RANSAC- RANdom SAmple Consensus

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CIVE 497 – CIVE 700: Smart Structure Technology

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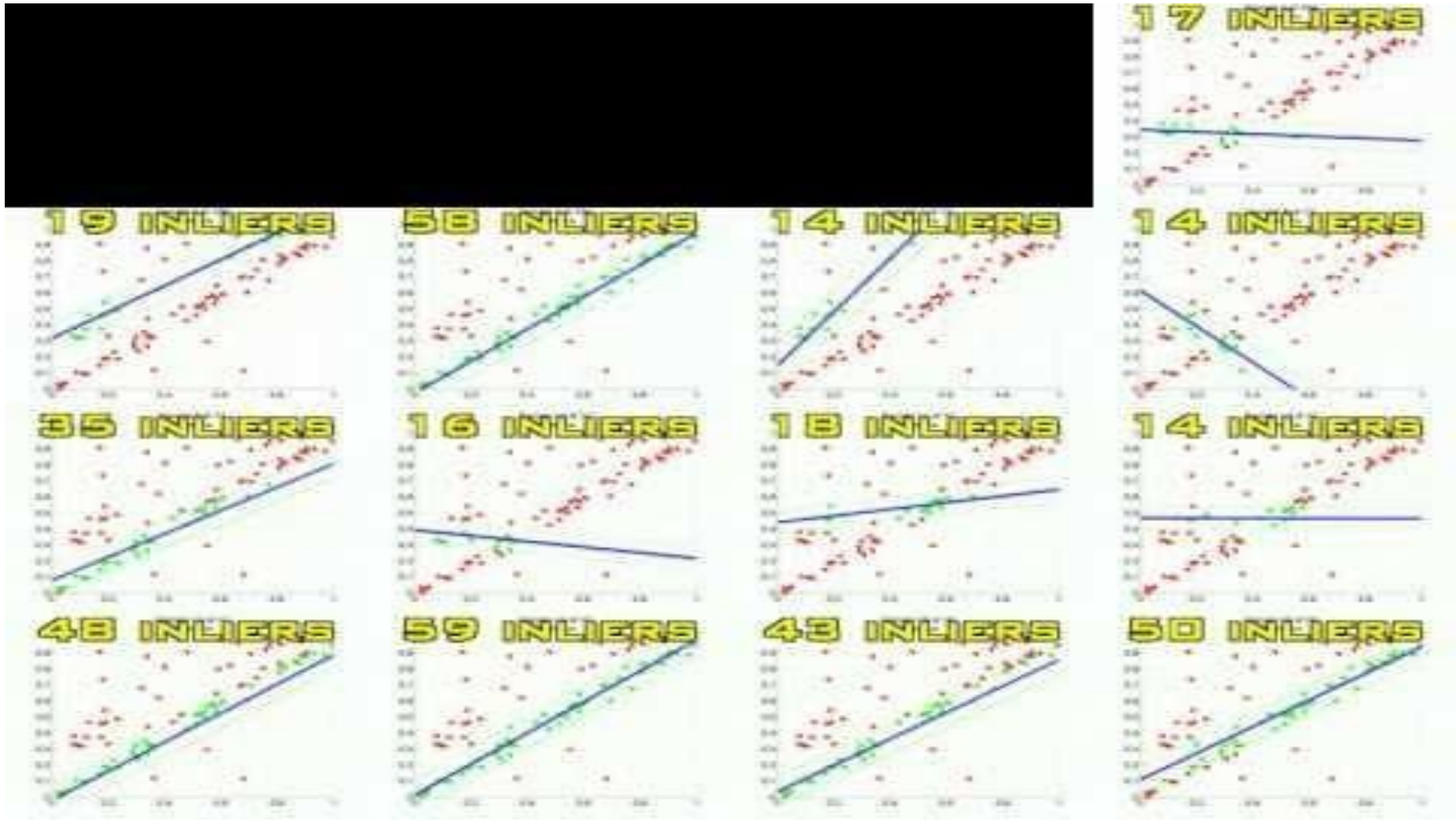
Overview

- If we know which points (with errors) belong to the line, how do we find the “optimal” line parameters?
  Least squares

- What if there are outliers?
  Robust fitting, RANSAC

- What if we’re not even sure it’s a line?
  Model selection (topics in machine learning)

RANSAC Song

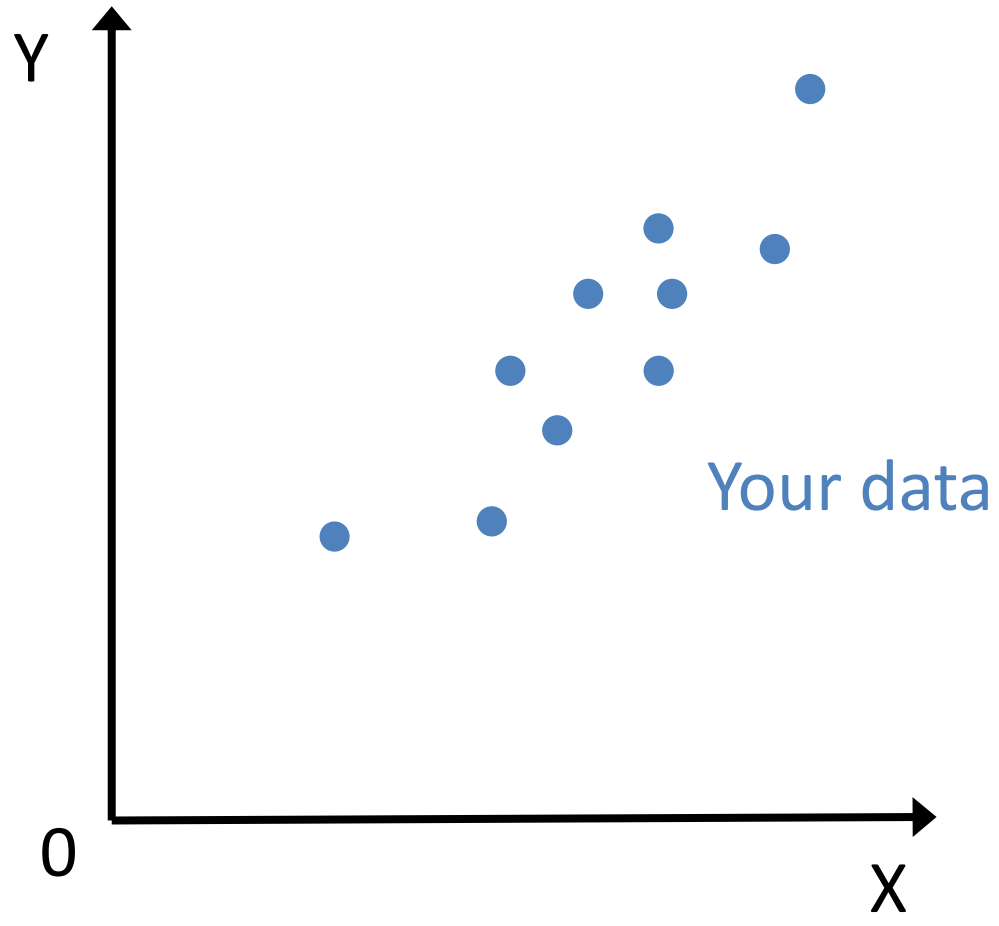

17 INLIERS
19 INLIERS
58 INLIERS
14 INLIERS
14 INLIERS
35 INLIERS
16 INLIERS
18 INLIERS
14 INLIERS
48 INLIERS
59 INLIERS
43 INLIERS
50 INLIERS

Line Fitting

Data (measurement): $(x_1, y_1), \ldots, (x_n, y_n)$

Known model: Line ($y_i = m\,x_i + b$)

We will find m and b.

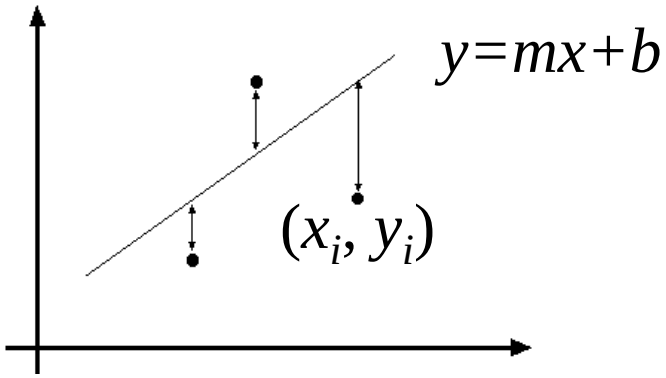


Least Squares Line Fitting (Approach 1)

Data (measurement): $(x_1, y_1), \ldots, (x_n, y_n)$

Model: Line ($y_i = m x_i + b$)

Task: Find (m, b)



Minimize $E = \sum_{i=1}^{n} (y_i - mx_i - b)^2$

$$\frac{\partial(E)}{\partial m} = -2\sum_{i=1}^{n}[y_i - mx_i - b]x_i = 0 \qquad \mathrm{b}\sum_{i=1}^{n} x_i + m\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$\frac{\partial(E)}{\partial b} = -2\sum_{i=1}^{n}[y_i - mx_i - b] = 0 \qquad \mathrm{nb} + m\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

Least Squares Line Fitting (Approach 1) (Continue)

$$\mathrm{nb} + m\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$\mathrm{b}\sum_{i=1}^{n} x_i + m\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$\rightarrow \quad \begin{bmatrix} \sum_{i=1}^{n} x_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$\begin{bmatrix} \sum_{i=1}^{n} x_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}^{-1}$$

$$= \frac{1}{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i - n\sum_{i=1}^{n} x_i^2} \begin{bmatrix} \sum_{i=1}^{n} x_i & -n \\ -\sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}$$

Least Squares Line Fitting (Approach 1) (Continue)

$$\begin{bmatrix} \sum_{i=1}^{n} x_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}^{-1} = \frac{1}{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i - n \sum_{i=1}^{n} x_i^2} \begin{bmatrix} \sum_{i=1}^{n} x_i & -n \\ -\sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$\mathrm{m} = \frac{\sum_{i=1}^{n} x_i y_i - 1/n(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i)}{\sum_{i=1}^{n} x_i^2 - 1/n\left(\sum_{i=1}^{n} x_i\right)^2}$$

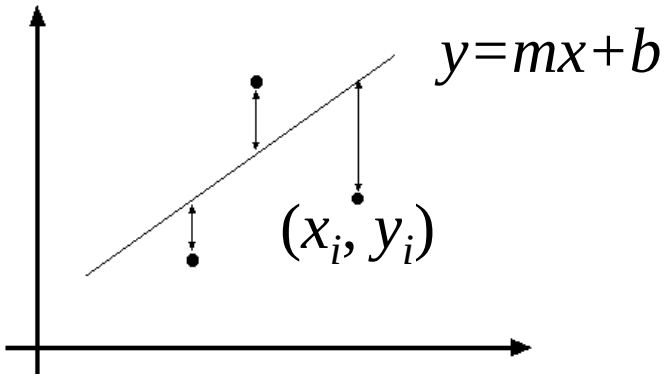
$$\mathrm{b} = \frac{1/n(\sum_{i=1}^{n} y_i)\left(\sum_{i=1}^{n} x_i^2\right) - 1/n \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 - 1/n\left(\sum_{i=1}^{n} x_i\right)^2}$$

Least Squares Line Fitting (Approach 2)

Data (measurement): $(x_1, y_1), \ldots, (x_n, y_n)$

Model: Line ($y_i = m x_i + b$)

Task: Find (m, b)



