Projective Geometry & Homography

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CIVE 497 – CIVE 700: Smart Structure Technology

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

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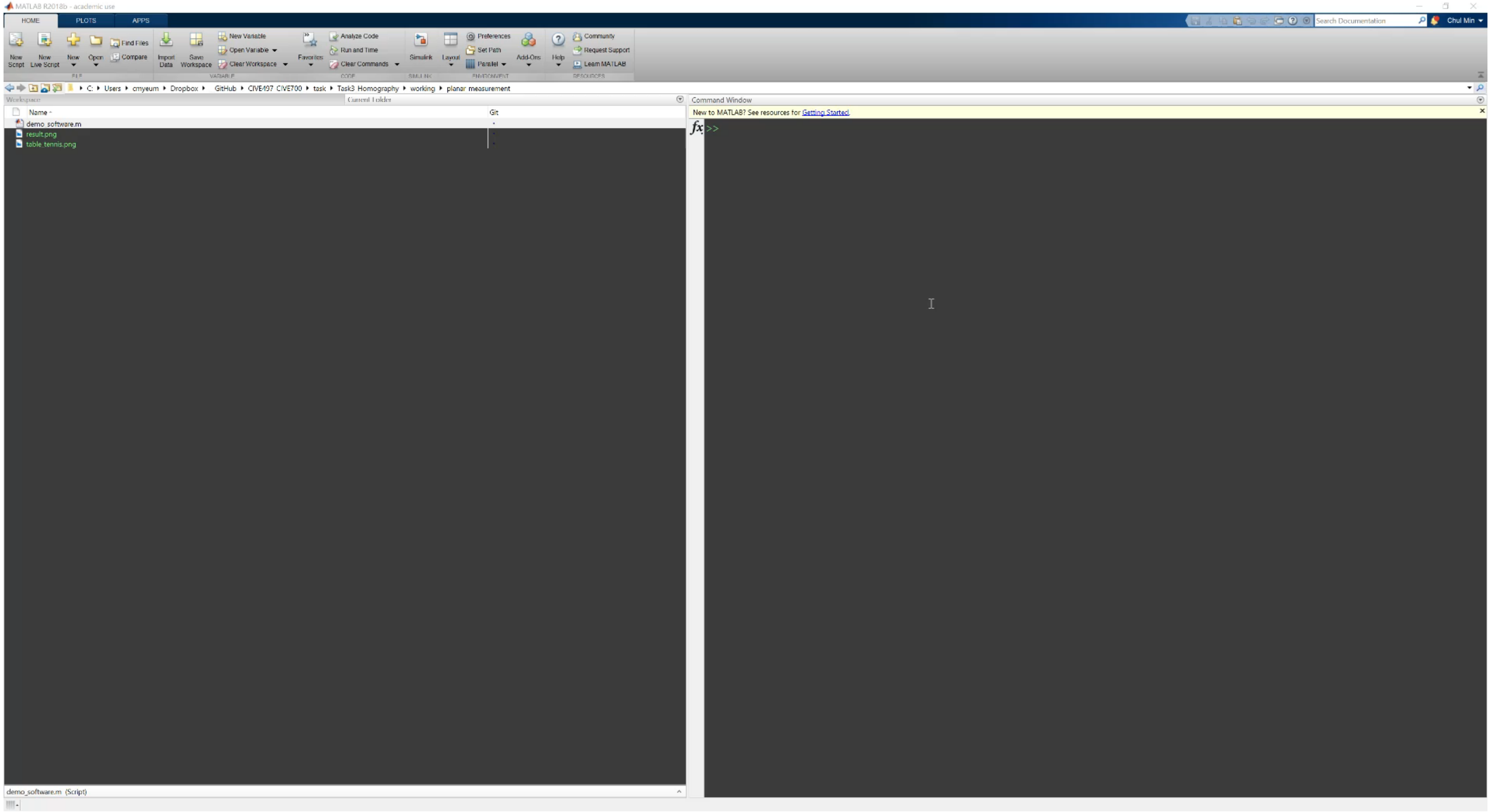
ON 3D-CameraMeasure App

Android: https://play.google.com/store/apps/details?id=com.potatotree.on3dcamerameasure

Measurement Demo

Measurement Demo (Continue)



Reference

We will study this topic using

ECE 661: Computer Vision (by Avinash Kak)

- Lecture 2: World 2D: Representing and Manipulating Points, Lines And Conics Using Homogeneous Coordinates
- Lecture 3: World 2D: Projective Transformations and Transformation Groups
- Lecture 4: Characterization of Distortions Caused by Projective Imaging and the Principle of Point/Line Duality
- Lecture 5: Estimating a Plane-to-Plane Homography with Angle-to-Angle and Point-to-Point Correspondences

Course website: https://engineering.purdue.edu/kak/computervision/ECE661Folder/

Lecture 2: World 2D: Representing and Manipulating Points, Lines And Conics Using Homogeneous Coordinates

2-1: Point in the Homogeneous Coordinate

An arbitrary homogeneous vector representative of a point is of the form $\mathbf{x} = (x_1, x_2, x_3)^T$, representing the point $(x_1/x_3, x_2/x_3)^T$ in $\mathbb{R}^2$.

Example) $\mathbf{x_1} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$ $\mathbf{x_2} = \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$ $\mathbf{x_3} = \begin{pmatrix} 5k \\ 3k \\ k \end{pmatrix}, k \neq 0$ **<u>up to a scale</u>**

$\mathbf{x_1}, \mathbf{x_2}$, and $\mathbf{x_3}$ indicate the same point of (5, 3) in $\mathbb{R}^2$

$\mathbb{R}^n$: n-dimension real coordinate system

2-1: Line in the Homogeneous Coordinate (HC)

$$ax + by + c = 0$$

Line equation in $\mathbb{R}^2$

$$\mathrm{l} = (a, b, c)^\intercal$$

Line representation in HC

Example) $\mathrm{l}_{\mathbf{1}} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$ $\quad \mathrm{l}_{\mathbf{2}} = \begin{pmatrix} 9 \\ 12 \\ 9 \end{pmatrix}$ $\quad \mathrm{l}_{\mathbf{3}} = \begin{pmatrix} 3k \\ 4k \\ 3k \end{pmatrix}, k \neq 0$

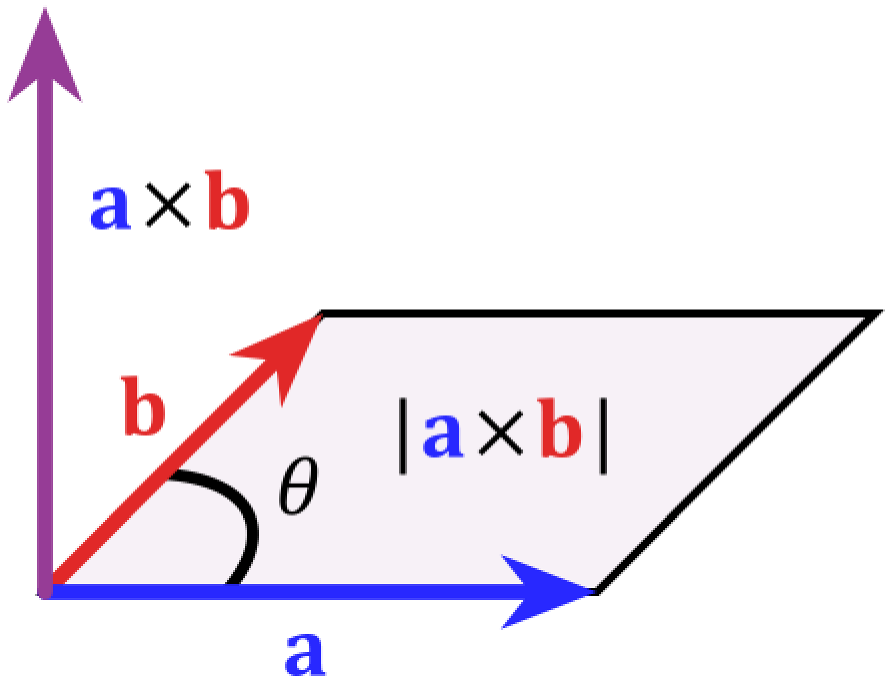
$\mathbf{x_1}, \mathbf{x_2}$, and $\mathbf{x_3}$ indicate the same line of $3x + 4y + 3 = 0$ in $\mathbb{R}^2$

2-1: Points and Lines in the Homogeneous Coordinate (HC)

$$ax + by + c = 0 \qquad\qquad (x_1, x_2, x_3)(a, b, c)^\mathsf{T} = 0$$

The point $\mathbf{x}$ lies on the line $\mathbf{l}$ if and only if $\mathbf{x}\mathbf{l}^\mathsf{T} = \mathbf{l}\mathbf{x}^\mathsf{T} = 0$.

