Problem 1: Sampling (10 points)

(a) What is the difference between a continuous (or analogue) and discrete (or digital) signals?

(b) Plot a 6 Hz sine wave with high sampling rate (nearly analog signal). Please connect sampled points and plot only four cycles of the wave.

(c) Plot the 3 Hz sine wave after sampling with 10 Hz. Please do not connect sampled points and plot the sampled data for four cycles of the wave.

(d) Plot the 6 Hz sine wave after sampling with 11 Hz. Please do not connect sampled points and plot only four cycles of the wave. Do you think that you can measure this wave? **NO**

(e) Plot the 6 Hz sine wave after sampling with 12 Hz. Please do not connect sampled points and plot only four cycles of the wave. Do you think that you can measure this wave if you add a phase angle (φ) on this sine wave? for example, the wave is $\sin(2\pi ft + \varphi)$. **NO**

(f) Do you think that you can measure this wave if you add a dc signal on this sine wave? for example, the wave is $\sin(2\pi ft) + d$.

NO

Problem 2: Aliasing (15 points)

(a) A 6 Hz sine wave is sampled at 8 Hz. Compute the alias frequency that can be represented in the resulting sampled signal. Plot the wave and sampled points.

(b) A 15 Hz sine wave is sampled at 15 Hz. Compute the alias frequency that can be represented in the resulting sampled signal. Plot the wave and sampled points.

(c) Assume that the measured signal has a combination of periodic signals:

$$y(t) = A1sin2\pi(25)t + A2sin2\pi(75)t + A3sin2\pi(125)t$$

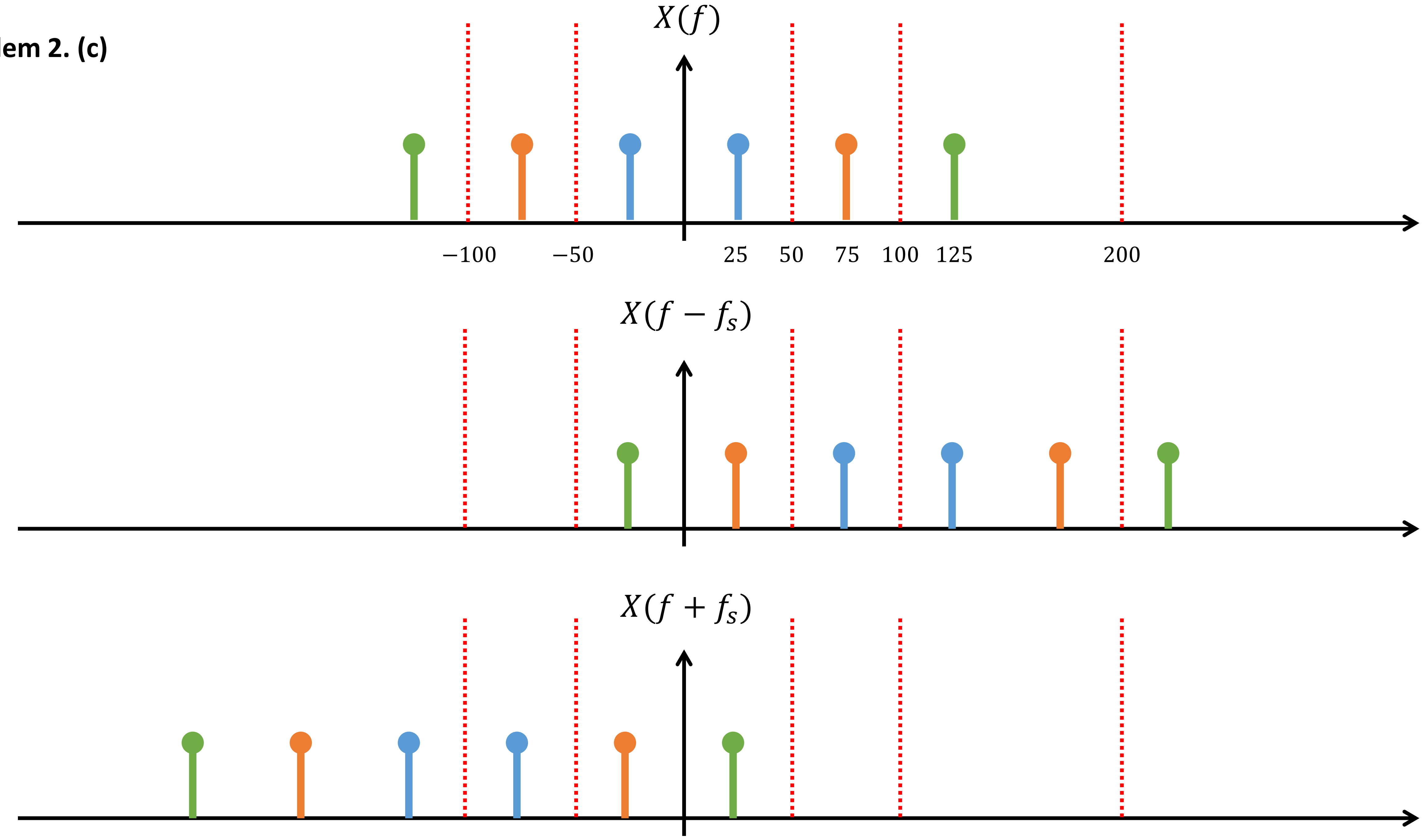
If the signal is sampled at 100 Hz, determine the frequency content of the resulting discrete response signal.

