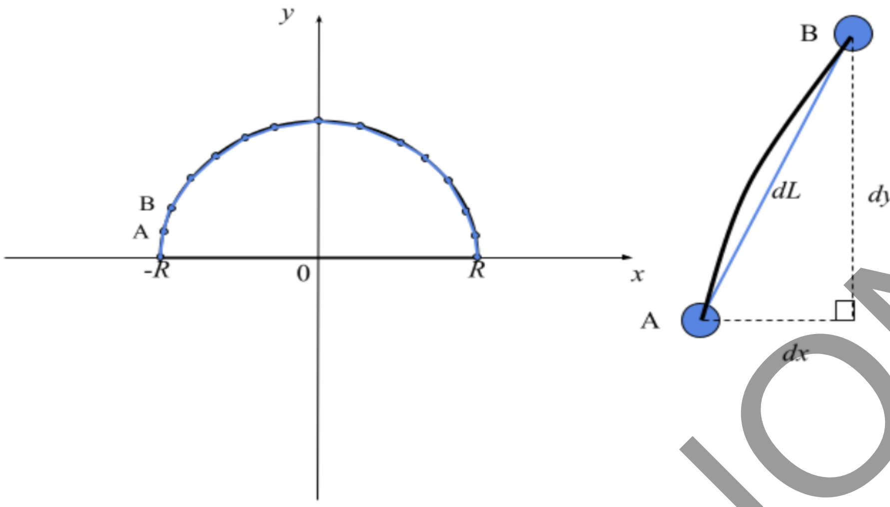


Figure 3: Approximation with 15 points; drawn on Google Drawings by author

Take the arbitrary points A and B with coordinates (x_A, y_A) and (x_B, y_B) respectively. The distance, or arclength, between those two points, dL is given by

$$dL_i = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \tag{2.3}$$

We can denote $x_B - x_A$ as dx and $y_B - y_A$ as dy. Thus, we can rewrite eq. (2.3) as,

$$dL = \sqrt{(dx)^2 + (dy)^2} \tag{2.4}$$

Factoring a dx out of the square root,

$$\begin{aligned} dL &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{(dx)^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right]} \\ dL &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \end{aligned} \tag{2.5}$$

The length L of the semicircle would then be given by adding up every single dL that is formed on the semicircle. Fig. 3 shows that if we were to approximate our semicircle with i points as

$i \to \infty$, we can start to better approximate the arclength. In this case, dx and dy then reduce to infinitesimally small differentials. Thus, the arclength L is given by,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} dL_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{dy_i}{dx}\right)^2} dx \tag{2.6}$$

Eq. (2.6) is the definition of a Riemann integral which reduces to the following definite integral,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{2.7}$$

This result is known as the arclength formula and can be used to calculate the length between two points of any function. In fact, eq. (2.7) is termed a *functional* as it maps a function to a real number (Graves). By taking in a function as an input, it outputs a number which corresponds to the arclength of that function between two points, assuming we have numerical bounds of integration and not a variable as is our case below.

In order to calculate the length of the circle, we must double the arclength of the semicircle shown in fig. 2. As we want the length between $-R$ and R, these will be the bounds of integration. Substituting eq. (2.2) into eq. (2.7) and multiplying by 2 for the full circumference we get,

$$\begin{aligned}
S_2(R) = C &= 2\int_{-R}^{R} \sqrt{1 + \left[\frac{d}{dx}\left(\sqrt{R^2 - x^2}\right)\right]^2} dx \\
&= 2\int_{-R}^{R} \sqrt{1 + \left[\frac{1}{2}(-2x)\left(R^2 - x^2\right)^{-\frac{1}{2}}\right]^2} dx \quad \text{(Via chain rule)} \\
&= 2\int_{-R}^{R} \sqrt{1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2} dx \\
&= 2\int_{-R}^{R} \sqrt{1 + \frac{x^2}{R^2 - x^2}} dx
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_{-R}^{R} \sqrt{\frac{R^2 - x^2 + x^2}{R^2 - x^2}}dx \quad \text{(Taking LCM)} \\
&= 2\int_{-R}^{R} \sqrt{\frac{R^2}{R^2 - x^2}}dx \\
&= 2\int_{-R}^{R} \frac{R}{\sqrt{R^2 - x^2}}dx \\
&= 2R\left[\arcsin\left(\frac{x}{R}\right)\right]_{-R}^{R} \\
&= 2R\left[\arcsin\left(\frac{R}{R}\right) - \arcsin\left(\frac{-R}{R}\right)\right] \\
&= 2R[\arcsin(1) - \arcsin(-1)] \\
&= 2R\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] \quad \left(\text{Range of the arcsine function is } \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right) \\
S_2(R) &= 2\pi R
\end{aligned}
\tag{2.8}
$$

It should be noted that there is an element of *circular* reasoning in the above calculation as the integral evaluates to $\arcsin(1)$ and $\arcsin(-1)$ which evaluate to $\pm\frac{\pi}{2}$. As π is defined as the ratio of a circle's circumference to its diameter (Bogart), using π to derive the circumference of a circle is effectively trying to prove an axiom. Nonetheless, it is insightful to see how a relatively advanced field like calculus enables us to develop core equations in mathematics!

2.1.2 Deriving $V_2(R)$

A 2-ball must also include all the points within the circle itself. Thus, we inscribe infinitely many smaller circles in order to "touch" every point on the interior. The circles should be so close that the they are an infinitesimally small length dr away from each other.

