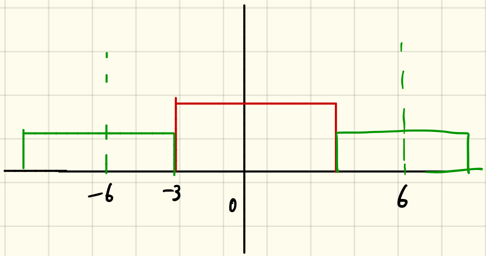
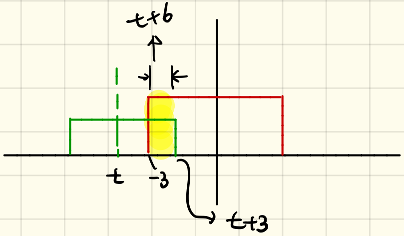
Problem 1.

$$f(t) = \begin{cases} 3, & 3 > |t| \\ 0, & \text{otherwise} \end{cases}$$

$$g(t) = \begin{cases} 2, & 3 > |t| \\ 0, & \text{otherwise} \end{cases}$$

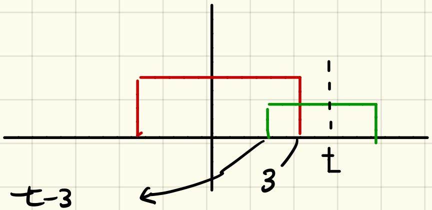


$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau .$$



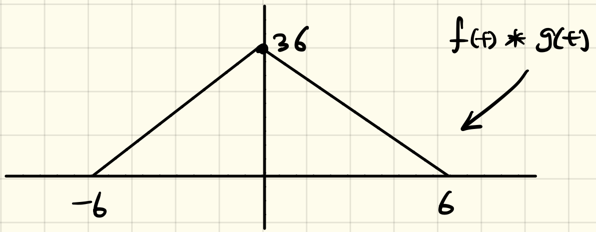
(i) $-6 < t < 0$

$$f * g = (t+6) \cdot 2 \cdot 3 = 6t + 36$$



(ii) $0 \le t < 6$

$$f * g = (6-t) \cdot 2 \cdot 3 = 36 - 6t .$$



problem2.

$$x(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

$$f(t), g(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = f(t) * g(t)$$

$$X(f) = \mathcal{F}(x(t)) = F(t) \cdot G(t)$$

$x(t) = a$ for $|t| < b$
$= 0$ for $|t| > b$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-b}^{b} a e^{-i2\pi f t} dt = \frac{2ab \sin 2\pi f b}{2\pi f b}$$

$y(t) = 1$ for $|t| < \frac{1}{2}$

otherwise

$$Y(f) = \frac{\sin \pi f}{\pi f}$$

$$\therefore X(f) = \frac{(\sin \pi f)^2}{(\pi f)^2}$$

problem 2

(continue)

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{i2\pi ft} dt \\
&= \int_{-1}^{1} (1+|t|) e^{i2\pi ft} dt \\
&= \int_{-1}^{0} (1+t) e^{i2\pi ft} dt + \int_{0}^{1} (1-t) e^{i2\pi ft} dt \\
&= \int_{-1}^{0} e^{i2\pi ft} dt + \int_{-1}^{0} te^{i2\pi ft} dt + \int_{0}^{1} e^{i2\pi ft} dt - \int_{0}^{1} te^{i2\pi ft} dt \\
&= \int_{-1}^{1} e^{i2\pi ft} dt + \int_{-1}^{0} te^{i2\pi ft} dt - \int_{0}^{1} te^{i2\pi ft} dt \\
&= \left[\frac{e^{i2\pi ft}}{i2\pi f}\right]_{-1}^{1} + \left[\frac{i2\pi ft - 1}{(i2\pi f)^2} \times e^{i2\pi ft}\right]_{-1}^{0} - \left[\frac{i2\pi ft - 1}{(i2\pi f)^2} \times e^{i2\pi ft}\right]_{0}^{1} \\
&= \left[\frac{e^{i2\pi f}}{i2\pi f} - \frac{e^{-i2\pi f}}{i2\pi f}\right] + \left[\frac{-1}{(i2\pi f)^2} - \frac{i2\pi f + 1}{(i2\pi f)^2} \times e^{-i2\pi f}\right] - \left[\frac{i2\pi f - 1}{(i2\pi f)^2} \times e^{i2\pi f} + \frac{1}{(i2\pi f)^2}\right] \\
&= \frac{e^{i2\pi f} - e^{-i2\pi f}}{i2\pi f} - \frac{2}{(i2\pi f)^2} - \frac{e^{i2\pi f} - e^{-i2\pi f}}{i2\pi f} + \frac{e^{i2\pi f} + e^{-i2\pi f}}{(i2\pi f)^2} \\
&= \frac{1}{2\pi^2 f^2} - \frac{1}{2\pi^2 f^2} \times \frac{e^{i2\pi f} + e^{-i2\pi f}}{2} \\
&= \frac{1}{2\pi^2 f^2}(1 - \cos(2\pi f)) \\
&= \frac{1}{2\pi^2 f^2}(1 - (1 - 2\sin^2(\pi f)) \\
&= \frac{1}{2\pi^2 f^2}(2\sin^2(\pi f)) \\
&= \frac{\sin^2(\pi f)}{\pi^2 f^2} \\
&= \operatorname{sinc}^2(f)
\end{aligned}
$$