Review: Cross Product



$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\ &= a_1b_1(\mathbf{i}\times\mathbf{i}) + a_1b_2(\mathbf{i}\times\mathbf{j}) + a_1b_3(\mathbf{i}\times\mathbf{k}) +\\ &\quad a_2b_1(\mathbf{j}\times\mathbf{i}) + a_2b_2(\mathbf{j}\times\mathbf{j}) + a_2b_3(\mathbf{j}\times\mathbf{k}) +\\ &\quad a_3b_1(\mathbf{k}\times\mathbf{i}) + a_3b_2(\mathbf{k}\times\mathbf{j}) + a_3b_3(\mathbf{k}\times\mathbf{k})\end{aligned}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = &\; -a_1b_1\mathbf{0} + a_1b_2\mathbf{k} - a_1b_3\mathbf{j}\\ &\; -a_2b_1\mathbf{k} - a_2b_2\mathbf{0} + a_2b_3\mathbf{i}\\ &\; +a_3b_1\mathbf{j} - a_3b_2\mathbf{i} - a_3b_3\mathbf{0}\\ = &\;(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

https://en.wikipedia.org/wiki/Cross_product

Example: Cross Product

$$x_1 = [1\ 3\ 1], \quad x_2 = [2\ 1\ 2]$$

Compute $x_1 \times x_2$

$$x_1 \times x_2 = \begin{vmatrix} i & j & k \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

```
>> x1 = [1 3 1];
>> x2 = [2 1 2];
>> cross(x1,x2)

ans =

     5     0    -5
```

2-1: Points and Lines in the Homogeneous Coordinate (HC)

Given any two lines $\mathbf{l_1} = (a_1, b_1, c_1)$ and $\mathbf{l_1} = (a_2, b_2, c_2)$, the point ($\mathbf{x}$) of intersection of the two lines :

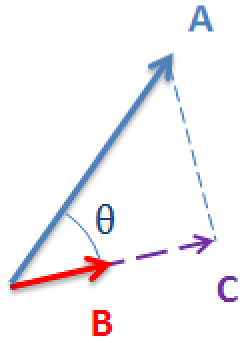
$$\mathbf{x} = \mathrm{l}_1 \times \mathrm{l}_2$$

Given any two points $\mathbf{x_1} = (x_1, y_1, z_1)$ and $\mathbf{x_2} = (x_2, y_2, z_2)$, the line ($\mathbf{l}$) that passes through the two points :

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

2-2: Prove the Relationship using the Triple Scalar Identity

Dot product

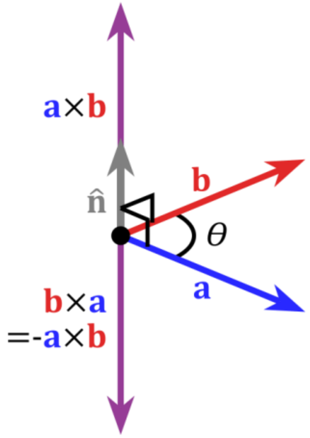


$$A \cdot B = |A||B|\cos(\theta)$$

if the magnitude of B is 1, then…

$$C = A \cdot B = |A|\cos(\theta)$$

Cross product



$$\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0$$

$$\mathbf{l_1 x = l_2 x = 0}$$

$$\mathbf{l_1 \cdot (l_1 \times l_2) = l_2 \cdot (l_1 \times l_2) = 0}$$

$$\mathbf{x = l_1 \times l_2}$$

Quiz 1

Q1: When $a = [2\ 4\ 2], b = [0\ 5\ 5]$, compute $a \times \mathrm{b}$

Q2: Line passes through two points (0,1) and (1,2)

Two-point form [edit]

Given two different points (x_1, y_1) and (x_2, y_2), there is exactly one line that passes through them. There are several ways to write a linear equation of is line.

If $x_1 \neq x_2$, the slope of the line is $\frac{y_2 - y_1}{x_2 - x_1}$. Thus, a point-slope form is[3]

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Quiz 1

Two-point form [edit]

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$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

```
pt1 = [0 1];
pt2 = [1 2];

slope = (pt2(2)-pt1(2))/(pt2(1)-pt1(1));
l1 = [slope -1 -slope*pt1(1)+pt1(2)]
l2 = cross([pt1 1], [pt2 1])
```

Given any two points $\mathbf{x_1} = (x_1, y_1, z_1)$ and $\mathbf{x_2} = (x_2, y_2, z_2)$, the line ($\mathbf{l}$) that passes through the two points :

$$\mathbf{l} = \mathbf{x_1} \times \mathbf{x_2}$$

```
l1 = 1x3
     1    -1     1

l2 = 1x3
    -1     1    -1
```

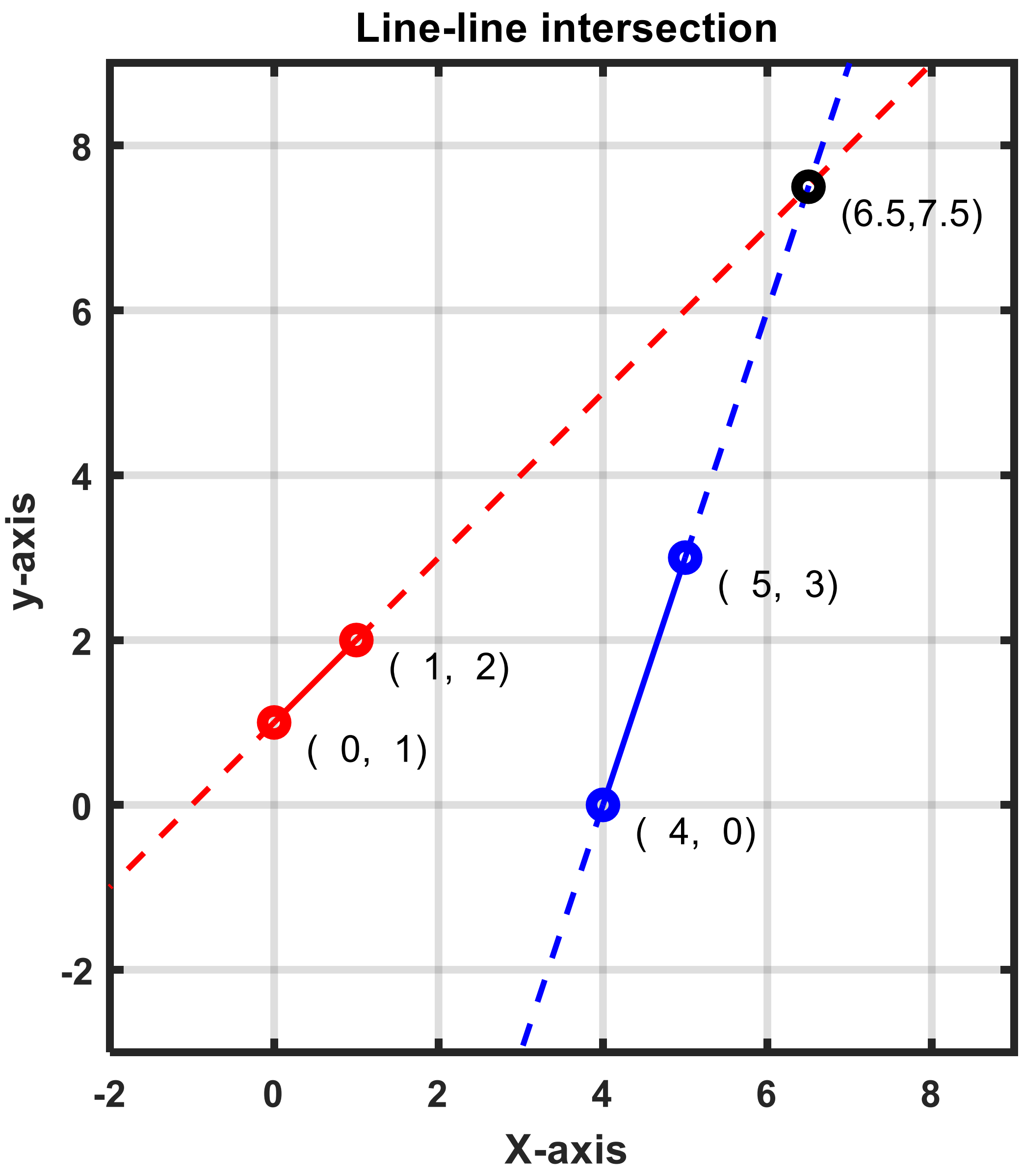
https://en.wikipedia.org/wiki/Linear_equation

Quiz 2

Intersection point (p_x, p_y) of two lines l_1 and l_2

l_1 passes through two distinct points, (0,1) and (1,2)
l_2 passes through two distinct points, (4,0) and (5,3)

Quiz 2: Graph



Line-line intersection
(6.5,7.5)
(5, 3)
(1, 2)
(0, 1)
(4, 0)
y-axis
X-axis
8
6
4
2
0
-2
-2
0
2
4
6
8

Quiz 2: Method 1

Given two points on each line [edit]