Minimize $$E = \sum_{i=1}^{n} (y_i - m x_i - b)^2$$

$$E = \|Y - XB\|^2 \quad \text{where} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad B = \begin{bmatrix} m \\ b \end{bmatrix}$$

$$E = \|Y - XB\|^2 = (Y - XB)^T (Y - XB) = Y^T Y - 2(XB)^T Y + (XB)^T (XB)$$

Least Squares Line Fitting (Approach 2) (Continue)

$$E = \|Y - XB\|^2 = (Y - XB)^T (Y - XB) = Y^T Y - 2(XB)^T Y + (XB)^T (XB)$$

$$\frac{dE}{dB} = 2X^T XB - 2X^T Y = 0 \qquad \boxed{X^T XB = X^T Y}$$

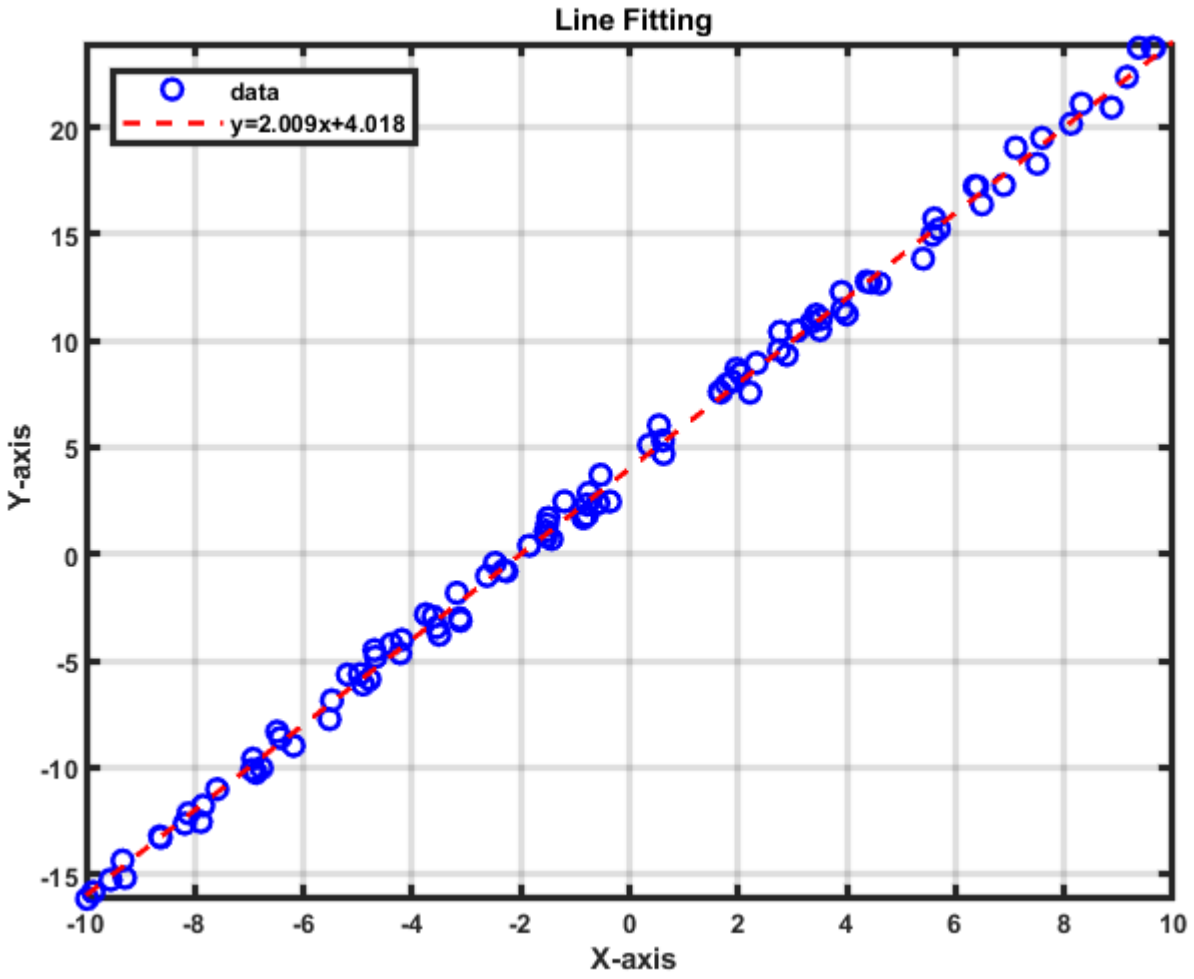
*L*east squares solution to $XB = Y$

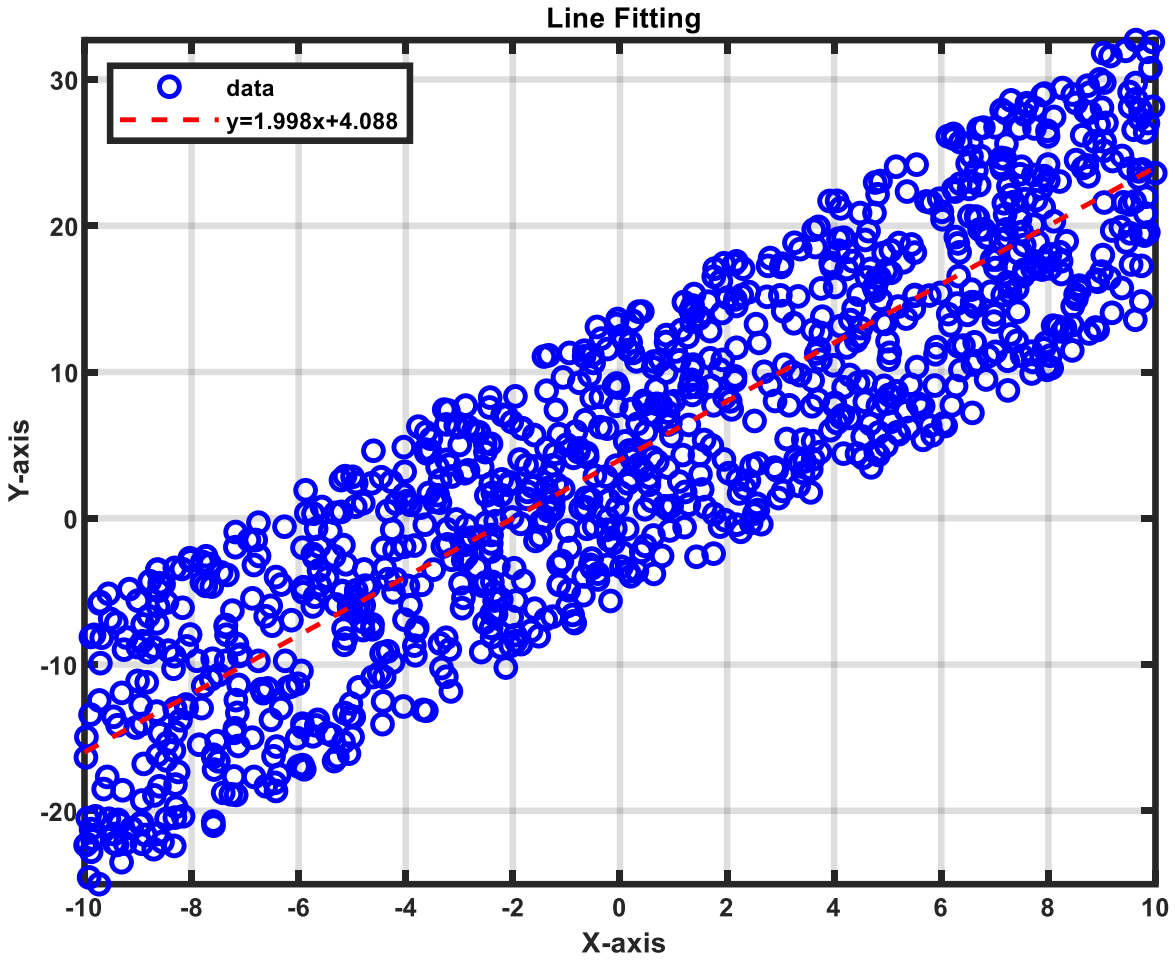
$$\text{where} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad B = \begin{bmatrix} m \\ b \end{bmatrix}$$

$$B \approx X^* Y \qquad \text{where } X^* = (X'X)^{-1}X'$$

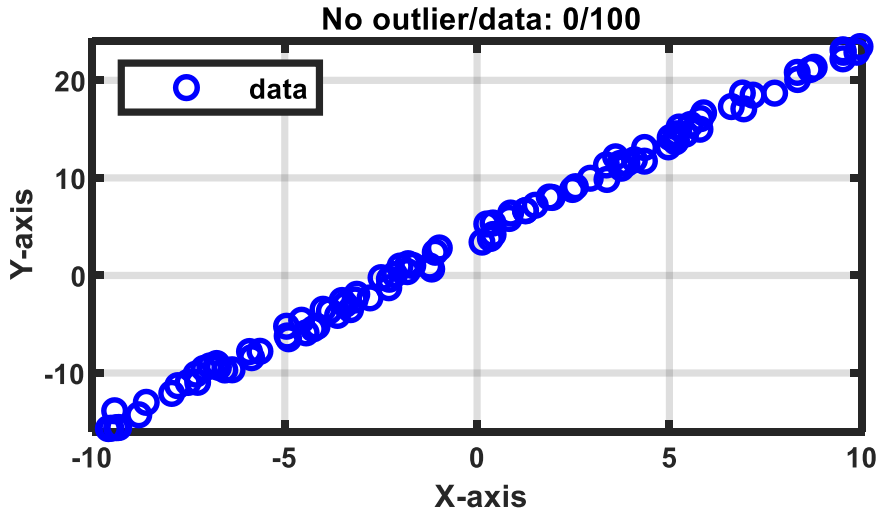
Pseudoinverse (when m>n)

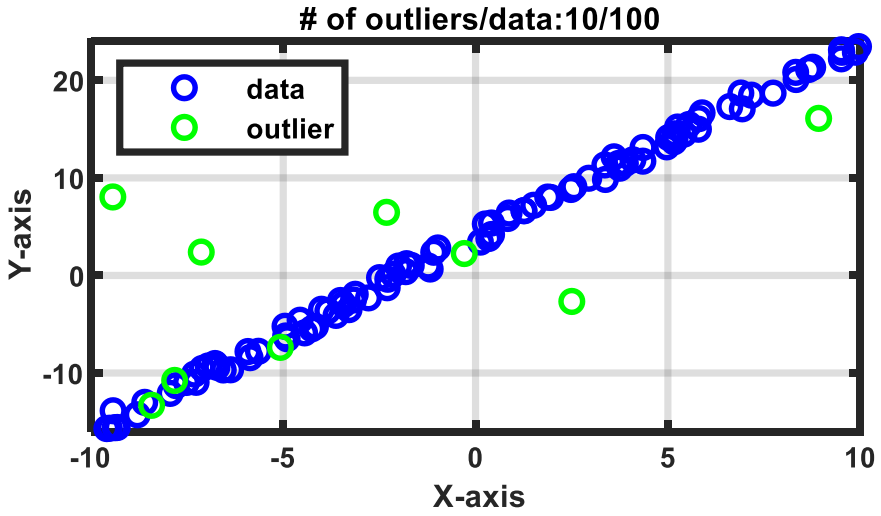
Least Square Line Fitting (Random Error)

