B

Dynamics of Discrete Systems

B.1 INTRODUCTION

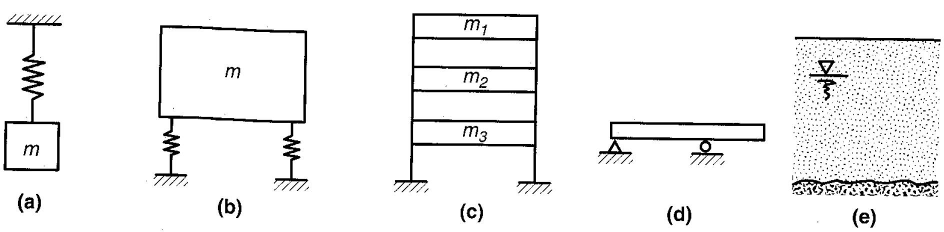
Many vibrating systems consist of discrete elements such as masses and springs, or can at least be idealized as such. For most practical problems of structural dynamics, the structure is idealized as a system of rigid masses connected by massless springs. Even continuous systems such as soil deposits have been idealized as assemblages of many discrete elements, though that approach is seldom taken any more. Since the geotechnical earthquake engineer often provides input to the structural engineer, a firm understanding of the dynamic response of discrete systems is required. Also, many of the concepts and terminologies used in geotechnical earthquake engineering analyses are analogous to those of discrete system dynamics and are more easily introduced in that framework.

This appendix introduces the dynamics of discrete systems. It begins with very simple systems, and adds complicating factors such as damping, base motion, and nonlinearity. Analytical and numerical solutions in the time domain and frequency domain are presented. Finally, the response of multiple-degree-of-freedom systems is introduced. While many of the basic concepts of structural dynamics are presented, much more complete treatments may be found in a number of structural dynamics texts (e.g., Clough and Penzien, 1975; Paz, 1980; Berg, 1989; Chopra, 1995).

B.2 VIBRATING SYSTEMS

Vibrating systems can be divided into two broad categories: *rigid systems* and *compliant systems.* A rigid system is one in which no strains occur. All points within a rigid system move in phase with each other, and the description of rigid-body motion is a relatively simple matter of kinematics. In compliant systems, however, different points within the system may move differently (and out of phase) from each other. A given physical system may behave very nearly as a rigid system under certain conditions and as a compliant system under other conditions. Since neither soils nor structures are rigid, the dynamic response of compliant systems is central to the study of soil and structural dynamics and to earthquake engineering.

Compliant systems can be characterized by the distribution of their mass. *Discrete systems* are those whose mass can be considered to be concentrated at a finite number of locations, where the mass of a *continuous system* is distributed throughout the system. The number of independent variables required to describe the position of all the significant masses of a system is the number of *dynamic degrees of freedom* of the system. Systems of interest in earthquake engineering may have anywhere from 1 to an infinite number of degrees of freedom. Figure B.1 illustrates several commonly encountered systems with varying numbers of degrees of freedom (DOF). Discrete systems have a finite number of degrees of freedom; the number of degrees of freedom of a continuous system is infinite. Certain types of analyses idealize continuous systems as discrete systems with large numbers of degrees of freedom, and other types represent discrete systems with many degrees of freedom as continuous systems.



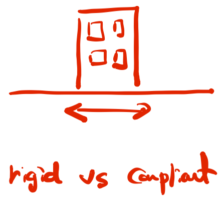


Figure B.1 Vibrating systems with various numbers of degrees of freedom: (a) one DOF, vertical translation; (b) two DOF, vertical translation and rocking; (c) three DOF, horizontal translation; (d) infinite DOF; (e) infinite DOF.

B.3 SINGLE-DEGREE-OF-FREEDOM SYSTEMS

A discrete system whose position can be described completely by a single variable is known as a *single-degree-of-freedom* (SDOF) system. That single degree of freedom may represent translational displacement, as in the SDOF systems of Figure B.2a–c, or rotational displacement, as in the case of the pendulum of Figure B.2d.

A typical SDOF system is one in which a rigid *mass*, m, is connected in parallel to a *spring* of stiffness, k, and a *dashpot* of viscous damping coefficient, c, and subjected to some external load, $Q(t)$, as shown in Figure B.3. The spring and dashpot are assumed to be massless and the displacement origin to coincide with the static equilibrium position.

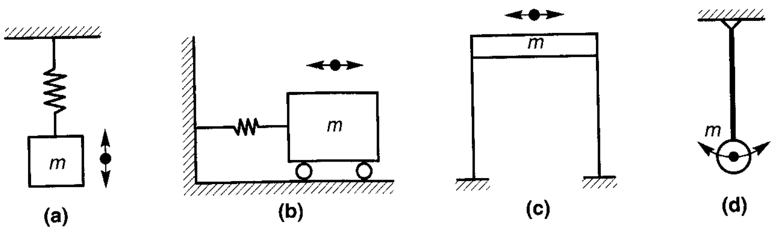


Figure B.2 Various SDOF systems. The degrees of freedom are (a) vertical translation, (b) and (c) horizontal translation, and (d) rotation.

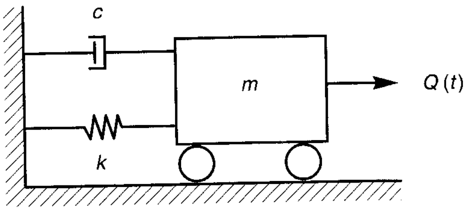


Figure B.3 Damped SDOF system subjected to external dynamic load, $Q(t)$.

B.4 EQUATION OF MOTION FOR SDOF SYSTEM

Many SDOF systems are acted upon by externally applied loads. In earthquake engineering, dynamic loading often results from another source—movement of the supports of the system. The dynamic response of a SDOF system such as that shown in Figure B.3 is governed by an *equation of motion*. The equation of motion can be derived in a number of ways; a simple, force equilibrium approach will be used here.

B.4.1 Equation of Motion: External Loading

When a dynamic load is applied to the mass of a SDOF system (Figure B.3), the tendency for motion is resisted by the inertia of the mass and by forces that develop in the dashpot and spring. Thus the external load, $Q(t)$, acting in the positive x-direction is opposed by three forces (Figure B.4) that act in the negative x-direction: the *inertial force*, f_I, the *viscous damping force*, f_D, and the *elastic spring force*, f_S. The equation of motion can be expressed in terms of the dynamic equilibrium of these forces:

$$f_I(t) + f_D(t) + f_S(t) = Q(t) \tag{B.1}$$

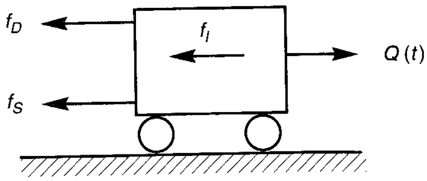


Figure B.4 Dynamic forces acting on mass from Figure B.3.

These forces can also be expressed in terms of the motion of the mass. Newton's second law states that the inertial force acting on a mass is equal to its rate of change of momentum, which for a system of constant mass produces

$$f_I(t) = \frac{d}{dt}\left(m\frac{du(t)}{dt}\right) = m\frac{d^2u(t)}{dt^2} = m\ddot{u}(t) \tag{B.2a}$$

For a viscous dashpot, the damping force is proportional to the velocity of the mass:

$$f_D(t) = c\frac{du(t)}{dt} = c\dot{u}(t) \tag{B.2b}$$

and the force provided by the spring is simply the product of its stiffness and the amount by which it is displaced

$$f_S(t) = ku(t) \tag{B.2c}$$

The behavior of these forces is illustrated graphically in Figure B.5. The inertial force is proportional to the acceleration and the constant of proportionality is the mass. Similarly, the viscous damping force and the elastic spring force are proportional to the velocity and displacement with the damping and spring coefficients serving as the respective constants of proportionality.