First we consider the intersection of two lines L_1 and L_2 in 2-dimensional space, with line L_1 being defined by two distinct points (x_1, y_1) and (x_2, y_2), and line L_2 being defined by two distinct points (x_3, y_3) and (x_4, y_4).[1]

```
denom = (0-1)*(0-3)-(1-2)*(4-5);
numerx = (0*2-1*1)*(4-5)-(0-1)*(4*3-0*3);
numery = (0*2-1*1)*(0-3)-(1-2)*(4*3-0*3);

px = numerx/denom;
py = numery/denom;
```

```
(px,py) = (6.50, 7.50)
```

$$(P_x, P_y) = \left(\frac{(x_1 y_2 - y_1 x_2)(x_3 - x_4) - (x_1 - x_2)(x_3 y_4 - y_3 x_4)}{(x_1 - x_2)(y_3 - y_4) - (y_1 - y_2)(x_3 - x_4)}, \frac{(x_1 y_2 - y_1 x_2)(y_3 - y_4) - (y_1 - y_2)(x_3 y_4 - y_3 x_4)}{(x_1 - x_2)(y_3 - y_4) - (y_1 - y_2)(x_3 - x_4)} \right)$$

https://en.wikipedia.org/wiki/Line%E2%80%93line_intersection

Quiz 2: Method 2

Given any two lines $\mathbf{l_1} = (a_1, b_1, c_1)$ and $\mathbf{l_1} = (a_2, b_2, c_2)$, the point ($\mathbf{x}$) of intersection of the two lines :

$$\mathbf{x} = \mathbf{l_1} \times \mathbf{l_2}$$

Given any two points $\mathbf{x_1} = (x_1, y_1, z_1)$ and $\mathbf{x_2} = (x_2, y_2, z_2)$, the line ($\mathbf{l}$) that passes through the two points :

$$\mathbf{l} = \mathbf{x_1} \times \mathbf{x_2}$$

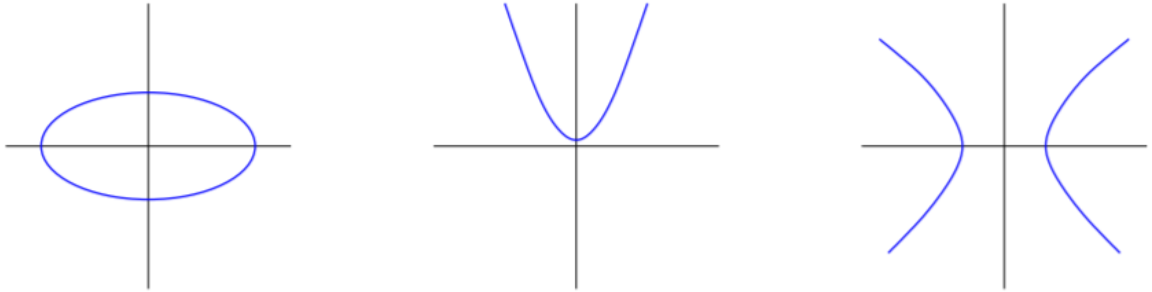
```
l1 = cross([0,1,1]',[1,2,1]');
l2 = cross([4,0,1]',[5,3,1]');
x = cross(l1,l2);
px = x(1)/x(3);
py = x(2)/x(3);
```

```
>> [px py]

ans =

    6.5000    7.5000
```

2-3: Conic



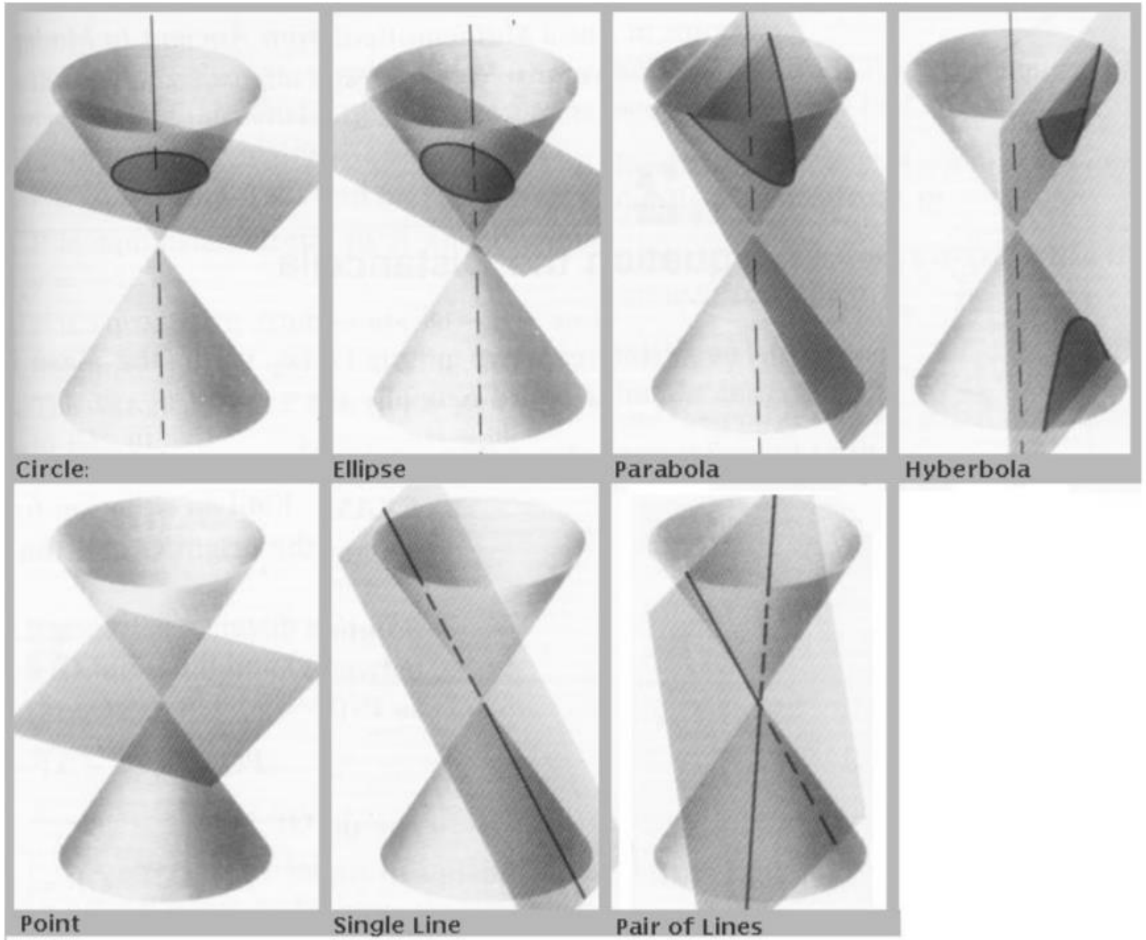
$(x/2)^2 + y^2 = 1$ ellipse

$y = x^2$ parabola

$x^2 - y^2 = 1$ hyperbola

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

A conic has five degrees of freedom in general



x and y can be represented by the conic equation

2-3: Conics in HC

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

(x, y) in $\mathbb{R}^2$

$$a\left(\frac{x_1}{x_3}\right)^2 + b\frac{x_1}{x_3}\frac{x_2}{x_3} + c\left(\frac{x_2}{x_3}\right)^2 + d\left(\frac{x_1}{x_3}\right) + e\left(\frac{x_2}{x_3}\right) + f$$

(x_1, x_2, x_3) in $\mathbb{HC}$

$$= ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

2-3: Conic in HC (Continue)

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

(x_1, x_2, x_3) in $\mathbb{HC}$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\,\mathbf{x} = \mathbf{0}$$

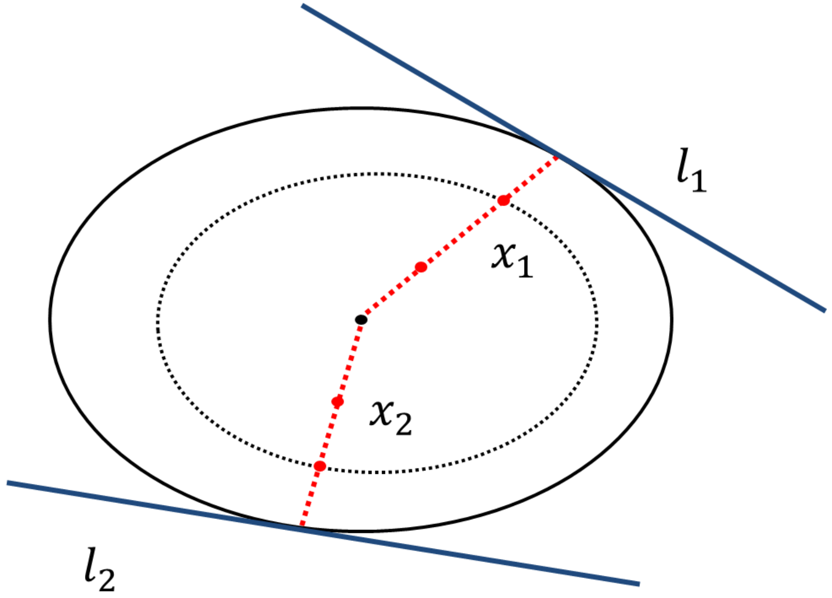
where $\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$