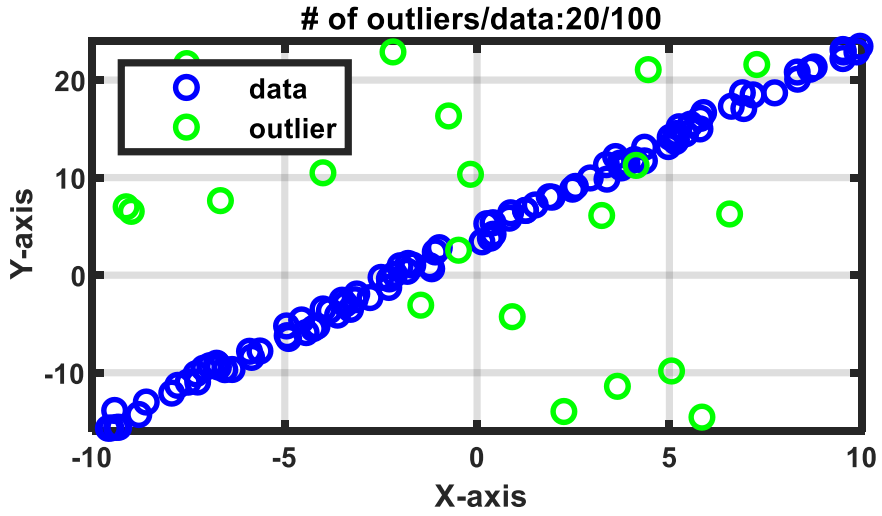


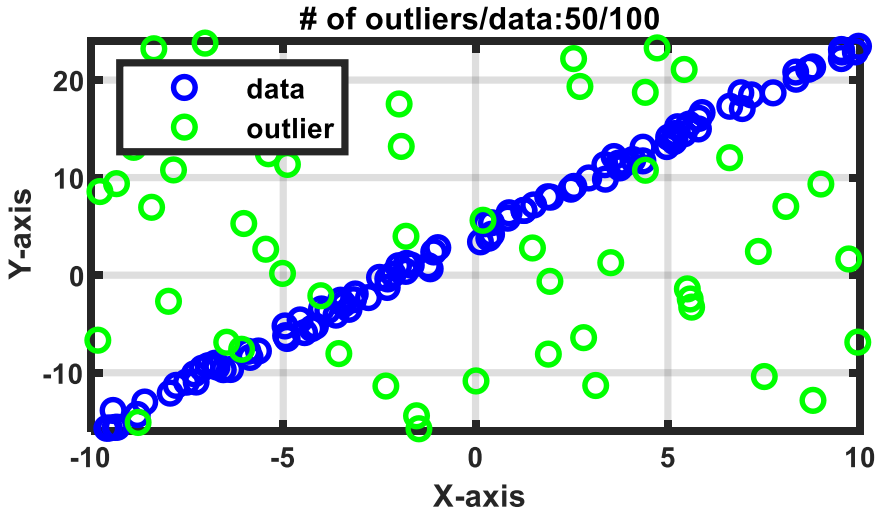


What if There are Outliers in Your Data

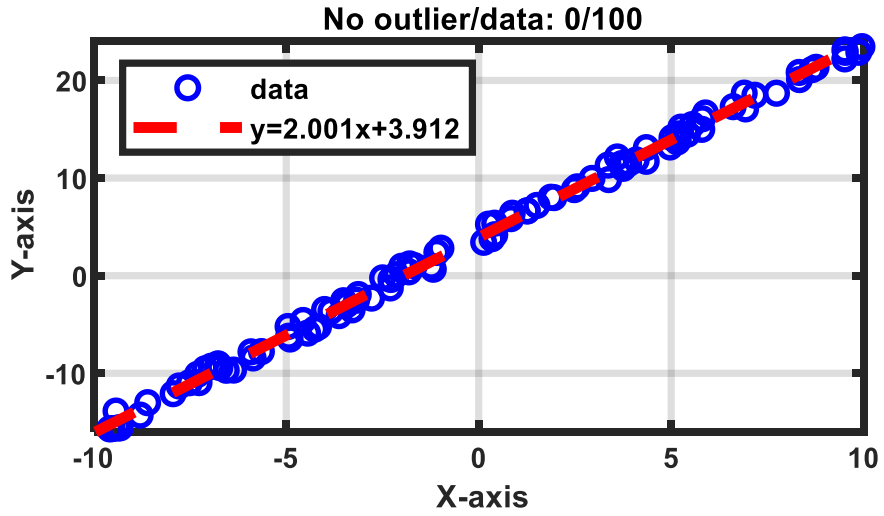


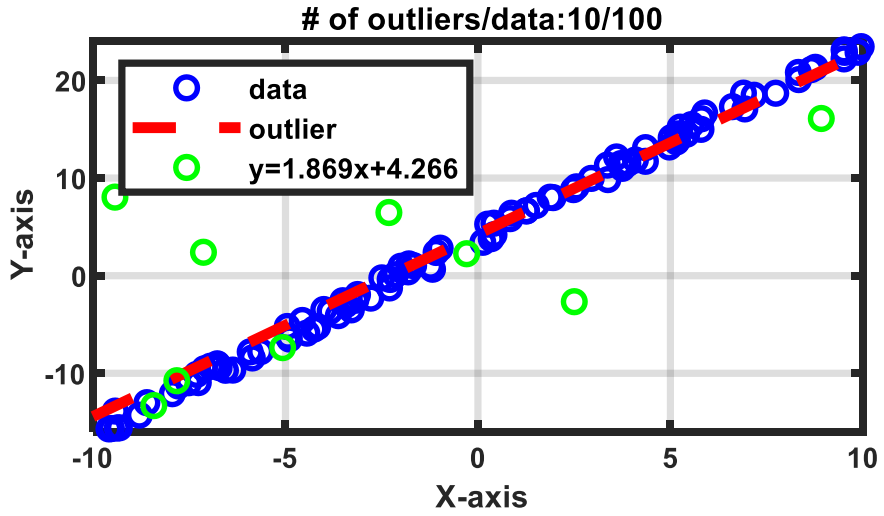


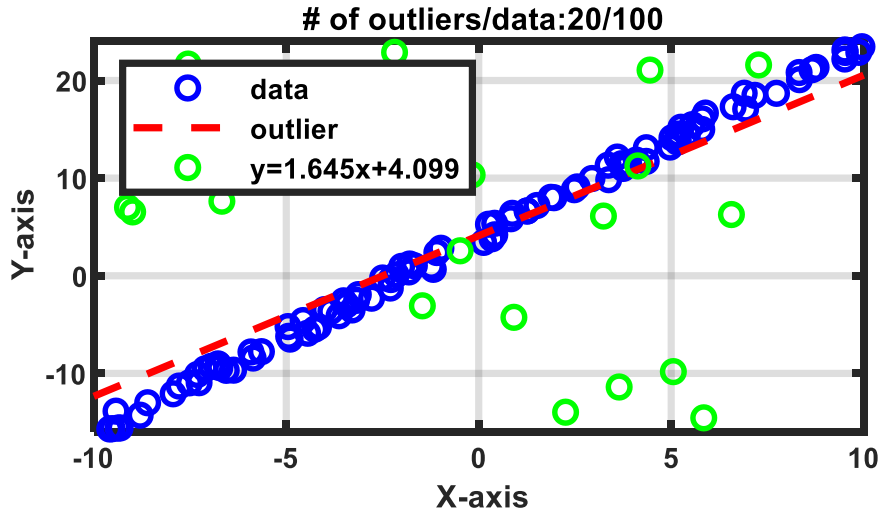


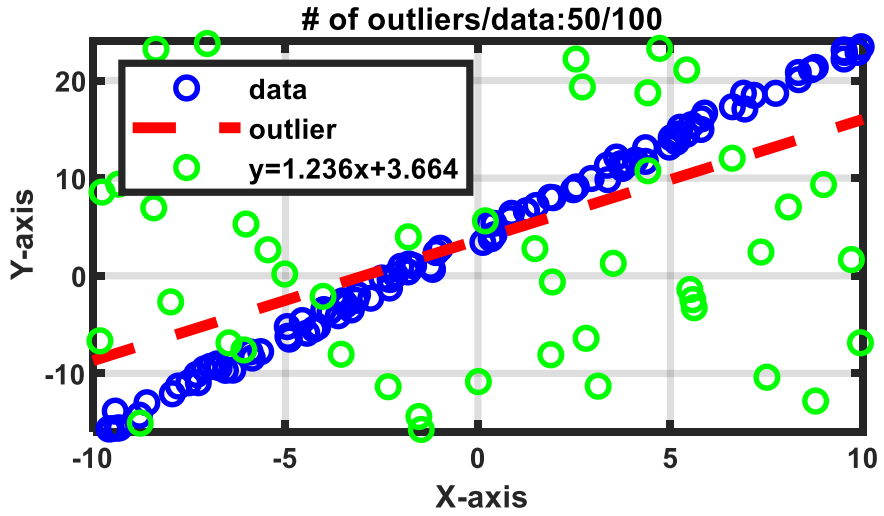


What if There are Outliers in Your Data (Continue)