Substituting equations (B.2) into equation (B.1), the equation of motion for the SDOF system can be written as

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = Q(t) \tag{B.3}$$

This second-order differential equation is commonly used to describe the behavior of oscillating systems ranging from the mechanical systems considered in earthquake engineering problems to electrical circuits. The differential equation of motion is linear (i.e., all of its terms have constant coefficients). This linearity allows a closed-form analytical solution to be readily obtained and, importantly, it allows the principle of superposition to be used. When any of the coefficients are not constant, the behavior is not linear and the solution becomes considerably more difficult. In most cases, the response of nonlinear systems must be evaluated numerically (Section B.7).

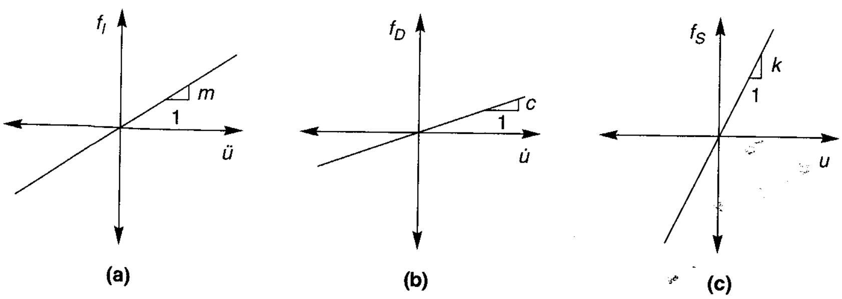


Figure B.5 Variation of (a) inertial, (b) viscous, and (c) elastic forces with acceleration, velocity, and displacement, respectively.

B.4.2 Equation of Motion: Vibration of Supports (Base Shaking)

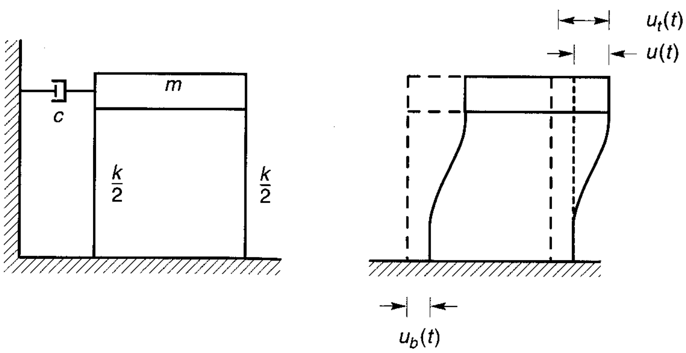
For earthquake engineering problems, dynamic loading often results from vibration of the supports of a system rather than from dynamic external loads. To evaluate the response of such systems, it is necessary to develop an equation of motion for loading caused by base shaking. Consider the damped SDOF system shown in Figure B.6a. When subjected to dynamic base shaking, $u_b(t)$, it will deform into a configuration that might look like that shown in Figure B.6b at a particular time, t. The total displacement of the mass, $u_t(t)$, can be broken down as the sum of the base displacement, $u_b(t)$, and the displacement of the mass relative to the base, $u(t)$. The inertial force will depend on the total acceleration of the mass, while the viscous damping and elastic spring forces will depend on the relative velocity and displacement, respectively. Using the notation shown in Figure B.6b, the equation of motion can be written as

$$m\ddot{u}_t + c\dot{u} + ku = 0$$

or substituting $\ddot{u}_t(t) = \ddot{u}_b(t) + \ddot{u}(t)$ and rearranging,

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_b \tag{B.4}$$

In other words, the response of the system to base shaking is equivalent to the response that the system would have if its base was fixed and the mass was subjected to an external load $Q(t) = -m\ddot{u}_b(t)$. Thus any solutions for the response of an SDOF system subjected to external load can be used to evaluate the response of the system to base shaking.



(a) (b)

Figure B.6 Damped SDOF system subjected to base shaking.

B.5 RESPONSE OF LINEAR SDOF SYSTEMS

In order to evaluate the dynamic response of a linear SDOF system, the differential equation of motion must be solved. There are several types of conditions under which the dynamic response of SDOF systems are commonly calculated. *Forced vibration* occurs when the mass is subjected to some external loading, $Q(t)$. The loading may be periodic or nonperiodic and it may correspond to an actual physical force applied to the mass or to some known level of base shaking. *Free vibration* occurs in the absence of external loading or base shaking. It may result from the release of the mass from some initial displacement or may occur after some transient forced vibration has ended. The following sections will develop solutions to the equation of motion for cases in which damping is and is not present, and for cases in which external loading is and is not present. The resulting four permutations of these conditions are

1. Undamped free vibrations: $c = 0, \quad Q(t) = 0$
2. Damped free vibrations: $c > 0, \quad Q(t) = 0$
3. Undamped forced vibrations: $c = 0, \quad Q(t) \neq 0$
4. Damped forced vibrations: $c > 0, \quad Q(t) \neq 0$

The solution of the equation of motion for each of these conditions will be presented in turn.

B.5.1 Undamped Free Vibrations

A SDOF system undergoes free vibration when it oscillates without being acted upon by any external loads. When damping is not present ($c = 0$) the equation of motion (for undamped free vibration) reduces to

$$m\ddot{u} + ku = 0 \tag{B.5}$$

or after dividing both sides by the mass,

$$\ddot{u} + \frac{k}{m}u = 0 \tag{B.6}$$

The solution to this simple differential equation can be found in any elementary text on differential equations as

$$u = C_1 \sin\sqrt{\frac{k}{m}}t + C_2 \cos\sqrt{\frac{k}{m}}t \tag{B.7}$$

where the values of the constants C_1 and C_2 depend on the initial conditions of the system. The quantity $\sqrt{k/m}$ is very important—it represents the *undamped natural circular frequency* of the system

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{B.8}$$

Then the *natural frequency*, f_0, and *natural period of vibration*, T_0, can be written as

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{B.9}$$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \tag{B.10}$$

Substituting equation (B.8) into the solution for the equation of motion [equation (B.7)] yields

$$u = C_1 \sin \omega_0 t + C_2 \cos \omega_0 t \tag{B.11}$$

which indicates that an undamped system in free vibration will oscillate harmonically at its undamped natural frequency. C_1 and C_2, can be evaluated by assuming the initial ($t = 0$) conditions to be represented by an initial displacement, u_0, and initial velocity, $\dot{u}_0$. Then

$$u_0 = C_1 \sin(0) + C_2 \cos(0) = C_2$$

$$\dot{u}_0 = \omega_0 C_1 \cos(0) - \omega_0 C_2 \sin(0) = \omega_0 C_1$$

Therefore, $C_1 = \dot{u}_0/\omega_0$ and $C_2 = u_0$, so the complete solution to the undamped free vibration response of an SDOF system is given by

$$u = \frac{\dot{u}_0}{\omega_0} \sin \omega_0 t + u_0 \cos \omega_0 t \tag{B.12}$$

The response of such a system is shown in Figure B.7.

