Signal Processing I (Fourier Series)

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Reference

We will cover some key topics in Chapters 3 $\sim$ 6 of the following reference:

Shin, K., & Hammond, J. K. (2008). Fundamentals of Signal Processing: for Sound and Vibration Engineers, John Wiley & Sons.

Chapter 3: Fourier Series

Chapter 4: Fourier Integrals (Fourier Transform) and Continuous-Time Linear Systems

Chapter 5: Time Sampling and Aliasing

Chapter 6: The Discrete Fourier Transform

Discrete Fourier Transform Using Fast Fourier Transform

A **F**ast **F**ourier **T**ransform (FFT) is an algorithm that computes the **Discrete** **F**ourier **T**ransform (DFT) of a sequence, or its inverse (IDFT). Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa. It manages to reduce the complexity of computing the DFT from $O(n^2)$, which arises if one simply applies the definition of DFT, to $O(n\ log\ n)$, where n is the data size.

`fft` in MATLAB

fft

Fast Fourier transform

Syntax

```
Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)
```

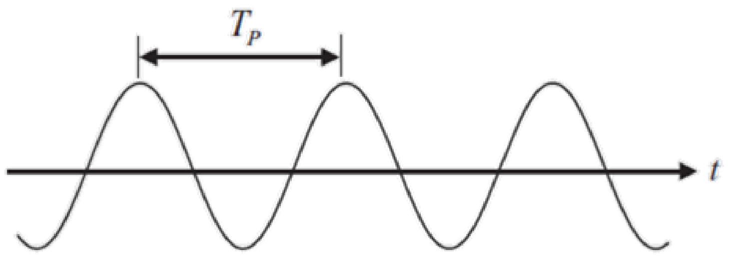
`Y = fft(X)` computes the **Discrete Fourier Transform** (DFT) of X using a Fast Fourier Transform (FFT) algorithm

Periodic Signal

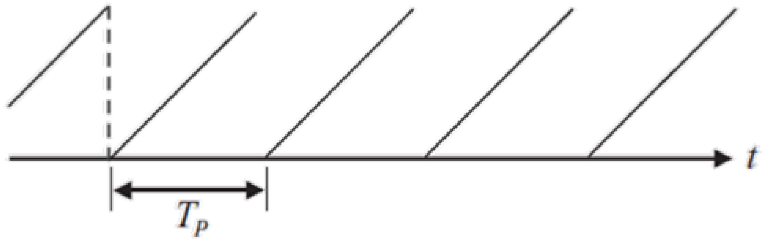
Periodic signals are defined as those whose waveform repeats exactly at regular time intervals. The mathematical definition of periodicity implies that the periodic behaviour of the wave is unchanged for all time. This is expressed as

$$x(t) = x(t + nT_p) \qquad n = \pm1, \pm2, \pm3, \ldots$$

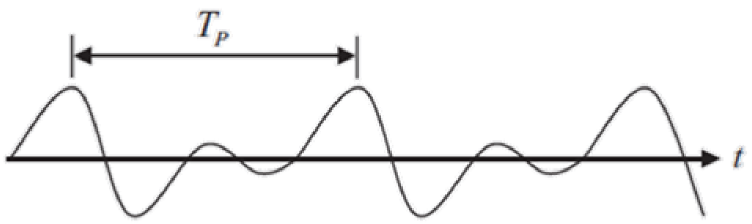
Periodic



Single sinusoidal signal



Triangular signal



General periodic signal

Example: Sinusoidal Signal

The simplest example is a sinusoidal signal

$$x(t) = Xsin(wt + \emptyset) = Xsin(2\pi ft + \emptyset)$$

where X is amplitude,

w is a circular (angular) frequency in radians per unit time (rad/s),

f is a (cyclical) frequency in cycles per unit time (Hz),

$\emptyset$ is phase angle with respect to the time origin in radians.

Q: What is the period of this signal?

