Signal Processing II (Fourier Transform)

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Fourier Series of a Periodic Signal

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t) e^{-i2\pi nt/T_p} dt$$

Let's represent the Fourier series of a periodic signal, where the interval of integration is defined from $-T_p/2$ to $T_p/2$ for convenience:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p} \qquad c_n = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t) e^{-i2\pi nt/T_p} dt$$

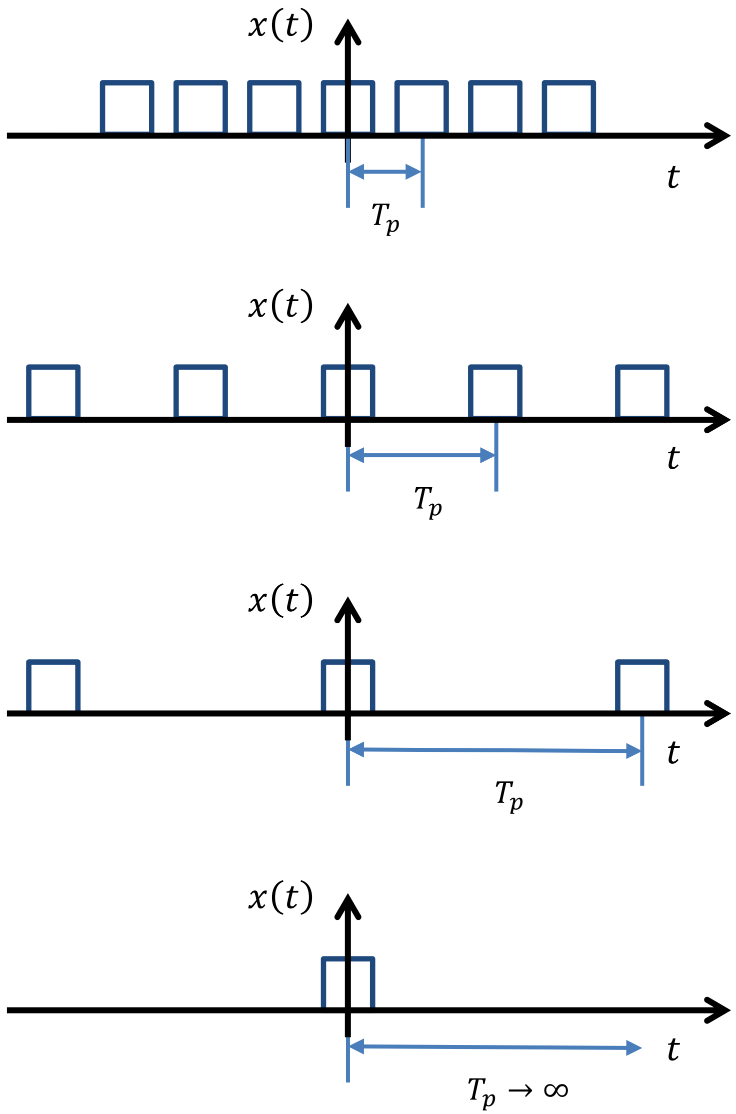
$$\boldsymbol{w = \frac{2\pi}{T_p} = 2\pi f}$$

Fourier Integral for Non-periodic Functions

The fundamental frequency $f = 1/T_p$ becomes smaller and smaller and all other frequencies ($f_n = nf$), being multiples of the fundamental frequency, are more densely packed on the frequency axis. Their separation is assumed to be $1/T_p = \Delta f$. $\Delta f \rightarrow 0$ as $T_p \rightarrow \infty$.

$$c_n = \lim_{T_p \to \infty} \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-i2\pi nt/T_p} dt$$

$$= \lim_{\Delta f \to 0} \Delta f \int_{-T_p/2}^{T_p/2} x(t) e^{-i2\pi nt\Delta f} dt$$



Fourier Integral for Non-periodic Functions (Continue)

$$c_n = \lim_{T_p \to \infty} \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-i2\pi nt/T_p} dt = \lim_{\Delta f \to 0} \Delta f \int_{-T_p/2}^{T_p/2} x(t) e^{-i2\pi nt\Delta f} dt$$

$$\lim_{\Delta f \to 0} \left(\frac{c_n}{\Delta f} \right) = \lim_{\Delta f \to 0} \int_{-T_p/2}^{T_p/2} x(t) e^{-i2\pi f_n t} dt = \lim_{\Delta f \to 0} \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt$$

We write the equation at the right as

$$X(f_n) = \lim_{\Delta f \to 0} \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt$$

Since $\Delta f \to 0$, the frequencies f_n become a continuum, so we write f instead of f_n.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

Fourier integral or Fourier transform of $x(t)$

Fourier Integral for Non-periodic Functions (Continue)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p} \qquad \lim_{\Delta f \to 0}\left(\frac{c_n}{\Delta f}\right) = \lim_{\Delta f \to 0} \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt = \lim_{\Delta f \to 0} X(f_n)$$

$x(t)$ can be rewritten as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p} = \lim_{\Delta f \to 0} \sum_{n=-\infty}^{\infty} \Delta f X(f_n) e^{i2\pi f_n t}$$

This can be represented in a continuous form as

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

Fourier integral pair or Fourier transform pair of $x(t)$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df$$

Fourier Integral Pair (Fourier Transform)

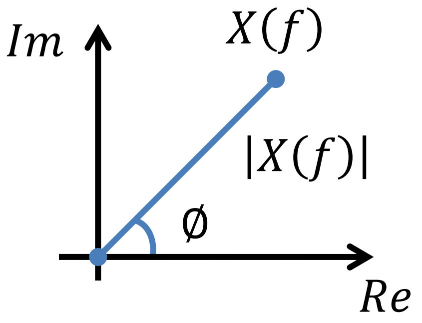
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

The Fourier transform (FT) decomposes a function of time (a signal) into its constituent frequencies. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time.

https://en.wikipedia.org/wiki/Fourier_transform

Comments on the Fourier Integral

1. $X(f)$ is complex and can be represented as



$$X(f) = X_{Re}(f) + i\,X_{Im}(f) = |X(f)|e^{i\emptyset(f)}$$

2. Fourier transformation using f and w

$$\boldsymbol{w = \frac{2\pi}{T_p} = 2\pi f}$$

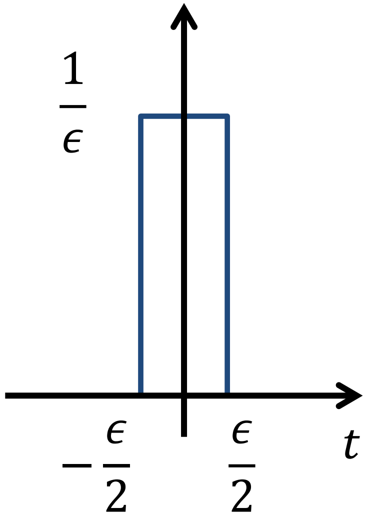
$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{iwt}dw$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt \qquad X(w) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt$$