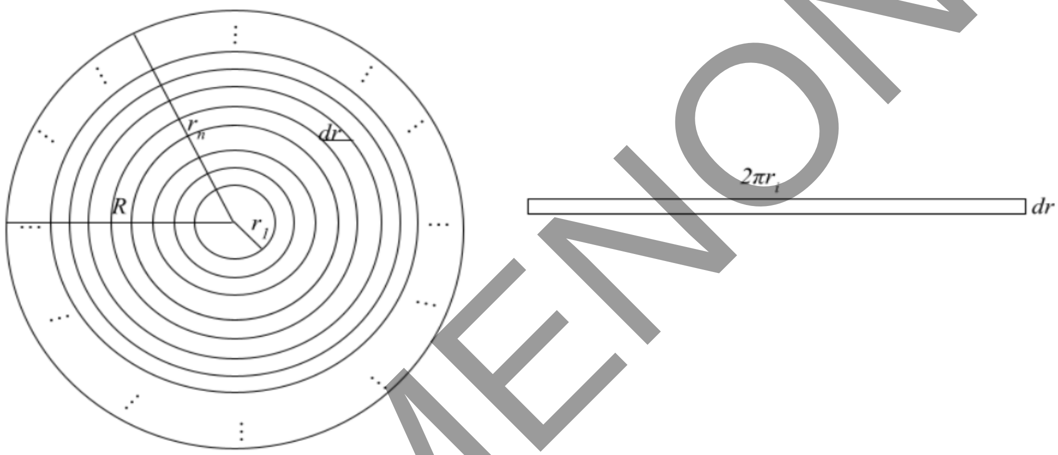


Figure 4: Infinitely many inscribed circles and rectangle element (not to scale); drawn on Google Drawings by author

The radius of the n^{th} circle away from the centre is denoted as r_n as shown in fig. 4. If an arbitrary i^{th} circle was chosen, and was "cut" across the top and unfolded to form a rectangle, we would get an infinitely thin rectangle element as shown in fig. 4 with width being the circumference of the i^{th} circle and length dr. The area of this rectangle is then $2\pi r_i dr$. It should thus be sensible that the sum of every single area element within the circle should add up to the area of the circle. Ideally. we would want infinitely many inscribed circles in order to end up with the original circle. Thus, we get that the volume of the 2-sphere is,

$$V_2(R) = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i dr \tag{2.9}$$

As the sum of the radii of the smaller inscribed circles should be the radius R of the main circle, we get eq. (2.9) to simply be the definition of the following definite integral as we integrate from the centre of the 2-ball to the outside,

$$\begin{aligned} V_2(R) &= \int_0^R 2\pi r dr \\ &= 2\pi \left[\frac{r^2}{2}\right]_0^R \\ &= \pi(R)^2 - \pi(0)^2 \\ V_2(R) &= \pi R^2 \end{aligned} \tag{2.10}$$

Thus, it is shown that the volume of the 2-ball is simply given by the area of a circle!

2.2 3-ball

2.2.1 Deriving $S_3(R)$

A 3-ball, as mentioned in section 1, is a traditional solid ball (think of a marble for example) while a 3-sphere is simply the sphere that envelopes the ball. As we did in section 2.1, the surface area of the 3-ball will first be derived. A sphere can be formed by revolving a semicircle 2π radians about the x-axis.

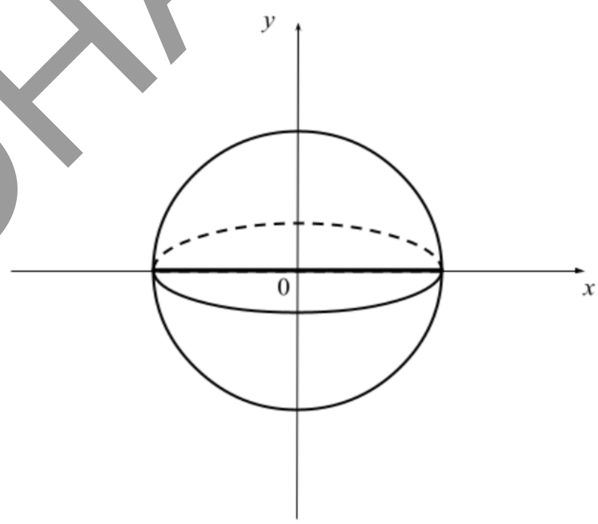


Figure 5: Sphere formed by revolution of semicircle; drawn on Google Drawings by author

If we take two infinitesimally close points on the circle as shown below and revolve them around the x-axis along with the semicircle, we end up getting a frustrum element.

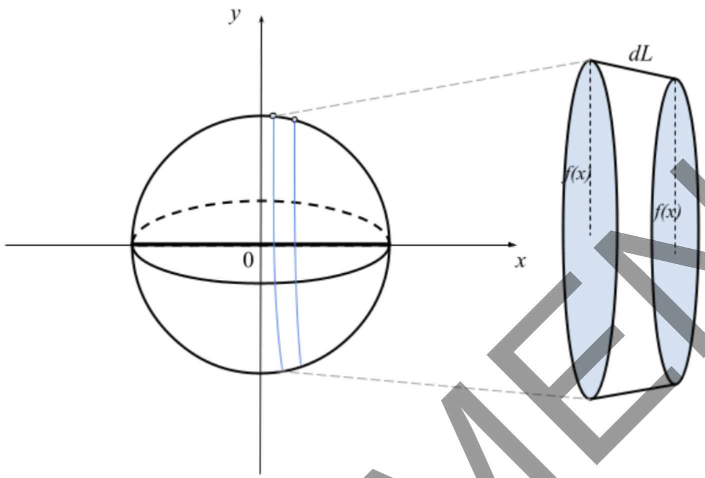


Figure 6: Revolved sphere and frustrum element; drawn on Google Drawings by author

The radius of the ends of the frustrum is simply the distance from the x-axis to the edge, which is defined as the function $y = f(x)$ itself as shown in fig. 6. The slant length of the frustrum is then the distance between the two points on the revolved surface, this corresponds to the arclength of the function between the two points which we term dL. Consider just the frustrum element,

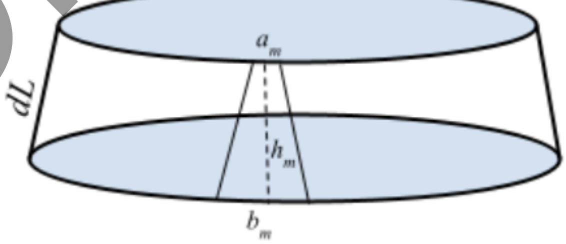


Figure 7: Frustrum element; drawn on Google Drawings by author

The frustrum element can be divided into m trapezium. The length of the parallel sides of the mth trapezium is denoted by a_m and b_m while the lateral height is denoted by h_m. The area of this trapezium is then,

$$A_m = \frac{a_m + b_m}{2} h_m \tag{2.11}$$

Given that there are m trapezium, the total area of all the trapezium is

$$mA_m = m\frac{a_m + b_m}{2} h_m \tag{2.12}$$

We will then get the surface area of the trapezium if m tends to infinity. Thus,

$$A_{trapezium} = \lim_{m\to\infty}\left(m\frac{a_m + b_m}{2} h_m\right) \tag{2.13}$$