(d) Assume that the measured signal has a combination of periodic signals:

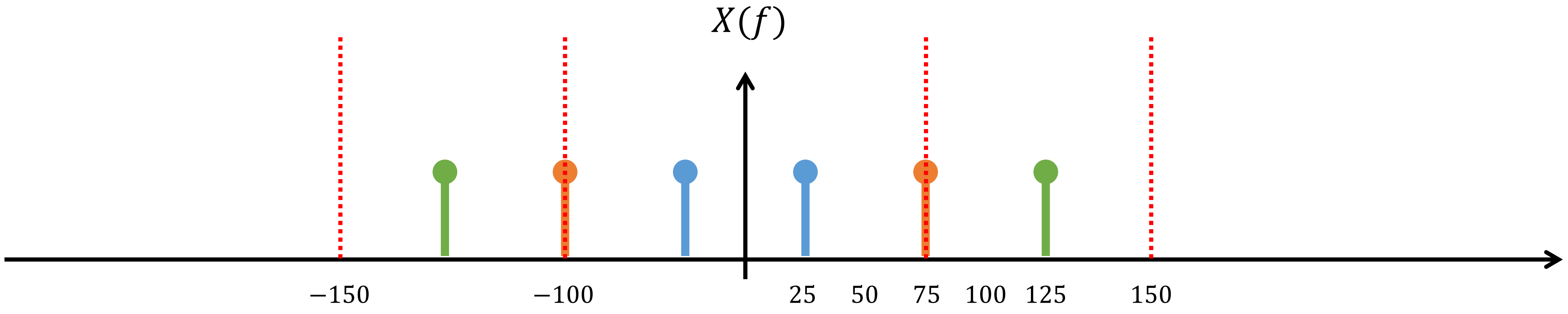
$$y(t) = A1sin2\pi(25)t + A2sin2\pi(75)t + A3sin2\pi(125)t$$