Referring back to equation (A.5), the free vibration response can also be expressed as

$$u = A \sin(\omega_0 t + \phi) \tag{B.13}$$

where the amplitude, A, and phase angle, ϕ, are given by

$$A = \sqrt{u_0^2 + \left(\frac{\dot{u}_0}{\omega_0}\right)^2}$$

$$\phi = \tan^{-1} \frac{u_0 \omega_0}{\dot{u}_0}$$

The solution to the equation of motion of an undamped system indicates that the response of the system depends on its initial displacement and velocity. Note that the amplitude remains constant with time. Because no energy is lost in an undamped system, it will continue to oscillate forever. Obviously, truly undamped systems do not exist in the real world; however, some systems can have such low damping that their response over short periods of time may approximate that of an undamped system.

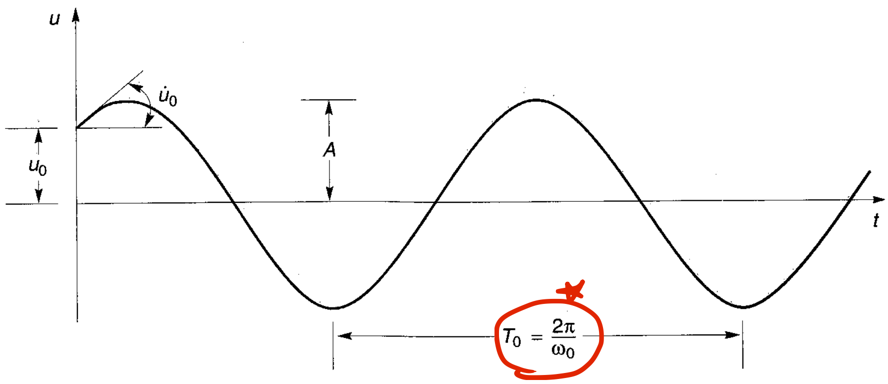


Figure B.7 Time history of displacement for undamped free vibration with initial displacement u_0 and initial velocity $\dot{u}_0$.

Example B.1

The SDOF structure shown in Figure EB.1a consists of a 10-kip weight supported by a massless column. Application of a 5-kip static horizontal force to the weight produces a horizontal deflection of 0.04 in. Compute (a) the natural circular frequency, (b) the natural period of vibration, and (c) the time history of response if the horizontal force was suddenly removed.

Figure EB.1a

Solution (a) The problem statement indicates that the stiffness of the column is

$$k = \frac{5 \text{ kips}}{0.04 \text{ in.}} = 125 \text{ kips/in.}$$

The natural circular frequency is given by

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{(125 \text{ kips/in.})(12 \text{ in./ft})(32.2 \text{ ft/sec}^2)}{10 \text{ kips}}} = 69.5 \text{ rad/sec}$$

(b) The natural period would be

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi \text{ rad}}{69.5 \text{ rad/sec}} = 0.09 \text{ sec}$$

(c) The horizontal force produced a static deflection of 0.04 in. Consequently, the initial conditions for free vibration would be

$$u_0 = 0.04 \text{ in.} \qquad \dot{u}_0 = 0$$

Then

$$u(t) = \frac{\dot{u}_0}{\omega_0}\sin \omega_0 t + u_0 \cos\omega_0 t = (0.04 \text{ in.}) \cos(69.5t)$$

The response is plotted in Figure EB.1b.

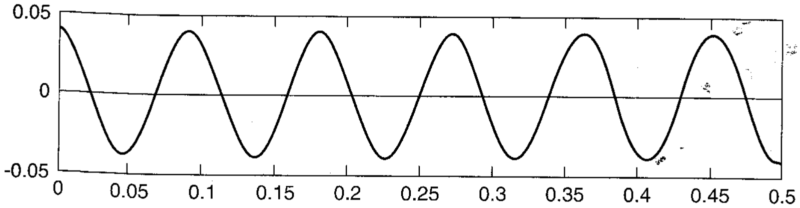


Figure EB.1b

B.5.2 Damped Free Vibrations

$c > 0$, $Q(t) = 0$

In real systems, energy may be lost as a result of friction, heat generation, air resistance, or other physical mechanisms. Hence the free vibration response of a damped SDOF system will diminish with time. For damped free vibrations, the equation of motion is written as

$$m\ddot{u} + c\dot{u} + ku = 0 \qquad \text{(B.14)}$$

or, dividing by m and substituting [from equation (B.17)] $k = m\omega_0^2$, we have

$$\ddot{u} + 2\frac{c}{2\sqrt{km}}\omega_0 u + \omega_0^2 u = 0 \qquad \text{(B.15)}$$

The quantity $2\sqrt{km}$, called the *critical damping coefficient*, c_c, allows the *damping ratio*, ξ, to be defined as the ratio of the damping coefficient to the critical damping coefficient, that is,

$c_c = 2\sqrt{km}$

$$\xi = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_0} = \frac{c\omega_0}{2k} \qquad \text{(B.16)}$$

$\xi = \frac{c}{c_c}$ damping ratio.

With this notation, the equation of motion can be expressed as

$$\ddot{u} + 2\xi\omega_0\dot{u} + \omega_0^2 u = 0 \qquad \text{(B.17)}$$

The solution of this differential equation of motion depends on the value of the damping ratio. When $\xi < 100\%$ ($c < c_c$), the system is said to be *underdamped*. When $\xi = 100\%$ ($c = c_c$) the system is *critically damped*, and when $\xi > 100\%$ ($c > c_c$) the system is *overdamped*. Separate solutions must be obtained for each of the three cases, but structures of interest in earthquake engineering are virtually always underdamped.

For the case in which damping is less than critical, the solution to the equation of motion is of the form

$$u = e^{-\xi\omega_0 t}\left[C_1 \sin(\omega_0\sqrt{1-\xi^2}\,t) + C_2 \cos(\omega_0\sqrt{1-\xi^2}\,t)\right] \qquad \text{(B.18)}$$

Note the exponential term by which the term in brackets is multiplied. This exponential term gets smaller with time and eventually approaches zero, indicating that the response of an underdamped system in free vibration decays exponentially with time. The rate of decay depends on the damping ratio—for small ξ the response decays slowly and for larger ξ the response decays more quickly. Defining the *damped natural circular frequency* of the system as $\omega_d = \omega_0\sqrt{1-\xi^2}$ the solution can be expressed as

$$u = e^{-\xi\omega_0 t}(C_1 \sin\omega_d t + C_2 \cos\omega_d t) \qquad \text{(B.19)}$$

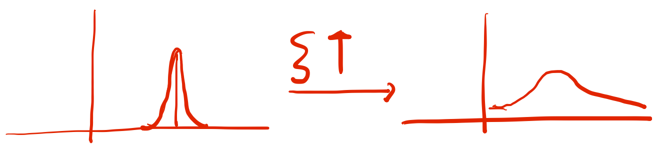
The natural frequency of a damped system is always lower than that of an undamped system, and it decreases with increasing damping ratio.