Dirac Delta Function

The Dirac delta function is denoted by $\delta(t)$

$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)dt = 1$$



$$\delta(t) = \frac{1}{\epsilon} \text{ for } -\frac{\epsilon}{2} < t < \frac{\epsilon}{2}$$
$$= 0 \text{ otherwise}$$

$$\longrightarrow$$

$$\delta(t) = \infty \text{ for } t = 0$$
$$= 0 \text{ otherwise}$$

Properties

$$\int_{-\infty}^{\infty} x(t)\delta(t-a)dt = x(a)$$

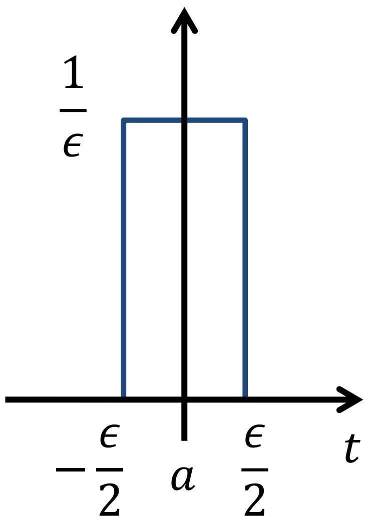
$$\int_{-\infty}^{\infty} e^{\pm i2\pi at}dt = \delta(a)$$

Proof: Dirac Delta Function

Supplement

$$\int_{-\infty}^{\infty} x(t)\delta(t-a)dt = \int_{a-\varepsilon}^{a+\varepsilon} x(t)\delta(t-a)dt = \int_{a-\varepsilon}^{a+\varepsilon} x(a)\delta(t-a)dt$$

$$= x(a)\int_{a-\varepsilon}^{a+\varepsilon} \delta(t-a)dt = x(a)$$



Q. What $\int_{-\infty}^{\infty} e^{\pm i2\pi at}dt$ mean by ?

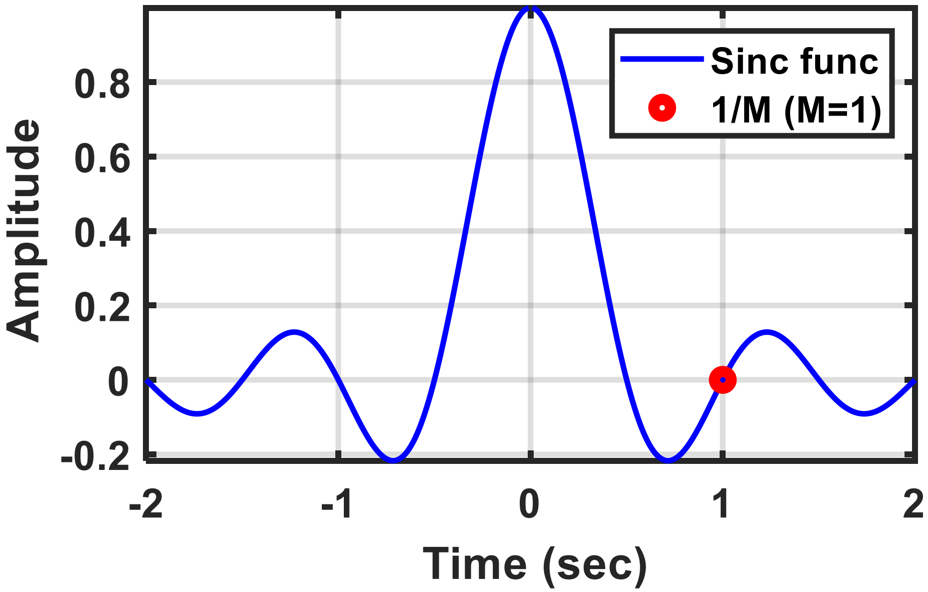
$$\int_{-\infty}^{\infty} e^{\pm i2\pi at}dt = \lim_{M\to\infty}\left(\int_{-M}^{M}(cos2\pi at \pm i\ sin2\pi at)dt\right)$$

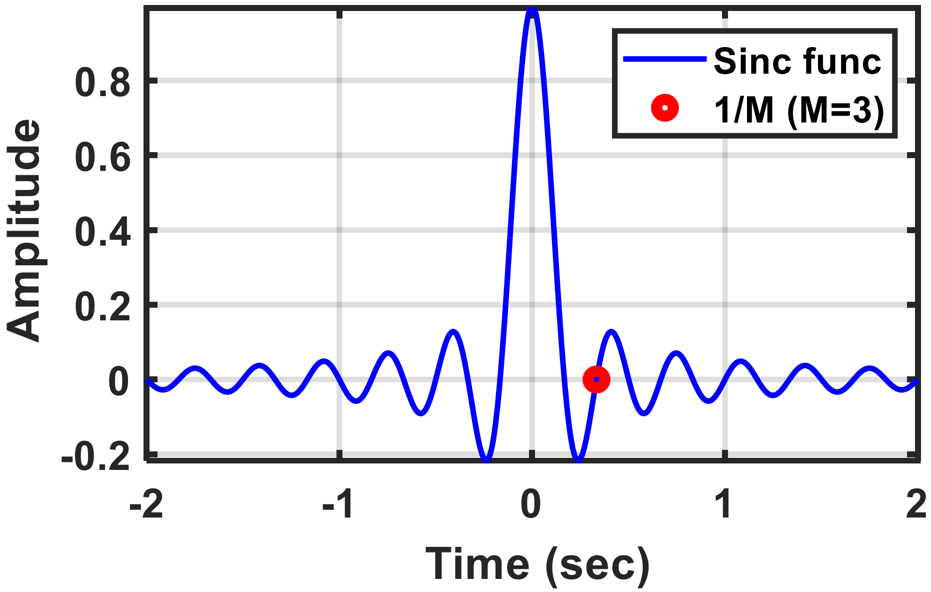
Sinc function

$$= \lim_{M\to\infty}\left(\int_{-M}^{M}(cos2\pi at)dt\right) = \lim_{M\to\infty}\frac{sin2\pi at}{2\pi a}\Bigg|_{-M}^{M} = \lim_{M\to\infty} M\boxed{\frac{sin2\pi aM}{2\pi aM}} = \delta(a)$$

Sinc Function

$$sinc(x) \equiv \begin{cases} 1 & for\ x = 0 \\ \dfrac{\sin 2\pi Mx}{2\pi Mx} & otherwise \end{cases}$$





Q. How does the graph look like if M increases?

Table of Fourier Transform Pairs

No.	Time function $x(t)$	Fourier transform $X(f)$	$X(\omega)$
1	$\delta(t)$	1	1
2	1	$\delta(f)$	$2\pi\,\delta(\omega)$
3	A	$A\delta(f)$	$2\pi A\delta(\omega)$
4	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	$\pi\,\delta(\omega)+\frac{1}{j\omega}$
5	$\delta(t-t_0)$	$e^{-j2\pi f t_0}$	$e^{-j\omega t_0}$
6	$e^{j2\pi f_0 t}$ or $e^{j\omega_0 t}$	$\delta(f-f_0)$	$2\pi\,\delta(\omega-\omega_0)$
7	$\cos(2\pi f_0 t)$ or $\cos(\omega_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
8	$\sin(2\pi f_0 t)$ or $\sin(\omega_0 t)$	$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
9	$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2+4\pi^2 f^2}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
10	$\frac{1}{\alpha^2+t^2}$	$\frac{\pi}{\alpha}e^{-\alpha 2\pi\|f\|}$	$\frac{\pi}{\alpha}e^{-\alpha\|\omega\|}$
11	$x(t)=e^{-\alpha t}u(t)$	$\frac{1}{\alpha+j2\pi f}$	$\frac{1}{\alpha+j\omega}$
12	$x(t)=A \quad \|t\|<T$; $=0 \quad \|t\|>T$	$2AT\frac{\sin(2\pi f T)}{2\pi f T}$	$2AT\frac{\sin(\omega T)}{\omega T}$
13	$2Af_0\frac{\sin(2\pi f_0 t)}{2\pi f_0 t}$ or $A\frac{\sin(\omega_0 t)}{\pi t}$	$X(f)=A \quad \|f\|<f_0$; $=0 \quad \|f\|>f_0$	$X(\omega)=A \quad \|\omega\|<\omega_0$; $=0 \quad \|\omega\|>\omega_0$
14	$\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$ or $\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$	$\sum_{n=-\infty}^{\infty} c_n\delta(f-nf_0)$	$2\pi\sum_{n=-\infty}^{\infty} c_n\delta(\omega-n\omega_0)$
15	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$	$\frac{2}{j\omega}$
16	$\frac{1}{t}$	$-j\pi\,\mathrm{sgn}(f)$	$-j\pi\,\mathrm{sgn}(\omega)$

