Signal Processing II

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Fourier Series of a Periodic Signal

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt}$$

$$c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-i2\pi nt/T_p}dt$$

Let's represent the Fourier series of a periodic signal, where the interval of integration is defined from $-T_p/2$ to $T_p/2$ for convenience:

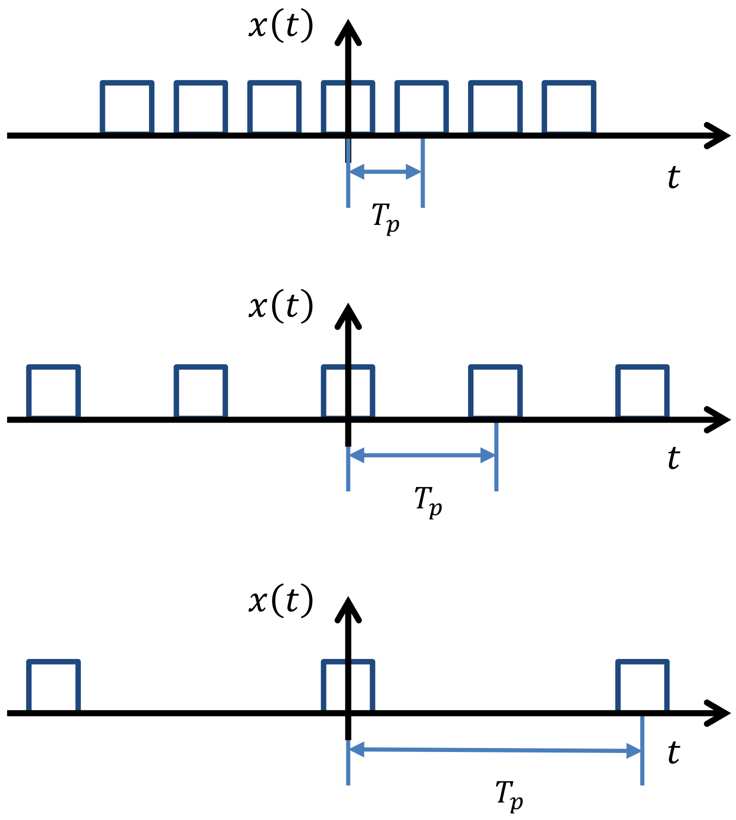
$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}$$

$$c_n = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)e^{-i2\pi nt/T_p}dt$$

$$\boldsymbol{w = \frac{2\pi}{T_p} = 2\pi f}$$

Fourier Integral for Non-periodic Functions

The fundamental frequency $f = 1/T_p$ becomes smaller and smaller and all other frequencies ($f_n = nf$), being multiples of the fundamental frequency, are more densely packed on the frequency axis. Their separation is assumed to be $1/T_p = \Delta f$. $\Delta f \to 0$ as $T_p \to \infty$.



$$c_n = \lim_{T_p \to \infty} \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-i2\pi nt/T_p} dt$$

$$= \lim_{\Delta f \to 0} \Delta f \int_{-\infty}^{\infty} x(t) e^{-i2\pi nt\Delta f} dt$$

Fourier Integral for Non-periodic Functions (Continue)

$$c_n = \lim_{\Delta f \to 0} \Delta f \int_{-\infty}^{\infty} x(t) e^{-i2\pi nt/T_p} dt = \lim_{\Delta f \to 0} \Delta f \int_{-\infty}^{\infty} x(t) e^{-i2\pi n \Delta f t} dt$$

$$= \lim_{\Delta f \to 0} \Delta f \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt$$

$$\lim_{\Delta f \to 0} \left(\frac{c_n}{\Delta f} \right) = \lim_{\Delta f \to 0} \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt$$

Fourier Integral for Non-periodic Functions (Continue)

$$\lim_{\Delta f \to 0} \left(\frac{c_n}{\Delta f} \right) = \lim_{\Delta f \to 0} \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt$$

Assuming the limits exist, we write this as

$$X(f_n) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt$$

Since $\Delta f \to 0$, the frequencies f_n become a continuum, so we write f instead of f_n.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

Fourier integral or Fourier transform of $x(t)$

Fourier Integral for Non-periodic Functions (Continue)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}$$

$$\lim_{\Delta f \to 0} \left(\frac{c_n}{\Delta f} \right) = \lim_{\Delta f \to 0} \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt = \lim_{\Delta f \to 0} X(f_n)$$

$x(t)$ can be rewritten as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p} = \lim_{\Delta f \to 0} \sum_{n=-\infty}^{\infty} \Delta f X(f_n) e^{i2\pi f_n t}$$

This can be represented in a continuous form as

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$$

Fourier Integral Pair (Fourier Transform)

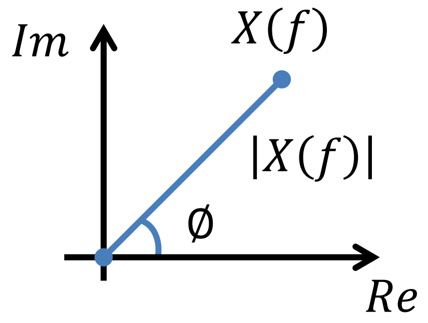
$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \qquad X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

The Fourier transform (FT) decomposes a function of time (a signal) into its constituent frequencies. This is similar to the way a musical chord can be expressed in terms of the volumes and frequencies of its constituent notes. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time.

https://en.wikipedia.org/wiki/Fourier_transform

Comments on the Fourier Integral

1. $X(f)$ is complex and can be represented as



$$X(f) = X_{Re}(f) + i\,X_{Im}(f) = |X(f)|e^{i\emptyset(f)}$$

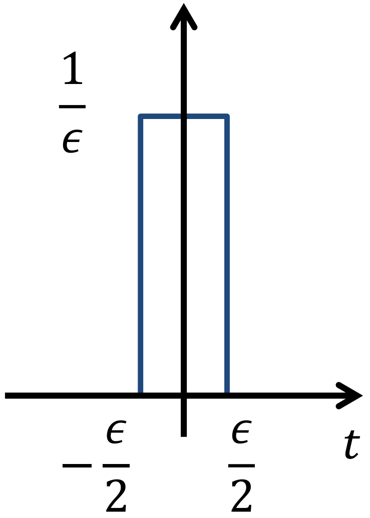
2. Fourier transformation using f and w

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{iwt}dw$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt \qquad X(w) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt$$

Dirac Delta Function

The Dirac delta function is denoted by $\delta(t)$



$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\begin{aligned}\delta(t) &= \frac{1}{\epsilon} \text{ for } -\frac{\epsilon}{2} < t < \frac{\epsilon}{2} \\ &= 0 \text{ otherwise}\end{aligned} \quad \Longrightarrow \quad \begin{aligned}\delta(t) &= \infty \text{ for } t = 0 \\ &= 0 \text{ otherwise}\end{aligned}$$

