Signal Processing I (Fourier Series)

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Reference

We will cover some key topics in Chapters 3 ~ 6 of the following reference:

Shin, K., & Hammond, J. K. (2008). Fundamentals of Signal Processing: for Sound and Vibration Engineers, John Wiley & Sons.

Chapter 3: Fourier Series

Chapter 4: Fourier Integrals (Fourier Transform) and Continuous-Time Linear Systems

Chapter 5: Time Sampling and Aliasing

Chapter 6: The Discrete Fourier Transform

Discrete Fourier Transform Using Fast Fourier Transform

A **F**ast **F**ourier **T**ransform (FFT) is an algorithm that computes the **Discrete** **F**ourier **T**ransform (DFT) of a sequence, or its inverse (IDFT). Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa. It manages to reduce the complexity of computing the DFT from $O(n^2)$, which arises if one simply applies the definition of DFT, to $O(n\ log\ n)$, where n is the data size.

`fft` in Python

https://docs.scipy.org/doc/scipy/reference/generated/scipy.fft.fft.html#scipy.fft.fft

scipy.fft.fft

```
scipy.fft.fft(x, n=None, axis=-1, norm=None, overwrite_x=False, workers=None, *, plan=None)
```
[source]

Compute the 1-D discrete Fourier Transform.

This function computes the 1-D *n*-point discrete Fourier Transform (DFT) with the efficient Fast Fourier Transform (FFT) algorithm [1].

Parameters: **x** : ***array_like***

Input array, can be complex.

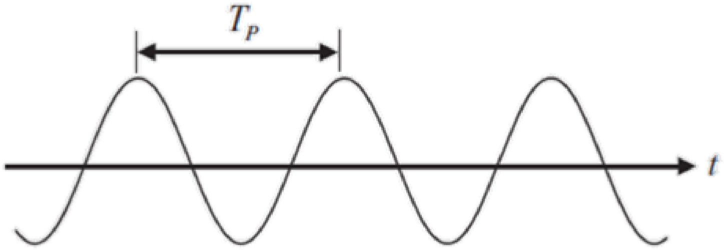
n : ***int, optional***

Length of the transformed axis of the output. If *n* is smaller than the length of the input, the input is cropped. If it is larger, the input is padded with zeros. If *n* is not given, the length of the input along the axis specified by *axis* is used.

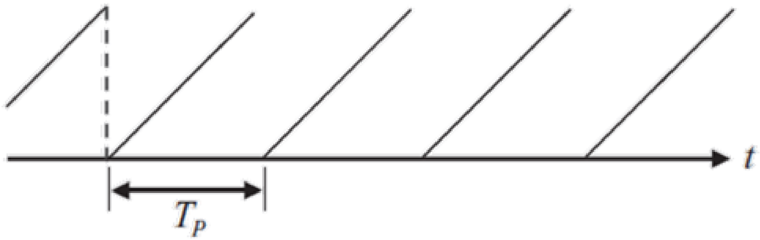
Periodic Signal

Periodic signals are defined as those whose waveform repeats exactly at regular time intervals. The mathematical definition of periodicity implies that the periodic behavior of the wave is unchanged for all time. This is expressed as

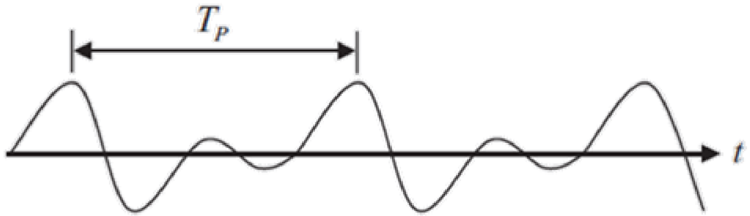
$$x(t) = x(t + nT_p) \qquad n = \pm 1, \pm 2, \pm 3, \ldots \qquad \text{Periodic}$$



Single sinusoidal signal



Triangular signal



General periodic signal

Example: Sinusoidal Signal

The simplest example is a sinusoidal signal

$$x(t) = Xsin(wt + \emptyset) = Xsin(2\pi ft + \emptyset)$$

where X is amplitude,

w is a circular (angular) frequency in radians per unit time (rad/s),

f is a (cyclical) frequency in cycles per unit time (Hz),

$\emptyset$ is phase angle with respect to the time origin in radians.

Q: What is the period of this signal?

