Camera Model

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UNIVERSITY OF WATERLOO
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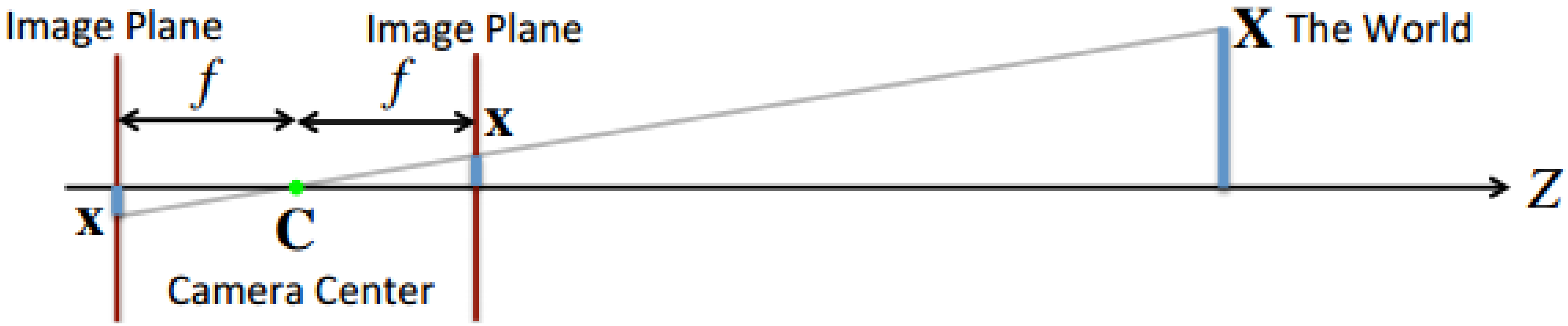
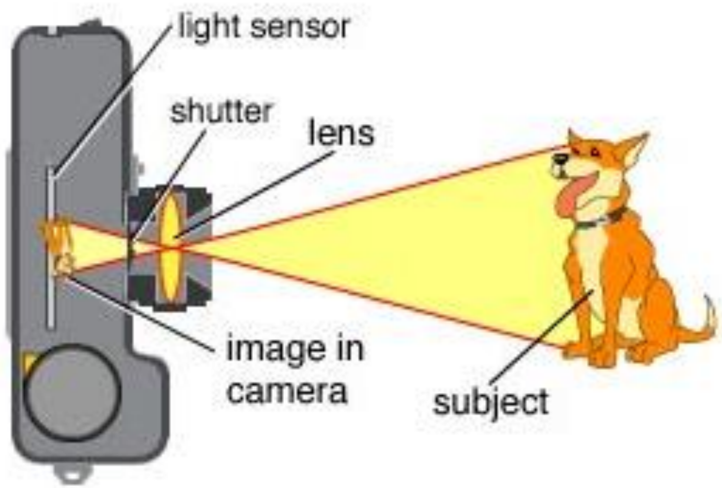
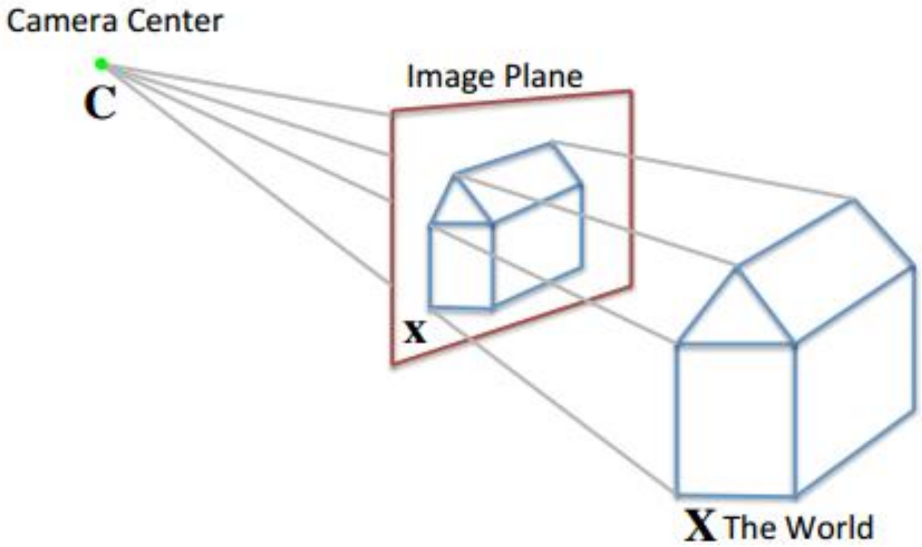
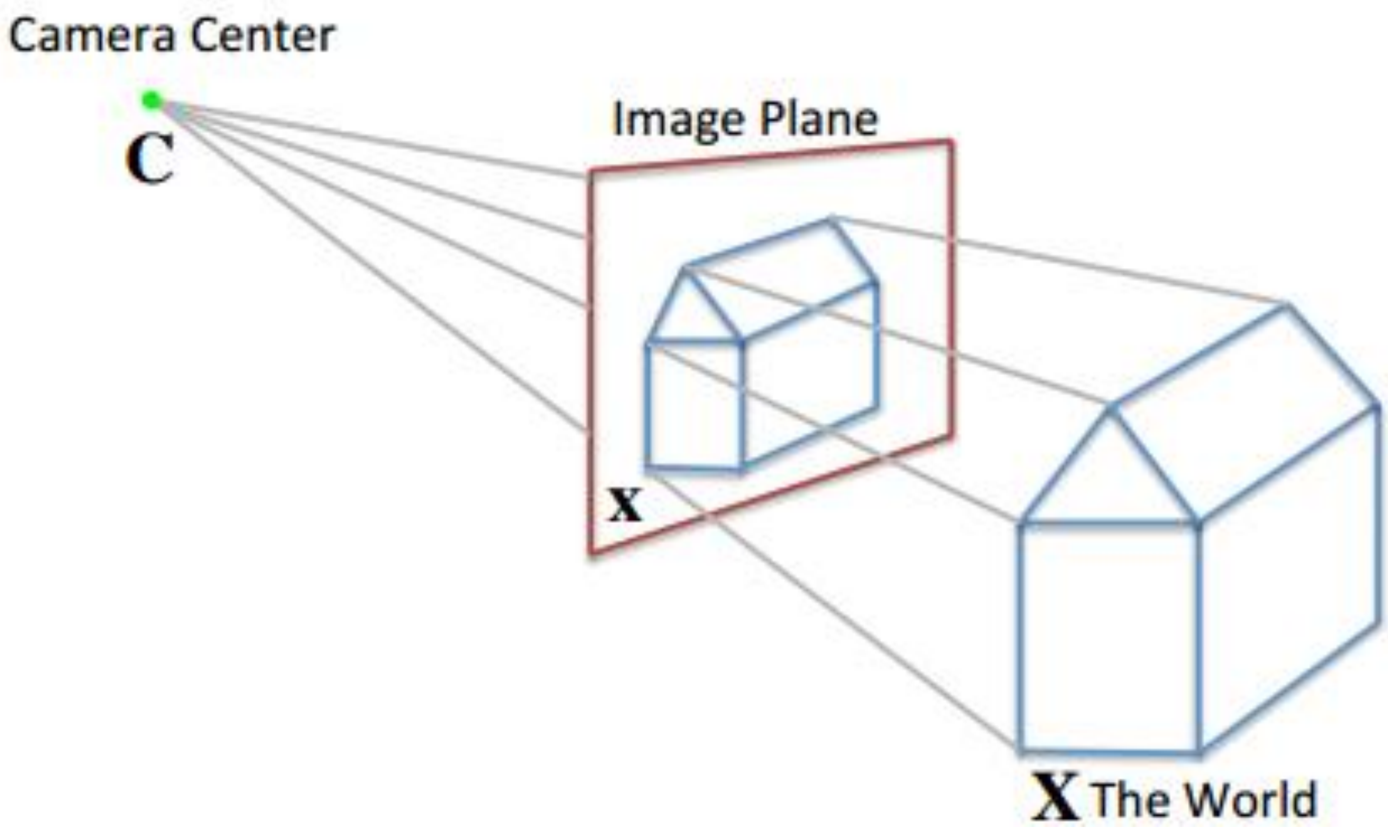
(Review) Pinhole Camera Model