The coefficients C_1 and C_2 can be determined from the initial conditions in the same manner as for the undamped case. The initial displacement and velocity are

$$u_0 = e^{-\xi\omega_0(0)}[C_1\sin(0) + C_2\cos(0)] = C_2$$

$$\dot{u}_0 = e^{-\xi\omega_0(0)}[\omega_d C_1\cos\omega_d(0) - \omega_d C_2\sin\omega_d(0)] - \xi\omega_0 e^{-\xi\omega_0(0)}[C_1\sin\omega_d(0) + C_2\cos\omega_d(0)]$$
$$= \omega_d C_1 - \xi\omega_0 C_2$$

Initial Condition



Therefore, $C_1 = (\dot{u}_0 + \xi\omega_0 u_0)/\omega_d$ and $C_2 = u_0$, so the solution for damped free vibrations can be expressed as

$$u = e^{-\xi\omega_0 t}\left(\frac{\dot{u}_0 + \xi\omega_0 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right) \tag{B.20}$$

The free vibration response of an underdamped system is shown in Figure B.8. Note the exponential decay of displacement amplitude with time. The ratio of the amplitudes of any two successive peaks will be

$t_n = \frac{2\pi n}{w_d}$, $t_{n+1} = \frac{2\pi(n+1)}{w_d}$

$$\frac{u_n}{u_n+1} = \exp\left(2\pi\xi\frac{\omega_0}{\omega_d}\right) \tag{B.21}$$

Defining the *logarithmic decrement* as $\delta = \ln(u_n/u_{n+1})$; then

$\frac{U_n}{U_{n+1}} = \exp(-\xi w_0 t_n)/\exp(-\xi w_0 t_{n+1})$

$$\delta = 2\pi\xi\frac{\omega_0}{\omega_d} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{B.22}$$

$= \exp(-\xi w_0 (t_n - t_{n+1}))$

Rearranging allows the damping ratio to be determined from the logarithmic decrement

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{B.23}$$

δ ← measurement

find ξ

For small values of δ, $\xi \approx \delta/2\pi$. Therefore, a simple way to estimate the damping ratio of an SDOF system is to perform a *free vibration test*, in which the logarithmic decrement is measured when a system is displaced by some initial displacement, u_0, and released with initial velocity $\dot{u}_0 = 0$.

$\xi = \frac{\delta}{2\pi}$

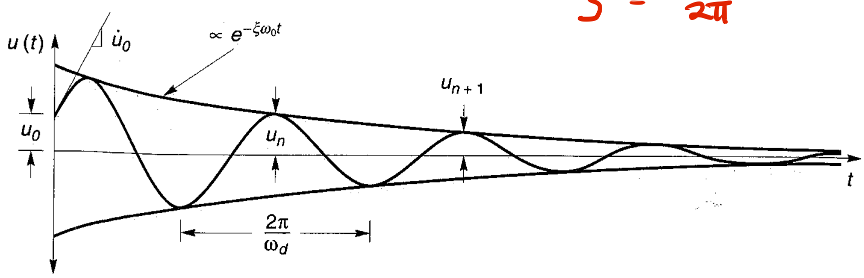


Figure B.8 Time history of damped free vibration with initial displacement u_0 and initial velocity $\dot{u}$.

Example B.2

The structure shown in Figure EB.2a is released from an initial displacement of 1 cm with an initial velocity of −5 cm/sec. Compute (a) the damped natural frequency and (b) the time history of response of the mass.

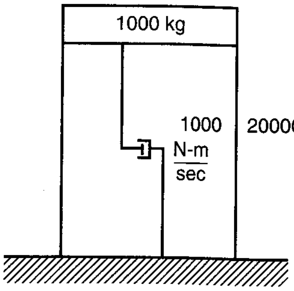


Figure EB2.a

Solution (a) The undamped natural frequency is

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20000 \text{ N/m}}{1000 \text{ kg}}} = 0.71 \text{ Hz}$$

and the damping ratio is

$$\xi = \frac{c}{2\sqrt{km}} = \frac{1000 \text{ N-m/sec}}{2\sqrt{(20000 \text{ N/m})(1000 \text{ kg})}} = 0.118$$

Then

$$f_d = f_0\sqrt{1-\xi^2} = (0.71 \text{ Hz})\sqrt{1-(0.118)^2} = 0.70 \text{ Hz}$$

(b) The undamped and damped natural circular frequencies will be $\omega_0 = 2\pi f_0 = 4.47$ rad/sec and $\omega_d = f_d/2\pi f_d = 4.44$ rad/sec, respectively. From equation (B.20), the displacement response is

$$u = e^{-\xi\omega_0 t}\left(\frac{\dot{u}_0 + \xi\omega_0 u_0}{\omega_d}\sin\omega_d t + u_0 \cos\omega_d t\right)$$

$$= \exp[-(0.118)(4.47)t]\left[\frac{-0.05 + (0.118)(4.47)(0.01)}{4.44}\sin(4.44t) + (1)\cos(4.44t)\right]$$

$$= e^{-0.527t}[\cos(4.44t) - 0.010\sin(4.44t)]$$

which is plotted in Figure EB.2b.

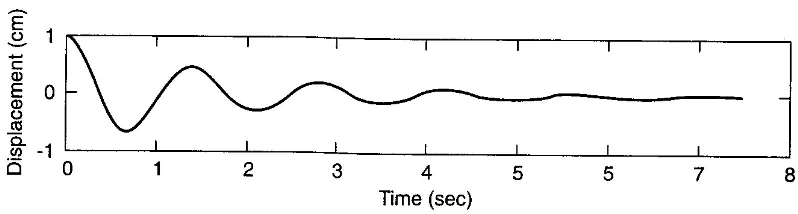


Figure EB.2b

B.5.3 Response of SDOF Systems to Harmonic Loading

A SDOF system is said to undergo forced vibration when acted upon by some external dynamic force, $Q(t)$. Dynamic loading may come from many different sources and may be

harmonic

periodic or nonperiodic. For problems of soil and structural dynamics, the response to harmonic loading is very important. One form of simple harmonic loading $Q(t)$ can be expressed as $Q(t) = Q_0 \sin \overline{\omega} t$, where Q_0 is the amplitude of the harmonic load and $\overline{\omega}$ is the circular frequency at which the load is applied.

B.5.3.1 Undamped Forced Vibrations

The equation of motion for an undamped system subjected to such simple harmonic loading is

$$m\ddot{u} + ku = Q_0 \sin \overline{\omega} t \tag{B.24}$$

The general solution to this equation of motion is given by the sum of the *complementary solution* (for the homogeneous case in which the right side of the equation is zero) and the *particular solution* [which must satisfy the right side of equation (B.24)].

The homogeneous equation is

$$m\ddot{u} + ku = 0$$

so the complementary solution is simply the solution to the undamped free vibration problem

$$u_c(t) = C_1 \sin \omega_0 t + C_2 \cos \omega_0 t \tag{B.25}$$

The portion of the response described by the complementary solution is that which results from the initial conditions of the system. It consists of a simple harmonic oscillation at the undamped natural frequency of the system.

The particular solution describes the portion of the response caused by the external loading. This portion of the response can be assumed to be of the same form and to be in phase with the harmonic loading; thus

$$u_p(t) = U_0 \sin \overline{\omega} t \tag{B.26}$$

where U_0 is the amplitude of the harmonic response. Substituting equation (B.26) into equation (B.24) yields

$$-m\overline{\omega}^2 U_0 \sin \overline{\omega} t + kU_0 \sin \overline{\omega} t = Q_0 \sin \overline{\omega} t \tag{B.27}$$

Substituting $k/m = \omega_0^2$ and rearranging gives

$$U_0 = \frac{Q_0/k}{1 - \overline{\omega}^2/\omega_0^2} = \frac{Q_0/k}{1 - \beta^2} \tag{B.28}$$

where $\beta = \overline{\omega}/\omega_0$ is referred to as the *tuning ratio*. Now the general solution of the equation of motion can be obtained by combining the complementary and particular solutions:

$$u(t) = u_c(t) + u_p(t) = C_1 \sin \omega_0 t + C_2 \cos \omega_0 t + \frac{Q_0/k}{1 - \beta^2} \sin \overline{\omega} t \tag{B.29}$$