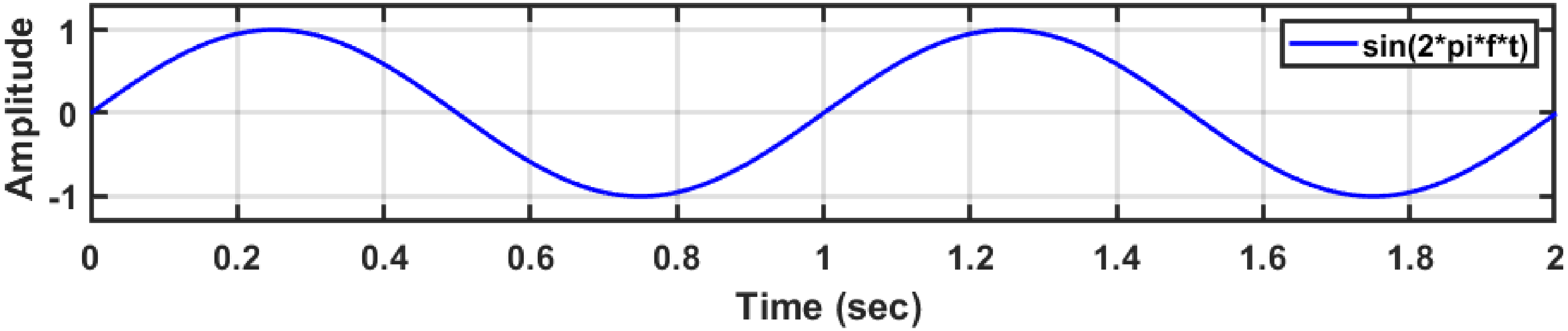
Q: Does the phase change the period?