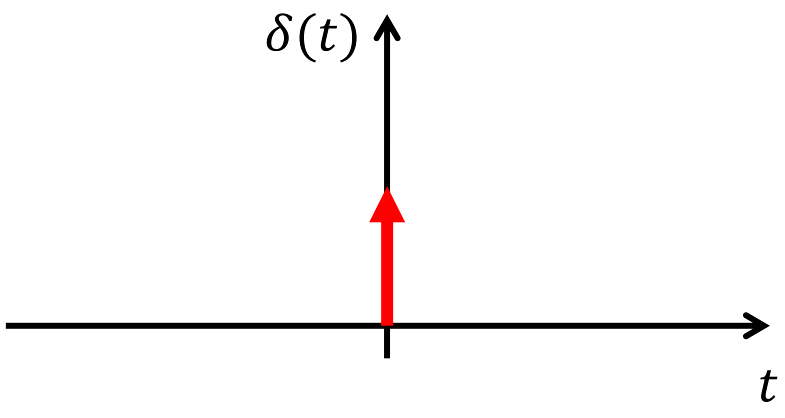
Example: Dirac Delta Function

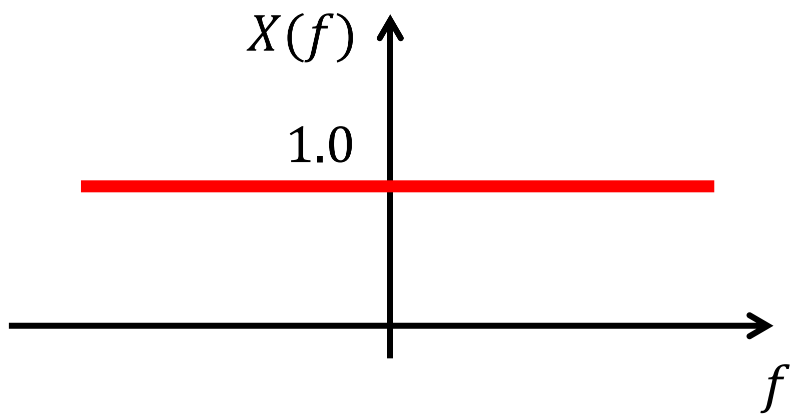
$$\delta(t) = \infty \text{ for } t = 0$$
$$= 0 \quad \text{otherwise}$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-i2\pi f t} dt = e^{-i2\pi f \cdot 0} = 1$$

Direct Delta property

$$\int_{-\infty}^{\infty} x(t)\delta(t-a)dt = x(a)$$





Example: Exponentially Decaying Symmetric Function

$$x(t) = e^{-\lambda|t|}, \qquad \lambda > 0$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} e^{-\lambda|t|} e^{-i2\pi f t} dt$$

$$= \int_{-\infty}^{0} e^{\lambda t} e^{-i2\pi f t} dt + \int_{0}^{\infty} e^{-\lambda t} e^{-i2\pi f t} dt$$

$$= \int_{-\infty}^{0} e^{\lambda t - i2\pi f t} dt + \int_{0}^{\infty} e^{-\lambda t - i2\pi f t} dt$$

$$= \frac{1}{\lambda - i2\pi f} e^{(\lambda - i2\pi f)t} \Big|_{-\infty}^{0} + \frac{1}{-\lambda - i2\pi f} e^{-(\lambda + i2\pi f)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{\lambda - i2\pi f} - \frac{1}{-\lambda - i2\pi f} = \frac{\lambda + i2\pi f}{\lambda^2 + 4\pi^2 f^2} + \frac{\lambda - i2\pi f}{\lambda^2 + 4\pi^2 f^2} = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$

Fourier Transform

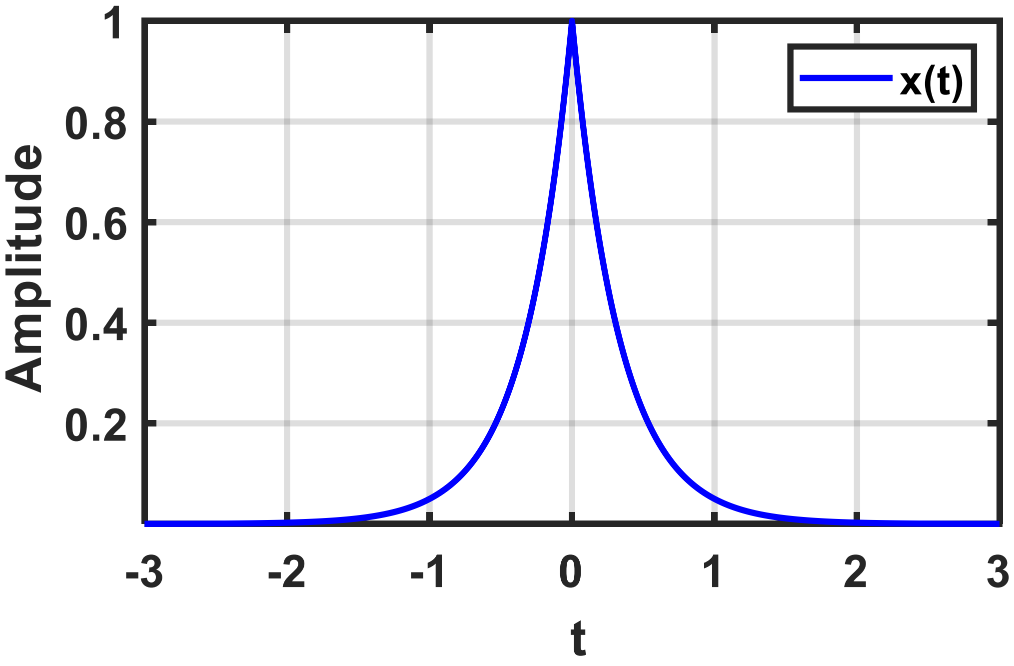
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$$

