Signal Processing I

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Reference

We will cover some key topics in Chapters 3 ~ 6 of the following reference:

Shin, K., & Hammond, J. K. (2008). Fundamentals of Signal Processing: for Sound and Vibration Engineers, John Wiley & Sons.

Chapter 3: Fourier Series

Chapter 4: Fourier Integrals (Fourier Transform) and Continuous-Time Linear Systems

Chapter 5: Time Sampling and Aliasing

Chapter 6: The Discrete Fourier Transform

Fast Fourier Transform

A fast Fourier transform (FFT) is an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT). Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa. It manages to reduce the complexity of computing the DFT from $O(n^2)$, which arises if one simply applies the definition of DFT, to $O(n \, log \, n)$, where n is the data size.

FFT in Matlab

fft

Fast Fourier transform

Syntax

```
Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)
```

Y = fft(X) computes the discrete Fourier transform (DFT) of X using a fast Fourier transform (FFT) algorithm

Y = fft(X,n) returns the n-point DFT. If no value is specified, Y is the same size as X.

Periodic Signals and Fourier Series

Periodic signals can be analyzed using Fourier series. The basis of Fourier analysis of **a periodic signal** is the representation of such a signal by adding together sine and cosine functions of appropriate frequencies, amplitudes, and relative phases. For a single sine wave

$$x(t) = Xsin(wt + \emptyset) = Xsin(2\pi ft + \emptyset)$$

where X is amplitude,

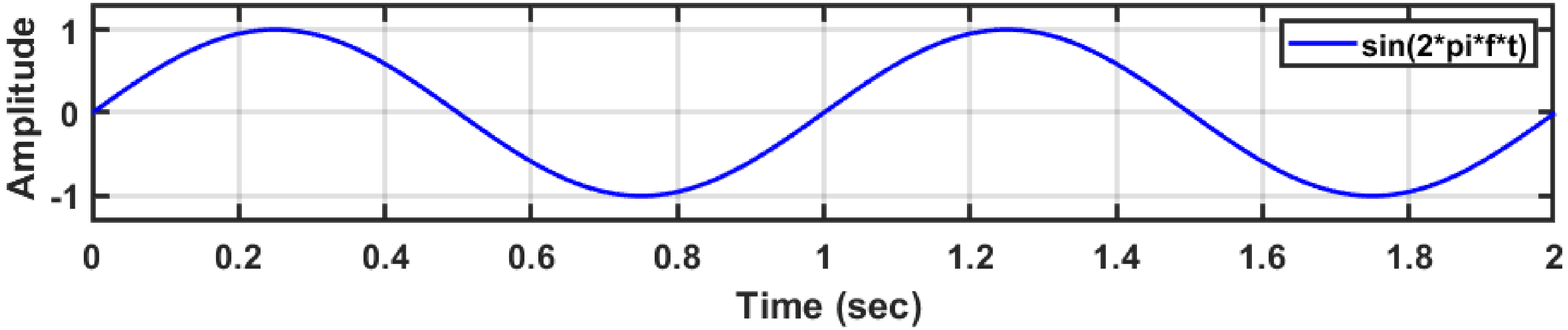
w is a circular (angular) frequency in radians per unit time (rad/s),

f is a (cyclical) frequency in cycles per unit time (Hz),

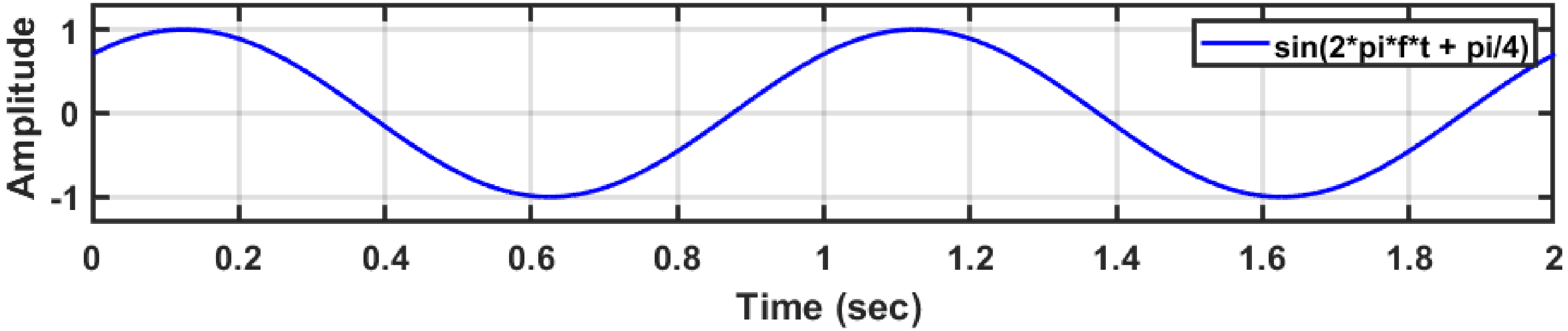
$\emptyset$ is phase angle with respect to the time origin in radians.

