SYLLABUS

Engineering Mathematics-I

Subject Code: 15MAT11 IA Marks: 20
Hours/Week: 04 Exam. Hours: 03
Total Hours: 50 Exam. Marks: 80

Course Objectives

To enable students to apply knowledge of Mathematics in various engineering fields by making hem to learn the following:

- nth derivatives of product of two functions and polar curves.
- Partial derivatives.
- Vectors calculus.
- Reduction formulae of integration to solve First order differential equations
- Solution of system of equations and quadratic forms.

Module –1

Differential Calculus -1:

Determination of nth order derivatives of Standard functions - Problems. Leibnitz's theorem (without proof) - problems.
Polar Curves - angle between the radius vector and tangent, angle between two curves, Pedal equation for polar curves. Derivative of arc length - Cartesian, Parametric and Polar forms (without proof) - problems. Curvature and Radius of Curvature – Cartesian, Parametric, Polar and Pedal forms(without proof) and problems. **10hrs**

Module –2

Differential Calculus -2

Taylor's and Maclaurin's theorems for function of o ne variable(statement only)- problems. Evaluation of Indeterminate forms.
Partial derivatives – Definition and simple problems, Euler's theorem(without proof) – problems, total derivatives, partial differentiation of composite functions-problems, Jacobians-definition and problems .
10hrs

Module –3

Vector Calculus:

Derivative of vector valued functions, Velocity, Acceleration and related problems, Scalar and Vector point functions.Definition Gradient, Divergence, Curl- problems . Solenoidal and Irrotational vector fields. Vector identities - div (F A), curl (F A),curl (grad F), div (curl A).
10hrs

Module- 4

Integral Calculus:

Reduction formulae $\int \sin^n x\, dx$ $\int \cos^n x\, dx$ $\int \sin^n x \cos^m x\, dx$,, (m and n are positive integers), evaluation of these integrals with standard limits (0 to л/2) and problems.

Differential Equations:

Solution of first order and first degree differential equations – Exact, reducible to exact and Bernoulli's differential equations. **Applications-** orthogonal trajectories in Cartesian and polar forms. Simple problems on Newton's law of cooling. **10hrs**

Module –5

Linear Algebra Rank of a matrix by elementary transformations, solution of system of linear equations - Gauss- elimination method, Gauss- Jordan method and Gauss-Seidel method. Rayleigh's power method to find the largest Eigen value and the corresponding Eigen vector. Linear transformation, diagonalisation of a square matrix, Quadratic forms, reduction to Canonical form **10hrs**

COURSE OUTCOMES

On completion of this course students are able to

- Use partial derivatives to calculate rates of change of multivariate functions
- Analyse position, velocity and acceleration in two or three dimensions using the calculus of vector valued functions
- Recognize and solve first order ordinary differential equations, Newton's law of cooling
- Use matrices techniques for solving systems of linear equations in the different areas of linear algebra.

Engineering Mathematics – I

MODULE I

DIFFERENTIAL CALCULUS-I

CONTENTS:

SUCCESSIVE DIFFERENTIATION

In this lesson, the idea of differential coefficient of a function and its successive derivatives will be discussed. Also, the computation of n^{th} derivatives of some standard functions is presented through typical worked examples.

1. Introduction:- Differential calculus (DC) deals with problem of calculating rates of change. When we have a formula for the distance that a moving body covers as a function of time, DC gives us the formulas for calculating the body's **velocity** and **acceleration** at any instant.

- **Definition of derivative of a function y = f(x):-**

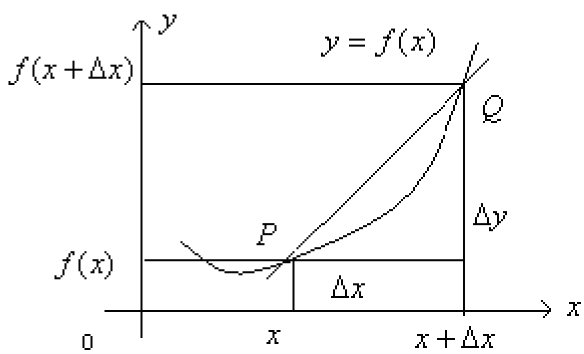


Fig.1. Slope of the line PQ is $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

The derivative of a function y = f(x) is the function $f'(x)$ whose value at each x is defined as

$$\frac{dy}{dx} = f'(x) = \text{Slope of the line PQ (See Fig.1)}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{-------- (1)}$$

$$= \lim_{\Delta x\to 0} \text{(Average rate change)}$$

= Instantaneous rate of change of f at x provided the limit exists.

The instantaneous velocity and acceleration of a body (moving along a line) at any instant *x* is the derivative of its position co-ordinate *y = f(x)* w.r.t x, i.e.,

$$\text{Velocity} = \frac{dy}{dx} = f'(x) \quad \text{--------- (2)}$$

And the corresponding acceleration is given by

$$\text{Acceleration} = \frac{d^2y}{dx^2} = f''(x) \quad \text{---------- (3)}$$

Successive Differentiation:-

The process of differentiating a given function again and again is called as **Successive differentiation** and the results of such differentiation are called **successive derivatives.**

- The higher order differential coefficients will occur more frequently in spreading a function all fields of scientific and engineering applications.
- Notations:
  i. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots$, n[th] order derivative: $\frac{d^ny}{dx^n}$
  ii $f'(x)$, $f''(x)$, $f'''(x)$, ……, n[th] order derivative: $f^n(x)$
  iii Dy, D^2y, D^3y, ……….., n[th] order derivative: D^ny
  iv y', y'', y''', ……, n[th] order derivative: $y^{(n)}$
  v. $y_1, y_2, y_3 \ldots$, n[th] order derivative: y_n
- <u>**Successive differentiation – A flow diagram**</u>

Input function: $y = f(x)$ $\longrightarrow$ Operation d/dx $\longrightarrow$ Output function $y' = \frac{df}{dx} = f'(x)$ (first order derivative)

Input function $y' = f'(x)$ $\longrightarrow$ Operation d/dx $\longrightarrow$ Output function $y'' = \frac{d^2f}{dx^2} = f''(x)$ (second order derivative)

Input function $y'' = f''(x)$ $\longrightarrow$ Operation d/dx $\longrightarrow$ Output function $y''' = \frac{d^3f}{dx^3} = f'''(x)$ (third order derivative)

Input function $y^{n-1} = f^{n-1}(x)$ $\longrightarrow$ Operation d/dx $\longrightarrow$ Output function $y^n = \frac{d^nf}{dx^n} = f^n(x)$ (nth order derivative)

Calculation of n^th derivatives of some standard functions

- Below, we present a table of n^th order derivatives of some standard functions for ready reference.

Sl. No	y = f(x)	$y_n = \frac{d^n y}{dx^n} = D^n y$
1	e^{mx}	$m^n e^{mx}$
2	a^{mx}	$m^n (\log a)^n a^{mx}$
3	$(ax+b)^m$	i. $m(m-1)(m-2)\ldots(m-n+1)a^n (ax+b)^{m-n}$ for all m. ii. 0 if $m<n$ iii. $(n!)a^n$ if $m=n$ iv. $\frac{m!}{(m-n)!}x^{m-n}$ if $m<n$
4	$\frac{1}{(ax+b)}$	$\frac{(-1)^n n!}{(ax+b)^{n+1}} a^n$
5.	$\frac{1}{(ax+b)^m}$	$\frac{(-1)^n (m+n-1)!}{(m-1)!(ax+b)^{m+n}} a^n$
6.	$\log(ax+b)$	$\frac{(-1)^{n-1}(n-1)!}{(ax+b)^n} a^n$
7.	$\sin(ax+b)$	$a^n \sin(ax+b+n\pi/2)$
8.	$\cos(ax+b)$	$a^n \cos(ax+b+n\pi/2)$
9.	$e^{ax}\sin(bx+c)$	$r^n e^{ax}\sin(bx+c+n\theta)$, $r=\sqrt{a^2+b^2}$ $\theta=\tan^{-1}(b/a)$
10.	$e^{ax}\cos(bx+c)$	$r^n e^{ax}\cos(bx+c+n\theta)$, $r=\sqrt{a^2+b^2}$ $\theta=\tan^{-1}(b/a)$

- We proceed to illustrate the proof of some of the above results, as only the above functions are able to produce a **<u>sequential change</u>** from one derivative to the other. Hence, in general we cannot obtain readymade formula for nth derivative of functions other than the above.

1. Consider e^{mx}. Let $y = e^{mx}$. Differentiating w.r.t x, we get

$y_1 = me^{mx}$. Again differentiating w.r.t x, we get

$y_2 = m(me^{mx}) = m^2 e^{mx}$

Similarly, we get

$y_3 = m^3 e^{mx}$

$y_4 = m^4 e^{mx}$

…………….

And hence we get

$y_n = m^n e^{mx}$ $\therefore \frac{d^n}{dx^n}\left[e^{mx}\right] = m^n e^{mx}$.

2. $(ax+b)^m$

let $y=(ax+b)^m$ Differentiating w.r.t x,

$y_1 = m(ax+b)^{m-1}a$. Again differentiating w.r.t x, we get

$y_2 = m(m-1)(ax+b)^{m-2}a^2$

Similarly, we get

$y_3 = m(m-1)(m-2)(ax+b)^{m-3}a^3$

…………………………………….

And hence we get

$y_n = m(m-1)(m-2)\ldots\ldots(m-n+1)(ax+b)^{m-n}a^n$ for all m.