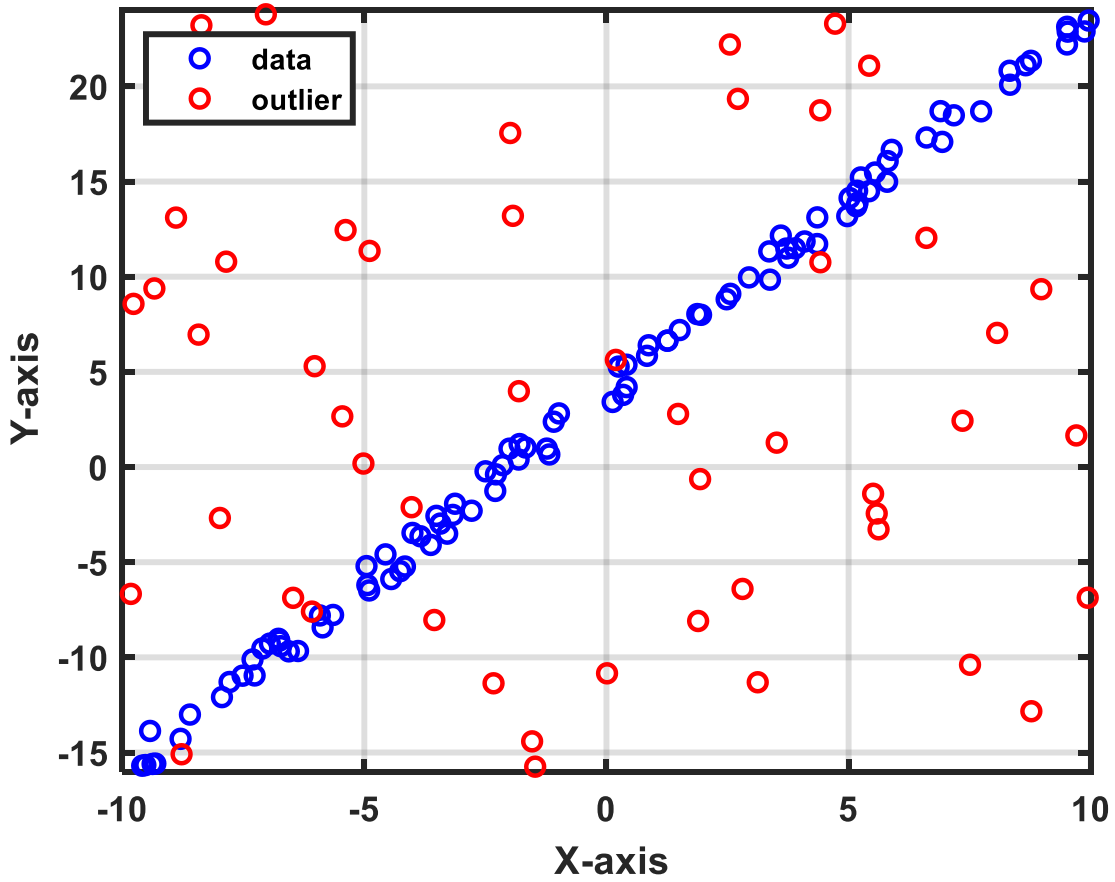


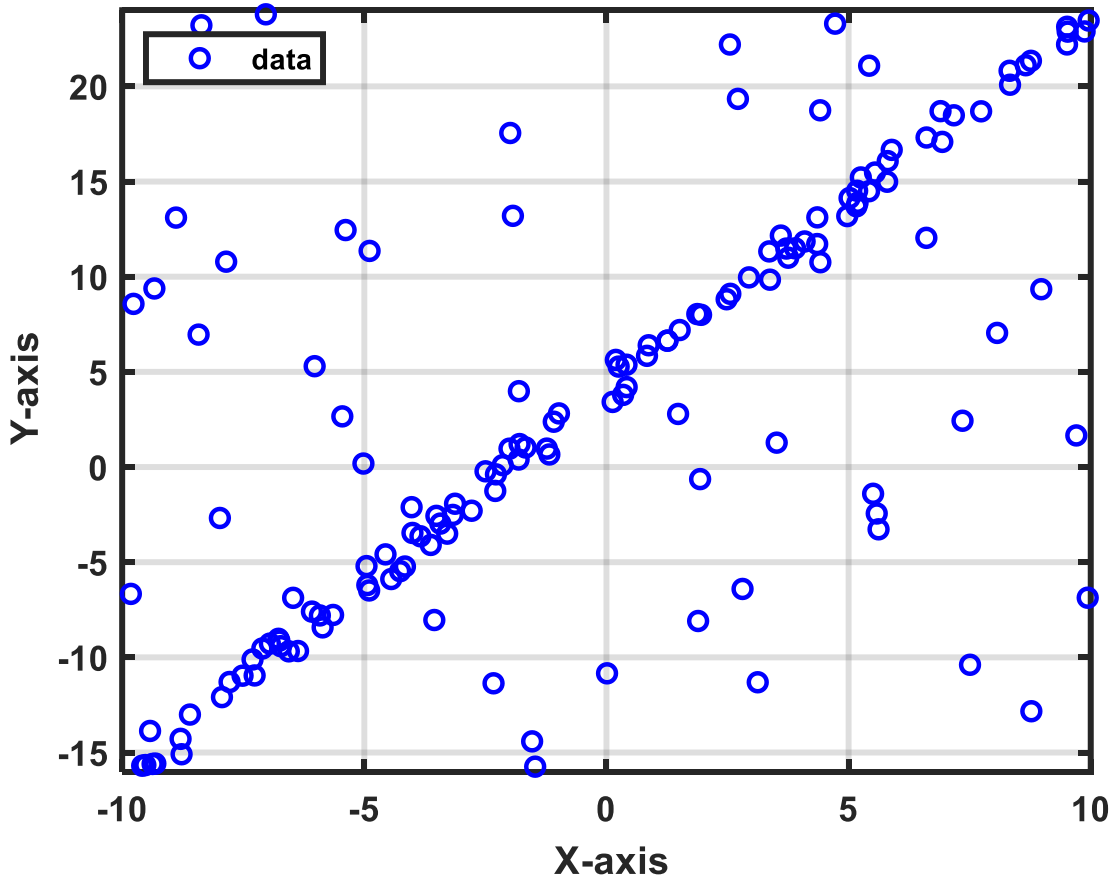






What's Your Solution?



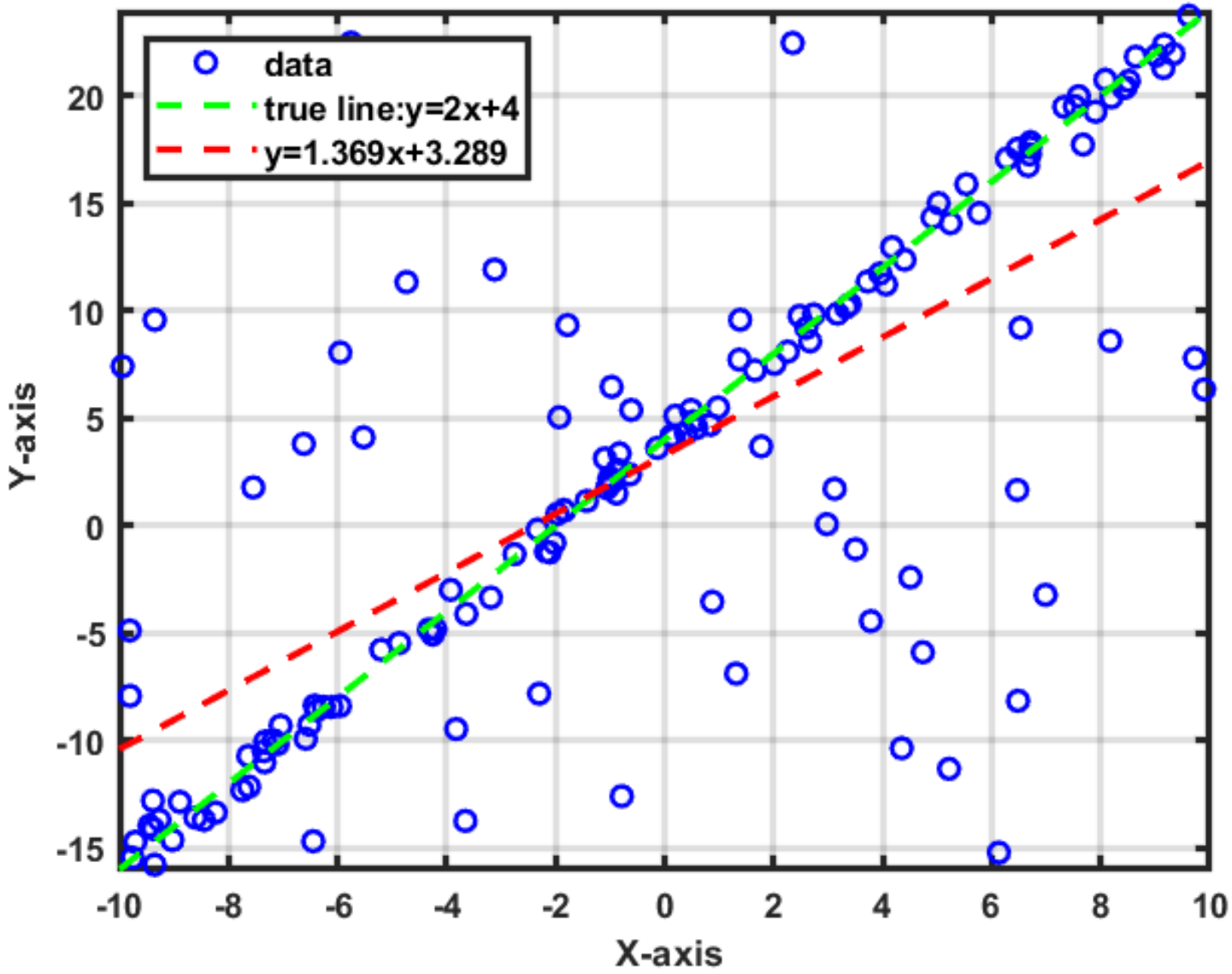


We do not know which points are inlier or outlier.

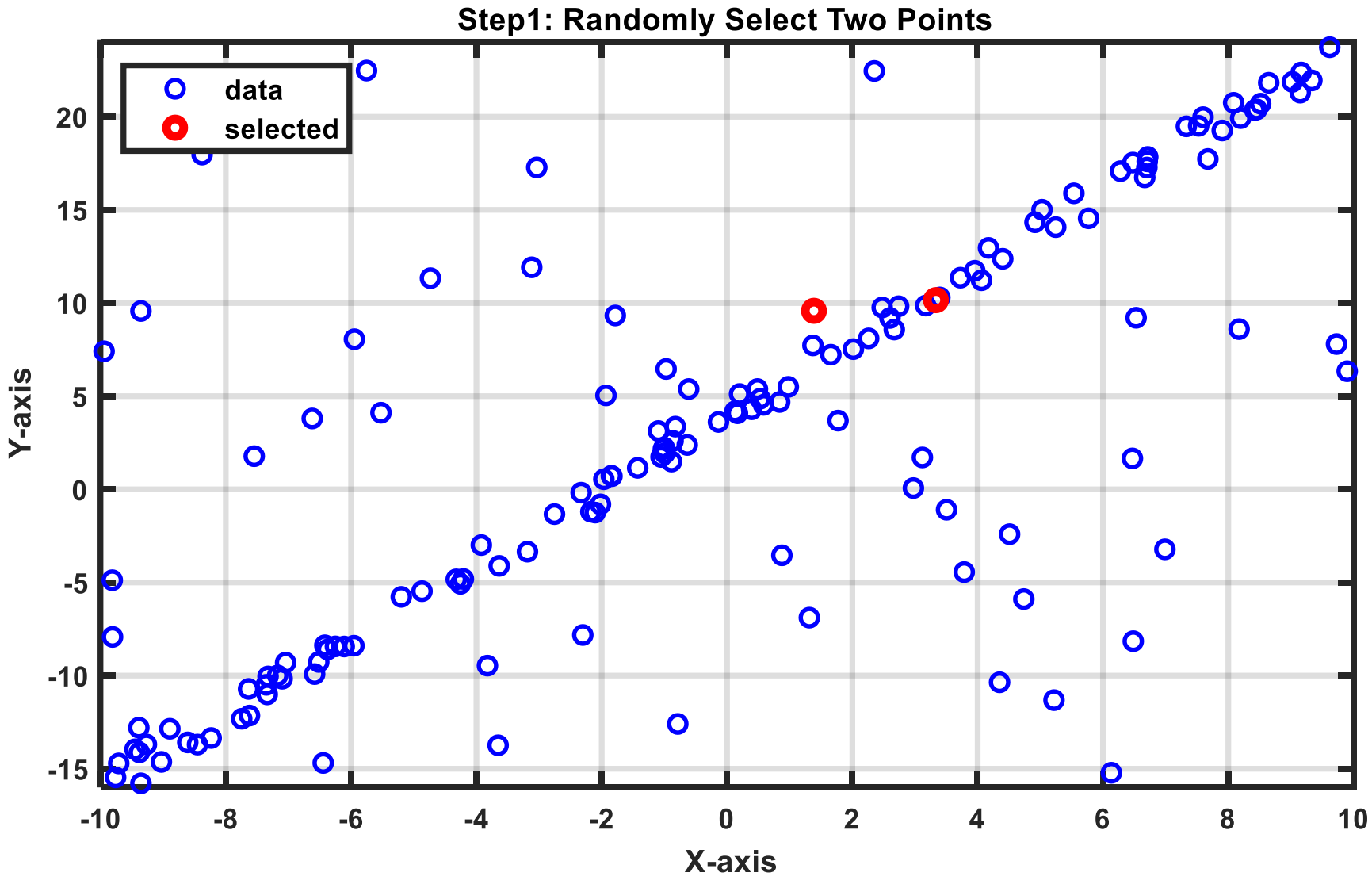
Random Sample Consensus (RANSAC)

Fundamental to RANSAC algorithm is using randomly-selected least amount of data to construct an estimate (parameters of a known mathematical model) and to then ascertain the number of inliers from the rest of the data. We accept inliers only if the error (difference between the estimate and data) does not exceed a threshold. We repeat this process many times and choose the best estimate having the greatest number of inliers.

