# Signal Processing III

**Chul Min Yeum**
Assistant Professor
Civil and Environmental Engineering
University of Waterloo, Canada

**CIVE 497 – CIVE 700: Smart Structure Technology**

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

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# Fourier Integral Pair (Fourier Transform)

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \qquad X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

The Fourier transform (FT) decomposes a function of time (a signal) into its constituent frequencies. This is similar to the way a musical chord can be expressed in terms of the volumes and frequencies of its constituent notes. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time.

https://en.wikipedia.org/wiki/Fourier_transform

# Properties of Fourier Transforms

**1. Time scaling**

$$F\big(x(at)\big) = \frac{1}{|a|} X(\frac{f}{a})$$

*Inverse spreading relationship*

**2. Time reversal**

$$F\big(x(-t)\big) = X(-f)$$

**3. Differentiation**

$$F\big(\dot{x}(t)\big) = i2\pi f X(f)$$

**4. Time shifting**

$$F\big(x(t-t_0)\big) = e^{-i2\pi f t_0} X(f)$$

*Only phase shift !*

**Sine wave**

**5. Modulation**

$$F\big(x(t)e^{i2\pi f_0 t}\big) = X(f - f_0)$$

$$F(x(t)\cos(2\pi f_0 t)\,) = \frac{1}{2}[X(f - f_0)\ + X(f + f_0)\,]$$

# Properties of Fourier Transforms (Convolution)

$$F(h(t) * x(t)) = H(f)X(f)$$

Where the convolution of the two functions $h(t)$ and $x(t)$ is defined as

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

It is a function of time !

# Proof of Convolution Relationship

$$\boxed{F(h(t) * x(t)) = H(f)X(f)}$$

**Proof:** $$F\big(h(t) * x(t)\big) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau\right]\exp^{-i2\pi f t}dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(v)\exp^{-i2\pi f(\tau+v)}\,d\tau\,dv \qquad v = t - \tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\exp^{-i2\pi f(\tau)}\,d\tau \int_{-\infty}^{\infty} x(v)\exp^{-i2\pi f(v)}\,dv = H(f)X(f)$$

$$\boxed{F(x(t)) = X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi f t}dt}$$

$$\boxed{h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

# Proof of Convolution Relationship (Continue)

$$F(x(t)w(t)) = X(f) * W(f)$$

**Proof:** $F(x(t)w(t)) = \int_{-\infty}^{\infty} x(t)\, w(t) \exp^{-i2\pi f t}\, dt$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f_1) \exp^{i2\pi f_1 t}\, W(f_2) \exp^{i2\pi f_2 t} \exp^{-i2\pi f t}\, df_1\, df_2\, dt$$

$$= \int_{-\infty}^{\infty} X(f_1) \int_{-\infty}^{\infty} W(f_2) \int_{-\infty}^{\infty} \exp^{-i2\pi(f-f_1-f_2)t}\, dt\, df_2\, df_1$$

$$= \int_{-\infty}^{\infty} X(f_1) \int_{-\infty}^{\infty} W(f_2)\delta(f - f_1 - f_2) df_2 df_1 = \int_{-\infty}^{\infty} X(f_1) W(f - f_1) df_1 = X(f) * W(f)$$

$$F(x(t)) = X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

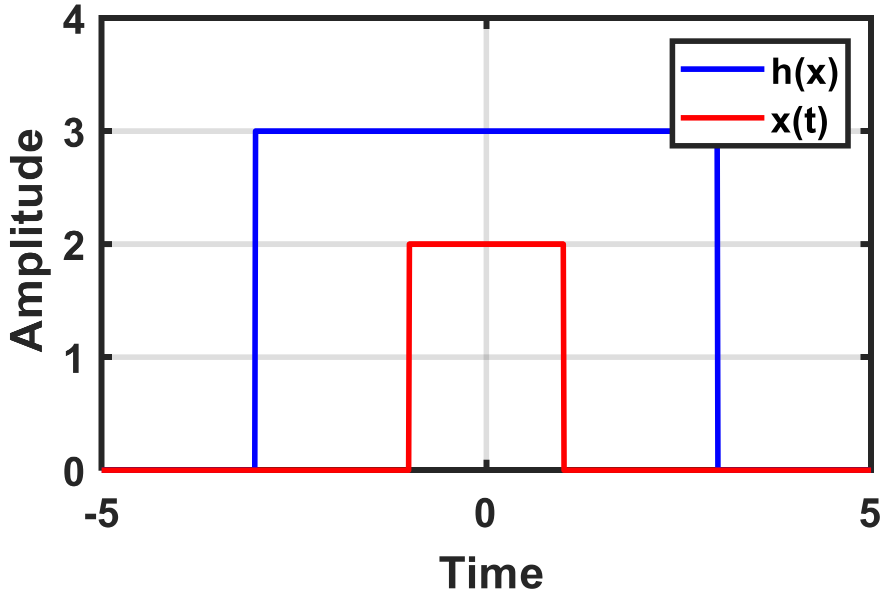
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$$

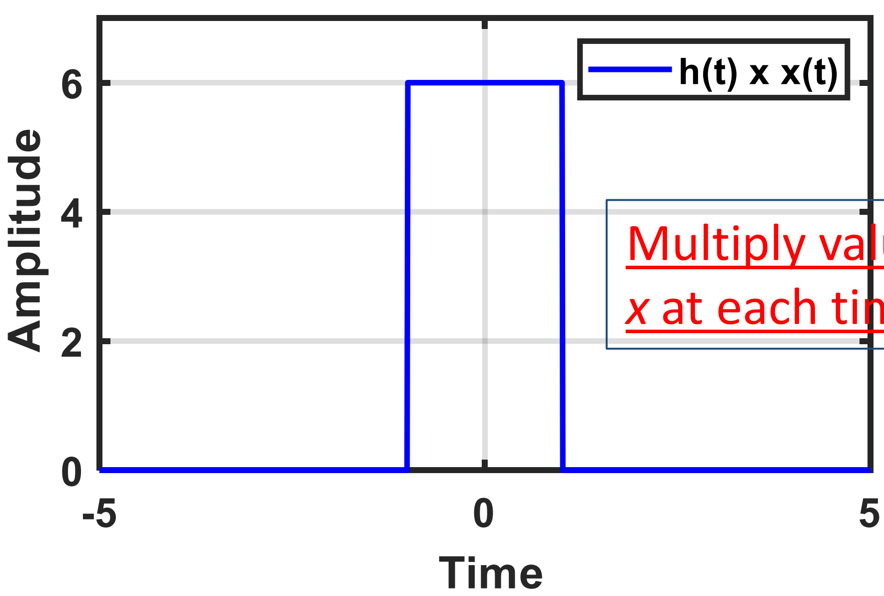
# Example: Convolution 1 (Square Wave)

