Problem 1: Sampling (10 points)

(a) What is the difference between continuous (or analogue) and discrete (or digital) signals?

(b) Plot a 10 Hz cosine wave with a high sampling rate (nearly analog signal). Please connect the sampled points and plot only four cycles of the wave.

(c) Plot the 10 Hz cosine wave after sampling with 40 Hz. Please plot the sampled data for four cycles of the wave (**do not connect** the sampled points).

(d) Plot the 10 Hz cosine wave after sampling with 15 Hz. Please plot only four cycles of the wave (**connect** the sampled points). Do you think that you can get a true frequency of the wave after sampling?

(e) Do you think that you can measure the frequency of the sampled wave in (d) if you add a phase angle to the original wave? For example, the original wave is $\cos(2\pi ft + \varphi)$. Explain your answer.

(f) Do you think that you can measure the frequency of the sampled wave in (d) if you add a dc signal to the original wave? For example, the original wave is $\cos(2\pi ft) + d$. Explain your answer.

(g) What sampling rate do we use to measure the 10

Task 02. Signal processing 1.

Prob.1.

(a) Analogue : Continuos & unlimited signal.

Discrete : Sampled finite signal.

(d) Plot the 10 Hz cosine wave after sampling with 15 Hz. Please plot only four cycles of the wave (**connect** the sampled points). Do you think that you can get a true frequency of the wave after sampling?

(e) Do you think that you can measure the frequency of the sampled wave in (d) if you add a phase angle to the original wave? For example, the original wave is cos(2πft + φ). Explain your answer.

(f) Do you think that you can measure the frequency of the sampled wave in (d) if you add a dc signal to the original wave? For example, the original wave is cos(2πft) + d. Explain your answer.

(g) What sampling rate do we use to measure the 10 Hz cosine wave?

(d) No. f_N : 7.5 Hz Nyquist frequency.

because. $10 > f_N$

(e) No,

it's nothing to do with ϕ and d.

(f) No,

(g) More than 20Hz.

Problem 2: Aliasing (20 points)

> Aliasing is the phenomenon that the high frequency contents will be ***falsely*** represented by a low frequency and this false frequency is called **alias frequency**.

(a) A 8 Hz sine wave is sampled at 10 Hz. Compute the alias frequency that results from the sampling of the original wave. Plot 12 cycles of the original wave, and overlay the sampled points and connect the points (with a different colour). Please confirm that the sampled signal appears as oscillating with the alias frequency that you found.

(b) A 8 Hz sine wave is sampled at 24 Hz. Compute the alias frequency that results from the sampling of the original wave. Plot 12 cycles of the original wave, and overlay the sampled points and connect the points (with a different colour). Please confirm that the sampled signal appears as oscillating with the alias frequency that you found.

(c) Assume that the measured signal has a combination of threeperiodic signals:

$y(t) = A1cos2\pi(15)t + A2cos2\pi(45)t + A3cos2\pi(75)t$

If the signal is sampled at 120 Hz, determine the frequency content of the sampled signal.

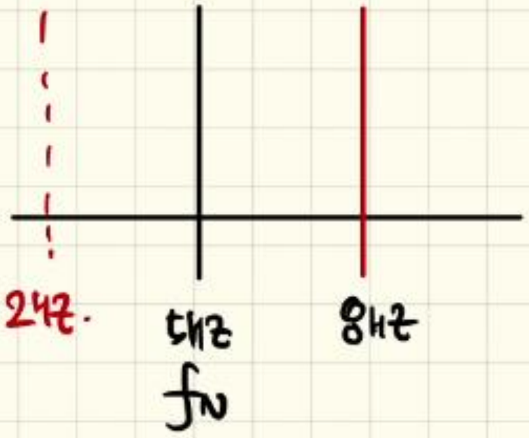
(d) Assume that the measured signal has a combination of periodic signals:

$y(t) = A1cos2\pi(15)t + A2cos2\pi(45)t + A3cos2\pi(75)t$

If the signal is sampled at 160 Hz, determine the frequency content of the sampled signal.

