Task 02: Signal Processing I

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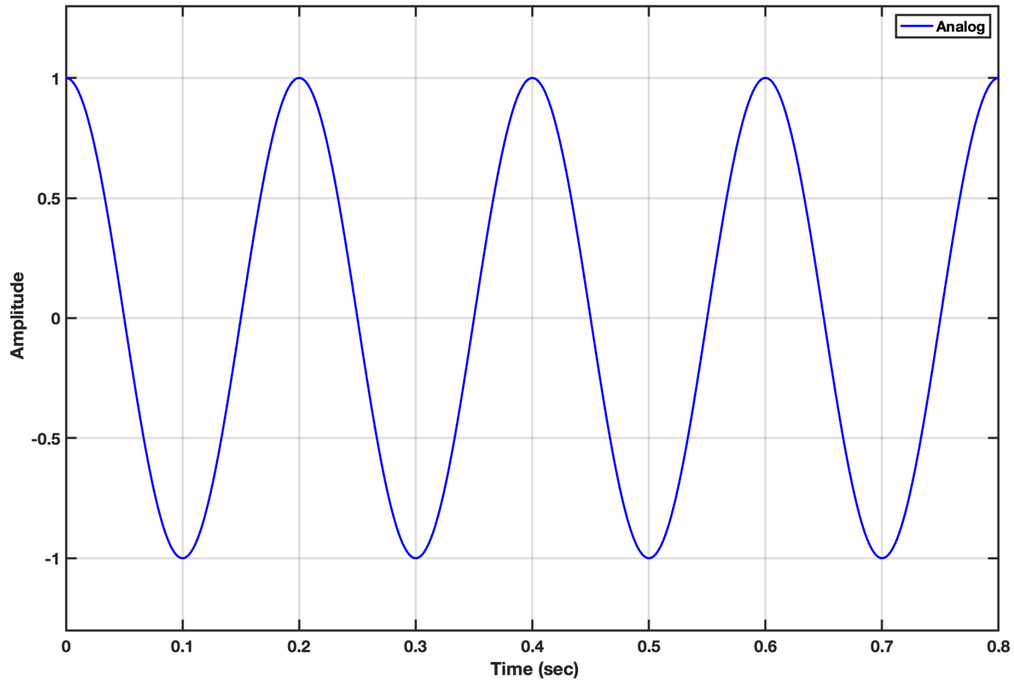
Problem 1: Sampling (10 points)

(a) What is the difference between continuous (or analogue) and discrete (or digital) signals?

Analogue signals are continuous with variating amplitudes and frequencies. Digital signals are the distinct values of the original signals, and grouped together to simulate a continuous signal.

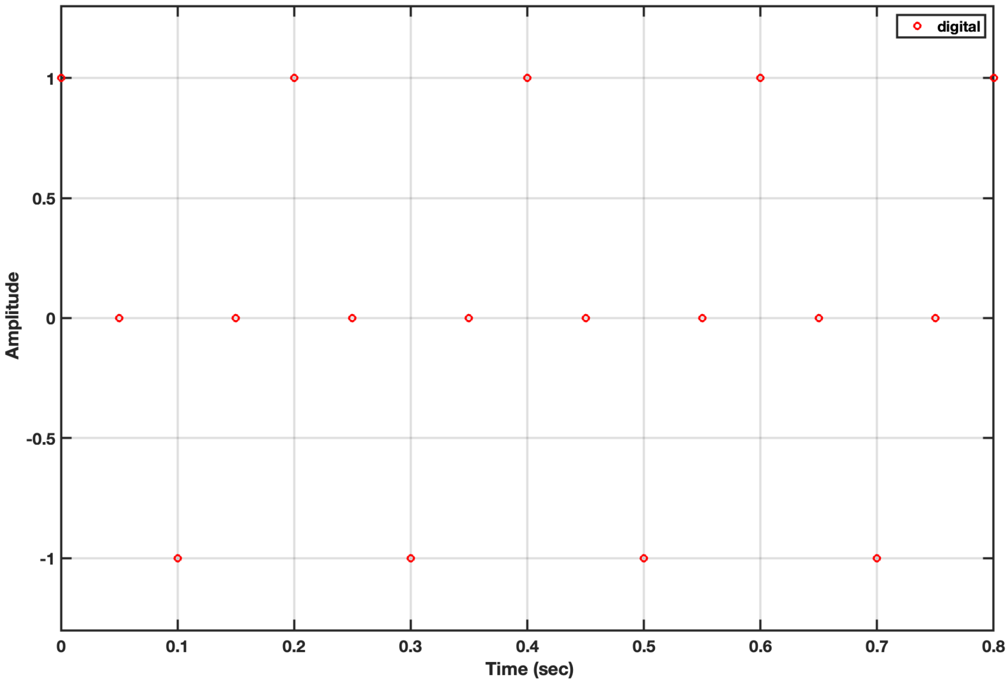
(b) Plot a 5 Hz cosine wave with a high sampling rate (nearly analog signal). Please connect the sampled points and plot only four cycles of the wave.

```
clear; clc ; close all;

syms t

f = 5;
y = cos(2*pi*f*t);

% this signal is assumed to be analog (very high sampling rate).
ncyle = 0.8;
Fsa = 1000; % # of samples per a second
ta = 0:1/Fsa:ncyle;
ya = subs(y, t, ta);

fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

plot(ta,ya,'-b', 'linewidth', 2);
legend('Analog'); axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.3 1.3]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```



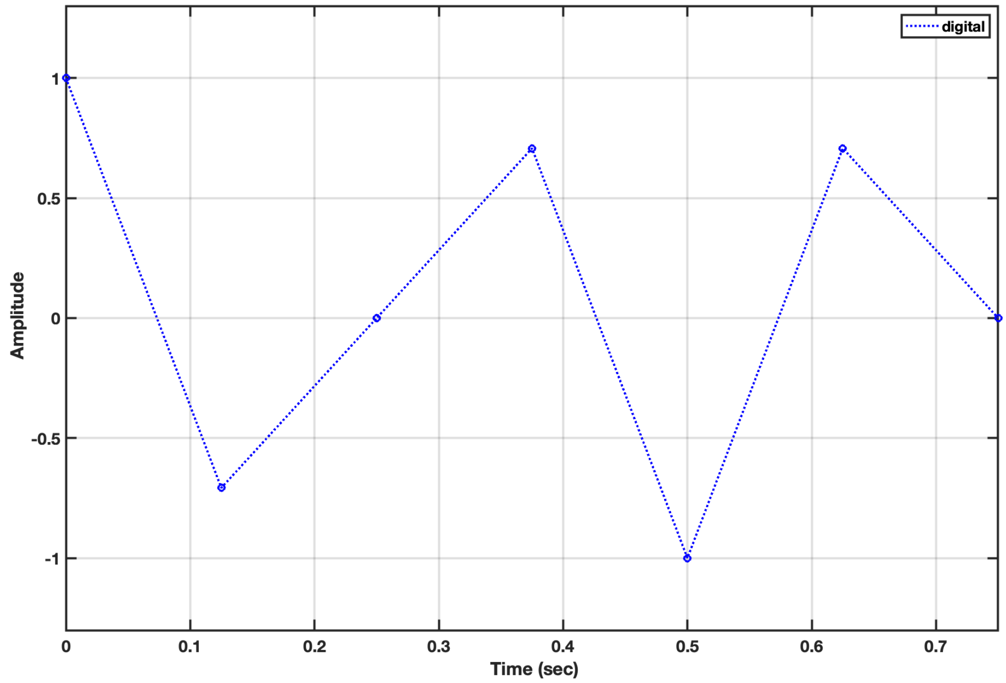
(c) Plot the 5 Hz cosine wave after sampling with 20 Hz. Please plot the sampled data for four cycles of the wave (do not connect the sampled points).