Example: Periodic Signals and Fourier Series

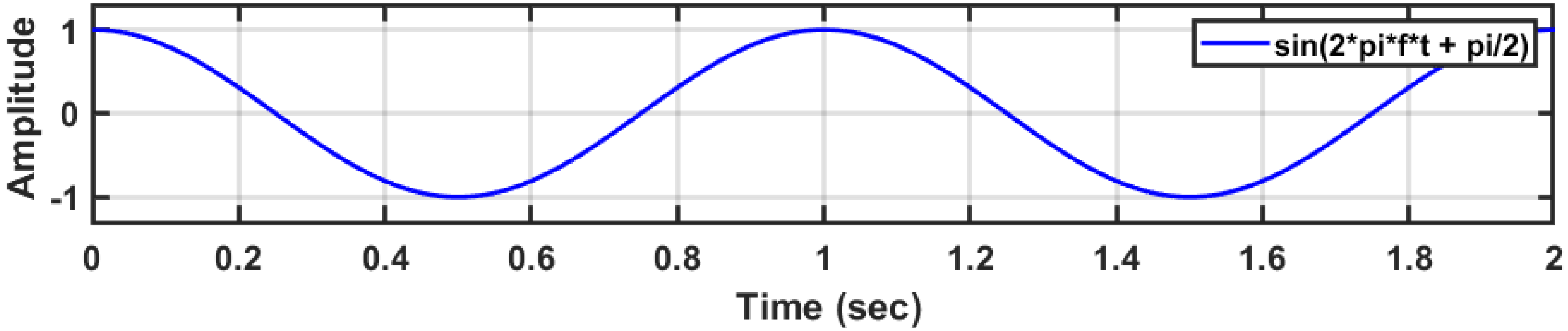
$sin(2\pi ft)$



$sin(2\pi ft + \pi/4)$



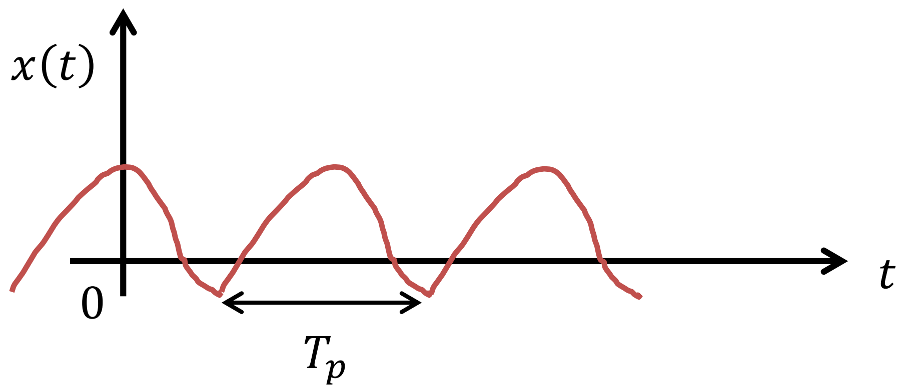
$sin(2\pi ft + \pi/2)$



Frequency (f): 1Hz

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. It decomposes any periodic function or periodic signal into the weighted sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines.



