Signal Processing III (Discrete Fourier Transform)

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Fourier Integral Pair (Fourier Transform)

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

The Fourier transform (FT) decomposes a function of continuous time (a signal) into its constituent frequencies. Inverse Fourier transform is to reconstruct the signals using frequencies and its amplitude.

Properties of Fourier Transforms

1. Time scaling

$$F\big(x(at)\big) = \frac{1}{|a|}X(\frac{f}{a})$$

Q: What does this mean by?

2. Time reversal

$$F\big(x(-t)\big) = X(-f)$$

3. Differentiation

$$F\big(\dot{x}(t)\big) = i2\pi f X(f)$$

4. Time shifting

$$F\big(x(t-t_0)\big) = e^{-i2\pi f t_0}X(f)$$

Only phase shift !

Sine wave

5. Modulation

$$F\big(x(t)e^{i2\pi f_0 t}\big) = X(f-f_0)$$

$$F(x(t)\cos(2\pi f_0 t)\,) = \frac{1}{2}[X(f-f_0)\ + X(f+f_0)\,]$$

Properties of Fourier Transforms (Convolution)

$$F(h(t) * x(t)) = H(f)X(f)$$

$$F(x(t)w(t)) = X(f) * W(f)$$

Where the convolution of the two functions $h(t)$ and $x(t)$ is defined as

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

It is a function of time !

Proof of Convolution Relationship

Supplement

$$\boxed{F(h(t) * x(t)) = H(f)X(f)}$$

Proof: $$F\big(h(t) * x(t)\big) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau\right]\exp^{-i2\pi f t}dt$$

Fubini's Theorem

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(v)\exp^{-i2\pi f(\tau+v)}\,d\tau\,dv \qquad v = t-\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\exp^{-i2\pi f(\tau)}\,d\tau \int_{-\infty}^{\infty} x(v)\exp^{-i2\pi f(v)}\,dv = H(f)X(f)$$

$$\boxed{F(x(t)) = X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi f t}dt}$$

Fourier integral

$$\boxed{h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

Conovlution

Proof of Convolution Relationship (Continue)

Supplement

$$F(x(t)w(t)) = X(f) * W(f)$$

Proof: $$F(x(t)w(t)) = \int_{-\infty}^{\infty} x(t)\, w(t) \exp^{-i2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f_1) \exp^{i2\pi f_1 t}\, W(f_2) \exp^{i2\pi f_2 t} \exp^{-i2\pi f t}\, df_1\, df_2\, dt$$

$$= \int_{-\infty}^{\infty} X(f_1) \int_{-\infty}^{\infty} W(f_2) \int_{-\infty}^{\infty} \exp^{-i2\pi(f - f_1 - f_2)t}\, dt\, df_2\, df_1$$

$$= \int_{-\infty}^{\infty} X(f_1) \int_{-\infty}^{\infty} W(f_2)\delta(f - f_1 - f_2) df_2 df_1 = \int_{-\infty}^{\infty} X(f_1) W(f - f_1) df_1 = X(f) * W(f)$$

$$F(x(t)) = X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$$

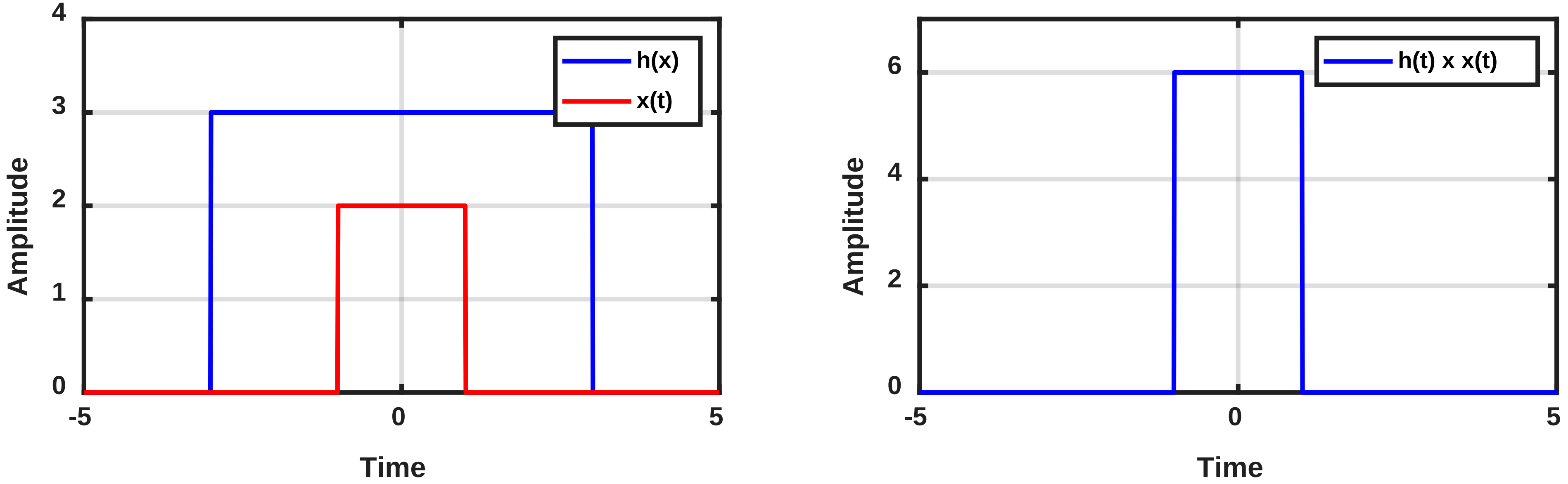
Example: Convolution 1 (Square Wave)

$h(t) = 3 \quad if \; -3 < t < 3$. Otherwise, $h(t) = 0$

$x(t) = 2 \quad if \; -1 < t < 1$. Otherwise, $x(t) = 0$

$$h(t)x(t)$$

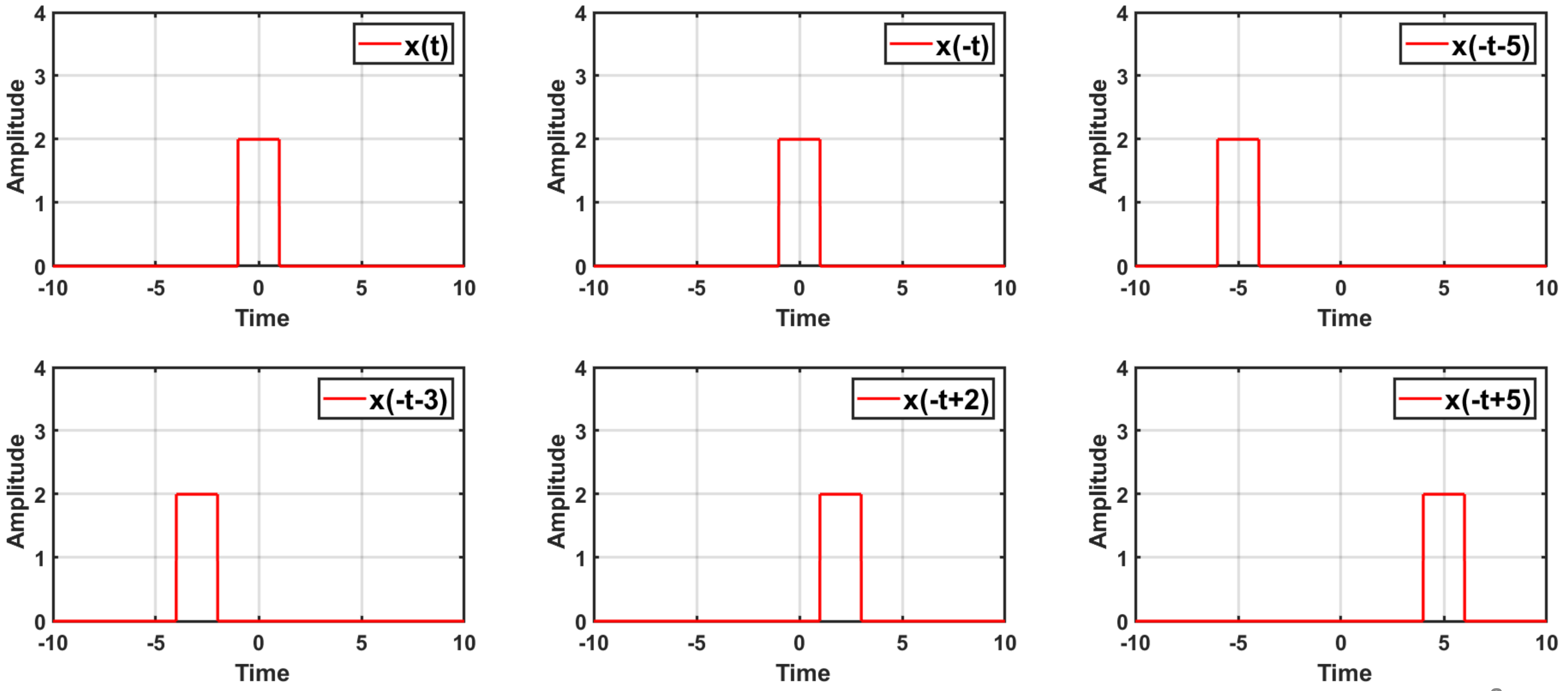
Multiply values of *h* and *x* at each time point



Tutorial

Time Shifting

Supplement



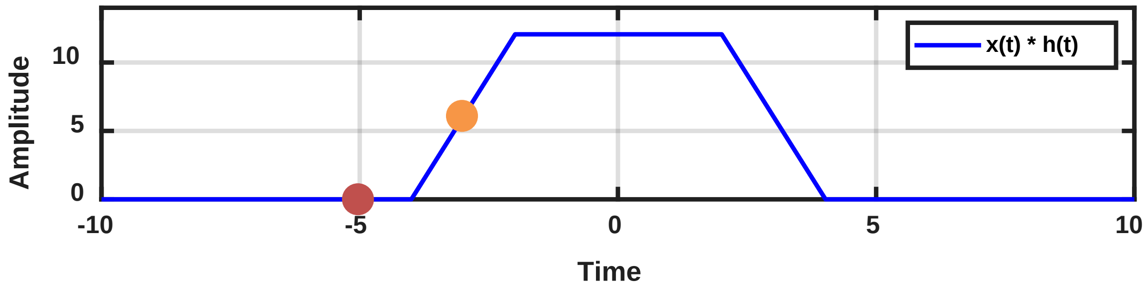
Tutorial

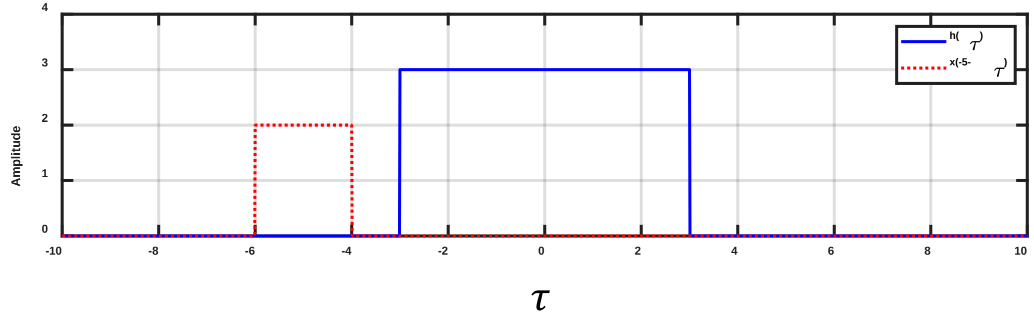
Example: Convolution 1 (Square Wave) (Continue)

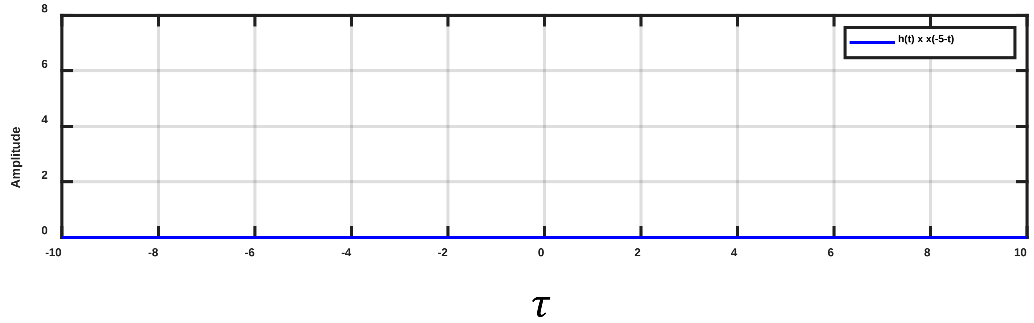
Tutorial

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Convolution

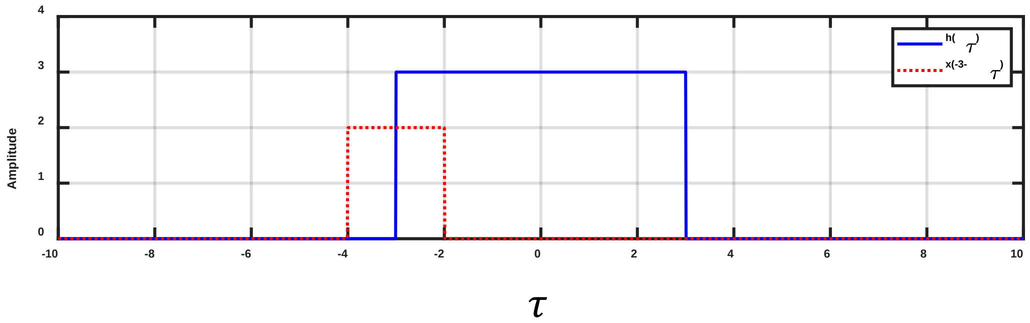


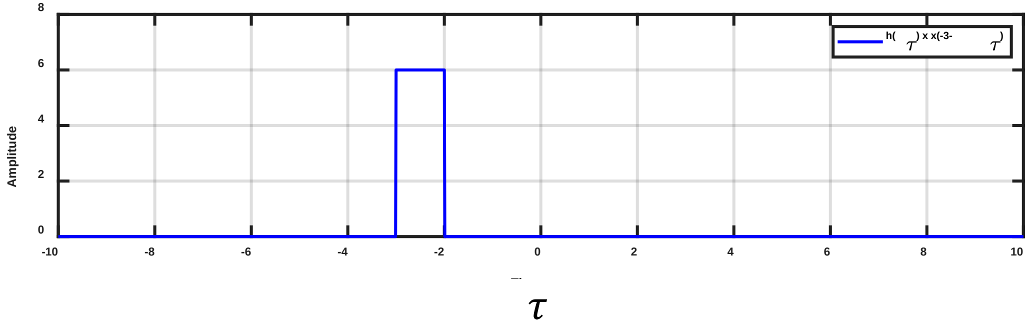




$$\int_{-\infty}^{\infty} h(\tau)x(-5-\tau)\,d\tau = 0$$

It is for finding a value of x(t)*h(t) at t= -5





$$\int_{-\infty}^{\infty} h(\tau)x(-3-\tau)\,d\tau = 6$$

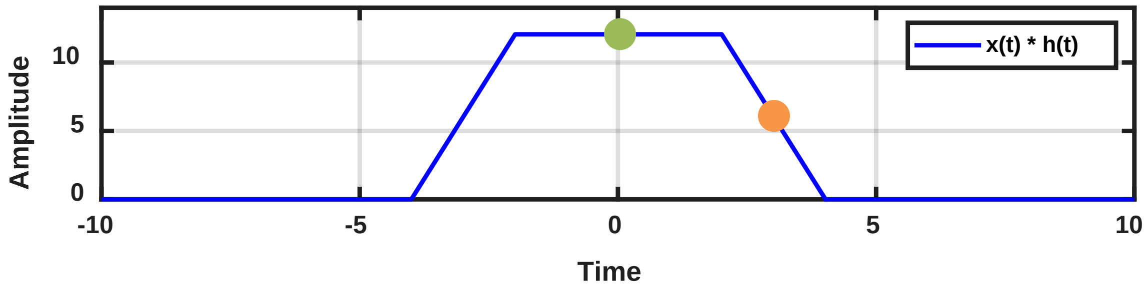
It is for finding a value of x(t)*h(t) at t= -3