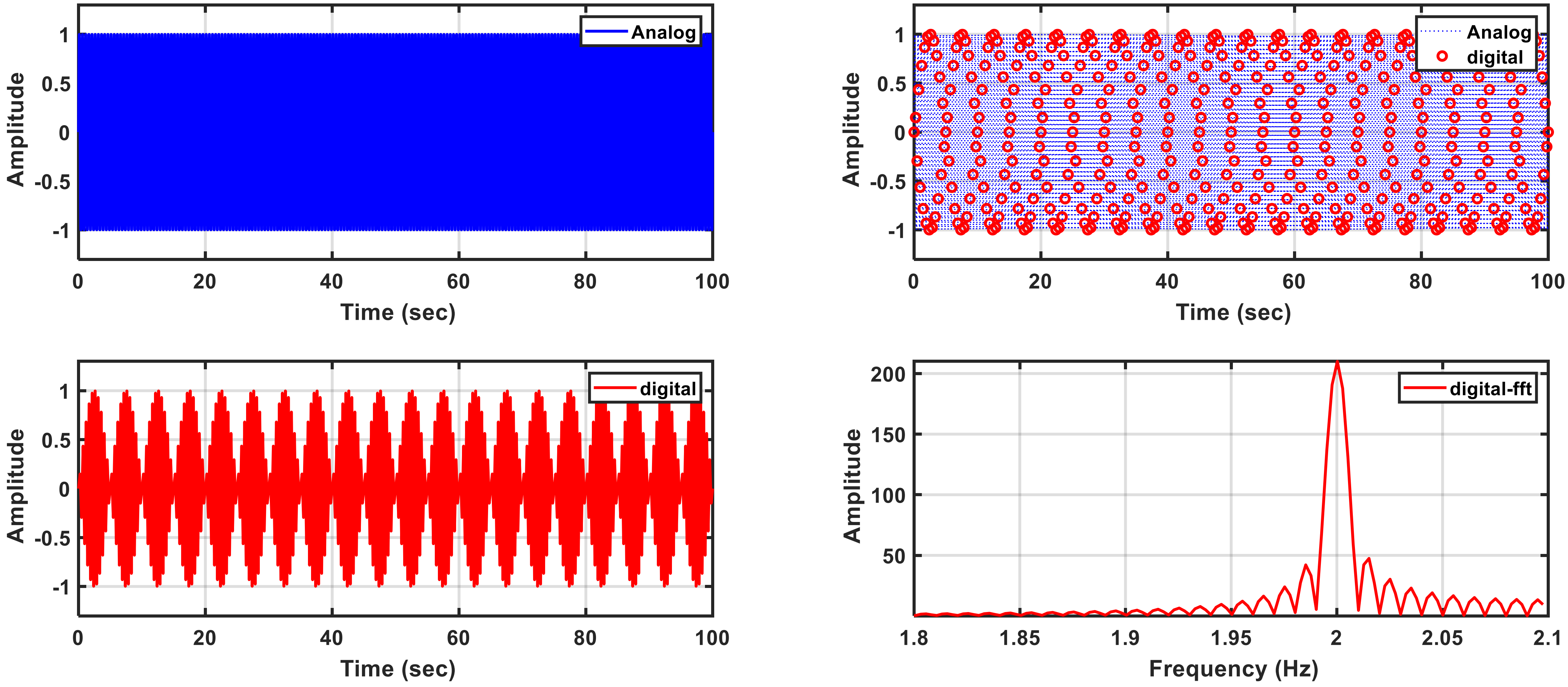
If the signal is sampled at 150 Hz, determine the frequency content of the resulting discrete response signal.

Problem 2. (c)



Problem 2. (d)





Signal frequency: 2Hz
Sampling frequency: 4.2 Hz

prob_2_d.m

Problem 4: Fourier Series 1 (15 points)

(a) Plot a **wave1** sampled with a 50 Hz sampling rate. The **wave1** is

$$y = sin(2\pi f_0 t) \cdot sin(2\pi f_0 t)$$

where $f_0 = 15$. Please connect sampled points and plot only ten cycles of the wave.

(b) Derive a Fourier series (general form) of analytic **wave1**. You should find analytic equations for coefficients of a_0, a_n, and b_n.

(c) Derive a Fourier series (complex form) of analytic **wave1**.You should find an analytic equations for a coefficient of c_n.

(d) Derive a Fourier series (general form) of analytic **wave2**:

$$y = sin(2\pi f_0 t) \cdot sin(2\pi f_0 t) + 5$$

You should find analytic equations for coefficients of a_0, a_n, and b_n.

(e) Please compare the results of (b) and (d) and explain their difference.

Problem 4

(b) $y(t) = \sin^2(2\pi f_0 t)$

$$= \frac{1}{2}\left(1 - \cos 4\pi f_0 t\right)$$

$$= \frac{1}{2}\left(1 - \cos \frac{2\pi \cdot 2t}{T_p}\right)$$

$T_p = 1/f_0$

$y(t) = y(t + nT_p)$

$= y(t + n/f_0)$

n : integer.