Case (i) If $m=n$ (m-positive integer),then the above expression becomes

$$y_n = n(n-1)(n-2)\ldots\ldots3.2.1\,(ax+b)^{n-n}a^n$$

i.e. $y_n = (n!)a^n$

Case (ii) If m<n,(i.e. if n>m) which means if we further differentiate the above expression, the right hand site yields zero. Thus $D^n\left[(ax+b)^m\right]=0 \quad if\ (m<n)$

Case (iii) If m>n, then $y_n = m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n}a^n$ becomes

$$= \frac{m(m-1)(m-2)\ldots(m-n+1)(m-n)!}{(m-n)!}(ax+b)^{m-n}a^n$$

. i.e $y_n = \dfrac{m!}{(m-n)!}(ax+b)^{m-n}a^n$

3. $\dfrac{1}{(ax+b)^m}$

Let $y = \dfrac{1}{(ax+b)^m} = (ax+b)^{-m}$

Differentiating w.r.t x

$y_1 = -m(ax+b)^{-m-1}a = (-1)m(ax+b)^{-(m+1)}a$

$y_2 = (-1)m\left[-(m+1)\right](ax+b)^{-(m+1)-1}a = (-1)^2 m(m+1)(ax+b)^{-(m+2)}a^2$

Similarly, we get $y_3 = (-1)^3 m(m+1)(m+2)(ax+b)^{-(m+3)}a^3$

$y_4 = (-1)^4 m(m+1)(m+2)(m+3)(ax+b)^{-(m+4)}a^4$

………………………………

$y_n = (-1)^n m(m+1)(m+2)\ldots(m+n-1)(ax+b)^{-(m+n)}a^n$

This may be rewritten as

$$y_n = \frac{(-1)^n(m+n-1)(m+n-2)\ldots(m+1)m(m-1)!}{(m-1)!}(ax+b)^{-(m+n)}a^n$$

or $y_n = \dfrac{(-1)^n(m+n-1)!}{(m-1)!(ax+b)^{m+n}}a^n$

4. $\dfrac{1}{(ax+b)}$

Putting $m=1,$ in the result

$$D^n\left[\frac{1}{(ax+b)^m}\right]=\frac{(-1)^n(m+n-1)\,!}{(m-1)!(ax+b)^{m+n}}a^n$$

we get $D^n\left[\dfrac{1}{(ax+b)}\right]=\dfrac{(-1)^n(1+n-1)\,!}{(1-1)!(ax+b)^{1+n}}a^n$

or $D^n\left[\dfrac{1}{(ax+b)}\right]=\dfrac{(-1)^n n\,!}{(ax+b)^{1+n}}a^n$

Find the nth derivative of the following examples

1. (a) $\log(9x^2-1)$ (b) $\log\left[(4x+3)e^{5x+7}\right]$ (c) $\log_{10}\sqrt{\dfrac{(3x+5)^2(2-3x)}{(x+1)^6}}$

Sol: (a) Let $y=\log(9x^2-1)=\log\left\{(3x+1)(3x-1)\right\}$

$$y=\log(3x+1)+\log(3x-1) \qquad (\because\ \log(AB)=\log A+\log B)$$

$$\therefore\ y_n=\frac{dn}{dx^n}\left\{\log(3x+1)\right\}+\frac{dn}{dx^n}\left\{\log(3x-1)\right\}$$

$$\text{i.e } y_n=\frac{(-1)^{n-1}(n-1)!}{(3x+1)^n}(3)^n+\frac{(-1)^{n-1}(n-1)!}{(3x-1)^n}(3)^n$$

(b) Let $y=\log\left[(4x+3)e^{5x+7}\right]=\log(4x+3)+\log e^{5x+7}$

$$=\log(4x+3)+(5x+7)\log_e e \qquad (\because\ \log A^B=B\log A)$$

$$\therefore\ y=\log(4x+3)+(5x+7) \qquad (\because\ \log_e e=1)$$

$$\therefore\ y_n=\frac{(-1)^{n-1}(n-1)!}{(4x+3)^n}(4)^n+0$$

$$\left(\begin{array}{l}\because\ D(5x+6)=5\\ D^2(5x+6)=0\\ D^n(5x+1)=0\ (n>1)\end{array}\right)$$

(c) Let $y=\log_{10}\sqrt{\dfrac{(3x+5)^2(2-3x)}{(x+1)^6}}$

$$=\frac{1}{\log_e 10}\left\{\sqrt{\frac{(3x+5)^2(2-3x)}{(x+1)^6}}\right\} \qquad \because\ \log_{10}X=\frac{\log_e X}{\log_e 10}$$

$$= \frac{1}{\log_e 10}\left\{\frac{1}{2}\log\left\{\frac{(3x+5)^2(2-3x)}{(x+1)^6}\right\}\right\} \qquad \because \log A^B = B\log A$$

$\because \log\left(\frac{A}{B}\right) = \log A - \log B$

$$= \frac{1}{2\log_e 10}\left\{\log(3x+5)^2 + \log(2-3x) - \log(x+1)^6\right\}$$

$$\therefore \quad y = \frac{1}{2\log_e 10}\left\{2\log(3x+5) + \log(2-3x) - 6\log(x+1)\right\}$$

Hence,

$$y_n = \frac{1}{2\log_e 10}\left\{2.\frac{(-1)^{n-1}(n-1)!}{(3x+5)^n}(3)^n + \frac{(-1)^{n-1}(n-1)!}{(2-3x)^n}(-3)^n - 6.\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}(1)^n\right\}$$

2. (a) $e^{2x+4} + 6^{2x+4}$ (b) $\cosh 4x + \cosh^2 4x$

(c) $e^{-x}\sinh 3x \cosh 2x$ (d) $\frac{1}{(4x+5)} + \frac{1}{(5x+4)^4} + (6x+8)^5$

Sol: (a) Let $y = e^{2x+4} + 6^{2x+4}$

$$= e^{2x}e^4 + 6^{2x}6^4$$

$$\therefore \quad y = e^4(e^{2x}) + 1296(6^{2x})$$

hence $$y_n = e^4\frac{dn}{dx^n}(e^{2x}) + 1296\frac{dn}{dx^n}(6^{2x})$$

$$= e^4\left\{2^n e^{2x}\right\} + 1296\left\{2^n(\log 6)^n 6^{2x}\right\}$$

(b) Let $y = \cosh 4x + \cosh^2 4x$

$$= \left(\frac{e^{4x}+e^{-4x}}{2}\right) + \left(\frac{e^{4x}+e^{-4x}}{2}\right)^2$$

$$= \frac{1}{2}\left\{e^{4x}+e^{-4x}\right\} + \frac{1}{4}\left\{(e^{4x})^2 + (e^{-4x})^2 + 2(e^{4x})(e^{-4x})\right\}$$

$$y = \frac{1}{2}\left\{e^{4x}+e^{-4x}\right\} + \frac{1}{4}\left\{e^{8x}+e^{-8x}+2\right\}$$

hence, $$y_n = \frac{1}{2}\left[4^n e^{4x} + (-4)^n e^{-4x}\right] + \frac{1}{4}\left[8^n e^{8n} + (-8)^n e^{-8n} + 0\right]$$

(c) Let $y = e^{-x}\sinh 3x \cosh 2x$

$$= e^{-x}\left\{\frac{e^{3x}-e^{-3x}}{2}\right\}\left\{\frac{e^{2x}+e^{-2x}}{2}\right\}$$

$$= \frac{e^{-x}}{4}\left\{(e^{3x}-e^{-3x})(e^{2x}+e^{-2x})\right\}$$

$$= \frac{e^{-x}}{4}\left\{ e^{5x} - e^{-x} + e^{x} - e^{-5x} \right\}$$

$$= \frac{1}{4}\left\{ e^{4x} - e^{-2x} + 1 - e^{-6x} \right\}$$

$$y = \frac{1}{4}\left\{ 1 + e^{4x} - e^{-2x} - e^{-6x} \right\}$$

Hence, $$y_n = \frac{1}{4}\left\{ 0 + (4)^n e^{4x} - (-2)^n e^{-2x} - (-6)^n e^{-6x} \right\}$$

(d) Let $y = \dfrac{1}{(4x+5)} + \dfrac{1}{(5x+4)^4} + (6x+8)^5$

Hence, $$y_n = \frac{dn}{dx^n}\left\{\frac{1}{(4x+5)}\right\} + \frac{dn}{dx^n}\left\{\frac{1}{(5x+4)^4}\right\} + \frac{dn}{dx^n}\left\{ (6x+8)^5 \right\}$$

$$= \frac{(-1)^n n!}{(4x+5)^{n+1}}(4)^n + \frac{(-1)^n (4+n-1)!}{(4-1)!(5x+4)^{4+n}}(5)^n + 0$$

i.e $$y_n = \frac{(-1)^n n!}{(4x+5)^{n+1}}(4)^n + \frac{(-1)^n (3+n)!}{3!(5x+4)^{n+4}}(5)^n$$

Evaluate

1. (i) $\dfrac{1}{x^2-6x+8}$ (ii) $\dfrac{1}{1-x-x^2+x^3}$ (iii) $\dfrac{x^2}{2x^2+7x+6}$

(iv) $\left(\dfrac{x+2}{x+1}\right) + \dfrac{1}{4x^2+12x+9}$ (v) $\tan^{-1}\left(x/a\right)$ (vi) $\tan^{-1} x$ (vii) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

Sol: (i) Let $y = \dfrac{1}{x^2-6x+8}$. The function can be rewritten as $y = \dfrac{1}{(x-4)(x-2)}$

This is proper fraction containing two distinct linear factors in the denominator.

So, it can be split into partial fractions as

$$y = \frac{1}{(x-4)(x-2)} = \frac{A}{(x-4)} + \frac{B}{(x-2)}$$ Where the constant A and B are found

as given below.

$$\frac{1}{(x-4)(x-2)} = \frac{A(x-2)+B(x-4)}{(x-4)(x-2)}$$

$$\therefore \quad 1 = A(x-2) + B(x-4) \text{ -------------(*)}$$

Putting x = 2 in (*), we get the value of B as $B = -\frac{1}{2}$

Similarly putting x = 4 in (*), we get the value of A as $A = \frac{1}{2}$

$$\therefore\ y = \frac{1}{(x-4)(x-2)} = \frac{(1/2)}{x-4} + \frac{(-1/2)}{x-2} \quad \text{Hence}$$

$$y_n = \frac{1}{2}\frac{d_n}{dx^n}\left(\frac{1}{x-4}\right) - \frac{1}{2}\frac{d_n}{dx^n}\left(\frac{1}{x-2}\right)$$

$$= \frac{1}{2}\left[\frac{(-1)^n n!}{(x-4)^{n+1}}(1)^n\right] - \frac{1}{2}\left[\frac{(-1)^n n!}{(x-2)^{n+1}}(1)^n\right]$$

$$= \frac{1}{2}(-1)^n n!\left[\frac{1}{(x-4)^{n+1}} - \frac{1}{(x-2)^{n+1}}\right]$$

(ii) Let $y = \frac{1}{1-x-x^2+x^3} = \frac{1}{(1-x)-x^2(1-x)} = \frac{1}{(1-x)(1-x^2)}$

ie $y = \frac{1}{(1-x)(1-x)(1+x)} = \frac{1}{(1-x)^2(1+x)}$

Though y is a proper fraction, it contains a repeated linear factor $(1-x)^2$ in its

denominator. Hence, we write the function as

$y = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{1+x}$ in terms of partial fractions. The constants A, B, C

are found as follows:

$$y = \frac{1}{(1-x)^2(1+x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{1+x}$$

ie $1 = A(1-x)(1+x) + B(1+x) + C(1-x)^2$ -------------(**)

Putting x = 1 in (**), we get B as $B = \frac{1}{2}$

Putting x = -1 in (**), we get C as $C = \frac{1}{4}$

Putting x = 0 in (**), we get $1 = A + B + C$

$\therefore\ A = 1 - B - C = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore\ A = \frac{1}{4}$