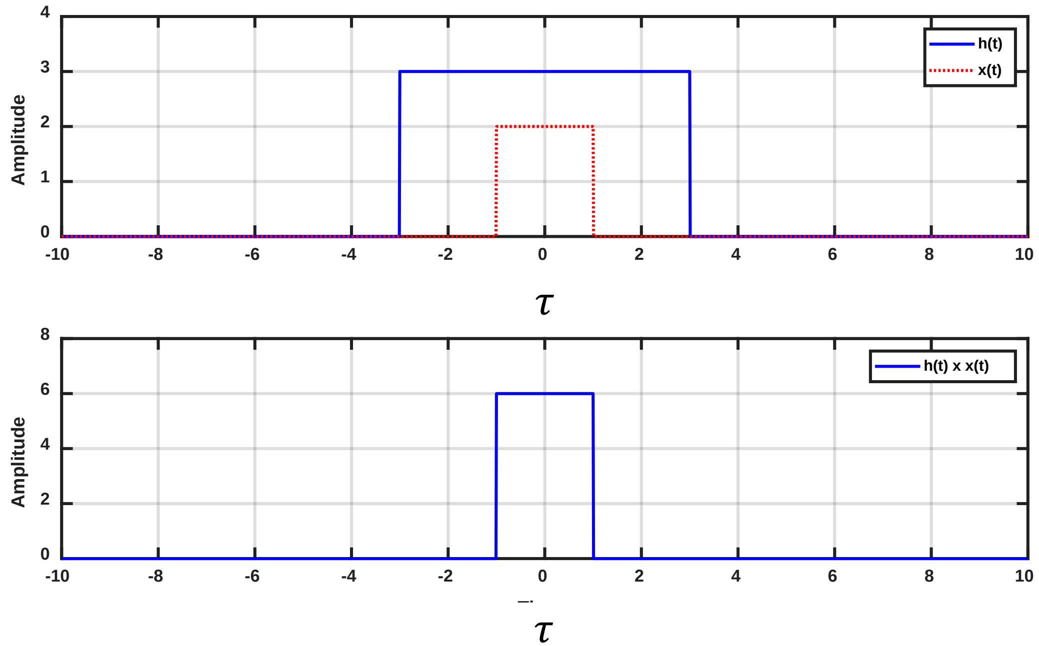
Example: Convolution 1 (Square Wave) (Continue)

Tutorial

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

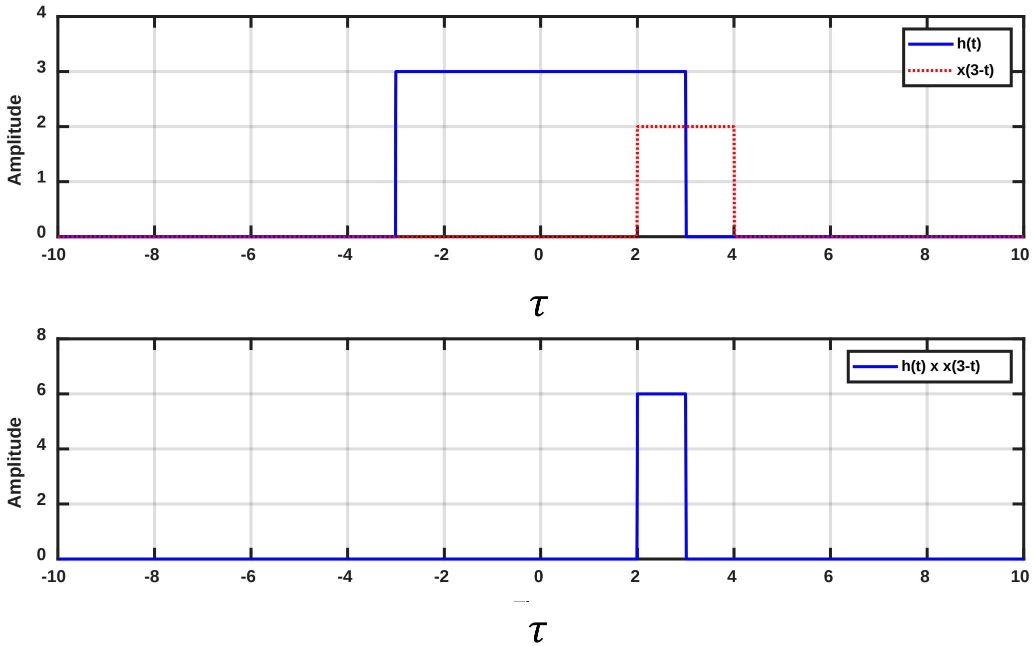
Convolution





$$\int_{-\infty}^{\infty} h(\tau)x(\tau)\, d\tau = 12$$

It is for finding a value of x(t)*h(t) at t= 0



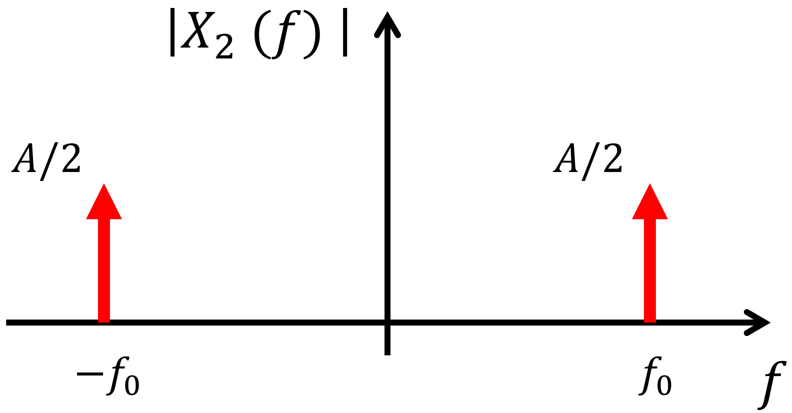
$$\int_{-\infty}^{\infty} h(\tau)x(3-\tau)\, d\tau = 6$$

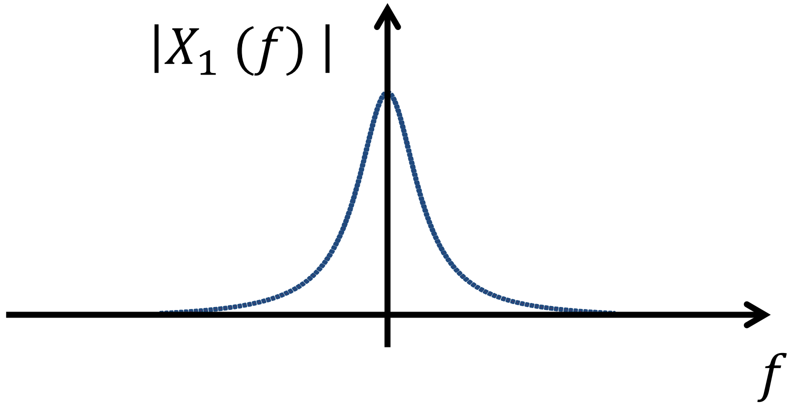
It is for finding a value of x(t)*h(t) at t= 3

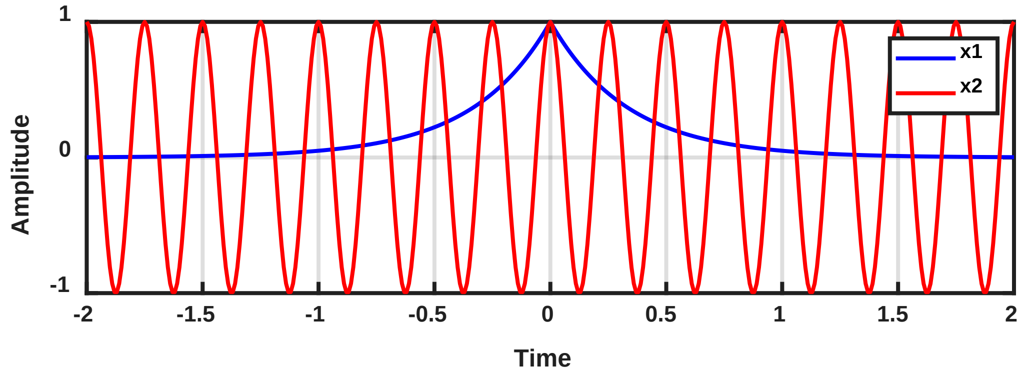
Example: Convolution 2 (Damped Symmetrically Oscillating Function)

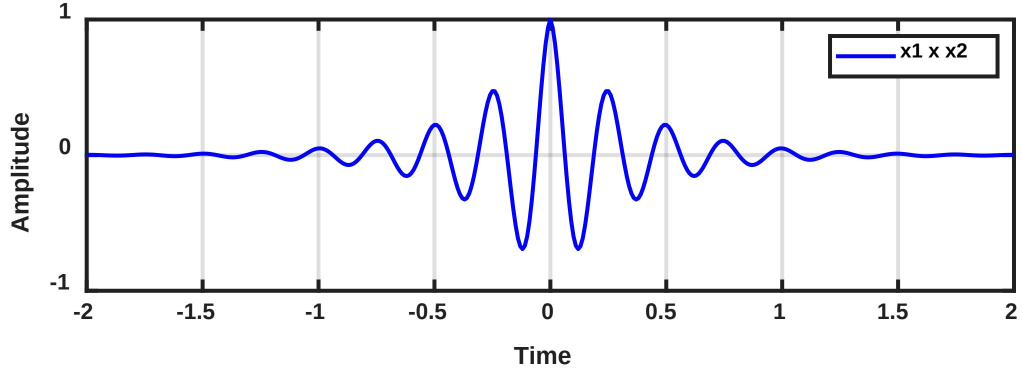
$$x_1(t) = e^{-a|t|},\ x_2(t) = cos2\pi f_0 t$$

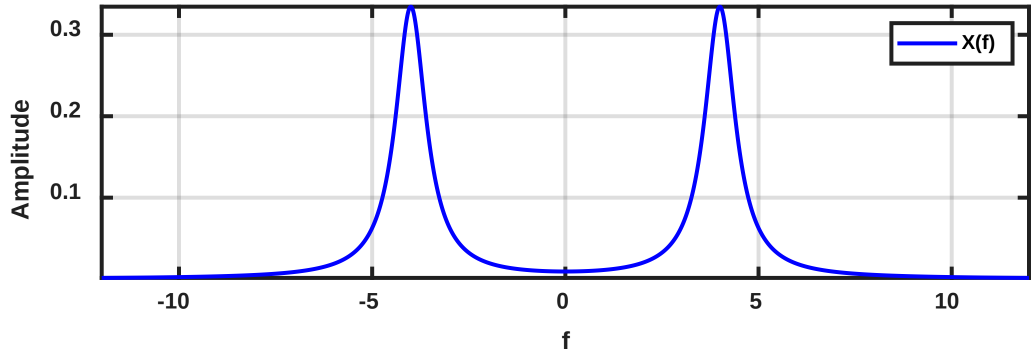
$$x(t) = x_1(t)x_2(t) = e^{-a|t|}cos2\pi f_0 t$$



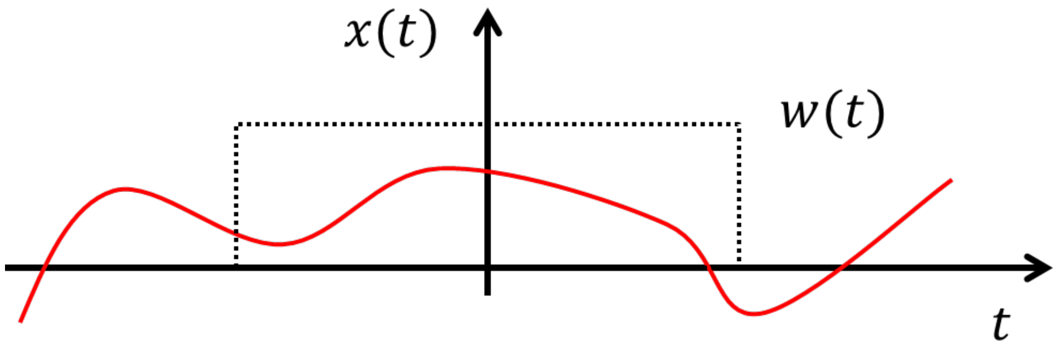








Effect of Data Truncation (Windowing) – Finite Signal



$w(t) = 1$ for $|t|$ <T/2
$= 0$ otherwise

Q: How to represent a finite signal?

$x(t)$ is known (or recorded) only for $-\frac{T}{2} < t < \frac{T}{2}$, denoted as $x_T(t)$

$$F(x_T(t)) = F(x(t)w(t)) = X(f) * W(f)$$

Fourier transform of the product of two-time signals is the convolution of their Fourier transforms.