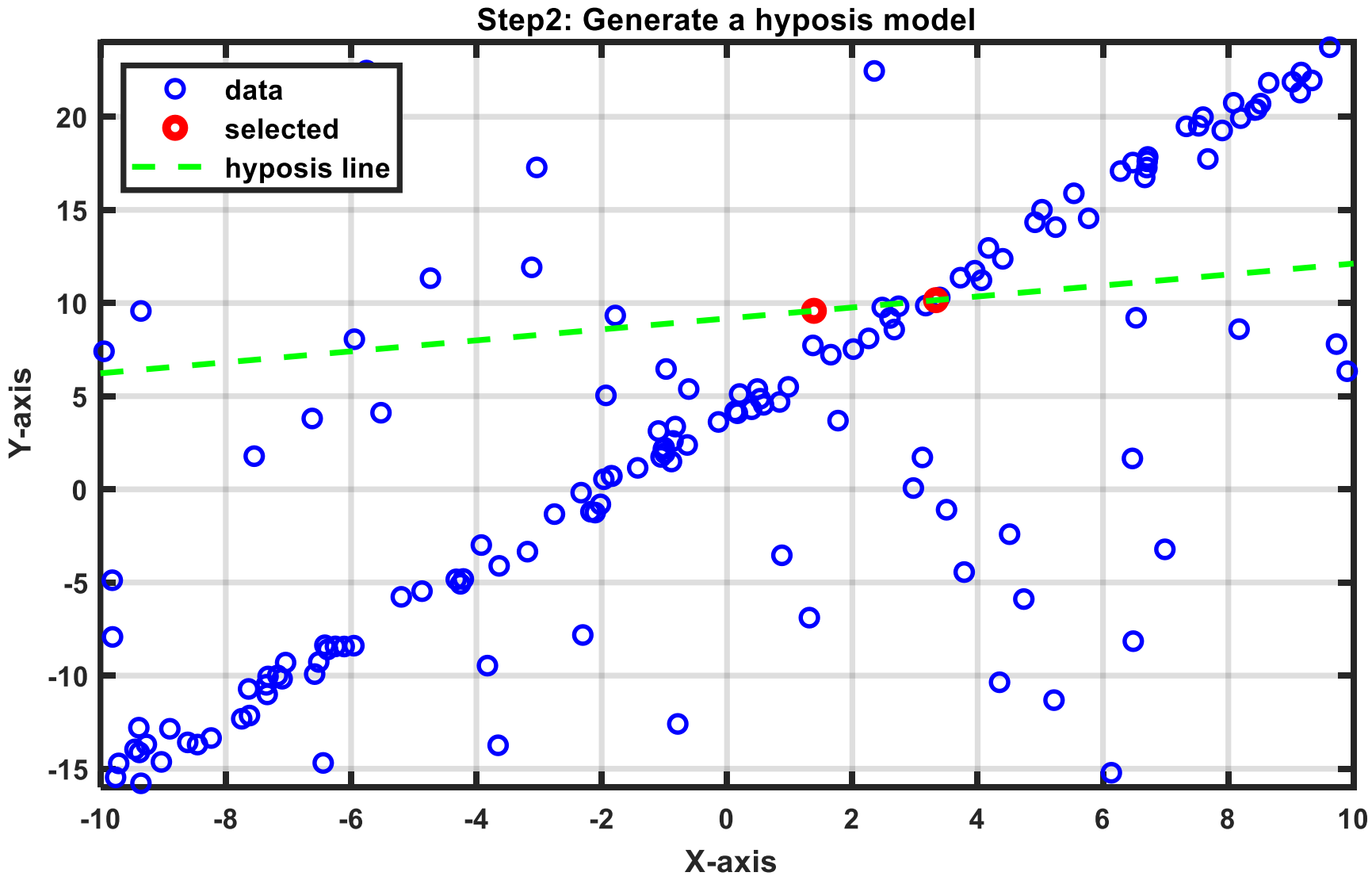
Linear Fitting using Dataset Having Outliers



Step 1: Randomly Select a Minimal Subset of Points

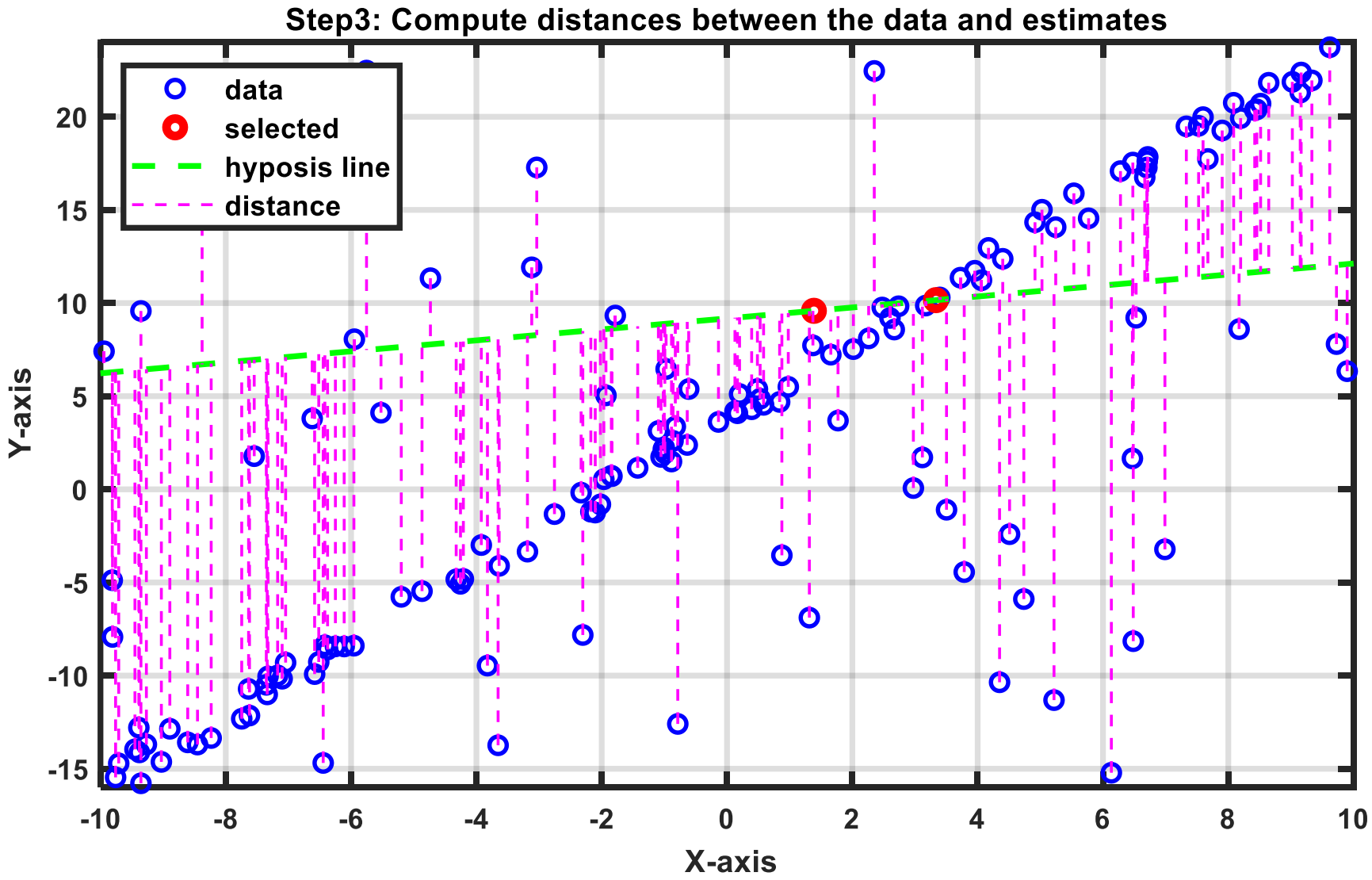


Step 2: Generate a Hypothesis Model

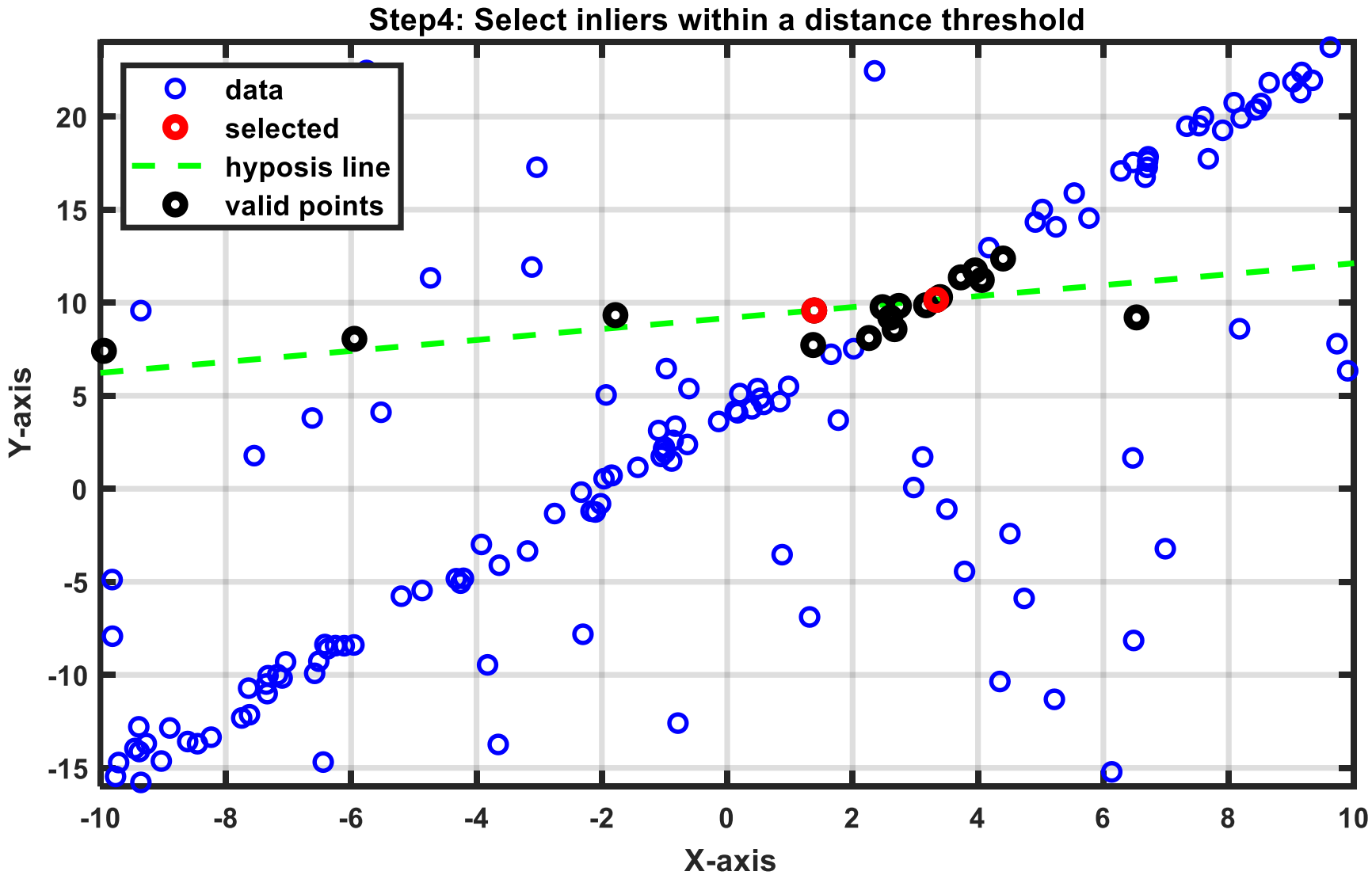


Tutorial

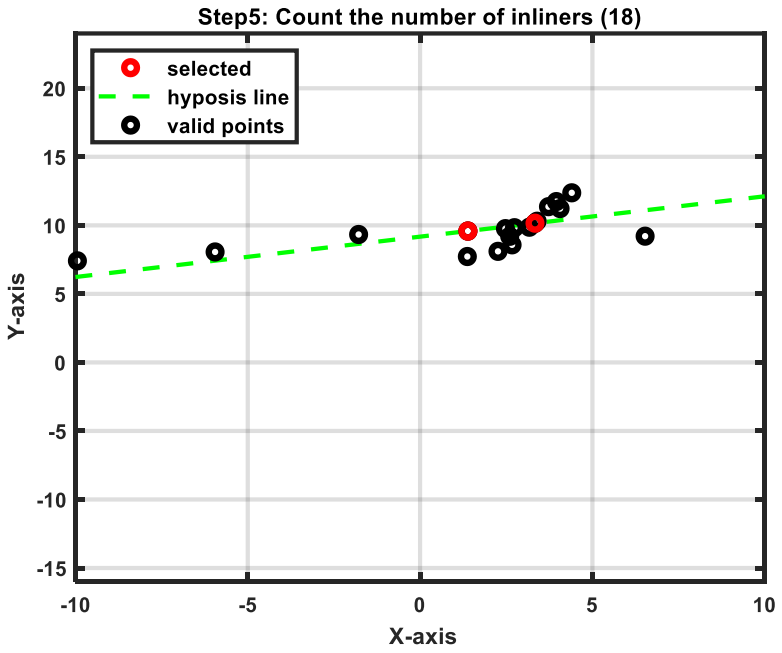
Step 3: Compute Distances between the Data and Estimates from the Hypothesis Model