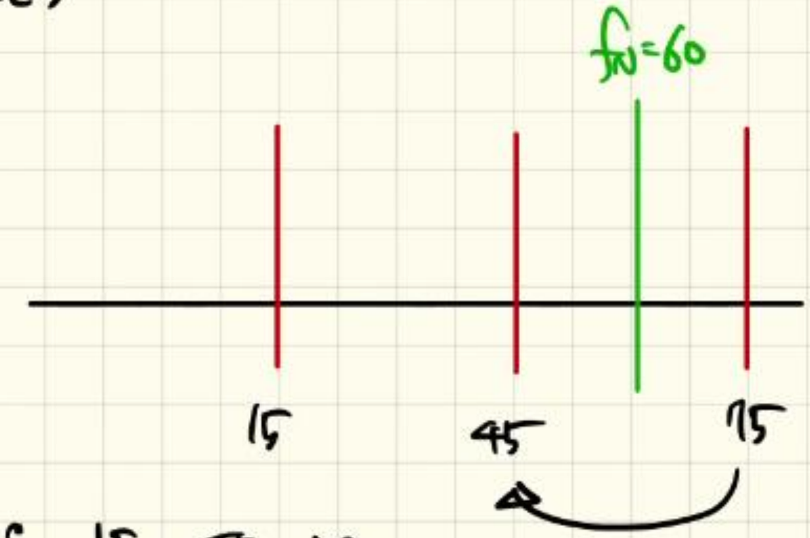
Prob 2.

(a).



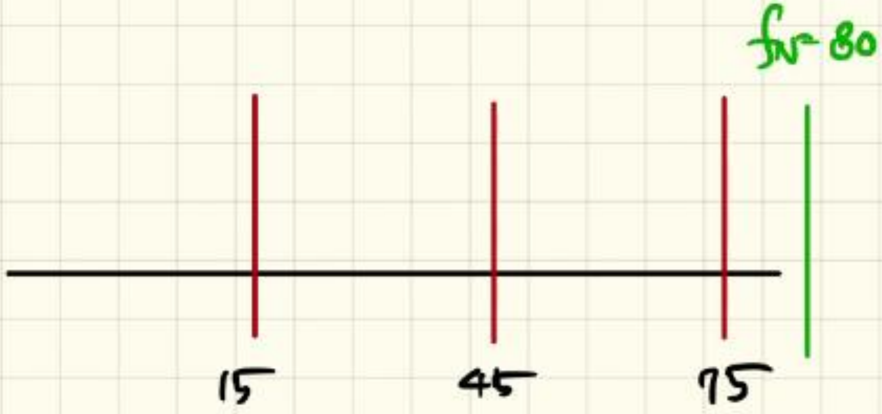
(b) No aliasing. It looks 8Hz wave. Which is the true signal frequency.

(c)



∴ 15, 45 Hz

(d)



∴ 15, 45, 75 Hz.

Problem 4: Fourier Series 1 (10 points)

(a) Plot a **wave1**. The **wave1** is

$$y = cos(2\pi f_0 t) \cdot cos(2\pi f_0 t)$$

where f_0 = 15. Please connect sampled points and plot only ten cycles of the wave (You can choose any sampling rate).

(b) Derive a Fourier series (general form) of analytic **wave1**. You should find an analytic equation for coefficients of a_0, a_n, and b_n.

(c) Derive a Fourier series (complex form) of analytic **wave1**. You should find an analytic equation for a coefficient of c_n.

(d) Derive a Fourier series (general form) of analytic **wave2**:

$$y = cos(2\pi f_0 t) \cdot cos(2\pi f_0 t) + 5$$

You should find analytic equations for coefficients of a_0, a_n, and b_n.

(e) Please compare the results of (b) and (d) and explain their difference.

prob 3. See Lectures

prob 4. * $2\cos\theta\cos\phi = \cos(\theta-\phi)+$

Product-to-sum[32]
$2\cos\theta\cos\varphi = \cos(\theta-\varphi)+\cos(\theta+\varphi)$
$2\sin\theta\sin\varphi = \cos(\theta-\varphi)-\cos(\theta+\varphi)$
$2\sin\theta\cos\varphi = \sin(\theta+\varphi)+\sin(\theta-\varphi)$
$2\cos\theta\sin\varphi = \sin(\theta+\varphi)-\sin(\theta-\varphi)$
$\tan\theta\tan\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{\cos(\theta-\varphi)+\cos(\theta+\varphi)}$
$\prod_{k=1}^{n}\cos\theta_k = \frac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1+\cdots+e_n\theta_n)$ where $S=\{1,-1\}^n$

(b) $y = \cos^2(2\pi f_0 t) = \frac{1}{2}\left(1 + \cos(4\pi f_0 t)\right)$

$$= \frac{1}{2}\left(1 + \cos\frac{2\pi\cdot 2t}{T_p}\right) \quad \text{where } f_0 = 1/T_p.$$

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$$a_0 = 1, \quad b_n = 0 \text{ for all } n, \quad a_n = \begin{cases} \frac{1}{2} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Complex Form of the Fourier Series