```
clear; clc ; close all;

syms t

f = 5;
y = cos(2*pi*f*t);

% this signal is digital (low sampling rate).
ncyle = 0.8;
Fsd = 20; % # of samples per a second
td = 0:1/Fsd:ncyle;
yd = subs(y, t, td);

fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

plot(td,yd,'or','linewidth',2);
legend('digital'); axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.3 1.3]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold')
```



(d) Plot the 5 Hz cosine wave after sampling with 8 Hz. Please plot only four cycles of the wave (connect the sampled points). Do you think that you can get a true frequency of the wave after sampling?

```
clear; clc ; close all;

syms t

f = 5;
y = cos(2*pi*f*t);

% this signal is digital (very low sampling rate).
ncyle = 0.8;
Fsd = 8; % # of samples per a second
td = 0:1/Fsd:ncyle;
yd = subs(y, t, td);

fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

plot(td,yd,':b','linewidth',2); hold on;
plot(td,yd,'ob','linewidth',2); hold off;
legend('digital'); axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.3 1.3]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold')
```



The sampling frequency is too low to capture the true frequency.

(e) Do you think that you can measure the frequency of the sampled wave in (d) if you add a phase angle to the original wave? For example, the original wave is cos(2πft + φ). Explain your answer.

No. Adding a phase angle to the original wave will only shift the wave form back and forth, and its frequency will still be remain unchanged. The under sampling problem still exists.

(f) Do you think that you can measure the frequency of the sampled wave in (d) if you add a dc signal to the original wave? For example, the original wave is cos(2πft) + d. Explain your answer.

No. Adding a dc signal will only shift the whole wave form upward by the amplitude d (or downward if d is negative), and its frequency will still be remain unchanged. The under sampling problem still exists.

(g) What sampling rate do we use to measure the 20 Hz sine wave?

To measure a 20Hz sine wave, the sampling rate should be at least 2 times of the sine wave, that is at least 40Hz. To be safe, 50Hz sampling rate would be more suitable to measure a 20Hz sine wave.

Problem 2: Aliasing (20 points)

(a) A 8 Hz sine wave is sampled at 12 Hz. Compute the alias frequency that results from the sampling of the original wave. Plot 20 cycles of the original wave, and overlay the sampled points and connect the points (with a different colour). Please confirm that the sampled signal appears as oscillating with the alias frequency that you found.

Signal frequency (f): 8Hz

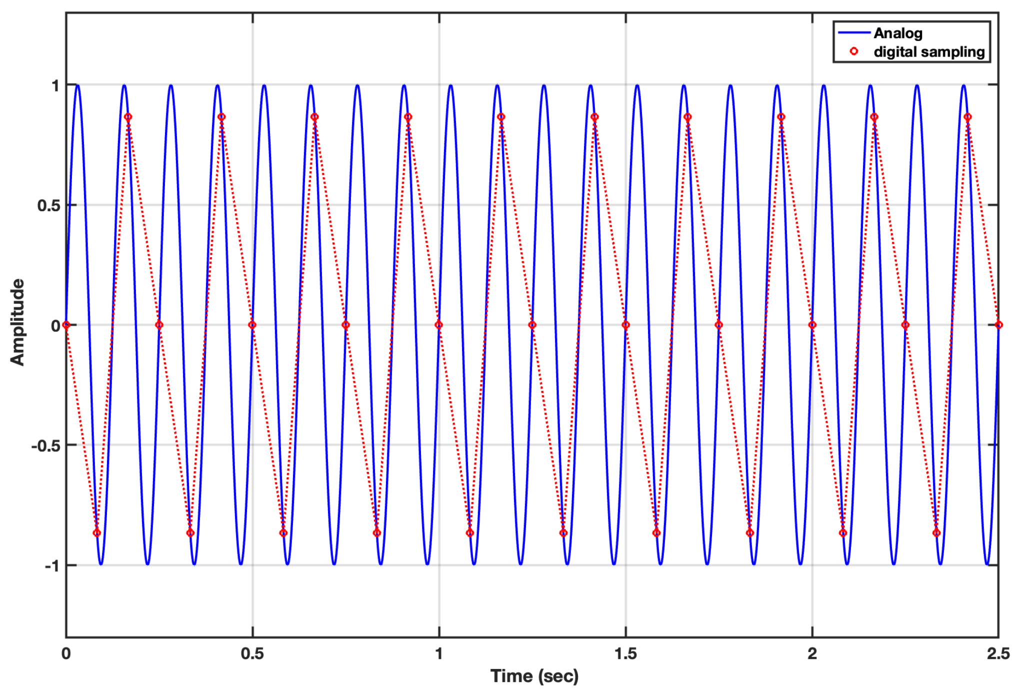
Sampling rate: 12Hz

Nyquist frequency (f_N): 12/2 = 6Hz

f = 1.33fN

From the folding diagram, the alias frequency is 0.67fN = 4Hz

```
clear; clc ; close all;

syms t

f = 8;
y = sin(2*pi*f*t);

% this signal is assumed to be analog (very high sampling rate).
ncyle = 2.5;
Fsa = 1000; % # of samples per a second
ta = 0:1/Fsa:ncyle;
ya = subs(y, t, ta);

fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

Fsd = 12; % # of samples per a second
td = 0:1/Fsd:ncyle;
yd = subs(y, t, td);

plot(ta,ya,'-b', 'linewidth', 2); hold on;
plot(td,yd,'or','linewidth',2);
plot(td,yd,':r','linewidth',2); hold off;
legend('Analog', 'digital sampling'); axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.3 1.3]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```



(c) Assume that the measured signal has a combination of three periodic signals $y(t) = A_1 \cos 2\pi(15)t + A_2 \cos 2\pi(45)t + A_3 \cos 2\pi(75)t$. If the signal is sampled at 110 Hz, determine the frequency content of the sampled signal.

The sampled signal contains 3 frequencies: 15Hz, 45Hz and 75Hz. As the sampling rate is 110Hz, Nyquist frequency is 55Hz.

The 75Hz in the signal is higher than the Nyquist frequency, aliasing occurs, which will be distorted to 35Hz after sampling.

The 15Hz and 45Hz in the signal are lower than the Nyquist frequency, no aliasing occurs, which will not be distorted. They will still be 15Hz and 45Hz after sampling.

(d) Assume that the measured signal has a combination of periodic signals $y(t) = A_1 \cos 2\pi(15)t + A_2 \cos 2\pi(45)t + A_3 \cos 2\pi(75)t$. If the signal is sampled at 170 Hz, determine the frequency content of the sampled signal.

The sampled signal contains 3 frequencies: 15Hz, 45Hz and 75Hz. As the sampling rate is 170Hz, Nyquist frequency is 85Hz.

The 15Hz, 45Hz and 75Hz in the signal are lower than the Nyquist frequency, no aliasing occurs, which will not be distorted. They will still be 15Hz, 45Hz and 75Hz after sampling.

Problem 3: Issues in Sampling (15 points)

(a) Please explain aliasing. When do they occur? How to avoid them?

When a continuous signal is sampled digitally, and the Nyquist frequency is lower than the original signal, because of inadequate sampling, a higher frequency is distorted and become a lower frequency signal.

To avoid aliasing, the sampling rate should be at least double the highest frequency of the signal that we want to capture.

(b) Please explain quantification errors. When do they occur? How to avoid them?

Due to the resolution of the sensor, the value of the original sample is constrained to a discrete set of value.

To mitigate the error, use a sensor with higher sensitivity.

(c) Please explain clipping errors. When do they occur? How to avoid them?

When a signal's peak amplitude exceeds the limit of the sensor, the peak of the signal is clipped to the limit of the sensor. This is clipping error.

To avoid clipping errors, choose the sensor with a limit that exceeds all the peaks of the signal.

(d) Please explain oversampling issues. When do they occur? What are the consequence of the oversampling?

Oversampling is recording information more than needed. Oversampling will make the file size very large.