Revisit: Continuous Signal

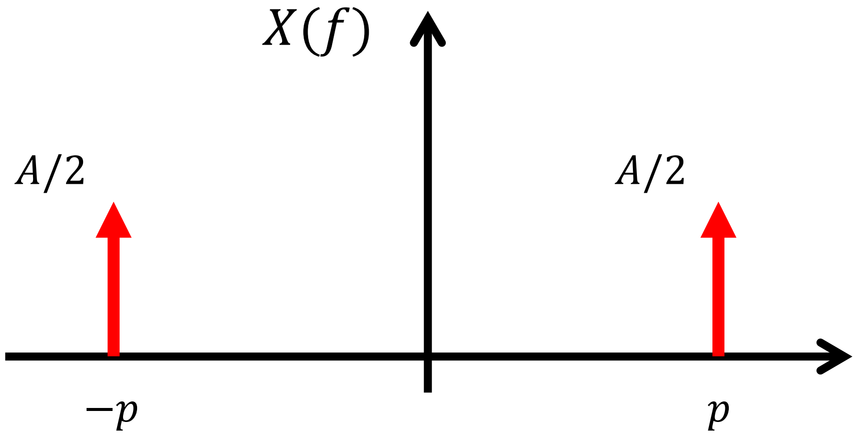
Supplement

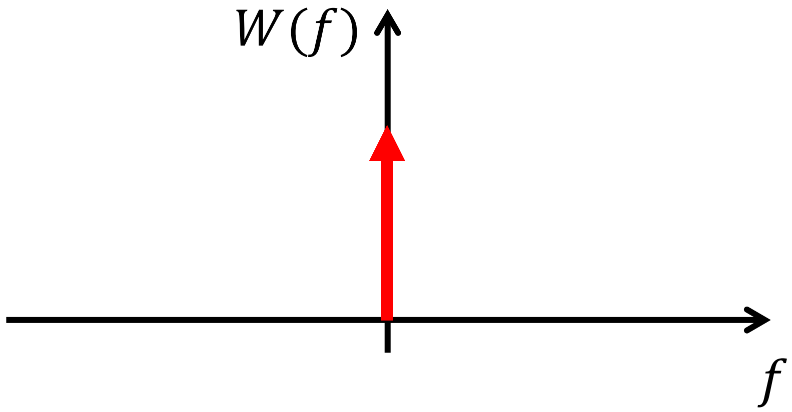
$$x(t) = A\cos 2\pi pt$$

$$w(t) = 1$$

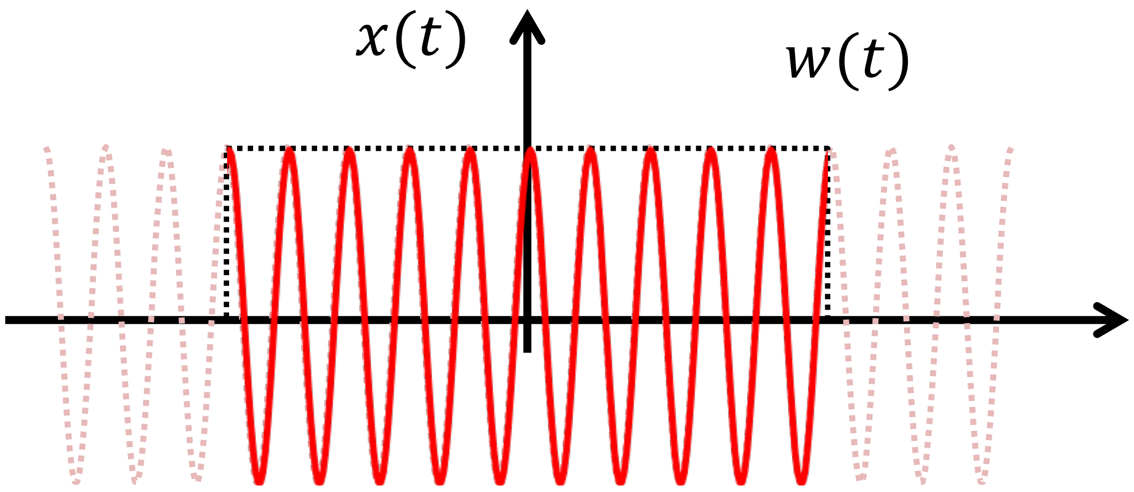
$$X(f) = \frac{A}{2}[\delta(f-p) - \delta(f+p)]$$

$$W(f) = \delta(f)$$

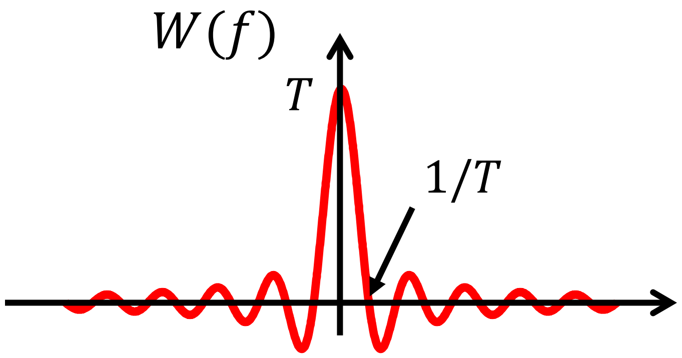




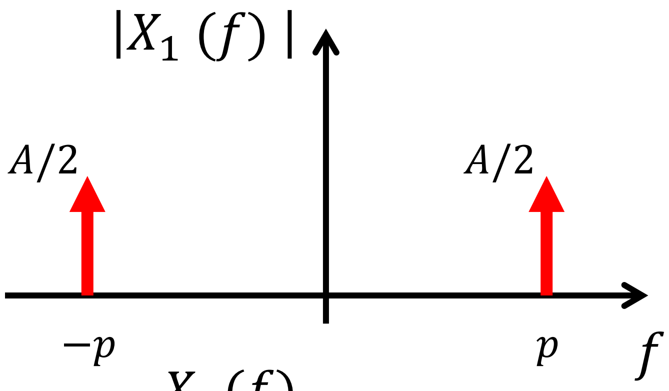
Truncated Sine Wave



$$W(f) = \frac{T \sin \pi f T}{\pi f T}$$



$$X(f) = \frac{A}{2}[\delta(f-p) - \delta(f+p)]$$



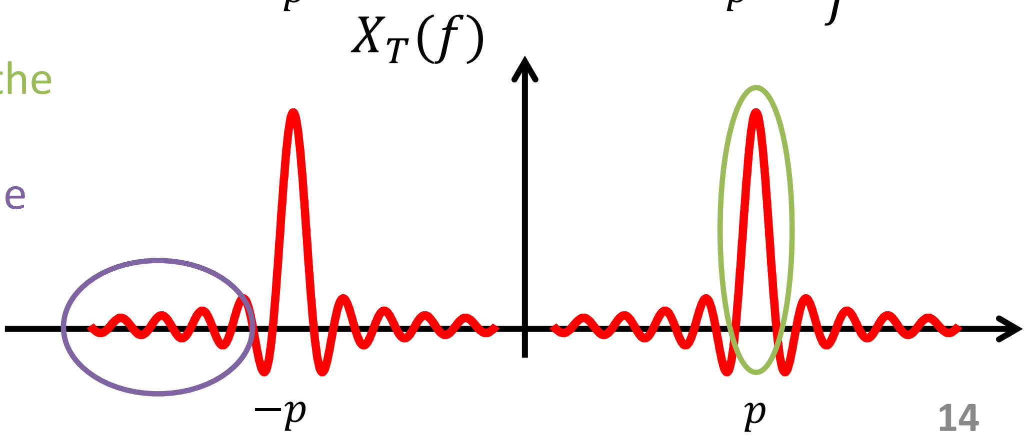
$$x(t) = A \cos 2\pi p t$$

$$\begin{aligned} w(t) &= a \ \text{ for } \ |t| < \mathrm{b} \\ &= 0 \ \ \text{ for } |t| > \mathrm{b} \end{aligned}$$

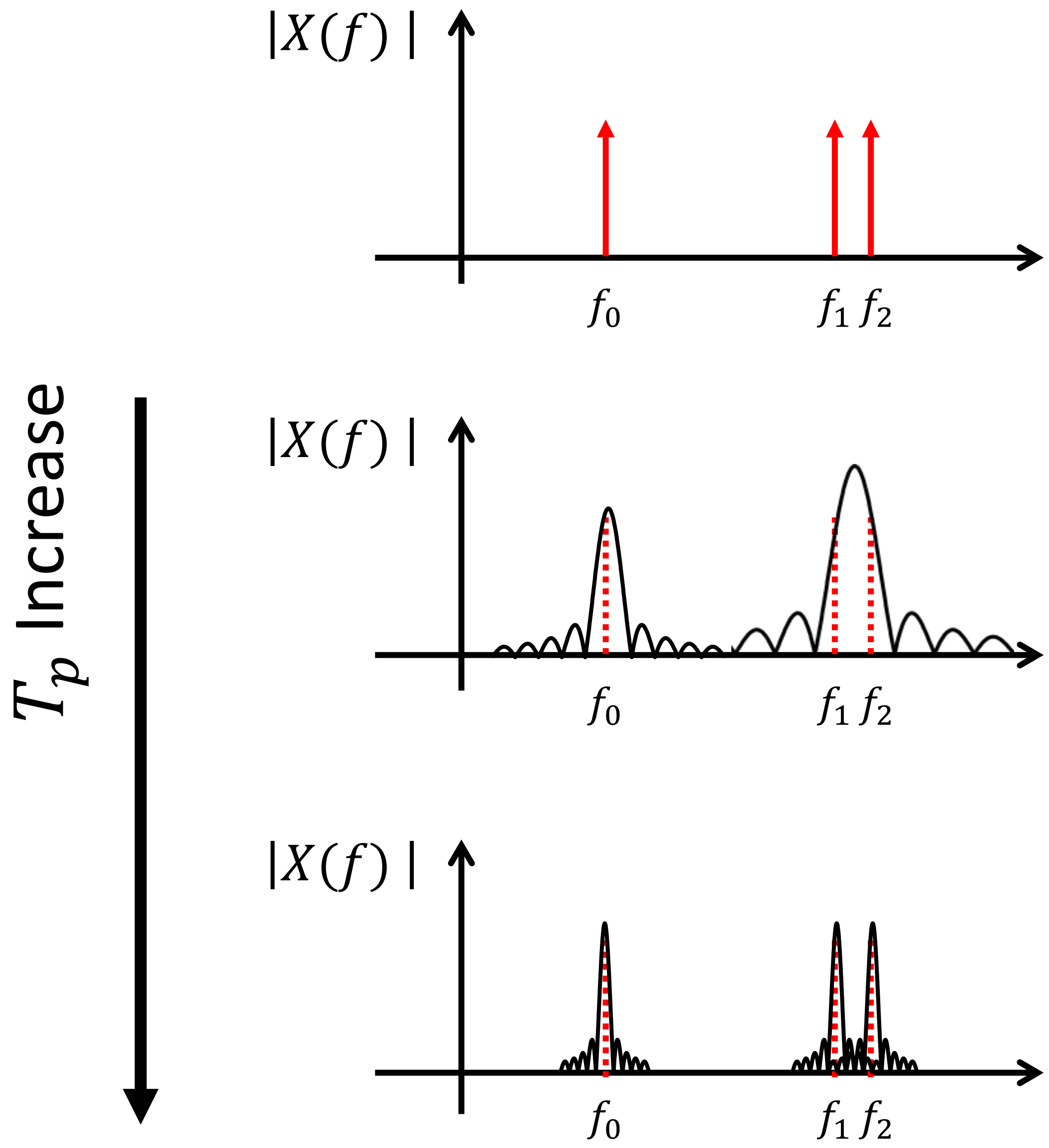
$$x_T(t) = x(t)w(t)$$

Smearing: Distortion due to the main lobe

Leakage: Distortion due to the side lobe



Effect of Data Truncation: Frequency Separation



$x(t)$ is the sum of three sine (or cosine) waves

Considerable smearing due to the spectral window

Three components are resolved but with considerable leakage at other frequencies

Window Function

Supplement

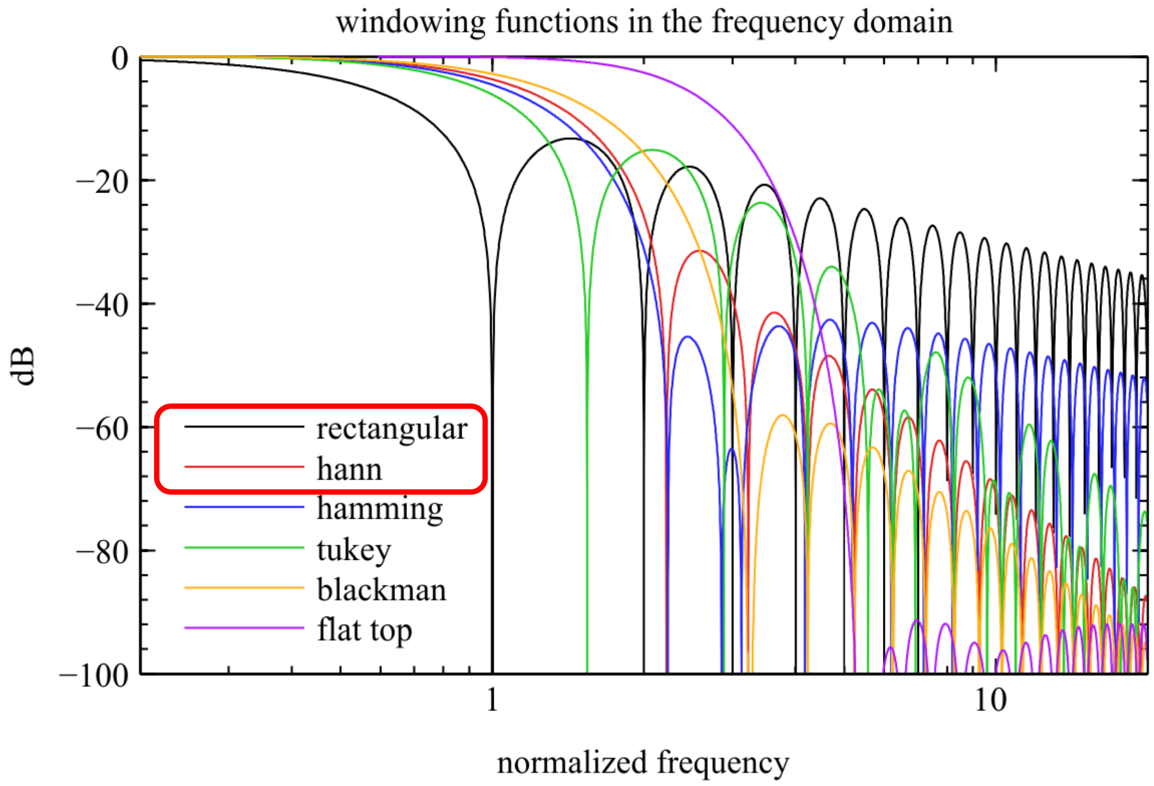
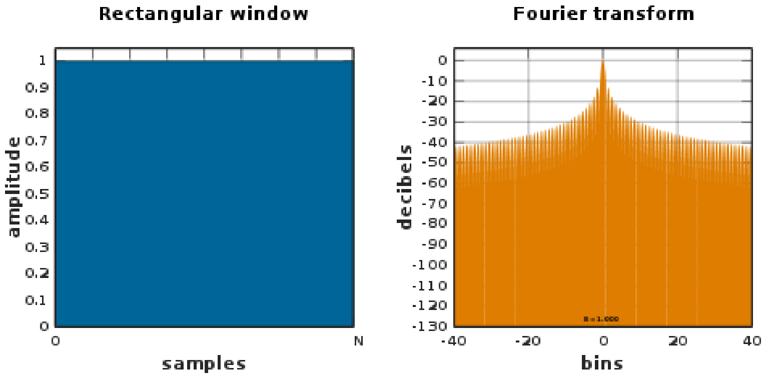
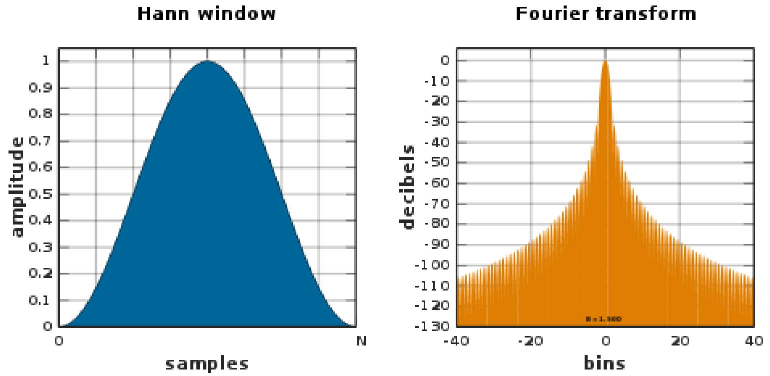
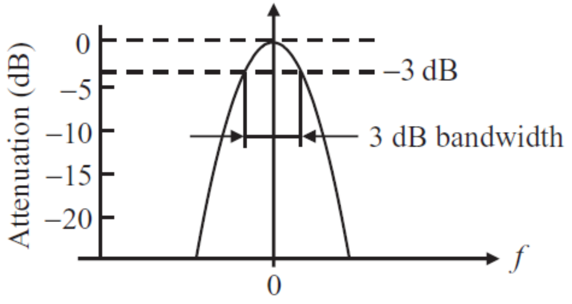