Hence, $y = \frac{(1/4)}{(1-x)} + \frac{(1/2)}{(1-x)^2} + \frac{(1/4)}{(1+x)}$

$$\therefore\ y_n = \frac{1}{4}\left[\frac{(-1)^n n!}{(1-x)^{n+1}}(1)^n\right] + \frac{1}{2}\left[\frac{(-1)^n (2+n-1)!}{(2-1)!(1-x)^{2+n}}(1)^n\right] + \frac{1}{4}\left[\frac{(-1)^n n!}{(1+x)^{n+1}}(1)^n\right]$$

$$= \frac{1}{4}(-1)^n n!\left[\frac{1}{(1-x)^{n+1}} + \frac{1}{(1+x)^{n+1}}\right] + \frac{1}{2}\left[\frac{(-1)^n (n+1)!}{(1-x)n+2}\right]$$

(iii) Let $y = \dfrac{x^2}{2x^2+7x+6}$ **(VTU July-05)**

This is an improper function. We make it proper fraction by actual division and later

spilt that into partial fractions.

i.e $x^2 \div (2x^2+7x+6) = \dfrac{1}{2} + \dfrac{(-\frac{7}{2}x-3)}{2x^2-7x+6}$

$\therefore\ y = \dfrac{1}{2} + \dfrac{-\frac{7}{2}x-3}{(2x+3)(x+2)}$ Resolving this proper fraction into partial fractions,

we get

$y = \dfrac{1}{2} + \left[\dfrac{A}{(2x+3)} + \dfrac{B}{(x+2)}\right]$. Following the above examples for finding A & B, we get

$$y = \frac{1}{2} + \left[\frac{\frac{9}{2}}{2x+3} + \frac{(-4)}{x+2}\right]$$

Hence, $y_n = 0 + \dfrac{9}{2}\left[\dfrac{(-1)^n n!}{(2x+3)^{n+1}}(2)^n\right] - 4\left[\dfrac{(-1)^n n!}{(x+2)^{n+1}}(1)^n\right]$

i.e $y_n = (-1)^n n!\left[\dfrac{\frac{9}{2}(2)^n}{(2x+3)^{n+1}} - \dfrac{4}{(x+2)^{n+1}}\right]$

(iv) Let $y = \underbrace{\dfrac{(x+2)}{(x+1)}}_{\text{(i)}} + \underbrace{\dfrac{x}{4x^2+12x+9}}_{\text{(ii)}}$

Here (i) is improper & (ii) is proper function. So, by actual division (i) becomes

$\left(\dfrac{x+2}{x+1}\right) = 1 + \left(\dfrac{1}{x+1}\right)$. Hence, y is given by

$y = 1 + \left(\dfrac{1}{x+1}\right) + \dfrac{1}{(2x+3)^2}$ $[\because (2x+3)^2 = 4x^2+12x+9]$

Resolving the last proper fraction into partial fractions, we get

$\dfrac{x}{(2x+3)^2} = \dfrac{A}{(2x+3)} + \dfrac{B}{(2x+3)^2}$. Solving we get

$$A = \tfrac{1}{2} \quad \text{and} \quad B = -\tfrac{3}{2}$$

$$\therefore \; y = 1 + \left(\frac{1}{1+x}\right) + \left[\frac{\tfrac{1}{2}}{(2x+3)} + \frac{-\tfrac{3}{2}}{(2x+3)^2}\right]$$

$$\therefore \; y_n = 0 + \left[\frac{(-1)^n n!}{(1+x)^n}(1)^n\right] + \frac{1}{2}\left[\frac{(-1)^n n!}{(2x+3)^{n+1}}(2)^n\right] - \frac{3}{2}\left[\frac{(-1)^n (n+1)!}{(2x+3)n+2}(2)^n\right]$$

(v) $\tan^{-1}\left(x/a\right)$

Let $y = \tan^{-1}\left(x/a\right)$

$$\therefore \; y_1 = \frac{1}{1+\left(x/a\right)^2}\left(\frac{1}{a}\right) = \frac{a}{x^2+a^2}$$

$$y_n = D^n y = D^{n-1}(y_1) = D^{n-1}\left(\frac{a}{x^2+a^2}\right)$$

Consider $\dfrac{a}{x^2+a^2} = \dfrac{a}{(x+ai)(x-ai)}$

$= \dfrac{A}{(x+ai)} + \dfrac{B}{(x-ai)}$, on resolving into partial fractions.

$= \dfrac{\left(-1/2i\right)}{(x+ai)} + \dfrac{\left(1/2i\right)}{(x-ai)}$, on solving for A & B.

$$\therefore \; D^{n-1}\left(\frac{a}{x^2+a^2}\right) = D^{n-1}\left(\frac{-\tfrac{1}{2i}}{x+ai}\right) + D^{n-1}\left(\frac{\tfrac{1}{2i}}{x-ai}\right)$$

$$= \left(-\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right] + \left(\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n}\right] \text{-----------}(*)$$

We take transformation $x = r\cos\theta \quad a = r\sin\theta$ where $r = \sqrt{x^2+a^2}$, $\theta = \tan^{-1}\left(\dfrac{a}{x}\right)$

$$x + ai = r\left(\cos\theta + i\sin\theta\right) = re^{i\theta}$$
$$x - ai = r\left(\cos\theta - i\sin\theta\right) = re^{-i\theta}$$

$$\frac{1}{\left(x-ai\right)^n} = \frac{1}{r^n e^{-in\theta}} = \frac{e^{in\theta}}{r^n}, \quad \frac{1}{\left(x+ai\right)^n} = \frac{e^{-in\theta}}{r^n}$$

now(*) is $y_n = \dfrac{\left(-1\right)^{n-1}\left(n-1\right)!}{2\,i\,r^n}\left[e^{in\theta} - e^{-in\theta}\right]$

$$y_n = \frac{\left(-1\right)^{n-1}}{2\,i\,r^n}\left(2\,i\sin n\theta\right) \Rightarrow \frac{\left(-1\right)^{n-1}\left(n-1\right)!}{r^n}\sin n\theta$$

(vi) Let $y=\tan^{-1}x$ .Putting a = 1 in Ex.(v) we get

y_n which is same as above with $r=\sqrt{x^2+1}$ $\theta=\tan^{-1}\left(1/x\right)$

$\theta=\cot^{-1}(x)$ *or* $x=\cot\theta$

$\therefore r=\sqrt{\cot^2\theta+1}=\operatorname{cosec}\theta\Rightarrow\dfrac{1}{r^n}=\dfrac{1}{\operatorname{cosec}^n\theta}=\sin^n\theta$

$D^n(\tan^{-1}x)=(-1)^{n-1}(n-1)!\ \sin^n\theta\sin n\theta$ *where* $\theta=\cot^{-1}x$

(vii) Let $y=\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

put $x=\tan\theta$ $\theta=\tan^{-1}x$

$$\therefore \quad y=\tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$=\tan^{-1}\left[\tan(\pi/4+\theta)\right] \quad \because \tan(\pi/4+\theta)=\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$=\pi/4+\theta=\frac{\pi}{4}+\tan^{-1}(x)$$

$$y=\frac{\pi}{4}+\tan^{-1}(x)$$

$$y_n=0+D^n(\tan^{-1}x)$$

$$=\left(-\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]+\left(\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n}\right]$$

<u>nth derivative of trigonometric functions:</u>

1. $\sin(ax+b)$.

Let $y=\sin(ax+b)$. Differentiating w.r.t x,

$y_1=\cos(ax+b).a$ As $\sin(x+\dfrac{\pi}{2})=\cos x$

We can write $y_1=a\sin(ax+b+\pi/2)$.

again differentiating w.r.t x, $y_2=a\cos(ax+b+\pi/2).a$

Again using $\sin(x+\dfrac{\pi}{2})=\cos x$,we get y_2 as

$$y_2=a\sin(ax+b+\pi/2+\pi/2).a$$

i.e. $y_2=a^2\sin(ax+b+2\pi/2)$.

Similarly, we get

$$y_3=a^3\sin(ax+b+3\pi/2).$$

$$y_4=a^4\sin(ax+b+4\pi/2).$$

$$y_n=a^n\sin(ax+b+n\pi/2).$$

2. $e^{ax}\sin(bx+c)$

Let $y = e^{ax}\sin(bx+c)$ ....(1)

Differentiating using product rule ,we get

$y_1 = e^{ax}\cos(bx+c)b + \sin(bx+c)ae^{ax}$

i.e. $y_1 = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$. For computation of higher order derivatives

it is convenient to express the constants 'a' and 'b' in terms of the constants r and

θ defined by $a = r\cos\theta$ & $b = r\sin\theta$,so that

$r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$ thus,

y_1 can be rewritten as

$y_1 = e^{ax}[(r\cos\theta)\sin(bx+c) + (r\sin\theta)\cos(bx+c)]$

or $y_1 = e^{ax}[r\{\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta\}]$

i.e. $y_1 = re^{ax}[\sin(bx+c+\theta)]$..........(2)

Comparing expressions (1) and (2), we write y_2 as

$y_2 = r^2e^{ax}\sin(bx+c+2\theta)$

$y_3 = r^3e^{ax}\sin(bx+c+3\theta)$

Continuing in this way, we get

$y_4 = r^4e^{ax}\sin(bx+c+4\theta)$

$y_5 = r^5e^{ax}\sin(bx+c+5\theta)$

....

$y_n = r^ne^{ax}\sin(bx+c+n\theta)$

$\therefore D^n[e^{ax}\sin(bx+c)] = r^ne^{ax}\sin(bx+c+n\theta)$ where

$r = \sqrt{a^2+b^2}$ & $\theta = \tan^{-1}(b/a)$

Solve the following:

1. (i) $\sin^2 x + \cos^3 x$ (ii) $\sin^3\cos^3 x$ (iii) $\cos x\cos 2x\cos 3x$
   (iv) $\sin x\sin 2x\sin 3x$ (v) $e^{3x}\cos 2x$ (vi) $e^{2x}(\sin^2 x + \cos^3 x)$

The following formulae are useful in solving some of the above problems.