$$x(t) = x(t + nT_p)$$ Periodic

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

Appendix

Fourier Coefficients

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt$$

$$a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt$$

$$b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

Basic Trigonometric Equations

$$\int_{-\pi}^{\pi} \cos nt \ dt = 0 \qquad \int_{-\pi}^{\pi} \sin nt \ dt = 0$$

$$\cos mt \cos nt = \frac{1}{2}[\cos(m+n)t + \cos(m-n)t]$$

$$\sin mt \sin nt = \frac{1}{2}[\cos(m-n)t - \cos(m+n)t]$$

$$\sin mt \cos nt = \frac{1}{2}[\sin(m+n)t + \sin(m-n)t]$$

Orthogonality of trigonometric functions

$$\int_{-\pi}^{\pi} \cos mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\boxed{\int_{-\pi}^{\pi} \sin mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ 0 \ if \ n = m \end{cases}}$$

Appendix

Derivation of the Fourier Coefficients

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_0^{T_p} x(t)dt = \boxed{\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt} = \frac{a_0}{2} + \underbrace{\cancel{\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right) dt}}_{0}$$

$$a_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right) dt} = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right) \cos\left(\frac{2\pi mt}{T_p}\right) dt$$

$$= \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} \overbrace{\cancel{\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right) \cos\left(\frac{2\pi mt}{T_p}\right)}}^{0} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) \cos\left(\frac{2\pi mt}{T_p}\right) dt = \frac{2a_m}{T_p}\int_{-T_p/2}^{T_p/2} \cos\left(\frac{2\pi mt}{T_p}\right) \cos\left(\frac{2\pi mt}{T_p}\right) dt = a_m$$

$$b_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right) dt}$$

$$A = \frac{2\pi t}{T_p}$$

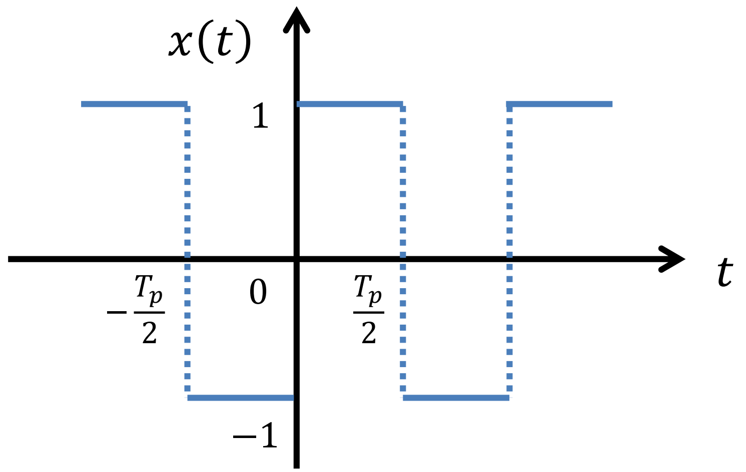
Example: Square Wave

$$x(t) = -1 \quad if \ -\frac{T_p}{2} < t < 0$$

$$x(t) = \quad 1 \quad if \ 0 < t < \frac{T_p}{2}$$

$$x(t + nT_p) = x(t)$$

$$where\, n = \pm 1, \pm 2, \ldots$$



$$\frac{a_0}{2} = \frac{1}{T_p}\int_0^{T_p} x(t)dt = 0$$

$$a_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi n t}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\cos\left(\frac{2\pi n t}{T_p}\right)dt + \int_0^{T_p/2} \cos\left(\frac{2\pi n t}{T_p}\right)dt\right] = 0$$

$$b_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi n t}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\sin\left(\frac{2\pi n t}{T_p}\right)dt + \int_0^{T_p/2} \sin\left(\frac{2\pi n t}{T_p}\right)dt\right] = \frac{2}{n\pi}(1 - cosn\pi)$$