The general solution must satisfy the initial conditions. From equation (B.29), the velocity can be written as

$$\dot{u}(t) = \frac{du}{dt} = \omega_0 C_1 \cos \omega_0 t - \omega_0 C_2 \sin \omega_0 t + \overline{\omega} \frac{Q_0/k}{1 - \beta^2} \cos \overline{\omega} t \tag{B.30}$$

$\beta = \frac{\overline{\omega}}{\omega_0}$ ← tuning ratio. $\omega_0 = \sqrt{\frac{k}{m}}$

For a given initial displacement, u_0, and initial velocity, $\dot{u}_0$,

$$u_0 = C_1 \sin \omega_0(0) + C_2 \cos \omega_0(0) + \frac{Q_0/k}{1-\beta^2} \sin \bar{\omega}(0) = C_2 \tag{B.31}$$

and

$$\dot{u}_0 = \omega_0 C_1 \cos \omega_0(0) - \omega_0 C_2 \sin \omega_0(0) + \bar{\omega}\frac{Q_0/k}{1-\beta^2} \cos \bar{\omega}(0) = \omega_0 C_1 + \bar{\omega}\frac{Q_0/k}{1-\beta^2} \tag{B.32}$$

from which

$$C_1 = \frac{\dot{u}_0 - \bar{\omega}\,[(Q_0/k)/(1-\beta^2)]}{\omega_0} = \frac{\dot{u}_0}{\omega_0} - \frac{Q_0\beta}{k\,(1-\beta^2)} \tag{B.33}$$

Now the general response can finally be written as

$$\boxed{u = \left[\frac{\dot{u}_0}{\omega_0} - \frac{Q_0\beta}{k(1-\beta^2)}\right]\sin \omega_0 t + u_0 \cos \omega_0 t + \frac{Q_0/k}{1-\beta^2} \sin \bar{\omega} t} \tag{B.34}$$

It is interesting to consider the case in which the system is initially at rest in its equilibrium position, (i.e., $u_0 = \dot{u}_0 = 0$). For this case the response is given by

$\beta = \dfrac{\bar{\omega}}{\omega_0}$

$$u = \frac{Q_0}{k}\frac{1}{1-\beta^2}(\sin \bar{\omega} t - \beta \sin \omega_0 t) \tag{B.35}$$

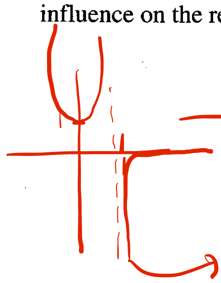
free vibration

forced vibration

which indicates that the response has two components. One component occurs in response to the applied loading and occurs at the frequency of the applied loading. The other is a free vibration effect induced by the initial conditions; it occurs at the natural frequency of the system. It is useful to realize that the term Q_0/k in equation (B.35) represents the displacement of the mass that would occur if the load Q_0 was applied statically. The term $1/(1-\beta^2)$ can then be thought of as a magnification factor that describes the amount by which the static displacement amplitude is magnified by the harmonic load. The magnification factor varies with the tuning ratio, β, as shown in Figure B.9. Note that the displacement amplitude is greater than the static displacement for loading frequencies lower than $\sqrt{2}\,\omega_0$. At higher loading frequencies, the displacement amplitude is less than the static displacement and can become very small at high frequencies. However, the response of an undamped SDOF system becomes very large as $\bar{\omega}$ approaches ω_0. When harmonic loading is applied at the natural frequency of an undamped SDOF system, the response goes to infinity indicating *resonance* of the system. However, since truly undamped systems do not exist, true resonance is never really achieved. The concept of the tuning ratio that relates the frequency of loading to the natural frequency of the system is an important one, as evidenced by its strong influence on the response.

$\dfrac{Q_0}{k}$

$y = \dfrac{1}{1-x^2}$



higher loading frequency

resonance.

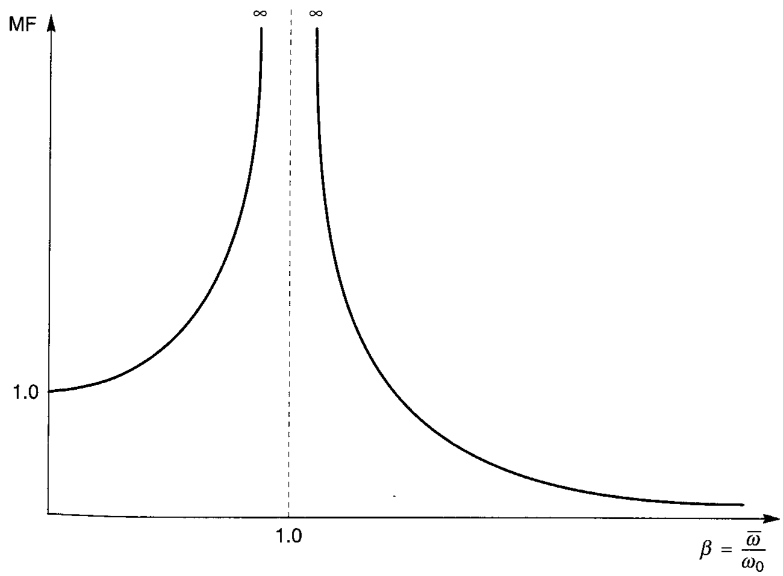


Figure B.9 Variation of magnification factor with tuning ratio for undamped SDOF system.

Example B.3

From an initial stationary state, the undamped SDOF system of Example B.1 is subjected to a harmonic base acceleration of $0.20g$ at a frequency of 2 Hz. Compute the response of the system.

Solution Expressing the base motion as

$$\ddot{u}_b(t) = (0.2)(32.2 \text{ ft/sec}^2) \sin 4\pi t = 6.44 \sin 4\pi t$$

the equivalent external force would be

$$Q(t) = -\frac{W}{g}\ddot{u}_b(t) = -\frac{10{,}000 \text{ lb}}{32.2 \text{ ft/sec}^2}(6.44 \text{ ft/sec}^2) \sin 4\pi t = -(2000 \text{ lb}) \sin 4\pi t$$

The tuning ratio would be

$$\beta = \frac{\bar{\omega}}{\omega_0} = \frac{2\pi \bar{f}}{\omega_0} = \frac{2\pi(2)}{69.5} = 0.181$$

Then, from equation (B.35),

$$\begin{aligned} u(t) &= \frac{Q_0}{k}\frac{1}{1-\beta^2}(\sin \bar{\omega}t - \beta \sin \omega_0 t) \\ &= \frac{-2 \text{ kips}}{1500 \text{ kips/ft}}\frac{1}{1-(0.181)^2}[\sin 4\pi t - 0.181 \sin (69.5t)] \\ &= 0.00138 \sin 4\pi t - 0.00025 \sin 69.5\text{t} \end{aligned}$$

which is plotted in Figure EB.3.