$$sin(\theta + 2n\pi) = \sin(\theta)$$

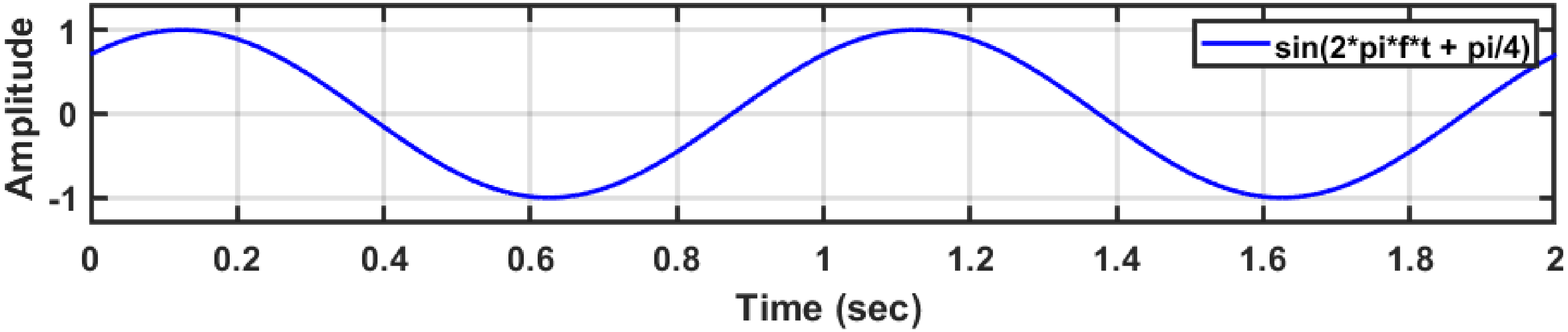
$$x(t) = x(t + nT_p)$$

Example: Sinusoidal Signals

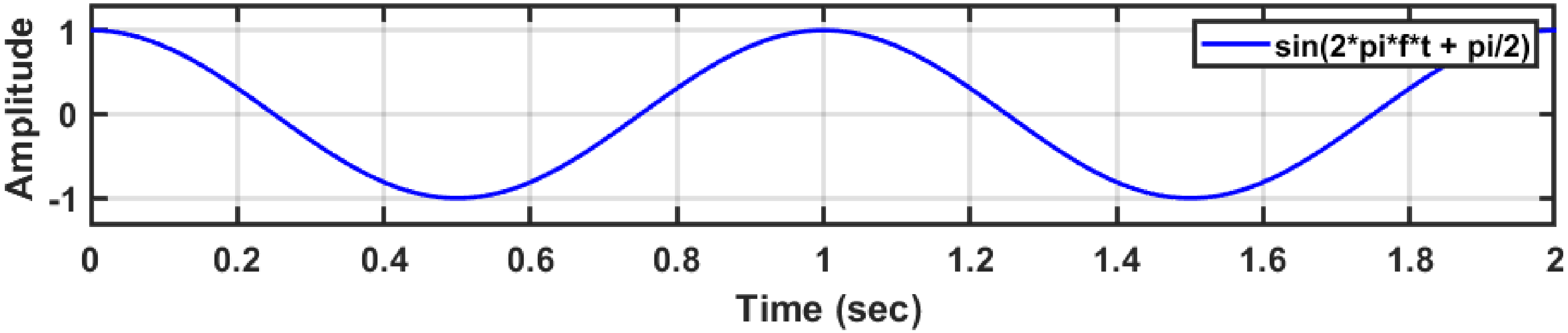
$sin(2\pi f_0 t)$



$sin(2\pi f_0 t + \pi/4)$



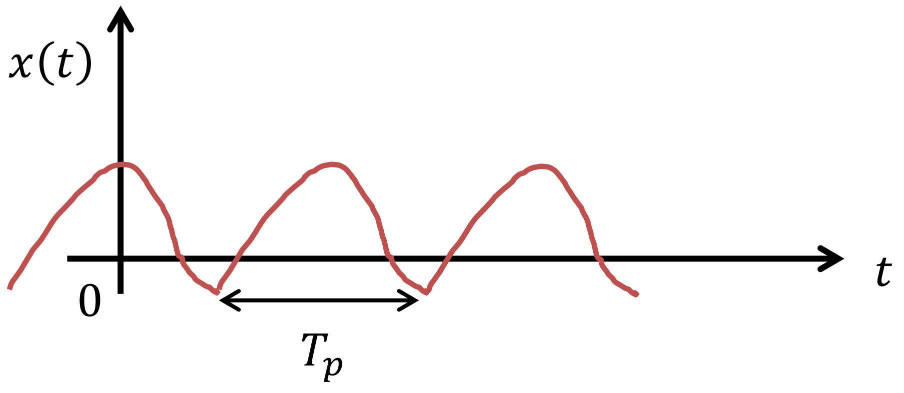
$sin(2\pi f_0 t + \pi/2)$



Frequency (f_0): 1Hz

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. The basis of Fourier analysis of **a periodic signal** is the representation of such a signal by adding together sine and cosine functions of appropriate frequencies and amplitudes. It decomposes **any** periodic function or periodic signal into the weighted sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines.



$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$$x(t) = x(t + mT_p)$$ Periodic

Fourier Coefficients

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

The coefficients are calculated from the following expressions:

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt$$

Q: What is the a_0?

$$a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt$$

$$b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

Basic Trigonometric Equations

Supplement

$$\int_{-\pi}^{\pi} \cos nt \ dt = 0 \qquad \int_{-\pi}^{\pi} \sin nt \ dt = 0$$

When n is a positive integer.

$$\cos mt \cos nt = \frac{1}{2}[\cos(m+n)\,t + \cos(m-n)\,t]$$

$$\sin mt \sin nt = \frac{1}{2}[\cos(m-n)\,t - \cos(m+n)\,t]$$

$$\sin mt \cos nt = \frac{1}{2}[\sin(m+n)\,t + \sin(m-n)\,t]$$

Orthogonality of trigonometric functions

$$\int_{-\pi}^{\pi} \cos mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\boxed{\int_{-\pi}^{\pi} \sin mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ 0 \ if \ n = m \end{cases}}$$

When n,m is a positive integer.

Derivation of the Fourier Coefficients

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \boxed{\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt} = \frac{a_0}{2} + \frac{1}{T_p}\cancel{\int_{-T_p/2}^{T_p/2} \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right) dt} \qquad 0$$

$$a_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right) dt} = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right) \cos\left(\frac{2\pi mt}{T_p}\right) dt$$

$$= \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} \overbrace{\cancel{\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right) \cos\left(\frac{2\pi mt}{T_p}\right)}}^{0} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) \cos\left(\frac{2\pi mt}{T_p}\right) dt = \frac{2a_m}{T_p}\int_{-T_p/2}^{T_p/2} \cos\left(\frac{2\pi mt}{T_p}\right) \cos\left(\frac{2\pi mt}{T_p}\right) dt = a_m$$

$$b_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t) \sin\left(\frac{2\pi mt}{T_p}\right) dt}$$

Numeric Integration

Supplement

```python
def fx(x):
    return x**3 - 6*x

n = 10000
a = 0
b = 3
dx = (b-a) / n # very narrow width

area_fx_num = 0

for ii in range(n):
    x_star = a + dx* ii
    area_fx_num += fx(x_star) * dx

area_fx_analy = -27/4
error_est = abs(area_fx_num - area_fx_analy)

print('The error of the numeric integration is ' + str(error_est))
```

$$f(x) = x^3 - 6x$$

```
error_est = 0.0014
```

```
The error of the numeric integration is 0.0013497975000120732
```