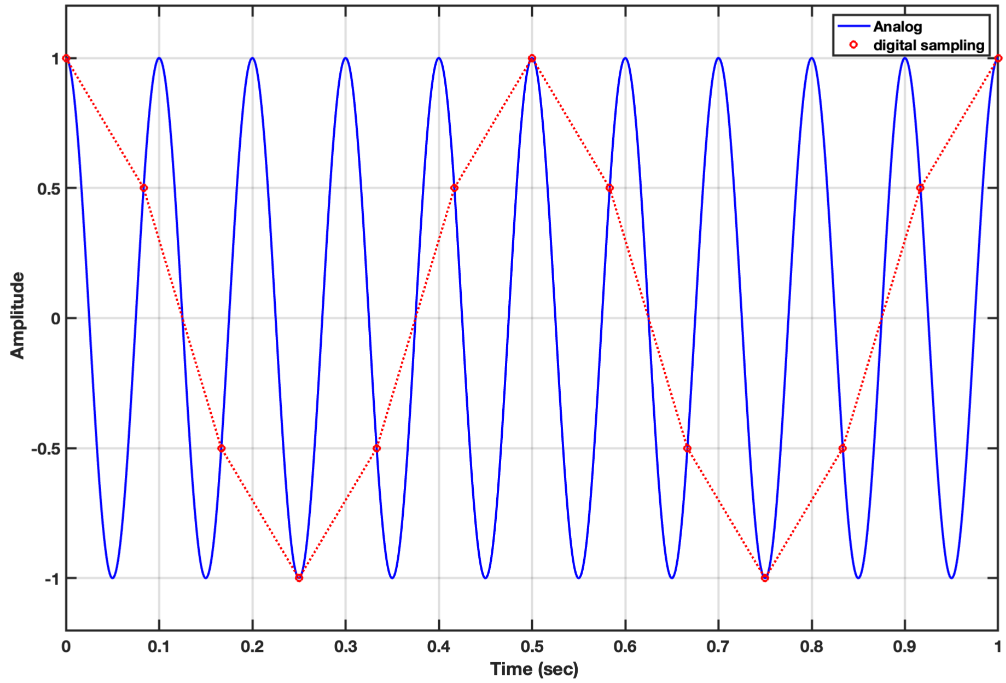
(e) Assume that a building vibrates with a 10 Hz sine wave $y = \cos(2\pi f_0 t)$ $f_0 = 10$, and you measured this vibration using your accelerometer and DAQ system. Please write a code to create three different sampled signals that are damaged by (1) aliasing, (2) quantization error, and (3) clipping error, respectively. Also, generate a signal having an oversampling issue. You should understand the

topics of *Aliasing*, *Quantization*, *Clipping*, and *Oversampling* to solve this problem. You need to explain why your sampled signals contain each of these errors/issues. You can assume any sampling rate, output range, or ADC resolution to generate these signals. Your code should plot these four signals.

(1) Aliasing

As the signal has a 10Hz frequency, to make aliasing happen, the Nyquist frequency has to be lower than 10Hz. By choosing 12Hz as the sampling rate, the Nyquist frequency is 6Hz. From the folding diagram, the original 10Hz frequency will be distorted as 2Hz.

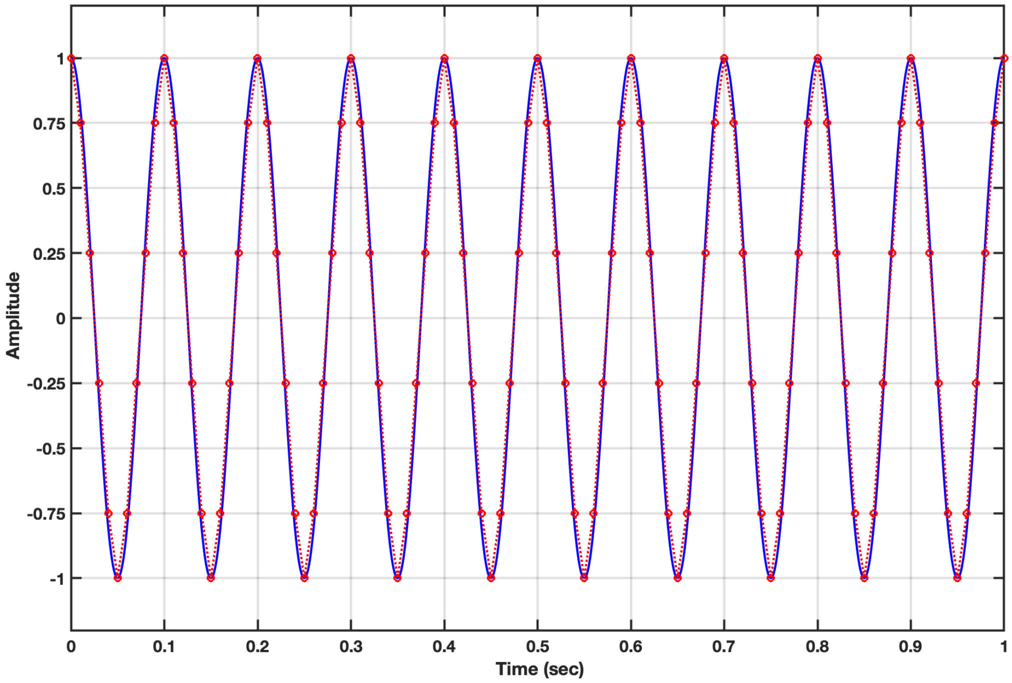
```
clear; clc ; close all;

syms t

f = 10;
y = cos(2*pi*f*t);

% this signal is assumed to be analog (very high sampling rate).
ncyle = 1;
Fsa = 1000; % # of samples per a second
ta = 0:1/Fsa:ncyle;
ya = subs(y, t, ta);

fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

Fsd = 12; % # of samples per a second
td = 0:1/Fsd:ncyle;
yd = subs(y, t, td);

plot(ta,ya,'-b', 'linewidth', 2); hold on;
plot(td,yd,'or','linewidth',2);
plot(td,yd,':r','linewidth',2); hold off;
legend('Analog', 'digital sampling'); axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.2 1.2]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```



(2) Quantization error

To show a quantization error, the resolution is set to 0.25.

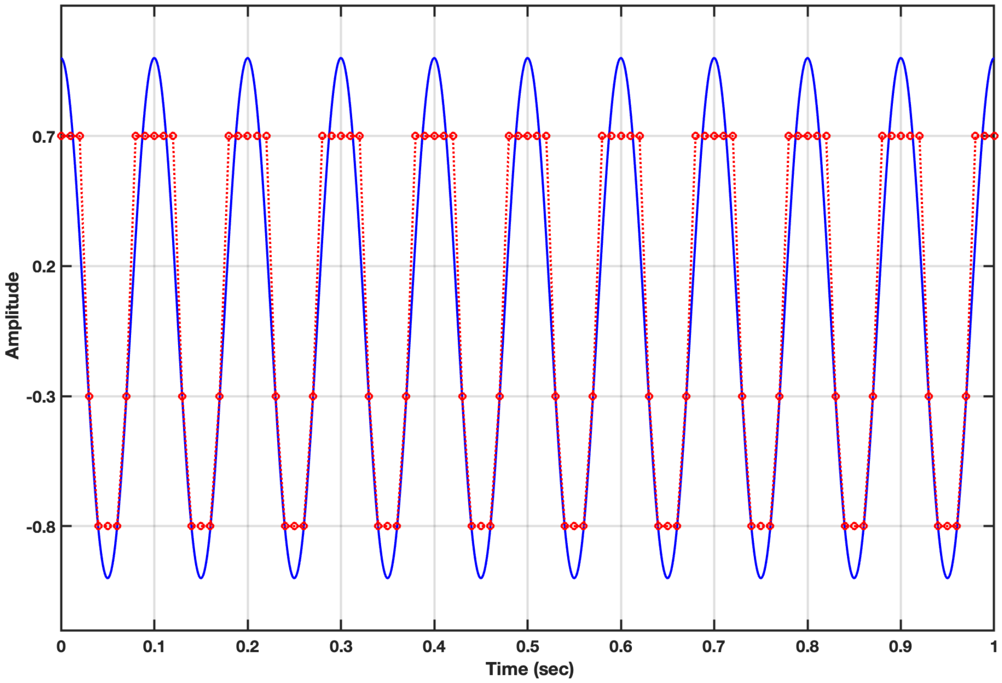
```
syms t

f = 10;
y = cos(2*pi*f*t);

fig2 = figure(2);
set(fig2,'Position', [100 100 1100 700]);