How to Model Projection



How to mathematically model the projection of 3D scenes on images

 Supplement

An arbitrary homogeneous vector representative of a point is of the form $\mathbf{x} = (x_1, x_2, x_3)^T$, representing the point $(x_1/x_3, x_2/x_3)^T$ in $\mathbb{R}^2$.

Line equation, $ax + by + c = 0$, in $\mathbb{R}^2$ is represented as $\mathrm{l} = (a, b, c)^\intercal$ in the homogeneous coordinate.

Representing Points in 3D

An arbitrary homogeneous vector representative of a point is of the form $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$, representing the point $(x_1/x_4, x_2/x_4, x_3/x_4)^T$ in $\mathbb{R}^3$.

Example)

$$\mathbf{x_1} = \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{x_2} = \begin{pmatrix} 10 \\ 6 \\ 4 \\ 2 \end{pmatrix} \quad \mathbf{x_3} = \begin{pmatrix} 5k \\ 3k \\ 2k \\ k \end{pmatrix}, k \neq 0$$

<u>up to a scale</u>

$\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ and $\mathbf{x_4}$ indicate the same point of (5, 3, 2) in $\mathbb{R}^3$

Representing Plane in 3D

$$\pi_1 x + \pi_2 y + \pi_3 z + \pi_4 = 0$$

Plane equation in $\mathbb{R}^3$

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^\intercal$$

Plane representation in HC

Example) $\boldsymbol{\pi_1} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}$ $\boldsymbol{\pi_2} = \begin{pmatrix} 4 \\ 2 \\ 8 \\ 6 \end{pmatrix}$ $\boldsymbol{\pi_3} = \begin{pmatrix} 2k \\ 1k \\ 4k \\ 3k \end{pmatrix}, k \neq 0$

$\boldsymbol{\pi_1}, \boldsymbol{\pi_2}$, and $\boldsymbol{\pi_3}$indicate the same plane of $2x + y + 4z + 3 = 0$ in $\mathbb{R}^3$

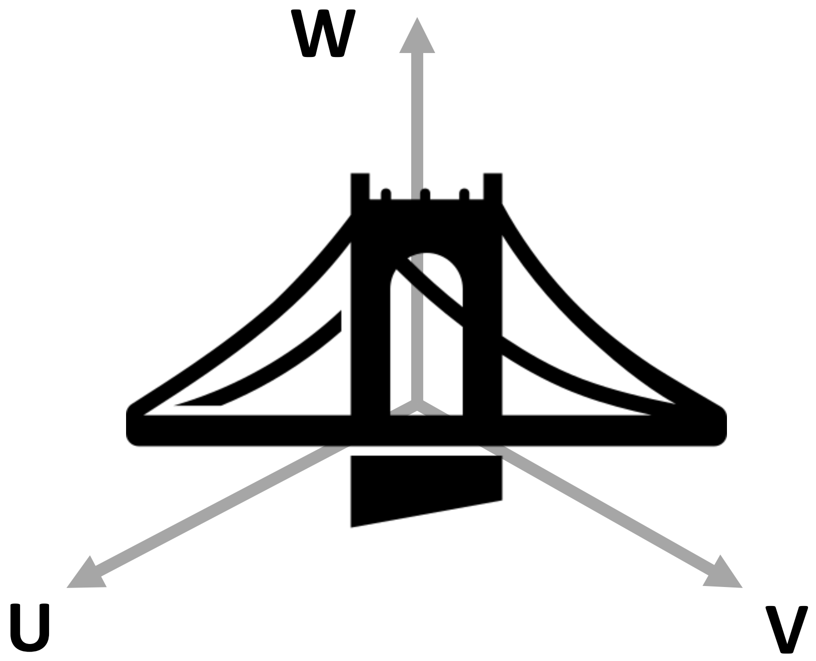
Image Geometry (World Coordinate)

Object of interest in World

Coordinate System (U, V, W)

Null Island (0°N 0°E)