Properties

$$\int_{-\infty}^{\infty} x(t)\delta(t-a)dt = x(a)$$

$$\int_{-\infty}^{\infty} e^{\pm i2\pi at}dt = \delta(a)$$

Proof: Dirac Delta Function

$$\int_{-\infty}^{\infty} e^{\pm i2\pi at}dt = \lim_{M\to\infty}\left(\int_{-M}^{M}(cos2\pi at \pm i\ sin2\pi at)dt\right)$$

$$= \lim_{M\to\infty}\left(\int_{-M}^{M}(cos2\pi at)dt\right) = \lim_{M\to\infty} 2\left.\frac{sin2\pi at}{2\pi a}\right|_{0}^{M}$$

$$= \lim_{M\to\infty} 2M\frac{sin2\pi aM}{2\pi aM} = \delta(a)$$

Sinc function

M = 0

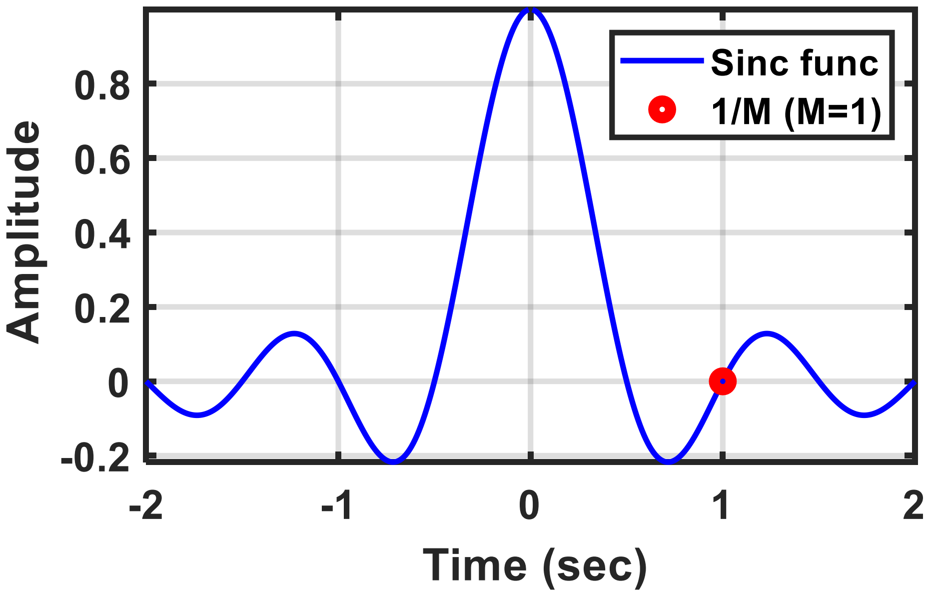
M ≠ 0

L'Hôpitals rule can be applied to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
If $f(x)$ and $g(x)$ are different functions and if $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is indeterminate, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Sinc Function

$$sinc(x) \equiv \begin{cases} 1 & for\ x = 0 \\ \dfrac{\sin Mx}{Mx} & otherwise \end{cases}$$



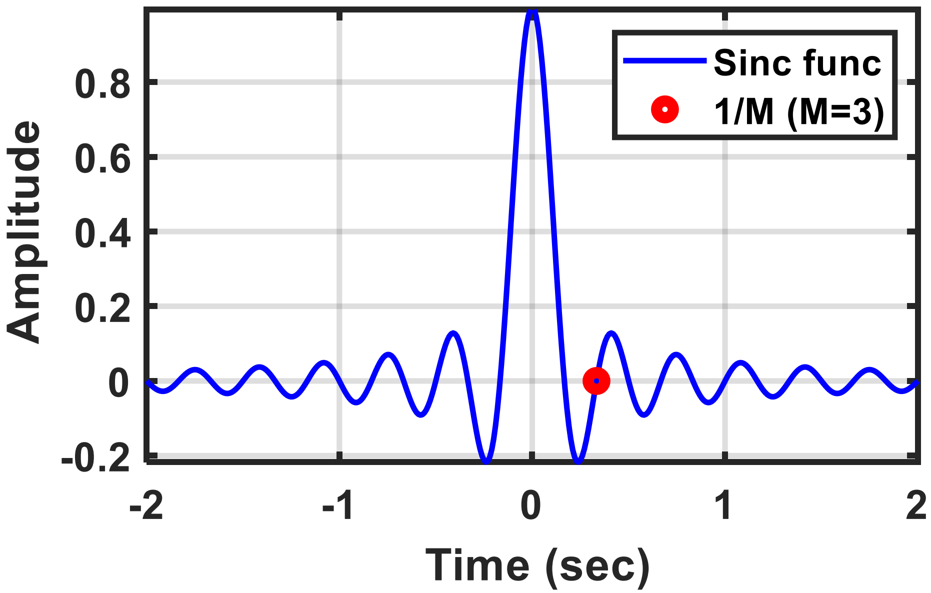


Table of Fourier Transform Pairs

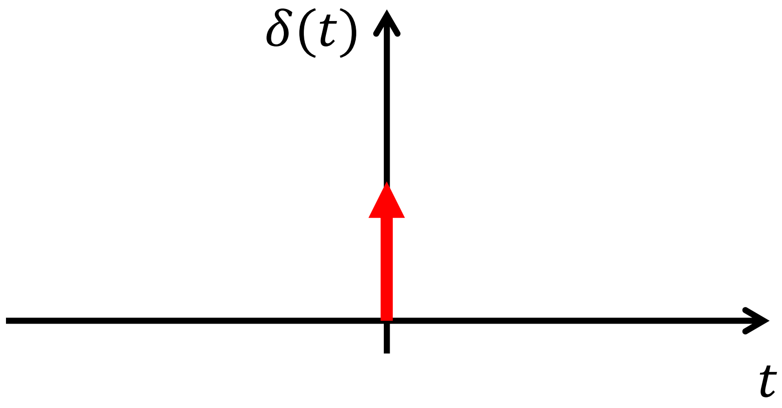
No.	Time function $x(t)$	Fourier transform $X(f)$	$X(\omega)$
1	$\delta(t)$	1	1
2	1	$\delta(f)$	$2\pi\delta(\omega)$
3	A	$A\delta(f)$	$2\pi A\delta(\omega)$
4	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	$\pi\delta(\omega)+\frac{1}{j\omega}$
5	$\delta(t-t_0)$	$e^{-j2\pi f t_0}$	$e^{-j\omega t_0}$
6	$e^{j2\pi f_0 t}$ or $e^{j\omega_0 t}$	$\delta(f-f_0)$	$2\pi\delta(\omega-\omega_0)$
7	$\cos(2\pi f_0 t)$ or $\cos(\omega_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
8	$\sin(2\pi f_0 t)$ or $\sin(\omega_0 t)$	$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
9	$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2+4\pi^2 f^2}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
10	$\frac{1}{\alpha^2+t^2}$	$\frac{\pi}{\alpha}e^{-\alpha 2\pi\|f\|}$	$\frac{\pi}{\alpha}e^{-\alpha\|\omega\|}$
11	$x(t)=e^{-\alpha t}u(t)$	$\frac{1}{\alpha+j2\pi f}$	$\frac{1}{\alpha+j\omega}$
12	$x(t)=A \quad \|t\|<T$; $=0 \quad \|t\|>T$	$2AT\frac{\sin(2\pi fT)}{2\pi fT}$	$2AT\frac{\sin(\omega T)}{\omega T}$
13	$2Af_0\frac{\sin(2\pi f_0 t)}{2\pi f_0 t}$ or $A\frac{\sin(\omega_0 t)}{\pi t}$	$X(f)=A \quad \|f\|<f_0$; $=0 \quad \|f\|>f_0$	$X(\omega)=A \quad \|\omega\|<\omega_0$; $=0 \quad \|\omega\|>\omega_0$
14	$\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$ or $\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$	$\sum_{n=-\infty}^{\infty} c_n\delta(f-nf_0)$	$2\pi\sum_{n=-\infty}^{\infty} c_n\delta(\omega-n\omega_0)$
15	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$	$\frac{2}{j\omega}$
16	$\frac{1}{t}$	$-j\pi\,\mathrm{sgn}(f)$	$-j\pi\,\mathrm{sgn}(\omega)$