C is the HC representation of a conic

2-4: Conic Property

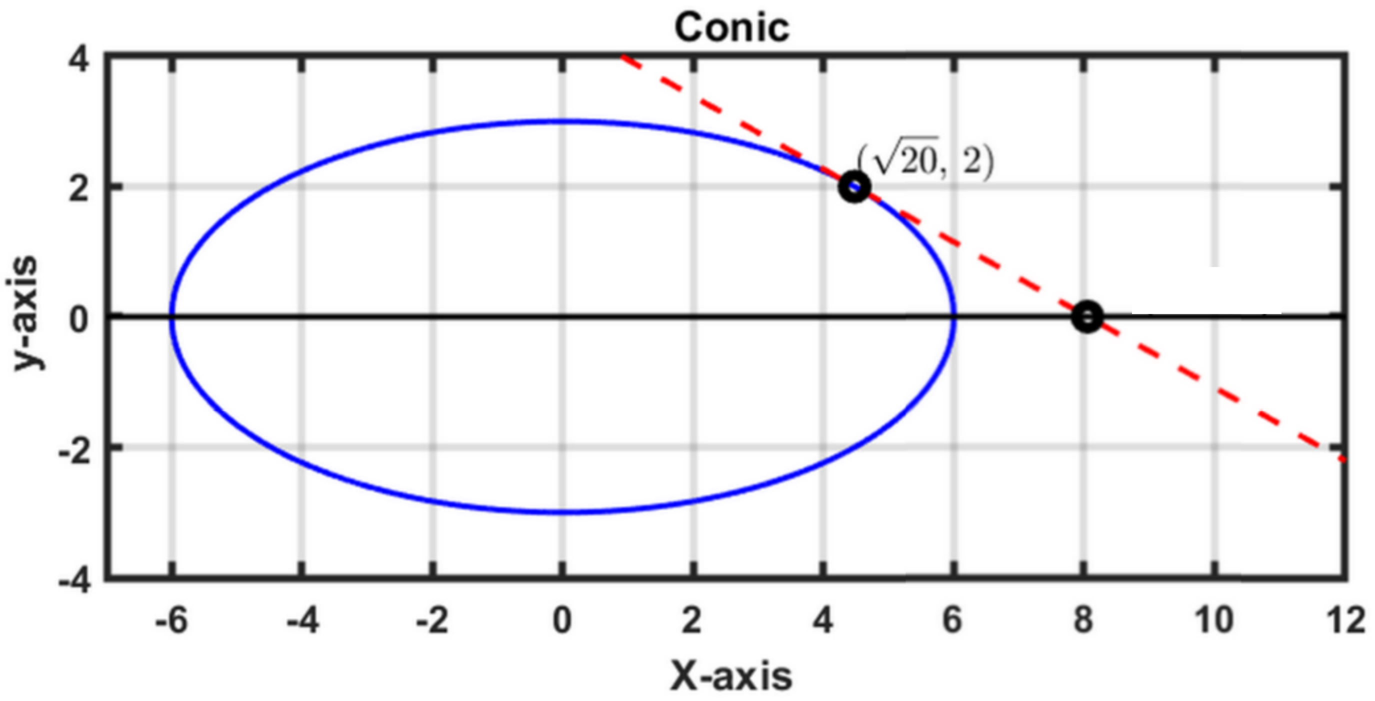
The HC representation of a conic gives us compact formulas for the tangent lines to a conic.

$$\mathbf{x}^\top \mathbf{l} = \mathbf{x}^\top \mathbf{C}\,\mathbf{x} = 0 \qquad\qquad \mathbf{l} = \mathbf{C}\,\mathbf{x}$$



Quiz3

Q2: Given a conic $\frac{x^2}{6^2} + \frac{y^2}{3^2} - 1 = 0$ (ellipse), compute a tagential line to this conic at ($\sqrt{20}\,, 2)$ and its intersection point with x-axis, $(p_x,\ p_y)$.



Quiz3: Solution

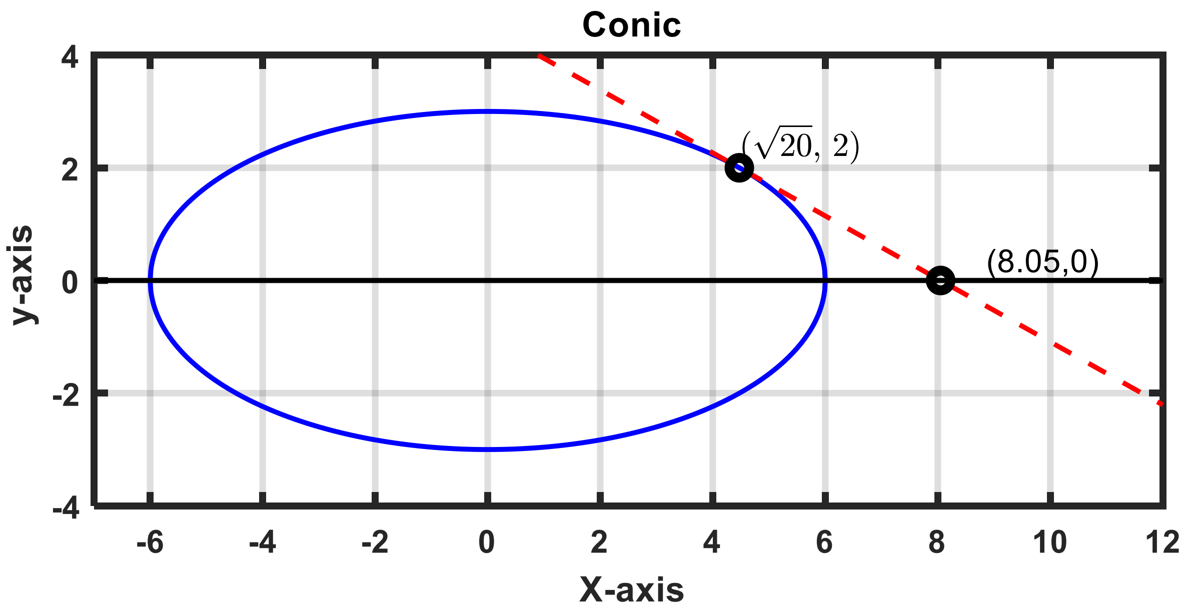
```
a = 1/6/6;
b = 0;
c = 1/3/3;
d = 0;
e = 0;
f = -1;

conic = [a b/2 d/2; b/2 c e/2; ...
d/2 e/2 f];

lmn = conic*[sqrt(20);2;1]

lxa  = [0 1 0]; % x-axis: y=0

x = cross(lmn, lxa);
px = x(1)/x(3);
py = x(2)/x(3);
```




```
(px,py) = (8.05, -0.00)
```

Summary

1. An arbitrary homogeneous vector representative of a point is of the form $\mathbf{x} = (x_1, x_2, x_3)^T$, representing the point $(x_1/x_3, x_2/x_3)^T$ in $\mathbb{R}^2$.

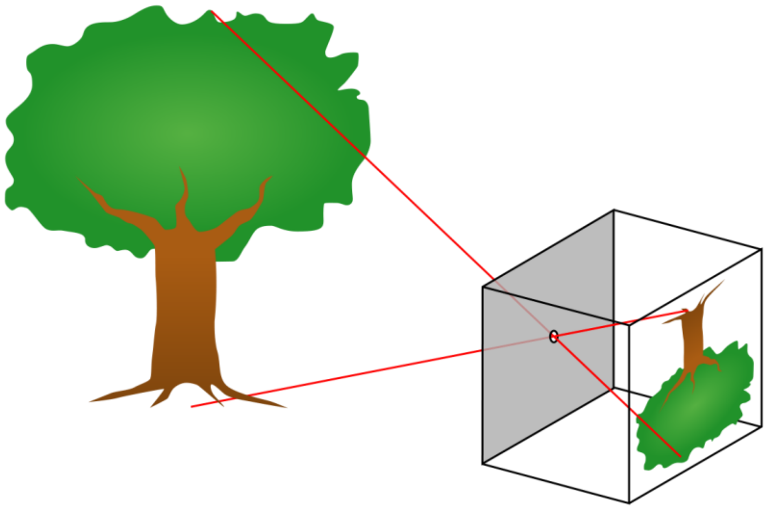
2. Line equation, $ax + by + c = 0$, in $\mathbb{R}^2$ is represented as $\mathrm{l} = (a, b, c)^\top$ in the homogeneous coordinate.

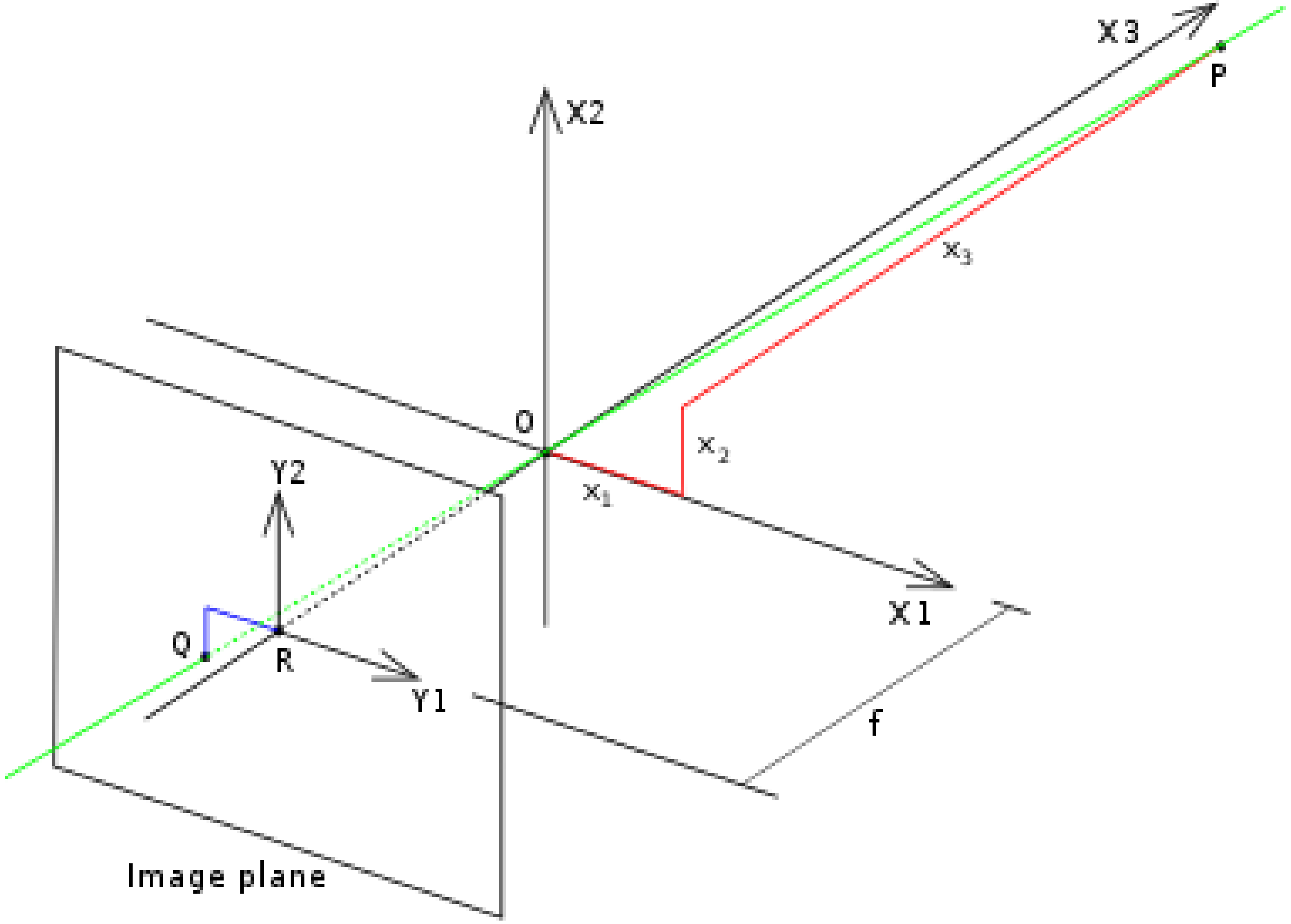
3. A conic, $ax^2 + bxy + cy^2 + dx + ey + f = 0$, in $\mathbb{R}^2$ become a 3x3 matrix, **C**, in the homogeneous coordinate,