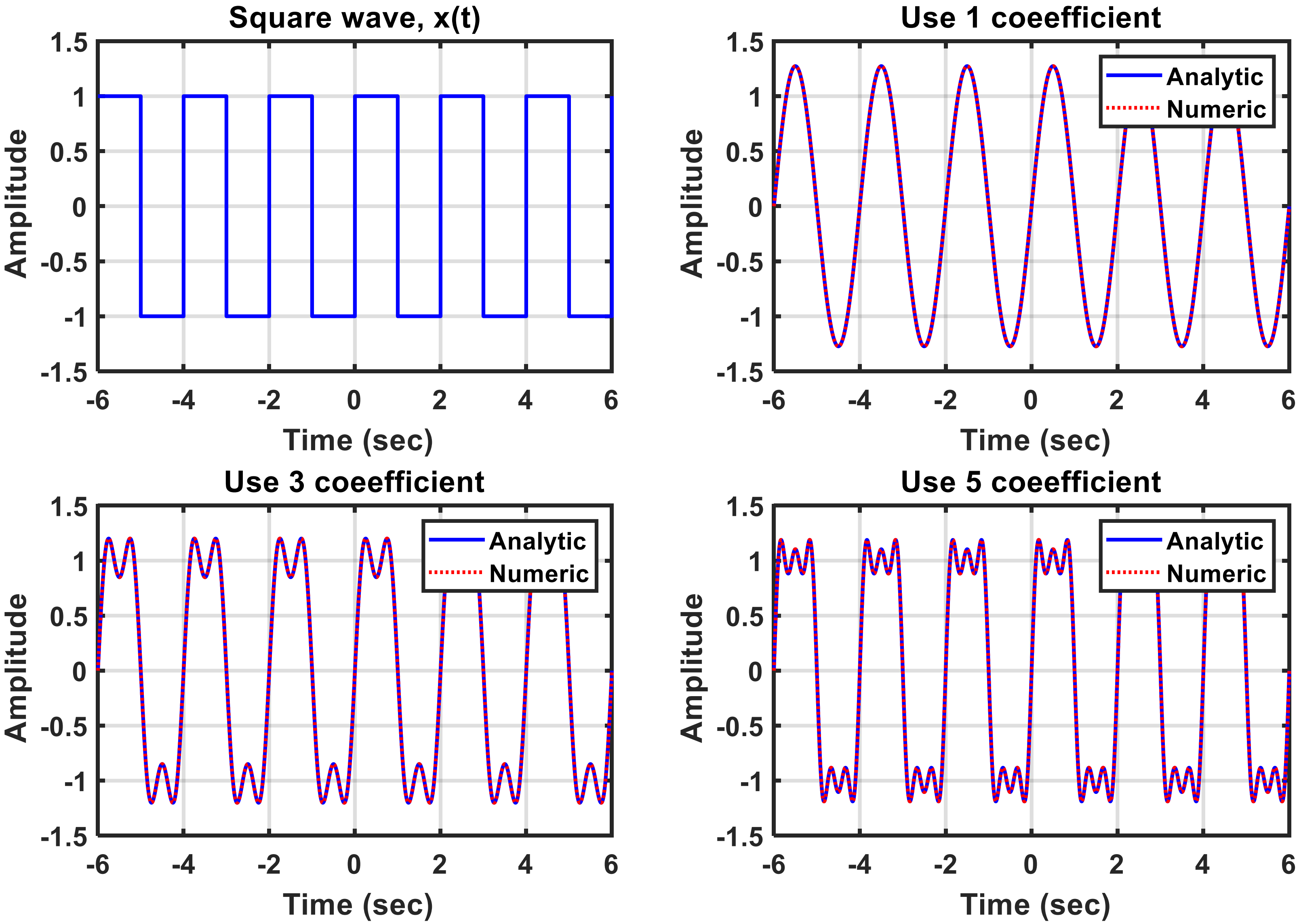
Example: Square Wave (Continue)

$x(t) = -1 \; if \; -1 < t < 0$

$x(t) = \;\; 1 \; if \; 0 < t < 1$

$x(t+2n) = x(t)$

$where \; n = \pm 1, \pm 2, \ldots$



Example: Square Wave – MATLAB Script (See Tutorial)

```
% the signal is assumed to be analog.
ncyle = 3;
Fsa = 1000; % # of samples per a second
Tp = 2;
t = -ncyle*Tp:1/Fsa:ncyle*Tp;
x = @(t) square(t*(2*pi)/Tp);

a0 = integral(x, -Tp/2, Tp/2);
nCoeff = 5;
a = zeros(nCoeff,1);
b = zeros(nCoeff,1);
for ii=1:nCoeff
    fun_a = @(t) x(t).*cos(2*pi*ii*t/Tp);
    a(ii) = integral(fun_a, -Tp/2, Tp/2);

    fun_b = @(t) x(t).*sin(2*pi*ii*t/Tp);
    b(ii) = integral(fun_b, -Tp/2, Tp/2);
end
```