$h(t) = 3 \quad if \ -3 < t < 3$. Otherwise, $h(t) = 0$

$$h(t)x(t)$$

$x(t) = 2 \quad if \ -1 < t < 1$. Otherwise, $x(t) = 0$

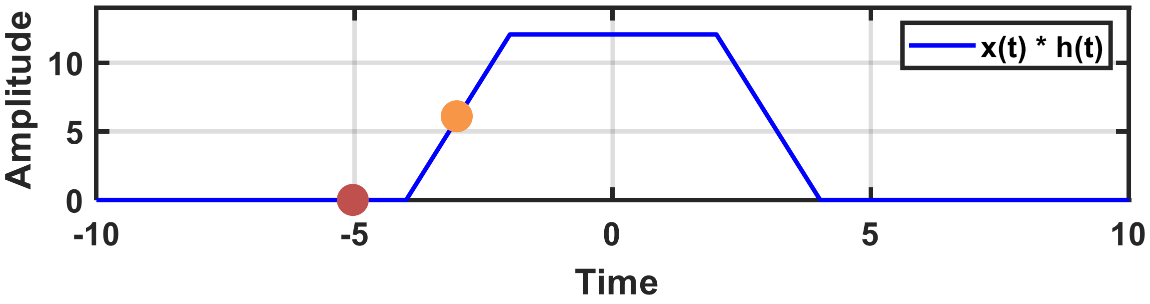


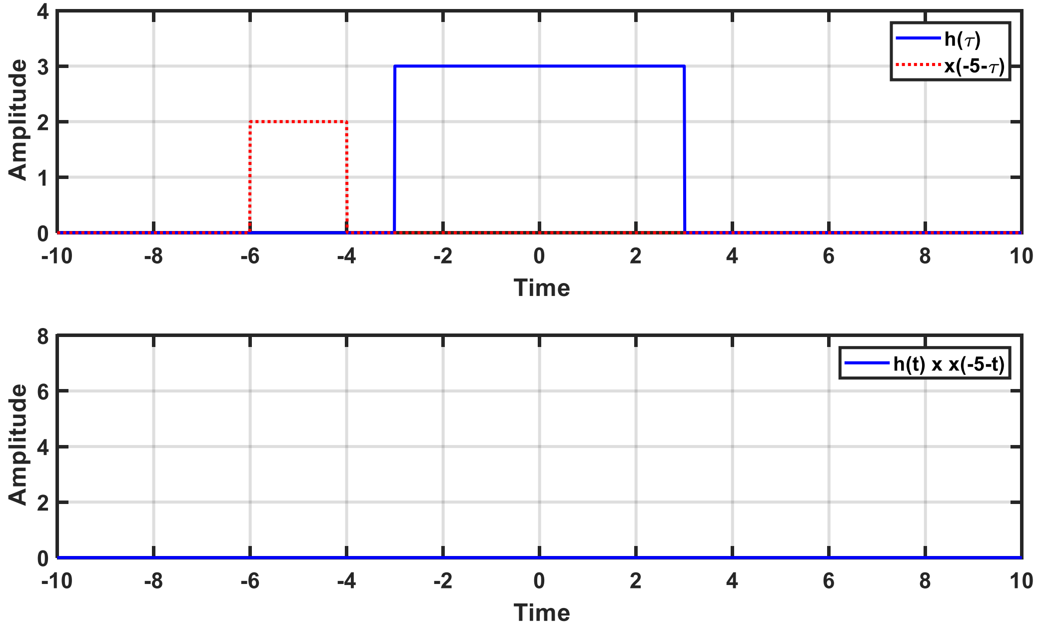



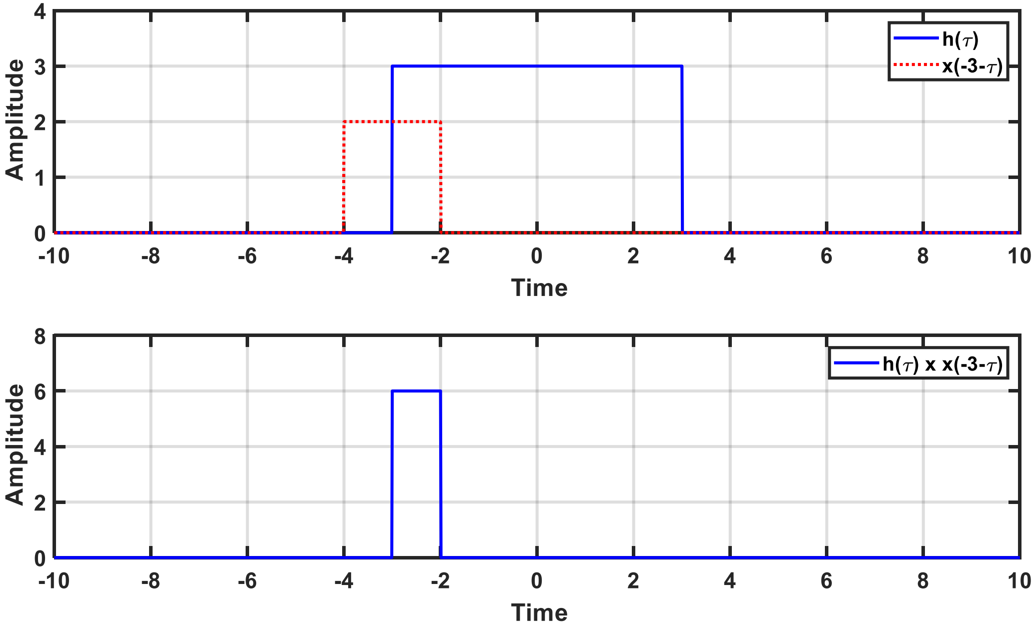

Multiply values of *h* and *x* at each time point

# Example: Convolution 1 (Square Wave) (Continue)

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$




$$\boxed{\int_{-\infty}^{\infty} h(\tau)x(-5-\tau)\,d\tau = 0}$$

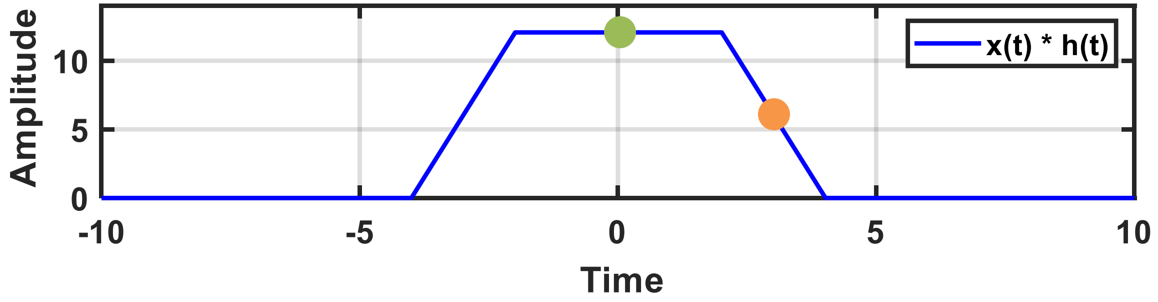
It is for finding a value of x(t)*h(t) at t= -5

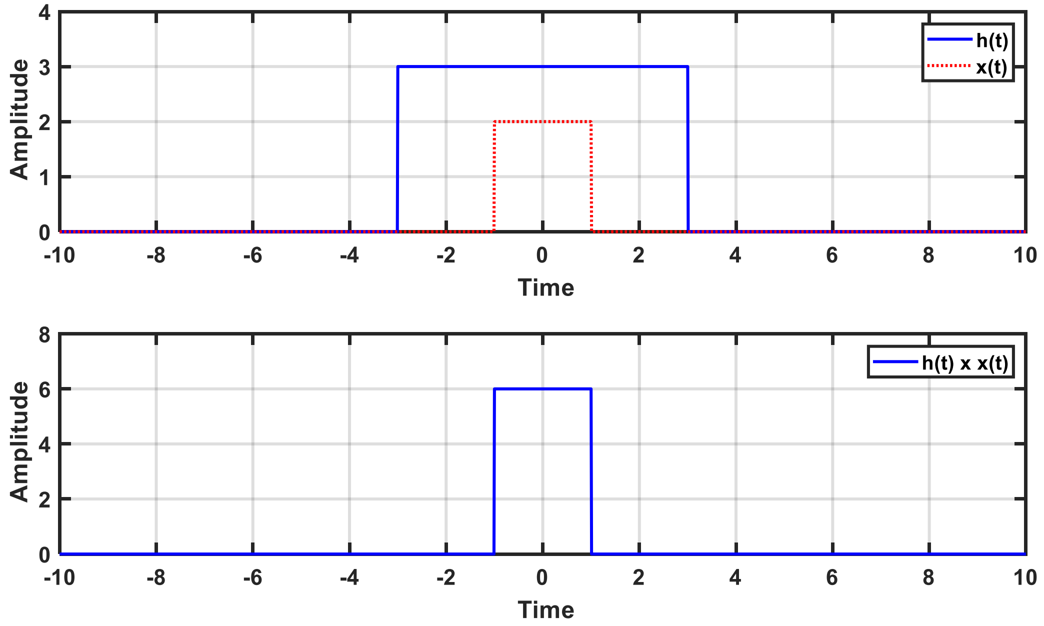
$$\boxed{\int_{-\infty}^{\infty} h(\tau)x(-3-\tau)\,d\tau = 6}$$

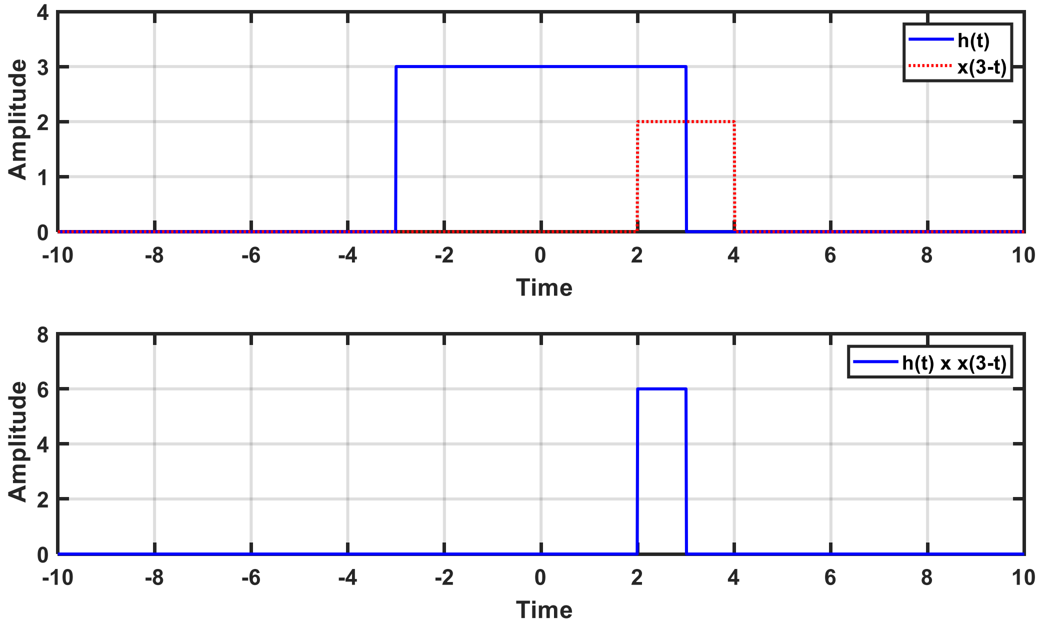
It is for finding a value of x(t)*h(t) at t= -3

# Example: Convolution 1 (Square Wave) (Continue)

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$




$$\int_{-\infty}^{\infty} h(\tau)x(\tau)\, d\tau = 12$$

It is for finding a value of x(t)*h(t) at t= 0

$$\int_{-\infty}^{\infty} h(\tau)x(3-\tau)\, d\tau = 6$$

It is for finding a value of x(t)*h(t) at t= 3

# Convolution (Visualization)

The Convolution Integral | Convolution Demo | A Systems Perspective | Evaluation of Convolution Integral | Laplace | Printable

## Convolution Demo and Visualization

This page can be used as part of a tutorial on the convolution of two signals. It lets the user visualize and calculate how the convolution of two functions is determined - this is ofen refered to as *graphical* convoluiton.

The tool consists of three graphs.

**Top graph:** Two functions, h(t) *(dashed red line)* and f(t) *(solid blue line)* are plotted in the topmost graph. As you choose new functions, these graphs will be updated.

**Middle graph:** The middle graph shows three separate functions and has an independent variable (i.e., x-axis) of $\lambda$. This is important - in this graph $\lambda$ varies and $t$ is constant. Shown are

- f(λ) *(solid blue line)*. Note: this looks just like f(t).
- h(t-λ) *(dashed red line)*. Note: this is reversed horizontally relative to the original (because of the minus sign on the independent variable, *i.e.*, *-λ*) and shifted horizontally (by the an amount equal to the constant *t*).
- the product, f(λ)·h(t-λ) *(pink line, and filled in with pink)*. Note: this is a function of $\lambda$, with $t$ being a constant). This function is shaded in pink as a reminder that the convolution is the *integral* of this function for this particular value of t. (i.e., the shaded area above 0 minus the shaded area below zero).
- the value of "t". You can change the value of t by clicking and dragging within the middle or bottom graph.

The variable λ does not appear in the final convolution, it is merely a dummy variable used in the convolution integral (see below).

**Bottom graph:** The bottom graph shows y(t), the convolution of h(t) and f(t), as well as the value of "t" specified in the middle graph (you can change the value of t by clicking and dragging within the middle or bottome graph).

$$y(t) = \int_{-\infty}^{+\infty} h(t-\lambda)\cdot f(\lambda)\cdot d\lambda = \int_{0^-}^{t} h(t-\lambda)\cdot f(\lambda)\cdot d\lambda$$

*Click here to see why the two integrals are equal despite the different limits of integration.*

The constant value of t from the middle graph is indicated by a black dot on the bottom two graphs. You can change the value of t by entering a value into the box below, or you can click and move the mouse horizontally in either of the two lower graphs.

3.86 t (S) -1 5

The value of the convolution at this time is y(3.86) = 0.02.