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

Lecture 3: World 2D: Projective Transformations and Transformation Groups

Pinhole Camera Model

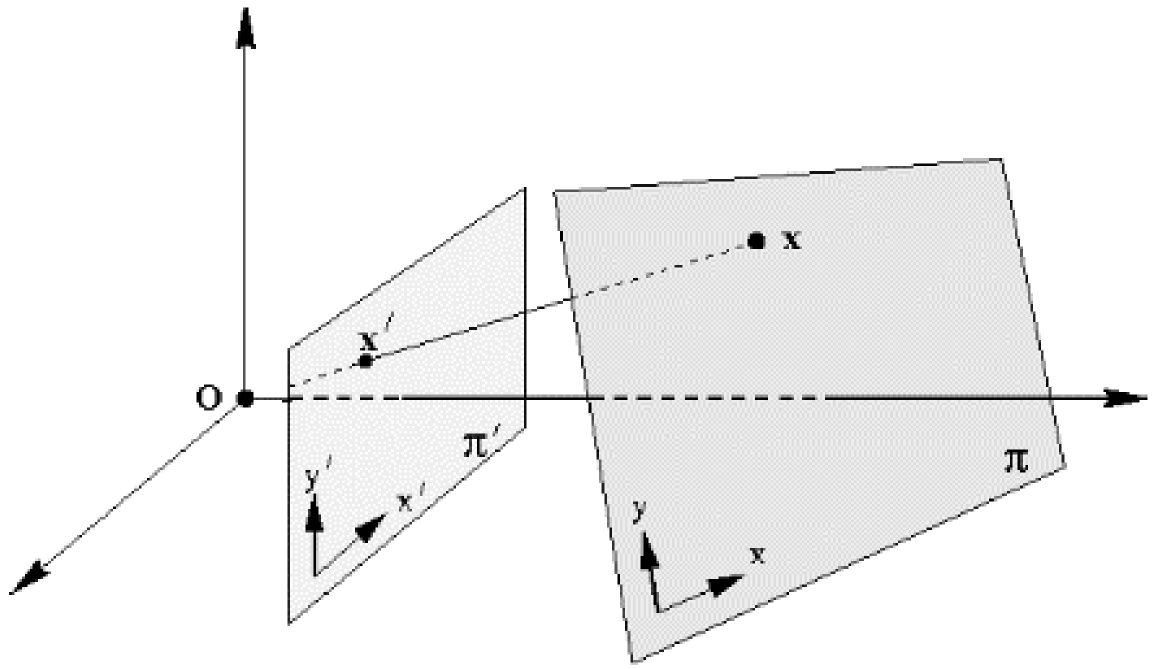




https://en.wikipedia.org/wiki/Pinhole_camera_model

3-1: Projective Transformation

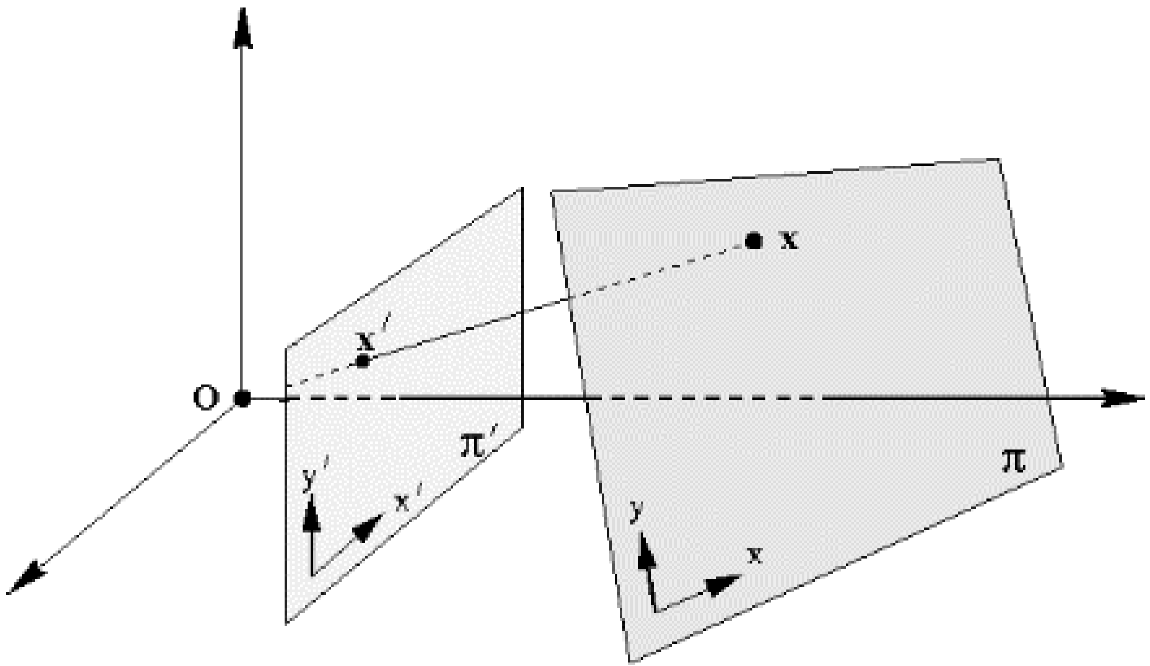
A **planar** projective transformation (homography) is a **linear** transformation on homogeneous 3-vectors, the transformation being represented by a **non-singular** 3x3 matrix H, as in



$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

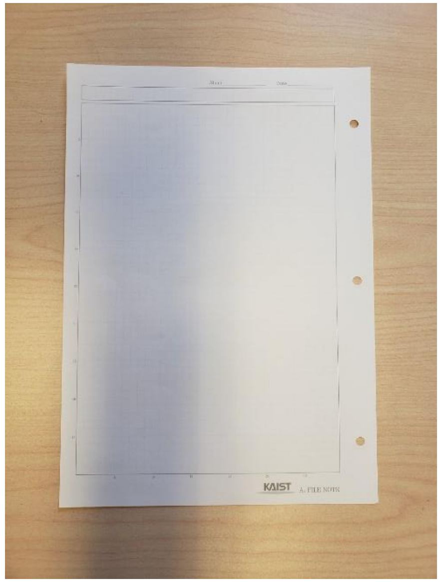
$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

3-1: Projective Transformation (continue)



In planar perspective transformation, all rays that join a scene point **x** with its corresponding image point **x**’ must pass through the same point that is referred to as the center of projection or the focal center. Obviously , an image formed with a planar perspective transformation will, in general, suffer from distortions including projective, affine, and similarity.

Example: Perspective Distortion



3-1: Property of a Homography

<u>**It always maps a straight line to a straight line.**</u>

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{l}'^{\top}\mathbf{x}' = \mathbf{l}'^{\top}\mathbf{H}\mathbf{x} = (\mathbf{l}'^{\top}\mathbf{H})\mathbf{x} = \mathbf{l}\mathbf{x} = 0$$

$$\mathbf{l}'^{\top}\mathbf{H} = \mathbf{l} \qquad \mathbf{l}' = \mathbf{H}^{-\top}\mathbf{l}$$

3-2: Hierarchy of Transformation

Projective group

subgroup of

Affine group

subgroup of

Similarity group

subgroup of

Euclidean group

(correct) A similarity transform is an affine transform.

(incorrect) ~~A projective transform is an affine transform.~~

(correct) An Euclidean transform is an affine transform.

(correct) An affine transform is an projective transform.

