Time-Frequency Analysis

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Vector Spaces

Definition (Vector space)

A vector space over $\mathbb{C}$ is a set V with addition and multiplication that satisfies $\forall u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$

1. $v + w = w + v$
2. $\alpha(\beta v) = (\alpha\beta)v$
3. $(v + w) + u = v + (w + u)$
4. $(\alpha + \beta)v = \alpha v + \beta v$
5. $\alpha(v + w) = \alpha v + \alpha w$
6. $v + 0 = v$
7. $v + (-v) = 0$
8. $1v = v$

Vector Spaces

Definition (Subspace)

A subspace is a nonempty subset of a vector space that is *closed under addition and scalar multiplication*, i.e., $S \subseteq V$ is a subspace of V if $\forall v, w \in S$ and $\alpha \in \mathbb{C}$

1. $v + w \in S$
2. $\alpha x \in S$

Norms

Definition (Norm)

A norm on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a real-valued function $\|\cdot\| : V \to \mathbb{R}$ with the following properties for any $v, w \in V$ and $\alpha \in \mathbb{C}$

1. $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$
2. $\|\alpha v\| = |\alpha| \|v\|$
3. $\|v + w\| \leq \|v\| + \|w\|$

A vector space endowed with a norm is called *normed vector space*.

Remark

1. $\|v - w\| \geq |\|v\| - \|w\||$
2. $\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$

Norms

Definition (Convergence in normed spaces)

A sequence of vectors $(v_0, v_1, \cdots)$ in a normed vector space V is said to converge to $v \in V$ when $\lim_{k \to +\infty} \|v - v_k\| = 0$, i.e., given $\epsilon > 0$, there exists a $K = K(\epsilon)$ such that

$$\|v - v_k\| < \epsilon, \ \forall k > K.$$

Norms

Definition (Cauchy sequence)

A sequence of vectors $(v_0, v_1, \cdots)$ in a normed vector space is called a Cauchy sequence when given $\epsilon > 0$, there exists a $K = K(\epsilon)$ such that

$$\|v_k - v_m\| < \epsilon, \ \forall k, m > K.$$

Lemma (Convergent sequences are Cauchy)

Assume that V is a normed vector space, and that $(v_0, v_1, \cdots)$ is a convergent sequence in V. Then $(v_0, v_1, \cdots)$ is a Cauchy sequence.

Banach Spaces

Definition (Banach space)

A normed vector space V with the property that each Cauchy sequence $(v_0, v_1, \cdots)$ in V converges toward some $v \in V$, is called a Banach space.

Banach Spaces

Examples:

- ℓ_p spaces

- L_p spaces

Inner Product Spaces

Definition (Inner product space)

An inner product of a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-valued (or real-valued) function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with the following properties for any $v, w, u \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$)

1. $\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, u \rangle$
2. $\langle v, w \rangle = \langle w, v \rangle^*$
3. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

Inner Product Spaces

Theorem (Cauchy-Schwarz' inequality)

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. Then,

$$|\langle v, w \rangle| \leq \langle v, v \rangle^{1/2} \langle w, w \rangle^{1/2}, \; \forall v, w \in V.$$

Inner Product Spaces

Lemma (Inner products induces the norm)

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. Then,

$$\|v\| = \langle v, v \rangle^{1/2}, v \in V,$$

defines a norm on V.

Hilbert Space

Definition (Hilbert space)

A vector space with an inner product $\langle \cdot, \cdot \rangle$, which is a Banach space with respect to $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$ is called a Hilbert space.

Hilbert Space

Examples:

- ℓ_2 space

- L_2 space

Orthogonality

Definition (Orthogonality)

Let H be a Hilbert space.

1. Two elements $v, w \in H$ are *orthogonal* if $\langle v, w\rangle = 0$ and we write $v \perp w$
2. A collection of vectors $\{v_k\}_{k\in\mathbb{N}}$ in H is an *orthogonal system* if $\langle v_k, v_\ell\rangle = 0,\ \forall k \neq \ell$
3. An orthogonal system $\{v_k\}_{k\in\mathbb{N}}$ for which $\|v_k\| = 1,\ \forall k \in \mathbb{N}$ is called an *orthonormal system*

Basis

Definition (Basis)

A set of vectors $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset V$, where $\mathcal{K}$ is countable, is called a *basis* for a normed vector space V when

- it is complete in V, i.e., for any $f \in V$, there exists a sequence $\alpha : \mathcal{K} \to \mathbb{C}$ such that

$$f = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k,$$

- for any $f \in V$, the sequence α is unique.