*This value is calculated numerically and may differ slightly from an exact value.*

You can select two functions, h(t) and f(t) to be convolved. You can also choose to show the complete solution or just the solution until a specified time. This is useful when learning because you can try to figure out what the convolution looks like to test your understanding.

Select h(t): Exp

Select f(t): Narrow Pulse

Show complete solution? ☑

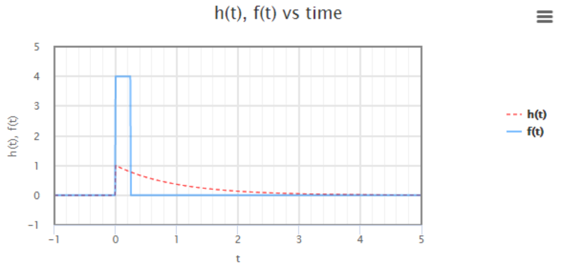
The functions used for the current example are *(note: all functions are implicitly multiplied by the unit step function, γ(t), so they are equal to 0 for t<0)*.

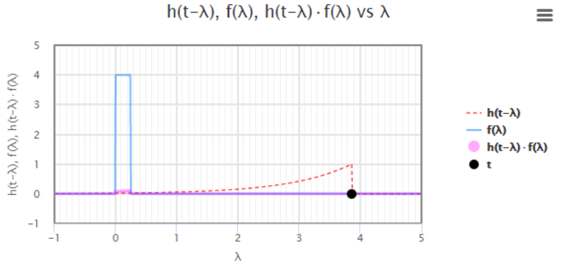
$h(t)=e^{-t}$,
and
f(t)= 4 (γ(t)-γ(t-0.25)).

There is no in depth explanation for this particular example. If you would like to see an example with a more detailed explanation, choose one of the options from the drop down box below.

To help your understanding, you can get an explanation of the convolution of select pairs of h(t) and f(t). Use the drop down below to select values two functions, and an explanation of the shape of the convolution will appear above.

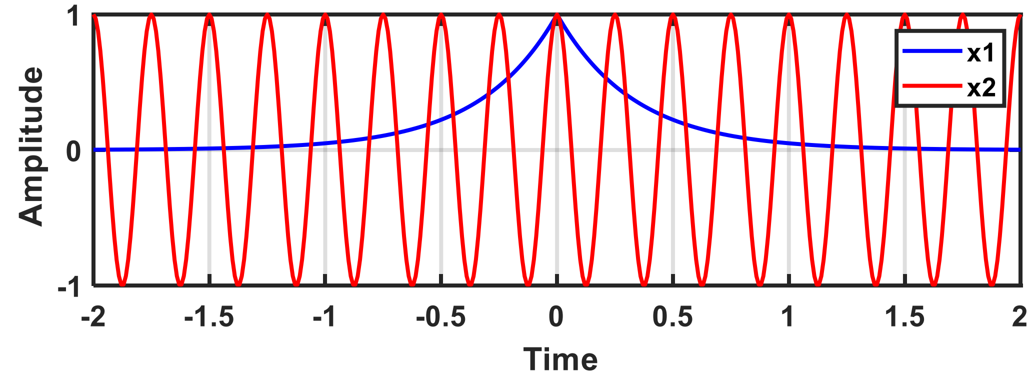
Select one.

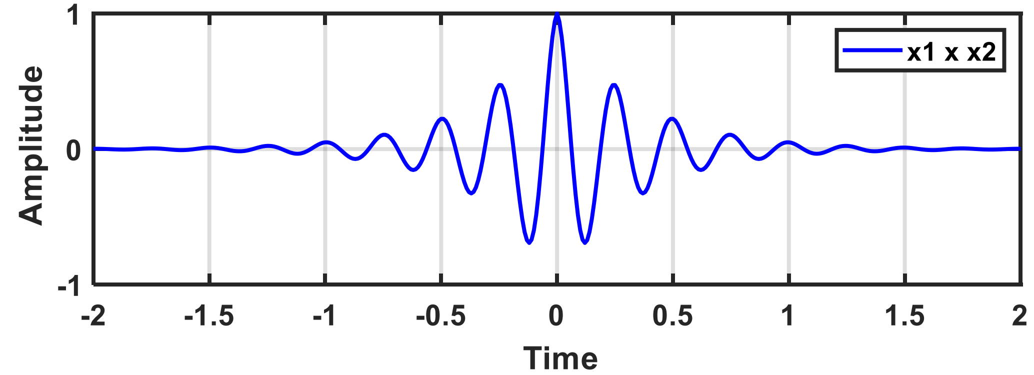


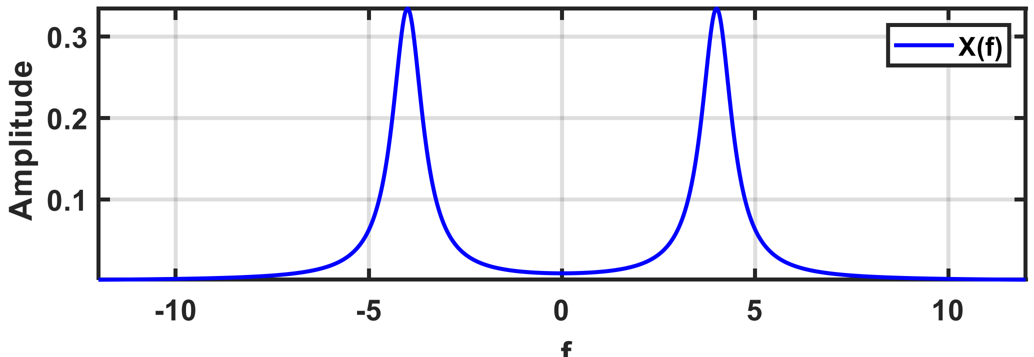




http://lpsa.swarthmore.edu/Convolution/CI.html

# Example: Convolution 2 (Damped Symmetrically Oscillating Function)

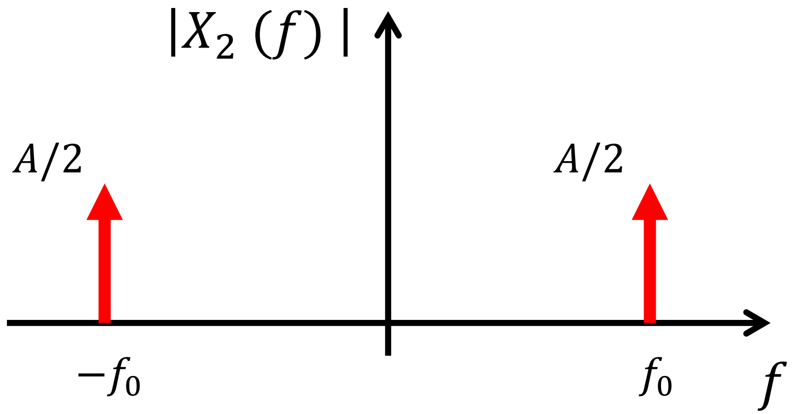


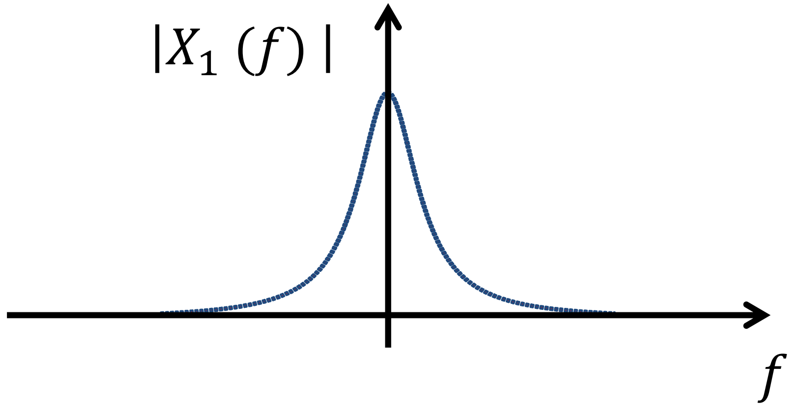



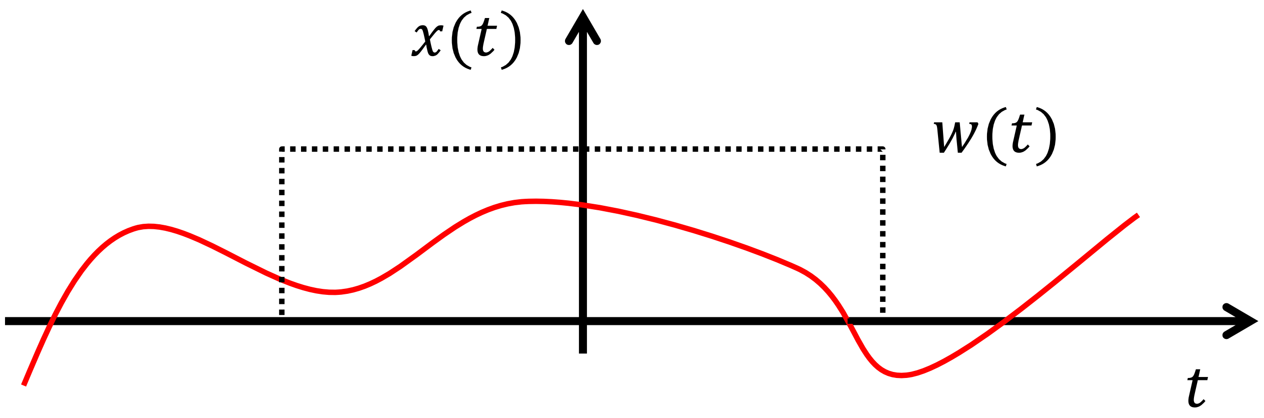

$$x_1(t) = e^{-a|t|},\ x_2(t) = cos2\pi f_0 t$$

$$x(t) = x_1(t)x_2(t) = e^{-a|t|}cos2\pi f_0 t$$



# Effect of Data Truncation (Windowing)




$w(t) = 1$ for $|t| < T/2$
$= 0$ otherwise

$x(t)$ is known (or recorded) only for $-\frac{T}{2} < t < \frac{T}{2}$, denoted as $x_T(t)$

$$F(x_T(t)) = F(x(t)w(t)) = X(f) * W(f)$$

Fourier transform of the product of two time signals is the convolution of their Fourier transforms.