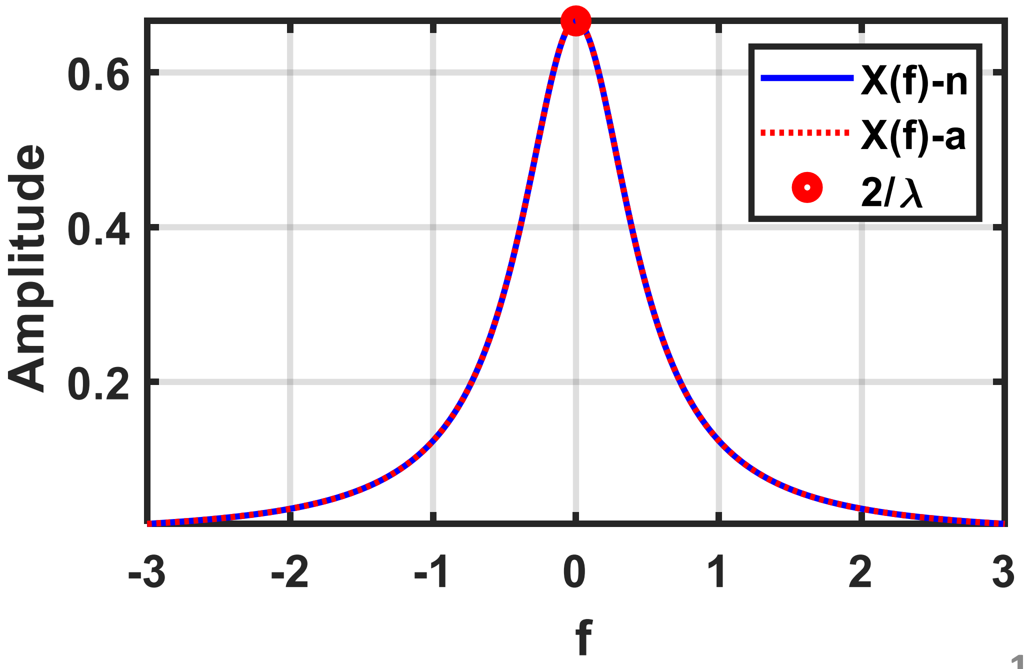
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

Example: Exponentially Decaying Symmetric Function (Continue)

$$x(t) = e^{-\lambda|t|}, \qquad \lambda > 0$$

$$X(f) = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$





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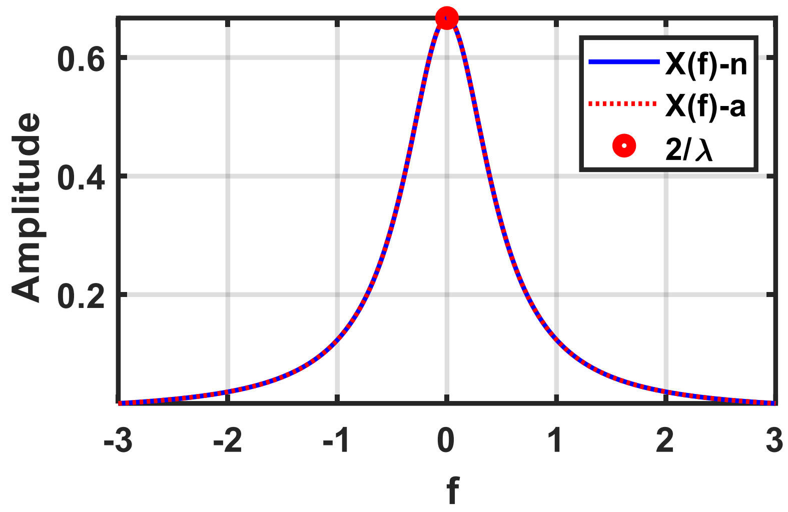
Example: Exponentially Decaying Symmetric Function (Continue)

$$x(t) = e^{-\lambda|t|}, \qquad \lambda > 0$$

$$X(f) = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$

$$X(0) = \frac{2\lambda}{\lambda^2} = \frac{2}{\lambda}$$

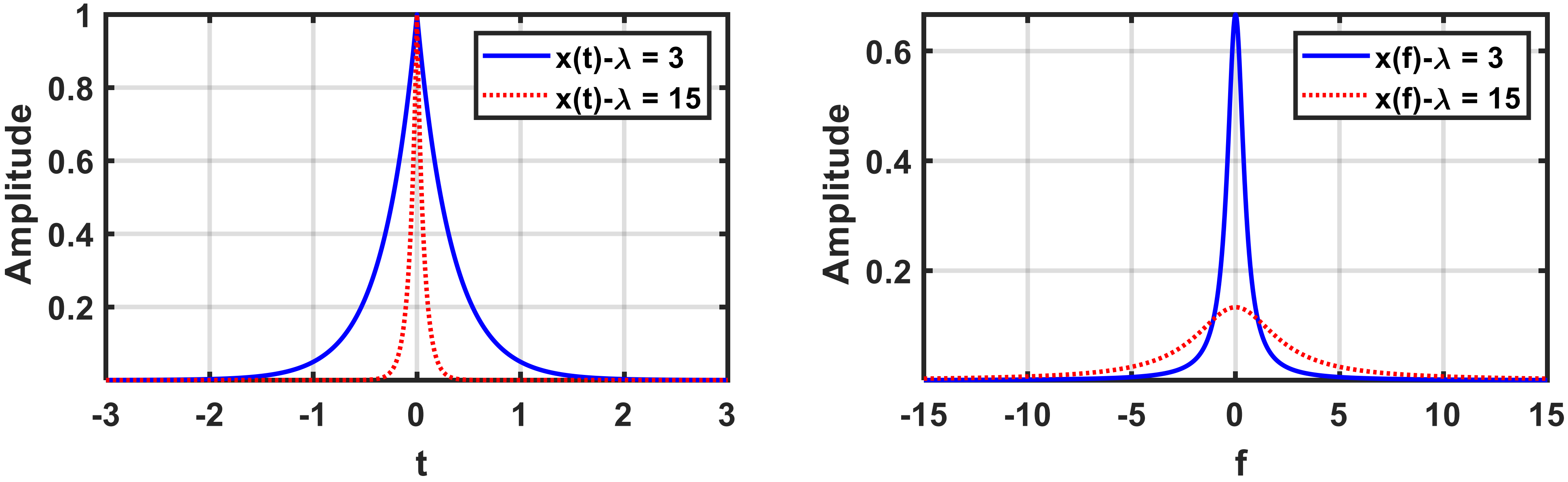
$$X\left(\frac{\lambda}{2\pi}\right) = \frac{2\lambda}{\lambda^2 + \lambda^2} = \frac{1}{\lambda} = \frac{X(0)}{2}$$



if λ is large then $X(f)$ is narrow in the time domain, but wide in the frequency domain and vice versa. This is an example of the so-called inverse spreading property of the Fourier transform, i.e. ***the wider in one domain, then the narrower in the other***.

Exponentially Decaying Symmetric Function with Different Lambda Values

if λ is large then $X(f)$ is narrow in the time domain, but wide in the frequency domain and vice versa. This is an example of the so-called **inverse spreading property** of the Fourier transform, i.e. the wider in one domain, then the narrower in the other.