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt \qquad a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi m t}{T_p}\right)dt \qquad b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi m t}{T_p}\right)dt$$

Example: Square Wave – MATLAB Script (Continue)

```
% numerical integration
sig_y_numeric = zeros(nCoeff, numel(t));
for ii=1:nCoeff
    if ii==1
        sig_y_numeric(ii,:) = a0/2 + a(ii)*cos(2*pi*ii*t/Tp) + b(ii)*sin(2*pi*ii*t/Tp);
    else
        sig_y_numeric(ii,:) = ...
            sig_y_numeric(ii-1,:) + a(ii)*cos(2*pi*ii*t/Tp) + b(ii)*sin(2*pi*ii*t/Tp);
    end
end
```

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$x(t) = \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T_p}\right) + b_1 \sin\left(\frac{2\pi t}{T_p}\right)$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{5} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

Example: Square Wave – MATLAB Script (Continue)

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right) \qquad a_n = 0 \qquad a_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt = 0 \qquad b_n = \frac{2}{n\pi}(1 - cosn\pi)$$

```
nCoeff = 5;
Tp = 2;
t = -ncyle*Tp:1/Fsa:ncyle*Tp;
sig_y_analytic = zeros(nCoeff, numel(t));
for ii=1:nCoeff
    sig_y_analytic(ii,:) = x_analytic(t, Tp, ii);
end
```

```
function x = x_analytic (t, Tp, n)
x = zeros(1, numel(t));
for ii=1:n
    x = x + 2/(ii*pi)*(1-cos(ii*pi))*sin(2*pi*ii*t/Tp);
end
end
```

Complex Form of the Fourier Series

Euler Formula

$$e^{iwt} = coswt + i\ sinwt \qquad e^{-iwt} = coswt - i\ sinwt \qquad coswt = \frac{1}{2}(e^{iwt} + e^{-iwt}) \qquad sinwt = \frac{1}{2}(e^{iwt} - e^{-iwt})$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos wnt + b_n \sin wnt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{iwnt} + e^{-iwnt}\right) + \frac{b_n}{2j}\left(e^{iwnt} - e^{-iwnt}\right) \qquad w = \frac{2\pi}{T_p}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{iwnt} + \sum_{n=1}^{\infty} c_n^* e^{-iwnt} \ \ where\ c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - jb_n}{2}, \qquad c_n^* = \frac{a_n + jb_n}{2}$$

$$c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad c_n^* = \frac{1}{T_p}\int_0^{T_p} x(t)e^{iwnt}dt = c_{-n}$$

Negative frequency term (c_{-n})