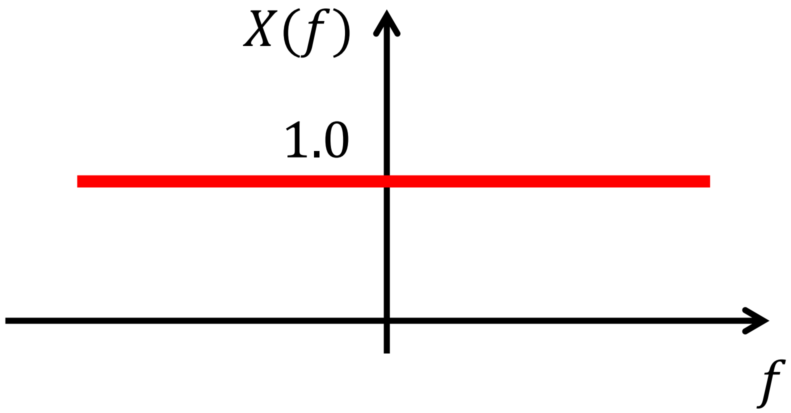
Example: Dirac Delta Function

$$\delta(t) = \infty \text{ for } t = 0$$
$$= 0 \quad \text{otherwise}$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-i2\pi f t} dt = e^{-i2\pi f \cdot 0} = 1$$

$$\int_{-\infty}^{\infty} x(t)\delta(t-a)dt = x(a)$$



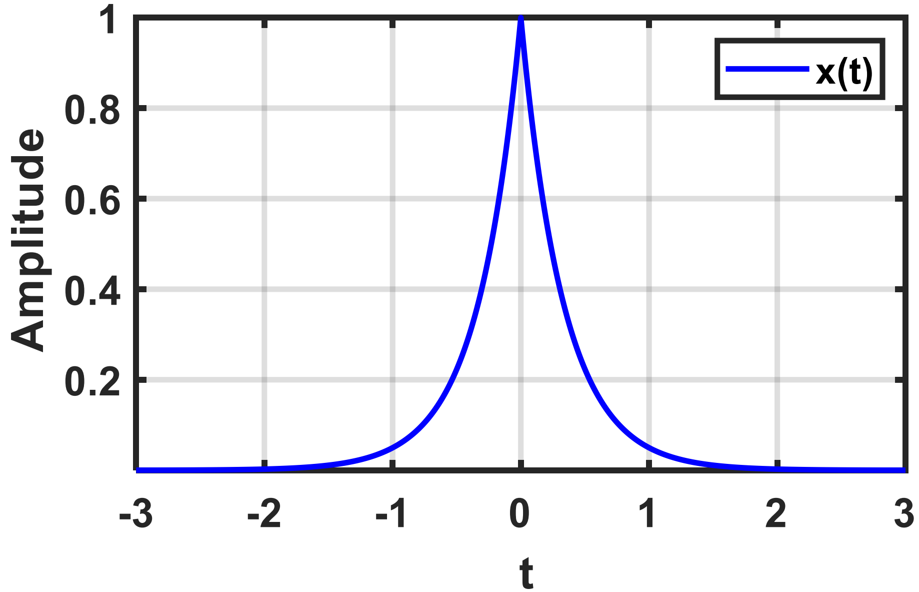


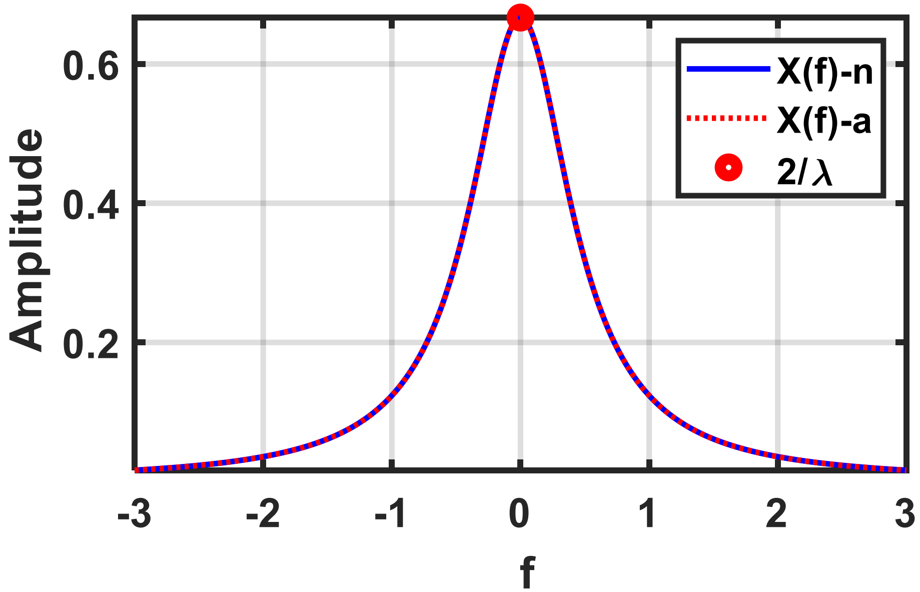
Example: Exponentially Decaying Symmetric Function

$$x(t) = e^{-\lambda|t|}, \qquad \lambda > 0$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} e^{-\lambda|t|}e^{-i2\pi ft}dt$$

$$= \int_{-\infty}^{0} e^{\lambda t}e^{-i2\pi ft}dt + \int_{0}^{\infty} e^{-\lambda t}e^{-i2\pi ft}dt = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$



