MACHINE LEARNING - I

LINEAR ALGEBRA - ASSIGNMENT - I

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1. Exercise: 1.17

GAUSS METHOD:

(a) Given,

Linear equations

$$① \Rightarrow 2x + 3y = 13$$
$$② \Rightarrow x - y = -1$$

Step 1:

$$① \Rightarrow \cancel{2x} + 3y = 13$$
$$2 \times ② \Rightarrow \cancel{2x} - 2y = -2$$
$$(-) \quad (+) \quad\quad (+)$$

$$5y = 15$$

$$y = \frac{15}{5} = \boxed{3}$$

Step 2:

Substitute $y = 3$ in ②

$$x - 3 = -1$$
$$x = 3 - 1 = 2$$
$$x = \boxed{2}$$

Solution for x, y is $\boxed{x=2}$, $\boxed{y=3}$

(b) GAUSS METHOD

Given,

$$① \Rightarrow \quad x \quad -z = 0$$
$$② \Rightarrow \quad 3x + y = 1$$
$$③ \Rightarrow \quad -x + y + z = 4$$

Steps:

$$① + ③ \Rightarrow \quad \cancel{x} - \cancel{z} = 0$$
$$-\cancel{x} + y + \cancel{z} = 4$$
$$\boxed{y = 4}$$

Step 2:

substitute $y = 4$ in ②

$$3x + 4 = 1$$
$$3x = 1 - 4 = -3$$
$$\boxed{x = -1}$$

Step 3:

Substitute $x = -1$ in ①

$$-1 - z = 0$$
$$\boxed{z = -1}$$

Solution for x, y, z is

$\boxed{x = -1}$, $\boxed{y = 4}$, $\boxed{z = -1}$

2. Exercise 1.18

classify the system in echelon form has unique solution, no solution or infinitely many solutions

(a) Given,

$$-3x + 2y = 0$$
$$-2y = 0$$

Above equations are → not contradictory
→ both the variables x and y are leading in the rows.

Hence, this system has $\boxed{\text{unique Solution}}$.

solution : $\boxed{x = 0, y = 0}$

(b) Given

$$x + y = 4$$
$$y - z = 0$$

In above two equations, x, y has leading rows but variables z does not have a leading row,
hence these system of equations can have $\boxed{\text{infinitely many solutions}}$.

Example : possible solution could be $\boxed{x = 2, y = 2, z = 2}$ $\boxed{x = 3, y = 1, z = 1}$ & so on etc.)

(c) Given,

$$x + y = 4$$
$$y - z = 0$$
$$0 = 0$$

Variables x, y have leading rows in the system of these equations but z does not have. Hence we can say that this has $\boxed{\text{infinitely Many Solutions}}$. The $\boxed{0=0}$ does not change this to unique solution. Hence this is same as previous problem.

(d)
$$x + y = 4$$
$$0 = 4$$

Clearly, the above system of equation are contradictory. That is there is no way possible for x and y to sum to 4 and also sum to 0 at the same time. Hence this system of equations has $\boxed{\text{No Solution}}$

(e)
$$3x + 6y + z = -0.5$$
$$-z = 2.5$$

variables x and z have leading rows by y does not have, hence this system of equation may have $\boxed{\text{infinitely many solutions}}$.

(f) $x - 3y = 2$

$0 = 0$

Variable x has leading row but variable y does not have. Hence this set of equations may have $\boxed{\text{infinitely many solutions}}$

Example $\Rightarrow$ $\boxed{x = 5, y = 1}$, $\boxed{x = 8, y = 2}$ etc & so on.

(g) $2x + 2y = 4 \Rightarrow 2(x + y = 2)$.

$y = 1$

$0 = 4$.

The above system of equation are contradictory. 2x + 2y cannot equate to 0 and 4 at the same time with $y = 1$. Hence this system has $\boxed{\text{no solution}}$

(h) $2x + y = 0$

Variable x has a leading row, but y does not hence the system can have $\boxed{\text{infinitely many solutions}}$,

Example $\Rightarrow$ $\boxed{x = 0, y = 0}$, $\boxed{x = 1, y = -2}$ etc & so on.

(i) $x - y = -1$

$0 = 0$

$0 = 4$

The above system of equation is contradictory, hence this has $\boxed{\text{no solution}}$.

(j) $x + y - 3z = -1$

$y - z = 2$

$z = 0$

$0 = 0$

$\boxed{x = -3, y = 2, z = 0}$

The above system is not contradictory & all the variables have leading rows hence, this has $\boxed{\text{unique solution}}$.