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi}{T_p}$$

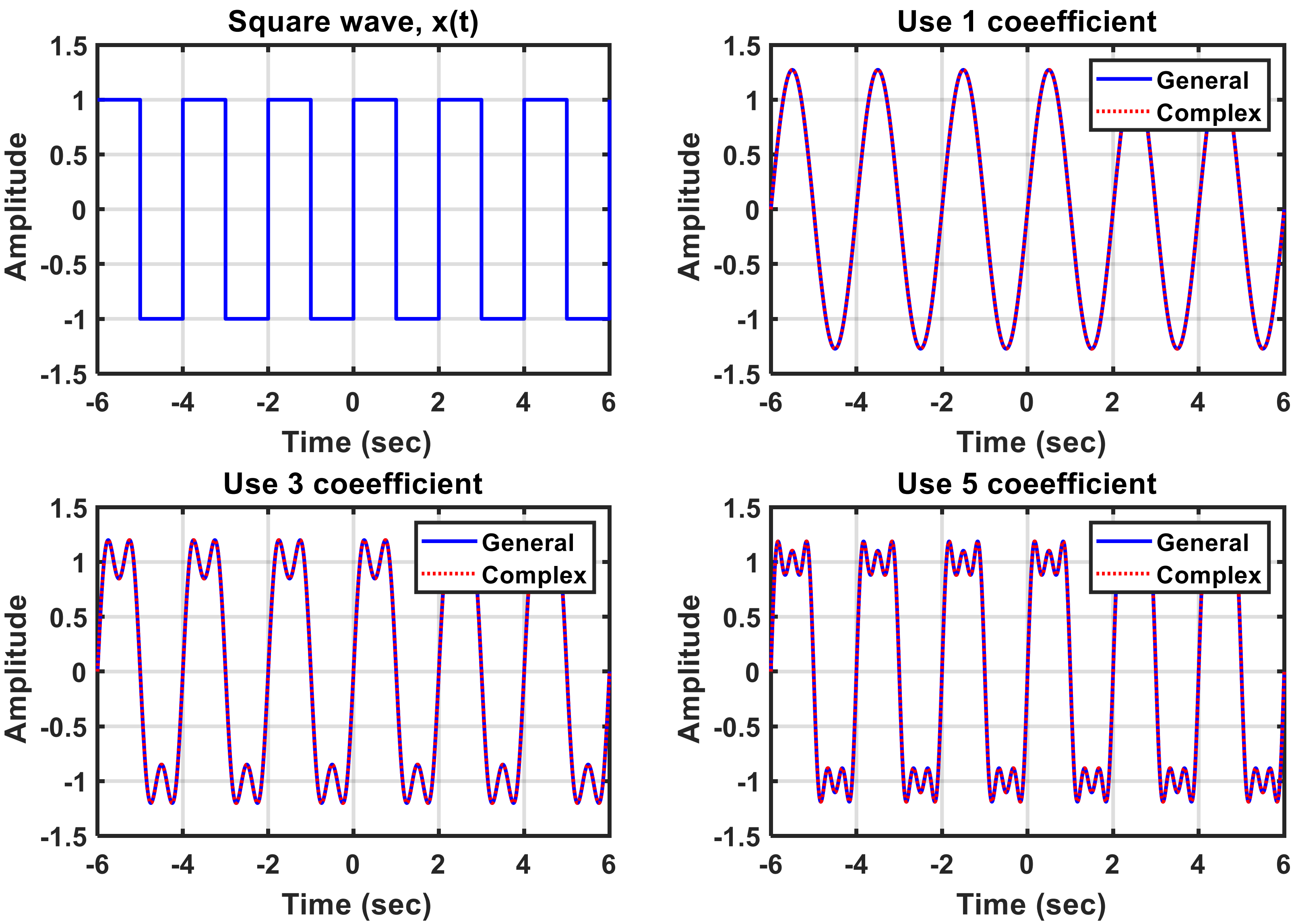
Example: Square Wave (Comparison of General and Complex Forms)

$$x(t) = -1 \; if \; -1 < t < 0$$

$$x(t) = \;\; 1 \; if \; 0 < t < 1$$

$$x(t + 2n) = x(t)$$

$$where \; n = \pm 1, \pm 2, \ldots$$



Summary

General form

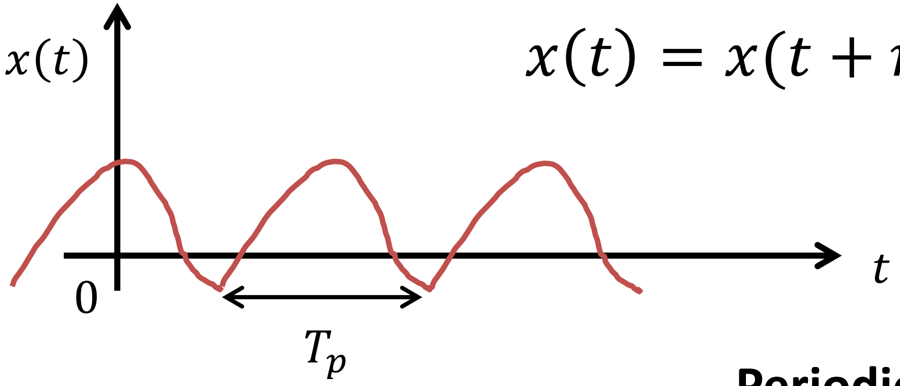
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt \qquad a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt \qquad b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

Complex form

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt}$$

$$c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi n}{T_p}$$



$$x(t) = x(t + nT_p)$$

Periodic signal