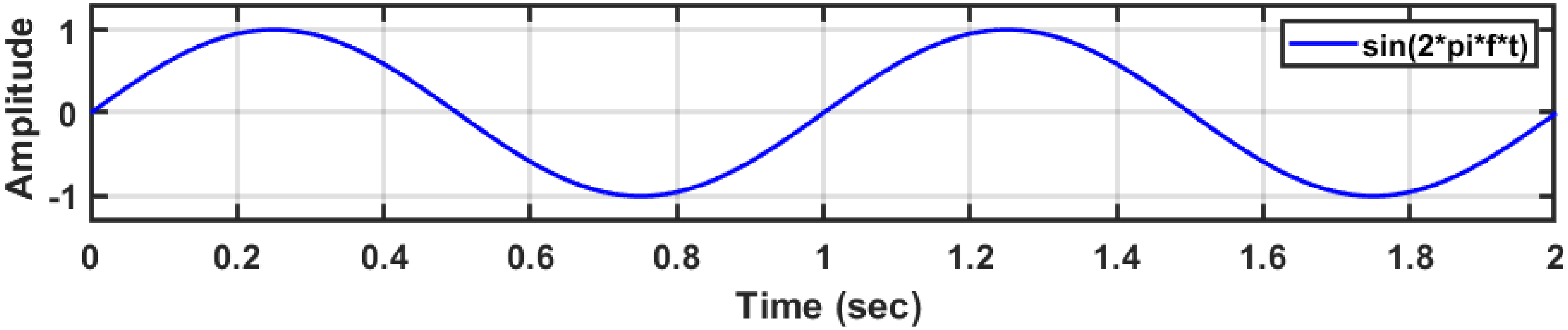
Q: Does the phase change the period?