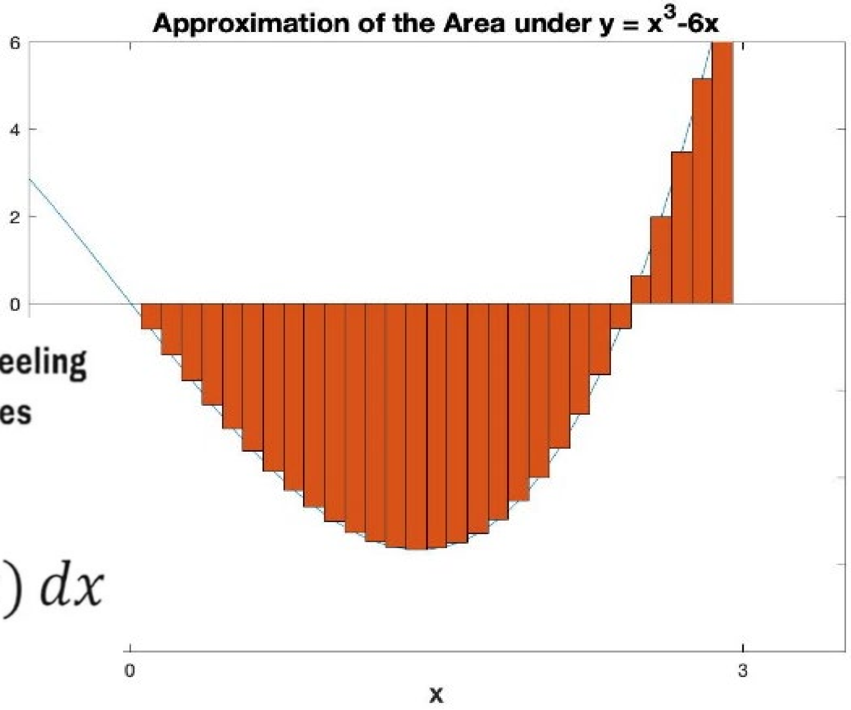
$$\int_0^3 f(x)dx = \frac{1}{4}x^4 - 3x^2\Big|_0^3 = \frac{81}{4} - 27 = -\frac{27}{4}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Riemann sum: approximation of an integral by a finite sum

Me peeling potatoes	My mum peeling potatoes
$\sum_{k=1}^{n} f(x_k).\Delta x$	$\int f(x)\,dx$



Tutorial

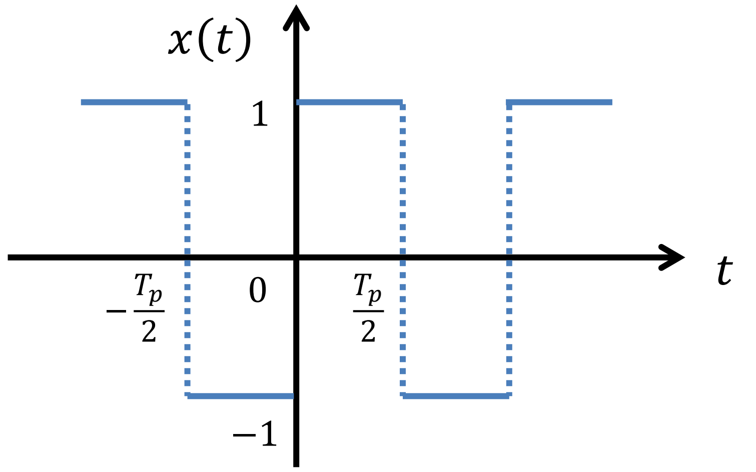
Example: Square Wave

$$x(t) = -1 \quad if \ -\frac{T_p}{2} < t \le 0$$

$$x(t) = \quad 1 \quad if \ 0 < t < \frac{T_p}{2}$$

$$x(t + nT_p) = x(t)$$

$$where \ n = \pm 1, \pm 2, \ldots$$



$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt = 0$$

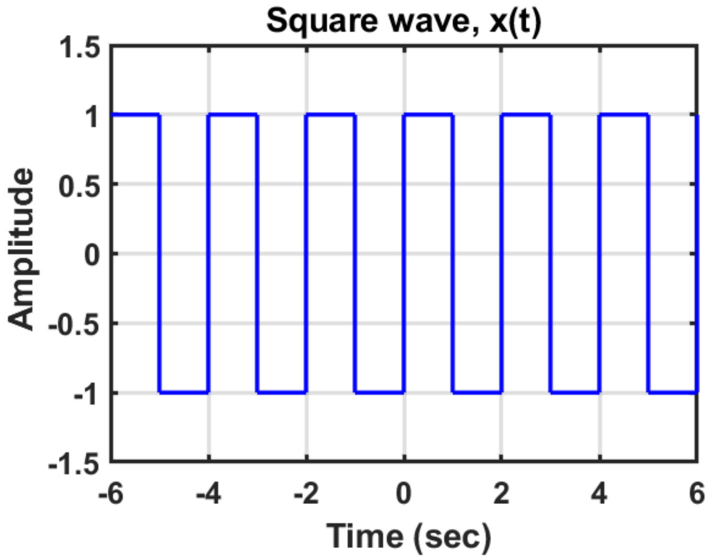
$$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cosn\pi) * \sin\left(\frac{2\pi nt}{T_p}\right)}$$