3. Exercise 2.19 :

Solve using matrix, give vector notation set

(a) Given

$$① \Rightarrow 2x + y - z = 1$$
$$② \Rightarrow 4x - y = 3$$

In Matrix notation,

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 4 & -1 & 0 & 3 \end{array}\right)$$

Reducing ~~Solving~~ ① & ②

$$\left.\begin{array}{l} ① \Rightarrow 2x + y - z = 1 \\ ② \Rightarrow 4x - y = 3 \end{array}\right\}$$ equation, now reducing the matrix $\Rightarrow$ multiply first row by -2 and adding row 2.

$② \Rightarrow ② - 2①$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ -4+4 & -2-1 & 2+0 & -2+3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -3 & 2 & 1 \end{array}\right)$$

From above, we can see that there are leading rows for both variable x & y, hence solve for y & x in terms of z.

Solving for y,

$$-3y + 2z = 1$$
$$-1 + 2z = 3y$$
$$y = \frac{-1}{3}(1-2z) = \frac{2}{3}z - \frac{1}{3}.$$

Solving for x,

$$2x + y - z = 1$$
$$2x - \frac{1}{3} + \frac{2}{3}z - z = 1$$
$$6x - 1 + 2z - 3z = 3$$
$$6x - z = 4.$$
$$6x = 4 + z.$$
$$x = \frac{4+z}{6} = \frac{2}{3} + \frac{1}{6}(z).$$

Therefore; Solution set in terms of z.

$$\boxed{\left\{ \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/6 \\ 2/3 \\ 1 \end{pmatrix} z \;\middle|\; z \in R \right\}}$$

(b). Given,

① $\Rightarrow$ $x - z = 1$

② $\Rightarrow$ $y + 2z - w = 3$

③ $\Rightarrow$ $x + 2y + 3z - w = 7$

In matrix form

$$\begin{array}{c} ① \Rightarrow \\ ② \Rightarrow \\ ③ \Rightarrow \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 1 & 2 & 3 & -1 & 7 \end{array}\right)$$

Reduction by Gauss method.

Subtracting Row (1) from Row (3) in place for Row (3)

$$(3)-(1) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 2 & 4 & -1 & 6 \end{array}\right)$$

Again, Subtracting Row 3 from 2 time of Row (2) to reduce $\boxed{\text{Row } 3}$

$$(3) - 2\times(2) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Now x, y, and w have leading rows, so we can represent other variables in terms of z

$x - z = 1$

$\boxed{x = 1 + z}$

$y + 2z - w = 3$

$y + 2z = 3$.

$2z = 3 - y$

$\boxed{z = \frac{3}{2} - \frac{1}{2}y}$

$\boxed{w = 0}$

$\boxed{y = 3 - 2z}$

$$\boxed{\text{Solution set: } \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z \;\middle|\; z \in \mathbb{R} \right\}}$$

(c) Given,

(1) $\Rightarrow$ $x - y + z = 0$

(2) $\Rightarrow$ $y + w = 0$

(3) $\Rightarrow$ $3x - 2y + 3z + w = 0$

(4) $\Rightarrow$ $-y - w = 0$

In matrix form,

$$\begin{array}{l} ① \Rightarrow \\ ② \Rightarrow \\ ③ \Rightarrow \\ ④ \Rightarrow \end{array} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 3 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{array}\right)$$

Reducing using Gauss method,

$$③ \Rightarrow ③ + -3 \times ① \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{array}\right)$$

now, $③ \Rightarrow ③ - ②$ & $④ \Rightarrow ④ + ②$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

After reducing we have leading rows for x & y and no leading rows for z & w.

$x - y + z = 0$

$x + w + z = 0$

$x = -w - z$.

$y + w = 0$.

$y = -w$

$$\boxed{\text{Solution set } \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} w \;\middle|\; z, w \in \mathbb{R} \right\}}$$

(d) Given;

① => $a + 2b + 3c + d - e = 1$

② => $3a - b + c + d + e = 3$

In matrix form,

$$\begin{array}{l} ① \Rightarrow \\ ② \Rightarrow \end{array} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & -1 & 1 \\ 3 & -1 & 1 & 1 & 1 & 3 \end{array}\right)$$

② => ② + $-3 \times$ ①

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & -1 & 1 \\ 0 & -7 & -8 & -2 & 4 & 0 \end{array}\right)$$

variables a, b has leading count, so now we use c, d, e to form solution set:

$a + 2b + 3c + d - e = 1$

$-7b - 8c - 2d + 4e = 0$

$$b = \frac{(8c + 2d - 4e)}{-7}$$

$$a + 2\left(\frac{8c + 2d - 4e}{-7}\right) + 3c + d - e = 1$$

$-7a + 16c + 4d - 8e - 21c - 7d + 7e = -7$

$-7a - 5c - 3d - e = -7$

$-7a = -7 + 5c + 3d + e$

$a = 1 + \left(-\frac{1}{7}\right)(5c) - \frac{3}{7}d - \frac{e}{7}$

Solution set :