$$sin(\theta + 2n\pi) = \sin(\theta)$$

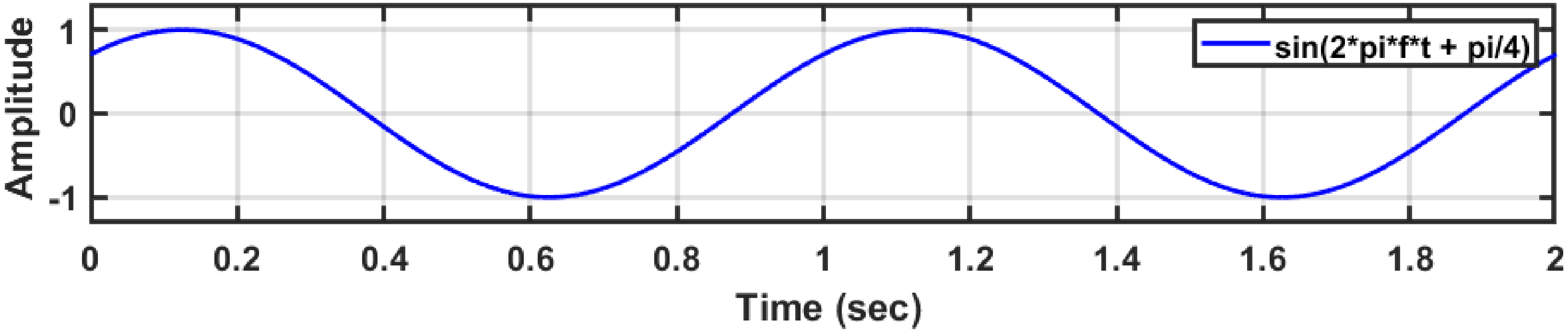
$$x(t) = x(t + nT_p)$$

Example: Sinusoidal Signals

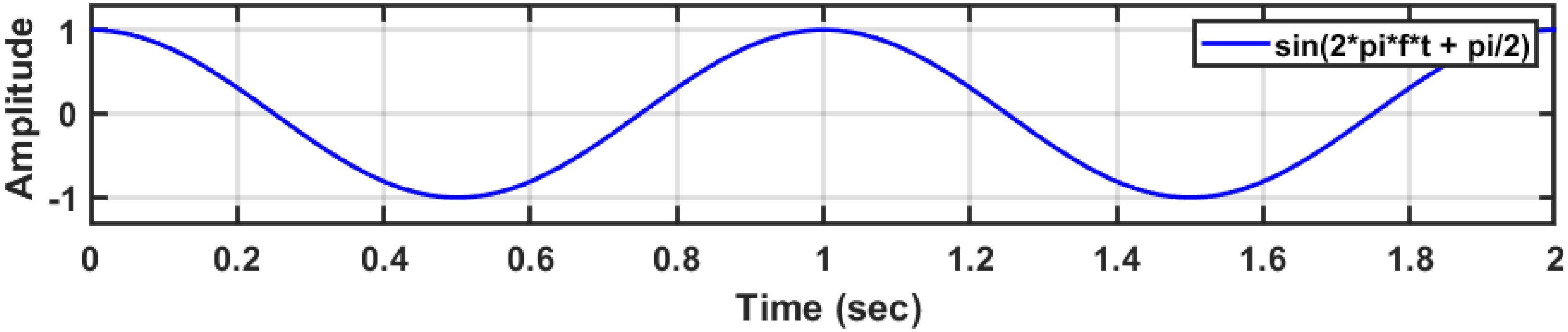
$sin(2\pi f_0 t)$



$sin(2\pi f_0 t + \pi/4)$



$sin(2\pi f_0 t + \pi/2)$

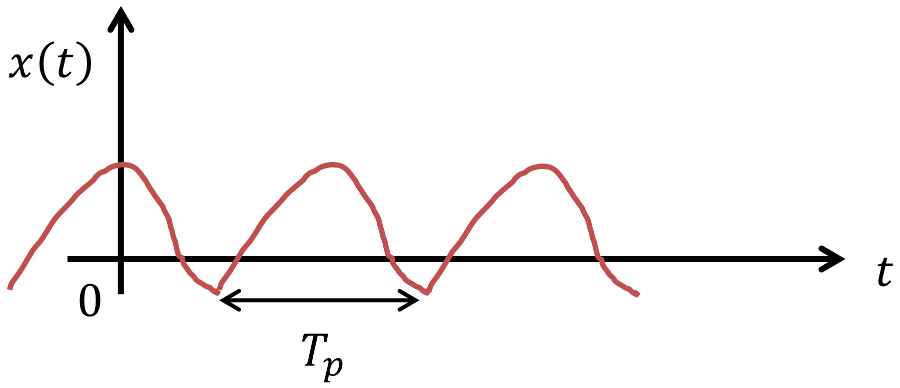


Frequency (f_0): 1Hz

[Tutorial]

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of <u>an infinite sum of sines and cosines.</u> Fourier series make use of the <u>orthogonality relationships</u> of the sine and cosine functions. The basis of Fourier analysis of **<u>a periodic signal</u>** is the representation of such a signal <u>by adding together sine and cosine functions of appropriate frequencies and amplitudes</u>. It decomposes **<u>any</u>** periodic function or periodic signal into the weighted sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines.



$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$$x(t) = x(t + mT_p)$$ Periodic

Fourier Coefficients

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

The coefficients are calculated from the following expressions:

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt$$

Q: What is the a_0?

$$a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi m t}{T_p}\right)dt$$

$$b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi m t}{T_p}\right)dt$$

Basic Trigonometric Equations

Supplement

$$\int_{-\pi}^{\pi} \cos nt \ dt = 0 \qquad \int_{-\pi}^{\pi} \sin nt \ dt = 0$$

When n is a non-zero integer.

$$\cos mt \cos nt = \frac{1}{2}[\cos(m+n)\,t + \cos(m-n)\,t]$$

$$\sin mt \sin nt = \frac{1}{2}[\cos(m-n)\,t - \cos(m+n)\,t]$$

$$\sin mt \cos nt = \frac{1}{2}[\sin(m+n)\,t + \sin(m-n)\,t]$$

Orthogonality of trigonometric functions

$$\int_{-\pi}^{\pi} \cos mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\boxed{\int_{-\pi}^{\pi} \sin mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ 0 \ if \ n = m \end{cases}}$$

When n,m is a non-zero integer.

Derivation of the Fourier Coefficients

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \boxed{\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt} = \frac{a_0}{2} + \underbrace{\cancel{\frac{1}{T_p}\int_{-T_p/2}^{T_p/2}\sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right) dt}}_{0}$$

$$a_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt} = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right)\cos\left(\frac{2\pi mt}{T_p}\right)dt$$

$$= \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} \overbrace{\cancel{\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right)\cos\left(\frac{2\pi mt}{T_p}\right)}}^{0} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right)\cos\left(\frac{2\pi mt}{T_p}\right)dt = \frac{2a_m}{T_p}\int_{-T_p/2}^{T_p/2}\cos\left(\frac{2\pi mt}{T_p}\right)\cos\left(\frac{2\pi mt}{T_p}\right)dt = a_m$$

$$b_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt}$$

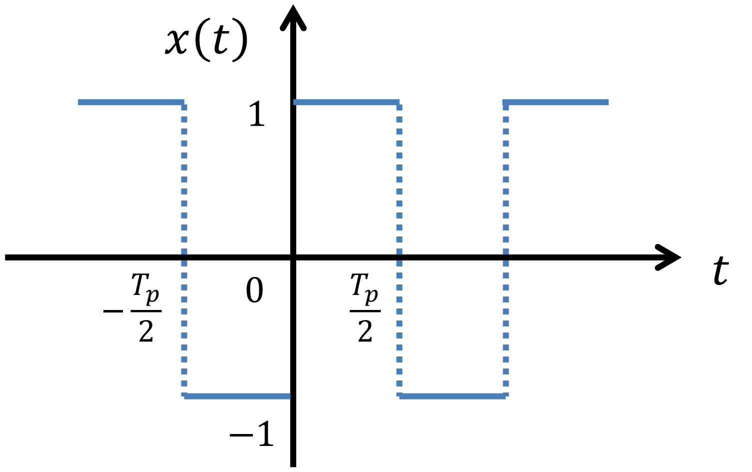
Example: Square Wave

$$x(t) = -1 \quad if \ -\frac{T_p}{2} < t \le 0$$

$$x(t) = \quad 1 \quad if \ 0 < t < \frac{T_p}{2}$$

$$x(t + nT_p) = x(t)$$

$$where\, n = \pm 1, \pm 2, \dots$$



$$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cosn\pi) * \sin\left(\frac{2\pi nt}{T_p}\right)}$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt = 0$$

$$a_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi nt}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\cos\left(\frac{2\pi nt}{T_p}\right)dt + \int_{0}^{T_p/2} \cos\left(\frac{2\pi nt}{T_p}\right)dt\right] = 0$$