Table 4.2 Properties of some window functions

Window (length T)	Highest side lobe (dB)	Asymptotic roll-off (dB/octave)	3 dB bandwidth	Noise bandwidth	First zero crossing (freq.)
Rectangular	−13.3	6	$0.89\frac{1}{T}$	$1.00\frac{1}{T}$	$\frac{1}{T}$
Bartlett (triangle)	−26.5	12	$1.28\frac{1}{T}$	$1.33\frac{1}{T}$	$\frac{2}{T}$
Hann(ing) (Tukey or cosine squared)	−31.5	18	$1.44\frac{1}{T}$	$1.50\frac{1}{T}$	$\frac{2}{T}$
Hamming	−43	6	$1.30\frac{1}{T}$	$1.36\frac{1}{T}$	$\frac{2}{T}$
Parzen	−53	24	$1.82\frac{1}{T}$	$1.92\frac{1}{T}$	$\frac{4}{T}$



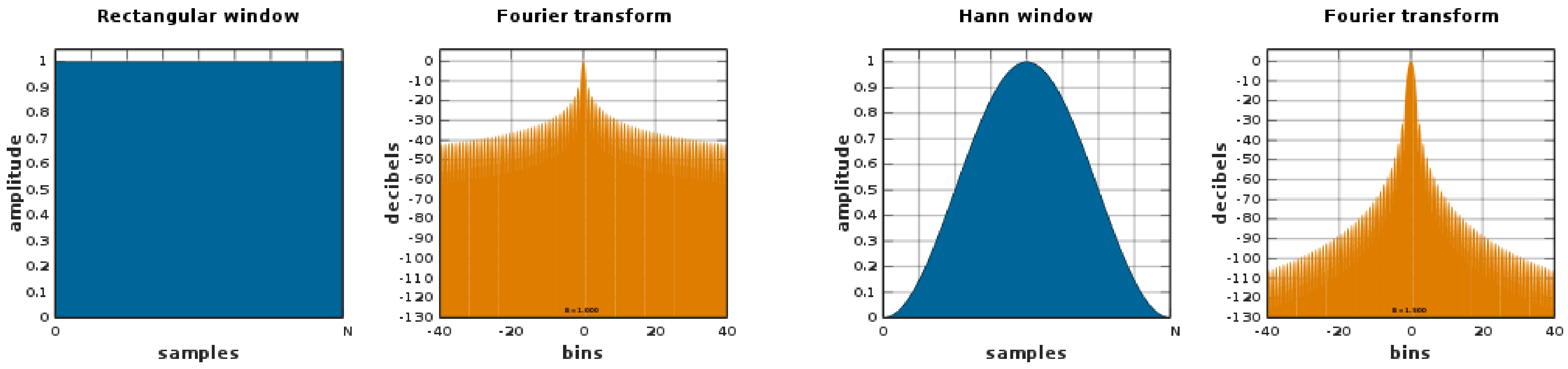




Example: Hanning Window vs Rectangular Window

$$x(t) = A_1 \sin 2\pi f_1 t + A_2 sin2\pi f_2 t + A_3 sin2\pi f_3 t$$

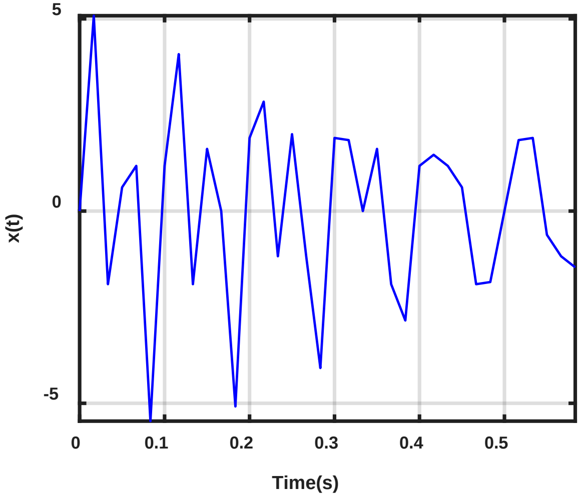
Amplitudes are $A_1 = A_2 = A_3 = 2$, which gives the magnitude '1' for each sinusoidal component in the frequency domain. The frequencies are chosen as $f_1 = 10$, $f_2 = 20$ and $f_3 = 21$.

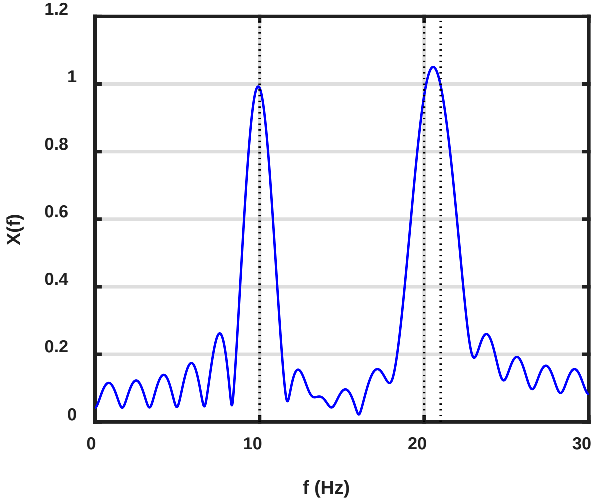


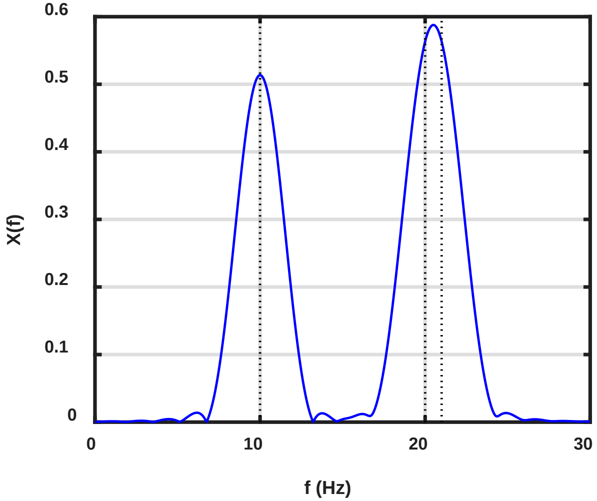
Example: Hanning Window vs Rectangular Window (Continue)

Original signal | **Rectangular** | **Hanning**

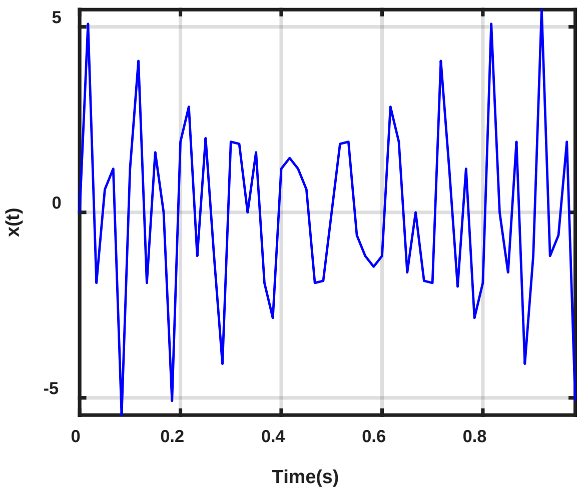
$T = 0.6s$

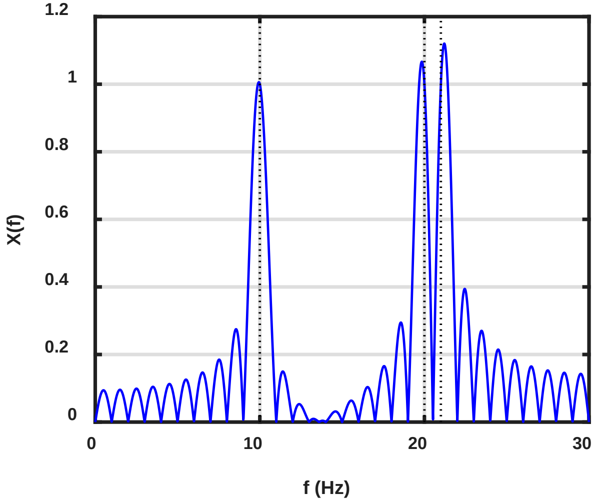


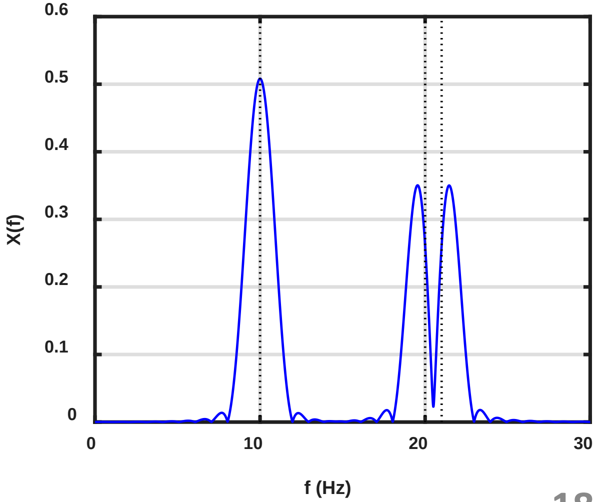




$T = 1s$







Black dotted lines indicate 10, 20, and 21 Hz

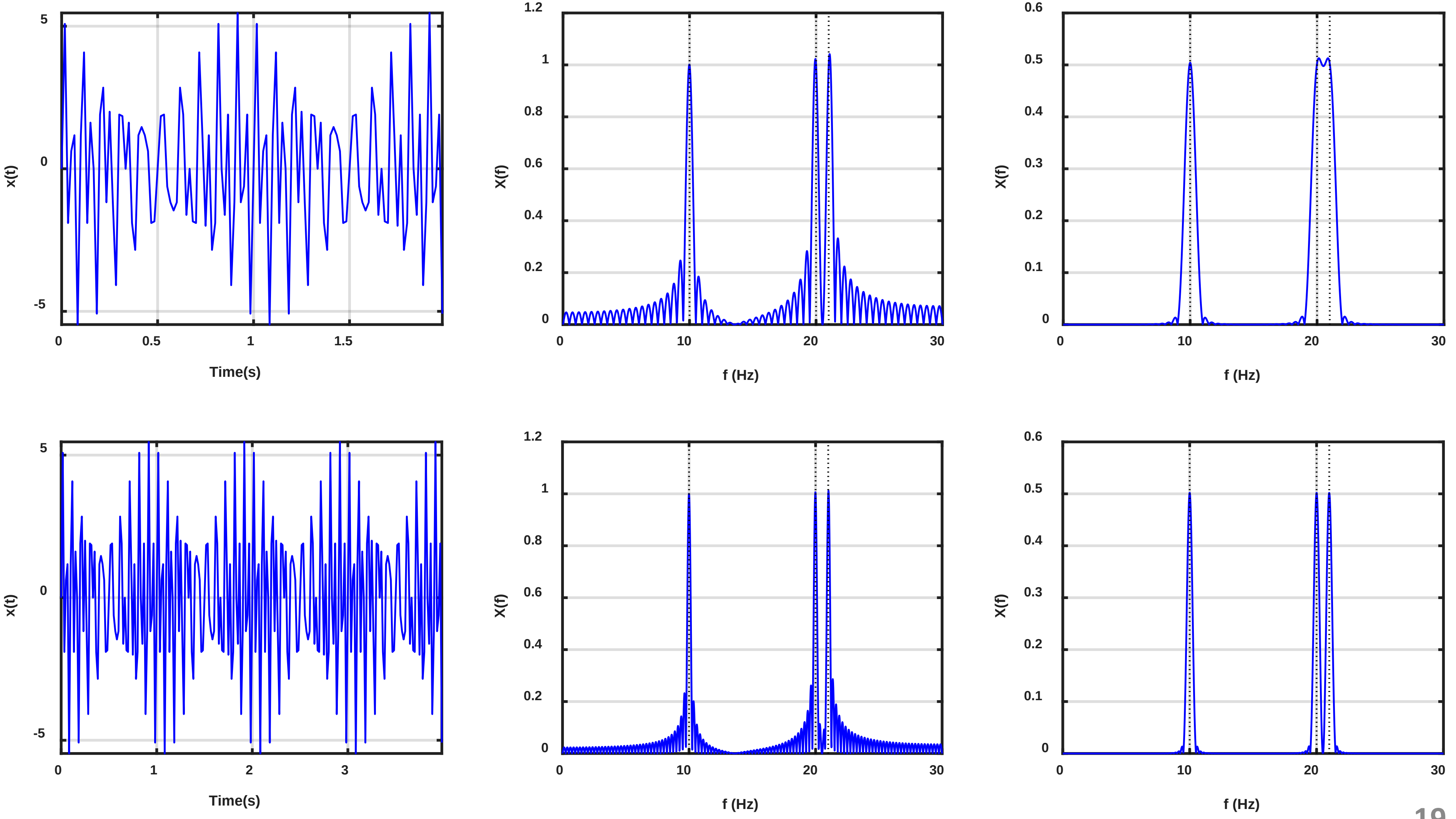
Tutorial

Example: Hanning Window vs Rectangular Window (Continue)

Original signal | Rectangular | Hanning

$T = 2s$

$T = 4s$



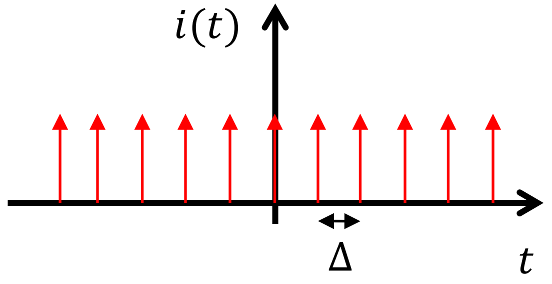
Black dotted lines indicate 10, 20, and 21 Hz

Tutorial

Impluse Train Modulation (Discretization)

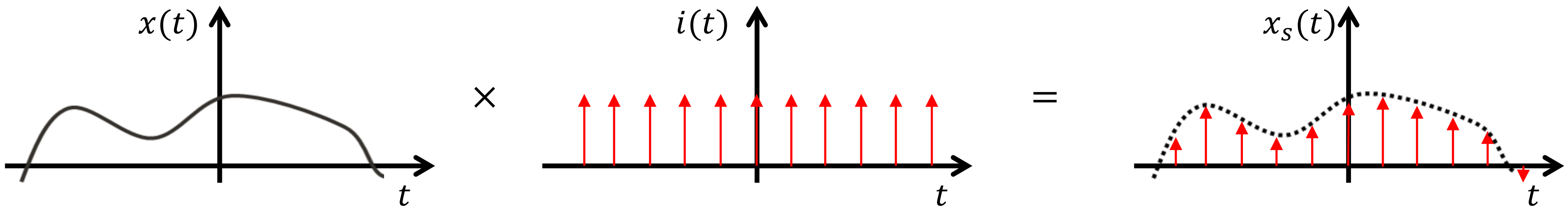
A 'train' of delta functions $i(t)$ is expressed as

$$i(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$$



$$f_s = \frac{1}{\Delta}$$

The sampling procedure can be illustrated as $x_s(t) = x(t)i(t) = x(n\Delta)$



$$\boxed{X_s(f)} = \int_{-\infty}^{\infty} \left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta) \right] e^{-i2\pi f t} dt = \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} \delta(t - n\Delta) dt \right] = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-i2\pi f n\Delta}$$

Q: What is the difference between $X_s(f)$ and $X(f)$?