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

orthogonality!!

$a_0 = 1$, $b_n = 0$ for all n, $a_n = \begin{cases} -\frac{1}{2} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$

Product-to-sum[32]
$2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$
$2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$
$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$
$2\cos\theta\sin\varphi = \sin(\theta+\varphi) - \sin(\theta-\varphi)$
$\tan\theta\tan\varphi = \frac{\cos(\theta-\varphi) - \cos(\theta+\varphi)}{\cos(\theta-\varphi) + \cos(\theta+\varphi)}$
$\prod_{k=1}^{n} \cos\theta_k = \frac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1 + \cdots + e_n\theta_n)$ where $S = \{1, -1\}^n$

Problem 4. (b)

https://en.wikipedia.org/wiki/List_of_trigonometric_identities

$$y(t) = \frac{1}{2}(1 - \cos 4\pi f_0 t) \qquad \underline{T_p = 1/2f_0}$$

$$= \frac{1}{2} - \frac{1}{2}\cos\frac{2\pi t}{T_p}$$

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$$\therefore \quad a_0 = 1, \quad b_n = 0, \quad a_n = \begin{cases} -\frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 4. (b)

Complex Form of the Fourier Series

Euler Formula

$$e^{iwt} = coswt + i\ sinwt \qquad e^{-iwt} = coswt - i\ sinwt \qquad coswt = \frac{1}{2}(e^{iwt} + e^{-iwt}) \qquad sinwt = \frac{1}{2}(e^{iwt} - e^{-iwt})$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos wnt + b_n \sin wnt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}(e^{iwnt} + e^{-iwnt}) + \frac{b_n}{2j}(e^{iwnt} - e^{-iwnt}) \qquad w = \frac{2\pi}{T_p}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{iwnt} + \sum_{n=1}^{\infty} c_n^* e^{-iwnt} \quad where\ \boxed{c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - jb_n}{2}, \qquad c_n^* = \frac{a_n + jb_n}{2}}$$

$$c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad c_n^* = \frac{1}{T_p}\int_0^{T_p} x(t)e^{iwnt}dt = c_{-n}$$

Negative frequency term (c_{-n})

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi}{T_p}$$

$n \neq 0 \Rightarrow \quad C_n = -\frac{1}{2T_p}\int_{-T_p/2}^{T_p/2} \cos 4\pi f_0 t \; e^{-i 2\pi f_0 t} \, dt$

$$C_n = -\frac{1}{2T_p}\underbrace{\int_{-T_p/2}^{T_p/2} \cos 4\pi f_0 t \cos 2\pi f_0 n t}_{\text{all } 0 \text{ expet } n=2,-2} - i \, \underbrace{\cancel{\cos 4\pi f_0 t \sin 2\pi f_0 n t}}_{\to 0 \text{ orthogonality}} \, dt$$

$$C_{2,-2} = -\frac{1}{2T_p}\int_{-T_p/2}^{T_p/2} \frac{1}{2}\left(1 + \cos 8\pi f_0 t\right) dt$$

$$= -\frac{1}{4T_p}\int_{-T_p/2}^{T_p/2} \underbrace{\cancel{\cos 8\pi f_0 t \, dt}}_{0} = -\frac{1}{4}$$

$C_0 = \frac{1}{2}$, $\quad C_n = \begin{cases} -\frac{1}{4} & n = 2, -2 \\ 0 & \text{otherwise.} \end{cases}$

another aproch.

$C_n = \frac{a_n + b_n i}{2}. \qquad C_0 = \frac{a_0}{2} = \frac{1}{2}$

$C_2 = \frac{a_2}{2} = -\frac{1}{4} \qquad C_{-2} = -\frac{1}{4}$

orthogonality!!

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$a_0 = 1$, $\quad b_n = 0$ for all n, $\quad a_n = \begin{cases} -\frac{1}{2} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$

Problem 4. (c)

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$a_0 = 1$, $b_n = 0$ for all n, $a_n = \begin{cases} -\frac{1}{2} & n = 2 \\ 0 & \text{otherwise} \end{cases}$.