(i) $\sin^2 x = \dfrac{1-\cos 2x}{2}$ $(ii)\cos^2 x = \dfrac{1+\cos 2x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$ $(iv)\cos 3x = 4\cos^3 x - 3\cos x$

(v) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(vi) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(vii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(viii) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Sol: (i) Let $y=\sin^2 x+\cos^3 x=\left(\dfrac{1-\cos 2x}{2}\right)+\dfrac{1}{4}(\cos 3x+3\cos x)$

$$\therefore y_n=\frac{1}{2}\left[-(2)^n\cos\left(2x+\frac{n\pi}{2}\right)\right]+\frac{1}{4}\left[3^n\cos\left(3x+\frac{n\pi}{2}\right)+3\cos\left(x+\frac{n\pi}{2}\right)\right]$$

(ii)Let $y=\sin^3 x\cos^3 x=\left(\dfrac{\sin 2x}{2}\right)^3=\dfrac{\sin^3 2x}{8}=\dfrac{1}{8}\left[\dfrac{-\sin 6x+3\sin 2x}{4}\right]$

$$=\frac{1}{32}\left[3\sin 2x-\sin 6x\right]$$

$$y_n=\frac{1}{32}\left[3.2^n\sin\left(2x+\frac{n\pi}{2}\right)-6^n\sin\left(6x+\frac{n\pi}{2}\right)\right]$$

(iii))Let y = $\cos 3x\cos x\cos 2x$

$$=\frac{1}{2}(\cos 4x+\cos 2x)\cos 2x=\frac{1}{2}\left[\cos 4x\cos 2x+\cos^2 2x\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}(\cos 6x+\cos 2x)+\frac{1-\cos 4x}{2}\right]$$

$$=\frac{1}{4}\cos 6x+\frac{\cos 2x}{4}+\frac{1}{4}(1-\cos 4x)$$

$$\therefore y_n=\frac{1}{4}6^n\cos\left(6x+\frac{n\pi}{2}\right)+\frac{2^n\cos\left(2x+\frac{n\pi}{2}\right)}{4}-\frac{4^n\cos\left(4x+\frac{n\pi}{2}\right)}{4}$$

(iv))Let y = $\sin 3x\sin x\, sn2x$

$$=\frac{1}{2}\left[\sin(2x)-\sin 4x\right]\sin 2x$$

$$=\frac{1}{2}\left[\sin^2 2x-\sin 4x\sin 2x\right]$$

$$=\frac{1}{2}\left[\frac{1-\cos 4x}{2}-\frac{1}{2}(\sin 2x-\sin 6x)\right]$$

$$=\left[\left(\frac{1-\cos 4x}{4}\right)-\frac{1}{4}(\sin 2x-\sin 6x)\right]$$

$$y_n=\frac{1}{4}\left[4^n\cos\left(4x+\frac{n\pi}{2}\right)-2^n\sin\left(2x+\frac{n\pi}{2}\right)+6^n\sin\left(6x+\frac{n\pi}{2}\right)\right]$$

(v) Let $y=e^{3x}\cos 2x$

$\therefore y_n=re^{3x}\cos(2x+n\theta)$

where $r=\sqrt{3^2+2^2}=\sqrt{13}$ & $\theta=\tan^{-1}\left(\dfrac{2}{3}\right)$

(vi) Let y = $e^{2x}\left(\sin^2 x+\cos^3 x\right)$

We know that $\left[\sin^2 x+\cos^3 x\right]=\frac{1-\cos 2x}{2}+\frac{1}{4}\left[\cos 3x+3\cos x\right]$

$\therefore$ y = $e^{2x}\left[\sin^2 x+\cos^3 x\right]=e^{2x}\left[\frac{1-\cos 2x}{2}\right]+\frac{e^{2x}}{4}\left[\cos 3x+3\cos x\right]$

$$\therefore y=\frac{1}{2}\left[e^{2x}-e^{2x}\cos 2x\right]+\frac{1}{4}\left[e^{2x}\cos 3x+3e^{2x}\cos x\right]$$

Hence,

$$y_n=\frac{1}{2}\left[2^n e^{2x}-r_1^n e^{2x}\cos(2x+n\theta_1)\right]+\frac{1}{4}\left[r_2^n e^{2x}\cos(3x+n\theta_2)+3r_3^n e^{2x}\cos(x+n\theta_3)\right]$$

where $r_1=\sqrt{2^2+2^2}=\sqrt{8}$; $r_2=\sqrt{2^2+3^2}=\sqrt{13}$; $r_3=\sqrt{2^2+1^2}=\sqrt{5}$

$\theta_1=\tan^{-1}\left(\frac{2}{2}\right)$; $\theta_2=\tan^{-1}\left(\frac{3}{2}\right)$; $\theta_3=\tan^{-1}\left(\frac{1}{2}\right)$;

Leibnitz's Theorem

Leibnitz's theorem is useful in the calculation of nth derivatives of product of two functions.

Statement of the theorem:

If *u* and *v* are functions of *x*, then

$$D^n(uv)=D^nuv+{}^nC_1D^{n-1}uDv+{}^nC_2D^{n-2}uD^2v+\ldots+{}^nC_rD^{n-r}uD^rv+\ldots uD^nv,$$

where $D=\frac{d}{dx}$, ${}^nC_1=n$, ${}^nC_2=\frac{n(n-1)}{2}$,........, ${}^nC_r=\frac{n!}{r!(n-r)!}$

Examples

1. If $x=\sin t, y=\sin pt$ *prove that*

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(p^2-n^2)y_n=0$$

Solution: Note that the function $y=f(x)$ is given in the parametric form with a parameter *t*.

So, we consider

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{p\cos pt}{\cos t} \qquad (p - \text{constant})$$

or $\left(\frac{dy}{dx}\right)^2=\frac{p^2\cos^2 pt}{\cos^2 t}=\frac{p^2(1-\sin^2 pt)}{1-\sin^2 t}=\frac{p^2(1-y^2)}{1-x^2}$

or $(1-x^2)y_1^2=p^2(1-y^2)$

So that $(1-x^2)y_1^2-p^2(1-y^2)$ Differentiating w.r.t. x,

$$\left[(1-x^2)(2y_1y_2)+y_1^2(-2x)\right]-p^2(-2yy_1)=0$$

$(1-x^2)y_2 - xy_1 + p^2y = 0$ --------------- (1) [$\div 2y_1$, throughout]

Equation (1) has second order derivative y_2 in it. We differentiate (1), n times, term wise,

using Leibnitz's theorem as follows.

$D^n\left[(1-x^2)y_2 - xy_1 - p^2y\right] = 0$

i.e $D^n\{(1-x^2)y_2\} - D^n\{xy_1\} + D^n\{p^2y\} = 0$ ---------- (2)

(a) (b) (c)

Consider the term (a):

$D^n\left[(1-x^2)y_2\right]$. Taking $u = y_2$ and $v = (1-x^2)$ and applying Leibnitz's theorem we get

$D^n\left[uv\right] = D^nuv + {}^nC_1D^{n-1}uDv + {}^nC_2D^{n-2}D^2v + {}^nC_3D^{n-3}uD^3v + \ldots$

i.e

$D^n\left[y_2(1-x^2)\right] = D^n(y_2).(1-x^2) + {}^nC_1D^{n-1}(y_2).D(1-x^2) + {}^nC_2D^{n-2}(y_2)D^2(1-x^2) + {}^nC_3D^{n-3}(y_2)D^3(1-x^2) +$

$= y_{(n)+2}(1-x^2) + ny_{(n-1)+2}.(-2x) + \frac{n(n-1)}{2\,!}y_{(n-2)+2}.(-2) + \frac{n(n-1)(n-2)}{3\,!}.y_{(n-3)+2}.(0) + \ldots$

$D^n\left[(1-x^2)y_2\right] = (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n$ ----------- (3)

Consider the term (b):

$D^n\left[xy_1\right]$. Taking $u = y_1$ and $v = x$ and applying Leibnitz's theorem, we get

$D^n\left[y_1(x)\right] = D^n(y_1).(x) + {}^nC_1D^{n-1}y_1.D(x) + {}^nC_2D^{n-2}(y_1).D^2(x) + \ldots.$

$= y_{(n)+1}.x + ny_{(n-1)+1} + \frac{n(n-1)}{2\,!}y_{(n-2)+2}(0) + \ldots.$

$D^n\left[xy_1\right] = xy_{n+1} + ny_n$ ---------- (4)

Consider the term (c):

$D^n(p^2y) = p^2D^n(y) = p^2y_n$ --------- (5)

Substituting these values (3), (4) and (5) in Eq (2) we get

$\left[(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n\right] - \left[xy_{n+1} + ny_n\right] + \left[p^2y_n\right] = 0$

ie $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n + ny_n - ny_n + p^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (p^2-n^2)y_n = 0$ as desired.

2. If $\sin^{-1}y = 2\log(x+1)$ or $y = \sin\left[2\log(x+1)\right]$ or $y = \sin\left[\log(x+1)^2\right]$ or $y = \sin\log(x^2+2x+1)$, show that $(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$

(VTU Jan-03)

Sol: Out of the above four versions, we consider the function as

$\sin^{-1}(y) = 2\log(x+1)$

Differentiating w.r.t x, we get

$$\frac{1}{\sqrt{1-y^2}}(y_1)=\left(\frac{2}{x+1}\right) \text{ ie } (x+1)y_1=2\sqrt{1-y^2}$$

Squaring on both sides

$$(x+1)^2 y_1^2 = 4(1-y^2)$$

Again differentiating w.r.t x,

$$(x+1)^2(2y_1y_2)+y_1^2(2(x+1))=4(-2yy_1)$$

or $\quad (x+1)^2 y_2+(x+1)y_1=-4y \quad (\div 2y_1)$

or $\quad (x+1)^2 y_2+(x+1)y_1+4y=0$ -----------*

Differentiating * w.r.t x, n-times, using Leibnitz's theorem,

$$\left\{D^n y_2(x+1)^2+nD^{n-1}(y_2)2(x+1)+\frac{n(n-1)}{2\,!}D^{n-2}(y_2)(2)\right\}+\left\{D^n(g_1)(x+1)+nD^{n-1}y_1(1)\right\}+4D^n y=0$$

On simplification, we get

$$(x+1)^2 y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0$$

3. If $x=\tan(\log y)$, then find the value of

$(1+x^2)y_{n+1}+(2nx-1)y_n+n(n-1)y_{n-1}$ **(VTU July-04)**

Sol: Consider $x=\tan(\log y)$

i.e. $\tan^{-1}x=\log y$ or $y=e^{\tan^{-1}x}$

Differentiating w.r.t x,

$$y_1=e^{\tan^{-1}x}.\frac{1}{1+x^2}=\frac{y}{1+x^2}$$

$\therefore (1+x^2)y_1=y \qquad ie\ (1+x^2)y_1-y=0$ -----------*

We differentiate * n-times using Leibnitz's theorem,

We get

$$D^n\left[(1+x^2)y_1\right]-D^n(y)=0$$

ie.

$$\left\{D^n(y_1)(1+x^2)+{}^nC_1D^{n-1}(y_1)D(1+x^2)+{}^nC_2D^{n-2}(y_1)D^2(1+x^2)+\dots\right\}-\left\{D^n y\right\}=0$$

$$\text{ie } \left\{y_{n+1}(1+x^2)+ny_n(2x)+\frac{n(n-1)}{2!}y_{n-1}(2)+0+\dots\right\}-y_n=0$$

$$(1+x^2)y_{n+1}+(2nx-1)y_n+n(n-1)y_{n-1}=0$$

4. If $y^{1/m}+y^{-1/m}=2x$, or $y=\left[x+\sqrt{x^2-1}\right]^m$ or $y=\left[x-\sqrt{x^2-1}\right]^m$

Show that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$ **(VTU Feb-02)**

Sol: Consider $y^{1/m}+y^{-1/m}=2x \quad \Rightarrow \quad y^{1/m}+\frac{1}{y^{1/m}}=2x$

$\Rightarrow \quad (y^{1/m})^2-2x(y^{1/m})+1=0$ Which is quadratic equation in $y^{1/m}$

$$\therefore\ y^{1/m} = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2-1}}{2} = \left(x \pm \sqrt{x^2-1}\right) \Rightarrow y^{1/m} = \left(x \pm \sqrt{x^2-1}\right)$$

$$\therefore\ y = \left(x \pm \sqrt{x^2-1}\right)^m$$

so, we can consider $y = \left[x + \sqrt{x^2-1}\right]^m$ or $y = \left[x - \sqrt{x^2-1}\right]^m$

Let us take $y = \left[x + \sqrt{x^2-1}\right]^m$

$$\therefore\ y_1 = m\left(x + \sqrt{x^2-1}\right)^{m-1}\left(1 + \frac{1}{2\sqrt{x^2-1}}(2x)\right)$$

$$y_1 = m\left(x + \sqrt{x^2-1}\right)^{m-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

or

$\left(\sqrt{x^2-1}\right)y_1 = my$. On squaring

$\left(x^2-1\right)y_1^2 = m^2y^2$.