(incorrect) ~~An similarity transform is an Euclidean transform.~~

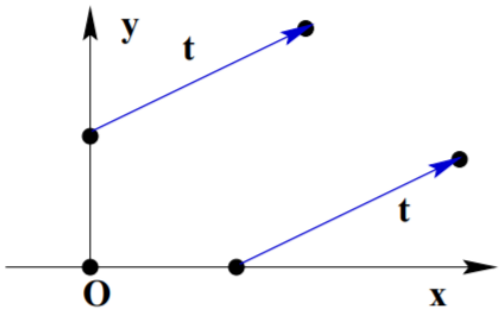
3-4: Geometric Transformation (Euclidean Transformation)

Rigid body motions

1. *Translation* — 2 dof in 2D

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

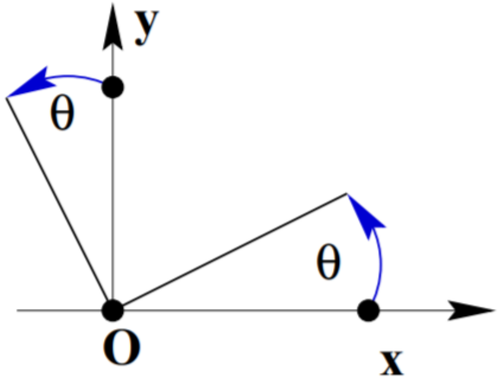


$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

2. *Rotation* — 1 dof in 2D

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$



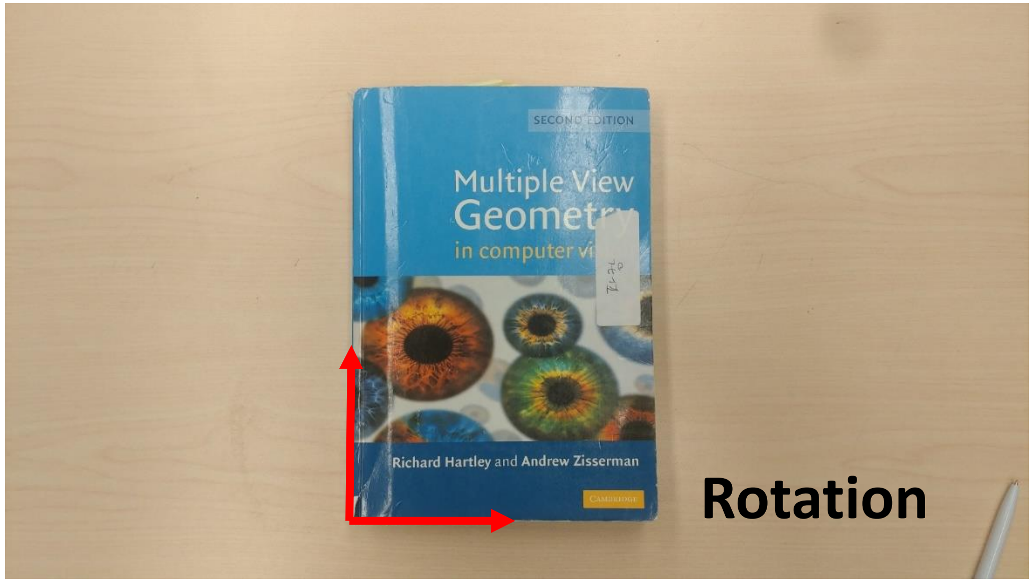
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

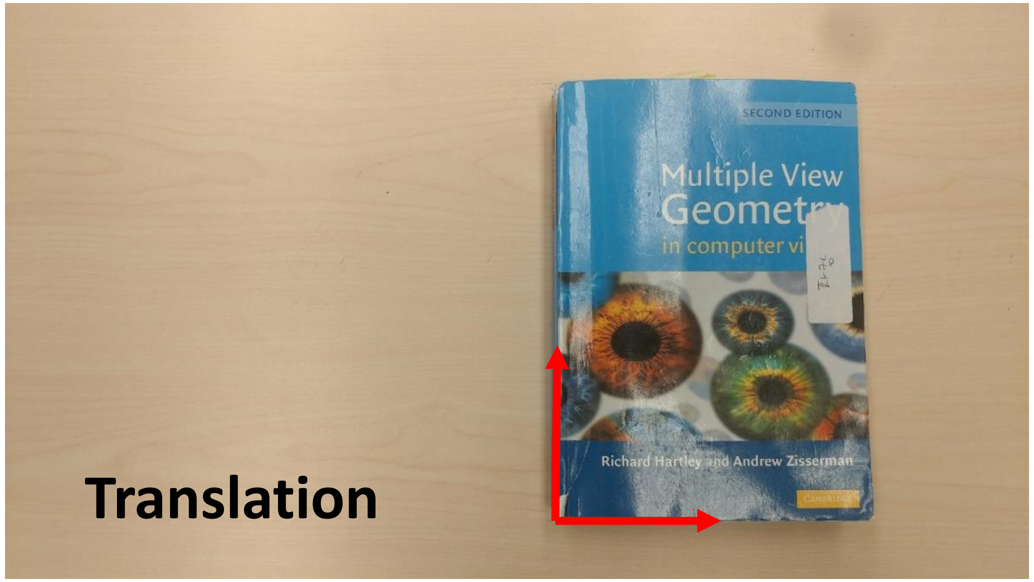
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} cos\theta & -sin\theta & t_x \\ sin\theta & cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

See tutorials

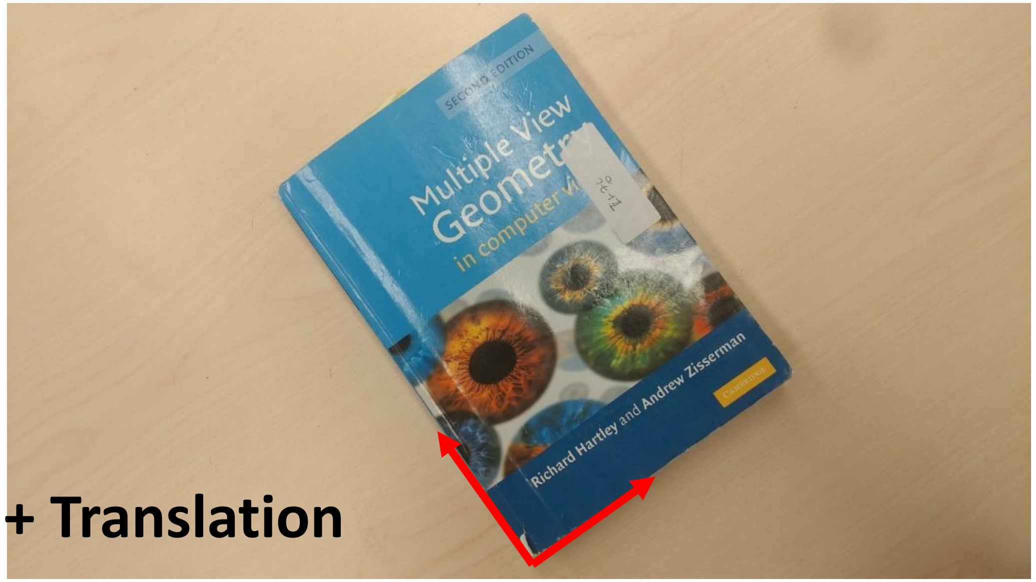
Example



Rotation



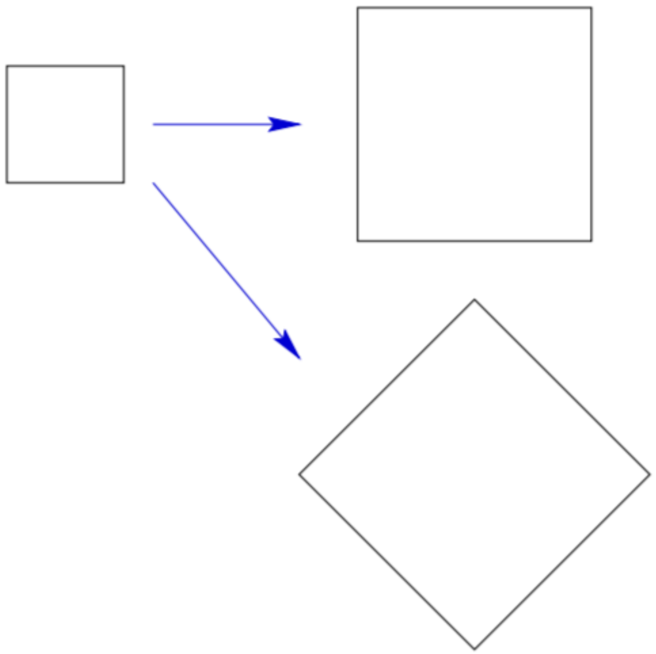
Translation



Rotation + Translation

3-4: Geometric Transformation (Similarity Transformation)

Preserve angles and ratios of lengths => Preserve "shape" (isotropic scale)