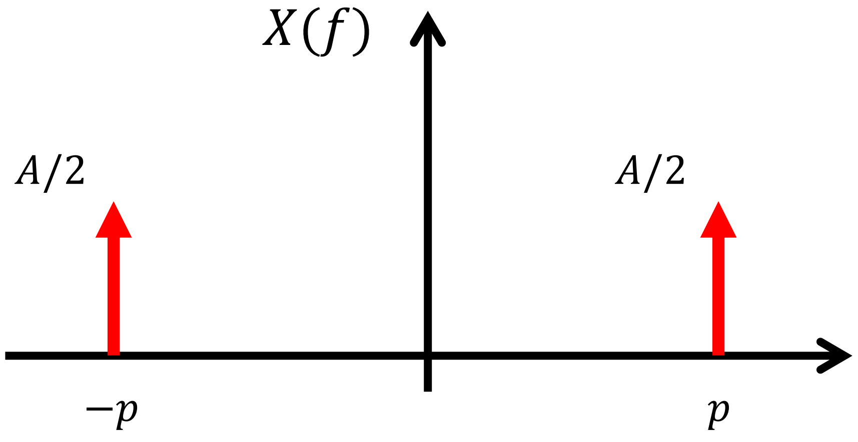
# Effect of Data Truncation (Windowing) (Continue)

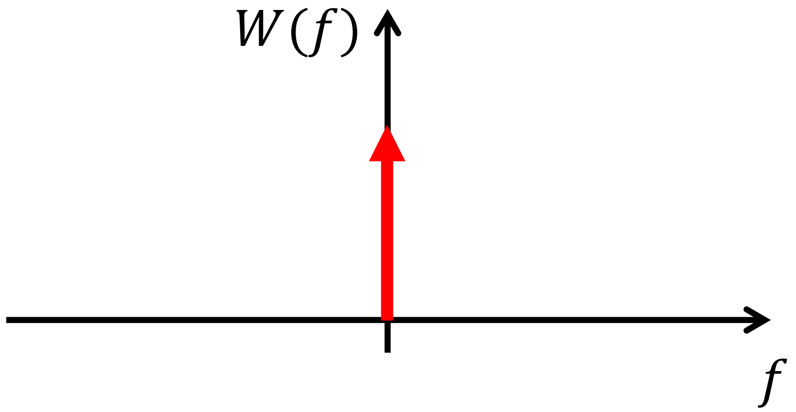
$$x(t) = A\cos 2\pi pt$$

$$w(t) = 1$$

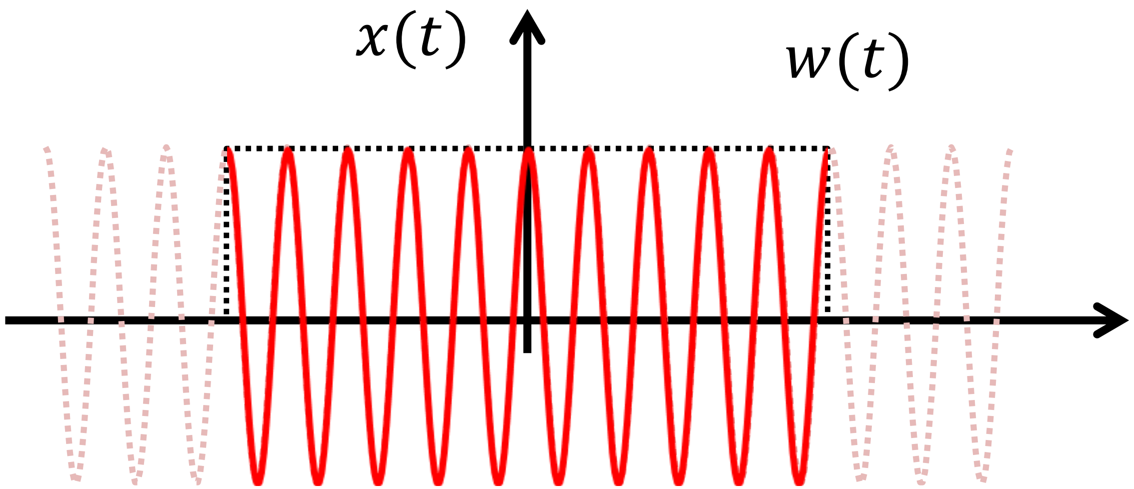
$$X(f) = \frac{A}{2}[\delta(f-p) - \delta(f+p)]$$

$$W(f) = \delta(f)$$



# Truncated Sine Wave

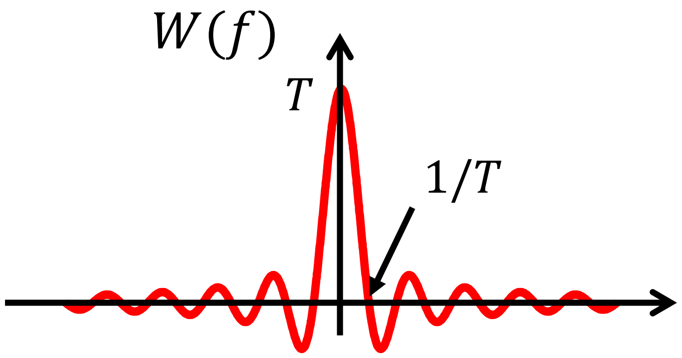



$$x(t) = A\cos 2\pi pt$$

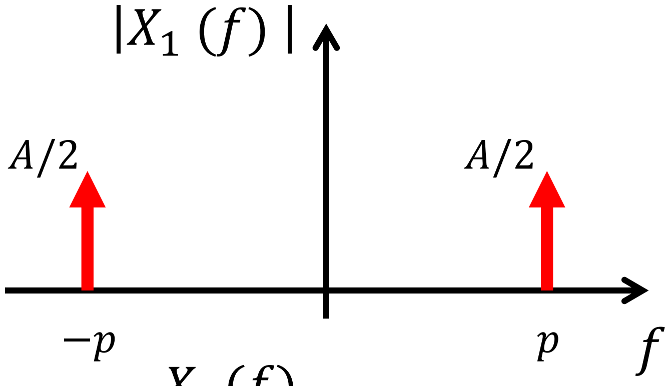
$$w(t) = a \text{ for } |t| < \text{b}$$
$$= 0 \text{ for } |t| > \text{b}$$