Again differentiating w.r.t x,

$$\left(x^2-1\right)2y_1y_2 + y_1^2(2x) = m^2(2yy_1)$$

or

$$\left(x^2-1\right)y_2 + xy_1 = m^2y \quad (\div 2y_1)$$

or

$$\left(x^2-1\right)y_2 + xy_1 - m^2y = 0 \qquad \text{------------}(*)$$

Differentiating (*) n- times using Leibnitz's theorem and simplifying, we get

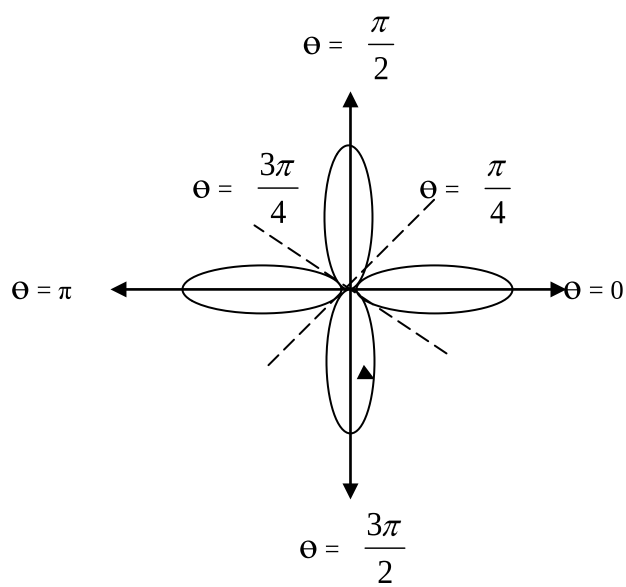
$$\left(x^2-1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2-m^2\right)y_n = 0$$

POLAR CURVES

Angle between Polar Curves:

Introduction:- We are familiar with Cartesian coordinate system for specifying a point in the xy – plane. Another useful system for similar purpose is Polar coordinate system, and the curves specified by these coordinates are referred to as polar curves.

- **A polar curve by name "three-leaved rose" is displayed below:**



- Any point P can be located on a plane with co-ordinates (r, θ) called **polar co-ordinates** of P where r = **radius vector** OP,(with pole 'O') θ = projection of $\overrightarrow{OP}$ on the **initial axis** OA.(See Fig.)
- The equation $r = f(\theta)$ is known as a **polar curve**.
- Polar coordinates (r, θ) can be related with Cartesian coordinates (x, y) through the relations
- **Fig.1. Polar coordinate system**
  $x = r\cos\theta$ & $y = r\sin\theta$.

<u>Theorem 1</u>: Angle between the radius vector and the tangent:

i.e., With usual notation prove that $\tan\phi = r\dfrac{d\theta}{dr}$

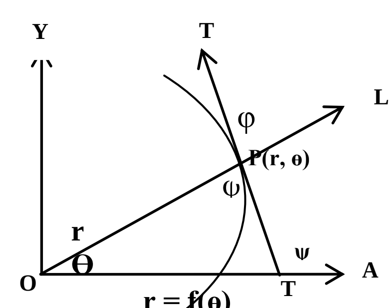


Fig.2. Angle between radius vector and the tangent

- **Proof:-** Let "ϕ" be the angle between the radius vector OPL and the tangent TPT^1 at the point `P` on the polar curve $r = f(\theta)$ (See fig.2)
  From Fig.2,

$$\psi = \theta + \phi$$

$$\tan\psi = \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

i.e. $\dfrac{dy}{dx} = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$................(1)

On the other hand, we have $x = r\cos\theta$; $y = r\sin\theta$ differentiating these, w.r.t θ,

$$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta\left(\frac{dr}{d\theta}\right) \;\&\; \frac{dy}{d\theta} = r(\cos\theta) + \sin\theta\left(\frac{dr}{d\theta}\right)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r(\cos\theta) + \sin\theta\left(\frac{dr}{d\theta}\right)}{r(-\sin\theta) + \cos\theta\left(\frac{dr}{d\theta}\right)} \textit{ dividing the Nr \& Dr by } \frac{dr}{d\theta}\cos\theta$$

$$\frac{dy}{dx} = \frac{r\left(d\theta/dr\right) + \tan\theta}{-\left(d\theta/dr\right)\tan\theta + 1}$$

i.e. $$\frac{dy}{dx} = \frac{\tan\theta + \left(r\,d\theta/dr\right)}{1-\tan\theta\left(r\,d\theta/dr\right)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Comparing equations (1) and (2)

we get $\tan\phi = r\,d\theta/dr$

- **Note that** $\cot\phi = \left(\frac{1}{r}\frac{dr}{d\theta}\right)$
- **<u>A Note on Angle of intersection of two polar curves</u>**:-

  If ϕ_1 and ϕ_2 are the angles between the common radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$ then the angle intersection of the curves is given by $|\phi_1 - \phi_2|$

<u>Theorem 2</u>: The length "p" of perpendicular from pole to the tangent in a polar curve

i.e.(i) $p = r\sin\phi$ **or (ii)** $\frac{1}{p^2} = \frac{1}{r^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

Proof:- In the Fig.3, note that ON = p, the length of the perpendicular from the pole to the tangent at p on $r = f(\theta)$ from the right angled triangle OPN,

$$\sin\phi = \frac{ON}{OP} \Rightarrow ON = (OP)\sin\phi$$

i.e. $p = r\sin\phi \ldots\ldots\ldots\ldots(i)$

Consider $\frac{1}{p} = \frac{1}{r\sin\phi} = \frac{1}{r}\operatorname{cosec}\phi$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1 + \cot 2\phi)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right]$$

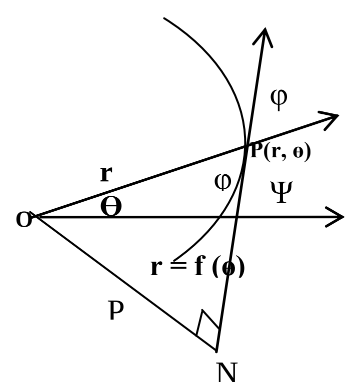


Fig.3 Length of the perpendicular from the pole to the tangent

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \text{...........}(ii)$$

Note:-If $u = \frac{1}{r}$, we get $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

In this session, we solve few problems on angle of intersection of polar curves and pedal equations.

Examples:-

Find the acute angle between the following polar curves

1. $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$ **(VTU-July-2003)**
2. $r = (\sin\theta + \cos\theta)$ and $r = 2\sin\theta$ **(VTU-July-2004)**
3. $r = 16\sec^2(\theta/2)$ and $r = 25\cos ec^2(\theta/2)$
4. $r = a\log\theta$ and $r = a/\log\theta$ **(VTU-July-2005)**
5. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Sol:

1. Consider

$r = a(1+\cos\theta)$

Diff w.r.t θ

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$\tan\phi_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= -\cot\theta/2$$

i.e $\tan\phi_1 = \tan(\pi/2 + \theta/2) \Rightarrow \phi_1 = (\pi/2 + \theta/2)$

Consider

$r = b(1-\cos\theta)$

Diff w.r.t θ

$$\frac{dr}{d\theta} = b\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta}$$

$$\tan\phi_1 = -\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= \tan\theta/2$$

$\tan\phi_1 = \tan\theta/2 \Rightarrow \phi_1 = \phi_2$

Angle between the curves

$$|\phi_1 - \phi_2| = \left|(\pi/2 + \theta/2) - \theta/2\right| = \pi/2$$

Hence, the given curves intersect orthogonally.

2. Consider

$r = (\sin\theta + \cos\theta)$

Diff w.r.t θ

Consider

$r = 2\sin\theta$

Diff w.r.t θ

$$\frac{dr}{d\theta}=\cos\theta-\sin\theta \qquad\qquad \frac{dr}{d\theta}=2\cos\theta$$

$$r\frac{d\theta}{dr}=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta} \qquad\qquad r\frac{d\theta}{dr}=\frac{2\sin\theta}{2\cos\theta}$$

$$\tan\phi_1=\frac{\tan\theta+1}{1-\tan\theta}\quad (\div\ \text{Nr \& Dr}\cos\theta) \qquad\qquad \tan\phi_2=\tan\theta$$

$$\text{i.e }\tan\phi_1=\frac{\tan\theta+1}{1-\tan\theta}=\tan\left(\pi/4+\theta\right) \qquad\qquad \Rightarrow\phi_2=\theta$$

$$\Rightarrow\phi_1=\pi/4+\theta$$

$$\therefore \text{Angle between the curves }=|\phi_1-\phi_2|=\left|\left(\pi/4+\theta\right)-\theta\right|=\pi/4$$

3. Consider $\qquad\qquad$ Consider

$$r=16\sec^2\left(\theta/2\right) \qquad\qquad r=25\operatorname{cosec}^2\left(\theta/2\right)$$