from

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W2020.

problem 3.

(a)

$X(f)$: Fourier transform of an original signal

$X_s(f)$: " of a sampled signal

$X(k)$: Discret fourier transform of a sampled signal.

(b)

- If f_s is not enogh high, aliasy is happened.

- $X_s(f)$ is repeated at every f_s.

$$X_s(f+r/\Delta) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi(f+r/\Delta)n\Delta} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta - i2\pi rn} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta} = X_s(f)$$

$X_s(f)$: Fourier transform of a <u>sampled (discretized) signal</u> (sequence)

$$X_s(f + r/\Delta) = X_s(f + r f_s) = X_s(f)$$

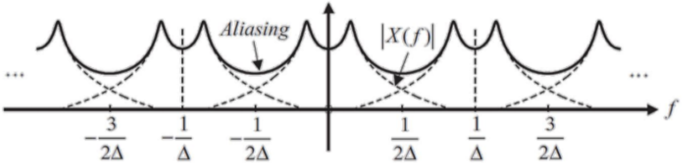
$\Rightarrow$ periodic function.

* This equation does not tell anything about the relationship with original signals.

Fourier Transform of a discrete sequence, $x_s(t)$

$$X_s(f) = I(f) * X(f) = \int_{-\infty}^{\infty} I(g)X(f-g)dg = \int_{-\infty}^{\infty} \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg$$

$$= \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{\Delta}\right) = \frac{1}{\Delta}\left(\dots + X\left(f-\frac{2}{\Delta}\right) + X\left(f-\frac{1}{\Delta}\right) + X(f) + X\left(f+\frac{1}{\Delta}\right) + \dots\right)$$



$X(f)$: fourier transform of the original signal $\Rightarrow$ a true Fourier spectrum of your signal.

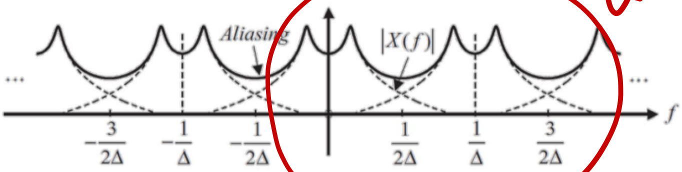
$$X_s(f) = \frac{1}{\Delta}\left(\cdots + X\left(f-\frac{2}{\Delta}\right) + X\left(f-\frac{1}{\Delta}\right) + \cdots \right)$$

$X_s(f)$ is the summation of $X(f)$ and shifted $X(f)$ with $n\frac{1}{\Delta}$
n : integer.