% this signal is assumed to be analog.
ncycle = 1;
Fsa = 1000; % # of samples per a second
ta = 0:1/Fsa:ncycle;
ya = subs(y, t, ta);

Fsd1 = 100; % # of samples per a second
td1 = 0:1/Fsd1:ncycle;
yd1 = subs(y, t, td1);

yd1_digit = interp1(-1:0.25:1,-1:0.25:1, double(yd1), 'nearest');

plot(ta,ya,'-b', 'linewidth', 2); hold on;
plot(td1,yd1_digit,':r','linewidth',2);
plot(td1,yd1_digit,'or','linewidth',2); hold off;
axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.2 1.2]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold', 'YTick', -1:0.25:1)
```



(3) Clipping error

To show a clipping error, the limit of the sensor is set to 0.8, so that the signal with amplitude exceeding 0.8 will be lost.

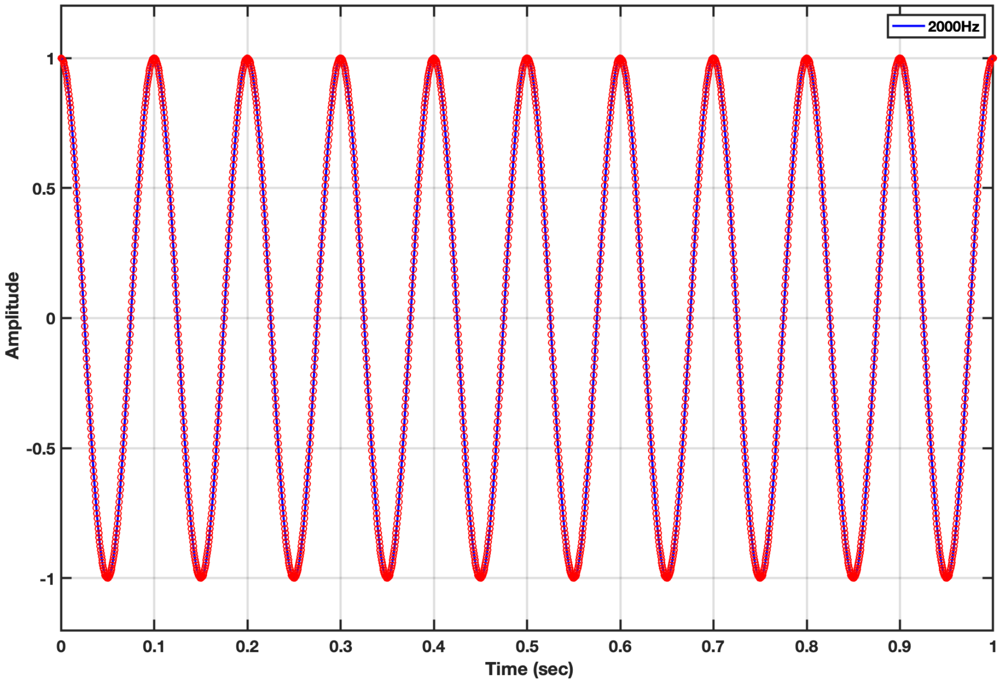
```
syms t

f = 10;
y = cos(2*pi*f*t);

fig2 = figure(2);
set(fig2,'Position', [100 100 1100 700]);

% this signal is assumed to be analog.
ncyle = 1;
Fsa = 1000; % # of samples per a second
ta = 0:1/Fsa:ncyle;
ya = subs(y, t, ta);

Fsd1 = 100; % # of samples per a second
td1 = 0:1/Fsd1:ncyle;
yd1 = subs(y, t, td1);

resol = 4/8;
yd1_digit = interp1(-0.8:resol:0.8,-0.8:resol:0.8, double(yd1*1.5), 'nearest','extrap');

plot(ta,ya,'-b', 'linewidth', 2); hold on;
plot(td1,yd1_digit,':r','linewidth',2);
plot(td1,yd1_digit,'or','linewidth',2); hold off;
axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.2 1.2]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold', 'YTick', -0.8:resol:0.8)
```



(4) Oversampling

To oversample the signal, the sampling rate is set to 2000Hz.

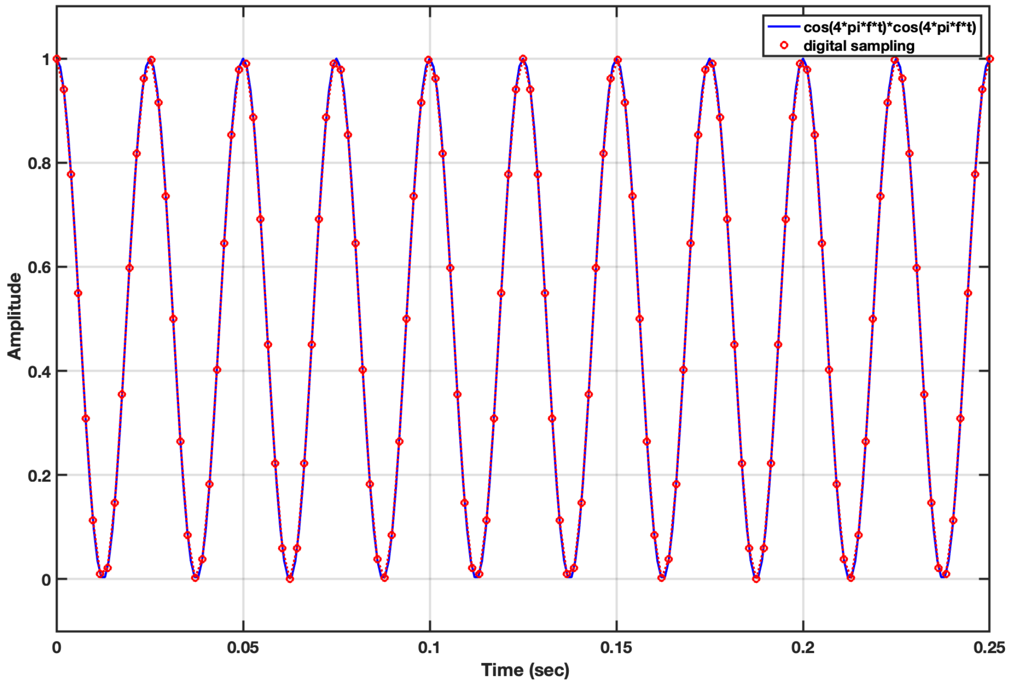
```
syms t

f = 10;
y = cos(2*pi*f*t);

% this signal is assumed to be analog (very high sampling rate).
ncyle = 1;
Fsa = 1000; % # of samples per a second
ta = 0:1/Fsa:ncyle;
ya = subs(y, t, ta);

fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);


Fsd2 = 2000; % # of samples per a second
td2 = 0:1/Fsd2:ncycle;
yd2 = subs(y, t, td2);

plot(ta,ya,'-b', 'linewidth', 2); hold on;
plot(td2,yd2,'or','linewidth',1); hold off;
legend('200Hz'); axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.2 1.2]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold')
```



Problem 4: Fourier Series 1 (10 points)

(a) Plot a wave1. The wave1 is $y = \cos(4\pi f_0 t) \cdot \cos(4\pi f_0 t)$ where $f_0 = 10$. Please connect sampled points and plot only ten cycles of the wave (You can choose any sampling rate).