Recognising that $\lim_{m\to\infty} a_m$ is simply the circumference of the top circle which has radius of the function y.

$$\therefore \lim_{m\to\infty} ma_m = 2\pi y \tag{2.14}$$

By the same argument,

$$\lim_{m\to\infty} mb_m = 2\pi y \tag{2.15}$$

Recognising from fig. 7 that,

$$\lim_{n\to\infty} mh_m = dL \tag{2.16}$$

Thus, the surface area of the frustrum element dS is given by,

$$dS = \frac{2\pi y + 2\pi y}{2} dL = 2\pi y dL \tag{2.17}$$

Given that in our revolution we would have k such frustrum elements we get that the surface area is best approximated as $k \to \infty$. Thus,

$$S = \lim_{k=\infty}\sum_{i=1}^{k} dS_k = \lim_{k=\infty}\sum_{i=1}^{k}(2\pi y_k dL) \tag{2.18}$$

By substituting in eq. (2.5), eq. (2.18) reduces to the integral,

$$S = \int_a^b 2\pi y dL = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{2.19}$$

It should be noted that we are once again using the arclength functional. If we take two points and determine the arclength between them, revolving those two points about the x-axis will result in a surface area formed—similar to how a volume of revolution is formed by manipulating an integral which represents the area under the curve. We are effectively going from a two dimensional analogue of a tool and extrapolating it to three dimensions, thus showing us how a mathematical tool can be extrapolated to have a wider use-case.

Substituting in eq. (2.1) for y and recognising from fig. 4 and fig. 5 that our bounds of integration are $-R$ and R,

$$\begin{aligned}
S_3(R) &= \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{1 + \left[\frac{d}{dx}\left(\sqrt{R^2 - x^2}\right)\right]^2} dx \\
&= \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{1 + \left[\frac{1}{2}\left(R^2 - x^2\right)^{-\frac{1}{2}}(-2x)\right]^2} dx \quad \text{(Via chain rule)} \\
&= \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2} dx \\
&= \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{1 + \frac{x^2}{R^2 - x^2}} dx \\
&= \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{\frac{R^2 - x^2 + x^2}{R^2 - x^2}} dx \quad \text{(Taking LCM)} \\
&= \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{\frac{R^2}{R^2 - x^2}} dx \\
&= \int_{-R}^{R} 2\pi R dx \\
&= \left[2\pi R x\right]_{-R}^{R} \\
&= 2\pi R^2 + 2\pi R^2
\end{aligned}$$

$$S_3(R) = 4\pi R^2 \tag{2.20}$$

Thus, it is proved that the surface area of a 3-ball is simply the surface area of a sphere!

2.2.2 Deriving $V_3(R)$

To find the volume of a 3-ball, we can apply a similar process as we did to find the volume of a 2-ball. By inscribing our 3-ball with infinitely many 3-balls of smaller radii, such that they are an infinitesimally small distance dr away from each other, we simply integrate from the centre to the surface of the 3-ball

$$\begin{aligned} V_3(R) &= \int_0^R 4\pi r^2 dr \\ &= \frac{4}{3}\pi \left[r^3\right]_0^R \\ &= \frac{4}{3}\pi[R^3 - 0] \\ V_3(R) &= \frac{4}{3}\pi R^3 \end{aligned} \tag{2.21}$$

Thus, it is proved that the volume of a 3-ball is simply the volume of a sphere!

2.3 Summary for the first two Dimensions

Table 1: Surface areas and volumes for $n = 2$ and $n = 3$

n	$S_n(R)$	$V_n(R)$
2	$2\pi R$	πR^2
3	$4\pi R^2$	$\frac{4}{3}\pi R^3$

Table 1 shows the surface areas and volumes that have been derived so far. There are some important patterns to notice. Firstly, there is a pattern between the dimension n and the exponent for $S_n(R)$ and $V_n(R)$. This allows us to state the theorem,

Theorem 1. *The relationship between the dimension and the exponent of $S_n(R)$ and $V_n(R)$ is given by*

$$S_n(R) \propto R^{n-1} \tag{2.22}$$

$$V_n(R) \propto R^n \tag{2.23}$$

By removing the proportionality symbol, we must add constants of proportionality,

$$S_n(R) = k_1 R^{n-1} \tag{2.24}$$

$$V_n(R) = k_2 R^n \tag{2.25}$$

where k_1 and k_2 are constants of proportionality, $(k_1, k_2) \in \mathbb{R}$. We can find the constants of proportionality by substituting $R = 1$,

$$S_n(1) = k_1(1^{n-1}) = k_1(1) = k_1$$

$$V_n(1) = k_2(1^n) = k_2(1) = k_2$$

Thus, we can rewrite eq. (2.24) and eq. (2.25) as

$$S_n(R) = S_n(1)R^{n-1} \tag{2.26}$$

$$V_n(R) = V_n(1)R^n \tag{2.27}$$

Now by considering the relationship between $S_n(R)$ and $V_n(R)$ for every value of n listed in table 1, it should be evident that $S_n(R)$ is simply the derivative of $V_n(R)$ with respect to R. Thus,

Theorem 2. *The relationship between $V_n(R)$ and $S_n(R)\ \forall\, n \in \mathbb{Z}^+$ is given by,*

$$S_n(R) = \frac{d}{dR} V_n(R) \tag{2.28}$$

Although, the proofs for theorem 1 and 2 are beyond the scope of this investigation, their application is still necessary and thus these theorems are presented.

2.4 Interlude - The Gamma Function

The gamma function is defined as ("Gamma Function"),

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \tag{2.29}$$

It is an extremely special function as it can generalise the concept of factorials, which is only defined for positive integers, to all real numbers (Weisstein). According to Orloff, the gamma function has the following properties which will be now be proved .

Theorem 3.

$$\Gamma(n+1) = n\Gamma(n) \tag{2.30}$$

Proof.