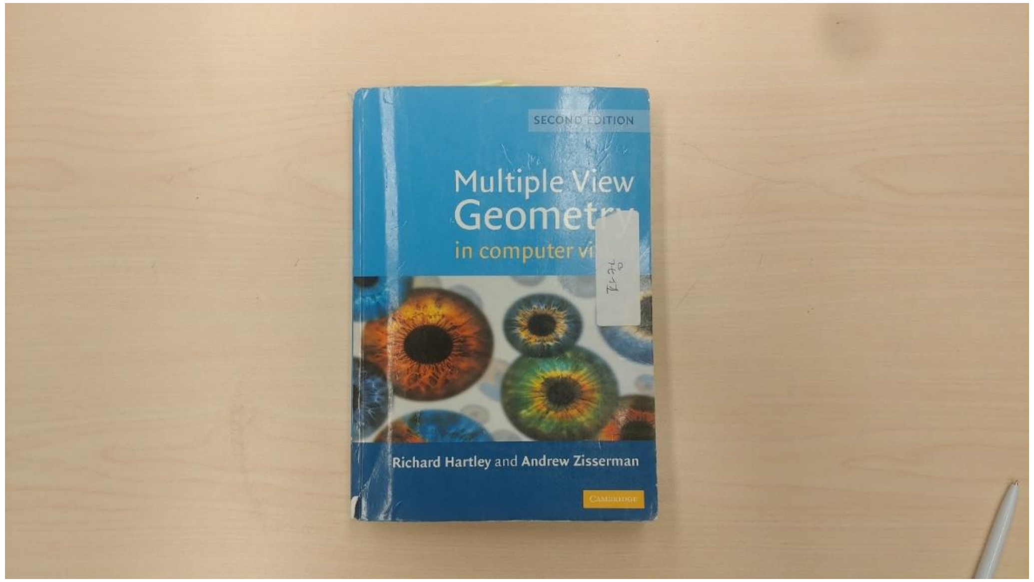
$$\mathbf{x}' = s\mathrm{R}\mathbf{x} + \mathbf{t}$$

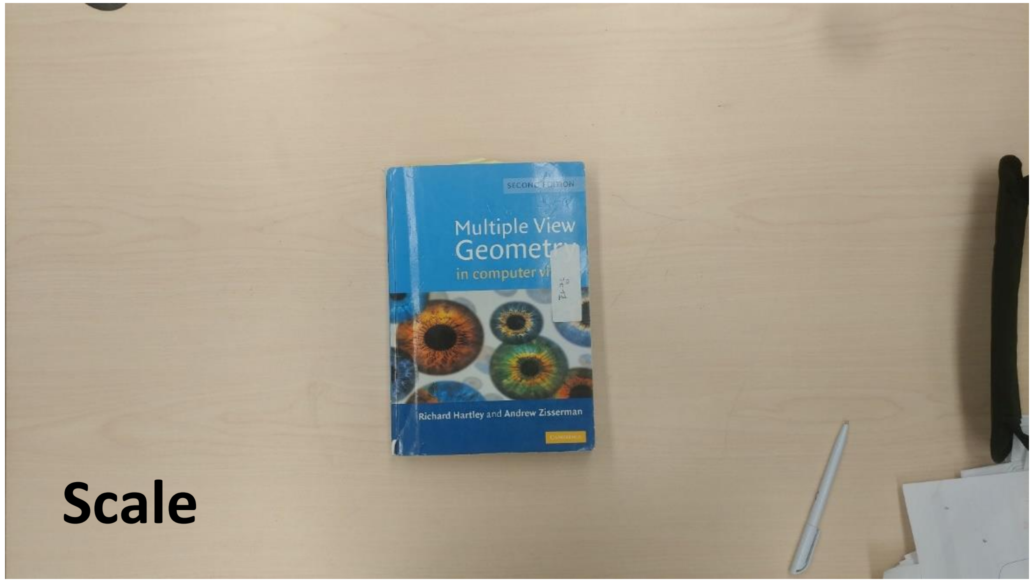
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} s * cos\theta & -s * sin\theta & t_x \\ s * sin\theta & s * cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

See tutorials

Example


SECOND EDITION
Multiple View
Richard Hartley and Andrew Zisserman


Multiple View
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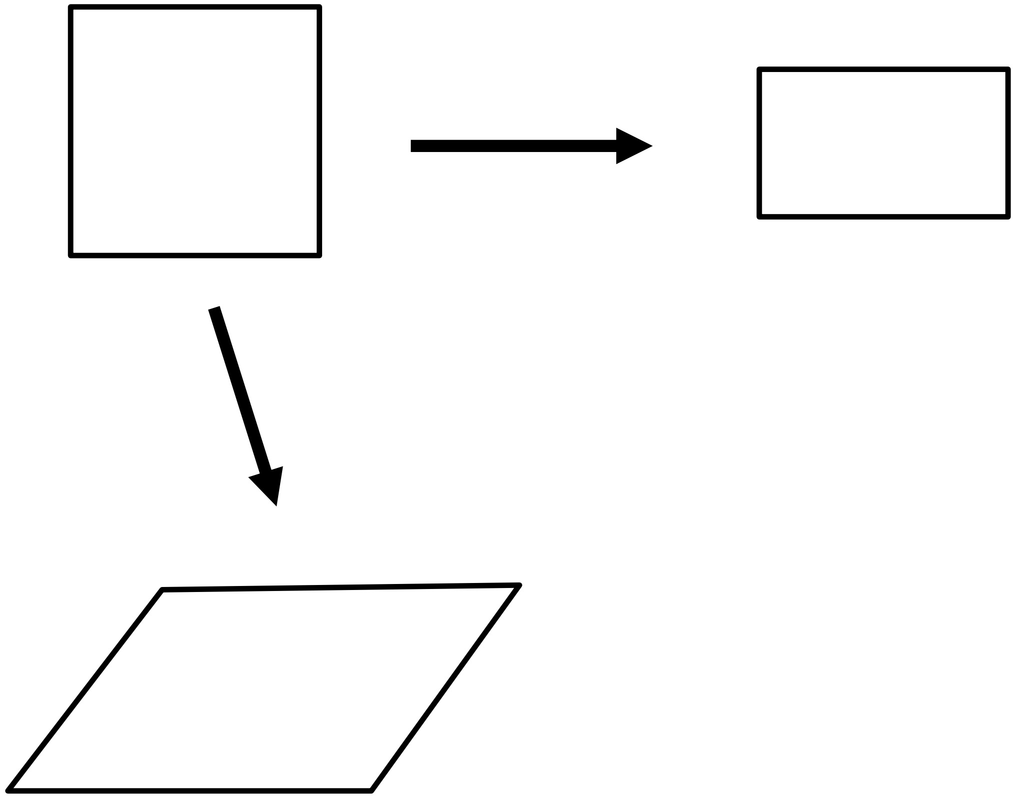
Scale

3-3 & 3-4: Geometric Transformation (Affine Transformation)

Keep parallel lines parallel.

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$



3-4: Algebraic Understanding on Affine Transformation

Singular value decomposition (SVD)

$$A=UDV^{T}=(UV^{T})(VDV^{T})=R(\theta)R(-\phi)DR(\phi)$$

U, V: Orthonormal matrix
D: Diagonal matrix

Product of two orthonormal matrices become an orthonormal matrix

$$Q^T Q = I$$

$$R^T R = I,$$

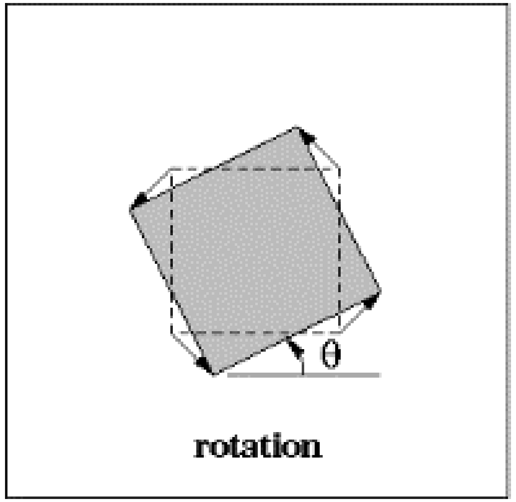
$$(QR)^T(QR) = R^T(Q^T Q)R = R^T R = I.$$

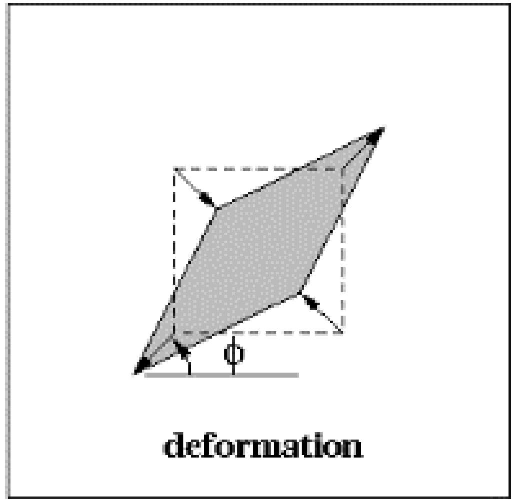
Every orthonormal matrix having determinant 1 acts as a rotation.

3-4: Algebraic Understanding on Affine Transformation

$$A = UDV^T = (UV^T)(VDV^T) = R(\theta)R(\phi)\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}R(\phi)$$

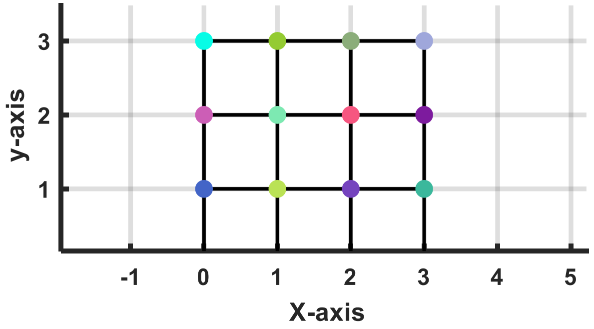
The affine matrix A is seen to be the concatenation of a rotation (by ϕ); a scaling by λ_1 and λ_2 respectively in the (rotated) x and y directions ; a rotation back (by $-\phi$); and finally another rotation (by θ). The only “new” geometry, compared to a similarity, is the non-isotropic scaling.