Analytic integration

$$b_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi nt}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\sin\left(\frac{2\pi nt}{T_p}\right)dt + \int_{0}^{T_p/2} \sin\left(\frac{2\pi nt}{T_p}\right)dt\right] = \frac{2}{n\pi}(1 - cosn\pi)$$

Numeric Integration


```
n = 10000;
a = 0;
b = 3;
dx = (b-a)/n;

area_fx = 0;

fx = @(x) x^3 - 6*x;
for ii=1:n
    x_star = a + dx*ii;
    area_fx = area_fx + fx(x_star)*dx;
end

error_est = area_fx - (-27/4)
```

```
error_est = 0.0014
```

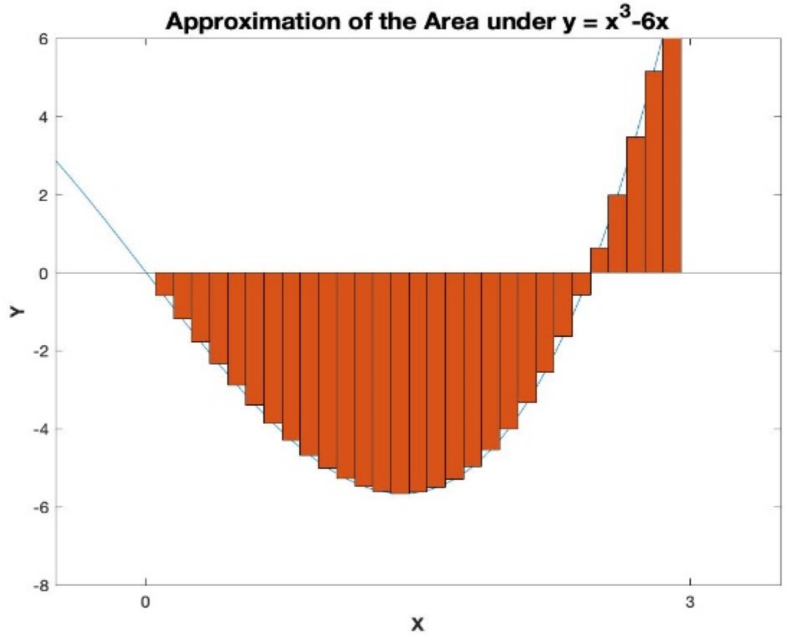
$$f(x) = x^3 - 6x$$

$$\int_0^3 f(x)dx = \frac{1}{4}x^4 - 3x^2\Big|_0^3 = \frac{81}{4} - 27 = -\frac{27}{4}$$

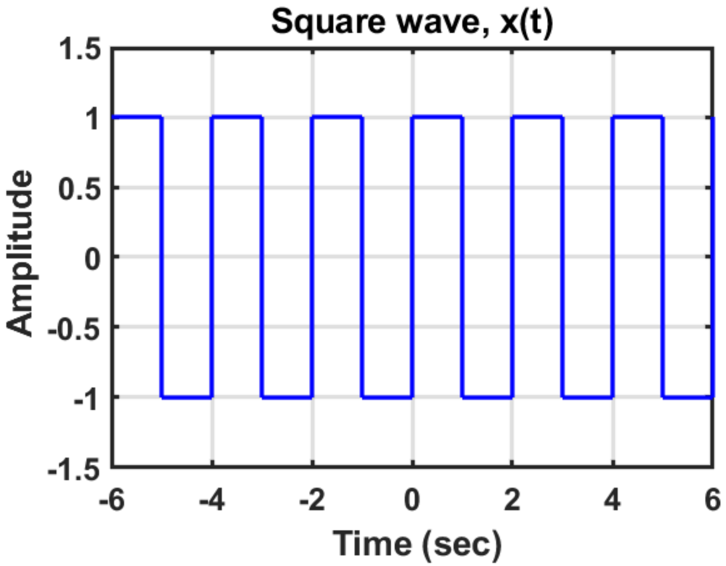
$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