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^{n+1-1} e^{-x} dx \\ &= \int_0^\infty x^n e^{-x} dx\end{aligned}$$

Integrating by parts where $u = x^n \implies \frac{du}{dx} = nx^{n-1}$ and $\frac{dv}{dx} = e^{-x} \implies v = -e^{-x}$,

$$\begin{aligned}\Gamma(n+1) &= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty -nx^{n-1} e^{-x} dx \\ &= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx \\ &= \left[\left(-x^n e^{-x}\right)_{x=\infty} - \left(-0^n e^{-0}\right)\right] + n\Gamma(n) \\ &= n\Gamma(n) \quad \left(\text{Noting } \lim_{x\to\infty} e^{-x} = 0\right)\end{aligned}$$

$\blacksquare$

Theorem 4.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{2.31}$$

Proof.

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx \\ &= \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \\ &= \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx\end{aligned}$$

Let $t = \sqrt{x} \implies \frac{dt}{dx} = \frac{1}{2\sqrt{x}} \implies dx = 2\sqrt{x}dt$. Substituting,

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty \frac{e^{-t^2}}{\sqrt{x}} 2\sqrt{x} dt \\ &= 2\int_0^\infty e^{-t^2} dt\end{aligned}$$

Let $f(t) = e^{-t^2}$. Testing for evenness,

$$f(t) = e^{-t^2}$$

$$f(-t) = e^{-(-t)^2} = e^{-t^2} = f(t)$$

Hence, $f(t)$ is even. Thus, by symmetry of the function we have,

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= 2\int_0^\infty e^{-t^2} dt \\ &= \int_{-\infty}^\infty e^{-t^2} dt\end{aligned}$$

The above integral is known as the integral of the gaussian function which evaluates elegantly to $\sqrt{\pi}$. The proof of this is in appendix B. Thus,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{2.32}$$

■

3 General Surface Areas and Volumes in n Dimensions

We are now equipped to consider the general equation for the surface area and volume of an n-ball.

3.1 Deriving $S_n(R)$ and $V_n(R)$

According to ("Hypersphere") the following equation is true,

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^n = \int_0^{\infty} e^{-R^2} S_n(R) dR \tag{3.1}$$

Note that since we have definite integrals, the variables in the integrand are dummy variables. This allows us to equate the integral with respect to x and R. The LHS involves two functions in terms of R, therefore we can rewrite $S_n(R)$ using eq. (2.26),

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^n &= \int_0^{\infty} e^{-R^2} S_n(1) R^{n-1} dR \\ &= S_n(1) \int_0^{\infty} e^{-R^2} R^{n-1} dR \end{aligned} \tag{3.2}$$

The LHS is simply the gaussian integral raised to the power of n. Thus eq. (3.2) can be rewritten as,

$$S_n(1) \int_0^{\infty} e^{-R^2} R^{n-1} dR = (\sqrt{\pi})^n = \pi^{\frac{n}{2}} \tag{3.3}$$

We want to find out $S_n(1)$ as this will allow us to extrapolate the surface area of an n-ball for any radius using eq. (2.26). We can then simply integrate to find the volume. Therefore,

$$S_n(1) = \frac{\pi^{\frac{n}{2}}}{\int_0^\infty e^{-R^2} R^{n-1} dR} \tag{3.4}$$

Evaluating the denominator using substitution: let $y = R^2 \Rightarrow \frac{dy}{dR} = 2R \Rightarrow dR = \frac{dy}{2R}$. Thus,

$$\int_0^\infty e^{-R^2} R^{n-1} dR = \int_0^\infty e^{-y} R^{n-1} \frac{dy}{2R} \tag{3.5}$$

Recognising from the substitution that $R = \sqrt{y}$,

$$\begin{aligned}\int_0^\infty e^{-y} R^{n-1} \frac{dy}{2R} &= \int_0^\infty e^{-y} (\sqrt{y})^{n-1} \frac{dy}{2\sqrt{y}} \\ &= \frac{1}{2} \int_0^\infty e^{-y} (\sqrt{y})^{n-2} dy \\ &= \frac{1}{2} \int_0^\infty e^{-y} y^{\frac{n}{2}-1} dy \end{aligned} \tag{3.6}$$

Notice that eq. (3.6) is simply $\Gamma(\frac{n}{2})$ where the y has been swapped with the x, which is irrelevant anyway as they are dummy variables. Hence,

$$\int_0^\infty e^{-R^2} R^{n-1} dR = \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \tag{3.7}$$

Substituting back into eq. (3.4),

$$S_n(1) = \frac{\pi^{\frac{n}{2}}}{\frac{1}{2}\Gamma(\frac{n}{2})} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{3.8}$$

We can now use eq. (2.26) to get the general equation for the surface area of an n-ball,

$$S_n(R) = \frac{2\pi^{\frac{n}{2}} R^{n-1}}{\Gamma(\frac{n}{2})} \tag{3.9}$$

We can now use theorem 2 to get the general equation for the volume of an n-ball,

$$\begin{aligned}
V_n(R) &= \int_0^R S_n(R)dR \\
&= \int_0^R \frac{2\pi^{\frac{n}{2}} R^{n-1}}{\Gamma(\frac{n}{2})} dR \\
&= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^R R^{n-1} dR \quad \text{(Pulling out the constants)} \\
&= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left[\frac{R^n}{n}\right]_0^R \\
&= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left(\left[\frac{R^n}{n}\right] - \left[\frac{0^n}{n}\right]\right) \\
V_n(R) &= \frac{2\pi^{\frac{n}{2}} R^n}{n\Gamma(\frac{n}{2})} \qquad (3.10)
\end{aligned}$$

Remarkably, eq. (3.9) and eq. (3.10) gives us the surface area and volume of a ball in any dimension! To find the equation for these quantities in a given dimension, we simply substitute in the dimension for the value of n.

3.2 Numerical Results

Although we have a general equation for the surface area and volume of an n-ball, it would be interesting to see what the equations look like for a given dimension. Thus, the first surface area and volume in the first ten dimensions were computed. Sample calculation for $n = 4$:

$$\begin{aligned}
S_4(R) &= \frac{2\pi^{\frac{4}{2}} R^{4-1}}{\Gamma(\frac{4}{2})} \\
&= \frac{2\pi^2 R^3}{\Gamma(2)} \qquad (3.11)
\end{aligned}$$

Evaluating $\Gamma(2)$,

$$\begin{aligned}
\Gamma(2) &= \Gamma(1+1) \\
&= 1 \cdot \Gamma(1) \text{ (From theorem 3)} \\
&= \int_0^\infty x^{1-1} e^{-x} dx \\
&= \int_0^\infty e^{-x} dx \\
&= \left[-e^{-x}\right]_0^\infty \\
&= \lim_{x \to \infty} (-e^{-x}) + e^0 \\
\Gamma(2) &= 1
\end{aligned} \tag{3.12}$$

Hence, $S_4(R) = 2\pi^2 R^3$.