Fourier Transform of a discrete sequence, $x_s(t)$

$$X_s(f) = I(f) * X(f) = \int_{-\infty}^{\infty} I(g)X(f-g)dg = \int_{-\infty}^{\infty} \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg$$

$$= \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{\Delta}\right) = \frac{1}{\Delta}\left(\dots + X\left(f-\frac{2}{\Delta}\right) + X\left(f-\frac{1}{\Delta}\right) + X(f) + X\left(f+\frac{1}{\Delta}\right) + \dots\right)$$



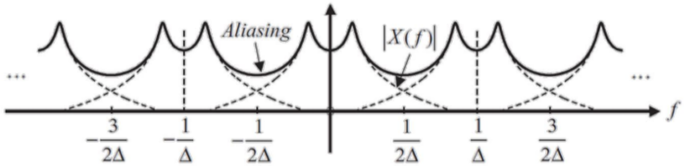
If There is no frequency component above $1/2\Delta$, there is no overlap between $X(f)$ and shifted $X(f)$ (here, $X(f-\frac{1}{\Delta})$)
(Not aliasing)

$\Rightarrow$ There is no aliasing !!!!

Fourier Transform of a discrete sequence, $x_s(t)$

$$X_s(f) = I(f) * X(f) = \int_{-\infty}^{\infty} I(g)X(f-g)dg = \int_{-\infty}^{\infty} \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} \delta\left(g - \frac{n}{\Delta}\right) X(f-g)dg = \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta\left(g - \frac{n}{\Delta}\right) X(f-g)dg$$

$$= \frac{1}{\Delta} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{\Delta}\right) = \frac{1}{\Delta}\left(\dots + X\left(f - \frac{2}{\Delta}\right) + X\left(f - \frac{1}{\Delta}\right) + X(f) + X\left(f + \frac{1}{\Delta}\right) + \dots\right)$$



If there are frequency components at above $1/2\Delta$, there is overlaps between $X(f)$ and shifted $X(f)$

$\Longrightarrow$ This is aliasing !!!

problem 4.

Inverse spreading property.

problem 5.

(c)(d) $f_s = 110$, $f_N = 55$ Hz

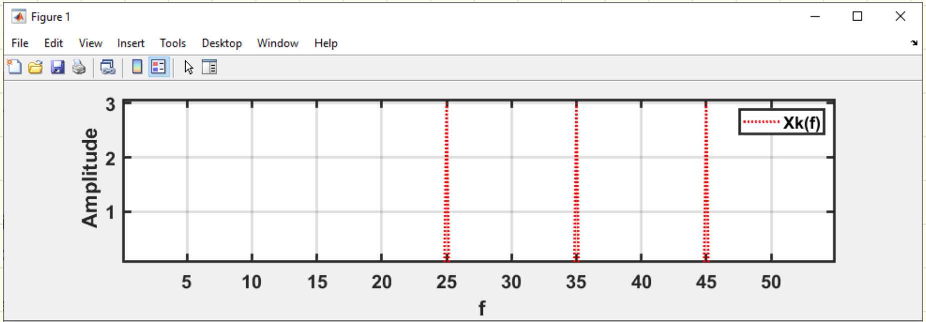
25, 45 , 35 Hz .

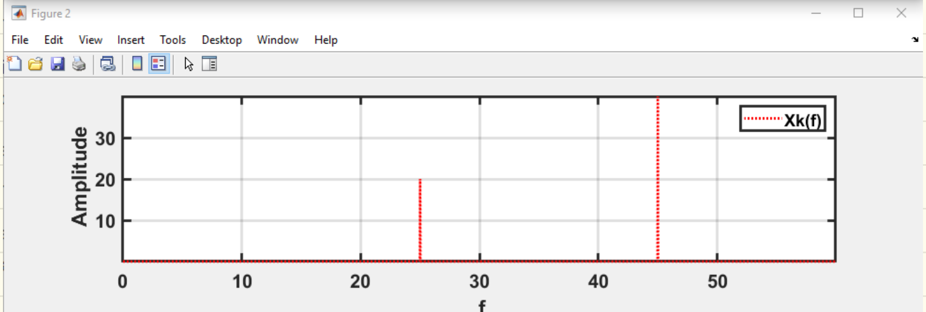
e) NO

$f_N = 60$

(f) NO, 25 , 45 ~~Hz~~.

(g) Yes.





120 Hz for 20 seconds, can you measure and extract all frequencies contained in the original signal,