```
clear; clc ; close all;

syms t

f = 10;
y = cos(4*pi*f*t)*cos(4*pi*f*t);

% the signal is assumed to be analog (very high sampling rate).
ncyle = 0.25;
fs = 1000; % # of samples per a second
ta = 0:1/fs:ncyle;
y1 = subs(y, t, ta);

Fsd = 512; % # of samples per a second
td = 0:1/Fsd:ncyle;
yd = subs(y, t, td);

% plot the signals
fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

plot(ta,y1,'-b', 'linewidth', 2);hold on;
plot(td,yd,'or','linewidth',2);
plot(td,yd,':r','linewidth',2); hold off;
legend('cos(4*pi*f*t)*cos(4*pi*f*t)','digital sampling');
axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-0.1 1.1]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```



(b) Derive a Fourier series (general form) of analytic wave1. You should find an analytic equation for coefficients of a0, an, and bn.

$$y = \cos(4\pi f_0 t) \cdot \cos(4\pi f_0 t)$$
$$= \cos^2(4\pi f_0 t)$$

Since $\cos^2 x = \frac{1+\cos 2x}{2}$

$$y = \cos^2(4\pi f_0 t)$$
$$= \frac{1+\cos(8\pi f_0 t)}{2}$$
$$= \frac{1}{2} + \frac{1}{2}\cos(2\pi(4f_0)t)$$

$For\ a_0$:

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{\frac{-T_p}{2}}^{\frac{T_p}{2}} x(t)\,dt$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{\frac{-T_p}{2}}^{\frac{T_p}{2}} \cos(4\pi f_0 t) \cdot \cos(4\pi f_0 t)\,dt$$

$$\frac{a_0}{2} = \frac{1}{T_p} \cdot \frac{T_p}{2} = 0.5$$

$$a_0 = 1$$

Compare wave 1 $y = \frac{1}{2} + \frac{1}{2}cos(2\pi(4f_0)t)$ with the general form of Fourier Series $y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$

Since $f = \frac{1}{T_p}$, Fourier Series $y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi(nf)t) + b_n \sin(2\pi(nf)t)$

$a_n = \binom{0.5\, if\ n=4}{0\, if\ n\neq 4}$
$b_n = 0$

(c) Derive a Fourier series (complex form) of analytic wave1. You should find an analytic equation for a coefficient of cn.

Let $w = 2\pi f$

$$
\begin{aligned}
y &= \cos^2(4\pi f_0 t) \\
&= \cos^2(2wt)
\end{aligned}
$$

Using Eular Formula $coswt = \frac{1}{2}(e^{iwt} + e^{-iwt})$:

$$
\begin{aligned}
y &= \cos^2(2wt) \\
&= \left(\frac{1}{2}(e^{i(2w)t} + e^{-i(2w)t})\right)^2 \\
&= \frac{1}{4}e^{i(4w)t} + \frac{1}{2} + \frac{1}{4}e^{-i(4w)t}
\end{aligned}
$$

Compare to the complex form of Fourier Series $y = \sum_{n=-\infty}^{\infty} c_n e^{iwnt},\ where\ w = \frac{2\pi}{T_p} = 2\pi f$

$c_n = \frac{1}{4}$, when $n = \pm 4$
$c_n = \frac{1}{2}$, when $n = 0$
$c_n = 0$, when $n \neq 0\ and\ \pm 4$

(d) Derive a Fourier series (general form) of analytic wave2: $y = \cos(4\pi f_0 t) \cdot \cos(4\pi f_0 t) + 5$. You should find analytic equations for coefficients of a0, an, and bn.

$$
\begin{aligned}
y &= \cos(4\pi f_0 t) \cdot \cos(4\pi f_0 t) + 5 \\
&= \cos^2(4\pi f_0 t) + 5 \\
&= \frac{1 + \cos(8\pi f_0 t)}{2} + 5 \\
&= 5\frac{1}{2} + \frac{1}{2}\cos(2\pi(4f_0)t)
\end{aligned}
$$

Compare wave 2 y with the general form of Fourier Series $y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi(nf)t) + b_n \sin(2\pi(nf)t)$

$\frac{a_0}{2} = 5\frac{1}{2}$
$a_0 = 11$

$a_n = \begin{cases} 0.5 \; if \; n=4 \\ 0 \; if \; n \neq 4 \end{cases}$

$b_n = 0$

(e) Please compare the results of (b) and (d) and explain their difference.

The difference between wave 1 and wave 2 is that there is an upward shift of the wave by 5. Other than the shift of the wave, the frequency, amplitude and wave form of both wave 1 and wave 2 are identical.

The coefficient a_0 in Fourier Series represents the permanent shift of the wave, therefore a_0 for wave 1 and wave 2 are different.

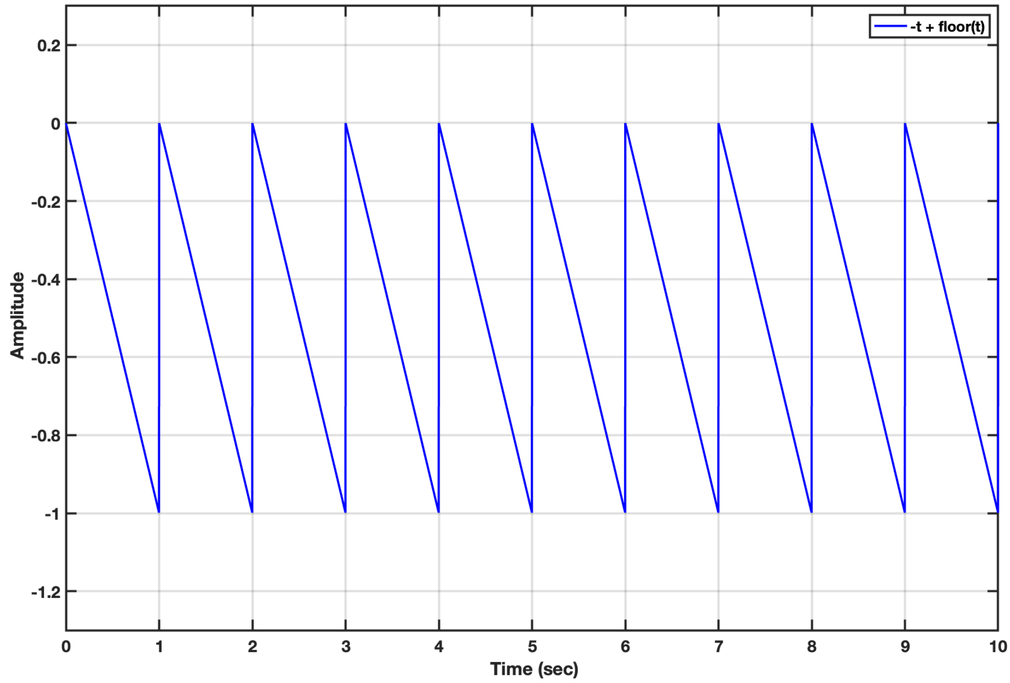
The coefficients a_n and b_n represents the frequency, amplitude and wave form. Since the wave form of wave 1 and wave 2 are the same, both a_n and b_n for wave 1 and wave 2 are also the same.