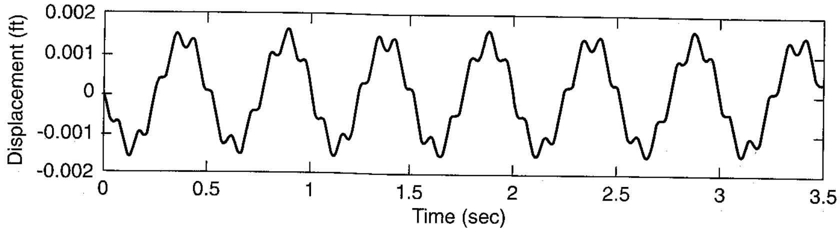


Figure EB.3

B.5.3.2 Damped Forced Vibrations

The most general case is that of a damped system subjected to forced harmonic loading. Each of the three prior cases can be considered as a subset of this one since their equations of motion can be obtained by setting various terms of the equation of motion for damped forced vibrations shown below to zero. The equation of motion for a damped SDOF system subjected to simple harmonic loading of the form $Q(t) = Q_0 \sin \bar{\omega} t$ is

$$m\ddot{u} + c\dot{u} + ku = Q_0 \sin \bar{\omega} t \tag{B.36}$$

After dividing by m and using the relationships $\xi = c/2m\omega_0$ and $\omega_0^2 = k/m$, equation (B.36) can be rewritten as

$$\ddot{u} + 2\xi\omega_0\dot{u} + \omega_0^2 u = \frac{Q_0}{m} \sin \bar{\omega} t \tag{B.37}$$

The complementary solution represents the damped free vibration response, which was expressed for an underdamped system by equation (B.19).

$$u_c(t) = e^{-\xi\omega_0 t}(C_1 \sin \omega_d t + C_2 \cos \omega_d t)$$

Since the response of a damped SDOF system is generally out of phase with the external loading, a harmonic particular solution of the form

$$u_p(t) = C_3 \sin \bar{\omega} t + C_4 \cos \bar{\omega} t \tag{B.38a}$$

can be assumed. The corresponding velocity and acceleration are

$$\dot{u}_p(t) = C_3\bar{\omega} \cos \bar{\omega} t - C_4\bar{\omega} \sin \bar{\omega} t \tag{B.38b}$$

$$\ddot{u}_p(t) = -\bar{\omega}^2 C_3 \sin \bar{\omega} t - \bar{\omega}^2 C_4 \cos \bar{\omega} t \tag{B.38c}$$

Substituting equations (B.38) into the equation of motion [equation (B.37)] and grouping the $\sin\bar{\omega}t$ and $\cos\bar{\omega}t$ terms gives

$$(C_3\omega_0^2 - C_3\bar{\omega}^2 - 2\xi\omega_0 C_4\bar{\omega}) \sin \bar{\omega} t + (C_4\omega_0^2 - C_4\bar{\omega}^2 + C_3\bar{\omega}2\xi\omega_0) \cos \bar{\omega} t = \frac{Q_0}{m} \sin \bar{\omega} t \tag{B.39}$$

Now, at the instances where $\bar{\omega}t = 0 + n\pi$ (where n is any positive integer), $\sin\bar{\omega}t = 0$ and $\cos\bar{\omega}t = 1$. Thus the relationship

$$C_4\omega_0^2 - C_4\bar{\omega}^2 + C_3\bar{\omega}2\xi\omega_0 = 0 \tag{B.40a}$$

must be satisfied. Further, at $\bar{\omega}t = \pi/2 + n\pi$, $\cos\bar{\omega}t = 0$ and $\sin\bar{\omega}t = 1$, which means that

$$C_3\omega_0^2 - C_3\bar{\omega}^2 - 2\xi\omega_0 C_4\bar{\omega} = \frac{Q_0}{m} \tag{B.40b}$$

must also be satisfied. Equations (B.40) represent two simultaneous equations with the two unknowns C_3 and C_4. Solving for the unknowns yields

$$C_3 = \frac{Q_0}{k}\frac{1-\beta^2}{(1-\beta^2)^2 + (2\xi\beta)^2} \tag{B.41a}$$

$$C_4 = \frac{Q_0}{k}\frac{-2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2} \tag{B.41b}$$

The general solution to the equation of motion for damped forced vibration can now be obtained by combining the complementary and particular solutions

$$\begin{aligned} u(t) &= e^{-\xi\omega_0 t}(C_1 \sin\omega_d t + C_2 \cos\omega_d t) \\ &\quad + \frac{Q_0}{k}\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}[(1-\beta^2)\sin\bar{\omega}t - 2\xi\beta\cos\bar{\omega}t] \end{aligned} \tag{B.42}$$

transient response

where the constants C_1 and C_2 depend on the initial conditions. There are several important characteristics of this solution. Note that the complementary solution (which represents the effects of the initial conditions) decays with time. The complementary solution therefore describes a *transient response* caused by the requirement of satisfying the initial conditions. After the transient response dies out, only the *steady-state response* described by the particular solution remains. The steady-state response occurs at the frequency of the applied harmonic loading but is out of phase with the loading.

steady state response

Example B.4

The SDOF system shown in Figure EB.4a is at rest when the sinusoidal load is applied. Determine the transient, steady state, and total motion of the system.

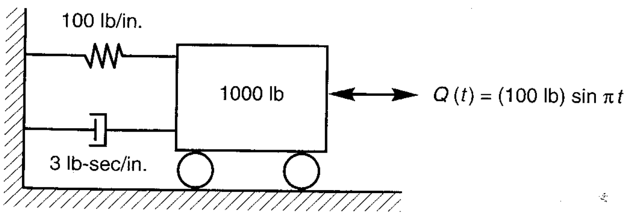


Figure EB.4a

Solution From equation (B.42), the total response is given by

$$u(t) = e^{-\xi\omega_0 t}(C_1 \sin \omega_d t + C_2 \cos \omega_d t) + \frac{Q_0}{k}\frac{1}{(1-\beta^2)^2+(2\xi\beta)^2}[(1-\beta^2)\sin \bar{\omega} t - 2\xi\beta \cos \bar{\omega} t]$$

For zero initial displacement,

$$\begin{aligned} u(t=0) &= 0 \\ &= e^{-\xi\omega_0(0)}[C_1 \sin \omega_d(0) + C_2 \cos \omega_d(0)] \\ &\quad + \frac{Q_0}{k}\frac{1}{(1-\beta^2)^2+(2\xi\beta)^2}[(1-\beta^2)\sin \bar{\omega}(0) - 2\xi\beta \cos \bar{\omega}(0)] \\ &= C_2 + \frac{Q_0}{k}\frac{-2\xi\beta}{(1-\beta^2)^2+(2\xi\beta)^2} \end{aligned}$$

or

$$C_2 = \frac{Q_0}{k}\frac{2\xi\beta}{(1-\beta^2)^2+(2\xi\beta)^2}$$

For zero initial velocity,

$$\begin{aligned} \dot{u} &= 0 \\ &= \omega_d^{-\xi\omega_0(0)}[C_1 \cos \omega_d(0) - C_2 \sin \omega_d(0)] - \xi\omega_0 e^{-\xi\omega_0 t}[C_1 \sin \omega_d(0) - C_2 \cos \omega_d(0)] \\ &\quad + \frac{Q_0}{k}\frac{\bar{\omega}}{(1-\beta^2)^2+(2\xi\beta)^2}[(1-\beta^2)\cos \bar{\omega}(0) + 2\xi\beta \sin \bar{\omega}(0)] \\ &= \omega_d C_1 - \xi\omega_0 C_2 + \frac{Q_0}{k}\frac{\bar{\omega}(1-\beta^2)}{(1-\beta^2)^2+(2\xi\beta)^2} \end{aligned}$$

or

$$C_1 = \frac{Q_0}{k}\frac{\bar{\omega}}{\omega_d}\frac{\beta^2-1}{(1-\beta^2)^2+(2\xi\beta)^2}$$