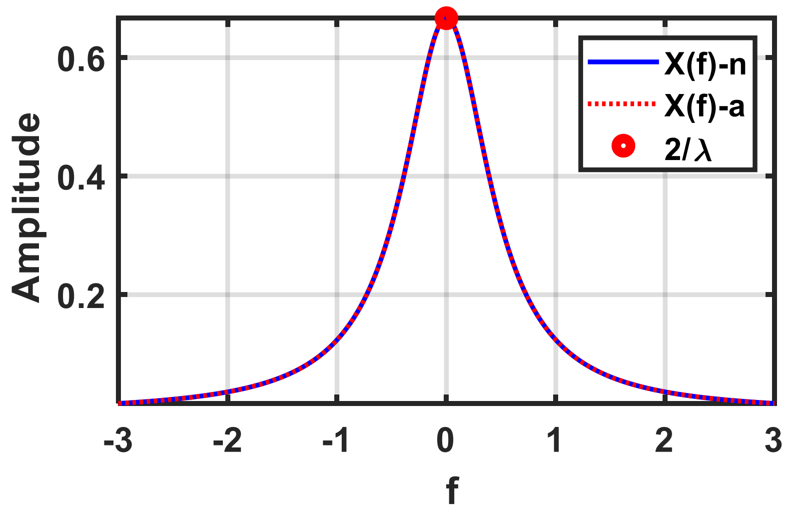


Example: Exponentially Decaying Symmetric Function (Continue)

$$x(t) = e^{-\lambda|t|}, \qquad \lambda > 0 \qquad\qquad X(f) = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$

$$X(0) = \frac{2\lambda}{\lambda^2} = \frac{2}{\lambda}$$

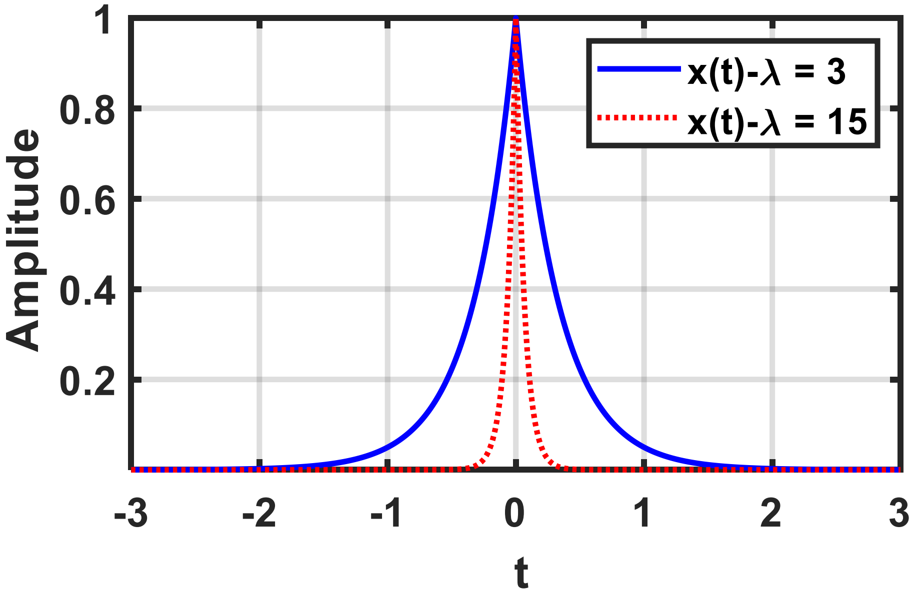
$$X\left(\frac{\lambda}{2\pi}\right) = \frac{2\lambda}{\lambda^2 + \lambda^2} = \frac{1}{\lambda} = \frac{X(0)}{2}$$

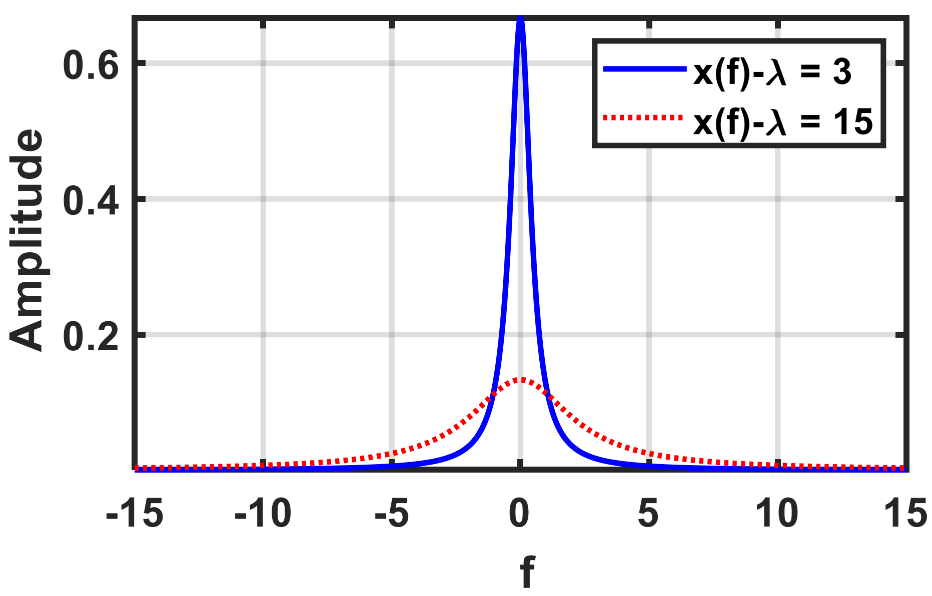


if λ is large then $X(f)$ is narrow in the time domain, but wide in the frequency domain and vice versa. This is an example of the so-called inverse spreading property of the Fourier transform, i.e. the wider in one domain, then the narrower in the other.

Exponentially Decaying Symmetric Function with Different Lambda Values

if λ is large then $X(f)$ is narrow in the time domain, but wide in the frequency domain and vice versa. This is an example of the so-called inverse spreading property of the Fourier transform, i.e. the wider in one domain, then the narrower in the other.