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Example: Sine Function

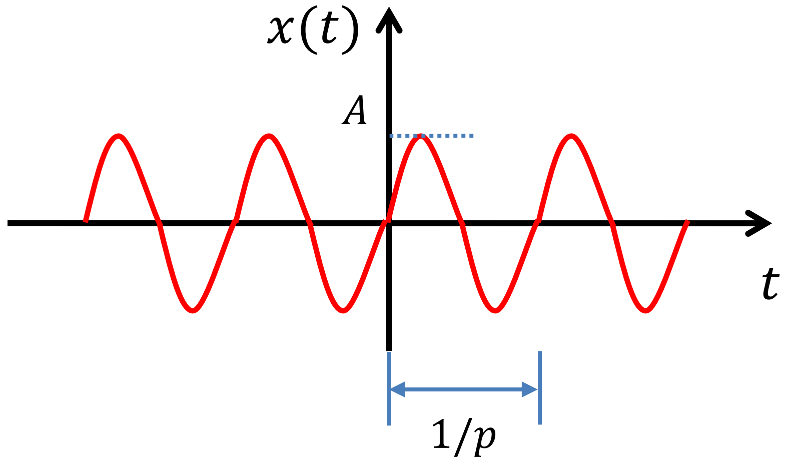
$$x(t) = A\sin 2\pi pt$$

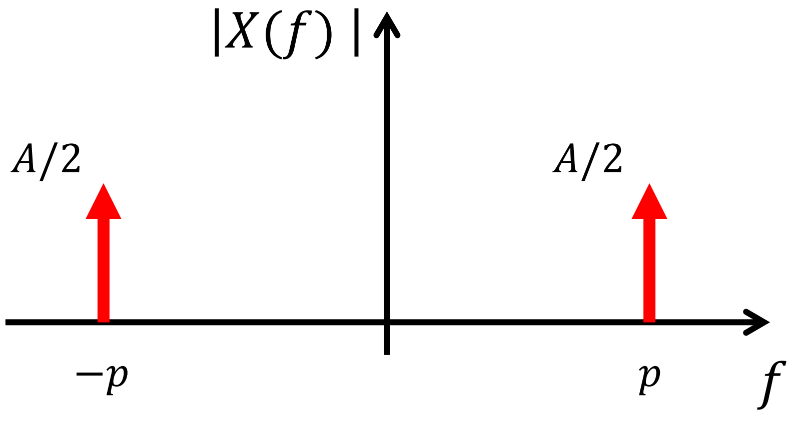
$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} A\sin 2\pi pt\, e^{-i2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} \frac{A}{2i}(e^{i2\pi pt} - e^{-i2\pi pt})e^{-i2\pi ft}dt$$

$$= \frac{A}{2i}\int_{-\infty}^{\infty} (e^{-i2\pi(f-p)t} - e^{--i2\pi(f+p)t})\, dt$$

$$= \frac{A}{2i}[\delta(f-p) - \delta(f+p)]$$



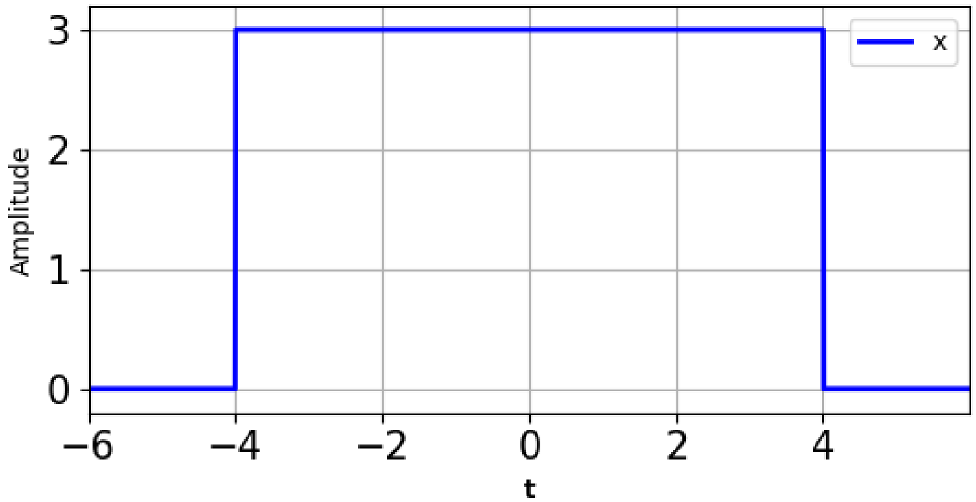


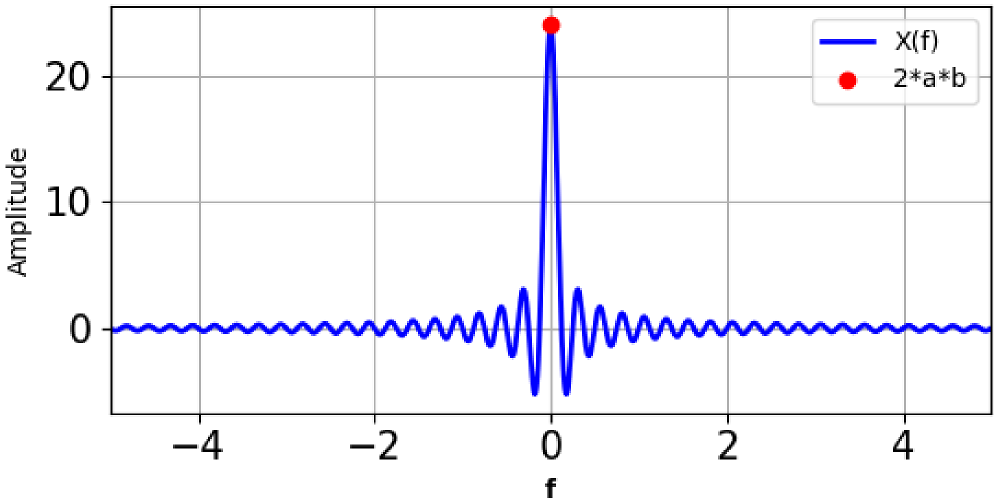
Example: Rectangular Function

$$x(t) = a \text{ for } |t| < \text{b}$$
$$= 0 \text{ for } |t| > \text{b}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-b}^{b} a e^{-i2\pi f t} dt = \frac{2ab \sin 2\pi f b}{2\pi f b}$$