Then the transient motion is given by

$$u_c(t) = \frac{Q_0}{k}\frac{1}{(1-\beta^2)^2+(2\xi\beta)^2}e^{-\xi\omega_0 t}\left[\frac{\bar{\omega}}{\omega_d}(\beta^2+2\xi^2-1)\sin \omega_d t + 2\xi\beta \cos \omega_d t\right]$$

and the steady-state motion by

$$u_p(t) = \frac{Q_0}{k}\frac{1}{(1-\beta^2)^2+(2\xi\beta)^2}[(1-\beta^2)\sin \bar{\omega} t - 2\xi\beta \cos \bar{\omega} t]$$

The total motion is the sum of the transient and steady-state motions. For the system shown in Figure EB.1a,

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{(100 \text{ lb/in.})\ (12 \text{ in./ft})\ (32.2 \text{ ft/sec/sec})}{1000 \text{ lb}}} = 6.22 \text{ rad/sec}$$

$$\xi = \frac{c}{2m\omega_0} = \frac{cg}{2W\omega_0} = \frac{(3 \text{ lb-sec/in.})(12 \text{ in./ft})(32.2 \text{ ft/sec/sec})}{2(1000 \text{ lb})(6.22 \text{ rad/sec})} = 0.093$$

$$\omega_d = \omega_0\sqrt{1-\xi^2} = \sqrt{1-(0.092)^2} = 6.19 \text{ rad/sec}$$

$$\beta = \frac{\bar{\omega}}{\omega_0} = \frac{\pi \text{ rad/sec}}{6.22 \text{ rad/sec}} = 0.505$$

Substituting these values into the solutions gives the response shown in Figure EB.4b

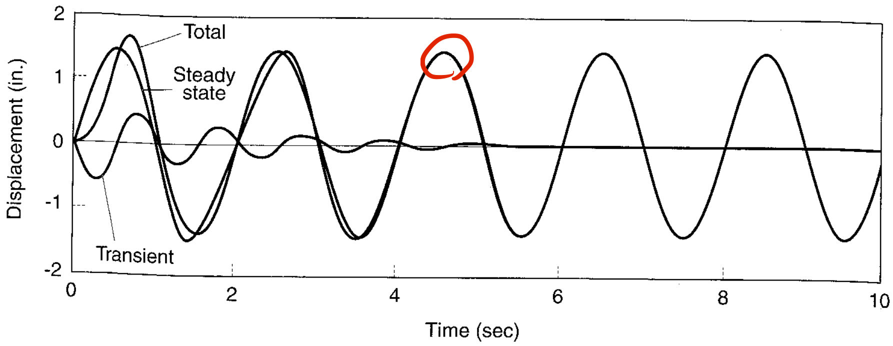


Figure EB.4b

The steady-state response could also be described by

$$u = A \sin(\bar{\omega} t + \phi) \tag{B.43}$$

where

$$A = \frac{Q_0}{k} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

$$\phi = \tan^{-1}\left(-\frac{2\xi\beta}{1-\beta^2}\right)$$

The steady-state response can be visualized with the aid of rotating vectors, both for the response and for the forces induced in the system, as shown in Figure B.10. Note that the spring, dashpot, and inertial forces act opposite to the displacement, velocity and acceleration vectors, and that the displacement lags the applied loading vector by the negative phase

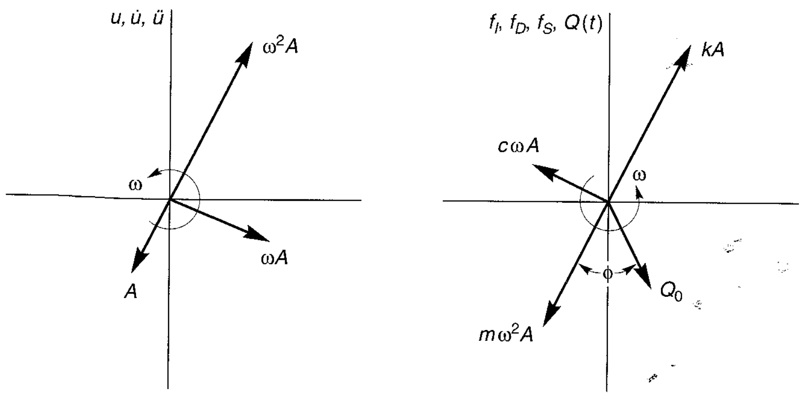
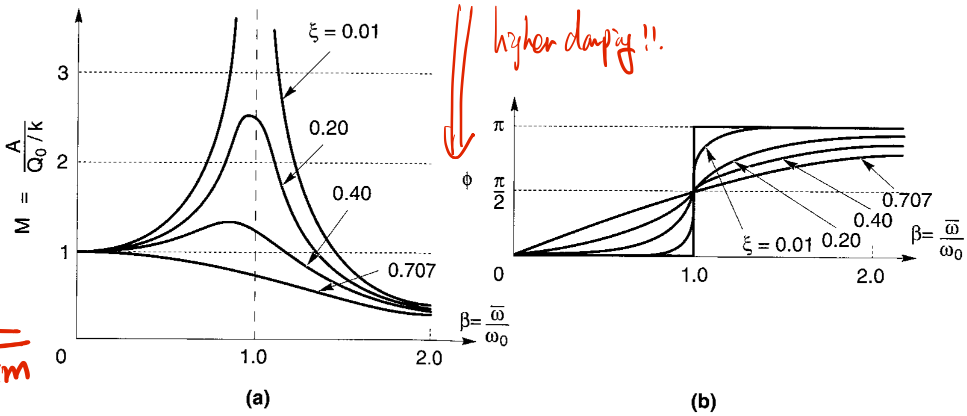


Figure B.10 Rotating vector representation of response and forces in vibrating SDOF system.



higher damping !!

$\xi = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$

Figure B.11 Variation of (a) magnification factor, and (b) phase angle with damping ratio and tuning ratio.

angle, ϕ. For harmonic loading the phase angle varies with both damping ratio and tuning ratio, as shown in Figure B.11a.

The influence of the tuning ratio can be illustrated by the use of the magnification factor, again defined as the ratio of the amplitude to the static displacement:

$$M = \frac{A}{Q_0/k} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \tag{B.44}$$

The variation of the magnification factor with tuning ratio and damping ratio is shown in Figure B.11b. The damping ratio influences the peak magnification factor and also the variation of magnification factor with frequency. The magnification factor curves broaden with increasing damping ratio. Note that the magnification is unbounded (resonance) only for $\xi = 0$ and $\beta = 1$. For nonzero damping, there is some maximum magnification, M_{max},

$$M_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}} \tag{B.45}$$

which occurs when the tuning ratio $\beta = \sqrt{1-2\xi^2}$. The shape of the magnification curve is obviously controlled by the damping ratio. Although a system with low damping may produce large magnification at a tuning ratio near 1, it will exhibit significant magnification over a smaller range of frequencies than a system with higher damping.

B.5.4 Response of SDOF Systems to Periodic Loading

The solutions for the response of a SDOF system to harmonic loading developed in the preceding section can be used to develop solutions for the more general case of periodic loading. As shown in Appendix A, periodic loading can be approximated by a Fourier series (i.e., as the sum of a series of harmonic loads). The response of a SDOF system to the periodic loading, using the principle of superposition, is simply the sum of the responses to each term in the loading series. The required calculations can be performed using trigonometric or exponential notation.