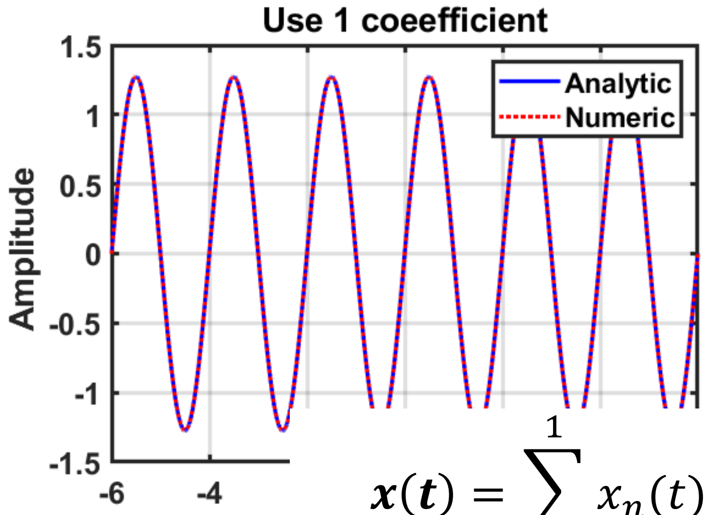
$$a_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi nt}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\cos\left(\frac{2\pi nt}{T_p}\right)dt + \int_{0}^{T_p/2} \cos\left(\frac{2\pi nt}{T_p}\right)dt\right] = 0$$

Analytic integration

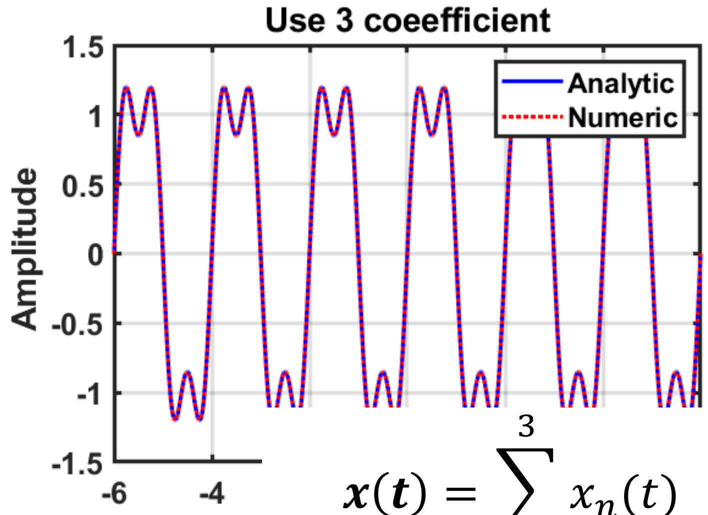
$$b_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi nt}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\sin\left(\frac{2\pi nt}{T_p}\right)dt + \int_{0}^{T_p/2} \sin\left(\frac{2\pi nt}{T_p}\right)dt\right] = \frac{2}{n\pi}(1 - cosn\pi)$$

Example: Square Wave (Continue)