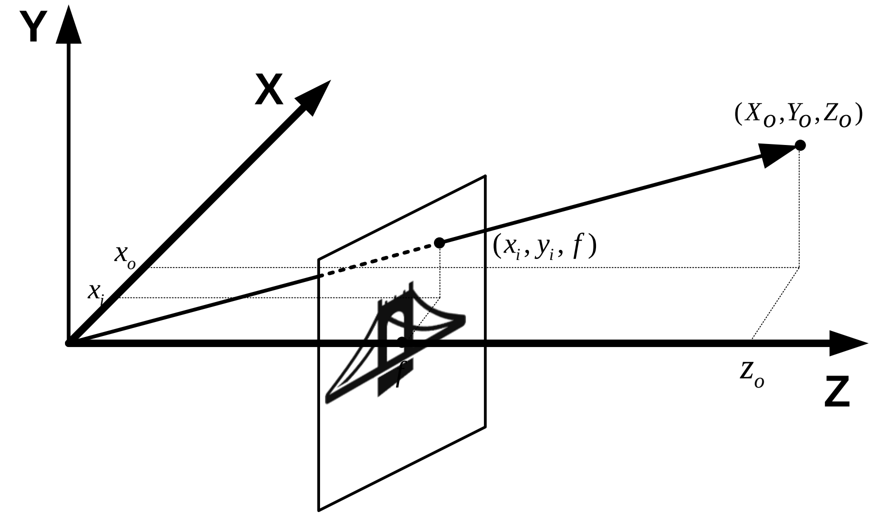
Custom coordinate



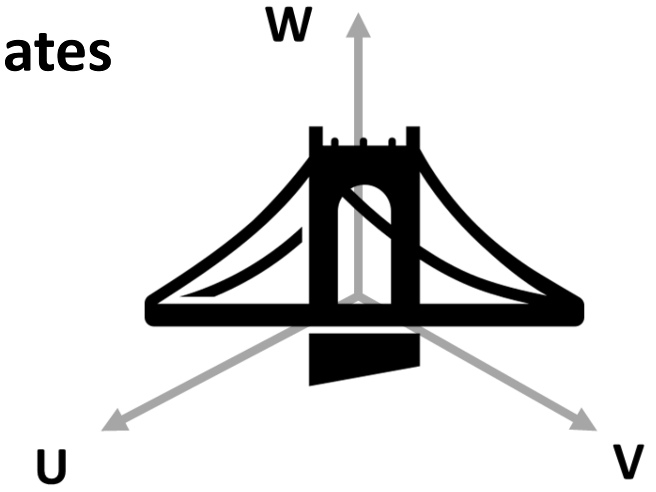
Q. Example of world coordinate?

Image Geometry (Pixel Coordinate)

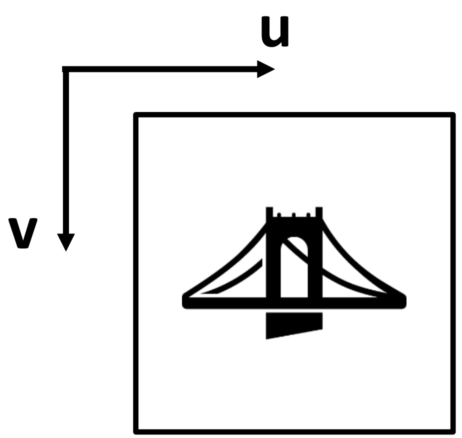
Camera coordinates



World coordinates

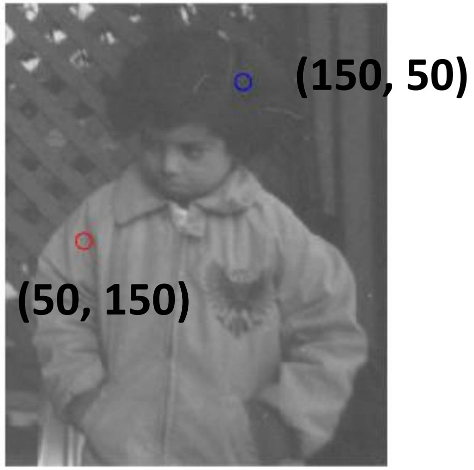


Pixel coordinates



Our image gets digitized

into pixel coordinates (u,v)



Forward Projection

World Coordinates		Camera Coordinates		Pixel Coordinates
U V W	⇨	**X Y Z**	⇨	**u v**

We will make a mathematical model to describe how 3D World points get projected into 2D pixel coordinates

Backward Projection

World Coordinates		Camera Coordinates		Film Coordinates		Pixel Coordinates
U V W	⇐	**X Y Z**	⇐	**x y**	⇐	**u v**

Much of vision concerns trying to derive backward projection equations to recover 3D scene structure from images.

Scene from images

Projection Matrix (Camera Matrix)