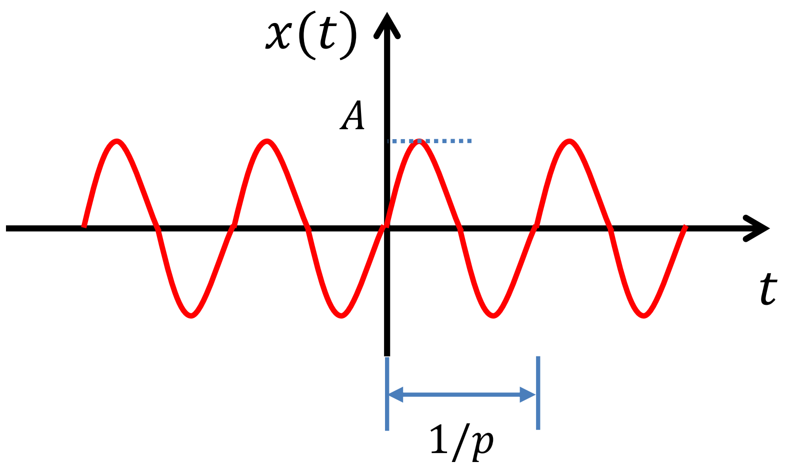


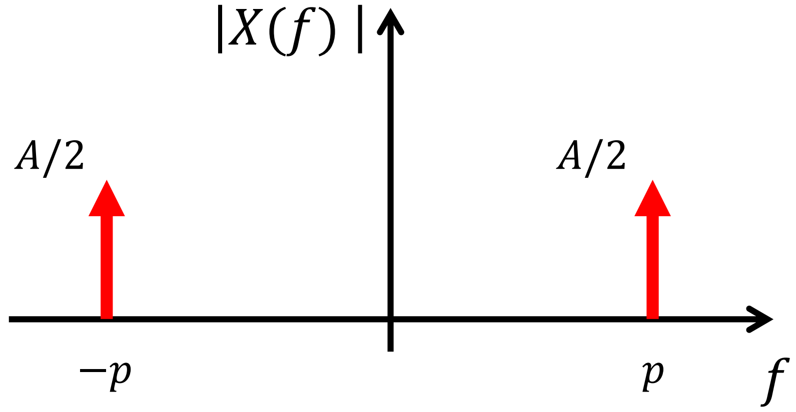
Example: Sine Function

$$x(t) = A \sin 2\pi p t$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} A \sin 2\pi p t \, e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} \frac{A}{2i} (e^{i2\pi p t} - e^{-i2\pi p t}) e^{-i2\pi f t} dt$$

$$= \frac{A}{2i} \int_{-\infty}^{\infty} (e^{-i2\pi(f-p)t} - e^{--i2\pi(f+p)t}) \, dt = \frac{A}{2i} [\delta(f-p) - \delta(f+p)]$$



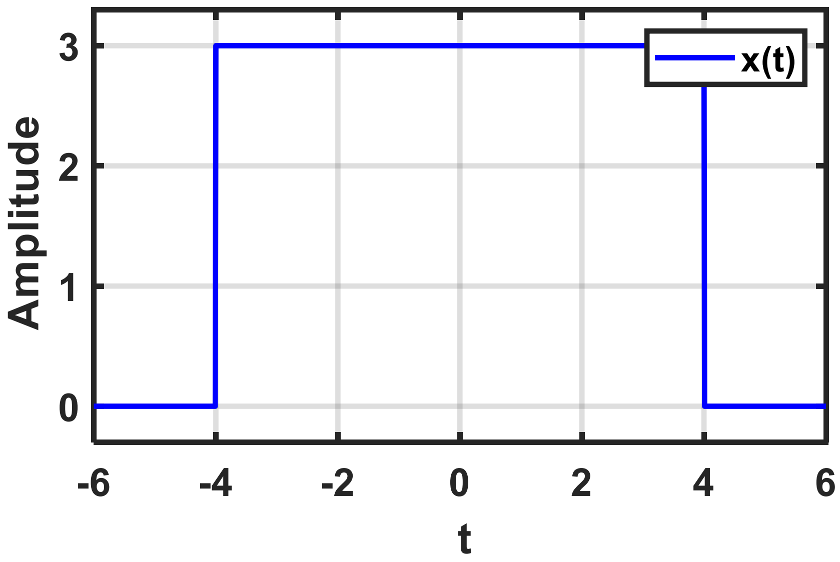


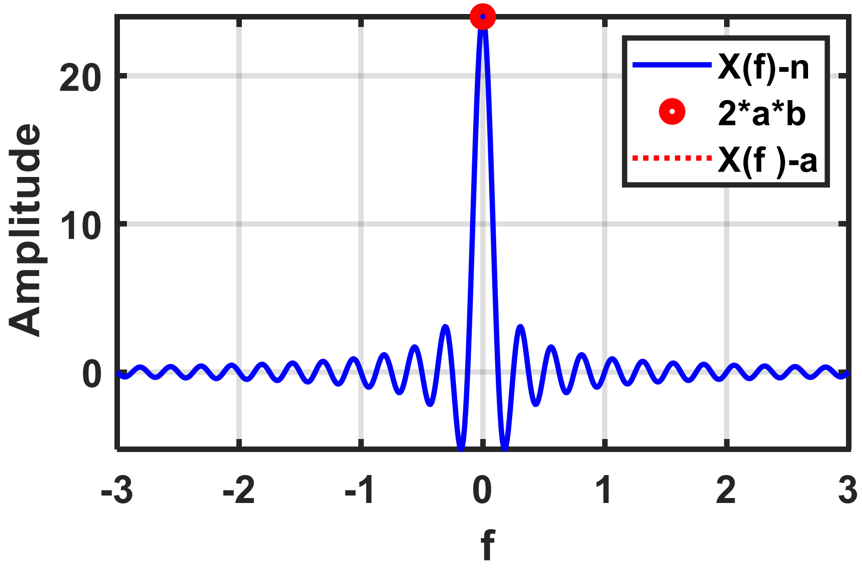
Example: Rectangular Function

$$x(t) = a \;\; \text{for } |t| < \text{b}$$
$$\;\;\;\; = 0 \;\; \text{for } |t| > \text{b}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-b}^{b} a e^{-i2\pi f t} dt = \frac{2ab \sin 2\pi f b}{2\pi f b}$$

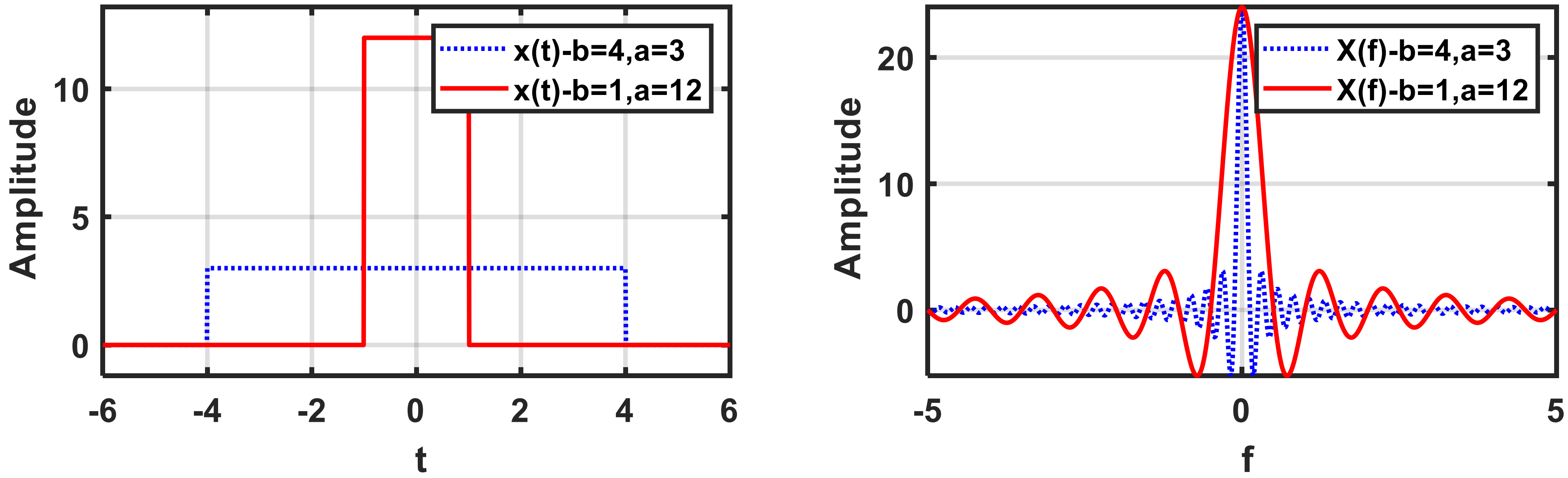
See Sinc function





Rectangular Function with Different Ranges

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-b}^{b} a e^{-i2\pi f t} dt = \frac{2ab \sin 2\pi f b}{2\pi f b}$$