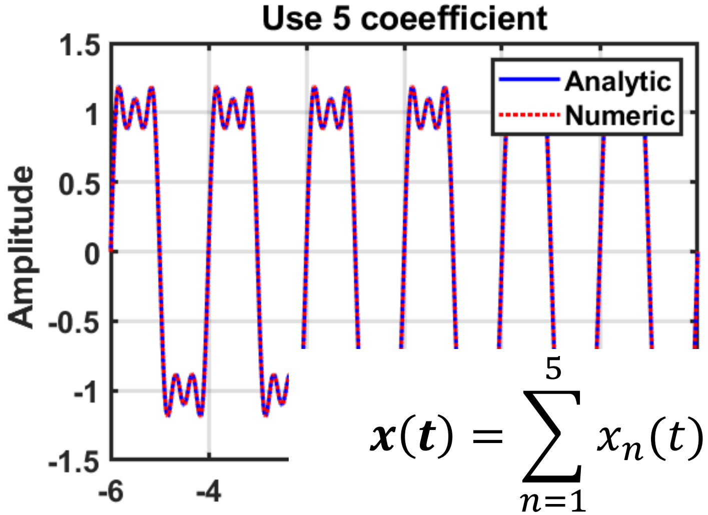




$$\boldsymbol{x(t)} = \sum_{n=1}^{1} x_n(t)$$



$$\boldsymbol{x(t)} = \sum_{n=1}^{3} x_n(t)$$



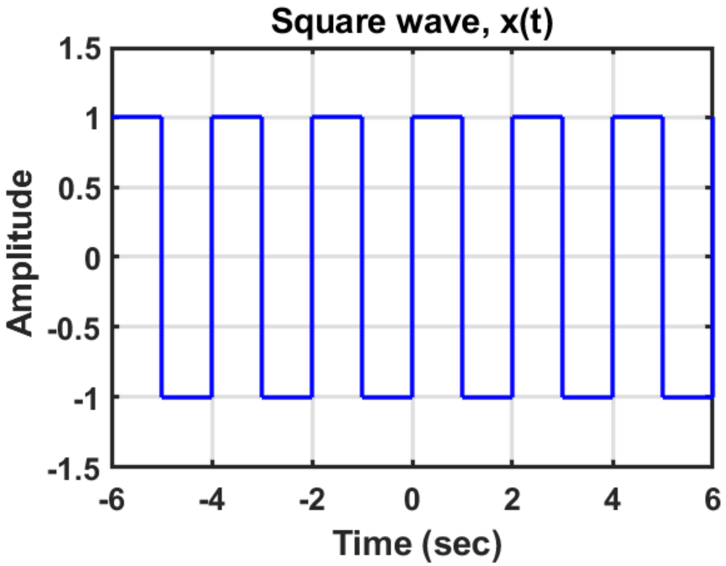
$$\boldsymbol{x(t)} = \sum_{n=1}^{5} x_n(t)$$

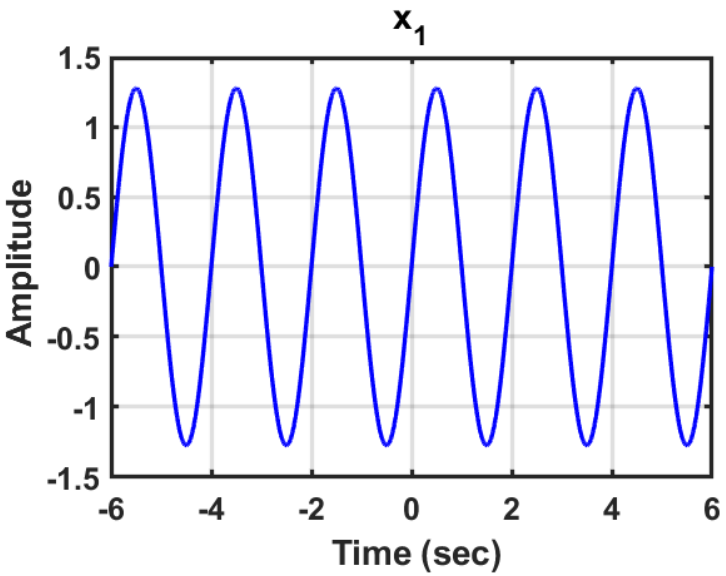
$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cos n\pi) * \sin\left(\frac{2\pi n t}{T_p}\right) = \sum_{n=1}^{\infty} x_n(t)$$

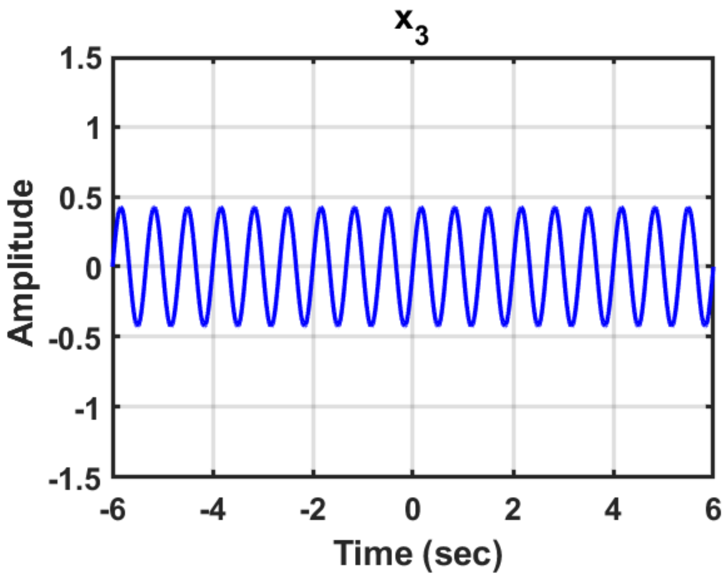
$$x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t) + \cdots$$

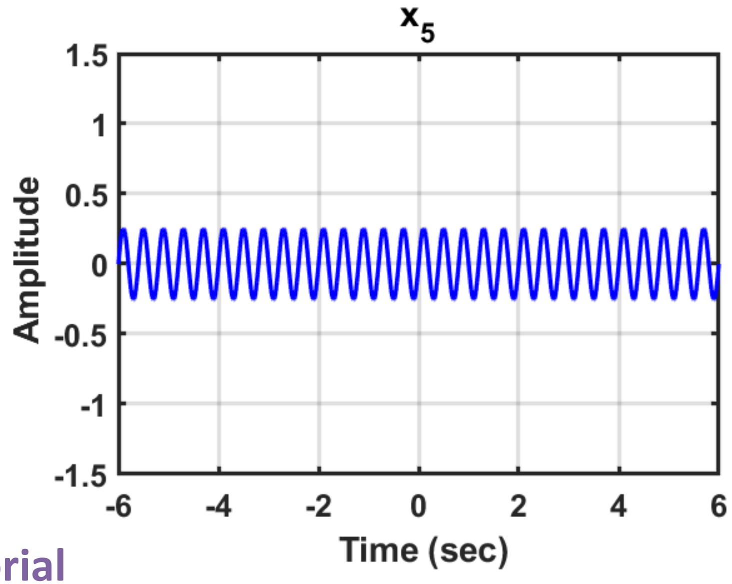
Tutorial

Example: Square Wave (Continue)