$$x_T(t) = x(t)w(t)$$

$$W(f) = \frac{T \sin \pi f T}{\pi f T}$$

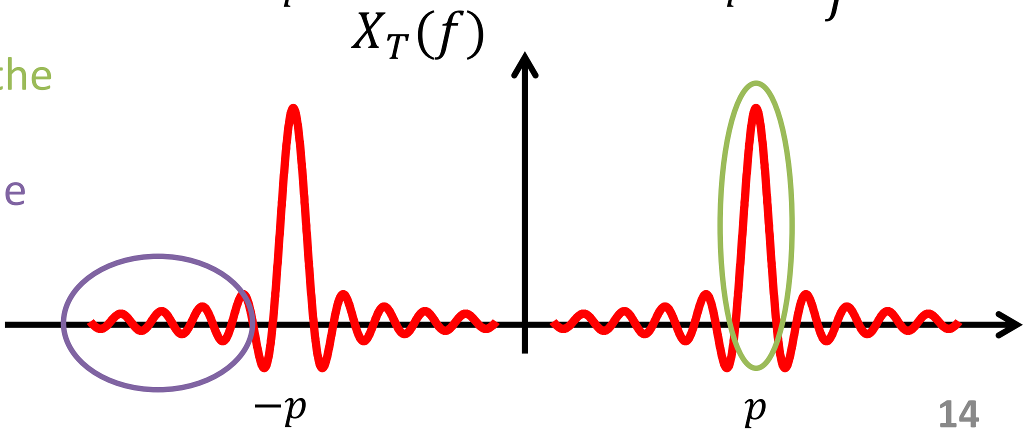



$$X(f) = \frac{A}{2}[\delta(f-p) - \delta(f+p)]$$

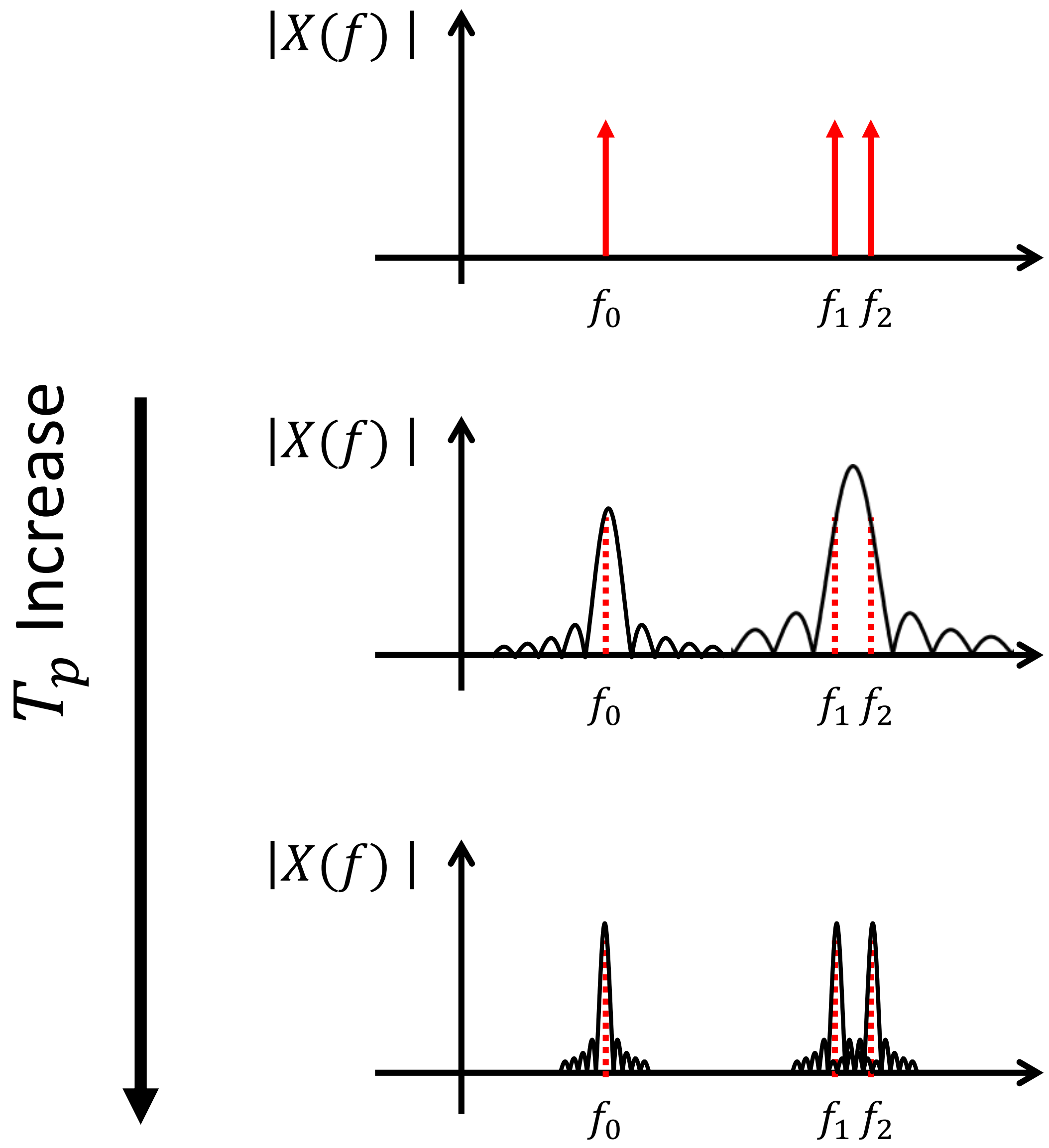



**Smearing**: Distortion due to the main lobe

**Leakage**: Distortion due to the side lobe



# Effect of Data Truncation: Frequency Separation




$x(t)$ is the sum of three sine (or cosine) waves

Considerable smearing due to the spectral window

Three components are resolved but with considerable leakage at other frequencies

# Window Function

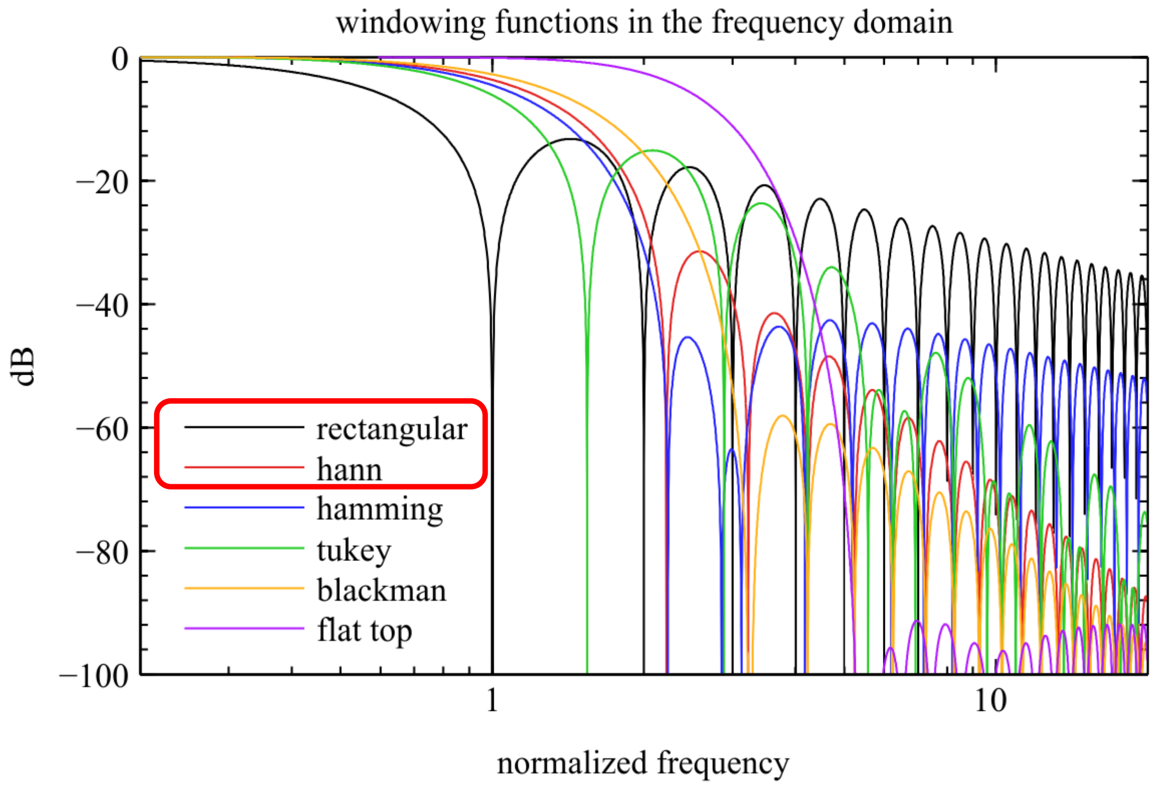
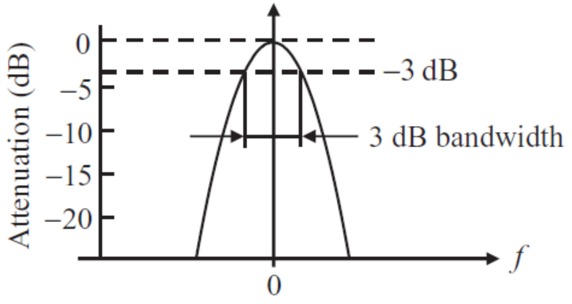



**Table 4.2** Properties of some window functions

| Window (length $T$) | Highest side lobe (dB) | Asymptotic roll-off (dB/octave) | 3 dB bandwidth | Noise bandwidth | First zero crossing (freq.) |
|---|---|---|---|---|---|
| Rectangular | −13.3 | 6 | $0.89\frac{1}{T}$ | $1.00\frac{1}{T}$ | $\frac{1}{T}$ |
| Bartlett (triangle) | −26.5 | 12 | $1.28\frac{1}{T}$ | $1.33\frac{1}{T}$ | $\frac{2}{T}$ |
| Hann(ing) (Tukey or cosine squared) | −31.5 | 18 | $1.44\frac{1}{T}$ | $1.50\frac{1}{T}$ | $\frac{2}{T}$ |
| Hamming | −43 | 6 | $1.30\frac{1}{T}$ | $1.36\frac{1}{T}$ | $\frac{2}{T}$ |
| Parzen | −53 | 24 | $1.82\frac{1}{T}$ | $1.92\frac{1}{T}$ | $\frac{4}{T}$ |

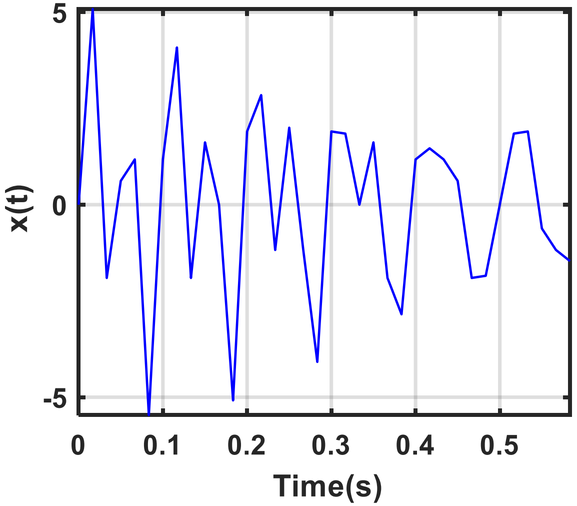
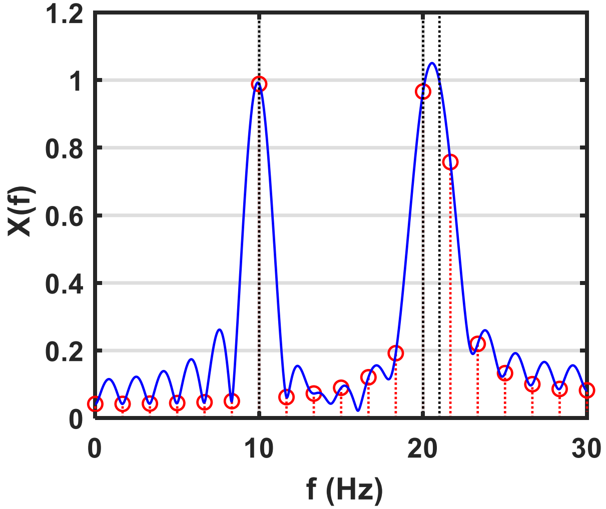
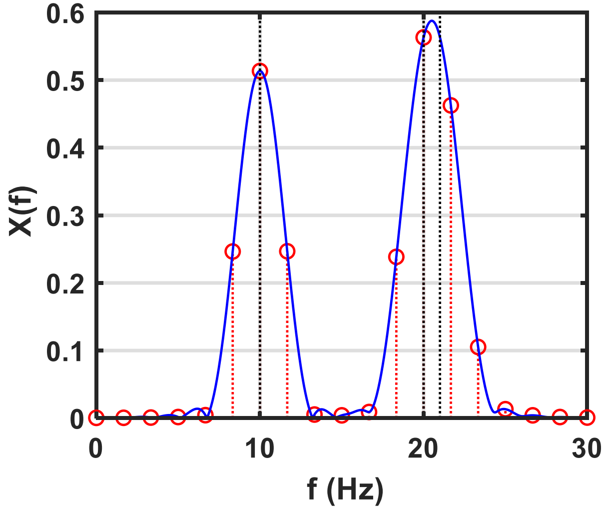
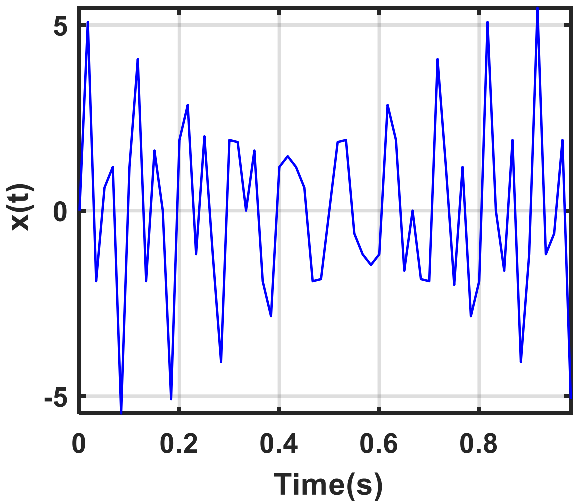
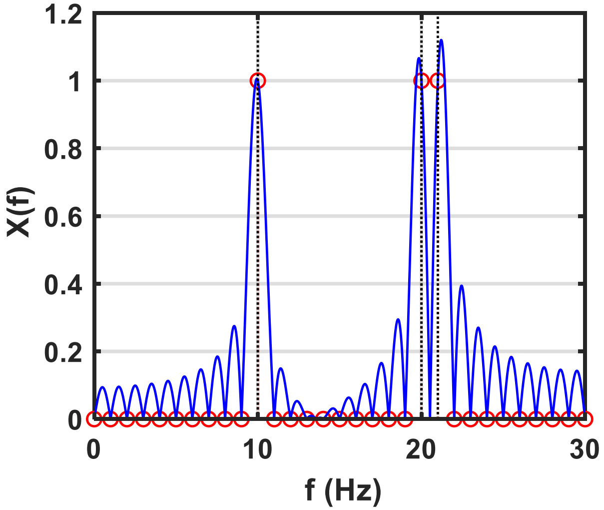
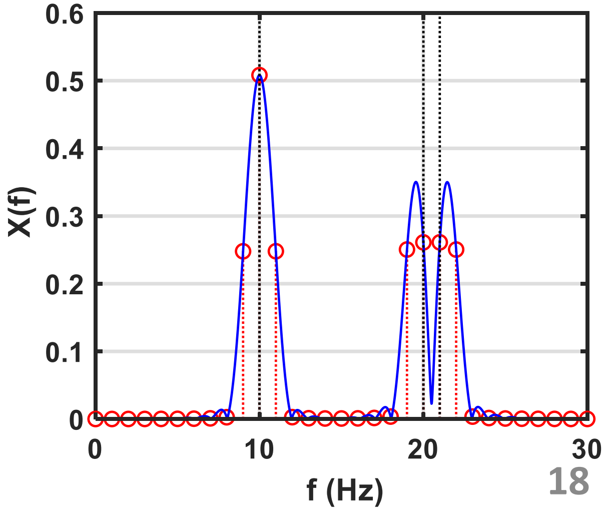