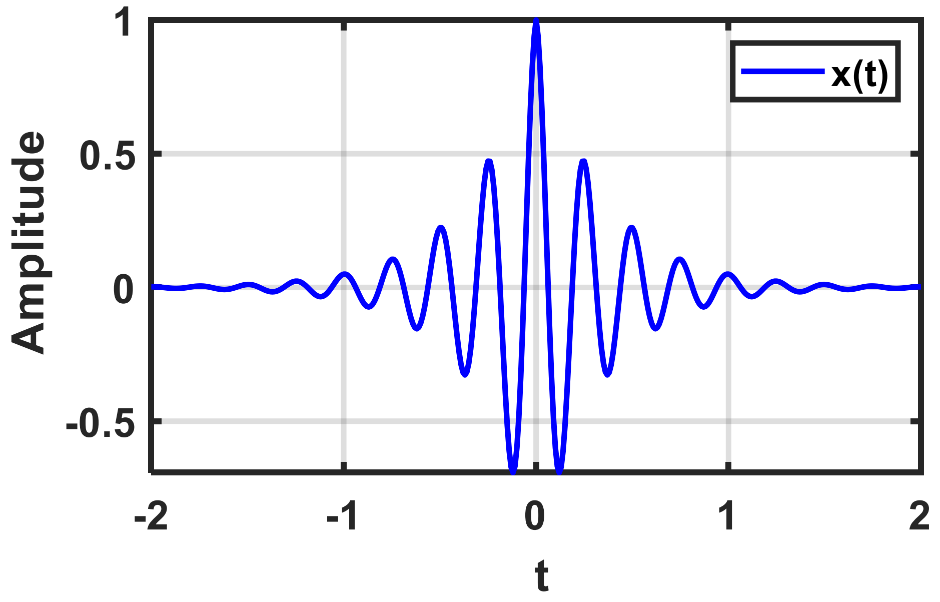
Example: Damped Symmetrically Oscillating Function

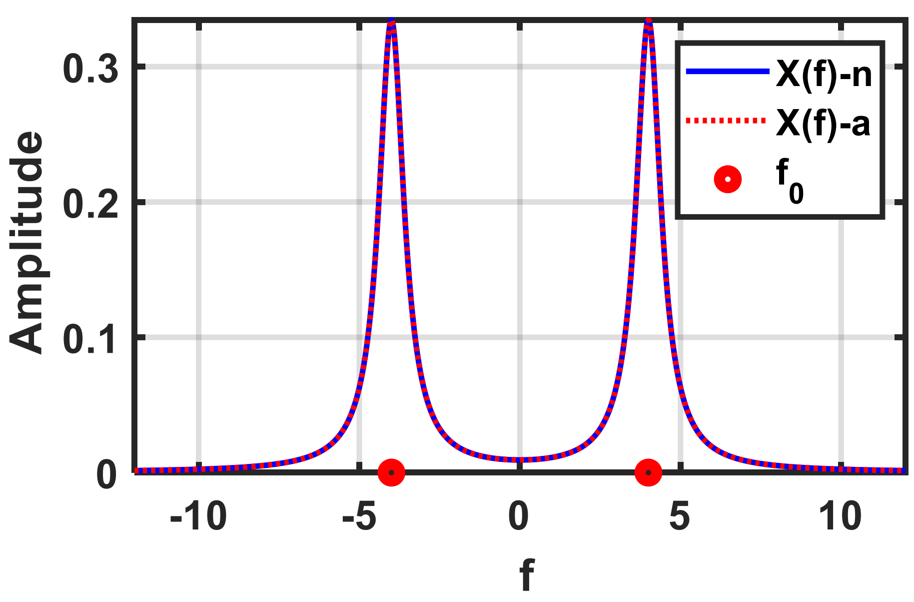
$$x(t) = e^{-a|t|} cos2\pi f_0 t$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} e^{-a|t|} cos2\pi f_0 t \, e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} e^{-a|t|} \frac{1}{2} (e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}) e^{-i2\pi f t} dt$$

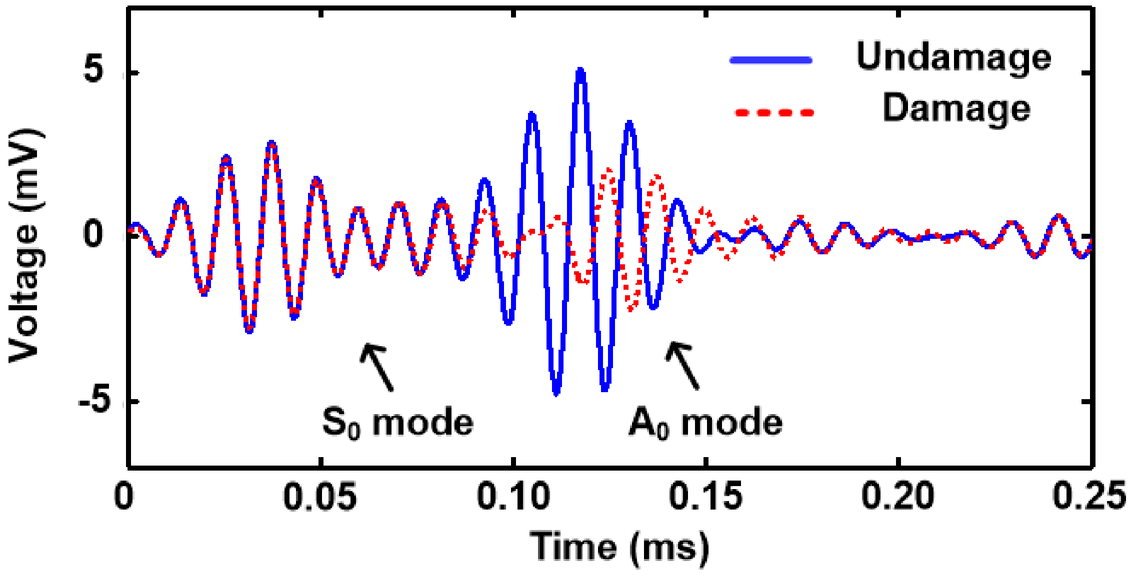
$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-a|t|} (e^{-i2\pi (f - f_0) t} + e^{-i2\pi (f + f_0) t}) \, dt = \frac{a}{a^2 + [2\pi(f - f_0)]^2} + \frac{a}{a^2 + [2\pi(f + f_0)]^2}$$