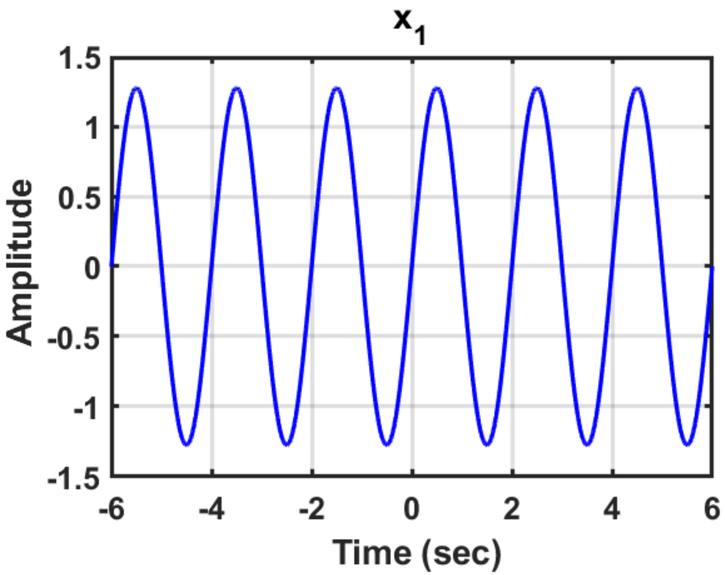
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

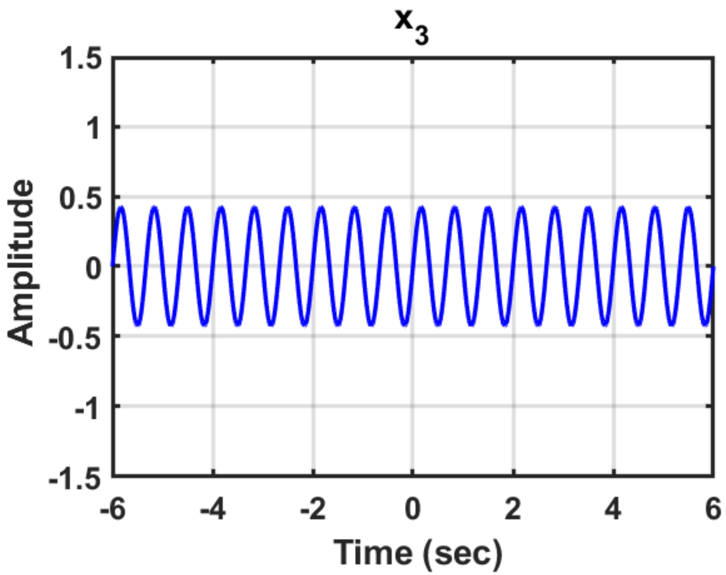
Riemann sum: approximation of an integral by a finite sum

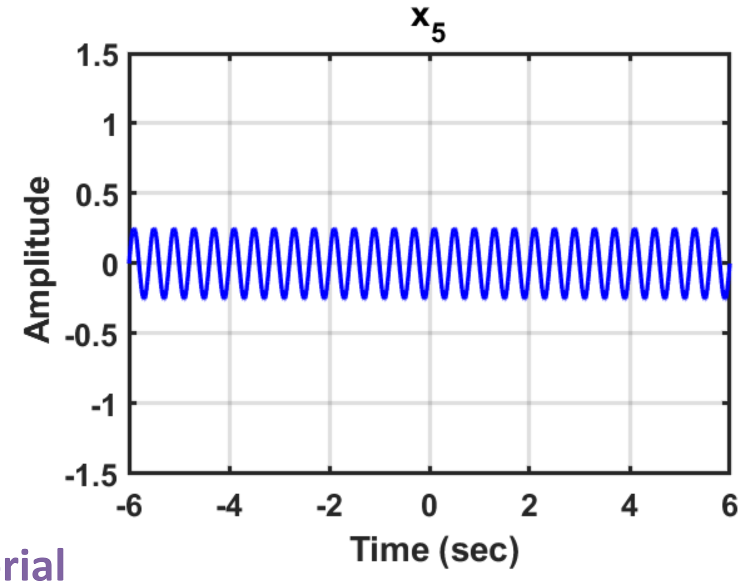


Example: Square Wave (Continue)