Calculating $V_4(R)$,

$$\begin{aligned}
V_4(R) &= \frac{2\pi^2 R^4}{4\Gamma(2)} \\
V_4(R) &= \frac{2\pi^2 R^4}{4} = \frac{1}{2}\pi^2 R^4
\end{aligned} \tag{3.13}$$

In fact, the surface areas and volumes for the first ten dimensions is shown in the table below. The derivation of these equations is shown in appendix A.

Table 2: $S_n(R)$ and $V_n(R)$ for $1 \le n \le 10$

n	$S_n(R)$	$V_n(R)$
1	2	$2R$
2	$2\pi R$	πR^2
3	$4\pi R^2$	$\frac{4}{3}\pi R^3$
4	$2\pi^2 R^3$	$\frac{1}{2}\pi^2 R^4$
5	$\frac{8}{3}\pi^2 R^4$	$\frac{8}{15}\pi^2 R^5$
6	$\pi^3 R^5$	$\frac{1}{6}\pi^3 R^6$
7	$\frac{16}{15}\pi^3 R^6$	$\frac{16}{105}\pi^3 R^7$
8	$\frac{1}{3}\pi^4 R^7$	$\frac{1}{24}\pi^4 R^8$
9	$\frac{32}{105}\pi^4 R^8$	$\frac{32}{945}\pi^4 R^9$
10	$\frac{1}{12}\pi^5 R^9$	$\frac{1}{120}\pi^5 R^{10}$

3.3 Interdimensional Comparisons of the Unit Ball

If we consider the volumes of the unit ball, that is the ball for which $R = 1$, and plot their volumes, $V_n(1)$, against the dimension n we get the following graph,

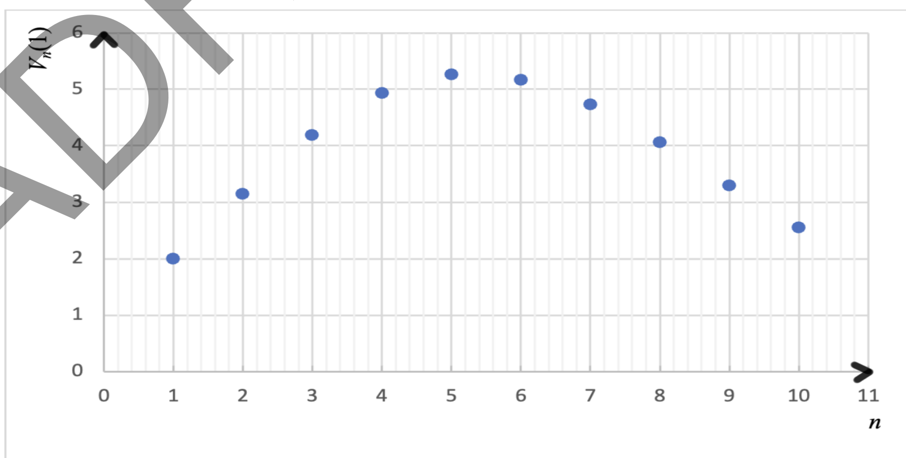


Figure 8: Volume of the unit ball for $1 \le n \le 10$; plotted on Microsoft Excel by author

We can clearly see that the volume of the unit ball increases from the first dimension to the fifth dimension, after which it starts to drop. This is extremely counter-intuitive. One might think that the volume would indefinitely increase as n tends to infinity. However, such assumptions are inappropriate as they are not defended by some form of mathematical proof. Interestingly, it is also inappropriate to compare the volume of the unit ball in between dimensions themselves. If we use a tool from the sciences called "dimensional analysis" where we consider the units of the volume, we can quickly see that that units of volume changes depending on dimensions. The unit for the volume comes entirely from the R term and its power as that is the only variable in the function. Hence, the volume of the unit 1-ball will have units of metres while the volume of the unit 5-ball will have units of metres^5. It is not sensible to compare quantities with different units as it would be the mathematical equivalent of comparing apples to oranges. To instead make a comparison, we would have to take the nth root for the n-ball in order to ensure consistent units. Thus, the first root of the unit 1-ball is simply 2 metres while the fifth root of the unit 5-ball $= \sqrt[5]{\frac{8}{15}\pi^2} \approx 1.393990117$ metres to ten significant figures as shown below in fig. 9. This might prompt us to ask whether there is a purpose in comparing between dimensions or whether the maxima at $n = 5$ is simply a mathematical coincidence.

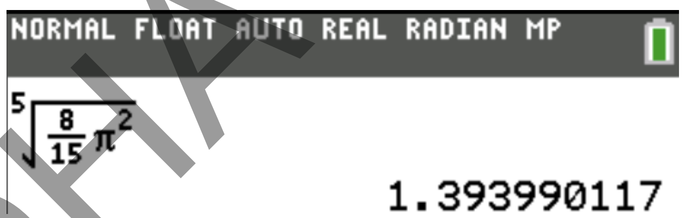


Figure 9: Fifth root of the volume of the unit 5-ball; Computed on TI-84 Plus CE

4 Future Work

This investigation solely focused on computing the volumes and surface areas in different dimensions and did not focus on potentially visualising them. An extension of this work could focus on the stereographic projection of the n-ball to get an understanding for how it might look in higher dimensions. Furthermore, the general equation for $S_n(R)$ and $V_n(R)$ will still yield an output for a fractional dimension. Thus, it might be prudent to consider mathematically if there can exist a fractional dimension of some sort. While this investigation focused on deriving the general expression for $S_n(R)$ and $V_n(R)$, they can also be expressed as recursive relationships where the value of the surface area and volume for an arbitrary dimension depends on the value of a previous dimension. This could be evident by the pattern in the exponent of π for both $S_n(R)$ and $V_n(R)$.

The application of abstract mathematics to generalise notions of "length" and "volume" is often done by a mathematical operation termed the Lebesgue measure of a set (Zakon). It could also be interesting to see how set theory can harmoniously collaborate with other fields of mathematics like differential geometry in order to help us obtain the same results.