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

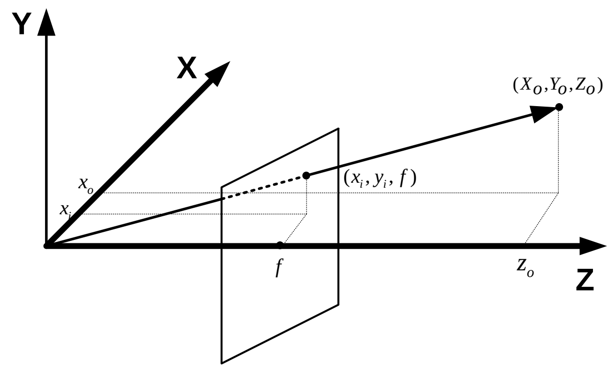
Homogenous coordinate system

$$\mathbf{x} = \mathbf{P}_i \mathbf{X}$$

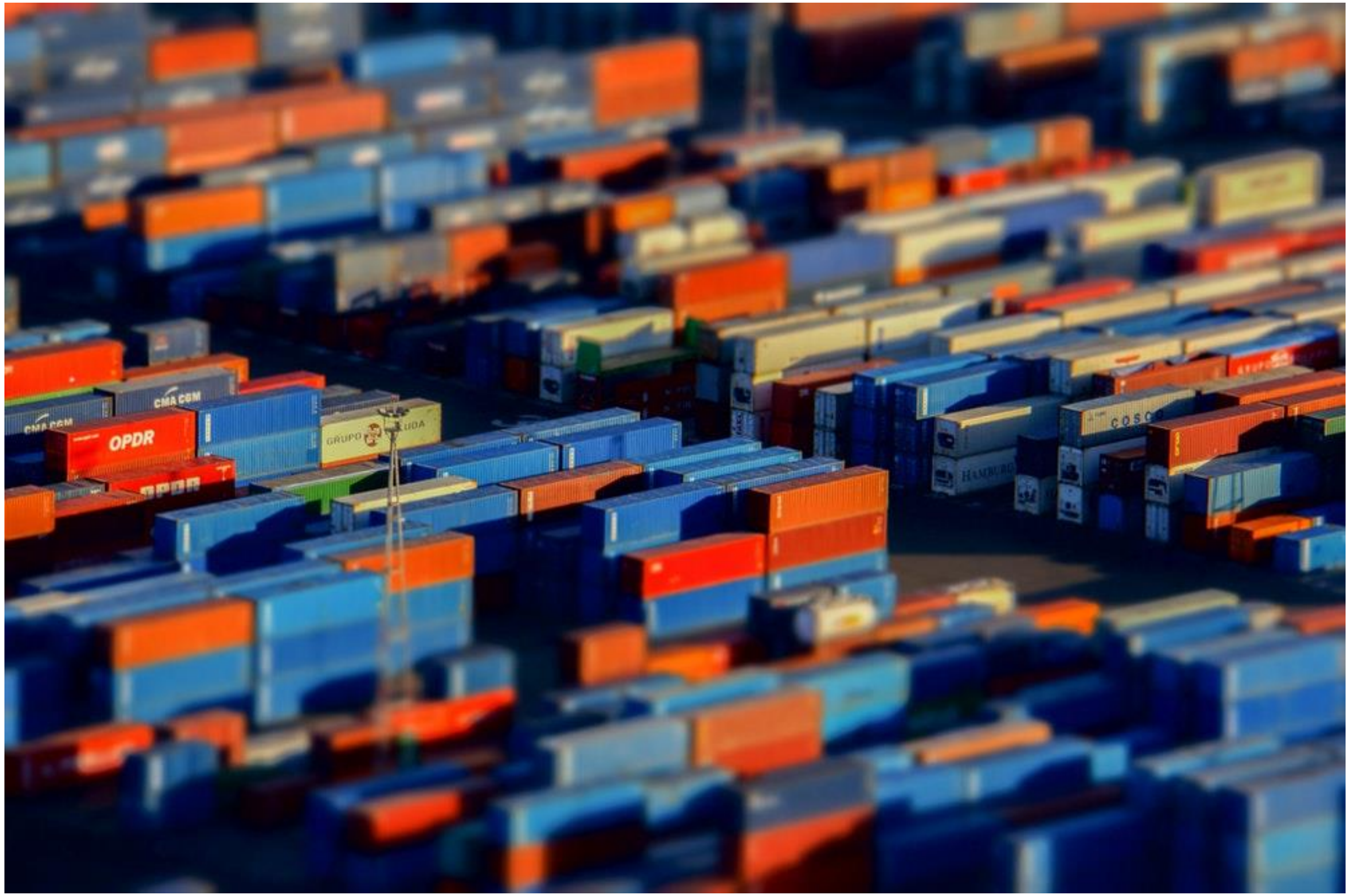
2D point **3D point**

- **If we knew a projection matrix in an image, we can compute the image point corresponding to the world point**
- **Although we knew an image points, we cannot find an unique world point.**