# Example: Hanning Window vs Rectangular Window

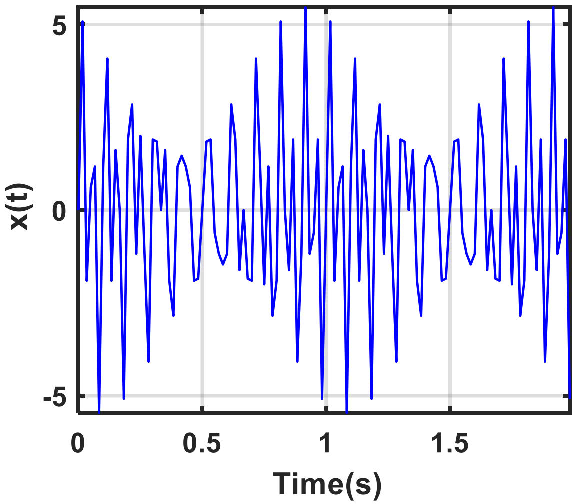
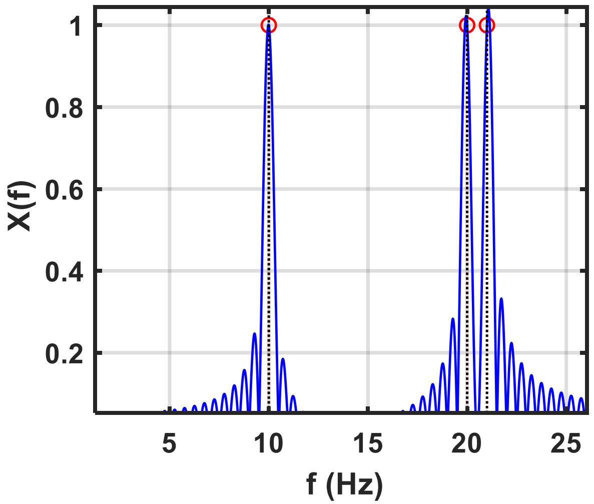
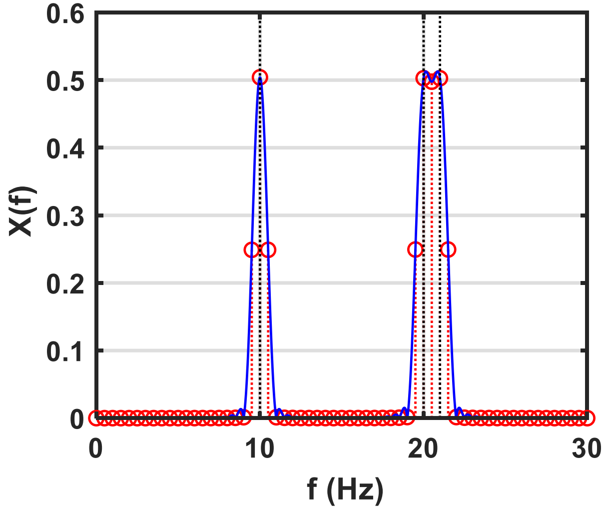
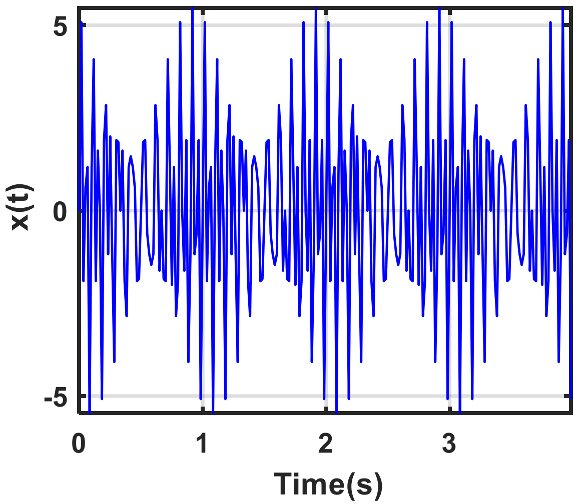
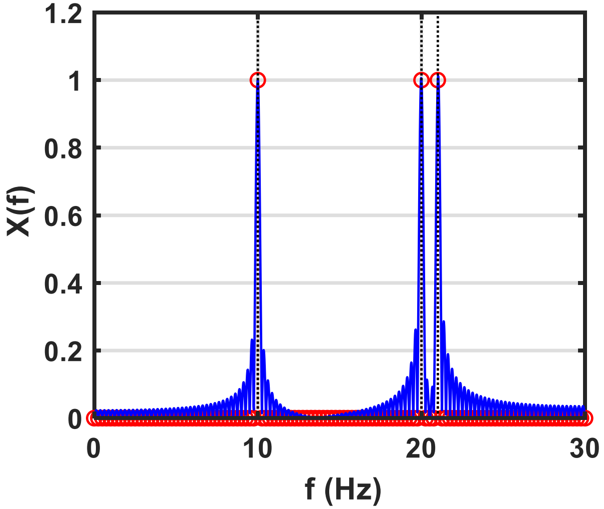
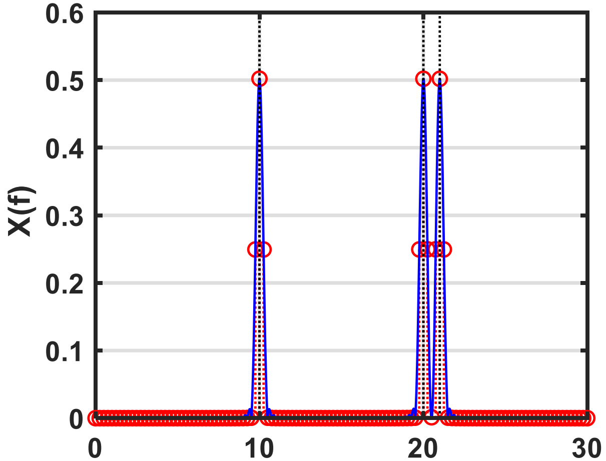
$$x(t) = A_1 \sin 2\pi f_1 t + A_2 sin2\pi f_2 t + A_3 sin2\pi f_3 t$$

Amplitudes are $A_1 = A_2 = A_3 = 2$, which gives the magnitude ‘1’ for each sinusoidal component in the frequency domain. The frequencies are chosen as $f_1 = 10$, $f_2 = 20$ and $f_3 = 21$.

# Example: Hanning Window vs Rectangular Window (Continue)

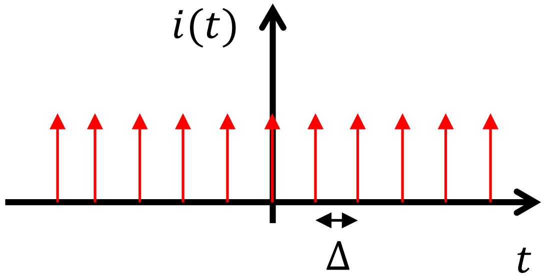
| | Original signal | Rectangular | Hanning |
|---|---|---|---|
| $T_p = 0.6$ |  |  |  |
| $T_p = 1$ |  |  |  |

# Example: Hanning Window vs Rectangular Window (Continue)

| | Original signal | Rectangular | Hanning |
|---|---|---|---|
| $T_p = 2$ |  |  |  |
| $T_p = 4$ |  |  |  |

# Impluse Train Modulation (Discretization)

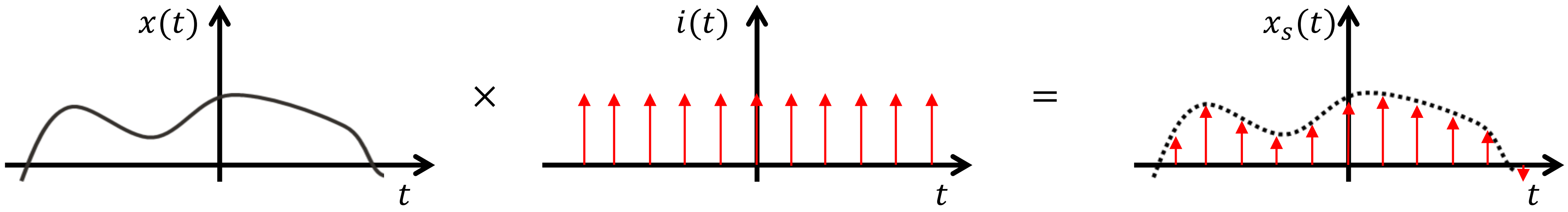
A 'train' of delta functions $i(t)$ is expressed as




$$i(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$$

$$f_s = \frac{1}{\Delta}$$

The sampling procedure can be illustrated as $x_s(t) = x(t)i(t)$




$$\boxed{X_s(f)} = \int_{-\infty}^{\infty} \left[ x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta) \right] e^{-i2\pi ft} dt = \sum_{n=-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} \delta(t - n\Delta) dt \right] = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-i2\pi fn\Delta}$$