5 Conclusion

In conclusion, this investigation focused on extending the notion of surface areas and volumes to higher dimensions. First, we derived the surface area and volumes of the n-ball for the first two dimensions by considering the arclength functional and surface area by revolution integral. We then introduced the gamma function which allowed us to derive the general equation for the surface area and volume of an n-ball. It is fascinating how surface area and volume, concepts that I was first introduced to in the sixth grade, have a rich mathematical background to their derivations and are a lot more complicated than they seem. I was pleasantly surprised to see that the equation for the surface area and volume of a sphere—equations I previously took for granted—can be derived relatively easily using the tools of calculus. As an aspiring Physics major, applied Math always called out to me. However, this investigation sparked my interest for the side of pure Math. The more theoretical side of Physics like String Theory often invoke higher dimensions to justify their mathematical framework (Sutter); Thus, even if notions of spheres and balls may not have a physical meaning for $n \neq 3$, it may have niche implications for certain areas in Physics.

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Appendix A Evaluating $S_n(R)$ and $V_n(R)$ for $n = 1$ and $5 \leq n \leq 10$

A.1 $n = 1$

$$\begin{aligned} S_1(R) &= \frac{2\pi^{\frac{1}{2}} R^{1-1}}{\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{2\pi^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \text{ (From theorem 4)} \\ \therefore S_1(R) &= 2 \end{aligned} \tag{A.1}$$

Finding the volume,

$$\begin{aligned} V_1(R) &= \frac{2\pi^{\frac{1}{2}} R^{1}}{1 \cdot \Gamma\left(\frac{1}{2}\right)} \\ &= 2R \end{aligned} \tag{A.2}$$

A.2 $n = 5$

$$S_5(R) = \frac{2\pi^{\frac{5}{2}} R^{5-1}}{\Gamma\left(\frac{5}{2}\right)}$$

Evaluating $\Gamma\left(\frac{5}{2}\right)$ by repeatedly applying theorem 3,

$$\begin{aligned}
\Gamma\left(\frac{5}{2}\right) &= \frac{3}{2}\Gamma\left(\frac{3}{2}\right) \\
&= \frac{1}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{3\pi^{\frac{1}{2}}}{4} \\
\therefore S_5(R) &= \frac{2\pi^{\frac{5}{2}}R^4}{\frac{3\pi^{\frac{1}{2}}}{4}} \\
S_5(R) &= \frac{8}{3}\pi^2R^4
\end{aligned} \tag{A.3}$$

Finding the volume,

$$\begin{aligned}
V_5(R) &= \frac{2\pi^{\frac{5}{2}}R^5}{5\Gamma\left(\frac{5}{2}\right)} \\
&= \frac{8}{15}\pi^2R^5
\end{aligned} \tag{A.4}$$

A.3 $n = 6$

$$\begin{aligned}
S_6(R) &= \frac{2\pi^{\frac{6}{2}}R^{6-1}}{\Gamma\left(\frac{6}{2}\right)} \\
&= \frac{2\pi^3R^5}{\Gamma(3)}
\end{aligned}$$

Evaluating $\Gamma(3)$ by repeatedly applying theorem 3,

$$\begin{aligned}
\Gamma(3) &= 2\Gamma(2) \\
&= 1\cdot 2\Gamma(1) \\
&= 2 \quad (\Gamma(1)\ \text{ evaluated in section 3.2)} \\
\therefore S_6(R) &= \pi^3R^5
\end{aligned} \tag{A.5}$$

Finding the volume,

$$\begin{aligned} V_6(R) &= \frac{2\pi^{\frac{6}{2}} R^6}{6\Gamma(3)} \\ &= \frac{1}{6}\pi^3 R^6 \end{aligned} \tag{A.6}$$

A.4 $n = 7$

$$\begin{aligned} S_7(R) &= \frac{2\pi^{\frac{7}{2}} R^{7-1}}{\Gamma\left(\frac{7}{2}\right)} \\ &= \frac{2\pi^{\frac{7}{2}} R^6}{\Gamma\left(\frac{7}{2}\right)} \end{aligned}$$

Evaluating $\Gamma\left(\frac{7}{2}\right)$ by repeatedly applying theorem 3,

$$\begin{aligned} \Gamma\left(\frac{7}{2}\right) &= \frac{5}{2}\Gamma\left(\frac{5}{2}\right) \\ &= \frac{3}{2}\cdot\frac{5}{2}\Gamma\left(\frac{3}{2}\right) \\ &= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\Gamma\left(\frac{1}{2}\right) \\ &= \frac{15\pi^{\frac{1}{2}}}{8} \\ \therefore S_7(R) &= \frac{2\pi^{\frac{7}{2}} R^6}{\frac{15\pi^{\frac{1}{2}}}{8}} \\ S_7(R) &= \frac{16}{15}\pi^3 R^6 \end{aligned} \tag{A.7}$$

Finding the volume,

$$
\begin{aligned}
V_7(R) &= \frac{2\pi^{\frac{7}{2}}R^7}{7\Gamma\left(\frac{7}{2}\right)} \\
&= \frac{2\pi^{\frac{7}{2}}R^7}{\frac{105\pi^{\frac{1}{2}}}{8}} \\
V_7(R) &= \frac{16}{105}\pi^3R^7 \qquad \text{(A.8)}
\end{aligned}
$$

A.5 $n = 8$

$$
\begin{aligned}
S_8(R) &= \frac{2\pi^{\frac{8}{2}}R^{8-1}}{\Gamma\left(\frac{8}{2}\right)} \\
&= \frac{2\pi^4R^7}{\Gamma(4)}
\end{aligned}
$$

Evaluating $\Gamma(4)$,

$$
\begin{aligned}
\Gamma(4) &= 3\Gamma(3) \\
&= 3 \cdot 2 = 6 \text{ (From section B.2)} \\
\therefore S_8(R) &= \frac{2\pi^4R^7}{6} \\
S_8(R) &= \frac{1}{3}\pi^4R^7 \qquad \text{(A.9)}
\end{aligned}
$$

Finding the volume,

$$
\begin{aligned}
V_8(R) &= \frac{2\pi^{\frac{8}{2}}R^8}{8\Gamma(4)} \\
&= \frac{2\pi^4R^8}{48} \\
V_8(R) &= \frac{1}{24}\pi^4R^8 \qquad \text{(A.10)}
\end{aligned}
$$