$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cos n\pi) * \sin\left(\frac{2\pi n t}{T_p}\right) = \sum_{n=1}^{\infty} x_n(t)$$

Tutorial

Example: Square Wave – Python Script

```python
# Make a square wave function
x = lambda t: square(t*(2*np.pi)/Tp)
```

```python
nCoeff = 5 # let's only compute five coefficients

# Analytic solution
# Assume that we know analytic solutions of an and bn (see above equations)
xn = np.zeros((nCoeff, len(t)))
for ii in range(nCoeff):
    xn[ii] = 2/(np.pi*(ii+1)) * (1-np.cos(np.pi*(ii+1))) * np.sin(2*np.pi*(ii+1)*t/Tp)

sig_y_analytic = np.cumsum(xn, axis=0)

# Numeric solution
# Assuem that an and bn should be computed from the equations of Fourier series

# Compute Fourier coefficients
# Compute a0
a0 = 1/Tp * quad(x, -Tp/2, Tp/2)[0]

# Compute an and bn
a = np.zeros(nCoeff)
b = np.zeros(nCoeff)
for ii in range(nCoeff):
    fun_a = lambda t: x(t) * np.cos(2*np.pi*(ii+1)*t/Tp)
    a[ii] = 2/Tp * quad(fun_a, -Tp/2, Tp/2)[0]

    fun_b = lambda t: x(t) * np.sin(2*np.pi*(ii+1)*t/Tp)
    b[ii] = 2/Tp * quad(fun_b, -Tp/2, Tp/2)[0]

# Numerical integration
sig_y_numeric = np.zeros((nCoeff, len(t)))
for ii in range(nCoeff):
    if ii == 0:
        sig_y_numeric[ii] = a0/2 + a[ii]*np.cos(2*np.pi*(ii+1)*t/Tp) + b[ii]*np.sin(2*np.pi*(ii+1)*t/Tp)
    else:
        sig_y_numeric[ii] = sig_y_numeric[ii-1] + a[ii]*np.cos(2*np.pi*(ii+1)*t/Tp) + b[ii]*np.sin(2*np.pi*(ii+1)*t/Tp)
```

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt \qquad a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt$$

$$b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} x_n(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - cosn\pi) * \sin\left(\frac{2\pi nt}{T_p}\right) = \sum_{n=1}^{\infty} x_n(t)$$

$$a_n = 0$$

$$a_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt = 0$$

$$b_n = \frac{2}{n\pi}(1 - cosn\pi)$$

Tutorial

Complex Form of the Fourier Series

Euler Formula

$$e^{iwt} = coswt + i\, sinwt \qquad e^{-iwt} = coswt - i\, sinwt \qquad coswt = \frac{1}{2}(e^{iwt} + e^{-iwt}) \qquad sinwt = \frac{1}{2j}(e^{iwt} - e^{-iwt})$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos wnt + b_n \sin wnt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}(e^{iwnt} + e^{-iwnt}) + \frac{b_n}{2j}(e^{iwnt} - e^{-iwnt})$$

$$\boldsymbol{w = \frac{2\pi}{T_p}}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt} = c_0 + \sum_{n=1}^{\infty} c_n e^{iwnt} + \sum_{n=1}^{\infty} c_n^* e^{-iwnt} \ \ where\ c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - jb_n}{2},$$