Impluse Train Modulation (Continue)

$$X_s(f + r/\Delta) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi(f+r/\Delta)n\Delta} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta-i2\pi rn} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta} = X_s(f)$$

r: integer

Q: Meaning?

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}$$

Multiplying both sides of equation by $e^{-i2\pi fr\Delta}$ and integrating w.r.t f

$$\int_{-1/2\Delta}^{1/2\Delta} X_s(f)e^{-i2\pi fr\Delta}df = \int_{-1/2\Delta}^{1/2\Delta}\left[\sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}\right]e^{-i2\pi fr\Delta}df = \sum_{n=-\infty}^{\infty}\int_{-1/2\Delta}^{1/2\Delta}\left[x(n\Delta)e^{-i2\pi fn\Delta}\right]e^{-i2\pi fr\Delta}df$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta)\int_{-1/2\Delta}^{1/2\Delta}\left[e^{-i2\pi f(n-r)\Delta}\right]df = x(r\Delta)\frac{1}{\Delta}$$

Use L'Hôpital's rule

$$f_S = \frac{1}{\Delta}$$

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta} \qquad x(n\Delta) = \Delta\int_{-1/2\Delta}^{1/2\Delta} X_s(f)e^{i2\pi fn\Delta}df$$

Q: What are these equation?

Supplement

$$\sum_{n=-\infty}^{\infty} x(n\Delta)\int_{-1/2\Delta}^{1/2\Delta}\left[e^{-i2\pi f(n-r)\Delta}\right]df = \sum_{n=-\infty}^{\infty} x(n\Delta)\left[\frac{1}{-i2\pi(n-r)\Delta}e^{-i2\pi f(n-r)\Delta}\right]_{-1/2\Delta}^{1/2\Delta}$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta)\frac{1}{-i2\pi(n-r)\Delta}\left[e^{-\pi(n-r)} - e^{\pi(n-r)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta)\frac{1}{\pi(n-r)\Delta}\left[e^{-\pi(n-r)} - e^{\pi(n-r)}\right]\Big/2i$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta)\frac{1}{\pi(n-r)\Delta}\sin(\pi(n-r))$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta)\frac{1}{\Delta}\cdot\frac{\sin(\pi(n-r))}{\pi(n-r)}$$

$$= x(r\Delta)\frac{1}{\Delta}$$

- $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

Euler's formula.

- $\lim_{x\to 0}\frac{\sin x}{x} = 1$

L'Hôpital's rule

- $\frac{\sin(\pi(n-r))}{\pi(n-r)} = 1$ when $n = r$

  " $= 0$ otherwise.

Link Between Fourier Transform of a Discrete Sequence and Continuous Signal

Fourier coefficients

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t) e^{-i2\pi nt/T_p} dt$$

Impluse train

$$i(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\Delta) \qquad c_n = \frac{1}{\Delta}\int_{-1/\Delta}^{1/\Delta} \sum_{m=-\infty}^{\infty} \delta(t - m\Delta)\, e^{-i2\pi nt/\Delta} dt == \frac{1}{\Delta}\int_{-1/\Delta}^{1/\Delta} \delta(t) e^{-i2\pi nt/\Delta} dt = \frac{1}{\Delta}$$

Fourier Transform of the impulse train

$$I(f) = F\big(i(t)\big) = \int_{-\infty}^{\infty} i(t) e^{-i2\pi ft} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T_p}} \cdot e^{-i2\pi ft} dt$$

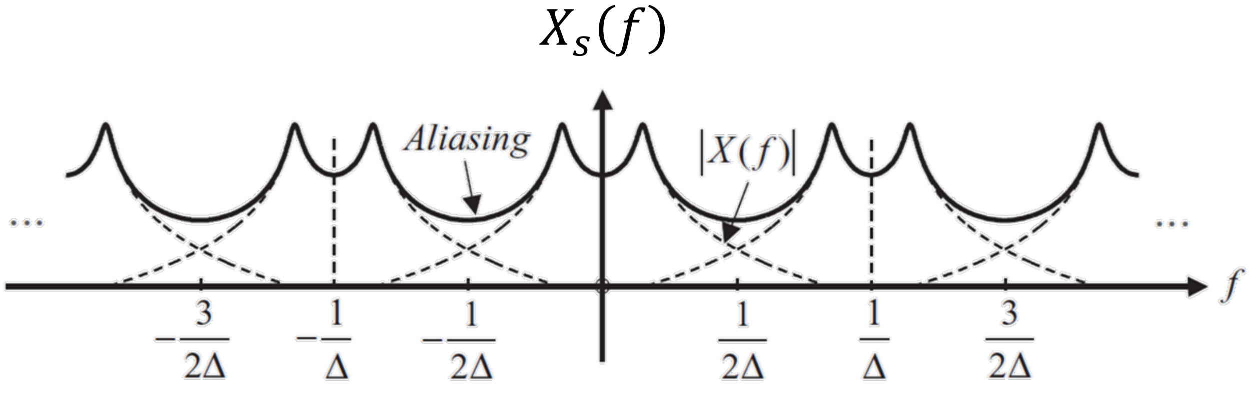
$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta} e^{\frac{i2\pi nt}{\Delta}} \cdot e^{-i2\pi ft} dt = \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{i2\pi nt}{\Delta}} \cdot e^{-i2\pi ft}\, dt = \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i2\pi\left(f - \frac{n}{\Delta}\right)t} dt = \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{\Delta})$$

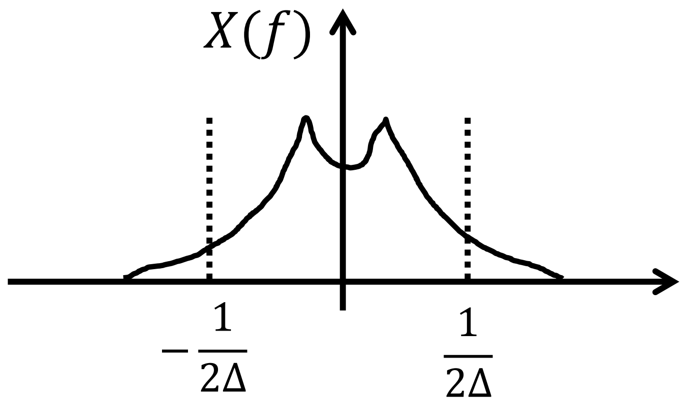
Link Between Fourier Transform of a Discrete Sequence and Continuous Signal (Continue)

Fourier Transform of a discrete sequence, $x_s(t) = x(t)i(t)$

$$X_s(f) = I(f) * X(f) = \int_{-\infty}^{\infty} I(g)X(f-g)dg = \int_{-\infty}^{\infty} \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg$$

$$= \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{\Delta}\right) = \frac{1}{\Delta}\left(\dots + X\left(f-\frac{2}{\Delta}\right) + X\left(f-\frac{1}{\Delta}\right) + X(f) + X\left(f+\frac{1}{\Delta}\right) + \dots\right)$$



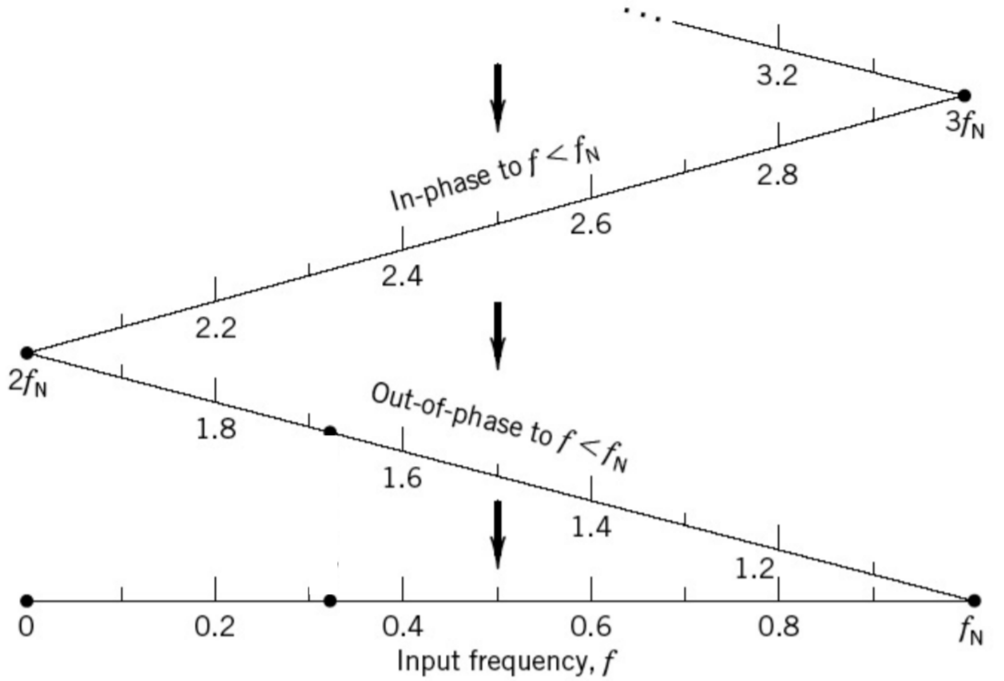


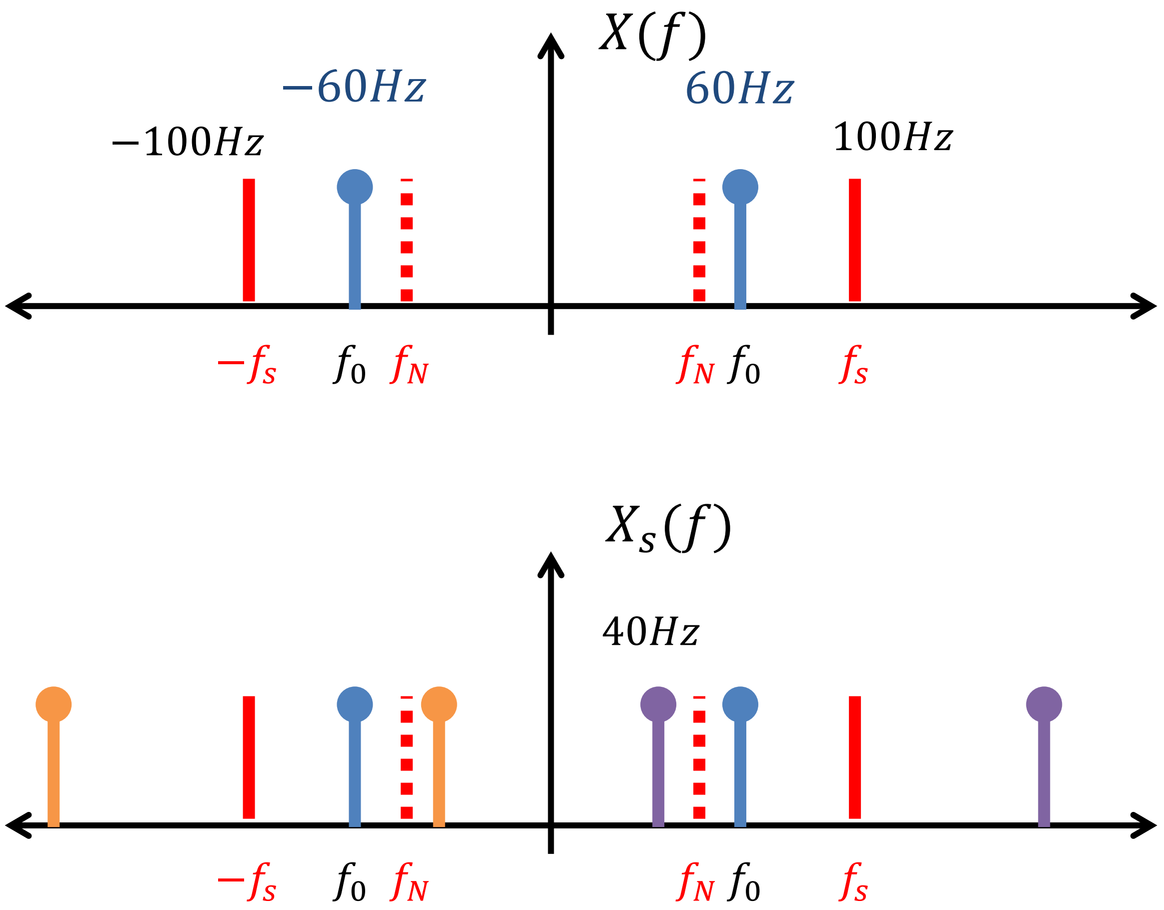
Q: What happened if Δ goes to a zero?