Diff w.r.t θ $\qquad\qquad$ Diff w.r.t θ

$$\frac{dr}{d\theta}=32\sec^2\left(\theta/2\right)\tan\left(\theta/2\right).1/2 \qquad\qquad \frac{dr}{d\theta}=-50\operatorname{cosec}^2\left(\theta/2\right)\cot\left(\theta/2\right).1/2$$

$$=16\sec\left(\theta/2\right)\tan\left(\theta/2\right) \qquad\qquad =-25\operatorname{cosec}^2\left(\theta/2\right)\cot\left(\theta/2\right)$$

$$r\frac{d\theta}{dr}=\frac{16\sec^2\left(\theta/2\right)}{16\sec^2\left(\theta/2\right)\tan\left(\theta/2\right)} \qquad\qquad r\frac{d\theta}{dr}=\frac{25\operatorname{cosec}^2\left(\theta/2\right)}{-25\operatorname{cosec}^2\left(\theta/2\right)\cot\left(\theta/2\right)}$$

$$\tan\phi_1=\cot\theta/2=\tan\left(\pi/2-\theta/2\right) \qquad\qquad \tan\phi_2=-\tan\theta/2=\tan\left(-\theta/2\right)$$

$$\Rightarrow\phi_1=\left(\pi/2-\theta/2\right) \qquad\qquad \Rightarrow\phi_2=-\theta/2$$

$$\text{Angle of intersection of the curves }=|\phi_1-\phi_2|=\left|\left(\pi/2-\theta/2\right)-\left(-\theta/2\right)\right|$$

$$=\pi/2$$

4. Consider $\qquad\qquad$ Consider

$$r=a\log\theta \qquad\qquad r=a/\log\theta$$

Diff w.r.t θ $\qquad\qquad$ Diff w.r.t θ

$$\frac{dr}{d\theta}=a/\theta \qquad\qquad \frac{dr}{d\theta}=-a/\left(\log\theta\right)^2.1/\theta$$

$$r\frac{d\theta}{dr}=a\log\theta\left(\theta/a\right) \qquad\qquad r\frac{d\theta}{dr}=-\left(a/\log\theta\right)\left(\frac{\left(\log\theta\right)^2\theta}{a}\right)$$

$$\tan\phi_1=\theta\log\theta.........(i) \qquad\qquad \tan\phi_2=-\theta\log\theta.........(ii)$$

We know that

$$\tan(\phi_1-\phi_2)=\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\,\tan\phi_2}$$

$$=\frac{\theta\log\theta-(-\theta\log\theta)}{1+(\theta\log\theta)(-\theta\log\theta)}$$

i.e $\tan(\phi_1-\phi_2)=\dfrac{2\theta\log\theta}{1-(\theta\log\theta)^2}$ ....(iii)

From the data: $a\log\theta=r=a/\log\theta\Rightarrow(\log\theta)^2=1\quad or\log\theta=\pm1$

As θ is acute, we take by $\theta=1\Rightarrow\theta=e\quad\|NOTE\|$

Substituting $\theta=e$ in (iii), we get

$$\tan(\phi_1-\phi_2)=\frac{2e\log e}{1-(e\log e)^2}=\left(\frac{2e}{1-e^2}\right)\qquad(\because\log_e^e=1)$$

$$\therefore|\phi_1-\phi_2|=\tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

5. Consider

$$r=\frac{a\theta}{1+\theta}\text{ as}$$

$$\frac{1}{r}=\frac{1+\theta}{a\theta}=\frac{1}{a}\left(1/\theta+1\right)$$

Diff w.r.t θ

$$-\frac{1}{r^2}\frac{dr}{d\theta}=\frac{1}{a}\left(1/\theta^2\right)$$

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{r}{a\theta^2}$$

$$r\frac{d\theta}{dr}=\frac{a\theta^2}{r}$$

$$\tan\phi_1=\frac{a\theta^2}{a\theta/(1+\theta)}$$

$$\therefore\tan\phi_1=\theta(1+\theta)$$

Consider

$$r=\frac{a\theta}{1+\theta^2}$$

$$\therefore(1+\theta^2)=a/r$$

Diff w.r.t θ

$$2\theta=-a/r^2\,\frac{dr}{d\theta}$$

$$\frac{-2r\theta}{a}=\frac{1}{r}\frac{dr}{d\theta}$$

i.e $r\dfrac{d\theta}{dr}=\dfrac{-a}{2r\theta}$

$$\tan\phi_2=-\frac{a}{2\theta}\left(\frac{1+\theta^2}{a}\right)$$

$$\tan\phi_2=-\frac{1}{2\theta}(1+\theta^2)$$

Now, we have

$$\frac{a\theta}{1+\theta}=r=\frac{a}{1+\theta^2}\Rightarrow a\theta(1+\theta^2)=a(1+\theta)$$

$$\text{or }\theta+\theta^3=1+\theta\Rightarrow\theta^3=1\text{ or }\theta=1$$

$$\therefore\tan\phi_1=2\;\&\;\tan\phi_2=(-1)$$

$$\text{Consider } \tan\left|(\phi_1-\phi_2)\right| = \left|\frac{\tan\phi_1-\tan\phi_2}{1+(\tan\phi_1)(\tan\phi_2)}\right|$$

$$= \left|\frac{2-(-1)}{1+(2)(-1)}\right| = |-3| = 3$$

$$\therefore |\phi_1-\phi_2| = \tan^{-1}(3)$$

<u>Pedal equations (p-r equations</u>):- Any equation containing only **p** & **r** is known as pedal equation of a polar curve.

<u>Working rules to find pedal equations</u>:-

(i) Eliminate r and ϕ from the Eqs.: (i) $r = f(\theta)$ & $p = r\sin\phi$

(ii) Eliminate only θ from the Eqs.: (i) $r = f(\theta)$ & $\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

- Find the pedal equations for the polar curves:-
  1. $\frac{2a}{r} = 1-\cos\theta$
  2. $r = e^{\theta\cot\alpha}$
  3. $r^m = a^m \sin m\theta + b^m \cos m\theta$ **(VTU-Jan-2005)**
  4. $l/r = 1+e\cos\theta$

Sol:

1. Consider $\frac{2a}{r} = 1-\cos\theta$ ……….(i)

Diff. w.r.t θ

$$2a\left(-1/r^2\right)\frac{dr}{d\theta} = \sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-r\sin\theta}{2a}$$

$$r\frac{d\theta}{dr} = -\frac{2a}{r}\frac{1}{\sin\theta}$$

$$\tan\phi = -\frac{(1-\cos\theta)}{\sin\theta} = -\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = -\tan(\theta/2)$$

$$\tan\phi = \tan(-\theta/2) \Rightarrow \phi = -\theta/2$$

Using the value of ϕ is $p = r\sin\phi$, we get

$$p = r\sin(-\theta/2) = -r\sin\theta/2 ....(ii)$$

Eliminating "θ" between (i) and (ii)

$$p^2 = r^2 \sin^2 \theta/2 = r^2\left(\frac{1-\cos\theta}{2}\right) = \frac{r^2}{2}\left(\frac{2a}{r}\right) \quad \text{[See eg: - (i)]}$$

$$p^2 = ar.$$

This eqn. is only in terms of p and r and hence it is the pedal equation of the polar curve.

2. Consider $r = e^{\theta \cot\alpha}$

Diff. w.r.t θ

$$\frac{dr}{d\theta} = e^{\theta\cot\alpha}(\cot\alpha) = r\cot\alpha \quad (\because r = e^{\theta\cot\alpha})$$

We use the equation

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}(r\cot\alpha)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}(\cot^2\alpha) = \frac{1}{r^2}(1+\cot^2\alpha) = \frac{1}{r^2}\operatorname{cosec}^2\alpha$$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\alpha$$

$p^2 = r^2/\operatorname{cosec}^2\alpha$ or $r^2 = p^2\operatorname{cosec}^2\alpha$ is the required pedal equation

3.Consider $r^m = a^m \sin m\theta + b^m \cos m\theta$

Diff. w.r.t θ

$$mr^{m-1}\frac{dr}{d\theta} = a^m(m\cos m\theta) + b^m(-m\sin m\theta)$$

$$\frac{r^m}{r}\frac{dr}{d\theta} = a^m \cos m\theta - b^m \sin m\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}$$

$$\cot\phi = \frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}$$

Consider $p = r\sin\phi$, $\frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}(1+\cot^2\phi)$$

$$=\frac{1}{r^2}\left[1+\left(\frac{a^m\cos m\theta-b^m\sin m\theta}{a^m\sin m\theta+b^m\cos m\theta}\right)^2\right]$$

$$=\frac{1}{r^2}\left[\frac{(a^m\sin m\theta+b^m\cos m\theta)^2+(a^m\cos m\theta-b^m\sin m\theta)^2}{(a^m\sin m\theta+b^m\cos m\theta)^2}\right]$$

$$\frac{1}{p^2}=\frac{1}{r^2}\left[\frac{a^{2m}+b^{2m}}{r^{2m}}\right]$$

$$\Rightarrow p^2=\frac{r^{2(m+1)}}{a^{2m}+b^{2m}}$$ is the required *p-r* equation

4. Consider $\frac{l}{r}=(1+e\cos\theta)$

Diff w.r.t θ

$$l\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)=-e\sin\theta\Rightarrow \frac{l}{r}\left(\frac{1}{r}\frac{dr}{d\theta}\right)=e\sin\theta$$

$$\frac{l}{r}(\cot\phi)=e\sin\theta$$

$$\therefore \cot\phi=\left(\frac{r}{l}\right)e\sin\theta$$

We have $\frac{1}{p^2}=\frac{1}{r^2}(1+\cot^2\phi)$ (see eg: 3 above)

Now $$\frac{1}{p^2}=\frac{1}{r^2}\left[\frac{l^2+e^2r^2\sin^2\theta}{l^2}\right]$$

$$=\frac{1}{r^2}\left(1+\frac{e^2r^2}{l^2}\sin^2\theta\right)$$

$$1+e\cos\theta=\frac{l}{r}\Rightarrow \qquad e\cos\theta=\frac{l-r}{r}$$

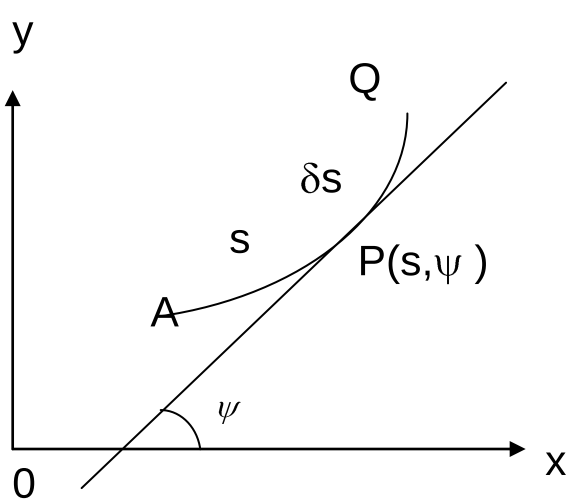
$$\cos\theta=\left(\frac{l-r}{re}\right)\Rightarrow \sin^2\theta=1-\cos^2\theta\Rightarrow =1-\left(\frac{l-r}{re}\right)^2$$

$$\frac{1}{p^2}=\frac{1}{r^2}\left[\frac{l^2+e^2r^2\left\{1-\left(\frac{l-r}{re}\right)^2\right\}}{l^2}\right]$$