# Impluse Train Modulation (Continue)

$$X_s(f + r/\Delta) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi(f+r/\Delta)n\Delta} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta - i2\pi rn} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta} = X_s(f)$$

$r$: integer

**meaning?**

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}$$

Multiplying both sides of equation by $e^{-i2\pi fr\Delta}$ and integrating w.r.t $f$

$$\int_{-1/2\Delta}^{1/2\Delta} X_s(f)e^{-i2\pi fr\Delta}df = \int_{-1/2\Delta}^{1/2\Delta}\left[\sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}\right]e^{-i2\pi fr\Delta}df = \sum_{n=-\infty}^{\infty}\int_{-1/2\Delta}^{1/2\Delta}\left[x(n\Delta)e^{-i2\pi fn\Delta}\right]e^{-i2\pi fr\Delta}df$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta)\int_{-1/2\Delta}^{1/2\Delta}\left[e^{-i2\pi f(n-r)\Delta}\right]df = x(r\Delta)\frac{1}{\Delta}$$

Use L'Hôpital's rule

$$f_s = \frac{1}{\Delta}$$

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}$$

$$x(n\Delta) = \Delta\int_{-1/2\Delta}^{1/2\Delta} X_s(f)e^{i2\pi fn\Delta}df$$

# Impluse Train Modulation (Continue) - Proof

$$\sum_{n=-\infty}^{\infty} x(n\Delta) \int_{-1/2\Delta}^{1/2\Delta} \left[ e^{-i2\pi f(n-r)\Delta} \right] df = \sum_{n=-\infty}^{\infty} x(n\Delta) \left[ \frac{1}{-i2\pi(n-r)\Delta} e^{-i2\pi f(n-r)\Delta} \right]_{-1/2\Delta}^{1/2\Delta}$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta) \frac{1}{-i2\pi(n-r)\Delta} \left[ e^{-\pi(n-r)} - e^{\pi(n-r)} \right]$$

- $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta) \frac{1}{\pi(n-r)\Delta} \left[ e^{-\pi(n-r)} - e^{\pi(n-r)} \right] / 2i$$

Euler's formula.

$$= \sum_{n=-\infty}^{\infty} x(n\Delta) \frac{1}{\pi(n-r)\Delta} \sin(\pi(n-r))$$

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

L'Hôpital's rule

$$= \sum_{n=-\infty}^{\infty} x(n\Delta) \frac{1}{\Delta} \cdot \frac{\sin(\pi(n-r))}{\pi(n-r)}$$

- $\frac{\sin(\pi(n-r))}{\pi(n-r)} = 1$ when $n = r$

  " $= 0$ otherwise.

$$= x(r\Delta) \frac{1}{\Delta}$$

# Link Between Fourier Transform of a Discrete Sequence and Continuous Signal

**Fourier coefficients**

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-i2\pi nt/T_p}dt$$

**Impluse train**

$$i(t) = \sum_{n=-\infty}^{\infty} \delta(t-n\Delta) \qquad c_n = \frac{1}{\Delta}\int_{-1/\Delta}^{1/\Delta} \sum_{n=-\infty}^{\infty} \delta(t-n\Delta)\, e^{-i2\pi nt/\Delta}dt = \frac{1}{\Delta}\int_{-1/\Delta}^{1/\Delta} e^{-i2\pi nt/\Delta}dt = \frac{1}{\Delta}$$

**Fourier Transform of the impulse train**

$$I(f) = F\big(i(t)\big) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T_p}} \cdot e^{-i2\pi ft}dt$$

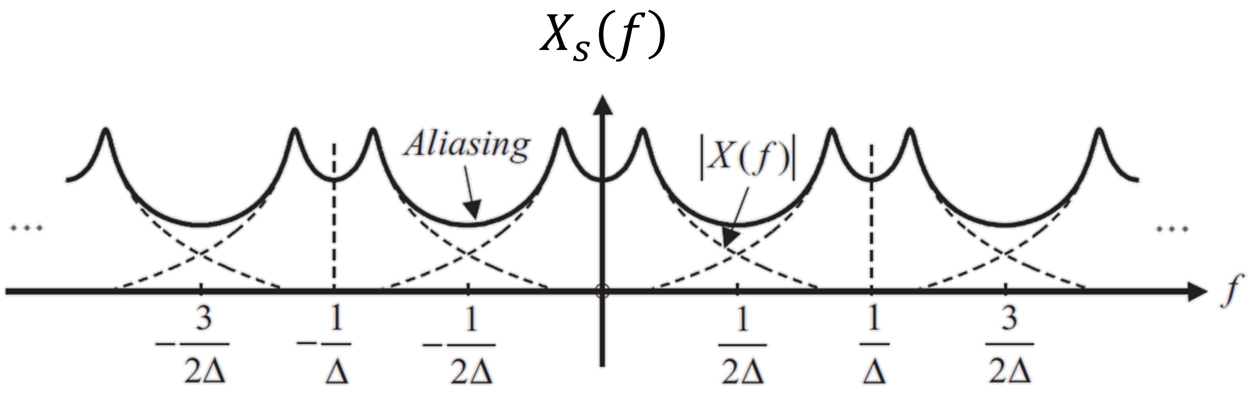
$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta} e^{\frac{i2\pi nt}{\Delta}} \cdot e^{-i2\pi ft}dt = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i2\pi nt}{\Delta}} \cdot e^{-i2\pi ft}\, dt = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi\left(f-\frac{n}{\Delta}\right)t}dt = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{\Delta})$$

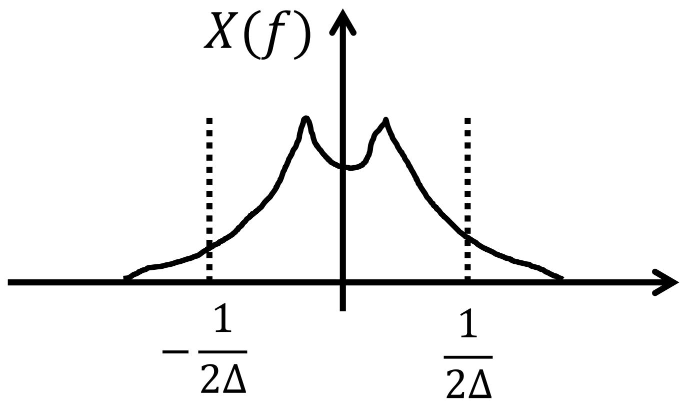
# Link Between Fourier Transform of a Discrete Sequence and Continuous Signal (Continue)

**Fourier Transform of a discrete sequence, $x_s(t)$**

$$X_s(f) = I(f) * X(f) = \int_{-\infty}^{\infty} I(g)X(f-g)dg = \int_{-\infty}^{\infty} \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg$$

$$= \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{\Delta}\right) = \frac{1}{\Delta}\left(\dots + X\left(f-\frac{2}{\Delta}\right) + X\left(f-\frac{1}{\Delta}\right) + X(f) + X\left(f+\frac{1}{\Delta}\right) + \ \dots\right)$$