Q. Meaning of a 3 x 4 matrix?



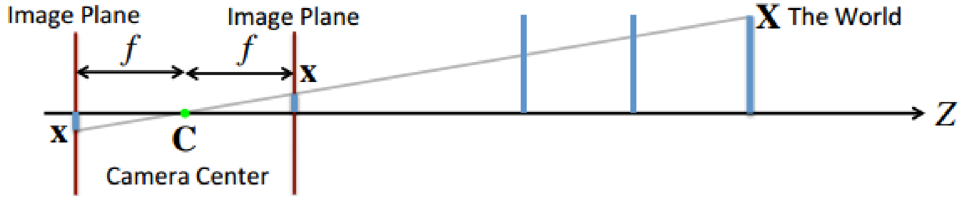
Example: Unknown Scale

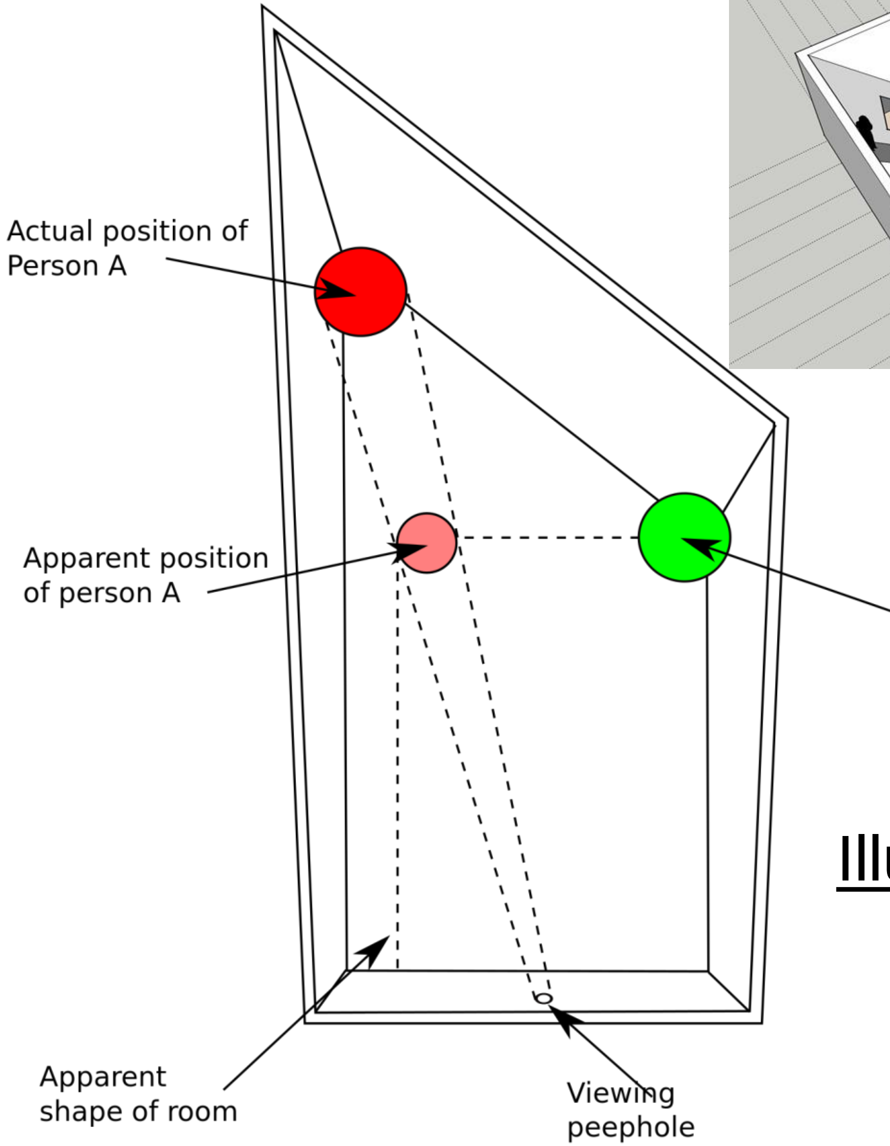


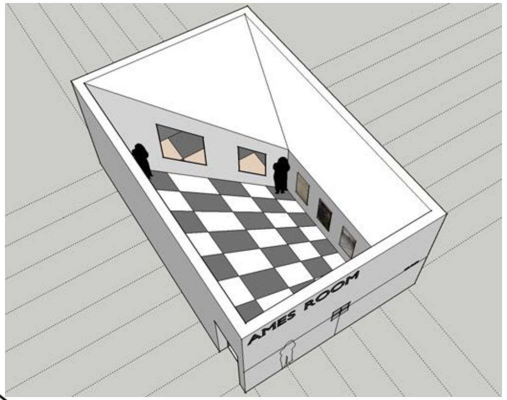
Example: Ames room

Supplement

Smaller when farther away



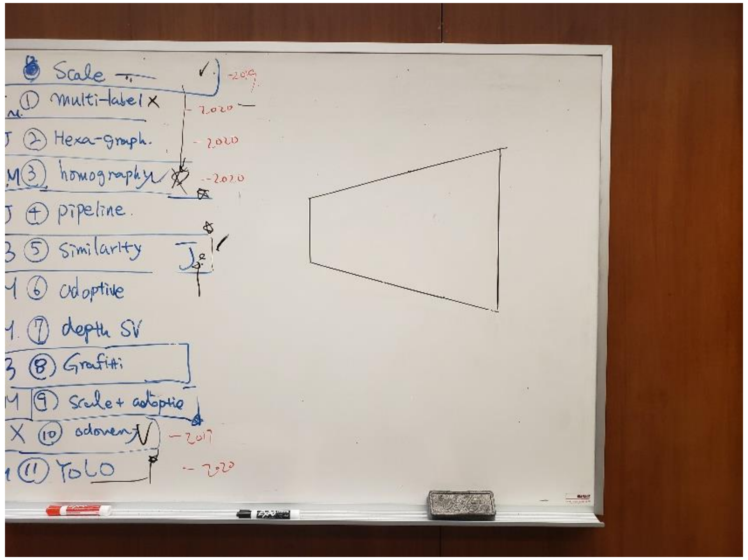


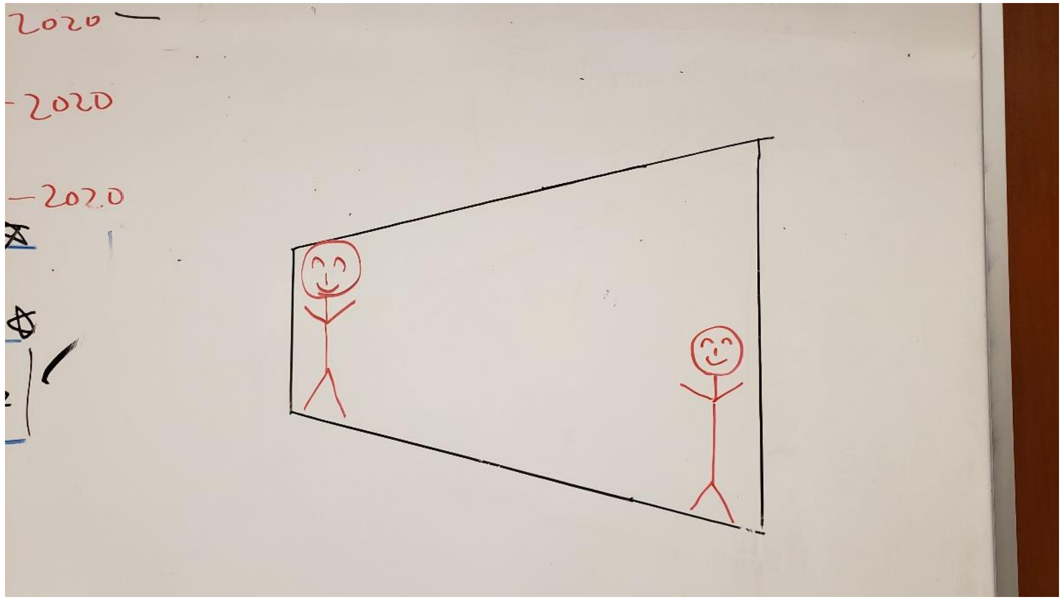


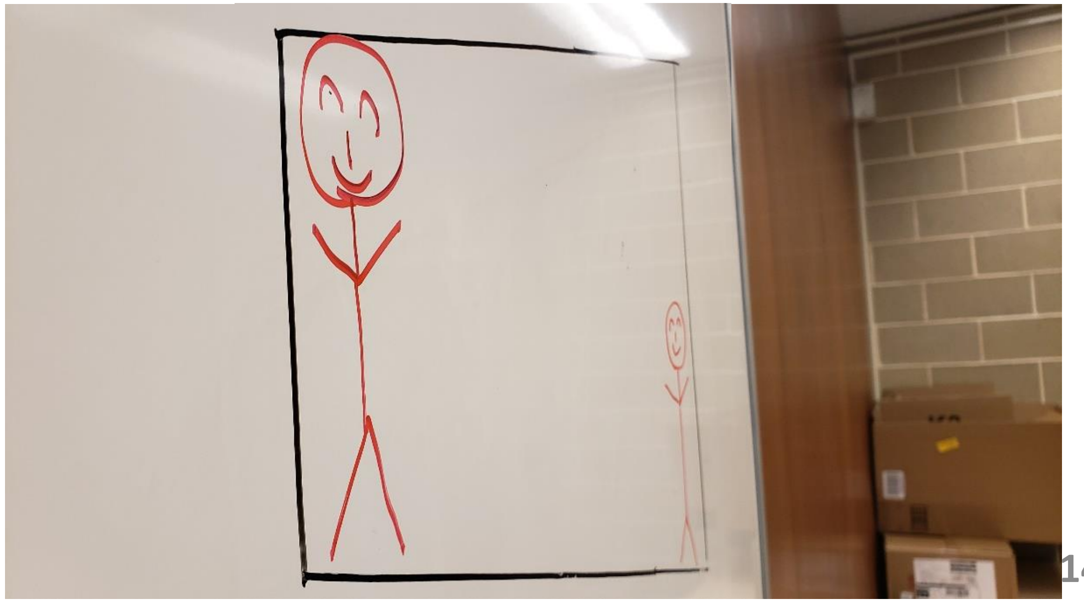
Illusion?