On simplification $$\frac{1}{p^2}=\left(\frac{e^2-1}{e^2}\right)+\frac{2}{lr}$$

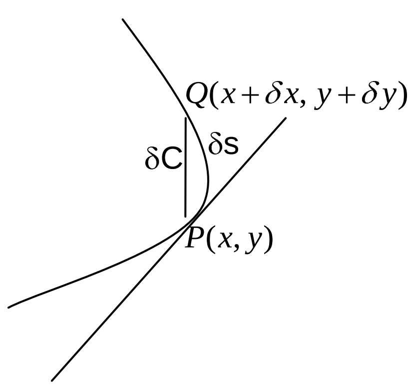
DERIVATIVES OF ARC LENGTH:

Consider a curve C in the XY plane. Let A be a fixed point on it. Let P and Q be two neighboring positions of a variable point on the curve C. If 's' is the distance of P from A measured along the curve then 's' is called the arc length of P. Let the tangent to C at P make an angle ψ with X-axis. Then (s,ψ) are called the intrinsic co-ordinates of the point P. Let the arc length AQ be $s + \delta s$. Then the distance between P and Q measured along the curve C is δs. If the actual distance between P and Q is δC. Then $\delta s = \delta C$ in the limit $Q \to P$ along C.



$$\text{i.e.} \quad \underset{Q \to P}{Lt} \frac{\delta s}{\delta C} = 1$$

Cartesian Form:



Let $y = f(x)$ be the Cartesian equation of the curve C and let $P(x, y)$ *and* $Q(x+\delta x, y+\delta y)$ be any two neighboring points on it as in fig.

Let the arc length $PQ = \delta s$ and the chord length $PQ = \delta C$. Using distance between two points formula we have $PQ^2 = (\delta C)^2 = (\delta x)^2 + (\delta y)^2$

$$\therefore \left(\frac{\delta C}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \quad or \quad \frac{\delta C}{\delta x} = \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}$$

$$\Rightarrow \frac{\delta s}{\delta x} = \frac{\delta s}{\delta C}\cdot\frac{\delta C}{\delta x} = \frac{\delta s}{\delta C}\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}$$

We note that $\delta x \to 0$ as $Q \to P$ along C, also that when $Q \to P, \dfrac{\delta s}{\delta C} = 1$

$\therefore$ When $Q \to P$ i.e. when $\delta x \to 0$, from (1) we get

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \to (1)$$

Similarly we may also write

$$\frac{\delta s}{\delta y} = \frac{\delta s}{\delta C}\cdot\frac{\delta C}{\delta y} = \frac{\delta s}{\delta C}\sqrt{1+\left(\frac{\delta x}{\delta y}\right)^2}$$

and hence when $Q \to P$ this leads to

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2} \quad \to (2)$$

Parametric Form: Suppose $x = x(t)$ *and* $y = y(t)$ is the parametric form of the curve C.

Then from (1)

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2} = \frac{1}{dx/dt}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\therefore \frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \quad \to (3)$$

<u>**Note**</u>: Since ψ is the angle between the tangent at P and the X-axis,

we have $\frac{dy}{dx} = \tan\psi$

$$\Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(y'\right)^2} = \sqrt{1+\tan^2\psi} = sec\psi$$

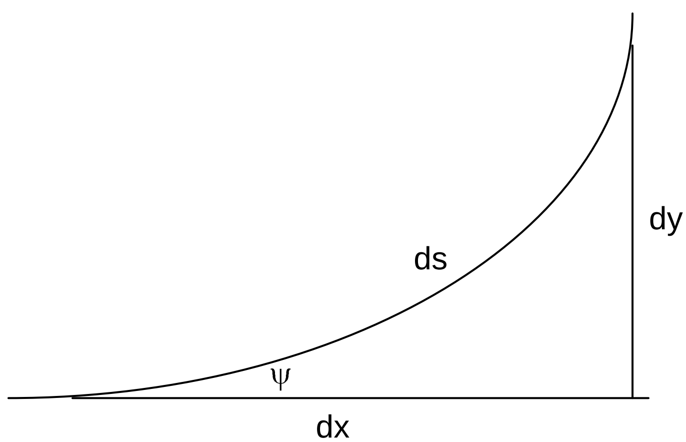
Similarly

$$\frac{ds}{dy} = \sqrt{1 + \frac{1}{\left(y'\right)^2}} = \sqrt{1+\frac{1}{\tan^2\psi}} = \sqrt{1+\cot^2\psi} = co\sec\psi$$

i.e. $\cos\psi = \frac{dx}{ds}$ *and* $sin\,\psi = \frac{dy}{ds}$

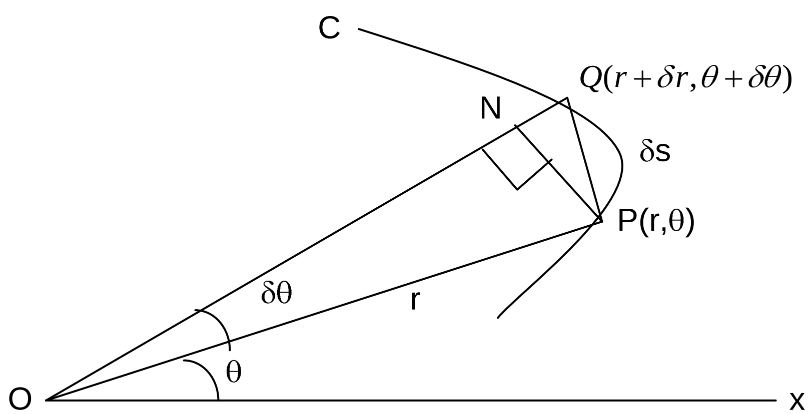
$$\therefore \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \Rightarrow \left(ds\right)^2 = \left(dx\right)^2 + \left(dy\right)^2$$

We can use the following figure to observe the above geometrical connections among $dx, dy, ds\ and\ \psi$.



Polar Curves:

Suppose $r = f(\theta)$ is the polar equation of the curve C and $P(r,\theta)$ *and* $Q(r+\delta r, \theta+\delta\theta)$ be two neighboring points on it as in figure:



Consider PN $\perp$ OQ.

In the right-angled triangle OPN, We have $sin\ \delta\theta = \frac{PN}{OP} = \frac{PN}{r} \Rightarrow PN = r\ sin\ \delta\theta = r\delta\theta$

since $\text{Sin}\delta\theta = \delta\theta$ when $\delta\theta$ *is very small*.

From the figure we see that, $cos\ \delta\theta = \frac{ON}{OP} = \frac{ON}{r} \Rightarrow ON = r\ cos\ \delta\theta = r(1) = r$

$\because \cos\delta\theta = 1\ when\ \ \delta\theta \to 0$

$$\therefore NQ = OQ - ON = (r+\delta r) - r = \delta r$$

$From\ \triangle PNQ,\ PQ^2 = PN^2 + NQ^2\ i.e,\ (\delta C)^2 = (r\delta\theta)^2 + (\delta r)^2$

$$\Rightarrow \frac{\delta C}{\delta\theta} = \sqrt{r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2} \quad \therefore \frac{\delta S}{\delta\theta} = \frac{\delta S}{\delta C} \cdot \frac{\delta C}{\delta\theta} = \frac{\delta S}{\delta C}\sqrt{r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2}$$

We note that $when\ Q \to P\ along\ the\ curve,\ \delta\theta \to 0\ also\ \frac{\delta S}{\delta C} = 1$

$$\therefore when\ Q \to P,\ \frac{dS}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \to (4)$$

$$Similarly,\quad (\delta C)^2 = (r\delta\theta)^2 + (\delta r)^2 \Rightarrow \frac{\delta C}{\delta r} = \sqrt{1 + r^2\left(\frac{\delta\theta}{\delta r}\right)^2}$$

$$and \quad \frac{\delta S}{\delta r} = \frac{\delta S}{\delta C} \cdot \frac{\delta C}{\delta r} = \frac{\delta S}{\delta C}\sqrt{1 + r^2\left(\frac{\delta\theta}{\delta r}\right)^2}$$

$$\therefore \text{ when } Q \to P, \text{ we get } \frac{dS}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \to (5)$$

Note:

We know that $\tan\phi = r\dfrac{d\theta}{dr}$

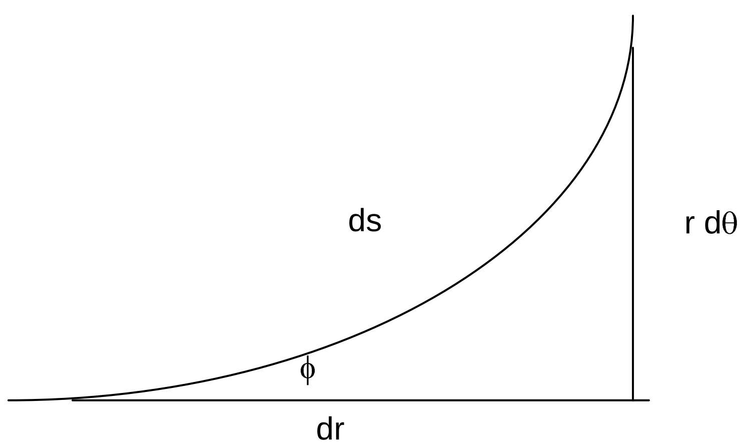
$$\therefore \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r^2 cot^2\phi} = r\sqrt{1 + cot^2\phi} = r co\sec\phi$$

Similarly

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} = \sqrt{1 + tan^2\phi} = sec\phi$$

$$\therefore \frac{dr}{ds} = cos\phi \quad and \quad \frac{d\theta}{ds} = \frac{1}{r} sin\phi$$

The following figure shows the geometrical connections among ds, dr, dθ and ϕ



Thus we have :

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \quad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}, \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \quad and \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Example 1: $\frac{\mathrm{ds}}{\mathrm{dx}}$ and $\frac{\mathrm{ds}}{\mathrm{dy}}$ for the curve $\mathrm{x}^{2/3} + y^{2/3} = a^{2/3}$

$$\mathrm{x}^{2/3} + y^{2/3} = a^{2/3} \Rightarrow \frac{2}{3}\mathrm{x}^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \quad \Rightarrow y' = \frac{\mathrm{x}^{\frac{-1}{3}}}{\mathrm{y}^{\frac{-1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Hence $\frac{\mathrm{ds}}{\mathrm{dx}} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}}$

$$= \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{a^{2/3}}{x^{2/3}}} = \left(\frac{a}{x}\right)^{1/3}$$