A.6 $n = 9$

$$S_9(R) = \frac{2\pi^{\frac{9}{2}} R^{9-1}}{\Gamma\left(\frac{9}{2}\right)}$$
$$= \frac{2\pi^{\frac{9}{2}} R^{8}}{\Gamma\left(\frac{9}{2}\right)}$$

Evaluating $\Gamma\left(\frac{9}{2}\right)$ by repeatedly applying theorem 3,

$$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right)$$
$$= \frac{5}{2} \cdot \frac{7}{2}\Gamma\left(\frac{5}{2}\right)$$
$$= \frac{35}{4} \cdot \frac{3\pi^{\frac{1}{2}}}{4} \quad \text{(From section B.1)}$$
$$= \frac{105\pi^{\frac{1}{2}}}{16}$$
$$\therefore S_9(R) = \frac{2\pi^{\frac{9}{2}} R^{8}}{\frac{105\pi^{\frac{1}{2}}}{16}}$$
$$S_9(R) = \frac{32}{105}\pi^4 R^8 \tag{A.11}$$

Finding the volume,

$$V_9(R) = \frac{2\pi^{\frac{9}{2}} R^{9}}{9\Gamma\left(\frac{9}{2}\right)}$$
$$= \frac{2\pi^{\frac{9}{2}} R^{9}}{\frac{945\pi^{\frac{1}{2}}}{16}}$$
$$V_9(R) = \frac{32}{945}\pi^4 R^9 \tag{A.12}$$

A.7 $n = 10$

$$S_{10}(R) = \frac{2\pi^{\frac{10}{2}} R^{10-1}}{\Gamma\left(\frac{10}{2}\right)}$$
$$= \frac{2\pi^5 R^9}{\Gamma(5)}$$

Evaluating $\Gamma(5)$,

$$\Gamma(5) = 4\Gamma(4)$$
$$= 4 \cdot 6 = 24 \text{ (From section B.4)}$$
$$\therefore S_{10}(R) = \frac{2\pi^5 R^9}{24}$$
$$S_{10}(R) = \frac{1}{12}\pi^5 R^9 \tag{A.13}$$

Finding the volume,

$$V_{10}(R) = \frac{2\pi^{\frac{10}{2}} R^{10}}{10\Gamma(5)} \tag{A.14}$$
$$= \frac{2\pi^5 R^{10}}{240}$$
$$V_{10}(R) = \frac{1}{120}\pi^5 R^{10} \tag{A.15}$$

Appendix B Evaluating the Gaussian Integral

The Gaussian integral was introduced in section 2.4 and is defined as,

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx \tag{B.1}$$

The method that will be used to evaluate this integral is attributed to the mathematician Siméon Denis Poisson (Bell), who is arguably most famous for the Poisson distribution. The application of this method to evaluate this integral was also taken from Bell.

We begin by squaring I to give,

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \end{aligned} \tag{B.2}$$

While it may seem absurd to suddenly express the integral in terms of y, we are reminded that as we have an improper definite integral, the variable used is irrelevant. The resulting equation can be simplified to,

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\ &= \int_{-\infty}^{\infty}\int_{\infty}^{-\infty} e^{-x^2} e^{-y^2} dxdy \end{aligned} \tag{B.3}$$

The proof for this is as follows,

Proof.

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \int_{-\infty}^{\infty}\int_{\infty}^{-\infty} e^{-x^2} e^{-y^2} dxdy$$

In the RHS, we have a double integral containing an x and y term. If we consider the integral with respect to x, the integral with respect to y is simply a constant which can be pulled out of the integral. Thus,

$$I^2 = \int_{-\infty}^{\infty}\int_{\infty}^{-\infty} e^{-x^2}e^{-y^2}dxdy$$

$\blacksquare$

We can now simplify the double integral using the addition law of indices,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\infty}^{-\infty} e^{-x^2}e^{-y^2}dxdy &= \int_{-\infty}^{\infty}\int_{\infty}^{-\infty} e^{-x^2-y^2}dxdy \\ &= \int_{-\infty}^{\infty}\int_{\infty}^{-\infty} e^{-\left(x^2+y^2\right)}dxdy \qquad \text{(B.4)}\end{aligned}$$

We now notice from eq. (2.1) that we have an equation for the circle in the exponent. Thus, it may be useful to convert our coordinate system into one that utilises the properties of the circle in order to evaluate the integral more effectively. Thus, we must now use polar coordinates.

B.1 Interlude - Polar Coordinates

The polar coordinate system offers another way to represent a point in space.

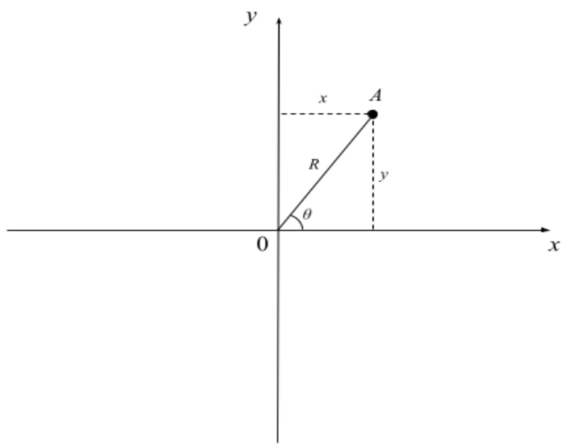

Figure 6: Arbitrary point on the Cartesian Coordinate System; drawn on Google Drawings

Consider an arbitrary point A with coordinates (x, y) on the Cartesian Coordinate System as shown by fig. 6. A point on the polar coordinate system is represented as (Strang):

$$(R, \theta) \tag{B.5}$$

where R is the length of the line from the origin to A while θ is the angle that A makes with the positive x-axis. We make the following conversions using basic right-angle trigonometry,

$$x = R\cos(\theta) \tag{B.6}$$

$$y = R\sin(\theta) \tag{B.7}$$

Using the Pythagoran theorem,

$$\begin{aligned} R &= \sqrt{x^2 + y^2} \\ &= \sqrt{[R\cos(\theta)]^2 + [R\sin(\theta)]^2} \\ &= \sqrt{R^2[\cos^2(\theta) + \sin^2(\theta)]} \\ &= \sqrt{R^2(1)} \\ &= R \end{aligned} \tag{B.8}$$

Once again utilising basic right angled trigonometry,

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{B.9}$$

The above method is the exact same way to convert a complex number from rectangular form to polar form—hence the name polar coordinates. Nevertheless, the use of polar coordinates in calculus shines with the concept of the area element. The area element is often used to convert a double integral, integrals used to interpret properties of functions of more than one variable ("What Is a", M408M), from cartesian coordinates into polar coordinates.