See Sinc function





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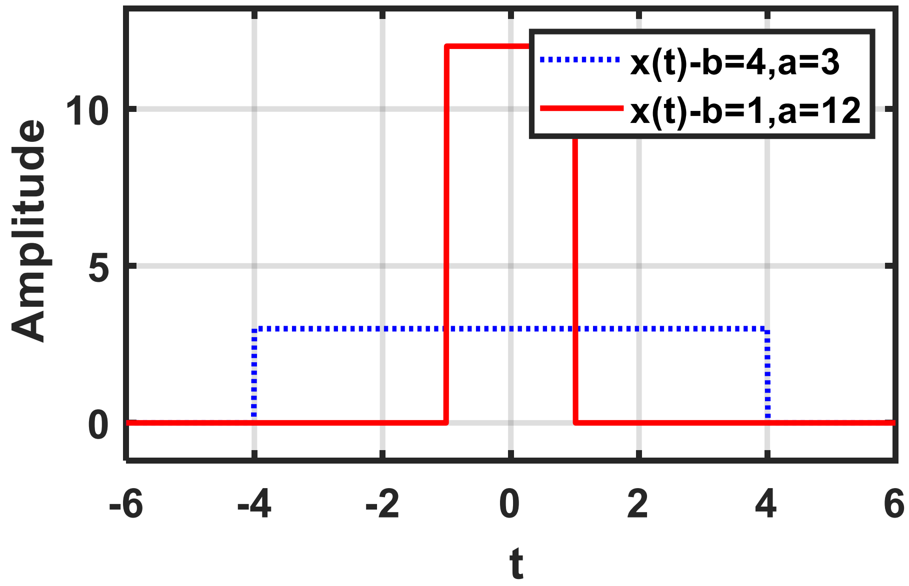
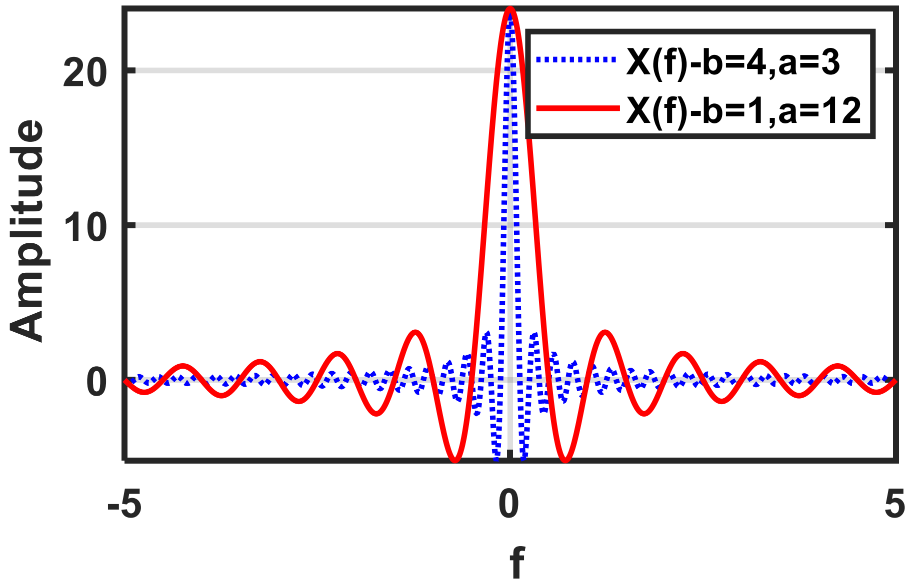
Rectangular Function with Different Ranges

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-b}^{b} ae^{-i2\pi ft}dt = \frac{2ab\sin 2\pi fb}{2\pi fb}$$





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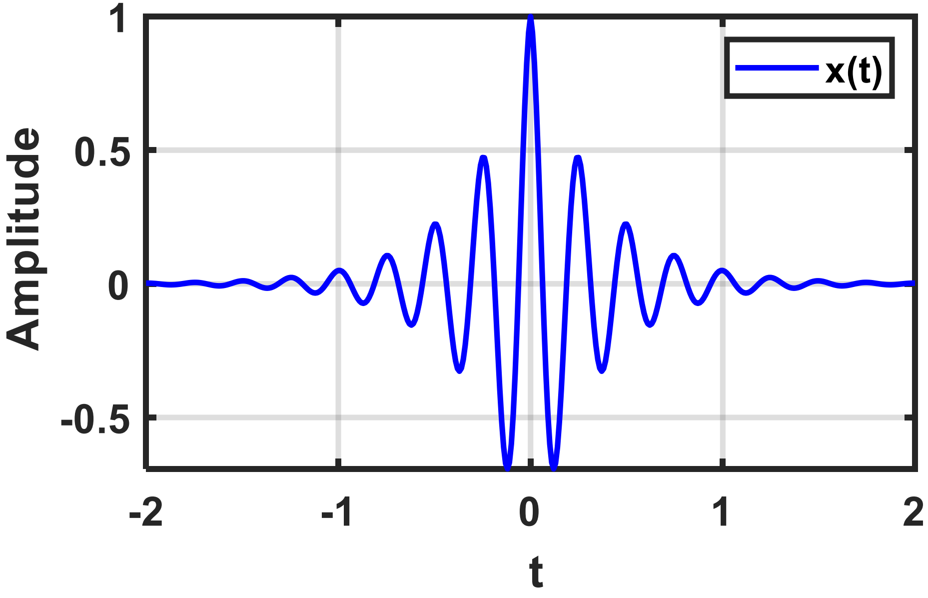
Example: Damped Symmetrically Oscillating Function

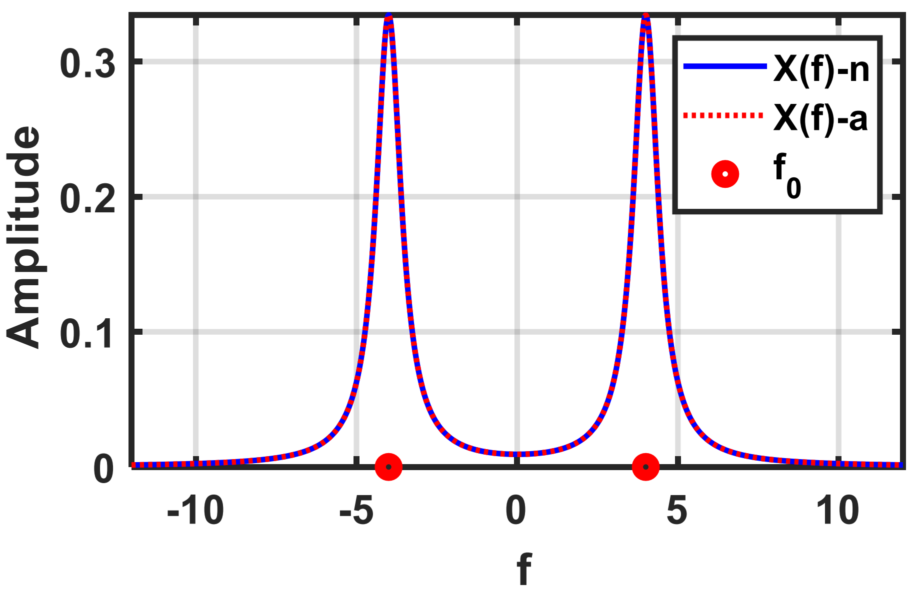
$$x(t) = e^{-a|t|} cos2\pi f_0 t$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} e^{-a|t|} cos2\pi f_0 t\, e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} e^{-a|t|} \frac{1}{2}(e^{i2\pi f_0 t} + e^{-i2\pi f_0 t})e^{-i2\pi ft}dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-a|t|}(e^{-i2\pi(f-f_0)t} + e^{-i2\pi(f+f_0)t})\, dt = \frac{a}{a^2 + [2\pi(f - f_0)]^2} + \frac{a}{a^2 + [2\pi(f + f_0)]^2}$$