Example: Wagon-Wheel Effect

$$X_s(f) = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{\Delta}\right) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

$$\frac{1}{\Delta}\left(\dots + X\left(f - \frac{2}{\Delta}\right) + X\left(f - \frac{1}{\Delta}\right) + X(f) + X\left(f + \frac{1}{\Delta}\right) + \dots\right)$$



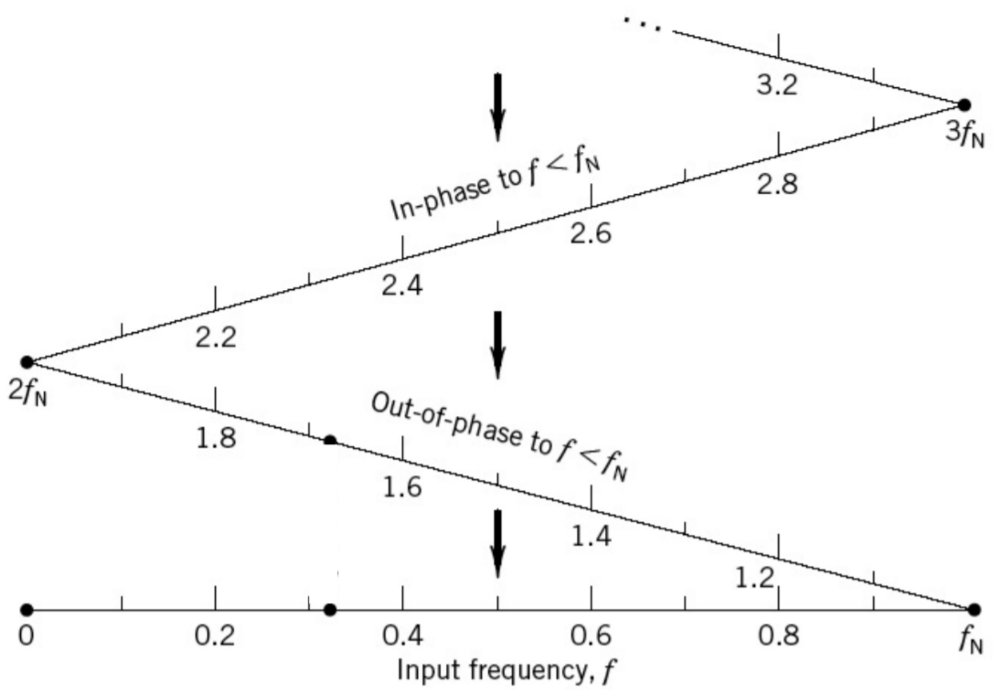


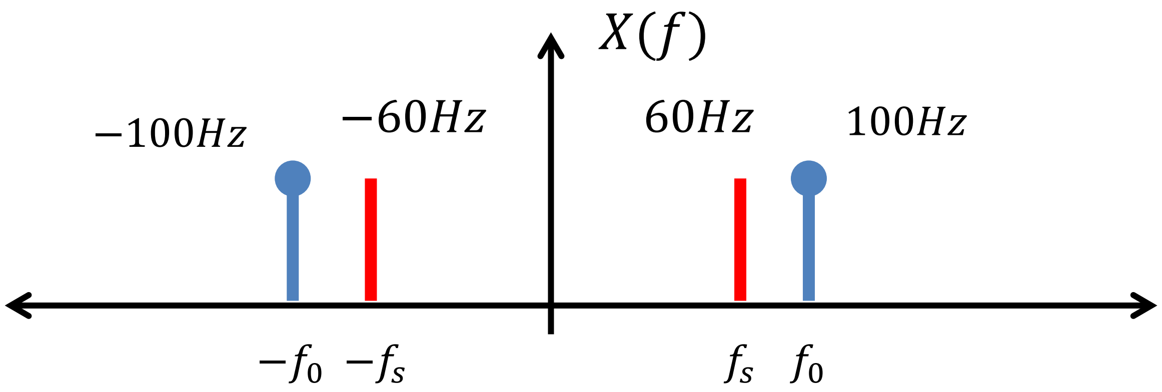
$$X_s(f) = X(f) + X(f - f_s) + X(f + f_s) + \dots$$

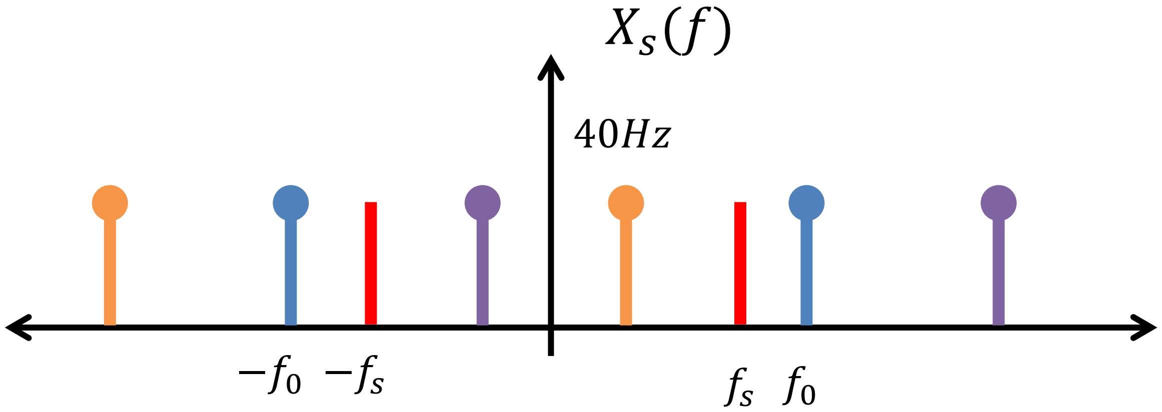
Example: Wagon-Wheel Effect 2

$$X_s(f) = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{\Delta}\right) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

$$\frac{1}{\Delta}\left(\ldots + X\left(f - \frac{2}{\Delta}\right) + X\left(f - \frac{1}{\Delta}\right) + X(f) + X\left(f + \frac{1}{\Delta}\right) + \ \ldots\right)$$



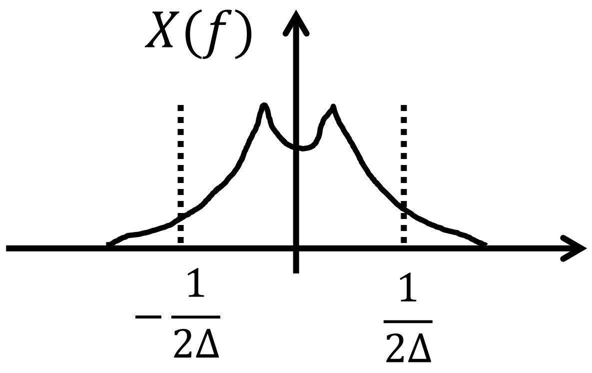


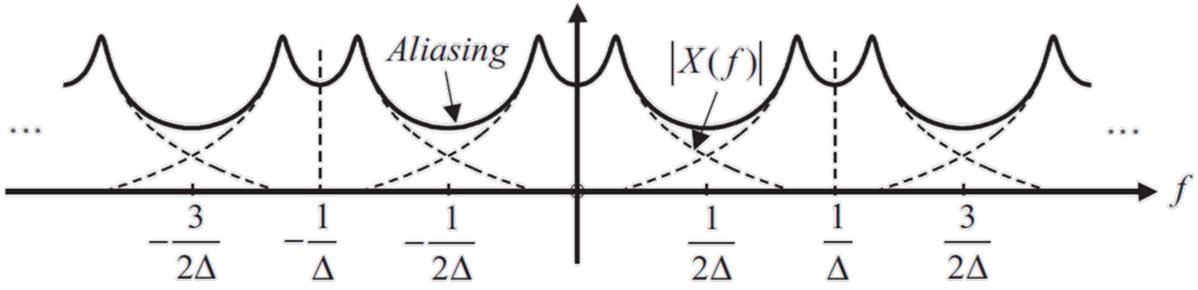


$$X_s(f) = X(f) + X(f - f_s) + X(f + f_s) + \ \ldots$$

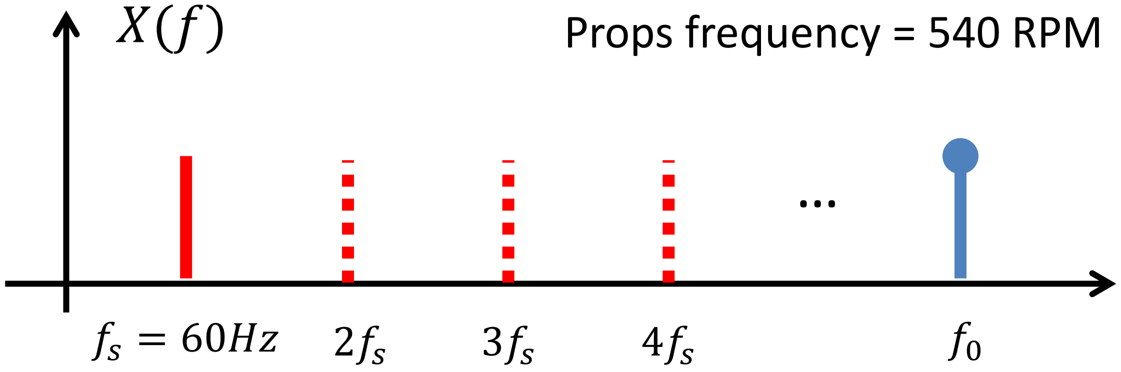
Revisit: Spaceship Helicopter

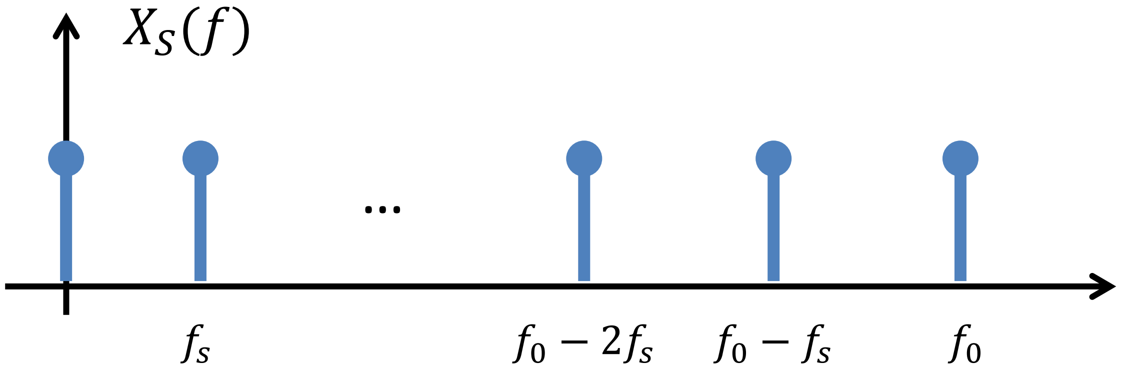
$$X_s(f) = \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{\Delta}\right) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$





Props frequency = 540 RPM





$f_0 = mf_s$ *where m=integer*

The frequency is wrapped around to zero frequency!!

Discrete Fourier Transform

So far, we have considered sequences that run over the range $-\infty < n < \infty$(n integer). For the special case where the sequence is of finite length (i.e. non-zero for a finite number of values) an alternative Fourier representation is possible called the **discrete Fourier transform (DFT)**.

It turns out that the DFT is a Fourier representation of a finite length sequence and is itself a sequence rather than a continuous function of frequency, and it corresponds to samples, **equally spaced in frequency**, of the Fourier transform of the signal. The DFT is fundamental to many digital signal processing algorithms.

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi f n\Delta}$$

Continuous in frequency

Repeated every 1/ Δ

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}nk}$$

DFT of a finite (sampled) sequence $x(n\Delta)$

$$f = \frac{k}{N\Delta}$$

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz (k integer)

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k)e^{i\frac{2\pi}{N}nk}$$

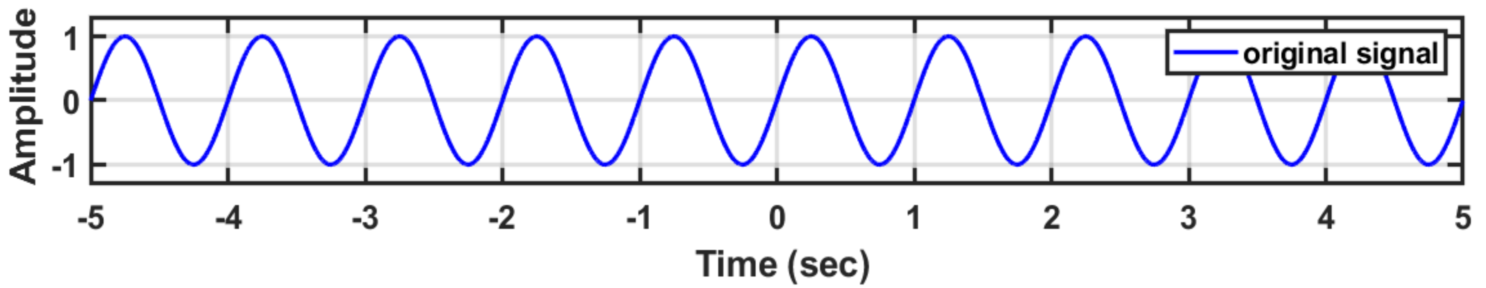
Inverse DFT

Q: How do we store frequency signals?

Discrete Fourier Transform (Continue)

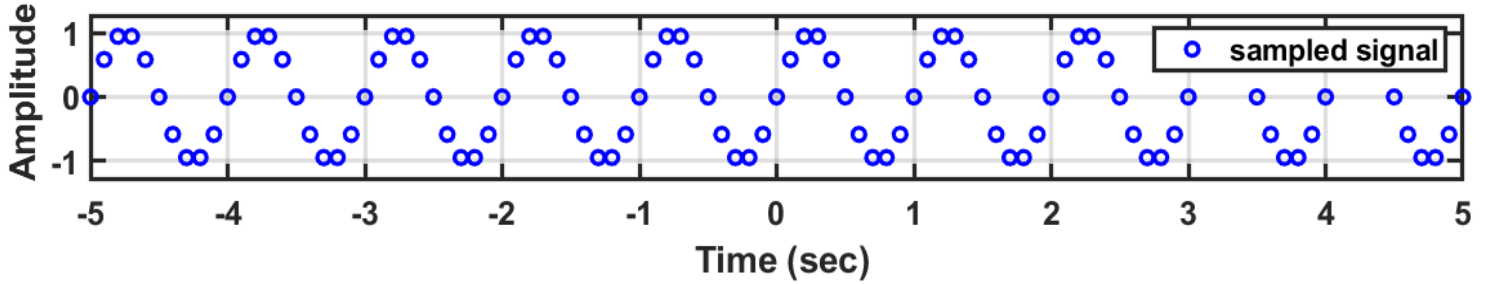
Fourier transform of a **continuous** time signal $x(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$



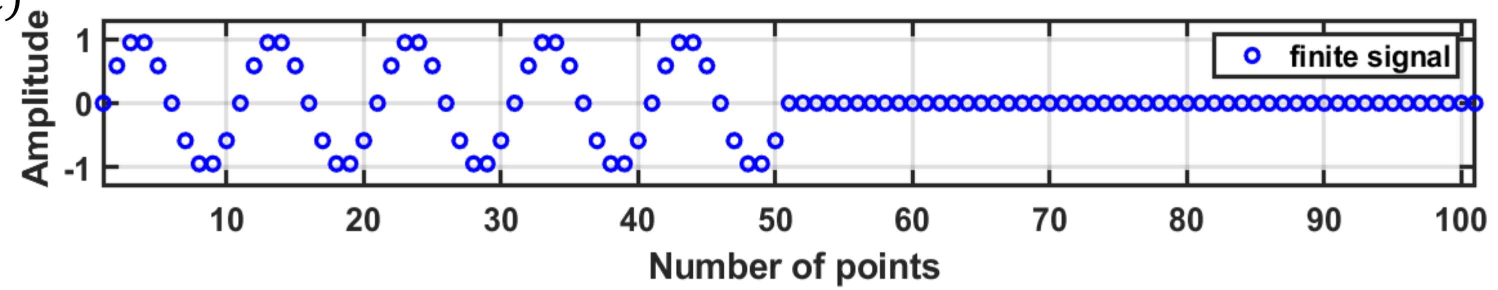
Fourier transform of a **discrete** sequence $x(n\Delta)$