Example: Chul Min’s Room

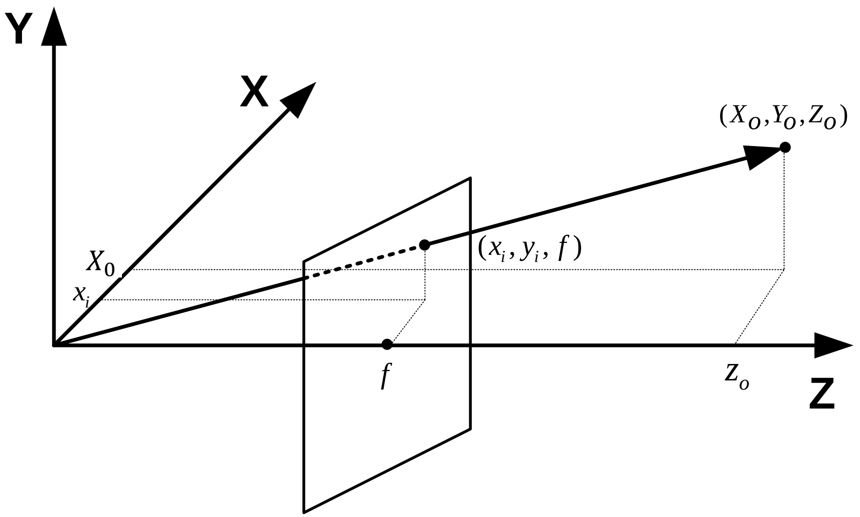
Supplement







Camera Model: A Simple Case

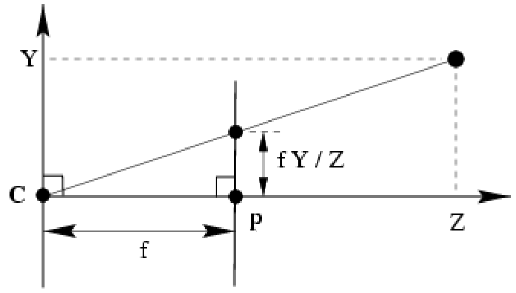


$$\frac{x_i}{f} = \frac{X_o}{Z_0} \quad , \quad \frac{y_i}{f} = \frac{Y_o}{Z_0}$$

$$x_i = f\frac{X_o}{Z_0} \quad , \quad y_i = f\frac{Y_o}{Z_0}$$

Up to scale !!!

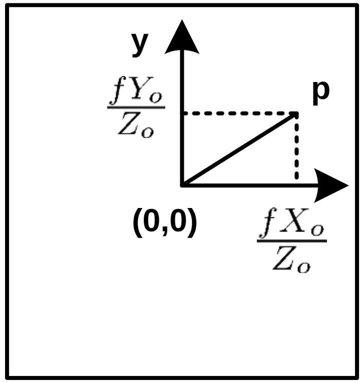
$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} fX_o \\ fY_o \\ Z_o \end{bmatrix} = \begin{bmatrix} \boxed{f} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

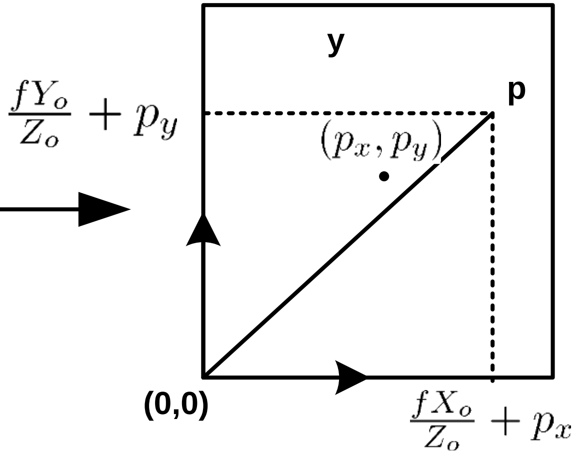


- Any point on the ray OP is projected onto a same image point.
- Only valid at homogeneous coordinate (up to scale)

1 Unknown

Camera Model: Pixel Coordinate w.r.t. the Image Origin





$$x_i = f\frac{X_o}{Z_0} + p_x \quad , \quad y_i = f\frac{Y_o}{Z_0} + p_y$$

$$\begin{bmatrix} fX_o + Z_0 p_x \\ fY_o + Z_0 p_y \\ Z_o \end{bmatrix} = \overbrace{\begin{bmatrix} \boxed{f} & 0 & \boxed{p_x} \\ 0 & f & \boxed{p_y} \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{K}} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix} = \mathbf{K} \left[\ \mathbf{I} \ \middle| \ \vec{0} \ \right] \vec{X}$$

If the image center (principal point) is not the center of the image,

3 Unknowns

Physical Scale VS Pixel Scale

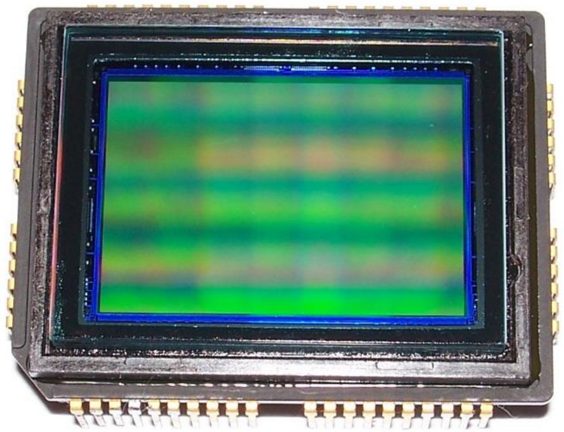
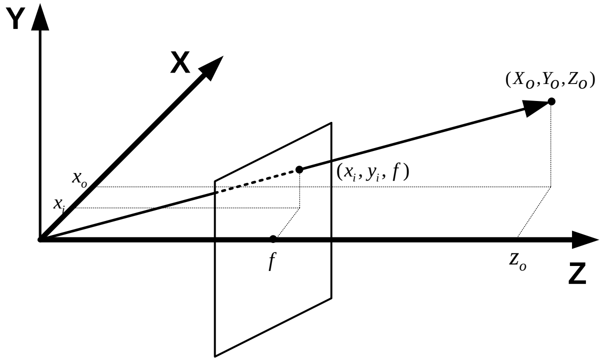


Image sensor



$$x_i = f\boxed{\frac{X_o}{Z_0}} + p_x \quad , \quad y_i = f\boxed{\frac{Y_o}{Z_0}} + p_y$$

Non-dimensional number **Non-dimensional number**

$$\begin{bmatrix} fX_o + Z_0 p_x \\ fY_o + Z_0 p_y \\ Z_o \end{bmatrix} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix} = \mathbf{K} \left[\ \mathbf{I} \mid \vec{0}\ \right] \vec{X}$$

When the projection matrix is represented by a pixel scale, the image points are also represented by a pixel scale.