Euler Formula

$$e^{iwt} = coswt + i\ sinwt \qquad e^{-iwt} = coswt - i\ sinwt \qquad coswt = \frac{1}{2}(e^{iwt} + e^{-iwt}) \qquad sinwt = \frac{1}{2j}(e^{iwt} - e^{-iwt})$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos wnt + b_n \sin wnt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}(e^{iwnt} + e^{-iwnt}) + \frac{b_n}{2j}(e^{iwnt} - e^{-iwnt})$$

$$\boldsymbol{w = \frac{2\pi}{T_p}}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt} = c_0 + \sum_{n=1}^{\infty} c_n e^{iwnt} + \sum_{n=1}^{\infty} c_n^* e^{-iwnt}$$

$$\boxed{where\ c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - jb_n}{2},}$$

$$\boxed{c_n^* = \frac{a_n + jb_n}{2}}$$

$$\boxed{c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad c_n^* = \frac{1}{T_p}\int_0^{T_p} x(t)e^{iwnt}dt = c_{-n}}$$

Negative frequency term (c_{-n})

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi}{T_p}$$

prob 4 (c).

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\omega n t} \qquad C_n = \frac{1}{T_p}\int_0^{T_p} x(t) e^{-i\omega t}\, dt \qquad \omega = \frac{2\pi}{T_p}$$

$$y = \cos^2(2\pi f_0 t) = \frac{1}{2}\left(1 + \cos(4\pi f_0 t)\right)$$

$$C_n = \frac{1}{T_p}\int_0^{T_p} \cos^2(2\pi f_0 t)\, e^{-i\omega n t}\, dt = \underbrace{\frac{1}{2T_p}\int_0^{T_p} e^{-i\omega n t}\, dt}_{0 \text{ except for } n=0} + \underbrace{\frac{1}{2T_p}\int_0^{T_p} \cos(4\pi f_0 t)\cos(\omega n t)}_{0 \text{ except for } n=2 \text{ or } n=-2} - \underbrace{\cancel{i\cos(4\pi f_0 t)\sin(\omega n t)}}_{\text{orthogonality}}\, dt$$

$$C_0 = \frac{1}{2T_p}\int_0^{T_p} dt = \frac{1}{2T_p}\cdot T_p = \frac{1}{2}.$$

$$C_2 = \frac{1}{2T_p}\int_0^{T_p} \cos(4\pi f_0 t)\cos(4\pi f_0 t)\, dt = \frac{1}{4T_p}\int_0^{T_p} 1 + \underbrace{\cancel{\cos(8\pi f_0 t)}}_{0}\, dt = \frac{1}{4}$$

$$\therefore\ C_0 = \frac{1}{2}, \quad C_n = \begin{cases} \frac{1}{4} & n = 2, -2 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos wnt + b_n \sin wnt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{iwnt} + e^{-iwnt}\right) + \frac{b_n}{2j}\left(e^{iwnt} - e^{-iwnt}\right)$$

$$w = \frac{2\pi}{T_p}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt} = c_0 + \sum_{n=1}^{\infty} c_n e^{iwnt} + \sum_{n=1}^{\infty} c_n^* e^{-iwnt}$$

$$\boxed{\text{where } c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - jb_n}{2},}$$

$$\boxed{c_n^* = \frac{a_n + jb_n}{2}}$$

$$\boxed{c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad c_n^* = \frac{1}{T_p}\int_0^{T_p} x(t)e^{iwnt}dt = c_{-n}}$$

Negative frequency term (c_{-n})

$$C_0 = \frac{a_0}{2}, \quad C_n = \frac{a_n - jb_n}{2}, \quad C_n^* = \frac{a_n + jb_n}{2}.$$

$$a_0 = 1, \quad b_n = 0 \text{ for all } n \quad a_n = \begin{cases} 1/2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore C_0 = \frac{1}{2}, \quad C_n = \frac{1}{4}, \quad C_n^* = \frac{1}{4}$$

$$\therefore C_0 = \frac{1}{2}, \quad C_n = \begin{cases} \frac{1}{4} & n = 2 \text{ or } -2 \\ 0 & \text{otherwise.} \end{cases}$$

$$y = \cos(2\pi f_0 t)^2 + 5$$

$$\frac{a_0}{2} = 5 + \frac{1}{2} \qquad \therefore a_0 = 11$$

$$b_n = 0$$

$$a_n = \begin{cases} \frac{1}{2} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

(d) Derive a Fourier series (general form) of analytic wave2:

$$y = cos(2\pi f_0 t) \cdot cos(2\pi f_0 t) + 5$$

You should find analytic equations for coefficients of a_0, a_n, and b_n.

(e) Please compare the results of (b) and (d) and explain their difference.

Problem 5: Fourier Series 2 (20 points)

Sawtooth wave: https://en.wikipedia.org/wiki/Sawtooth_wave

(a) Plot only ten cycles of a reverse sawtooth wave:

$$x(t) = -t + floor(t)$$

(b) Derive a Fourier series (general form) for a reverse sawtooth wave:

$$x(t) = -t + floor(t)$$

Please check the wikipedia link. You should find an analytic equation for coefficients of a_0, a_n, and b_n.