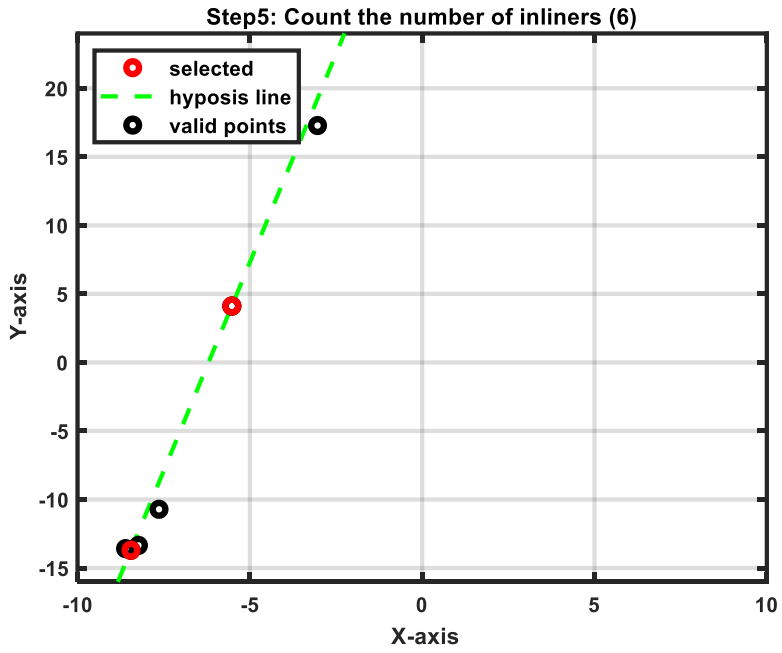


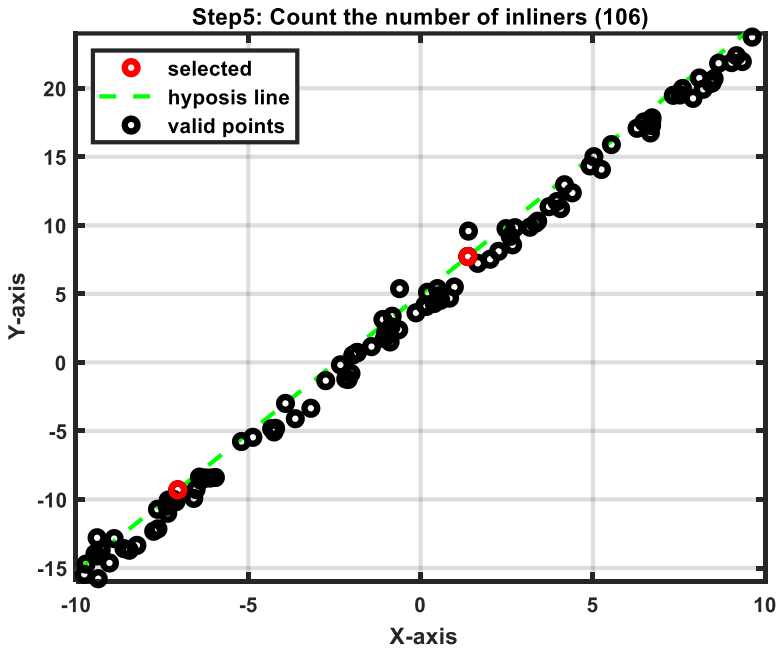
Step 4: Select Inliers that are within a Distance Threshold



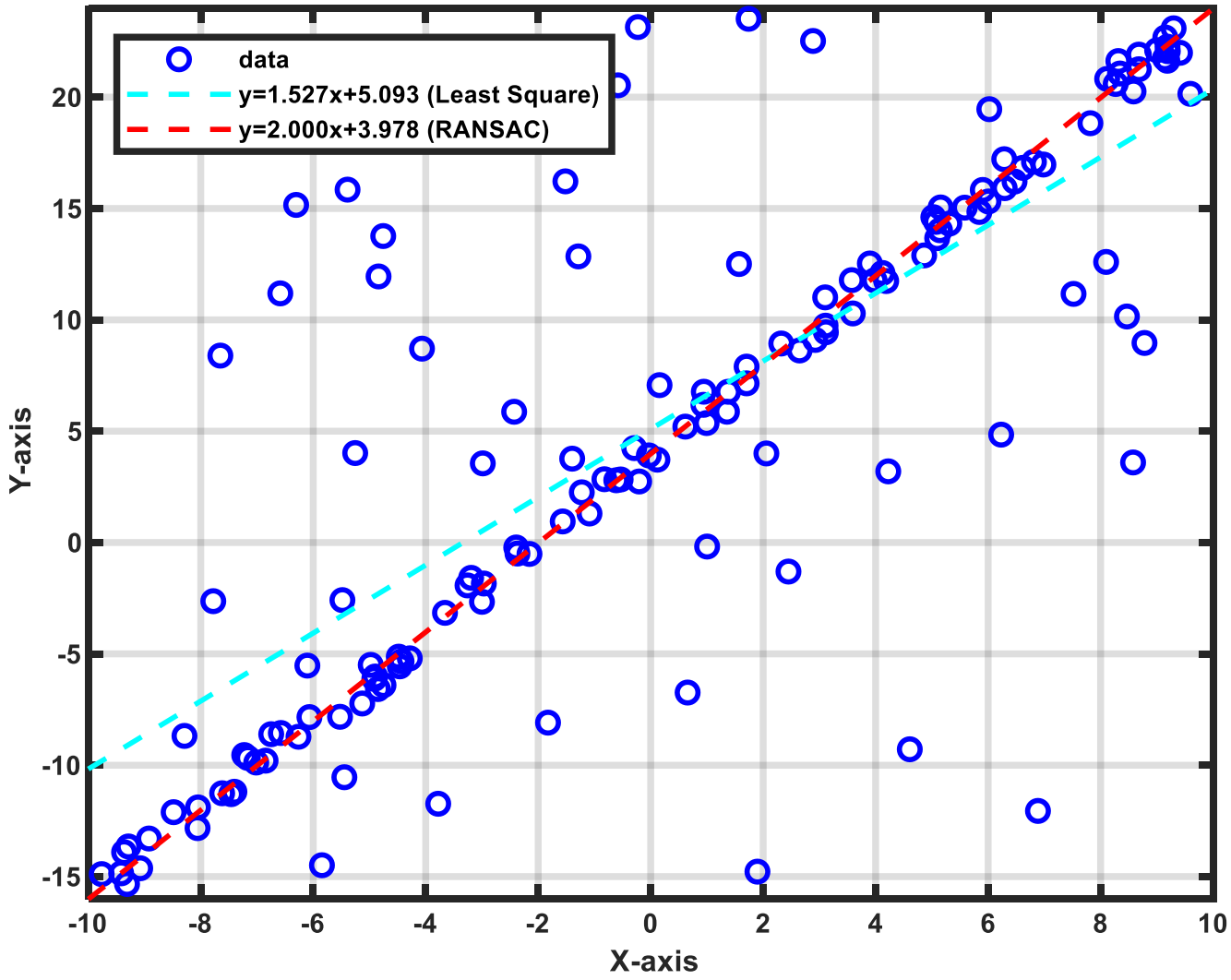
Step 5: Repeat This Process and Count the Number of Inliers in each Hypothesis Model







Demo: Line Fitting using RANSAC



Tutorial: ransac_demo.mlx

Three Important Parameters for the Use of RANSAC

δ: Decision threshold to construct the inlier set.

How to set an inlier threshold

N: Maximum number of trials to generate hypothesis models

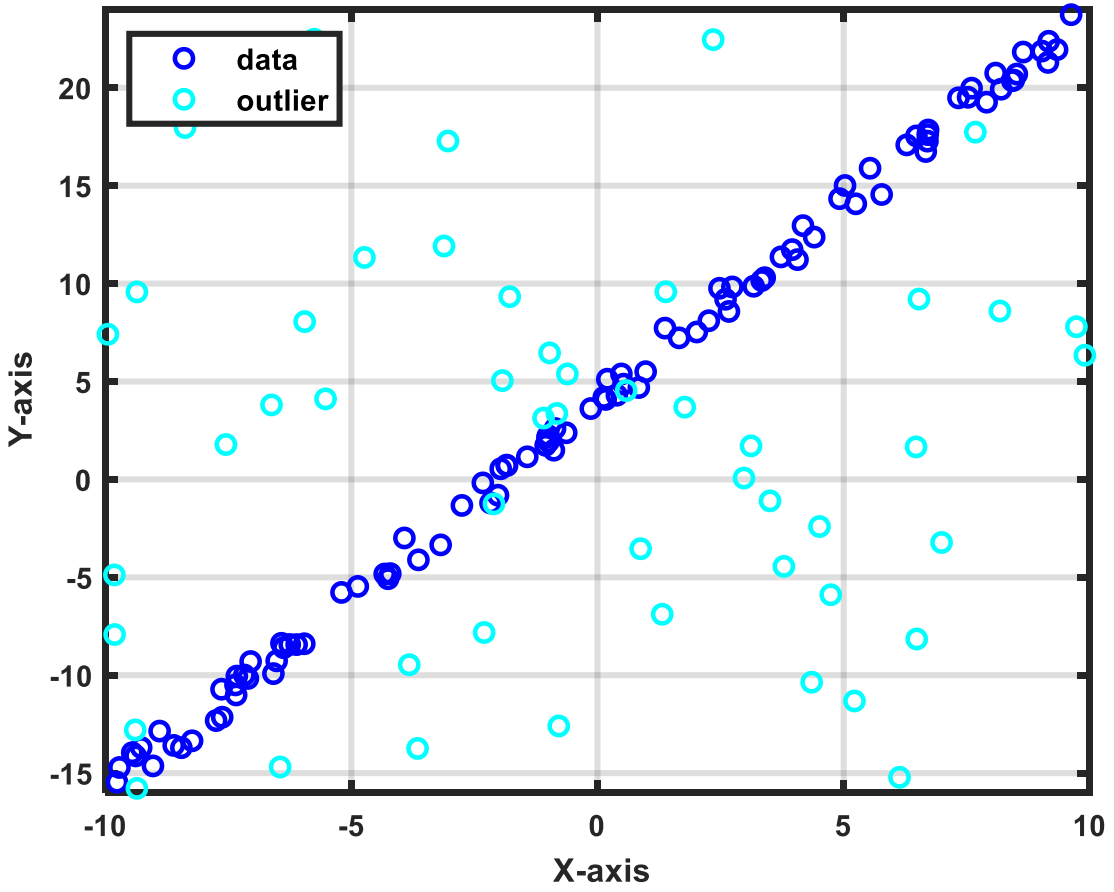
How many times of iterations?