Q. What is the assumption?

Let's have m_x pixel/unit length along X and m_y pixel/unit length along Y (because the pixel may not be square, but most case, it is square)

$$\begin{bmatrix} I \\ J \\ 1 \end{bmatrix} = \begin{bmatrix} m_x x_i \\ m_y y_i \\ 1 \end{bmatrix} = \begin{bmatrix} m_x f X_o + Z_o m_x p_x \\ m_y f Y_o + Z_o m_y p_y \\ Z_o \end{bmatrix} = \begin{bmatrix} m_x f X_o + Z_o x_0 \\ m_y f Y_o + Z_o y_0 \\ Z_o \end{bmatrix} = \begin{bmatrix} m_x f & 0 & x_0 & 0 \\ 0 & m_y f & y_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

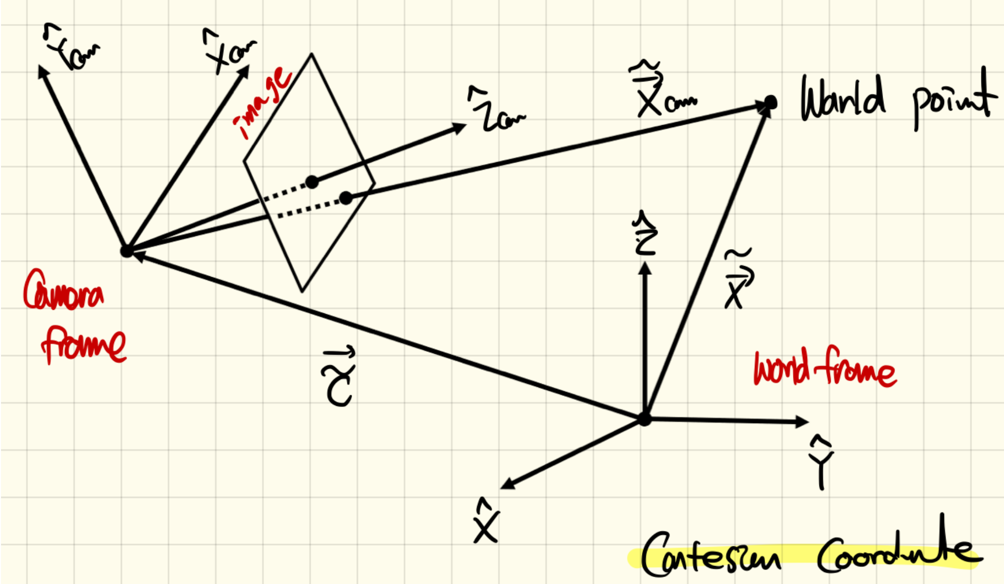
$$\begin{aligned} \vec{x} &= \mathbf{K} [\ \mathbf{I} \mid \vec{0}\ ] \vec{X} \\ &= \begin{bmatrix} m_x f & 0 & x_0 & 0 \\ 0 & m_y f & y_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vec{X} \end{aligned}$$

We do not know physical locations where the point is mapped on the image sensor, but we know their pixel locations.

where $\mathbf{K} = \begin{bmatrix} m_x f & 0 & x_0 \\ 0 & m_y f & y_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{\alpha} & 0 & \boxed{x_0} \\ 0 & \boxed{\beta} & \boxed{y_0} \\ 0 & 0 & 1 \end{bmatrix}$

4 Unknowns

Camera Model : Full Configuration



$$\tilde{X}_{cam} = R(\tilde{X} - \tilde{C}).$$

R : (3×3) $\tilde{C}$: 3×1 ①

$$\vec{X}_{cam} = \begin{pmatrix} \tilde{X}_{cam} \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ \vec{0}^T & 1 \end{bmatrix} \begin{pmatrix} \tilde{X} \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ \vec{0}^T & 1 \end{bmatrix} \vec{X}$$

$$\vec{x}_{cam} = K[I \mid \vec{0}]\, \vec{X}_{cam} = K[I \mid \vec{0}] \begin{bmatrix} R & -R\tilde{C} \\ \vec{0}^T & 1 \end{bmatrix} \vec{X}$$

$\vec{X}_{cam} = R\vec{X}$ World frame → Camera frame. Rotation

$\vec{X}_{cam} = \tilde{X} - \tilde{C}$ World frame → Camera frame Translation.

We move the origin first and rotate the axes.

②

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & *^1 \\ R_{21} & R_{22} & R_{23} & *^2 \\ R_{31} & R_{32} & R_{33} & *^3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} & *^1 \\ R_{21} & R_{22} & R_{23} & *^2 \\ R_{31} & R_{32} & R_{33} & *^3 \end{bmatrix} = [R \mid -R\vec{C}]$$

where $\begin{pmatrix} *^1 \\ *^2 \\ *^3 \end{pmatrix} = -R\tilde{C}$

$$\boxed{\begin{aligned} \vec{x}_{cam} &= K[R \mid -R\vec{C}]\, \vec{X} \\ &= KR[I \mid -\vec{C}]\, \vec{X} \end{aligned}}$$

Camera Model : Full Configuration (Continue)

Translation

$$\vec{x}_{cam} = \mathbf{K}\,[\,\mathbf{I}\,|\,\vec{0}\,]\,\vec{X}$$
$$= \mathbf{K}\,[\,\mathbf{I}\,|\,\vec{0}\,]\,(\tilde{X} - \vec{C})$$