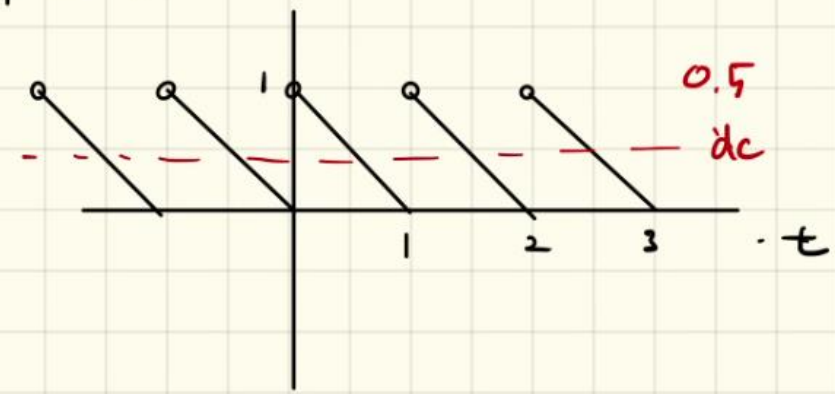
(c) Derive a Fourier series (complex form) for the same reverse sawtooth wave. You should find an analytic equation for a coefficient of c_n.

(d) Write a code to create and plot approximated sawtooth waves (# of coefficients (n) = 8) using the derived Fourier series in the general and complex forms. You should compare the waves from the general and complex forms.

(e) Write a code to find numerical Fourier coefficients in the general and complex forms and compare them with the analytic Fourier coefficients found in (b) and (c).

Please explain your answer and findings. Please study tutorial codes.

prob. 5



$T_p = 1$

$x(t) = -t + 1$

$$\boxed{uv = \int u\,dv + \int v\,du}$$

(b) $\frac{a_0}{2} = 0.5 \quad \therefore a_0 = 1$

$$a_n = \frac{2}{T_p}\int_0^{T_p} x(t)\cos\left(\frac{2\pi n t}{T_p}\right)dt$$

$$= 2\int_0^1 \underbrace{\cancel{\cos(2\pi n t)}}_{0} - x\cos(2\pi n t)\,dt$$

$u = -t$

$V = \frac{\sin(2\pi n t)}{2\pi n}$

$$= 2\left[-t\,\underbrace{\cancel{\frac{\sin(2\pi n t)}{2\pi n}}}_{0}\Big|_0^1 - \frac{1}{2\pi n}\underbrace{\cancel{\int_0^1 \sin 2\pi n t\,dt}}_{0}\right]$$

$= 0$

$$b_n = \frac{2}{T_p}\int_0^{T_p} x(t)\sin\left(\frac{2\pi n t}{T_p}\right)dt$$

$$= 2\left[t\,\frac{\cos(2\pi n t)}{2\pi n}\Big|_0^1 + \frac{1}{2\pi n}\underbrace{\cancel{\int_0^1 \cos 2\pi n t\,dt}}_{0}\right]$$

$$= 2\left(\frac{1}{2\pi n}\right) = \frac{1}{\pi n}$$

$$\boxed{\left.\begin{array}{l} a_0 = 1 \\ a_n = 0 \\ b_n = \frac{1}{n\pi} \end{array}\right\} \text{for all } n.}$$

(C) $C_n = \int_0^1 (-t+1) e^{-i\omega n t}\, dt = \int_0^1 -t e^{-i2\pi n t} + e^{-i2\pi n t}\, dt$

0 except for $n=0$

$C_0 = \int_0^1 -t + 1 = -\frac{1}{2}t^2 + t \Big|_0^1 = \frac{1}{2}$

$n \neq 0$

$C_n = \int_0^1 -t e^{-i2\pi n t} dt = \frac{-t e^{-i2\pi n t}}{-i2\pi n} \Big|_0^1 + \int_0^1 \frac{1}{i2\pi n} \cancel{e^{-i2\pi n t}}$

$\swarrow 0$

$u = -t$

$v = \frac{e^{-i2\pi n t}}{-i2\pi n}$

$= \frac{1}{i2\pi n}\left(e^{-i2\pi n}\right) = \frac{1}{i2\pi n}\left(\cancel{\cos 2\pi n} - i\cancel{\sin 2\pi n}\right)$

$\swarrow 1 \quad \swarrow 0$

$= \frac{1}{i2\pi n}$

$C_n = \begin{cases} 1/2 & n=0 \\ \frac{1}{i2\pi n} & n \neq 0 \end{cases}$

$\boxed{C_n = \frac{a_n - jb_n}{2} = -\frac{1}{2} j b_n = \frac{-j}{2n\pi} = \frac{1}{i2n\pi}}$