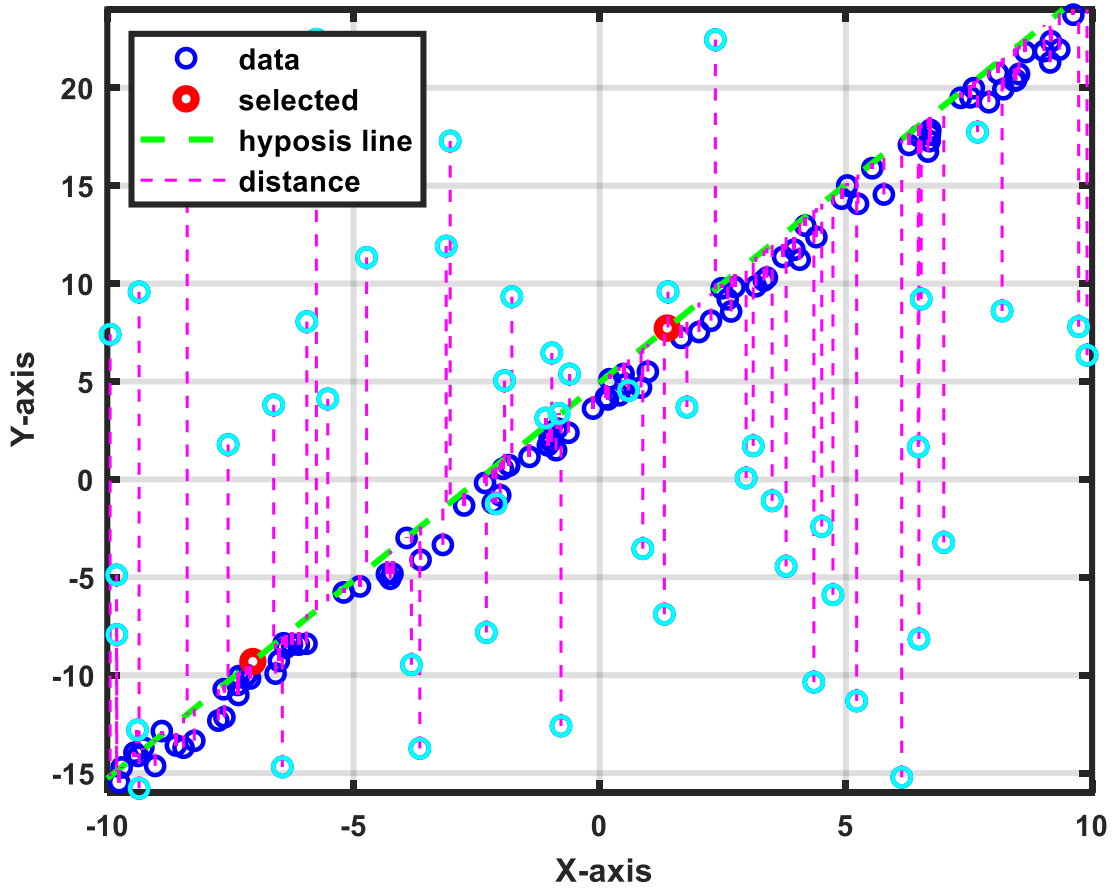
p: Confidence that the final solution finds the maximum number of inliers (finds a correct answer)

How confident to find a correct initial guess?

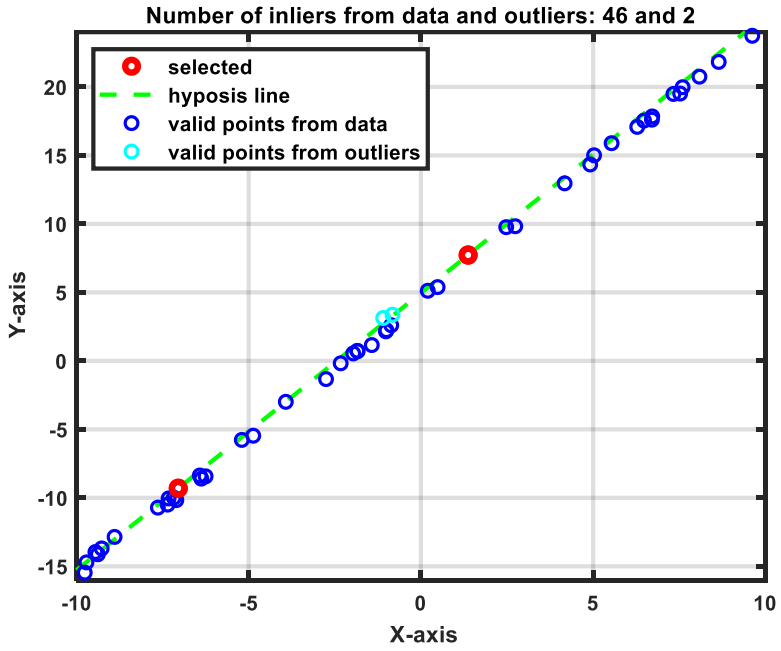
What are the issues when we set wrong parameters?

Parameter: Decision Distance Threshold (δ)

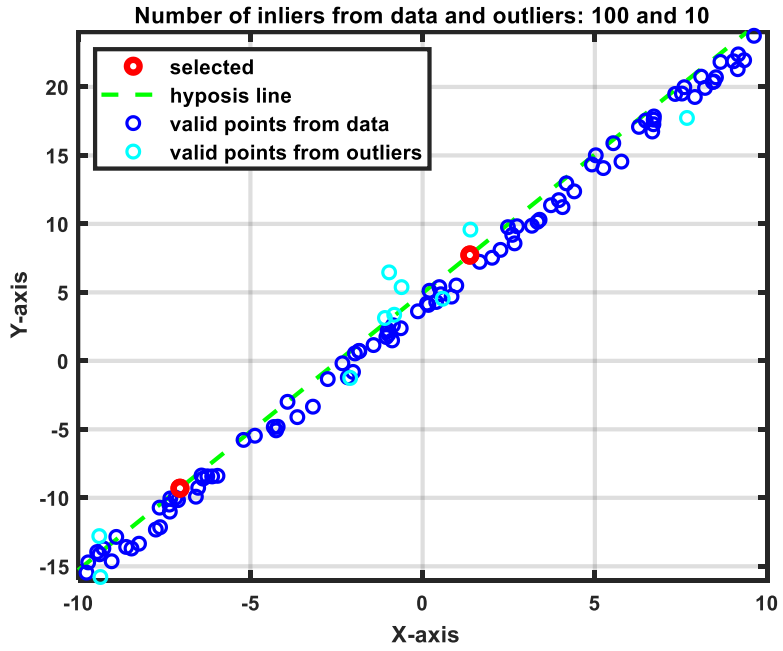




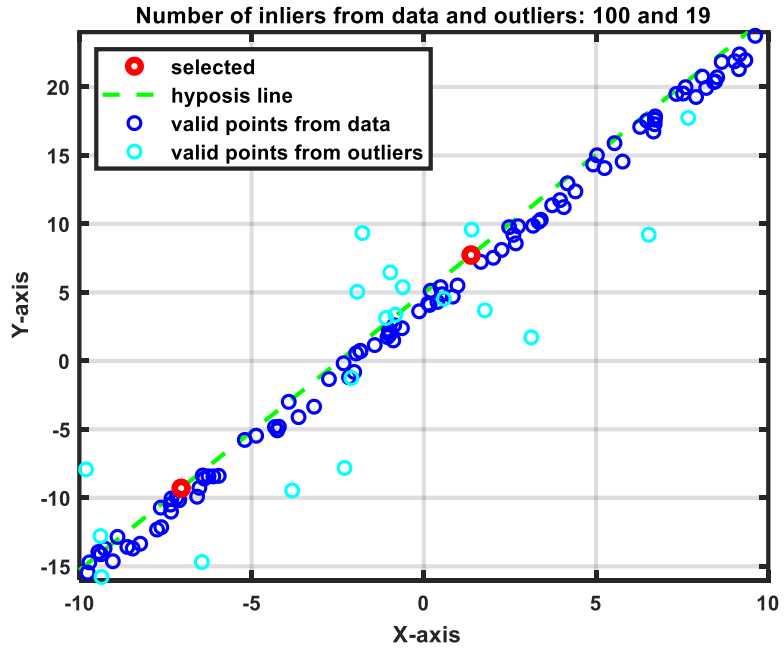
Parameter: Decision Distance Threshold (δ) (Continue)



$\delta = 1$



$\delta = 4$



$\delta = 10$

(Review) Point Correspondences for Estimating a Homography

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

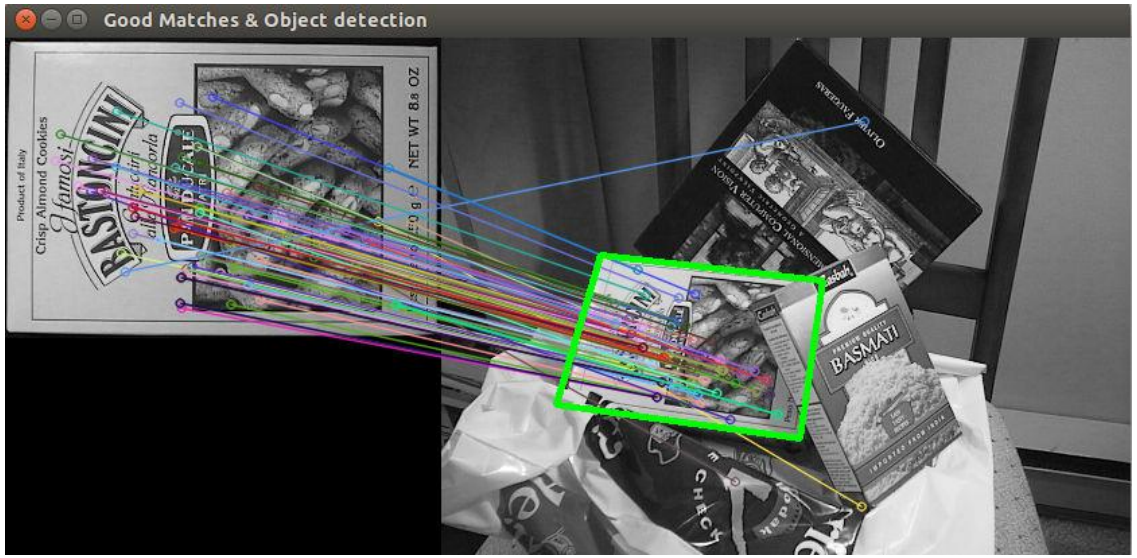
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{pmatrix} xh_{11} + yh_{12} + h_{13} \\ xh_{21} + yh_{22} + h_{23} \\ xh_{31} + yh_{32} + h_{33} \end{pmatrix}$$

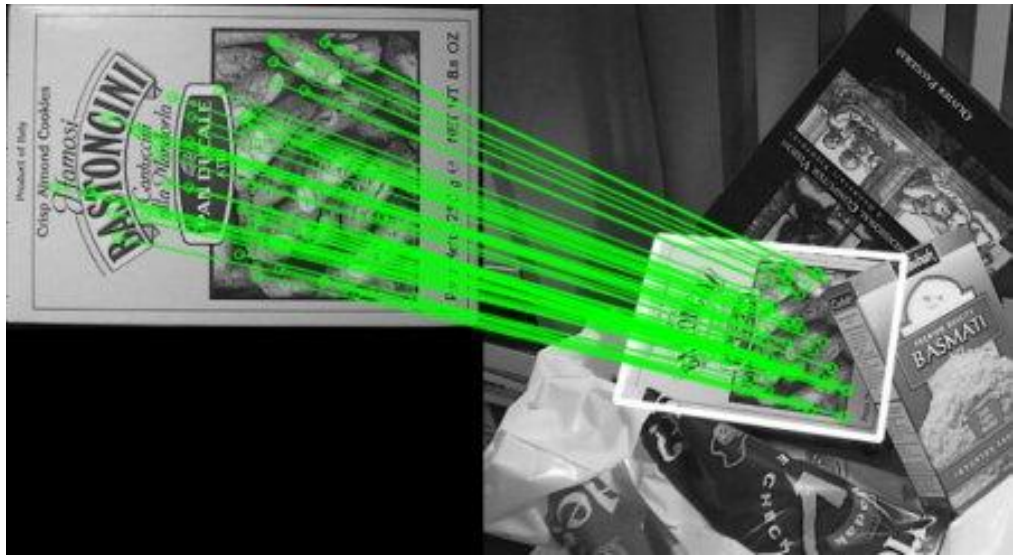
$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -xx' & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -xy' & -y'y & -y' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0$$

$$x' = \frac{xh_{11} + yh_{12} + h_{13}}{xh_{31} + yh_{32} + h_{33}}$$

$$y = \frac{xh_{21} + yh_{22} + h_{23}}{xh_{31} + yh_{32} + h_{33}}$$

Homography Estimation





Maximum Number of Trials (N) & Confidence (p)

Binomial combinations or all combinations

The binomial coefficient $\binom{n}{k}$ is the number of ways of picking k unordered outcomes from n.