convolution

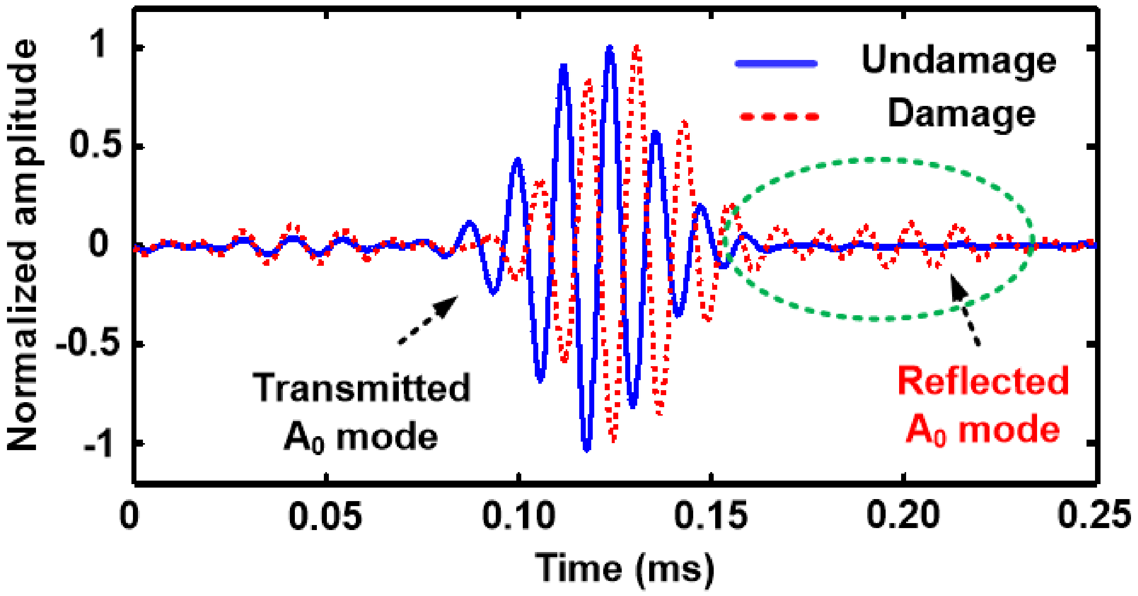




Example: Ultrasonic-based Damage Detection



Pitch-catch Lamb waves obtained from undamaged and damaged conditions

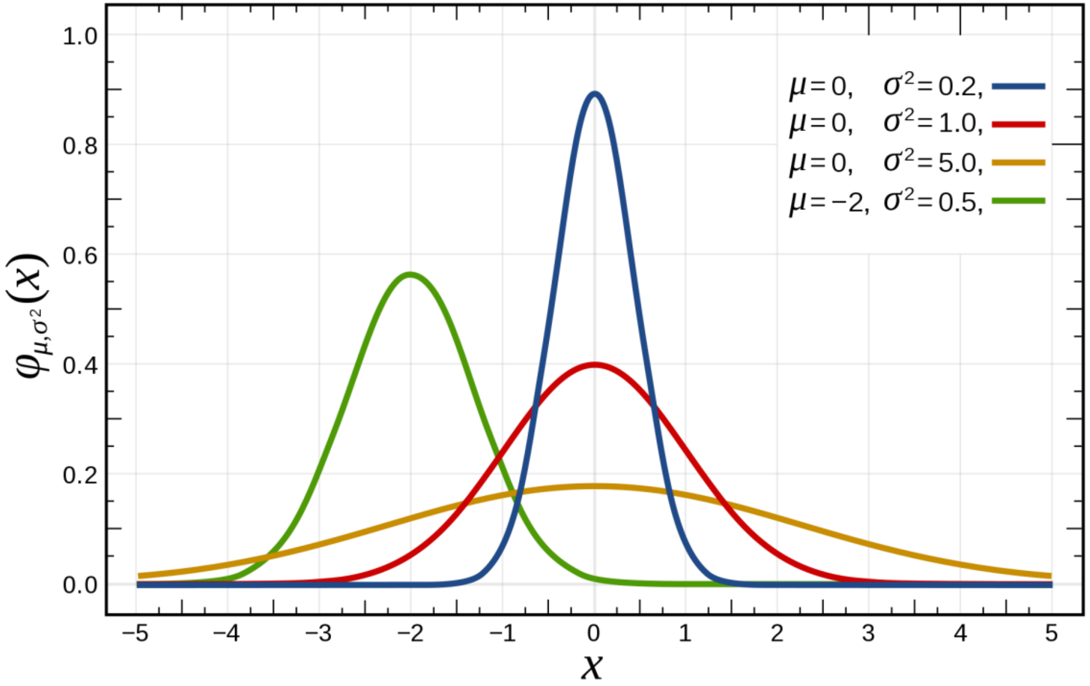


Extracted A_0 mode obtained from undamaged and damaged conditions

https://www.sciencedirect.com/science/article/pii/S0165212511000102

https://www.sciencedirect.com/science/article/pii/S0886779809001266

Gaussian Function



$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}}$$

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{\sigma}\right)^2}$$

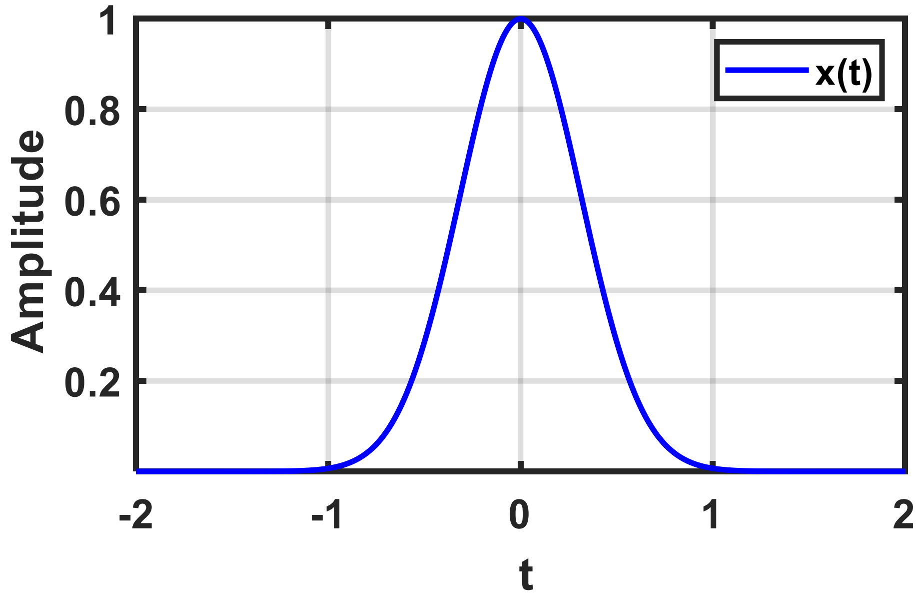
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ae^{-\frac{(x-b)^2}{2c^2}} dx = \sqrt{2\pi}a|c|$$