$M\ddot{u} + C\dot{u} + Ku = Q(t)$

$Q(t) = a_0 + a_1 \cos\omega_1 t$

$+ b_1 \sin\omega_1 t$

$+ \cdots\cdots$

B.5.4.1 Trigonometric Notation

From equation (A.11) a periodic load, $Q(t)$, can be expressed by the Fourier series

$$Q(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t$$

where the *Fourier coefficients* are

$$a_0 = \frac{1}{T_f}\int_0^{T_f} Q(t)\, dt$$

$$a_n = \frac{2}{T_f}\int_0^{T_f} Q(t) \cos \omega_n t\, dt$$

$$b_n = \frac{2}{T_f}\int_0^{T_f} Q(t) \sin \omega_n t\, dt$$

and $\omega_n = 2\pi n/T_f$. Using the steady-state portion of equation (B.42), the response to each sine term in the Fourier series is

$$u_{n,\sin}(t) = \frac{b_n}{k}\frac{1}{(1-\beta_n^2)^2 + (2\xi\beta_n)^2}[(1-\beta_n^2)\sin \bar{\omega} t - 2\xi\beta_n \cos \bar{\omega} t]$$ Sin loading

where $\beta_n = \omega_n T_f/2\pi$. In the same way, the steady-state response to each cosine term can be shown to be

$$u_{n,\cos}(t) = \frac{a_n}{k}\frac{1}{(1-\beta_n^2)^2 + (2\xi\beta_n)^2}[(1-\beta_n^2)\cos \bar{\omega} t + 2\xi\beta_n \sin \bar{\omega} t]$$ Cos loading

Since the steady-state response to the constant load term is the static displacement, $u_0 = a_0/k$, the total steady-state response is given by

$$u(t) = u_0 + \sum_{n=1}^{\infty} u_{n,\sin}(t) + u_{n,\cos}(t)$$

Initial displacement

$$= \frac{1}{k}\Bigg(a_0 + \sum_{n=1} \frac{1}{(1-\beta_n^2)^2 + (2\xi\beta_n)^2}\{[a_n 2\xi\beta_n + b_n(1-\beta_n^2)]\sin\omega_n t$$ (B.46)

$$+ [a_n(1-\beta_n^2) - b_n 2\xi\beta_n]\cos \omega_n t\}\Bigg)$$

2.5.4.2 Exponential Notation

Periodic loading can also be described by the Fourier series in exponential form. Using equation (A.15), a periodic load can be expressed as

$$Q(t) = \sum_{n=-\infty}^{\infty} q_n^* e^{i\omega_n t}$$

The complex Fourier coefficients, q_n^*, can be determined directly from $Q(t)$ as

$$q_n^* = \frac{1}{T_f}\int_0^{T_f} Q(t)e^{-i\omega_n t}dt$$

The response of a SDOF system loaded by the nth harmonic would be governed by the equation of motion

$$m\ddot{u}_n(t) + (c\dot{u}_n(t) + ku_n(t)) = q_n^* e^{i\omega_n t} \tag{B.47}$$

a single nth harmonic loading !!

The response of the system can be related to the loading by

$$u_n(t) = H(\omega_n)q_n^* e^{i\omega_n t} \tag{B.48}$$

where $H(\omega_n)$ is a *transfer function* [i.e., a function that relates one parameter (in this case, the displacement of the oscillator) to another (the external load)]. Substituting equation (B.48) into the equation of motion gives

$$-m\omega_n^2 H(\omega_n)q_n^* e^{i\omega_n t} + ic\omega_n H(\omega_n)q_n^* e^{i\omega_n t} + kH(\omega_n)\, q_n^* e^{i\omega_n t} = q_n^* e^{i\omega_n t}$$

or

$$H(\omega_n) = \frac{1}{-m\omega_n^2 + ic\omega_n + k} = \frac{1}{k(-\beta_n^2 + 2i\beta_n\xi + 1)} \tag{B.49}$$

Since $A^* = a + ib = Ae^{i\theta}$, where the *modulus*, $A = \sqrt{a^2 + b^2}$, and the *argument*, $\theta = \tan^{-1}(b/a)$, the transfer function can also be written as

$$H(\omega_n) = \frac{1/k}{\sqrt{(1-\beta_n^2)^2 + (2\xi\beta_n)^2}}\exp\left(i \tan^{-1}\frac{2\xi\beta_n}{\beta_n^2 - 1}\right)$$

Note the close relationship between the modulus of the transfer function and the magnification factor of equation (B.44). Because the transfer function can be used for any frequency in the series, the principle of superposition gives the total response as

$$u(t) = \sum_{n=-\infty}^{\infty} H(\omega_n)q_n^* e^{i\omega_n t} \tag{B.50}$$

→ input loading !!

Many different transfer functions can be developed. For example, a transfer function relating the acceleration of the SDOF system to the external load could have been developed just as easily. The advantages of the transfer function approach lie in its simplicity and in the ease with which it allows computation of the response to complicated loading patterns.

The transfer function may be viewed as a *filter* that acts upon some input signal to produce an output signal. In the case just considered, the input signal was the time history of loading, $Q(t)$, and the output was the displacement, $u(t)$. If the input signal has Fourier amplitude and phase spectra, $F_i(\omega_n)$ and $\phi_i(\omega_n)$, the Fourier amplitude spectra of the output signal will be given by

→ frequency response function.

$$F_o(\omega_n) = H(\omega_n)F_i(\omega_n) \tag{B.51a}$$

$$\phi_o(\omega_n) = H(\omega_n)\phi_i(\omega_n) \tag{B.51b}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi n t/T_p} \longrightarrow X(f) = \sum_{n=-\infty}^{\infty} C_n\, \delta\left(f - \frac{n}{T_p}\right)$$

Thus the procedure for Fourier analysis of SDOF system response can be summarized in the following steps:

1. Obtain the Fourier series for the applied loading (or base motion). In doing so, the loading (or base motion) is expressed as a function of frequency rather than a function of time.
2. Multiply the Fourier series coefficients by the appropriate value of the transfer function at each frequency, ω_n. This will produce the Fourier series of the output motion.
3. Express the output motion in the time domain by obtaining the inverse Fourier transform of the output motion.

It is precisely this approach that forms the backbone of several of the most commonly used methods for analysis of ground response and soil–structure interaction. These methods are presented in Chapter 7.

B.5.5 Response of SDOF Systems to General Loading

Not all loading is harmonic or even periodic. To determine the response of SDOF systems to general loading conditions, a more general solution of the equation of motion is required.

B.5.5.1 Response to Step Loading

Consider a damped SDOF system subjected to a step load of intensity, Q_0, which is applied instantaneously at $t = 0$ and removed instantaneously at $t = t_1$ as shown in Figure B.12. For $t \leq t_1$, the complementary solution to the equation of motion for this system [equation (B.19)],

$$u_c(t) = e^{-\xi\omega_0 t}[C_1 \sin \omega_d t + C_2 \cos \omega_d t]$$

describes the transient response of the system. The equation of motion for the steady-state condition is given by

$$m\ddot{u}_p + c\dot{u}_p + ku_p = Q_0$$

Since the applied load does not vary with time, the steady-state response will be a constant displacement,

$$u_p(t) = \frac{Q_0}{k}$$

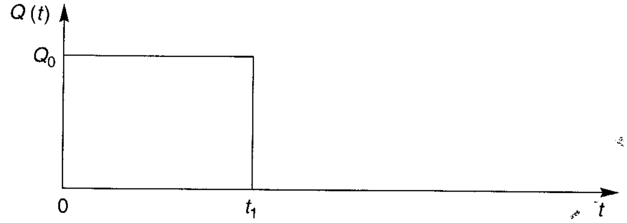


Figure B.12 Time history of step loading.