convolution





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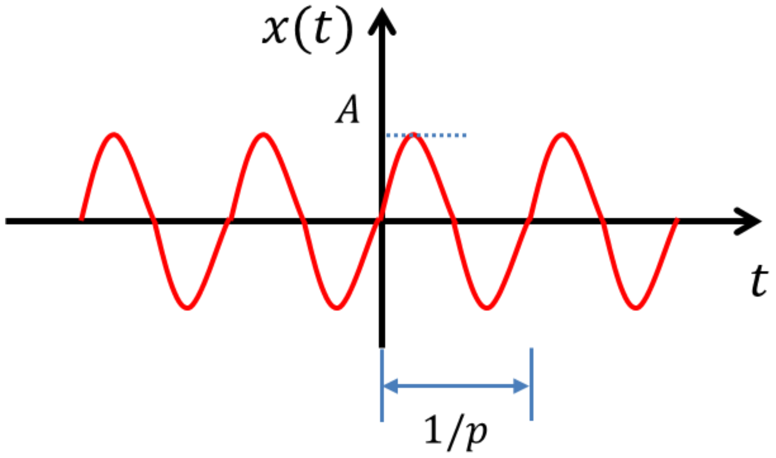
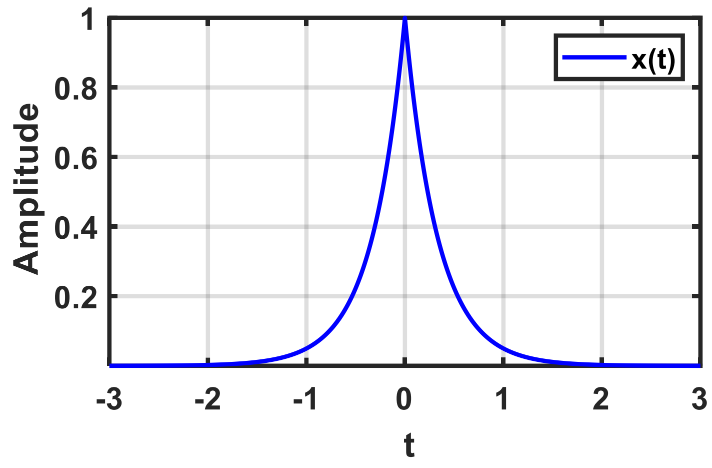
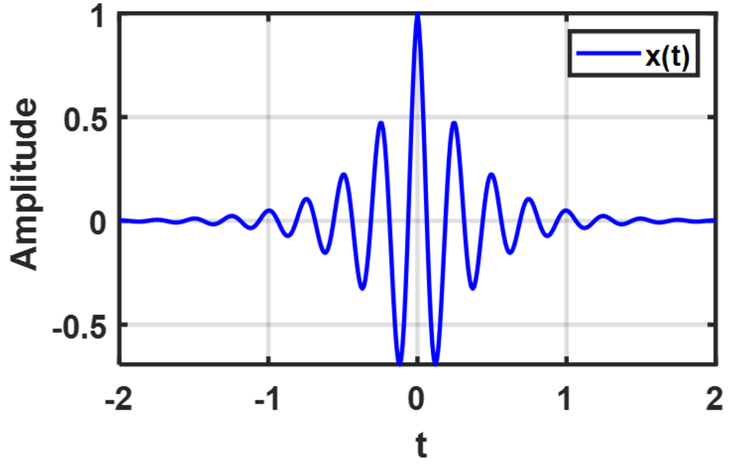
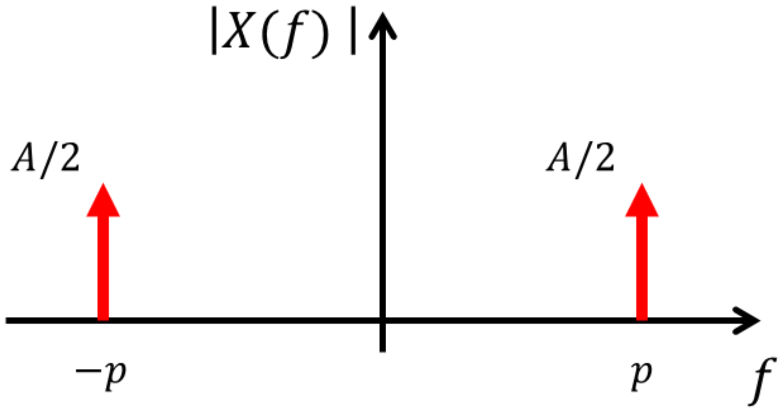
Can You See Any Pattern?