Basis

Definition (Orthonormal Basis)

A set of vectors $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$, where $\mathcal{K}$ is countable, is called a *orthonormal basis* for the Hilbert space H when

- it is a basis for H, and
- it is an orthonormal set, i.e., $\langle \varphi_i, \varphi_k \rangle = \delta_{i-k} \; \forall i, k \in \mathcal{K}$.

Basis

Theorem (Orthogonal Basis Expansion)

Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ *be an orthonormal basis for a Hilbert spaces* H. *The unique expansion expansion coefficients for any* $f \in H$ *are given by*

$$\alpha_k = \langle f, \varphi_k \rangle.$$

Synthesis with these coefficients yield

$$f = \sum_{k\in\mathcal{K}} \langle f, \varphi_k \rangle \varphi_k.$$

Basis

Theorem (Parseval Equality)

Let $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ *be an orthonormal basis for a Hilbert spaces* H*. The expansion coefficients satisfies the Parseval equality*

$$\|f\|^2 = \sum_{k \in \mathcal{K}} |\langle f, \varphi_k \rangle|^2 = \|\alpha\|^2.$$

The generalised Parseval equality:

$$\langle f, g \rangle = \langle \alpha, \beta \rangle.$$

Basis

Examples:

- Discrete Fourier basis for $\mathbb{C}^N$

- Fourier basis for $L_2([-\pi, \pi])$

Bandlimited Signals

Definition (Paley-Wiener space)

A function $f \in L_2(\mathbb{R})$ is bandlimited if its Fourier transform $\hat{f}$ has compact support. The Paley-Wiener space is the prototype bandlimited space, defined as

$$\mathrm{PW} := \left\{ f \in L_2(\mathbb{R}) : \mathsf{supp}\ \hat{f} \subset [-\pi, \pi] \right\},$$

Bandlimited Signals

Lemma

If $f \in \mathrm{PW}$, *then* $\hat{f} \in L_1(\mathbb{R})$

Bandlimited Signals

Theorem (Continuity of functions in PW)

Let $f \in \mathrm{PW}$. Then, f is equivalent to a continuous function.

Basis for PW

Theorem (Shannon sampling theorem)

The set $\{\mathrm{sinc}(\cdot - n)\}_{n\in\mathbb{Z}}$ *form an orthonormal basis for* PW. *If* $f \in \mathrm{PW}$, *then:*

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(t-n),$$

with two interpretations of convergence of the infinite series:

1. *The symmetric partial sum converges pointwise,*

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n)\,\mathrm{sinc}(t-n) = f(t),\ \forall t \in \mathbb{R}.$$

2. *The symmetric partial sum converges in* $L_2(\mathbb{R})$,

$$\lim_{N\to\infty} \int_{\mathbb{R}} \left| f(t) - \sum_{n=-N}^{N} f(n)\,\mathrm{sinc}(t-n) \right|^2 \mathrm{d}t = 0.$$

Basis for PW

- Orthogonality

Basis for PW

- Sampling and Fourier analysis

Shannon Sampling

Theorem (Shannon sampling theorem)

If a function f contains no frequencies higher than γ_N, it is completely described by giving its ordinates at a series of points spaced $\dfrac{1}{2\gamma_N}$ apart.

Maximally Bandlimited Signals

Theorem (Optimality of Shannon Sampling)

Suppose $f \notin \mathrm{PW}$, $\Pi_{\mathrm{PW}} f = \arg \min\limits_{g \in \mathrm{PW}} \|f - g\|_2^2$

$$\Pi_{\mathrm{PW}} f(t) = \sum_{n \in \mathbb{Z}} \langle f, \mathrm{sinc}(\cdot - n) \rangle \, \mathrm{sinc}(t - n).$$