Problem 5: Fourier Series 2 (20 points)

Sawtooth wave: https://en.wikipedia.org/wiki/Sawtooth_wave

(a) Plot only ten cycles of a reverse sawtooth wave: $x(t) = -t + floor(t)$

```
clear; clc ; close all;

syms t

y = -t + floor(t);

% the signal is assumed to be analog (very high sampling rate).
ncyle = 10;
fs = 1000; % # of samples per a second
ta = 0:1/fs:ncyle;
x = subs(y, t, ta);

% plot the signals
fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

plot(ta,x,'-b', 'linewidth', 2);
legend('-t + floor(t)'); axis tight;grid on;
ylabel('\bf Amplitude'); ylim([-1.3 0.3]);
xlabel('\bf Time (sec)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```



(b) Derive a Fourier series (general form) for a reverse sawtooth wave: $x(t) = -t + floor(t)$. Please check the wikipedia link. You should find an analytic equation for coefficients of a0, an, and bn.

Amplitude = -1

Period = 1

$x(t) = -t,\ 0 < t < 1$

$T_p = 1$

$$
\begin{aligned}
\frac{a_0}{2} &= \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x(t)\,dt \\
&= \int_0^1 -t\,dt \\
&= -\frac{t^2}{2}\Big|_0^1 \\
&= -\frac{1}{2} \\
a_0 &= -1
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x(t)\cos\left(\frac{2\pi nt}{T_p}\right)dt \\
&= 2\int_0^1 -t\cdot\cos(2\pi nt)\,dt
\end{aligned}
$$

Using integration by parts: $\int u\,dv = uv - \int v\,du$

Let $u = t,\ du = dt$

Let $dv = \cos(2\pi nt)\,dt,\ v = \frac{1}{2n\pi}\sin(2\pi nt)$

$a_n = 2\int_0^1 -t \cdot \cos(2\pi nt)\,dt$
$= -2[t \cdot \frac{1}{2n\pi}\sin(2\pi nt)|_0^1 - \int_0^1 \frac{1}{2n\pi}\sin(2\pi nt)\,dt]$
$= -2[0 + \frac{1}{2n\pi}\int_0^1 \sin(2\pi nt)\,dt]$
$= -2[\frac{1}{(2n\pi)^2} \cdot -\cos(2\pi nt)|_0^1]$
$= -2[0]$
$= 0$

$b_n = \frac{2}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x(t)\sin\left(\frac{2\pi nt}{T_p}\right)dt$
$= 2\int_0^1 -t \cdot \sin(2\pi nt)\,dt$

Using integration by parts: $\int u\,dv = uv - \int v\,du$
Let $u = t,\ du = dt$
Let $dv = \sin(2\pi nt)\,dt,\ v = -\frac{1}{2n\pi}\cos(2\pi nt)$

$b_n = 2\int_0^1 -t \cdot \sin(2\pi nt)\,dt$
$= -2[t \cdot -\frac{1}{2n\pi}\cos(2\pi nt)|_0^1 + \int_0^1 \frac{1}{2n\pi}\cos(2\pi nt)\,dt]$
$= -2[-\frac{1}{2n\pi} + \frac{1}{2n\pi}\int_0^1 \cos(2\pi nt)\,dt]$
$= -2[-\frac{1}{2n\pi} + \frac{1}{(2n\pi)^2} \cdot \sin(2\pi nt)|_0^1]$
$= -2[-\frac{1}{2n\pi} + \frac{1}{(2n\pi)^2} \cdot 0]$
$= \frac{1}{n\pi}$

Putting the values of the coefficients of a_0, a_n and b_n into the general form of Fourier Series:
$x(t) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\sin(2\pi nt)$

(c) Derive a Fourier series (complex form) for the same reverse sawtooth wave. You should find an analytic equation for a coefficient of cn.

$c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}\,dt,\ where\ w = \frac{2\pi}{T_p}$
Since $T_p = 1$
$c_n = -\int_0^1 t \cdot e^{-2\pi int}\,dt$
$= -\left[[t \cdot \frac{e^{-2\pi int}}{-2\pi in}]_0^1 - \int_0^1 e^{-2\pi int}\,dt\right]$
$= -\left[\frac{1}{-2\pi in} - 0\right]$
$= \frac{1}{2\pi in}, when\, n \neq 0$
When $n = 0$
$c_o = \frac{a_o}{2}$

$= -\frac{1}{2}$

Putting the values of the coefficient of c_n into the complex form of Fourier Series:

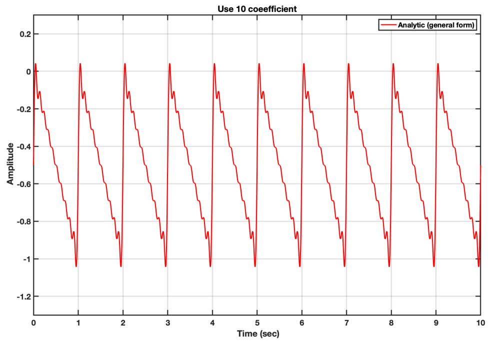
$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i n} e^{2\pi i n t},\ where\, n \neq 0$

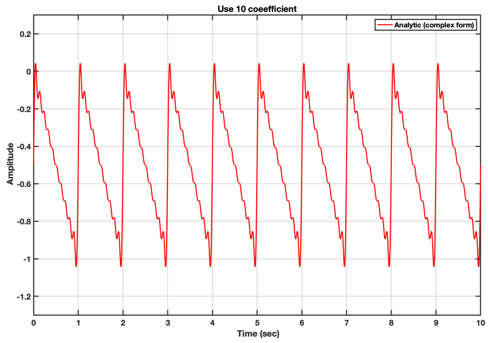
$x(t) = -\frac{1}{2} e^{2\pi i n t},\ where\, n = 0$

(d) Write a code to create and plot approximated sawtooth waves (# of coefficients (n) = 10) using the derived Fourier series in the general and complex forms. You should compare the waves from the general and complex forms.

```
% this signal is assumed to be analog.
ncyle = 10;
Fsa = 1000; % # of samples per a second (assumed to be infinite)
Tp = 1;
t = 0:1/Fsa:ncyle*Tp;

nCoeff = 10;

% analytic integration
xn = zeros(nCoeff, numel(t));

for ii=1:nCoeff
    if ii==1
        xn(ii,:) = -0.5 + (1/(ii*pi))*sin(2*pi*ii*t);
    else
        xn(ii,:) = xn(ii-1,:) + (1/(ii*pi))*sin(2*pi*ii*t);
    end
end

sig_y_analytic = xn;

% plot the signals
fig1 = figure(1);
set(fig1,'Position',  [100 100 1100 700]);

plot(t,sig_y_analytic(nCoeff,:),'-r', 'linewidth', 2);
axis tight;grid on; hold off;
title('Use 10 coeefficient')
ylabel('\bf Amplitude'); ylim([-1.3 0.3]);
xlabel('\bf Time (sec)');
legend('Analytic (general form)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');

% complex form
xc = zeros(nCoeff*2+1, numel(t));

count = 1;
for ii=-nCoeff:nCoeff
    if ii~=0
    xc(count,:) = (1/(2*pi*sqrt(-1)*ii))*exp(2*pi*sqrt(-1)*ii*t);
    else
        xc(count,:) = -1/2*exp(2*pi*sqrt(-1)*ii*t);
    end
    count = count + 1;
end
sig_y_analytic = sum(xc, 1);

% plot the signals
fig1 = figure(2);
set(fig1,'Position',  [100 100 1100 700]);

plot(t,sig_y_analytic,'-r', 'linewidth', 2);
axis tight;grid on; hold off;
title('Use 10 coeefficient')
ylabel('\bf Amplitude'); ylim([-1.3 0.3]);
```

```
xlabel('\bf Time (sec)');
legend('Analytic (complex form)');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```