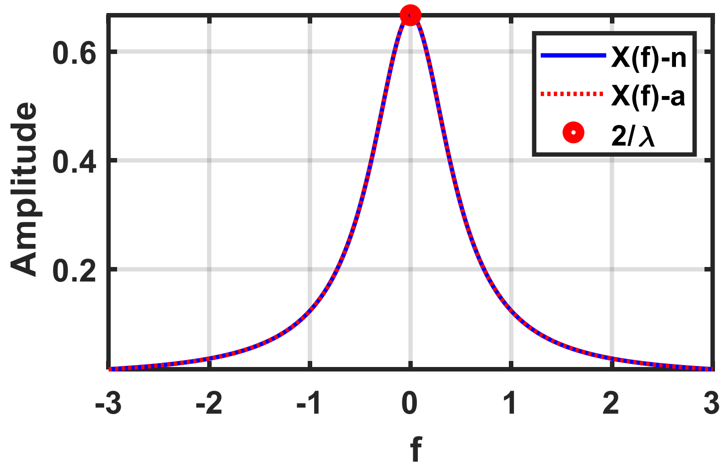
Supplement

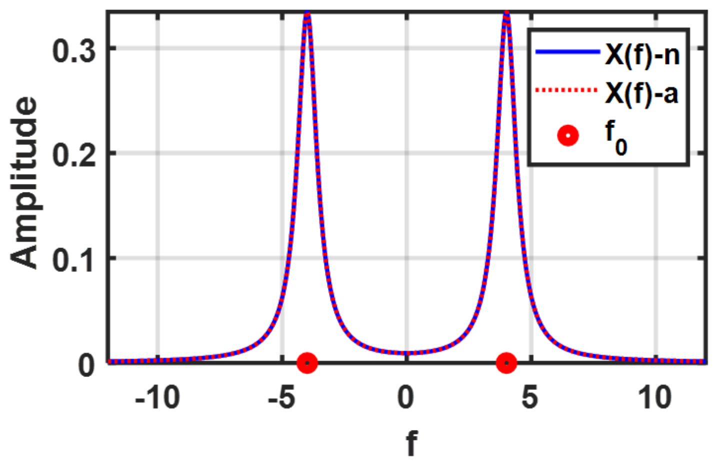












$$x(t) = A\cos 2\pi pt$$

$$x(t) = e^{-\lambda|t|}, \lambda > 0$$

$$x(t) = e^{-a|t|} cos 2\pi f_0 t$$

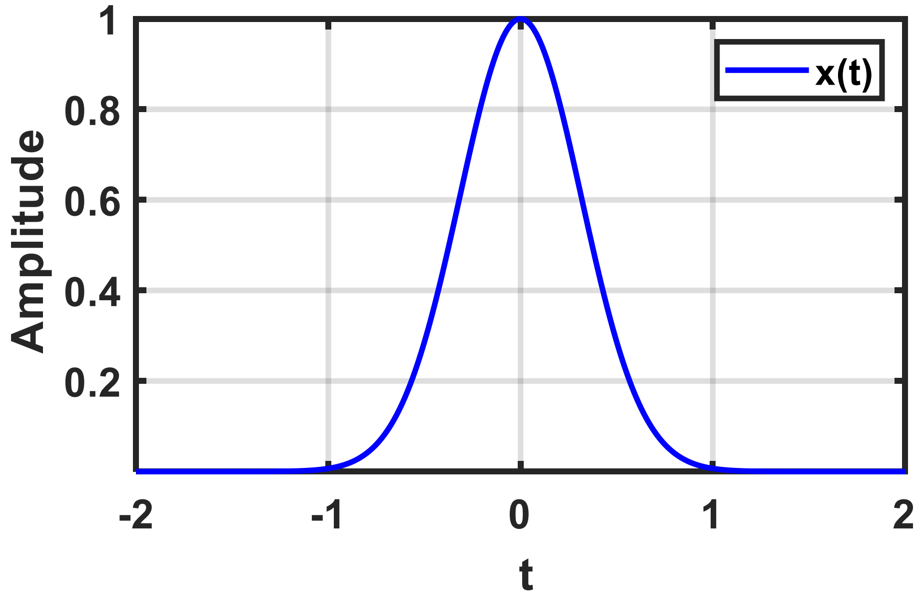
Tutorial

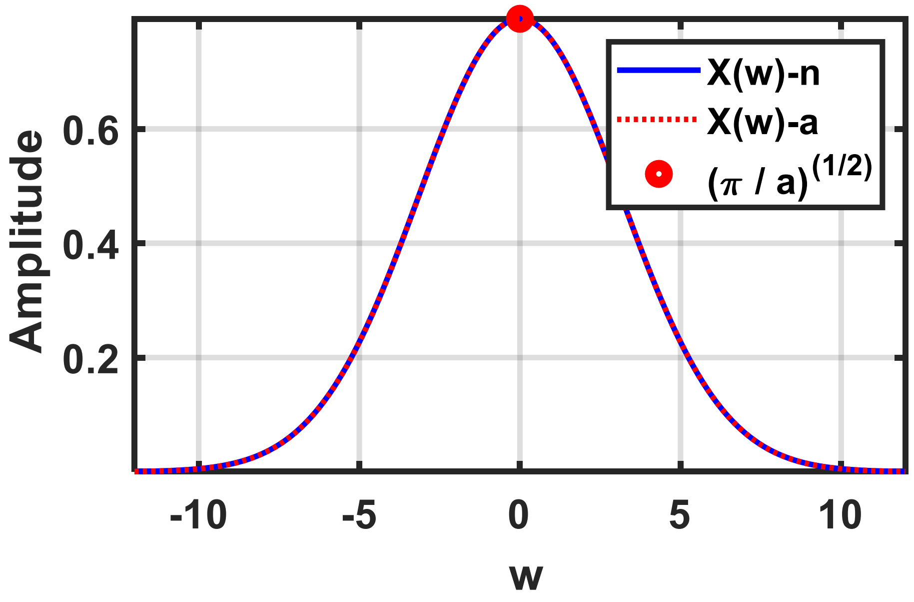
Example: Gaussian Function

$$x(t) = \boxed{e^{-at^2}}$$

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt = \int_{-\infty}^{\infty} e^{-at^2}e^{-iwt}dt = \int_{-\infty}^{\infty} e^{-a(t^2+\frac{iwt}{a})}dt = e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-a(t^2+\frac{iwt}{a}-\frac{w^2}{4a})}dt$$

$$= e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(t+i\left(\frac{w}{2a}\right)\right)^2}dt = e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-ay^2}dy = \boxed{\sqrt{\pi/a}\cdot e^{-\frac{w^2}{4a}}}$$





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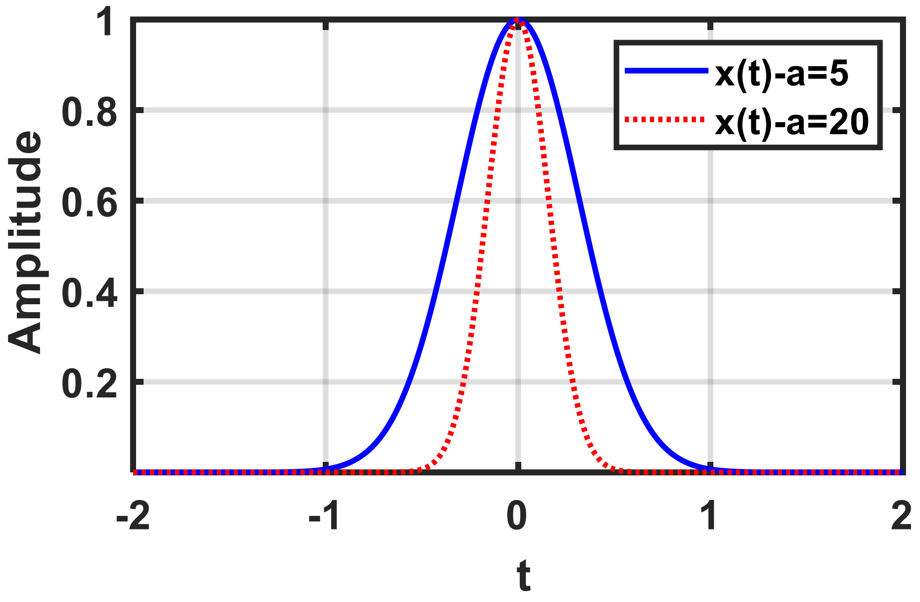
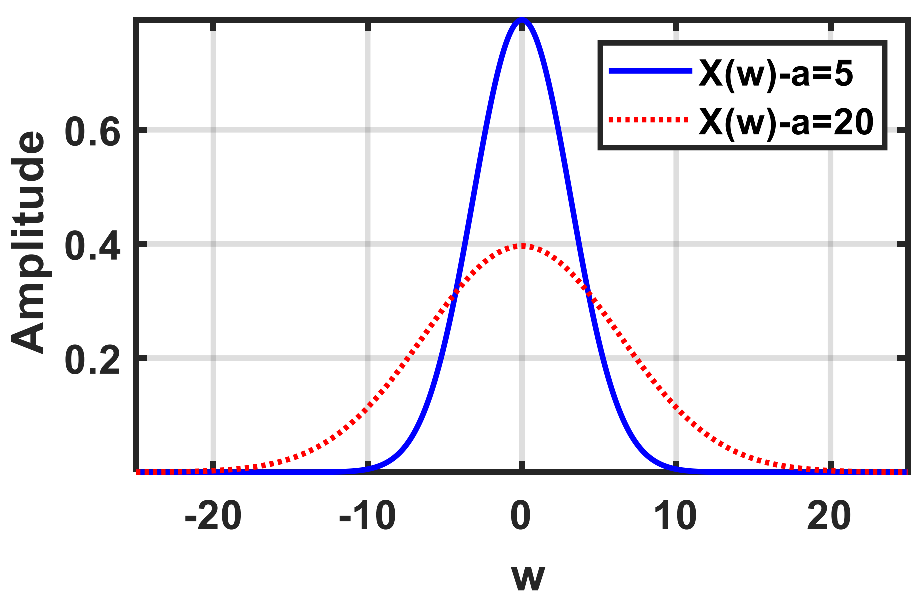
Gaussian Function with Different Variances

$$x(t) = e^{-at^2} \qquad\qquad X(w) = \sqrt{\pi/a} \cdot e^{-\frac{w^2}{4a}}$$

As a is increasing, then, $x(t)$ become narrower but $X(f)$ is wider. The wider in one domain, then the narrower in the other. → **Inverse spreading property of the Fourier transform**





Tutorial

Example: Fourier Transform of a Periodic Function

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}$$

$$\int_{-\infty}^{\infty} e^{\pm i2\pi at} dt = \delta(a)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}\, e^{-i2\pi ft} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} c_n e^{-i2\pi\left(f-\frac{n}{T_p}\right)t} dt = \sum_{n=-\infty}^{\infty} c_n \delta(f - \frac{n}{T_p})$$

Fourier transform of a periodic function is a series of delta functions scaled by c_n, and located at multiples of the fundamental frequency, $1/T_p$.