Consider an arbitrary function $f(x)$ in Cartesian coordinates as shown in fig. 7. The area under the curve could be approximated by splitting up the desired area into small rectangles, termed the area element, and adding up the area of each rectangle. If each rectangle has infinitesimally small width dx and infinitesimally small length dy, the area element in cartesian coordinates is simply the area of the rectangle.

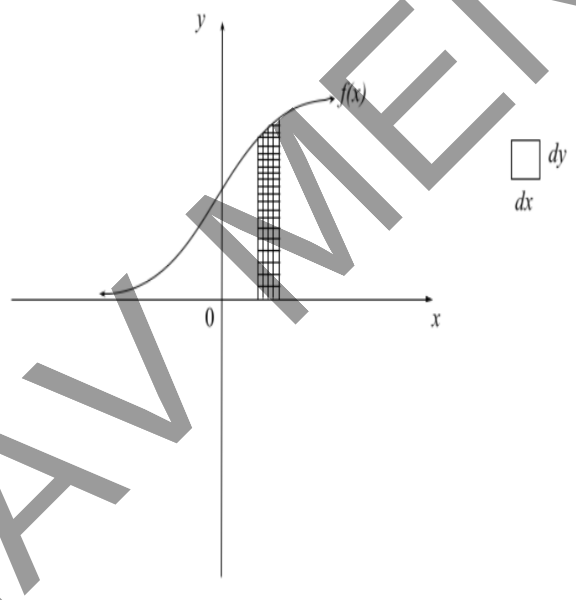

Figure 7: Arbitrary function along with enlarged area element; drawn on Google Drawings

Thus the area element dA is given by,

$$dA = dxdy \tag{B.10}$$

Now consider an arbitrary function in polar coordinates as shown below.

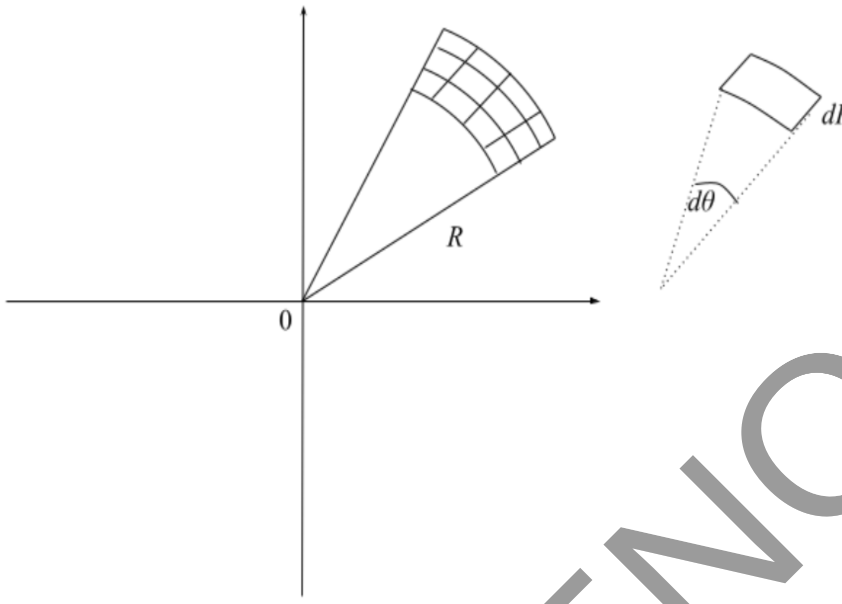

Figure 8: Arbitrary function along in polar coordinates with enlarged area element; drawn on Google Drawings

If we consider the change in angle of the area element to be infinitesimally small $d\theta$ and the side to be dR, we can approximate the area element to be rectangular in nature. Hence, the area would be the product of dR and the arclength of the sector that bounds $d\theta$, as shown in fig. 8. This arclength is simply the arclength of a sector with radius R and angle $d\theta$. Hence the area element in polar coordinates is ,

$$dA = RdRd\theta \tag{B.11}$$

B.2 Utilising Polar Coordinates to Evaluate the Gaussian Integral

Substituting eq. (2.1) into eq. (B.4) we have

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-R^2} dxdy \tag{B.12}$$

We can now convert our cartesian area element into our polar area element. The upper and lower bounds of the integral must change appropriately. The integral with respect to R will now have bounds 0 and ∞ as negative length cannot exist, thus negating the $-\infty$ bound. The integral with respect to θ will now have bounds 0 and 2π as that corresponds to one full rotation in polar coordinates.

Thus,

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-R^2} R\, dRd\theta \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\infty} Re^{-R^2} dR\right) \\ &= [\theta]_0^{2\pi}\left(\int_0^{\infty} Re^{-R^2} dR\right) \\ &= 2\pi \int_0^{\infty} Re^{-R^2} dR \end{aligned} \tag{B.13}$$

Evaluating the integral via substitution. Let $u = R^2 \implies du = 2RdR \implies dR = \frac{du}{2R}$. The bounds remain unchanged after the substitution as $0^2 = 0$ and $\infty^2 = \infty$. Thus,

$$\begin{aligned} I^2 &= 2\pi \int_0^{\infty} Re^{-u}\frac{du}{2R} \\ &= \pi \int_0^{\infty} e^{-u} du \\ &= [-\pi e^{-u}]_0^{\infty} \\ &= \lim_{u\to\infty}\left(-\pi e^{-u}\right) - \left(-\pi e^{0}\right) \\ &= 0 + \pi \quad \left(\text{From the fact that } \lim_{x\to\infty} e^{-x} = 0\right) \\ I^2 &= \pi \end{aligned} \tag{B.14}$$

Thus, the gaussian integral I is given by,

$$I = \sqrt{\pi} \tag{B.15}$$