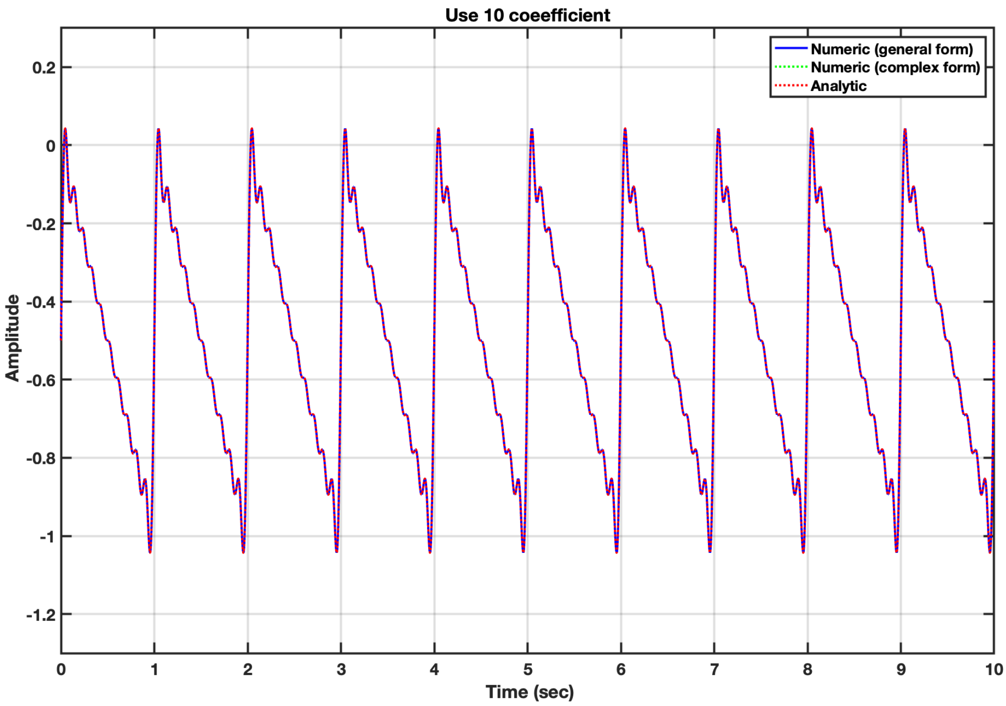




(e) Write a code to find numerical Fourier coefficients in the general and complex forms and compare them with the analytic Fourier coefficients found in (b) and (c).

```
clear; clc ; close all;

% this signal is assumed to be analog.
ncyle = 10;
Fsa = 1000; % # of samples per a second (assumed to be infinite)
Tp = 1;
t = 0:1/Fsa:ncyle*Tp;
x = @(t) -t + floor(t); % original signal

a0 = 2*1/Tp*integral(x, -Tp/2, Tp/2);

nCoeff = 10;
a = zeros(nCoeff,1);
b = zeros(nCoeff,1);
for ii=1:nCoeff
    fun_a = @(t) x(t).*cos(2*pi*ii*t/Tp);
    a(ii) = 2*integral(fun_a, -Tp/2, Tp/2);

    fun_b = @(t) x(t).*sin(2*pi*ii*t/Tp);
    b(ii) = 2*integral(fun_b, -Tp/2, Tp/2);
end

% numerical integration
sig_y_numeric = zeros(nCoeff, numel(t));
sig_y_analytic = zeros(nCoeff, numel(t));
for ii=1:nCoeff
    if ii==1
        sig_y_numeric(ii,:) = ...
            a0/2 + a(ii)*cos(2*pi*ii*t/Tp) + b(ii)*sin(2*pi*ii*t/Tp);
    else
        sig_y_numeric(ii,:) = ...
            sig_y_numeric(ii-1,:) + a(ii)*cos(2*pi*ii*t/Tp) + b(ii)*sin(2*pi*ii*t/Tp);
    end
end

% complex form
sig_y_complex_all = zeros(2*nCoeff+1, numel(t));
count = 1;
for ii=-nCoeff:nCoeff
    fun = @(t) x(t).*exp(-sqrt(-1)*2*pi*ii*t/Tp);
    sig_y_complex_all(count,:) = 1/Tp* integral(fun, 0, Tp)*exp(sqrt(-1)*2*pi*ii*t/Tp);
    count = count + 1;
end

sig_y_complex = zeros(nCoeff, numel(t));
```

```
for ii=1:nCoeff
    rowId = (nCoeff-ii+1):(nCoeff+ii+1);
    sig_y_complex(ii,:) = sum(sig_y_complex_all(rowId,:));
end

% analytic integration
xn = zeros(nCoeff, numel(t));

for ii=1:nCoeff
    if ii==1
        xn(ii,:) = -0.5 + (1/(ii*pi))*sin(2*pi*ii*t);
    else
        xn(ii,:) = xn(ii-1,:) + (1/(ii*pi))*sin(2*pi*ii*t);
    end
end
sig_y_analytic = xn;

% plot the signals
fig1 = figure(1);
set(fig1,'Position',  [100 100 1100 700]);

plot(t,sig_y_numeric(nCoeff,:),'-b', 'linewidth', 2); hold on;
plot(t,sig_y_complex(nCoeff,:),':g', 'linewidth', 2);
plot(t,sig_y_analytic(nCoeff,:),':r', 'linewidth', 2);
axis tight;grid on; hold off;
title('Use 10 coeefficient')
ylabel('\bf Amplitude'); ylim([-1.3 0.3]);
xlabel('\bf Time (sec)');
legend('Numeric (general form)','Numeric (complex form)','Analytic');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```



Problem 6: Fourier Transformation 1 (15 points)

Compute the Fourier transformation (integral) of the following functions and show the derivation process in detail:

(a) cosine wave $y = \cos(3\pi p_0 t)$

$$
\begin{aligned}
x(t) &= \cos(3\pi p_0 t) \\
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \cos(3\pi p_0 t) \cdot e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \tfrac{1}{2}(e^{i3\pi p_0 t} - e^{-i3\pi p_0 t}) \cdot e^{-i2\pi f t}\, dt \\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-i\pi(2f-3p_0)t} - e^{-i\pi(2f+3p_0)t}\, dt
\end{aligned}
$$

Using Dirac Delta Function $\int_{-\infty}^{\infty} e^{\pm i2\pi a t}\, dt = \delta(a)$

$$
\begin{aligned}
X(f) &= \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-i2\pi \frac{(2f-3p_0)}{2} t} - e^{-i2\pi \frac{(2f+3p_0)}{2} t}\, dt \\
&= \tfrac{1}{2} \left[\delta(\tfrac{2f-3p_0}{2}) - \delta(\tfrac{2f+3p_0}{2})\right]
\end{aligned}
$$

(b) cosine wave + dc (direct current) wave $y = \cos(3\pi p_0 t) + d$

$$
\begin{aligned}
x(t) &= \cos(3\pi p_0 t) + d \\
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} [\cos(3\pi p_0 t) + d] \cdot e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \cos(3\pi p_0 t) \cdot e^{-i2\pi f t} + d e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \tfrac{1}{2}(e^{i3\pi p_0 t} - e^{-i3\pi p_0 t}) \cdot e^{-i2\pi f t} + d e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \tfrac{1}{2} e^{-i\pi(2f-3p_0)t} - \tfrac{1}{2} e^{-i\pi(2f+3p_0)t} + d e^{-i2\pi f t}\, dt
\end{aligned}
$$

Using Dirac Delta Function $\int_{-\infty}^{\infty} e^{\pm i2\pi a t}\, dt = \delta(a)$

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} \tfrac{1}{2} e^{-i2\pi \frac{(2f-3p_0)}{2} t} - \tfrac{1}{2} e^{-i2\pi \frac{(2f+3p_0)}{2} t} + d e^{-i2\pi f t}\, dt \\
&= \tfrac{1}{2} \left[\delta(\tfrac{2f-3p_0}{2}) - \delta(\tfrac{2f+3p_0}{2})\right] + d\delta(f)
\end{aligned}
$$

(c) two cosine waves $y = \cos(3\pi p_0 t) + \cos(3\pi p_1 t)$

$$
\begin{aligned}
x(t) &= \cos(3\pi p_0 t) + \cos(3\pi p_1 t) \\
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} [\cos(3\pi p_0 t) + \cos(3\pi p_1 t)] \cdot e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \cos(3\pi p_0 t) \cdot e^{-i2\pi f t} + \cos(3\pi p_1 t) \cdot e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} \tfrac{1}{2}(e^{i3\pi p_0 t} - e^{-i3\pi p_0 t}) \cdot e^{-i2\pi f t} + \tfrac{1}{2}(e^{i3\pi p_1 t} - e^{-i3\pi p_1 t}) \cdot e^{-i2\pi f t}\, dt \\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-i\pi(2f-3p_0)t} - e^{-i\pi(2f+3p_0)t} + e^{-i\pi(2f-3p_1)t} - e^{-i\pi(2f+3p_1)t}\, dt
\end{aligned}
$$

Using Dirac Delta Function $\int_{-\infty}^{\infty} e^{\pm i2\pi a t}\, dt = \delta(a)$

$$
\begin{aligned}
X(f) &= \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-i2\pi \frac{(2f-3p_0)}{2} t} - e^{-i2\pi \frac{(2f+3p_0)}{2} t} + e^{-i2\pi \frac{(2f-3p_1)}{2} t} - e^{-i2\pi \frac{(2f+3p_1)}{2} t}\, dt \\
&= \tfrac{1}{2} \left[\delta(\tfrac{2f-3p_0}{2}) - \delta(\tfrac{2f+3p_0}{2}) + \delta(\tfrac{2f-3p_1}{2}) - \delta(\tfrac{2f+3p_1}{2})\right]
\end{aligned}
$$

Problem 7: Fourier Transformation 2 (10 points)

$$y(t) = e^{-a|t|}(b \cdot \cos 2\pi f_1 t + c \cdot \cos 2\pi f_2 t)$$

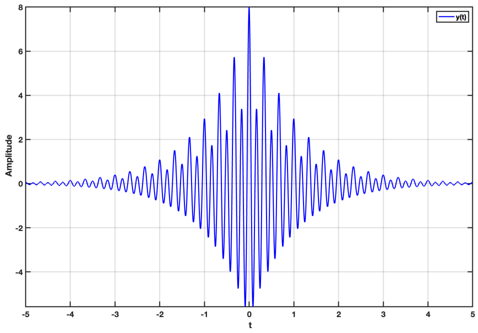
(a) Compute the Fourier transformation (integral) of the above function

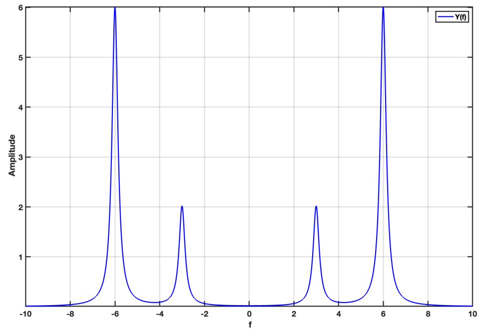
$$y(t) = e^{-a|t|}(b \cdot \cos 2\pi f_1 t + c \cdot \cos 2\pi f_2 t)$$

$$\begin{aligned}
Y(f) &= \int_{-\infty}^{\infty} y(t) e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} [e^{-a|t|}(b \cdot \cos 2\pi f_1 t + c \cdot \cos 2\pi f_2 t)] e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} [e^{-a|t|}(b \cdot \tfrac{1}{2}(e^{i2\pi f_1 t} + e^{-i2\pi f_1 t}) + c \cdot \tfrac{1}{2}(e^{i2\pi f_2 t} + e^{-i2\pi f_2 t}))] e^{-i2\pi f t}\, dt \\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} [e^{-a|t|}(b \cdot (e^{i2\pi f_1 t} + e^{-i2\pi f_1 t}) e^{-i2\pi f t} + c \cdot (e^{i2\pi f_2 t} + e^{-i2\pi f_2 t}) e^{-i2\pi f t})]\, dt \\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} [e^{-a|t|}(b \cdot (e^{-i2\pi (f-f_1) t} + e^{-i2\pi (f+f_1) t}) + c \cdot (e^{-i2\pi (f-f_2) t} + e^{-i2\pi (f+f_2) t}))]\, dt \\
&= \frac{ab}{a^2 + [2\pi(f-f_1)]^2} + \frac{ab}{a^2 + [2\pi(f+f_1)]^2} + \frac{ac}{a^2 + [2\pi(f-f_2)]^2} + \frac{ac}{a^2 + [2\pi(f+f_2)]^2}
\end{aligned}$$

(b) Plot y in time domain and frequency domain, where a = 1, b = 2, c= 6, f1 = 3, and f2 = 6

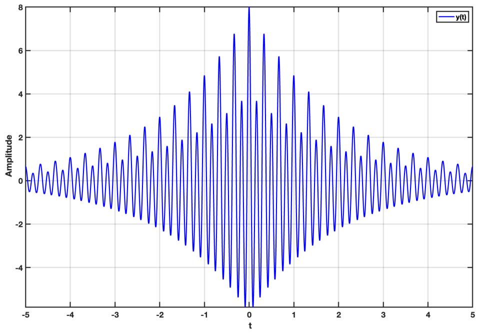
```
clear; clc ; close all;