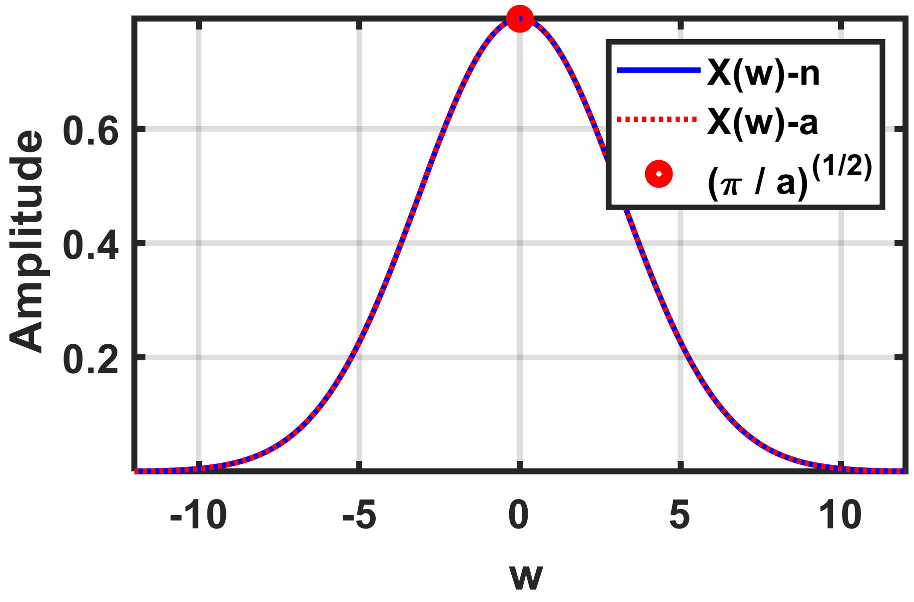
Example: Gaussian Function

$$x(t) = \boxed{e^{-at^2}}$$

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt = \int_{-\infty}^{\infty} e^{-at^2}e^{-iwt}dt = \int_{-\infty}^{\infty} e^{-a(t^2+\frac{iwt}{a})}dt = e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-a(t^2+\frac{iwt}{a}-\frac{w^2}{4a})}dt$$

$$= e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(t+i\left(\frac{w}{2a}\right)\right)^2}dt = e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-ay^2}dy = \boxed{\sqrt{\pi/a}\cdot e^{-\frac{w^2}{4a}}}$$

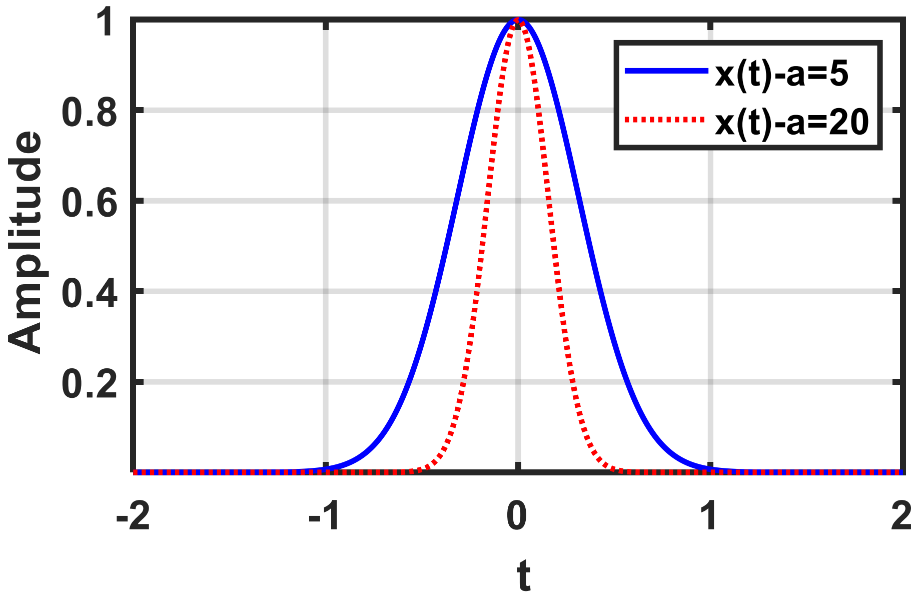


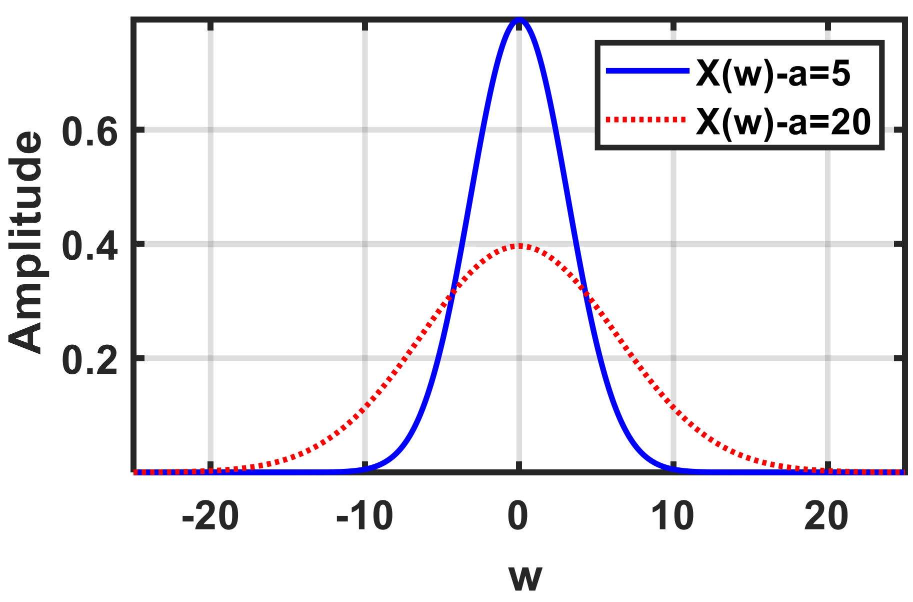


Gaussian Function with Different Variances

$$x(t) = e^{-at^2} \qquad X(w) = \sqrt{\pi/a} \cdot e^{-\frac{w^2}{4a}}$$

As a is increasing, then, $x(t)$ become narrower but $X(f)$ is wider. The wider in one domain, then the narrower in the other. → Inverse spreading property of the Fourier transform





Example: Fourier Transform of a Periodic Function

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}$$

$$\int_{-\infty}^{\infty} e^{\pm i2\pi at} dt = \delta(a)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}\, e^{-i2\pi ft} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} c_n e^{-i2\pi\left(f-\frac{n}{T_p}\right)t} dt = \sum_{n=-\infty}^{\infty} c_n \delta(f - \frac{n}{T_p})$$

Fourier transform of a periodic function is a series of delta functions scaled by c_n, and located at multiples of the fundamental frequency, $1/T_p$.