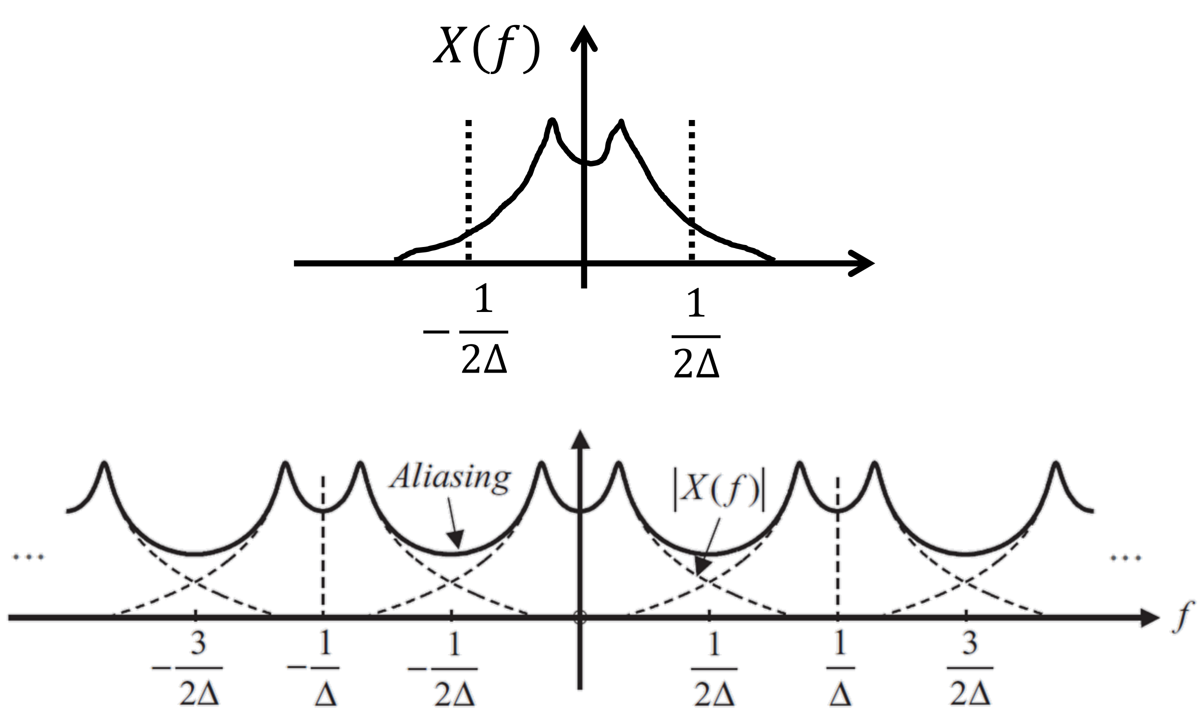


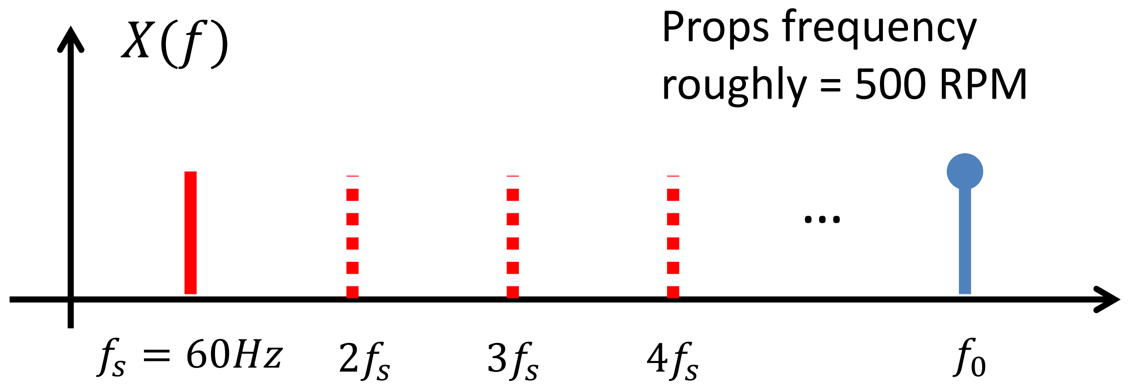



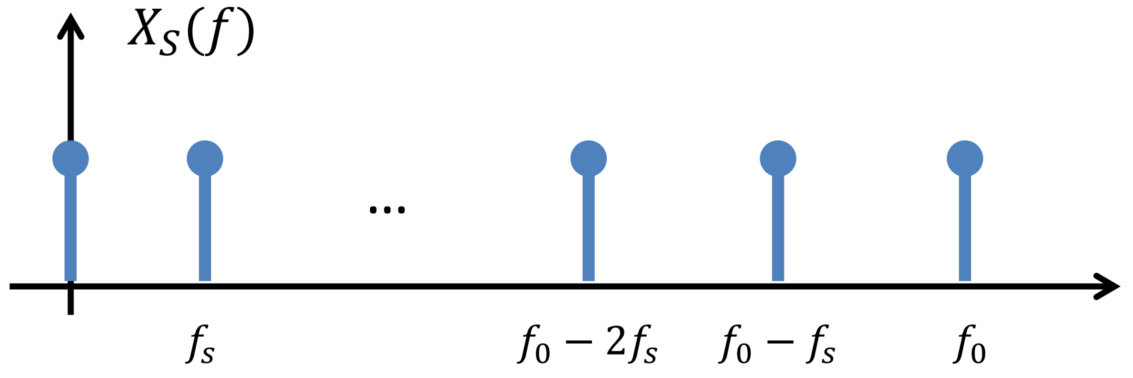

Q: What happened if Δ goes to a zero?

# Revisit: Spaceship Helicopter

$$X_s(f) = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{\Delta}\right) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$




$f_0 = mf_s$ *where m=integer*

The frequency is wrapped around to zero frequency!!

# Discrete Fourier Transform

So far we have considered sequences that run over the range $-\infty < n < \infty$(n integer). For the special case where the sequence is of finite length (i.e. non-zero for a finite number of values) an alternative Fourier representation is possible called the **discrete Fourier transform (DFT)**.

It turns out that the DFT is a Fourier representation of a finite length sequence and is itself a sequence rather than a continuous function of frequency, and it corresponds to samples, **equally spaced in frequency**, of the Fourier transform of the signal. The DFT is fundamental to many digital signal processing algorithms.

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi f n\Delta}$$

Continuous in frequency

Repeated every 1/ Δ

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}nk}$$

DFT of a finite (sampled) sequence $x(n\Delta)$

$$f = \frac{k}{N\Delta}$$

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz ($k$ integer)

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k)e^{i\frac{2\pi}{N}nk}$$

Inverse DFT

# Discrete Fourier Transform (Derivation)

Here is a Fourier transform of a discrete sequence $x(n\Delta)$

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi f n\Delta}$$

Let's drop the subscript s.

Let's consider only first N points and assume zero elsewhere.

$$X(f) = \sum_{n=0}^{N-1} x(n)e^{-i2\pi f n\Delta}$$

Note that this is still continuous in frequency

Let's evaluate this at frequencies $f = \frac{k}{N\Delta}$ where k is integer

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}kn}$$

**This is the DFT of a finite (sampled) sequence $x(n\Delta)$.**

The relationship between the Fourier transform of a sequence and the DFT of a finite length sequence can be expressed as

$$X(k) = X_s(f) \text{ evaluated at } f = \frac{k}{N\Delta} \text{ Hz (k is integer)}$$

# Fast Fourier Transform

A fast Fourier transform (FFT) is an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT). Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa. It manages to reduce the complexity of computing the DFT from $O(n^2)$, which arises if one simply applies the definition of DFT, to $O(n \, log \, n)$, where $n$ is the data size.

**FFT in Matlab**

fft

Fast Fourier transform

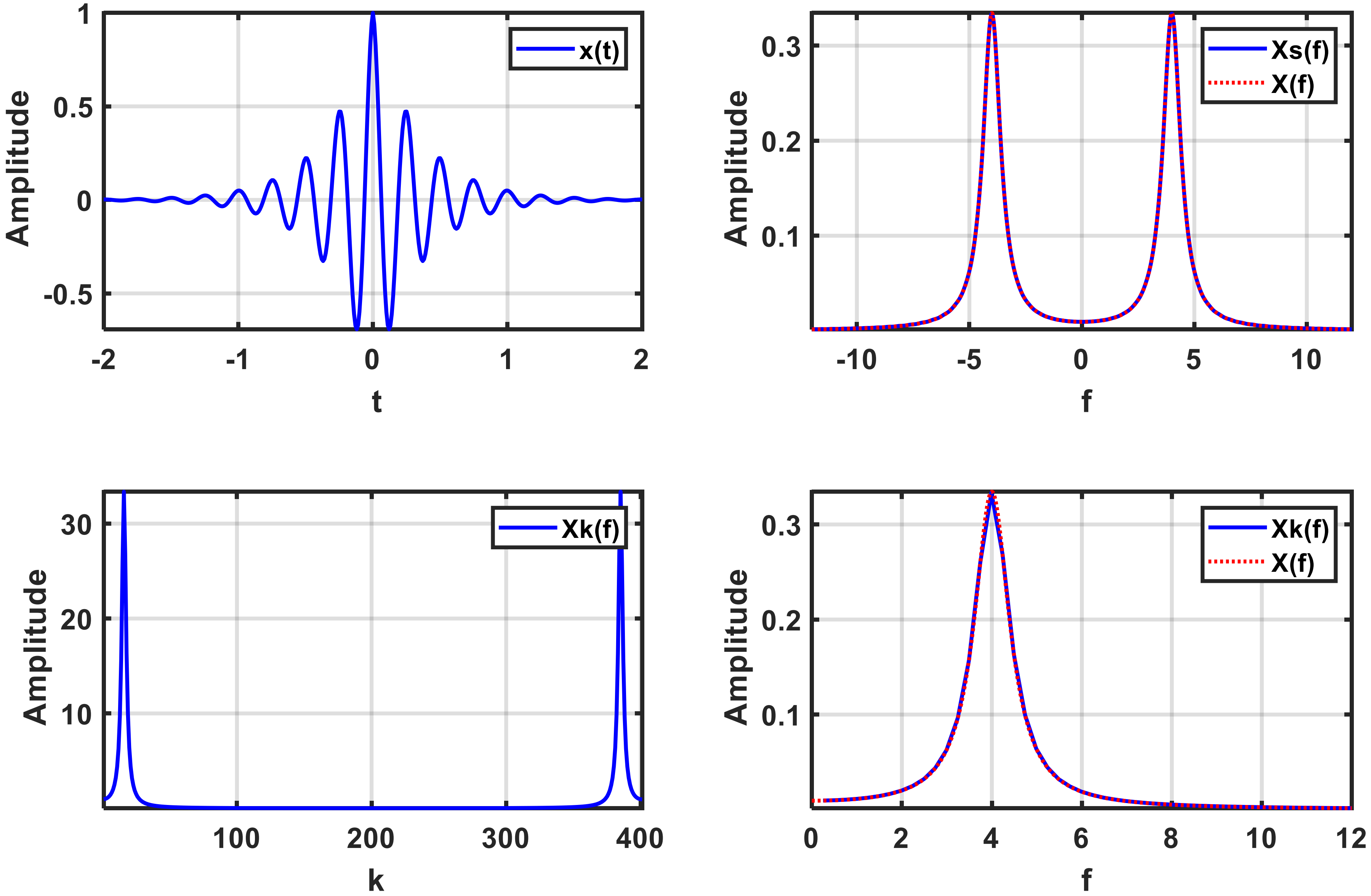
**Syntax**

```
Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)
```

Y = fft(X) computes the discrete Fourier transform (DFT) of X using a fast Fourier transform (FFT) algorithm

Y = fft(X,n) returns the n-point DFT. If no value is specified, Y is the same size as X.

# Discrete Fourier Transform (See a Tutorial)



# Discrete Fourier Transform (See a Tutorial): Continue

```
a = 3;
f0 = 4;

x1 = @(t) exp(-a.*abs(t));
x2 = @(t) cos(2*pi*f0*t);
x3 = @(t) x1(t).*x2(t);

fs = 100;
dt = 1/fs;
t = -2:dt:2-dt;

xn = x3(t);
nfft = numel(xn); % there is no zero-padding.
f = 1/(nfft*dt) * (0:nfft/2-1);

% from DFT equation
Xk_fun = @(k) sum(xn.*exp(-sqrt(-1)*2*pi/nfft*k*(1:nfft)),2);
Xk = Xk_fun([0:nfft/2-1]');

% use of a 'fft' function
X_fft = fft(xn, nfft);
```

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{N}kn}$$

$X(k) = X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz

```
plot(f,abs(Xk),'-b', 'linewidth', 2);hold on;
plot(f,abs(X_fft(1:nfft/2)),':r', 'linewidth', 2);hold off
legend('Xs(f)', 'X(f)'); axis tight;grid on;
ylabel('\bf Amplitude');
xlabel('\bf f');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```

# Zero-Padding

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{N}nk}$$

Y = fft(X, n) returns the n-point DFT. If no value is specified, Y is the same size as X.

- If X is a vector and the length of X is less than n, then X is padded with trailing zeros to length n.
- If X is a vector and the length of X is greater than n, then X is truncated to length n.

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz

frequency spacing $\frac{1}{N\Delta}$ Hz

# Zero-Padding (Continue)

$$\hat{x}(t) = x(n) \quad \text{for } 0 \le n \le N-1$$
$$= 0 \qquad \text{for } N \le n \le L-1$$

$$\hat{X}(k) = \sum_{n=0}^{L-1} \hat{x}(n) e^{-i\frac{2\pi}{L}nk} = \boxed{\sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{L}nk}}$$

- "Finer" spacing in the frequency domain. However, the zero padding does not increase the "true" resolution.
- Vibration problems, this can be used to obtain the fine detail near resonances

See the slide of "effect of data truncation"

# Example: Zero-Padding