syms t f

a = 1;
b = 2;
c = 6;
f1 = 3;
f2 = 6;

yt = exp(-a*abs(t))*(b*cos(2*pi*f1*t)+c*cos(2*pi*f2*t));
yf = a*b*(1/(a^2+(2*pi*(f-f1))^2)+1/(a^2+(2*pi*(f+f1))^2)) + a*c*(1/(a^2+(2*pi*(f-f2))^2)+1/(a^2+(2*pi*(f+f2))^2));

% the signal is assumed to be analog (very high sampling rate).
ncyle = 10;
fs = 1000; % # of samples per a second
ta = -ncyle:1/fs:ncyle;
yt = subs(yt, t, ta);

tf = -ncyle:1/fs:ncyle;
yf = subs(yf, f, tf);

% plot the signals
fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

plot(ta,yt,'-b', 'linewidth', 2);
legend('y(t)'); axis tight;grid on;
ylabel('\bf Amplitude');
xlabel('\bf t'); xlim([-5 5])
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');

fig2 = figure(2);
set(fig2,'Position', [100 100 1100 700]);

plot(tf,yf,'-b', 'linewidth', 2);
legend('Y(f)'); axis tight;grid on;
ylabel('\bf Amplitude');
xlabel('\bf f');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```

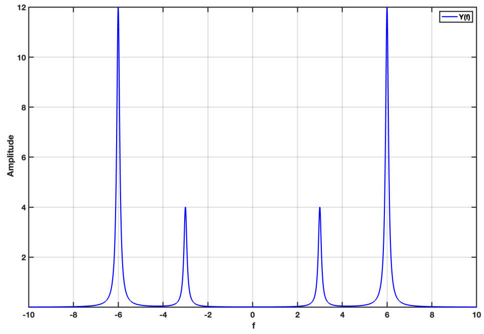




(c) Plot y in time domain and frequency domain, where a = 0.5, b = 2, c= 6, f1 = 3, and f2 = 6

```
clear; clc ; close all;

syms t f

a = 0.5;
b = 2;
c = 6;
f1 = 3;
f2 = 6;

yt = exp(-a*abs(t))*(b*cos(2*pi*f1*t)+c*cos(2*pi*f2*t));
yf = a*b*(1/(a^2+(2*pi*(f-f1))^2)+1/(a^2+(2*pi*(f+f1))^2)) + a*c*(1/(a^2+(2*pi*(f-f2))^2)+1/(a^2+(2*pi*(f+f2))^2));

% the signal is assumed to be analog (very high sampling rate).
ncyle = 10;
fs = 1000; % # of samples per a second
ta = -ncyle:1/fs:ncyle;
yt = subs(yt, t, ta);

tf = -ncyle:1/fs:ncyle;
yf = subs(yf, f, tf);

% plot the signals
fig1 = figure(1);
set(fig1,'Position', [100 100 1100 700]);

plot(ta,yt,'-b', 'linewidth', 2);
legend('y(t)'); axis tight;grid on;
ylabel('\bf Amplitude');
xlabel('\bf t'); xlim([-5 5])
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');

fig2 = figure(2);
set(fig2,'Position', [100 100 1100 700]);

plot(tf,yf,'-b', 'linewidth', 2);
legend('Y(f)'); axis tight;grid on;
ylabel('\bf Amplitude');
xlabel('\bf f');
set(gca,'fontsize',15,'linewidth',2,'fontweight','bold');
```





(d) Compare the graphs in (b) and (c)

The difference between (b) and (c) is the value of coefficient a, where the value of a in (c) is half of that in (b).

By reducing the value of a half from (b) to (c), the wave in time domain of (c) is wider than (b). Fourier transform has inverse spreading property, the wider the wave in time domain, the narrower in frequency domain. As a result, the wave in frequency domain of (c) has narrower and higher peaks than (b).