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}$$



Fourier transform of a **discrete finite** sequence $x(n)$

$$X_s(f) = \sum_{n=0}^{N-1} x(n)e^{-i2\pi fn\Delta}$$



Discrete Fourier transform of a **discrete finite** sequence $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}nk}$$

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz (k integer)

Tutorial

Fast Fourier Transform

A fast Fourier transform (FFT) is an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT). Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa. It manages to reduce the complexity of computing the DFT from $O(n^2)$, which arises if one simply applies the definition of DFT, to $O(n\ log\ n)$, where n is the data size.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{N}nk}$$

FFT in Matlab

fft

Fast Fourier transform

Syntax

```
Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)
```

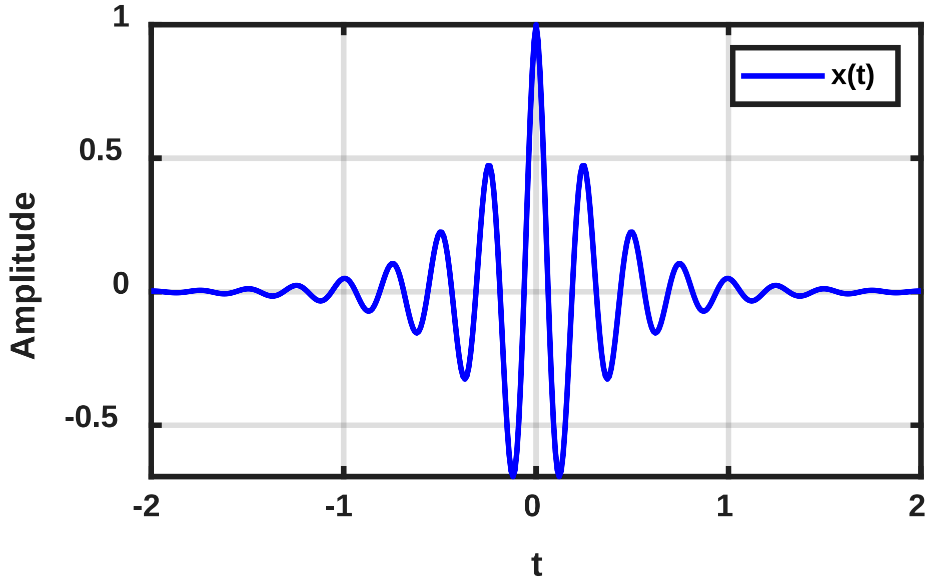
Y = fft(X) computes the discrete Fourier transform (DFT) of X using a fast Fourier transform (FFT) algorithm

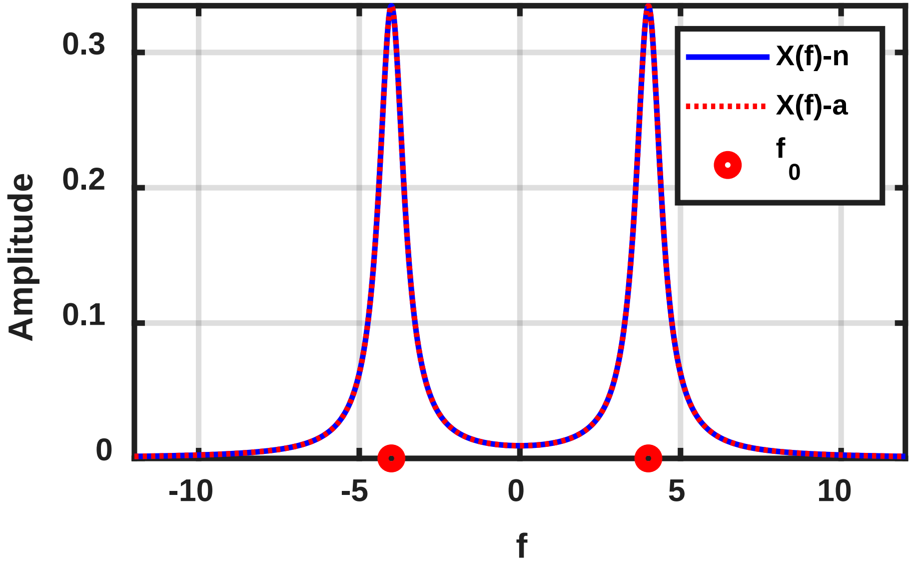
Y = fft(X,n) returns the n-point DFT. If no value is specified, Y is the same size as X.

Example: Fourier Transform of a Discrete Signal

$$x(t) = e^{-a|t|}cos2\pi f_0 t \quad \Rightarrow \quad X(f) = \frac{a}{a^2 + [2\pi(f - f_0)]^2} + \frac{a}{a^2 + [2\pi(f + f_0)]^2}$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi f t}dt$$





Note that –n and –a indicate numerical and analytical integration for Fourier transformation.

Example: Fourier Transform of a Discrete Signal (Continue)

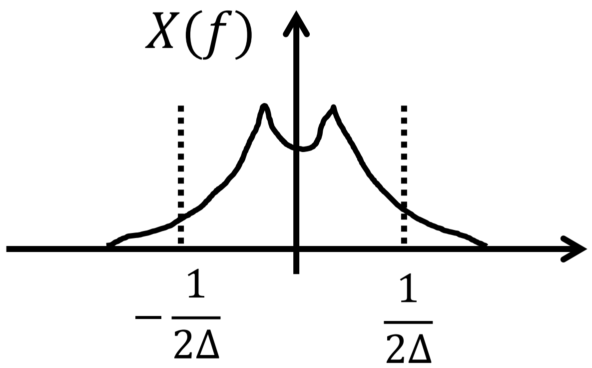
$$x(t) = e^{-a|t|}cos2\pi f_0 t \Longrightarrow X(f) = \frac{a}{a^2 + [2\pi(f - f_0)]^2} + \frac{a}{a^2 + [2\pi(f + f_0)]^2}$$

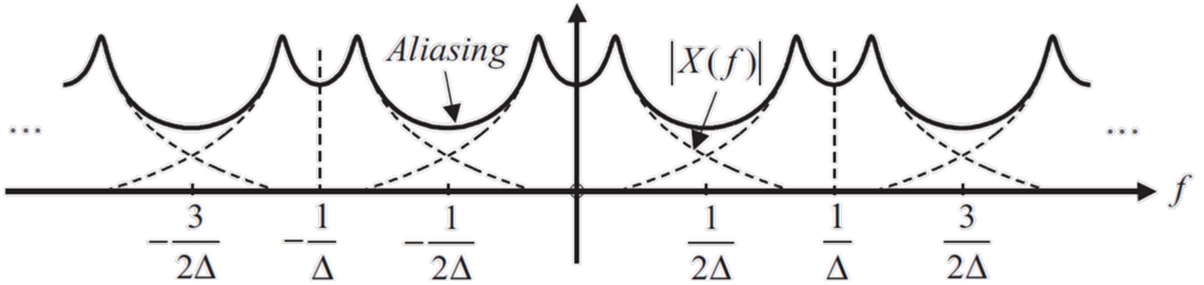
$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

$$x_s(t) = x(t)i(t) \Longrightarrow X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}$$

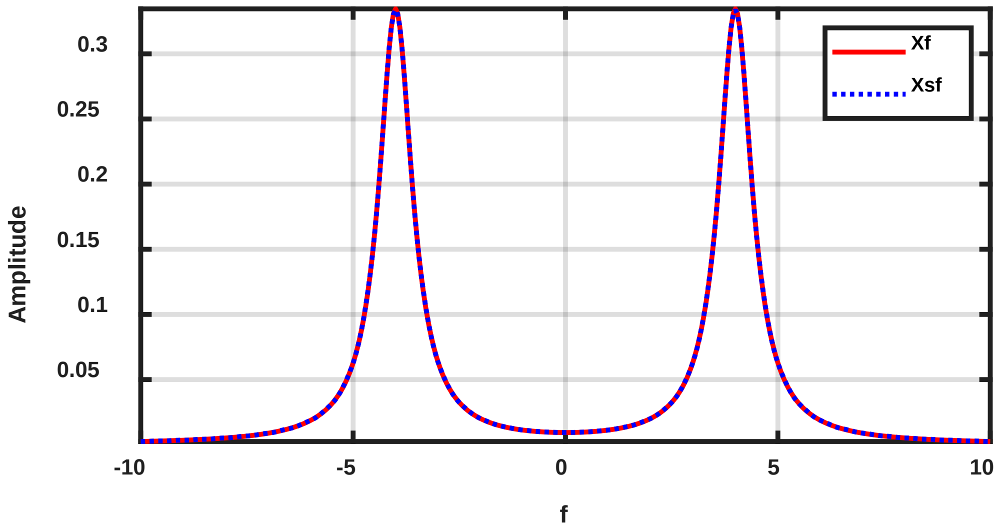
$$X_s(f) = I(f) * X(f) = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{\Delta}\right)$$

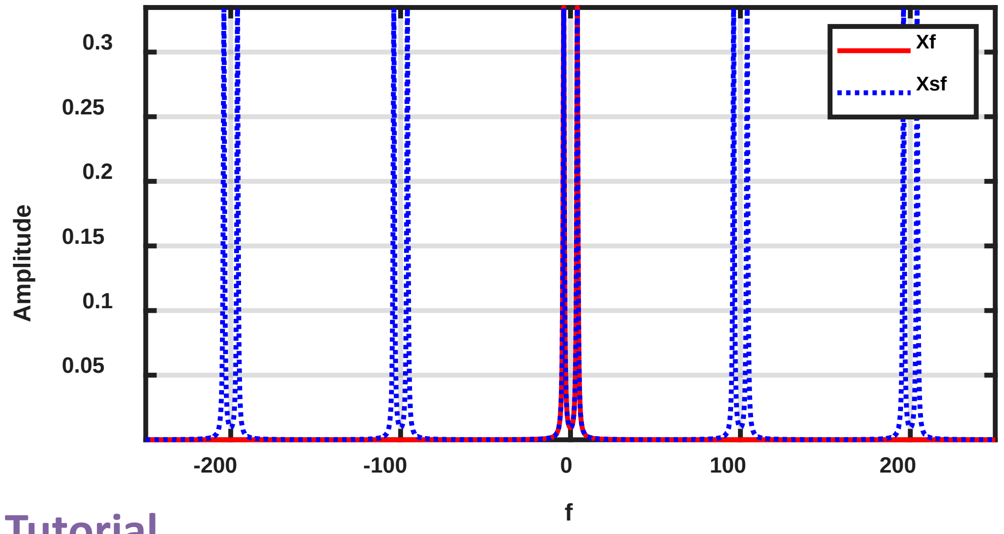
$$= \frac{1}{\Delta}\left(\dots + X\left(f - \frac{1}{\Delta}\right) + X(f) + X\left(f + \frac{1}{\Delta}\right) + \ \dots\right)$$





Example: Fourier Transform of a Discrete Signal (No Aliasing)



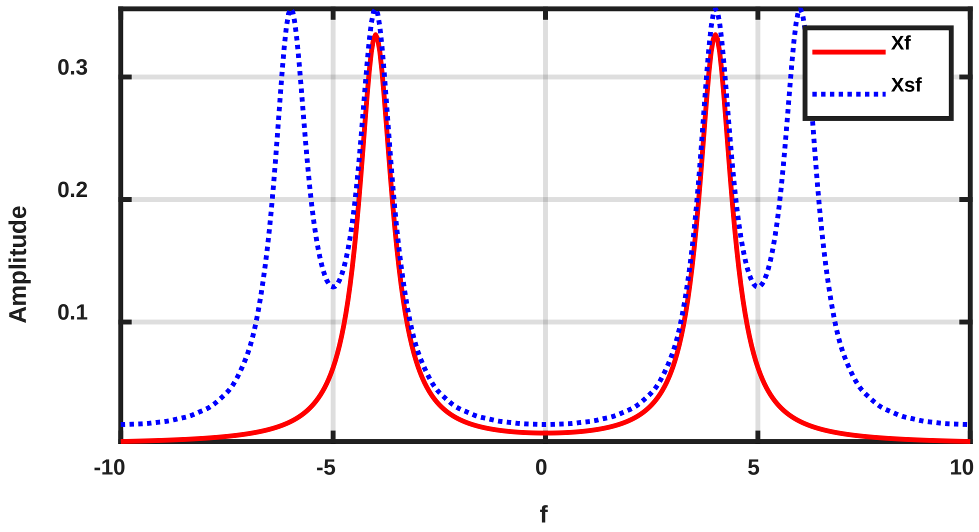


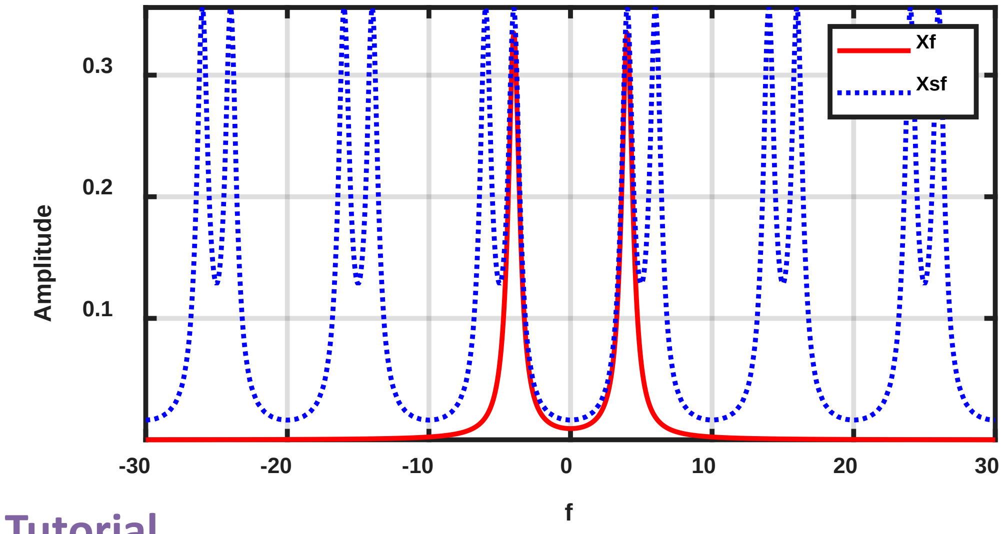

```
a = 3; f0 = 4;
x1 = @(t) exp(-a.*abs(t)).* cos(2*pi*f0*t);
fs = 100; dt = 1/fs; t = -2:dt:2-dt;

xn = x1(t);

Xf = @(f) a./(a^2+ (2*pi*(f-f0)).^2) + a./(a^2+ ...
(2*pi*(f+f0)).^2) ; % Xf
Xsf = @(f) 1/fs*sum(xn.*exp(-sqrt(-1)*2*pi*f*t), 2); % Xsf

fig1 = figure(1);
subplot(121); f = [-10:0.05:10]';
plot(f,abs(Xf(f)),'-r', 'linewidth', 2); hold on;
plot(f,abs(Xsf(f)),':b', 'linewidth', 2); hold off;

subplot(122); f = [-250:0.05:250]';
plot(f,abs(Xf(f)),'-r', 'linewidth', 2); hold on;
plot(f,abs(Xsf(f)),':b', 'linewidth', 2); hold off;
```