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -5/7 \\ -8/7 \\ 1 \\ 0 \\ 0 \end{pmatrix} c + \begin{pmatrix} -3/7 \\ -2/7 \\ 0 \\ 1 \\ 0 \end{pmatrix} d + \begin{pmatrix} -1/7 \\ -2/7 \\ 0 \\ 1 \\ 0 \end{pmatrix} e \;\middle|\; c, d, e \in \mathbb{R} \right\}}$$

Exercise 3.23

find characteristic equation and the eigen values & vectors

To find the characteristic equation, we bring to the following determinant matrix format,

$$\begin{vmatrix} 7-x & a \\ a & 7-x \end{vmatrix} = 0$$

(a) $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} 3-x & 0 \\ 8 & -1-x \end{vmatrix} = 0$$

$$\boxed{(3-x)(-1-x) = 0}$$

$\therefore x$ can have two roots, the eigen values can be

$\lambda_1 = 3, \lambda_2 = -1$.

Now, to find eigen vectors.

$$\begin{pmatrix} 3-x & 0 \\ 8 & -1-x \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we have two eigen values,

for $\lambda_1 = 3$,

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$8a_1 - 4a_2 = 0.$$

$$8a_1 = 4a_2.$$

$$a_1 = \frac{a_2}{2}.$$

Eigen space is $\left\{ \begin{pmatrix} a_2/2 \\ a_2 \end{pmatrix} \middle| a_2 \in C \right\}$

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} a_2/2 \\ a_2 \end{pmatrix} = 3 \begin{pmatrix} a_2/2 \\ a_2 \end{pmatrix}$$

The eigen vector associated with $\lambda_1 = 3$ is $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$\boxed{\text{Eigen vector ie; } \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

for $\lambda_2 = -1.$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4a_1 = 0$$

$$8a_1 = 0.$$

Eigen space is $\left\{ \begin{pmatrix} 0 \\ a_2 \end{pmatrix} \middle| a_2 \in C \right\}$

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ a_2 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ a_2 \end{pmatrix}$$

The eigen vector associated with $\boxed{\lambda_1 = -1 \text{ is } \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Exercise 3.23

(b) $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

Characteristic equation

$$\begin{vmatrix} 3-x & 2 \\ -1 & 0-x \end{vmatrix} = 0$$

$$(3-x)(-x) - (-2) = 0$$

$$-3x + x^2 + 2 = 0$$

$$\boxed{(x-2)(x-1) = 0}$$

Eigen values $\boxed{\lambda_1 = 2, \lambda_2 = 1}$

for $\lambda_1 = 2$,

$$\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a_1 + 2a_2 = 0$$

$$-a_1 - 2a_2 = 0$$

$\boxed{\text{Eigenspace } \left\{ \begin{pmatrix} -2a_2 \\ a_2 \end{pmatrix} \middle| a_2 \in \mathbb{C} \right\} \text{ is eigen vector } \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

for $\lambda_2 = 1$, $\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$2a_1 + 2a_2 = 0$$

$$-a_1 - a_2 = 0$$

$\boxed{\text{Eigenspace } \left\{ \begin{pmatrix} -a_2 \\ a_2 \end{pmatrix} \middle| a_2 \in \mathbb{C} \right\} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ is eigen vector}}$

Exercise 3.2 6

find characteristic equation of eigen value of associated eigen vector

(a) $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Characteristic equation,

$$\begin{vmatrix} 3-x & -2 & 0 \\ -2 & 3-x & 0 \\ 0 & 0 & 5-x \end{vmatrix} = 0$$

$$(3-x)(3-x)(5-x) = 0$$

$$(9-3x-3x+x^2)(5-x) = 0$$

$$45-15x-15x+15x^2-x^3+3x^2+3x^2-9x=0$$

$$x^3-11x^2+35x-25=0$$

$$\boxed{(x-1)(x-5)^2 = 0}$$

Eigen values $\lambda_1 = 1$, $\lambda_2 = 5$

For $\lambda_1 = 1$,

$$\begin{pmatrix} 3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3a_1 - 2a_2 + 0 = 0$$

$$-2a_1 + 2a_2 = 0$$

$$4a_3 = 0$$

$\boxed{a_3 = 0}$, $\boxed{a_1 = a_2}$

$$\boxed{\text{Eigen space } \left\{ \begin{pmatrix} a_2 \\ a_2 \\ 0 \end{pmatrix} \middle| \, a_2 \in \mathbb{C} \right\} \text{ \& Eigen vector } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$\lambda = 5$,