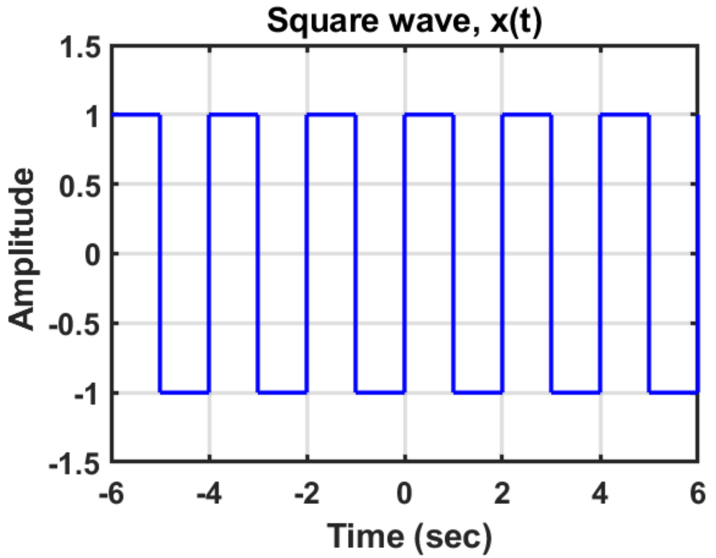


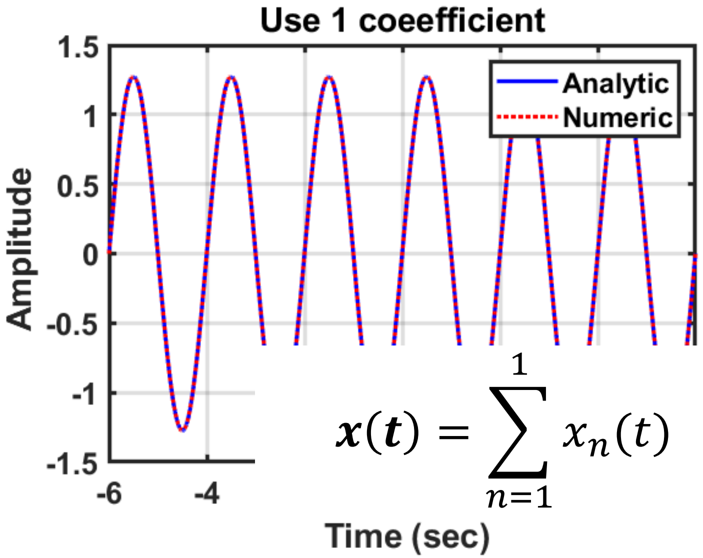


$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cosn\pi) * \sin\left(\frac{2\pi nt}{T_p}\right) = \sum_{n=1}^{\infty} x_n(t)$$

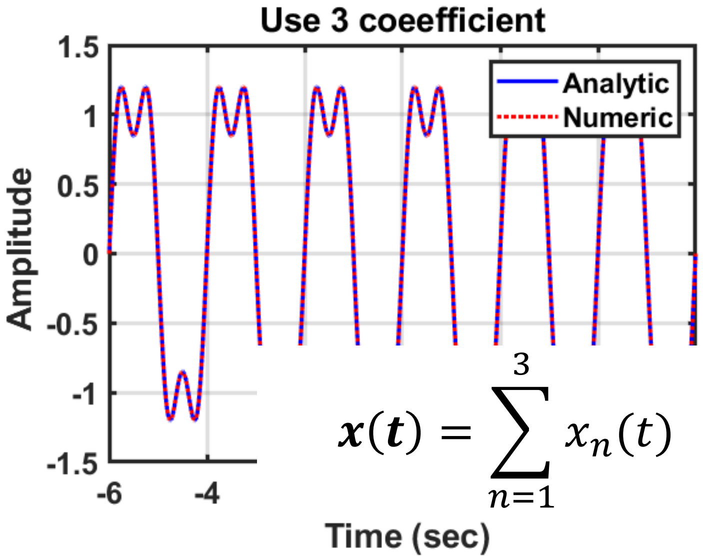
Tutorial

Example: Square Wave (Continue)

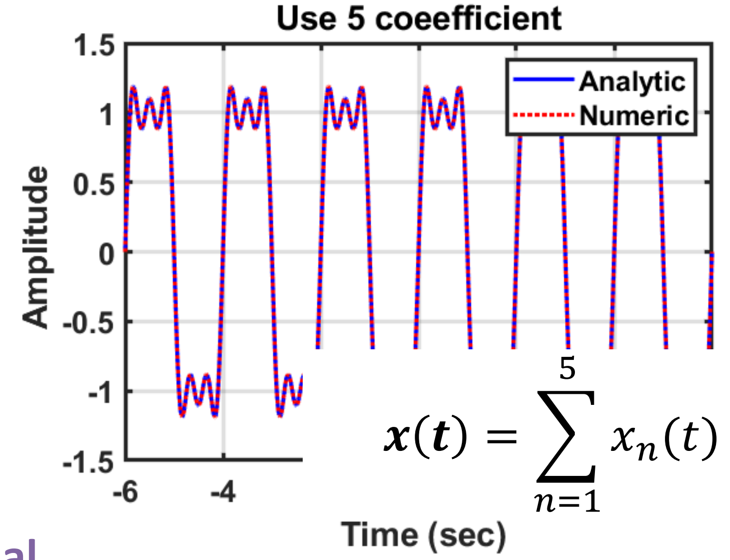




$$\boldsymbol{x(t)} = \sum_{n=1}^{1} x_n(t)$$



$$\boldsymbol{x(t)} = \sum_{n=1}^{3} x_n(t)$$



$$\boldsymbol{x(t)} = \sum_{n=1}^{5} x_n(t)$$

$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cosn\pi) * \sin\left(\frac{2\pi nt}{T_p}\right) = \sum_{n=1}^{\infty} x_n(t)$$