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi f n\Delta}$$

Tutorial

Example: Fourier Transform of a Discrete Signal (Aliasing)






```
a = 3; f0 = 4;
x1 = @(t) exp(-a.*abs(t)).* cos(2*pi*f0*t);
fs = 10; dt = 1/fs; t = -2:dt:2-dt;

xn = x3(t);

Xf = @(f) a./(a^2+ (2*pi*(f-f0)).^2) + a./(a^2+ ...
(2*pi*(f+f0)).^2) ; % Xf
Xsf = @(f) 1/fs*sum(xn.*exp(-sqrt(-1)*2*pi*f*t), 2); % Xsf

fig1 = figure(1);
subplot(121); f = [-10:0.05:10]';
plot(f,abs(Xf(f)),'-r', 'linewidth', 2); hold on;
plot(f,abs(Xsf(f)),':b', 'linewidth', 2); hold off;

subplot(122); f = [-30:0.05:30]';
plot(f,abs(Xf(f)),'-r', 'linewidth', 2); hold on;
plot(f,abs(Xsf(f)),':b', 'linewidth', 2); hold off;
```

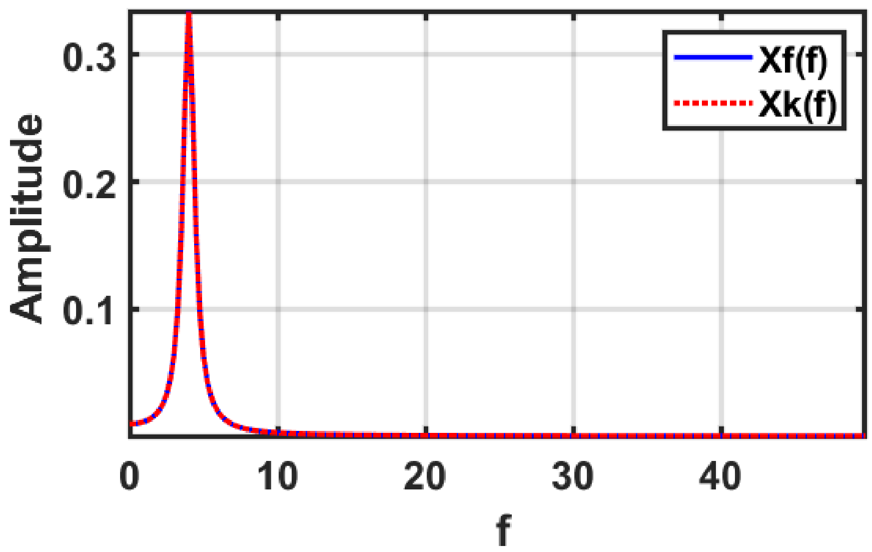
Example: Discrete Fourier Transform

```
a = 3; f0 = 4;
x1 = @(t) exp(-a.*abs(t)).* cos(2*pi*f0*t);
fs = 100; dt = 1/fs; t = -2:dt:2-dt;

xn = x1(t);

Xf = @(f) a./(a^2+ (2*pi*(f-f0)).^2) + a./(a^2+...
(2*pi*(f+f0)).^2); % Xf

% use of a 'fft' function
nfft = numel(xn);
Xk = fft(xn, nfft);

f = 1/(nfft*dt) * (0:nfft/2-1);
figure(1)
plot(f,abs(Xf(f)),'-b', 'linewidth', 2);hold on;
plot(f,1/fs*abs(Xk(1:(nfft/2))),':r', 'linewidth', 2);
```

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz (k integer)

$$X_s(f) = \sum_{n=0}^{N-1} x(n)e^{-i2\pi f n\Delta}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}nk}$$



Tutorial

Zero-Padding

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{N}nk}$$

Y = fft(X, n) returns the n-point DFT. If no value is specified, Y is the same size as X.

- If X is a vector and the length of X is less than n, then X is padded with trailing zeros to length n.
- If X is a vector and the length of X is greater than n, then X is truncated to length n.

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz

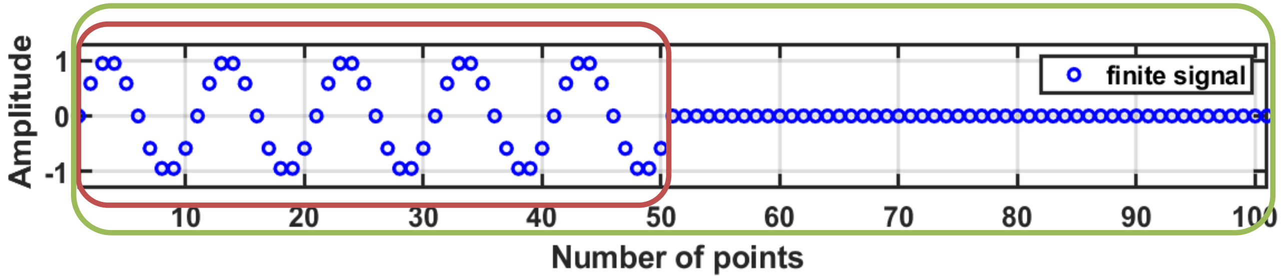
frequency spacing $\frac{1}{N\Delta}$ Hz

fft

Fast Fourier transform

Syntax

```
Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)
```



Zero-Padding (Continue)

$$\hat{x}(t) = x(n) \quad \text{for } 0 \leq n \leq N-1$$
$$= 0 \quad \text{for } N \leq n \leq L-1$$

$$\hat{X}(k) = \sum_{n=0}^{L-1} \hat{x}(n) e^{-i\frac{2\pi}{L}nk} = \boxed{\sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{L}nk}}$$

- "Finer" spacing in the frequency domain. However, the zero padding does not increase the "true" resolution.

- Vibration problems, this can be used to obtain the fine detail near resonances

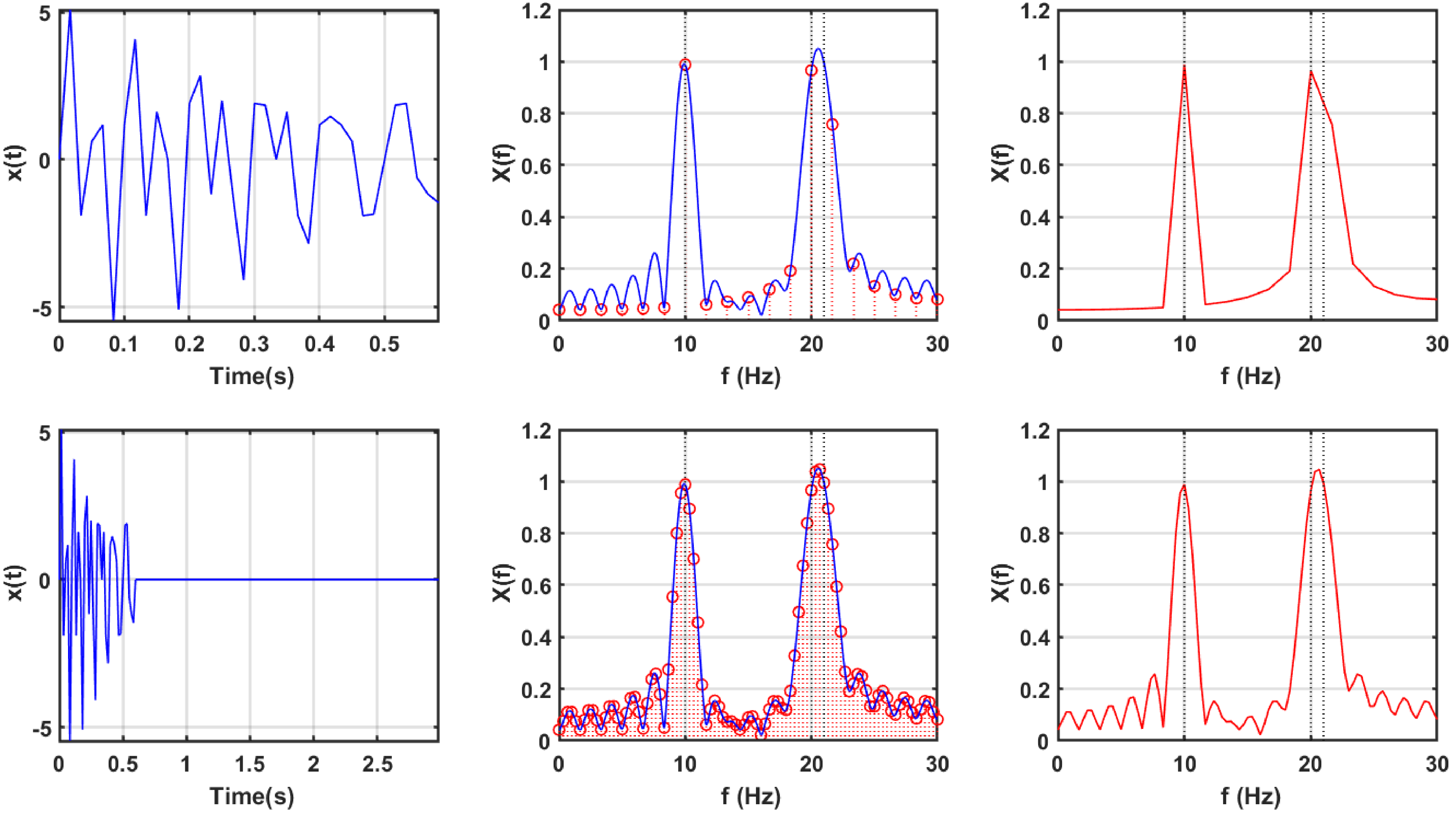
See the slide of "effect of data truncation"

Example: Zero-Padding Effect (Does not Increase a True Resolution)

Black dotted lines indicate 10, 20, and 21 Hz

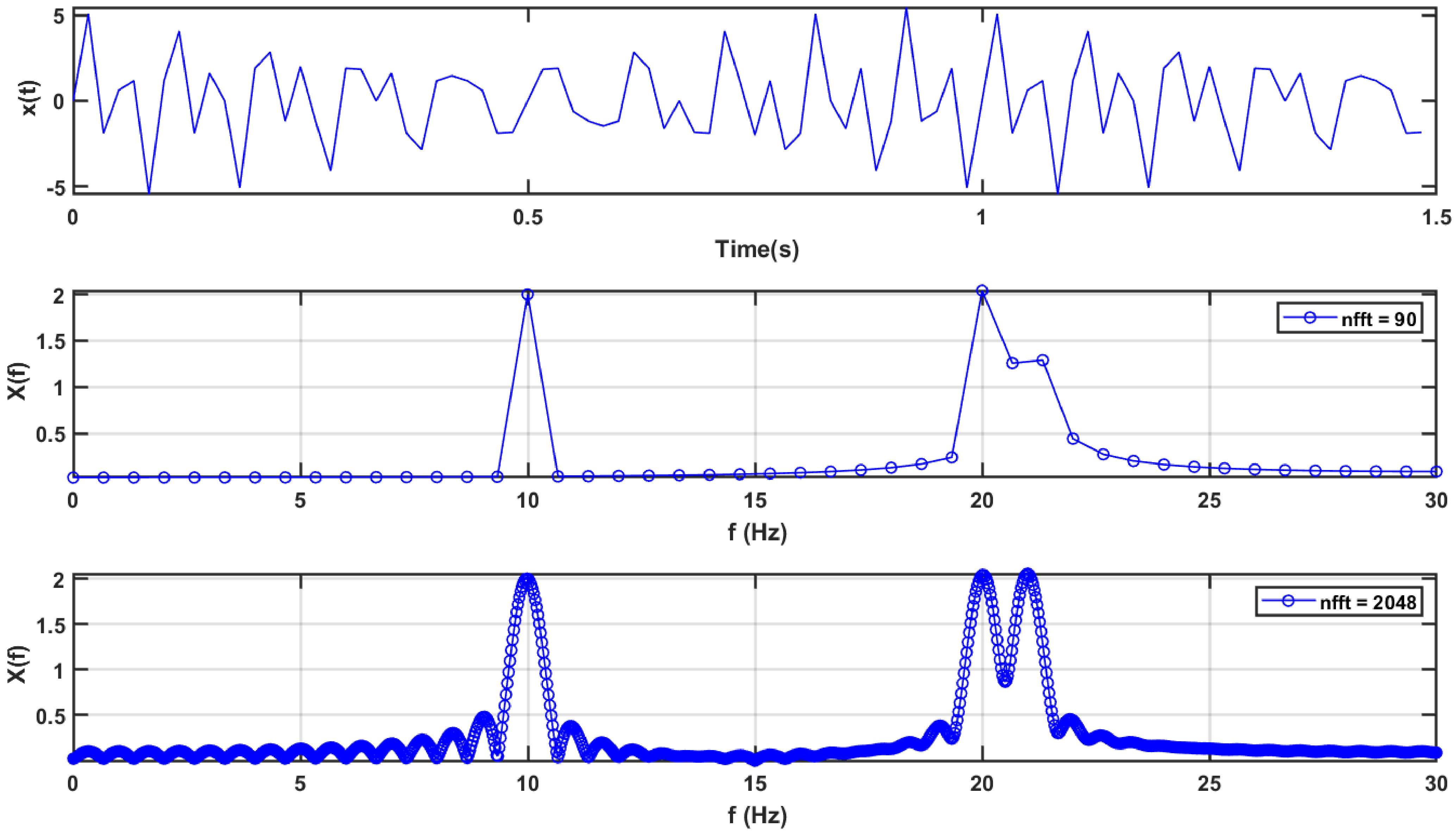
$T = 0.6$

$T = 0.6$
+ zero padding (2.4 s)



Tutorial

Example: Zero-Padding Effect (Still Useful for Obtaining the Detail)



Tutorial

Waveform VS FFT Frequency Resolution

The "**waveform frequency resolution**" is the minimum spacing between two frequencies that can be resolved. The "**FFT resolution**" is the number of points in the spectrum, which is directly proportional to the number points used in the FFT. **It is possible to have extremely fine FFT resolution, yet not be able to resolve two coarsely separated frequencies.** It is also possible to have fine waveform frequency resolution but have the peak energy of the sinusoid spread throughout the entire spectrum (this is called FFT spectral leakage).

The waveform frequency resolution is defined by the following equation:

$$\Delta R = \frac{1}{T}$$

where T is the time length of the signal with data. It's important to note here that you should not include any zero padding in this time! Only consider the actual data samples.

It's important to make the connection here that the discrete time Fourier transform (DTFT) or FFT operates on the data as if it were an infinite sequence with zeros on either side of the waveform. This is why the FFT has the distinctive sinc function shape at each frequency bin. You should recognize the waveform resolution equation 1/T is the same as the space between nulls of a sinc function.

The FFT resolution is defined by the following equation:

$$\Delta R = \frac{f_s}{N_{fft}}$$

https://www.bitweenie.com/listings/fft-zero-padding