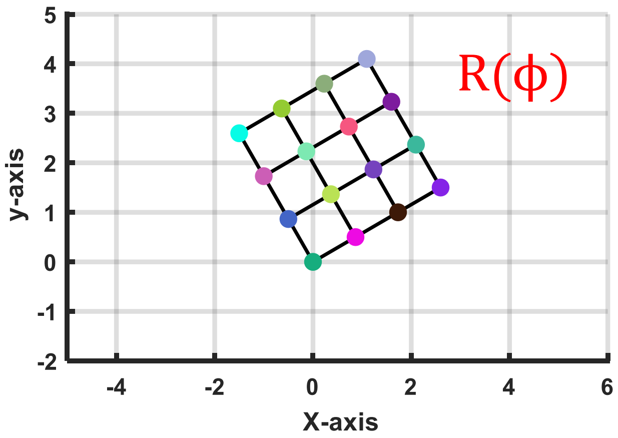


See tutorials

Example: Decomposition of Affine Transformation

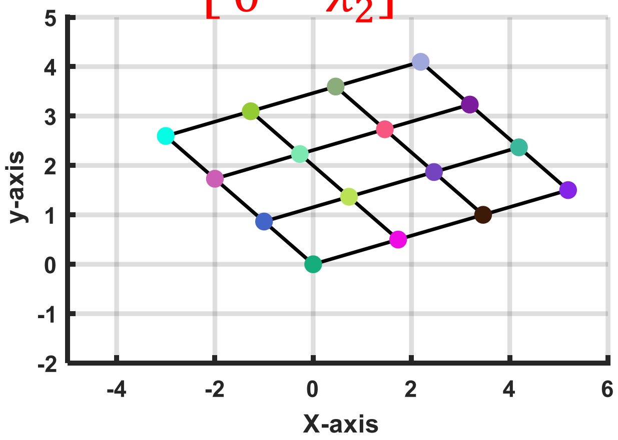


$$A = R(\theta) R(\phi) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R(\phi)$$

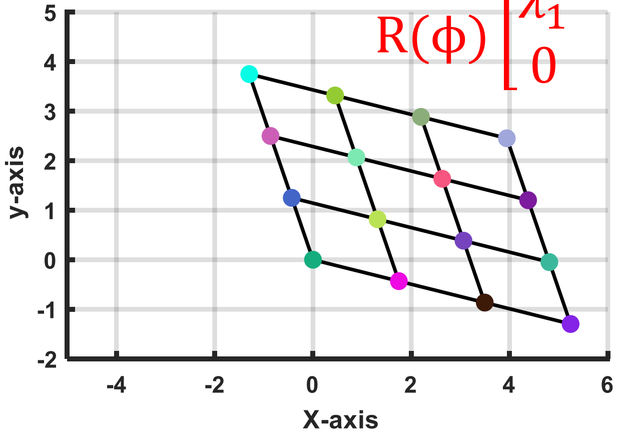


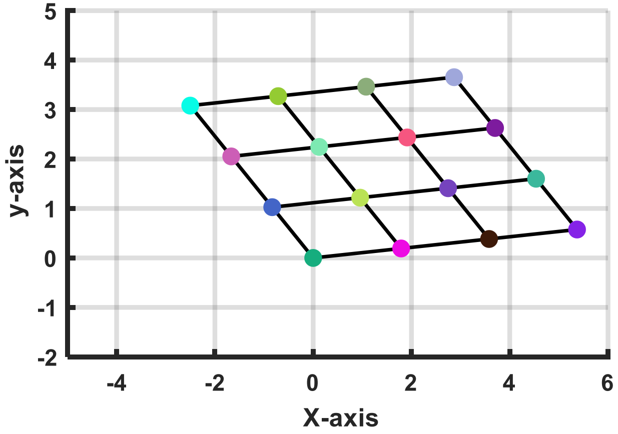
$R(\phi)$

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R(\phi)$



$R(\phi) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R(\phi)$

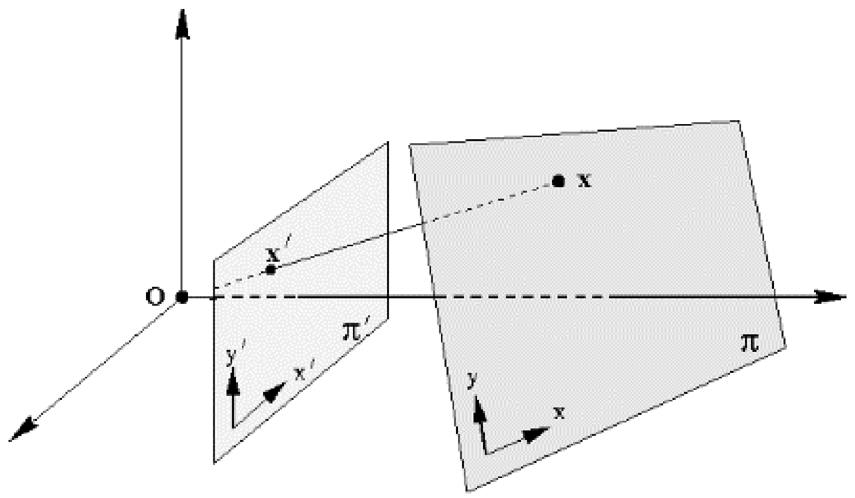


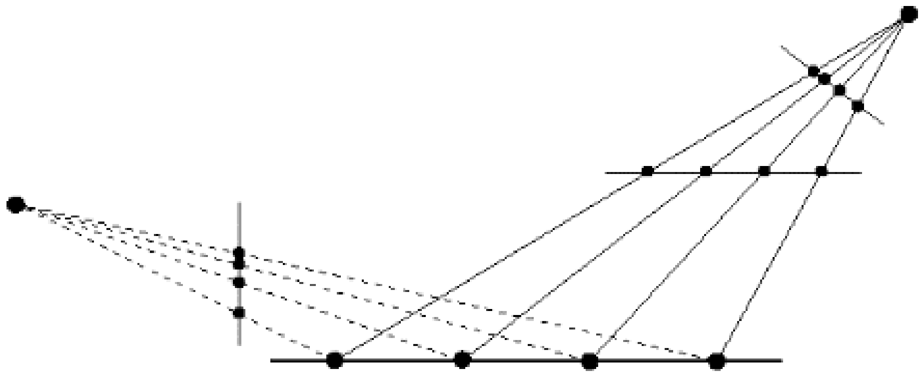


$R(\theta) R(\phi) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R(\phi)$

3-4: Geometric Transformation (Projective Transformation)

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$





$Cross\ ratio$

Point Correspondences for Estimating a Homography

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{pmatrix} xh_{11} + yh_{12} + h_{13} \\ xh_{21} + yh_{22} + h_{23} \\ xh_{31} + yh_{32} + h_{33} \end{pmatrix}$$

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -xx' & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -xy' & -y'y & -y' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0$$

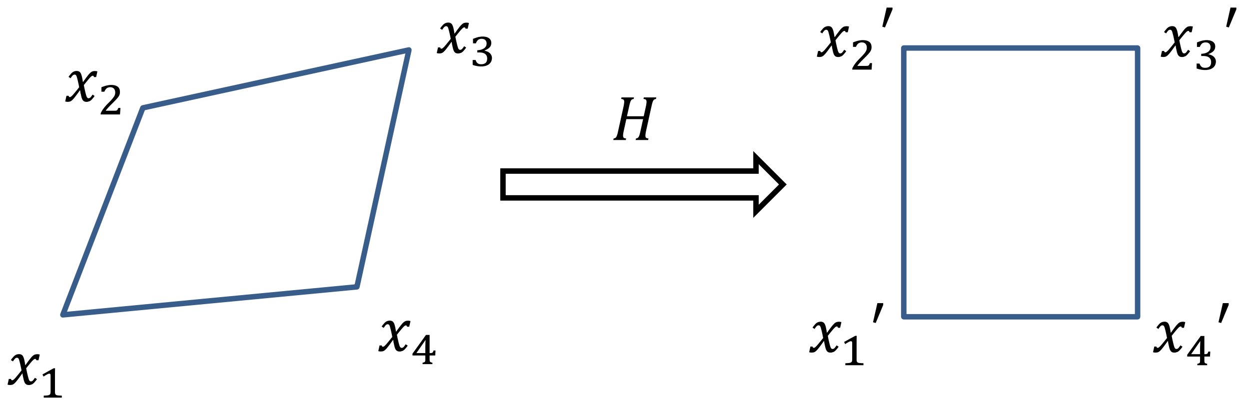
$$x' = \frac{xh_{11} + yh_{12} + h_{13}}{xh_{31} + yh_{32} + h_{33}}$$

$$y = \frac{xh_{21} + yh_{22} + h_{23}}{xh_{31} + yh_{32} + h_{33}}$$

Example

$$3x + 2y + z = 3 \qquad \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3$$

Point Correspondences for Estimating a Homography (Continue)



$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x_1' & -x_1'y_1 & -x_1' \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y_1' & -y_1'y_1 & -y_1' \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x_2' & -x_2'y_2 & -x_2' \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y_2' & -y_2'y_2 & -y_2' \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x_3' & -x_3'y_3 & -x_3' \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y_3' & -y_3'y_3 & -y_3' \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x_4' & -x_4'y_4 & -x_4' \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y_4' & -y_4'y_4 & -y_4' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0$$