(d) $y(t) = \sin(2\pi f_0 t)\,\sin(2\pi f_0 t) + 5$

non-periodic term

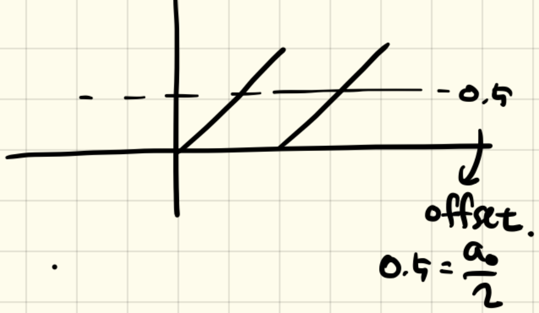
$b_n = 0$ for all n , $a_n = \begin{cases} -\frac{1}{2} & n=2 \\ 0 & \text{otherwise} \end{cases}$

non-periodic term $\Rightarrow \frac{1}{2} + 5 = \frac{a_0}{2} \quad \therefore a_0 = 11$

offset , DC term .

Problem 4. (d)

$T_p = 1 \qquad x(t) = t.$



(b) $a_0 = 1$

$$a_n = 2\int_0^1 t\cos 2\pi n t\,dt = 2\left[t\,\underbrace{\cancel{\frac{\sin 2\pi n t}{2\pi n}}}_{0}\Big|_0^1 - \frac{1}{2\pi n}\underbrace{\cancel{\int_0^1 \sin 2\pi n t\,dt}}_{0}\right]$$

$= 0$

$\sin 2\pi = \sin 0 = 0$ integration one period.

$$b_n = \frac{2}{T_p}\int_0^{T_p} t\sin 2\pi n t\,dt = 2\left[-\frac{t\cos 2\pi n t}{2\pi n}\Big|_0^1 - \frac{1}{2\pi n}\underbrace{\cancel{\int_0^1 \cos 2\pi n t\,dt}}_{0}\right]$$

$$= -\frac{1}{\pi n}$$

$a_0 = 1$, $a_n = 0$ for all n, $b_n = -\frac{1}{n\pi}$.

(c) $$C_n = \int_0^1 t\,e^{-i2\pi n t}\,dt$$

$$C_n = \frac{t\,e^{-i2\pi n t}}{-i2\pi n}\Big|_0^1 - \underbrace{\cancel{\int_0^1 \frac{1}{i2\pi n} e^{-i2\pi n t}\,dt}}_{0}$$

integration one period.

$$= \left(\frac{1}{-i2\pi n}\right) e^{-i2\pi n}$$

$$= \left(\frac{1}{-i2\pi n}\right)\left(\underbrace{\cos 2\pi n}_{1} - i\,\underbrace{\cancel{\sin 2\pi n}}_{0}\right)$$

for $n \neq 0$

$$= \frac{i}{2\pi n}$$

$C_0 = \frac{1}{2}$

$$\therefore \quad C_n = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{i}{2\pi n} & n \neq 0 \end{cases}$$

$$c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - jb_n}{2},$$

Problem 5

Problem 6: Fourier Transformation 1 (15 points)

Compute the Fourier transformation (integral) of the following functions and show the derivation process in detail:

(a) cosine wave

$$y = cos(2\pi p_0 t)$$

(b) cosine wave + dc (direct current) wave

$$y = cos(2\pi p_0 t) + d$$

(c) Gaussian function

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x)^2/2\sigma^2}$$

Problem 7: Fourier Transformation 2 (15 points)

$$y(t) = e^{-a|t|}(b \cdot cos2\pi f_1 t + c \cdot cos2\pi f_2 t)$$

(a) Compute the Fourier transformation (integral) of the above function

(b) Plot y in time domain and frequency domain, where a = 1, b = 2, c= 6, f_1 = 3, and f_2 = 6

(c) Plot y in time domain and frequency domain, where a = 0.5, b = 2, c= 6, f_1 = 3, and f_2 = 6