Example

$\binom{100}{2} = 4950$: 2 sample point picking among 100 data (line model)

$\binom{1000}{4} = 4e10$: 4 sample point picking among 1000 data (homography estimation)

Computationally expensive

Maximum Number of Trials (N) & Confidence (p) (Continue)

ϵ : Probability of the false correspondence

$1 - \epsilon$: Probability of the true correspondence

$(1 - \epsilon)^n$: Probability of all **n** true correspondences in a trial

$1 - (1 - \epsilon)^n$: Probability of **at least one false** correspondence in **n** correspondences in a trial

$(1 - (1 - \epsilon)^n)^N$: Probability of at least one false correspondence in n correspondences in N trial

$1 - (1 - (1 - \epsilon)^n)^N$: Probability of at least one trial that has all true n correspondence in N trial

$$\mathrm{p} = 1 - (1 - (1 - \epsilon)^n)^N$$

$$\ln(1 - \mathrm{p}) = \mathrm{N}\ln(1 - (1 - \epsilon)^n)$$

$$\mathrm{N} = \frac{\ln(1 - \mathrm{p})}{\ln(1 - (1 - \epsilon)^n)}$$

Parameter: Maximum Number of Trials (N) & Confidence (p) (Continue)

ϵ (= 0.5): Probability of the false correspondence

Homography!

$1 - \epsilon$ (= 0.5): Probability of the true correspondence

$(1 - \epsilon)^n$ (= 0.06): Probability of all 4 true correspondences in a trial

$1 - (1 - \epsilon)^n$ (= 0.94): Probability of at least one false correspondence in 4 correspondences in a trial

$(1 - (1 - \epsilon)^n)^N$ (= 0.0021): Probability of at least one false correspondence in 4 correspondences in 100 trial

$1 - (1 - (1 - \epsilon)^n)^N$ (= 0.9979) : Probability of at least one trial that has all true 4 correspondence in 100 trial

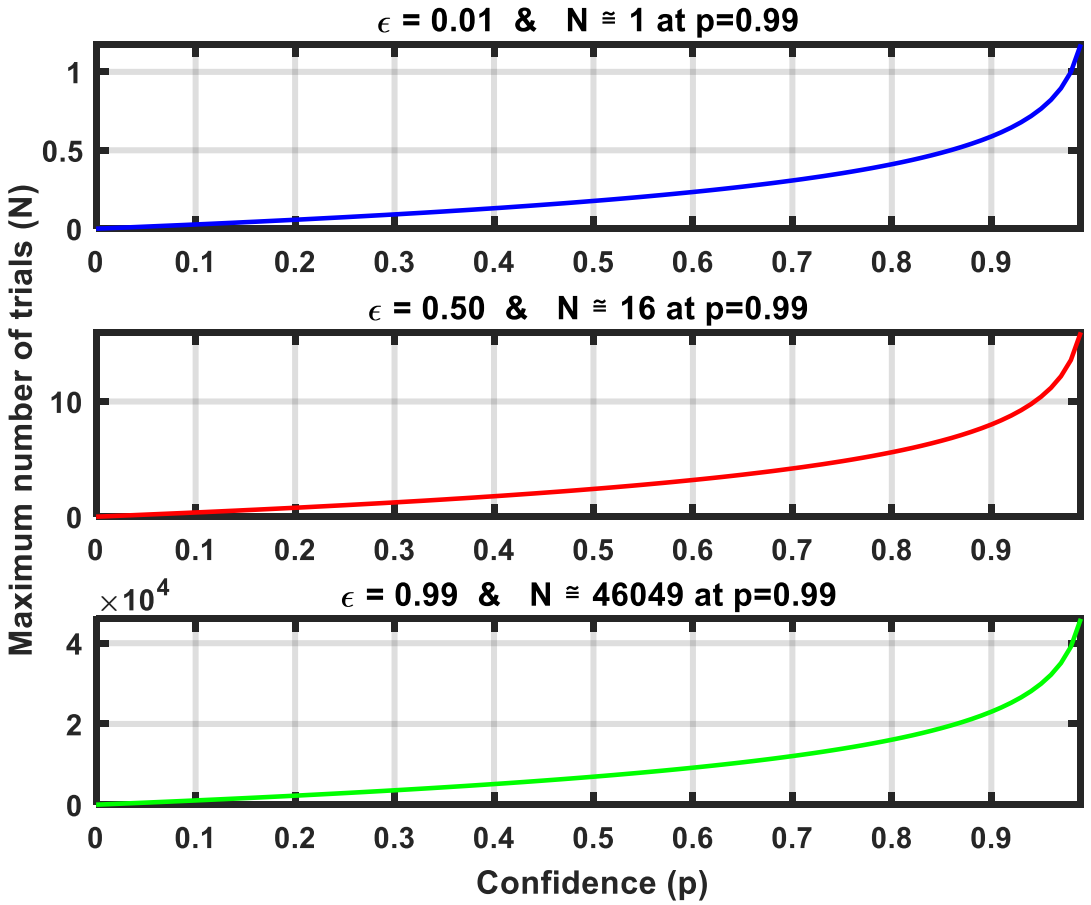
$$\mathrm{p} = 1 - (1 - (1 - \epsilon)^n)^N$$

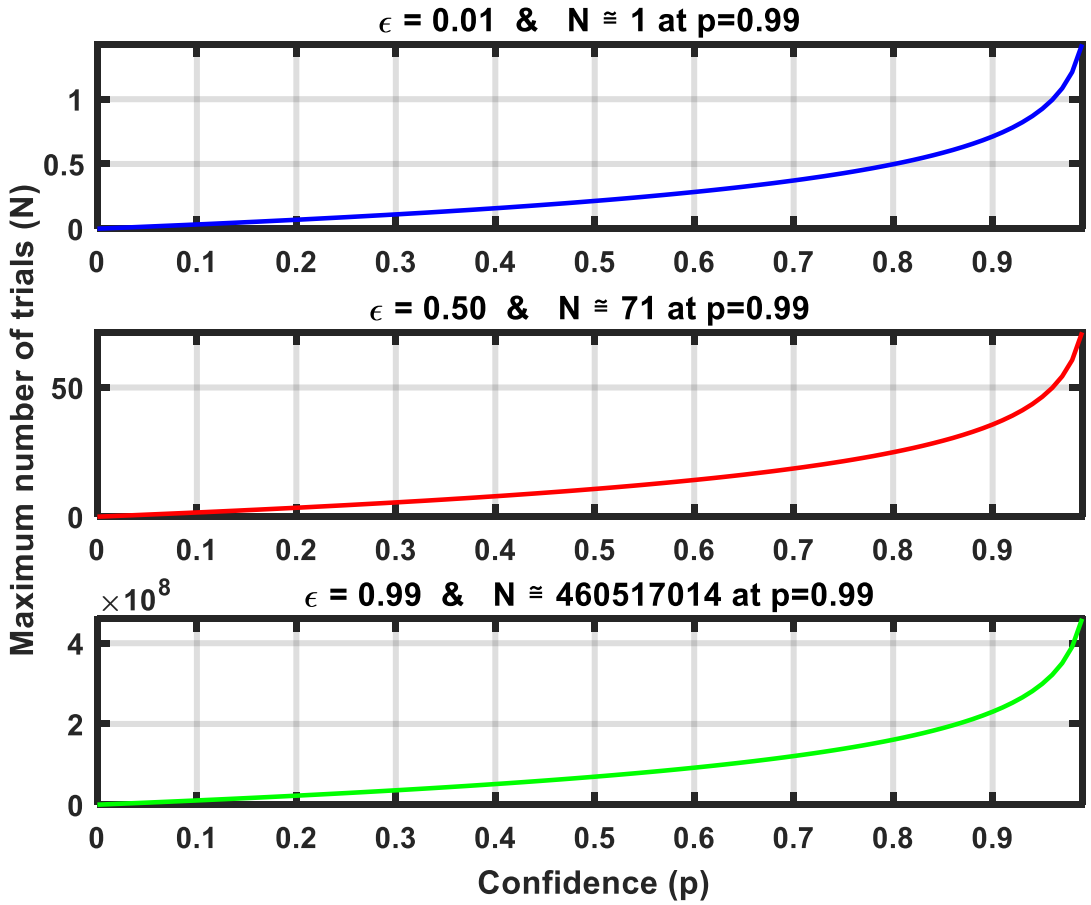
ϵ = 0.5 p = 0.99

$$\ln(1 - \mathrm{p}) = \mathrm{N}\ln(1 - (1 - \epsilon)^n)$$

$$\mathrm{N} = \frac{\ln(1 - \mathrm{p})}{\ln(1 - (1 - \epsilon)^n)} = \frac{\ln(1 - 0.99)}{\ln(1 - (1 - 0.5)^4)} \approx 71$$

Parameter: Maximum Number of Trials (N) & Confidence (p) (Continue)





Line: n=2

Homography: n=4

Advantages and Disadvantage

An advantage of RANSAC is its ability to do robust estimation[2] of the model parameters, i.e., it can estimate the parameters **with a high degree of accuracy even when a significant number of outliers are present in the data set**. A disadvantage of RANSAC is that there is **no upper bound on the time** it takes to compute these parameters (except exhaustion). When the number of iterations computed is limited the solution obtained may not be optimal, and it may not even be one that fits the data in a good way. In this way RANSAC offers a trade-off; by computing a greater number of iterations the probability of a reasonable model being produced is increased. Moreover, RANSAC is not always able to find the optimal set even **for moderately contaminated sets and it usually performs badly when the number of inliers is less than 50%.** Optimal RANSAC [3] was proposed to handle both these problems and is capable of finding the optimal set for heavily contaminated sets, even for an inlier ratio under 5%. Another disadvantage of RANSAC is that it requires the setting of **problem-specific thresholds.**

https://en.wikipedia.org/wiki/Random_sample_consensus

Slide Credits and References

- Lecture notes: Svetlana Lazebnik
- Lecture notes: S. Seitz
- Lecture notes: M. Pollefeys
- Lecture notes: A. Effros