$$x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t) + \cdots$$

Tutorial

Example: Square Wave – MATLAB Script (Numerical)

```
% the signal is assumed to be analog.
ncyle = 3;
Fsa = 1000; % # of samples per a second
Tp = 2;
t = (-ncyle*Tp): 1/Fsa :(ncyle*Tp);
x = @(t) square(t*(2*pi)/Tp); % org. sig.

a0 = integral(x, -Tp/2, Tp/2);
```

```
nCoeff = 5;
a = zeros(nCoeff,1);
b = zeros(nCoeff,1);
for ii=1:nCoeff
    fun_a = @(t) x(t).*cos(2*pi*ii*t/Tp);
    a(ii) = integral(fun_a, -Tp/2, Tp/2);

    fun_b = @(t) x(t).*sin(2*pi*ii*t/Tp);
    b(ii) = integral(fun_b, -Tp/2, Tp/2);
end
```

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt \qquad a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi m t}{T_p}\right)dt \qquad b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi m t}{T_p}\right)dt$$

Example: Square Wave – MATLAB Script (Numerical)

```
% numerical integration
sig_y_numeric = zeros(nCoeff, numel(t));
for ii=1:nCoeff
    if ii==1
        sig_y_numeric(ii,:) = a0/2 + a(ii)*cos(2*pi*ii*t/Tp) + b(ii)*sin(2*pi*ii*t/Tp);
    else
        sig_y_numeric(ii,:) = ...
            sig_y_numeric(ii-1,:) + a(ii)*cos(2*pi*ii*t/Tp) + b(ii)*sin(2*pi*ii*t/Tp);
    end
end
```

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} x_n(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

Example: Square Wave – MATLAB Script (Analytical)

```
nCoeff = 5;
Tp = 2;
t = (-ncyle*Tp): 1/Fsa :(ncyle*Tp);

xn = zeros(nCoeff, numel(t));
for ii=1:nCoeff
    xn(ii,:) = 2/(ii*pi)*(1-cos(ii*pi))*sin(2*pi*ii*t/Tp);
end
sig_y_analytic = cumsum(xn, 1);
```

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} x_n(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cosn\pi) * \sin\left(\frac{2\pi nt}{T_p}\right) = \sum_{n=1}^{\infty} x_n(t)$$

$$a_n = 0$$

$$a_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt = 0$$

$$b_n = \frac{2}{n\pi}(1 - cosn\pi)$$

Complex Form of the Fourier Series

Euler Formula

$$e^{iwt} = coswt + i\ sinwt \qquad e^{-iwt} = coswt - i\ sinwt \qquad coswt = \frac{1}{2}(e^{iwt} + e^{-iwt}) \qquad sinwt = \frac{1}{2j}(e^{iwt} - e^{-iwt})$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos wnt + b_n \sin wnt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}(e^{iwnt} + e^{-iwnt}) + \frac{b_n}{2j}(e^{iwnt} - e^{-iwnt})$$

$$\boldsymbol{w = \frac{2\pi}{T_p}}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt} = c_0 + \sum_{n=1}^{\infty} c_n e^{iwnt} + \sum_{n=1}^{\infty} c_n^* e^{-iwnt} \ \ where\ c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - jb_n}{2},$$