Translation & rotation

$$\vec{x}_{cam} = \mathbf{K}\,[\,\mathbf{I}\,|\,\vec{0}\,]\,\mathbf{R}\vec{X}$$
$$= \mathbf{K}\,[\,\mathbf{I}\,|\,\vec{0}\,]\,\mathbf{R}(\tilde{X} - \vec{C})$$

- **Rotation matrix (3)**
- **Camera center (3)**
- **Focal length (1~2)**
- **Principal points (2)**

9 (10) Unknowns

$$\vec{x}_{cam} = KR\,[\,I_{3\times3}\,|\,-\tilde{\vec{C}}\,]\,\vec{X}$$

Pixel Coordinate (homogeneous)

Camera internal parameter

World to Camera rotation (physical)

World to Camera translation (physical)

World point (homogeneous)

$$\vec{x}_{cam} = P\vec{X}.$$
$$= [p_1\ p_2\ p_3\ p_4]\,\vec{X}.$$

$$\mathbf{x} = \mathbf{K}[\mathbf{R}\quad \mathbf{t}]\mathbf{X}$$

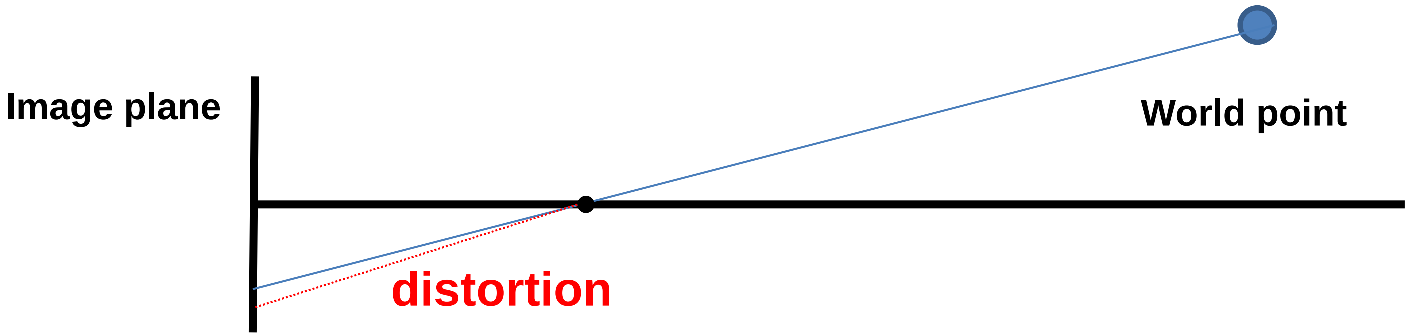
$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Radial distortion (due to optics of the lens)

$$r^2 = \|\mathbf{x}\|^2 = x^2 + y^2$$

$$\mathbf{x}' = (1 + \boxed{k_1} r^2 + \boxed{k_2} r^4)\mathbf{x}$$

before after



9 (10) + # of distortion parameters unknowns

Radial Distortion Model in MATLAB

Supplement

Any real world point $P(X, Y, Z)$ can be defined with respect to some 3-D world origin.

Relative to a camera lens, this 3-D point can be defined as p0, which is obtained by rotating and translating P.

$$p_0 = (x_0, y_0, z_0) = RP + t$$

The 3-D point p (0) is then projected into the camera's image plane as a 2D point, (x (1) , y (1)).

$$x_1 = \frac{x_0}{z_0}, \; y_1 = \frac{y_0}{z_0}$$

When a camera captures an image, it does not precisely capture the real points, but rather a slightly distorted version of the real points which can be denoted (x (2) , y (2)). The distorted points can be described using the following function:

$$x_2 = x_1 \left(1 + k_1 r^2 + k_2 r^4\right) + 2 p_1 x_1 y_1 + p_2 \left(r^2 + 2 {x_1}^2\right)$$

$$y_2 = y_1 \left(1 + k_1 r^2 + k_2 r^4\right) + 2 p_2 x_1 y_1 + p_1 \left(r^2 + 2 {y_1}^2\right)$$

where:

k (1) , k (2) = radial distortion coefficients of the lens

p (1) , p (2) = tangential distortion coefficients of the lens

r = $\sqrt{{x_1}^2 + {y_1}^2}$

https://www.mathworks.com/help/symbolic/examples/developing-an-algorithm-for-undistorting-an-image.html

Review All Terms

- Rotation matrix (3)
- Translation matrix (3)
- Focal length (2)
- Principal points (2)
- Lens distortion parameters (2~4)

Exterior (External)

Interior (Internal)

Example: Pix4D

Camera Model Name	Field used to type the camera model name. It is recommended to type the name as follow: *camera_name_focal_length_sensor_widthxsensor_height*
Image Width [pixel]	Image width in pixels.
Image Height [pixel]	Image height in pixels.
Focal Length [pixel]	Focal length in pixels (if defined in pixels, the mm value is automatically computed and added to the corresponding field).
Principal Point x [pixel]	Principal point x coordinate in pixels (if defined in pixels, the mm value is automatically computed and added to the corresponding field).
Principal Point y [pixel]	Principal point y coordinate in pixels (if defined in pixels, the mm value is automatically computed and added to the corresponding field).
Sensor Width [mm]	Sensor width in mm.
Sensor Height [mm]	Sensor height in mm.
Pixel Size [μm]	Pixel size in μm.
Focal Length [mm]	Focal length in mm (if defined in mm, the pixel value is automatically computed and added to the corresponding field).
Principal Point x [mm]	Principal point x coordinate in mm (if defined in mm, the pixel value is automatically computed and added to the corresponding field).
Principal Point y [mm]	Principal point y coordinate in mm (if defined in mm, the pixel value is automatically computed and added to the corresponding field).
Radial Distortion R1	Radial distortion R1 parameter of the lens (optional, it is recommended to leave the distortion parameters to 0).
Radial Distortion R2	Radial distortion R2 parameter of the lens (optional, it is recommended to leave the distortion parameters to 0).
Radial Distortion R3	Radial distortion R3 parameter of the lens (optional, it is recommended to leave the distortion parameters to 0).
Tangential Distortion T1	Tangential distortion T1 parameter of the lens (optional, it is recommended to leave the distortion parameters to 0).
Tangential Distortion T2	Tangential distortion T2 parameter of the lens (optional, it is recommended to leave the distortion parameters to 0).

Slide Credits and References

- Lecture notes: Robert Collins
- Hartley, Richard, and Andrew Zisserman. Multiple view geometry in computer vision. Cambridge university press, 2003.