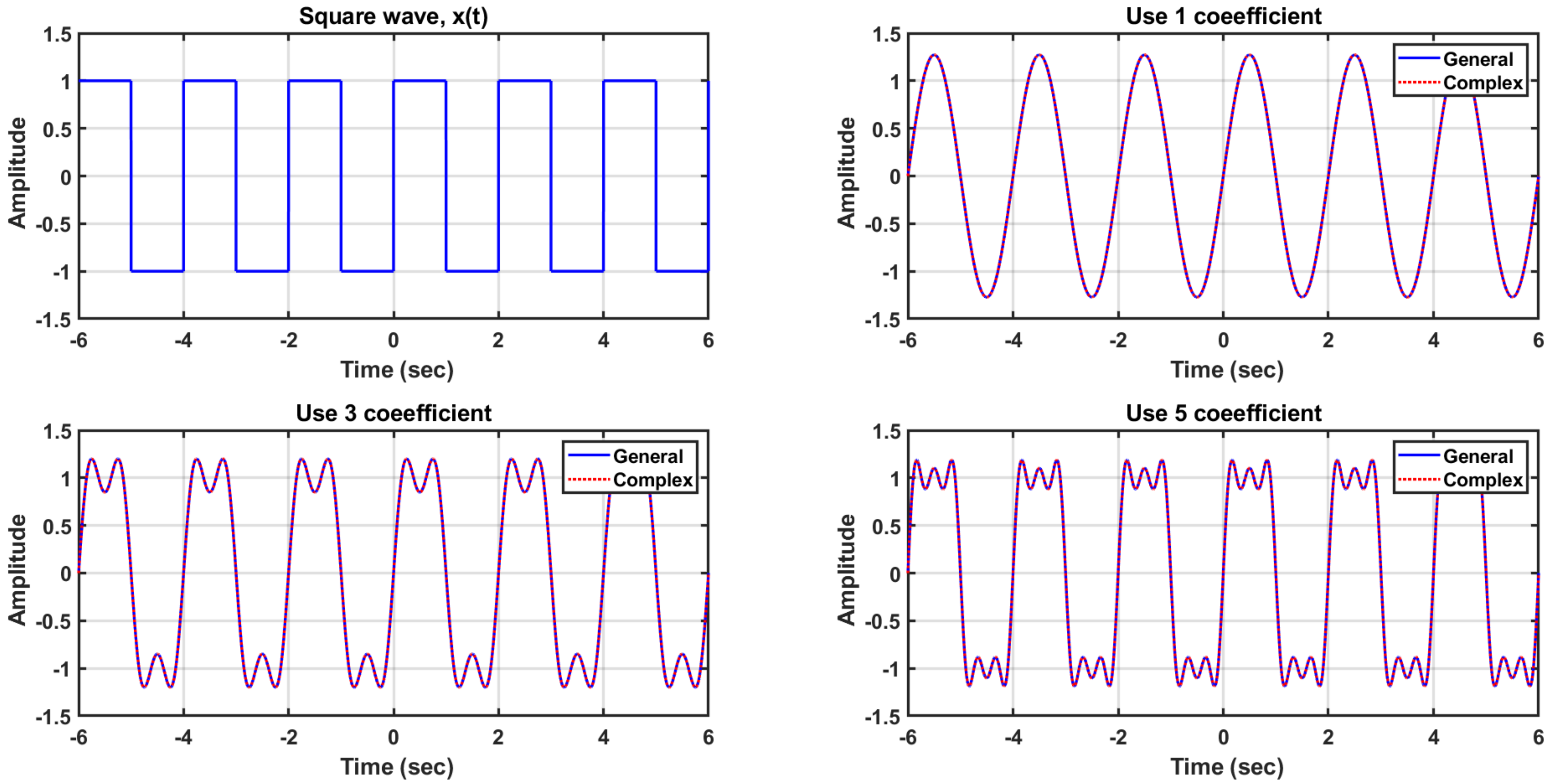
$$c_n^* = \frac{a_n + jb_n}{2}$$

$$\boxed{c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad c_n^* = \frac{1}{T_p}\int_0^{T_p} x(t)e^{iwnt}dt = c_{-n}}$$

Negative frequency term (c_{-n})

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi}{T_p}$$

Example: Square Wave (Comparison of General and Complex Forms)



Tutorial

Summary

General form

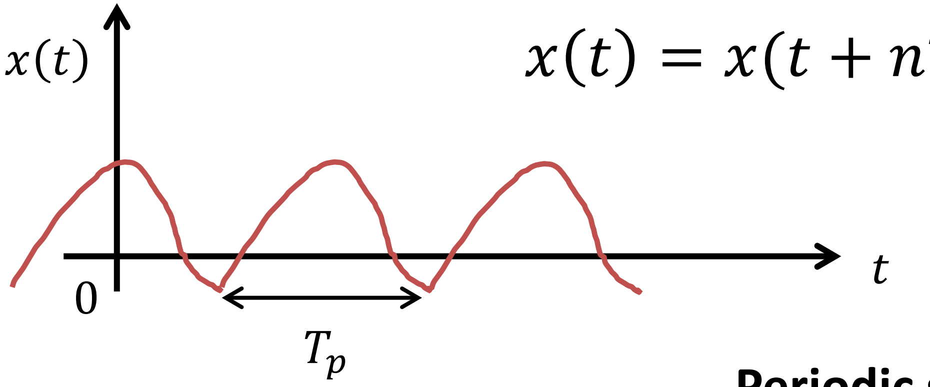
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt \qquad a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt \qquad b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

Complex form

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt}$$

$$c_n = \frac{1}{T_p}\int_{0}^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi n}{T_p}$$



$$x(t) = x(t + nT_p)$$

Periodic signal