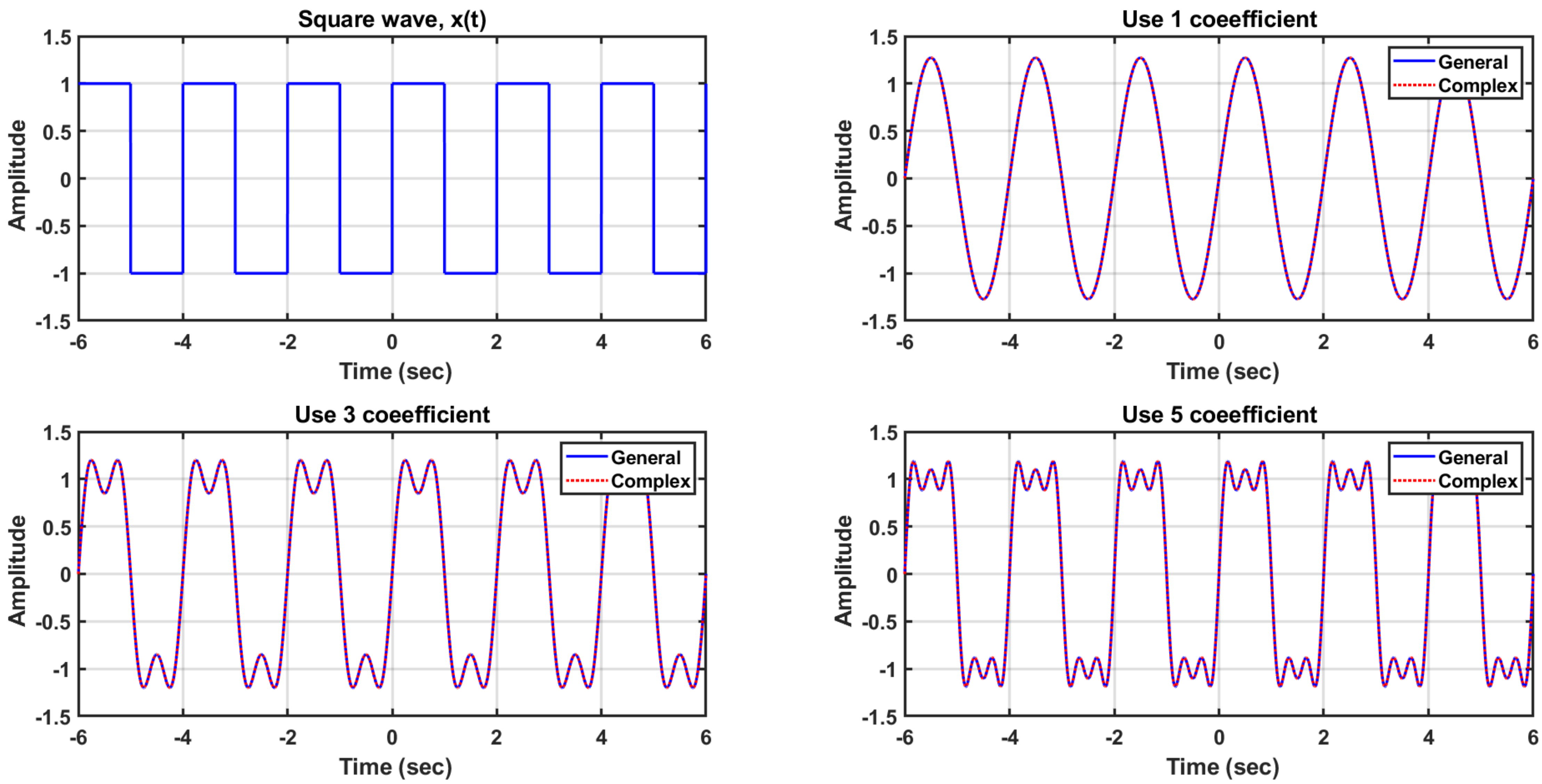
$$c_n^* = \frac{a_n + jb_n}{2}$$

$$\boxed{c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad c_n^* = \frac{1}{T_p}\int_0^{T_p} x(t)e^{iwnt}dt = c_{-n}}$$

Negative frequency term (c_{-n})

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi}{T_p}$$

Example: Square Wave (Comparison of General and Complex Forms)



Tutorial

Summary

General form

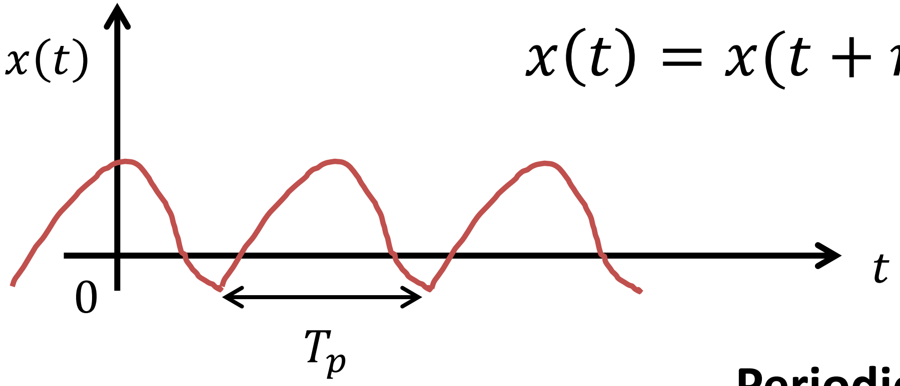
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt \qquad a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt \qquad b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

Complex form

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt}$$

$$c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi n}{T_p}$$



$$x(t) = x(t + nT_p)$$

Periodic signal