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2a_1 - 2a_2 = 0$$
$$-2a_1 - 2a_2 = 0$$
$$a_1 = -a_2 \quad , \quad 0 \cdot a_3 = 0$$

$$\boxed{\therefore \text{ Eigen space } \left\{ \begin{pmatrix} -a_2 \\ a_2 \\ 0 \end{pmatrix} \middle| \, a_2 \in \mathbb{C} \right\} \text{ \& eigenvector } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

(b)

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

Characteristic equation,

$$\begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 4 & -17 & 8-x \end{vmatrix} = 0$$

$$(-x)\ (-x)(8-x) = 0$$
$$(x^2)(8-x) - (17x + 4)$$
$$-17x + 4 + 8x^2 - x^3 = 0$$

$$-x^3+8x^2-17x+4=0$$

$$-1(x-4)(x^2-4x+1)=0$$

Possible eigen values are $\lambda_1=4$, $\lambda_2=2+\sqrt{3}$, $\lambda_3=2-\sqrt{3}$.

$\lambda_1=4$)

$$\begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 4 & -17 & 4 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4a_1+a_2=0$$

$$-4a_2+a_3=0$$

$$4a_1-17a_2+4a_3=0$$

$$-4a_1+\frac{a_3}{4}=0$$

$$a_3=4a_2$$

$$\boxed{a_2=\tfrac{1}{4}a_3}$$

$$\frac{a_3}{4}=4a_1$$

$$\boxed{a_1=\frac{1}{16}a_3}$$

Eigen space is $\left\{\begin{pmatrix} a_3/16 \\ a_3/4 \\ a_3 \end{pmatrix} \middle| a_3\in\mathbb{C}\right\}$

A eigen vector is $\begin{pmatrix} 1/16 \\ 1/4 \\ 1 \end{pmatrix}$ ie $\begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$

$\lambda_2 = 2+\sqrt{3}$

$$\begin{pmatrix} -2-\sqrt{3} & 1 & 0 \\ 0 & -2-\sqrt{3} & 1 \\ 4 & -17 & 6-\sqrt{3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2a_1 - \sqrt{3}a_1 + a_2 = 0$$

$$-2a_2 - \sqrt{3}a_2 + a_3 = 0$$

$$4a_1 - 17a_2 + a_3 6 - \sqrt{3}a_3 = 0$$

$$a_2(-2-\sqrt{3}) + a_3 = 0$$

$$a_3 = (2+\sqrt{3})a_2$$

$$\boxed{a_2 = \frac{1}{(2+\sqrt{3})} a_3}$$

$$(-2-\sqrt{3})a_1 + a_2 = 0$$

$$a_2 = (2+\sqrt{3})a_1$$

$$\frac{a_3}{(2+\sqrt{3})} = (2+\sqrt{3})a_1$$

$$\boxed{a_1 = \frac{1}{(2+\sqrt{3})^2} a_3}$$

Hence,

Eigen space $= \left\{ \begin{pmatrix} 1/(2+\sqrt{3})^2 \, a_3 \\ 1/(2+\sqrt{3}) \, a_3 \\ a_3 \end{pmatrix} \middle| \, a_3 \in \mathbb{C} \right\}$

Eigen vector $\begin{pmatrix} 1/(2+\sqrt{3})^2 \\ 1/(2+\sqrt{3}) \\ \end{pmatrix}$

$\lambda_3 = 2-\sqrt{3}$

$$\begin{pmatrix} -2+\sqrt{3} & 1 & 0 \\ 0 & -2+\sqrt{3} & 1 \\ 4 & -17 & \cancel{6}0+\sqrt{3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2a_1 + \sqrt{3}a_1 + a_2 = 0$

$a_1(-2+\sqrt{3}) + a_2 = 0$

$(-2+\sqrt{3})a_2 + a_3 = 0$

$4a_1 - 17a_2 + (\cancel{6}0 + \sqrt{3})a_3 = 0$

$a_3 = (2-\sqrt{3})a_2$

$$\boxed{a_2 = \frac{a_3}{(2-\sqrt{3})}}$$

$a_2 = (2-\sqrt{3})a_1$

$a_1 = \frac{a_2}{(2-\sqrt{3})}$

$$\boxed{a_1 = \frac{a_3}{(2-\sqrt{3})^2}}$$

Eigen space is $\left\{ \begin{pmatrix} a_3/(2-\sqrt{3})^2 \\ a_3/(2-\sqrt{3}) \\ a_3 \end{pmatrix} \middle| a_3 \in \mathbb{C} \right\}$

Eigenvector is $\begin{pmatrix} 1/(2-\sqrt{3})^2 \\ 1/(2-\sqrt{3}) \\ 1 \end{pmatrix}$

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