Similarly

$$\frac{\mathrm{ds}}{\mathrm{dy}} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{\ + \frac{x^{2/3}}{y^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{y^{2/3}}}$$

$$= \sqrt{\frac{a^{2/3}}{y^{2/3}}} = \left(\frac{a}{y}\right)^{1/3}$$

Example 2: *Find* $\frac{\mathrm{ds}}{\mathrm{dx}}$ for the curve $\mathrm{y} = \mathrm{a}\log\left(\frac{\mathrm{a}^2}{\mathrm{a}^2 - \mathrm{x}^2}\right)$

$$y = a\log a^2 - a\log\ \ a^2 - x^2 \ \ \Rightarrow \frac{dy}{dx} = -a\left(\frac{-2x}{a^2 - x^2}\right) = \frac{2ax}{a^2 - x^2}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4a^2x^2}{a^2 - x^2}}$$

$$= \sqrt{\frac{a^2 - x^2 \ \ + 4a^2x^2}{a^2 - x^2 \ \ ^2}} = \sqrt{\frac{a^2 + x^2 \ \ ^2}{a^2 - x^2}}$$

$$= \frac{a^2 + x^2}{a^2 - x^2}$$

Example 3: If $\mathrm{x} = \mathrm{ae}^{\mathrm{t}}\sin\mathrm{t}$, $\mathrm{y} = \mathrm{ae}^{\mathrm{t}}\cos\mathrm{t}$, find $\frac{ds}{dt}$

$$\mathrm{x} = \mathrm{ae}^{\mathrm{t}}\sin\mathrm{t} \Rightarrow \frac{dx}{dt} = \mathrm{ae}^{\mathrm{t}}\sin\mathrm{t} + \mathrm{ae}^{\mathrm{t}}\cos\mathrm{t}$$

$$y = ae^t \cos t \Rightarrow \frac{dy}{dt} = ae^t \cos t - ae^t \sin t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2e^{2t}\ (\cos t + \sin t)^2 + a^2e^{2t}\ (\cos t - \sin t)^2}$$

$$= ae^t\sqrt{2\ (\cos^2 t + \sin^2 t)} = a\sqrt{2}\ e^t \qquad \because (a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$

Example 4: If $x = a\left[\cos t + \log \tan \frac{t}{2}\right], y = a \sin t$, find $\frac{ds}{dt}$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{\sec^2 t/2}{2\cdot \tan t/2}\right] = a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\frac{1-\sin^2 t}{\sin t} = \frac{a\cos^2 t}{\sin t} = a\cos t \cdot \cot t$$

$$\frac{dy}{dt} = a\cos t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)}$$

$$= \sqrt{a^2\cos^2 t\ \cot^2 t + a^2\cos^2 t}$$

$$= \sqrt{a^2\cos^2 t\ (\cot^2 t + 1)}$$

$$= \sqrt{a^2\cos^2 t \cdot \mathrm{co}\sec^2 t} = \sqrt{a^2\cot^2 t}$$

$$= a\cot t$$

Example 5: If $x = a\cos^3 t$, $y = \sin^3 t$, find $\frac{ds}{dt}$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2\cos^4 t\ \sin^2 t + 9a^2\sin^4 t\ \cos^2 t}$$

$$= \sqrt{9a^2\cos^2 t\ \sin^2 t\ (\cos^2 t + \sin^2 t)} = 3a\sin t\cos t$$

Example 6: If $r^2 = a^2 \cos 2\theta$, Show that $r\dfrac{ds}{d\theta}$ is constant

$$r^2 = a^2 cos2\theta \Rightarrow 2r\frac{dr}{d\theta} = -2a^2 sin2\theta \quad \Rightarrow \frac{dr}{d\theta} = \frac{-a^2}{r} sin2\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + \frac{a^4}{r^2} sin^2 2\theta} = \frac{1}{r}\sqrt{r^4 + a^4 sin^2 2\theta}$$

$$\therefore r\frac{ds}{d\theta} = \sqrt{r^4 + a^4 sin^2 2\theta} = \sqrt{a^4 cos^2 2\theta + a^4 sin^2 2\theta}$$

$$= a^2\sqrt{cos^2\theta + sin^2 2\theta} = a^2 = constant \qquad \therefore r\frac{ds}{d\theta} \text{ is constant for } r^2 = a^2 cos2\theta$$

Example 7: For the curve $\theta = cos^{-1}\left(\dfrac{r}{k}\right) - \dfrac{\sqrt{k^2 - r^2}}{r}$, Show that $r\dfrac{ds}{dr}$ is constant.

$$\frac{d\theta}{dr} = \frac{-1}{\sqrt{1 - \frac{r^2}{k^2}}} \cdot \frac{1}{k} - \frac{r\left(\frac{-2r}{2\sqrt{k^2 - r^2}}\right) - \sqrt{k^2 - r}\,(1)}{r^2} = \frac{-1}{\sqrt{k^2 - r^2}} + \frac{r^2 + \ k^2 - r^2}{r^2\sqrt{k^2 - r^2}}$$

$$= \frac{-1}{\sqrt{k^2 - r^2}} + \frac{k^2}{r^2\sqrt{k^2 - r^2}} = \frac{-r^2 + k^2}{r^2\sqrt{k^2 - r^2}} \qquad = \frac{\sqrt{k^2 - r^2}}{r}$$

$$\therefore \frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)}$$

$$= \sqrt{1 + r^2\,\frac{k^2 - r^2}{r^2}} = \frac{\sqrt{r^2 + k^2 - r^2}}{r} = \frac{k}{r}$$

Hence $r\dfrac{ds}{dr} = k$ (constant)

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Example 8: For a polar curve $r = f\ \theta$ show that $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{r^2 - p^2}}, \dfrac{ds}{d\theta} = \dfrac{r^2}{p}$

We know that $cos\,\phi = \dfrac{dr}{ds}$ *and* $\dfrac{d\theta}{ds} = \dfrac{1}{r} sin\,\phi$

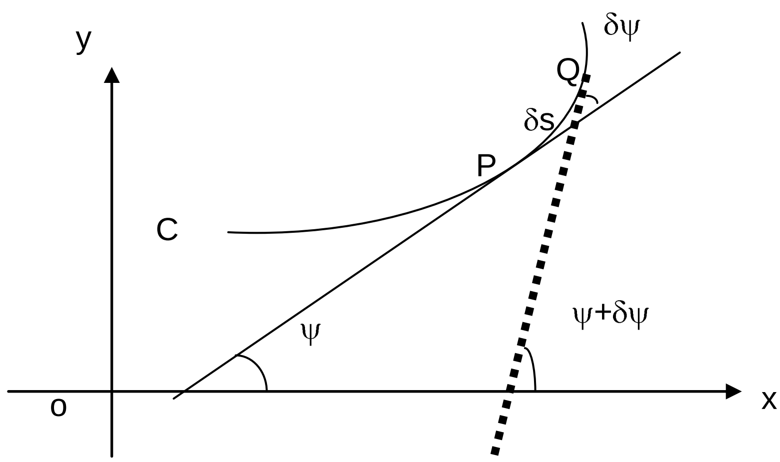
$$\therefore \frac{dr}{ds} = cos\varphi = \sqrt{1 - sin^2\phi} = \sqrt{1 - \frac{p^2}{r^2}} = \frac{\sqrt{r^2 - p^2}}{r} \qquad \because p = rsin\varphi$$

$$\therefore \frac{\mathrm{ds}}{\mathrm{dr}} = \frac{r}{\sqrt{r^2 - p^2}}$$

$$Also \quad \frac{\mathrm{ds}}{\mathrm{d}\theta} = \frac{r}{\sin\phi} = \frac{r}{p/r} = \frac{r^2}{p}$$

CURVATURE:

Consider a curve C in XY-plane and let P, Q be any two neighboring points on it. Let arc AP=s and arc PQ=δs. Let the tangents drawn to the curve at P, Q respectively make angles ψ and ψ+δψ with X-axis i.e., the angle between the tangents at P and Q is δψ. While moving from P to Q through a distance'δs', the tangent has turned through the angle 'δψ'. This is called the bending of the arc PQ. Geometrically, a change in ψ represents the bending of the curve C and the ratio $\frac{\delta\psi}{\delta s}$ represents the ratio of bending of C between the point P & Q and the arc length between them.



$\therefore$ Rate of bending of Curve at P is $\frac{d\psi}{ds} = \underset{Q\to P}{Lt}\frac{\delta\psi}{\delta s}$

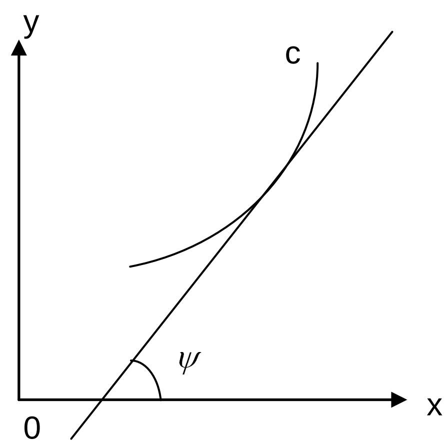
This rate of bending is called the curvature of the curve C at the point P and is denoted by κ (kappa). Thus $\kappa = \frac{\mathrm{d}\psi}{\mathrm{ds}}$ We note that the curvature of a straight line is zero since there exist no bending i.e. κ=0, and that the curvature of a circle is a constant and it is not equal to zero since a circle bends uniformly at every point on it

If $\kappa \neq 0$, then $\frac{1}{\kappa}$ is called the radius of curvature and is denoted by ρ (rho - Greek letter).

$$\therefore \rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$$

Radius of curvature in Cartesian form :

Suppose y = f(x) is the Cartesian equation of the curve considered in figure.



we have $y' = \frac{dy}{dx} = \tan\psi \Rightarrow y'' = \frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{d\psi}{dx} = 1+\tan^2\psi \ \frac{d\psi}{ds}\cdot\frac{ds}{dx}$

But we know that $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\therefore \frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]\cdot\frac{d\psi}{ds}\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{ds}{d\psi} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{\left[1+\ y'^{\,2}\right]^{3/2}}{y''}$$

This is the expression for radius of curvature in Cartesian form.

NOTE: We note that when y'=∞, we find ρ using the formula $\rho = \dfrac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\left(\dfrac{d^2x}{dy^2}\right)}$

Example 9: Find the radius of curvature of the curve $x^3+y^3 = 2a^3$ at the point (a, a).

$$x^3+y^3=2a^3 \Rightarrow 3x^2+3y^2\cdot y'=0 \;\; \Rightarrow y'=-\frac{x^2}{y^2} \;\; hence\ at\ \ (a,a),\ \ y'=-1$$

$$\therefore y''=-\left[\frac{y^2(2x)-x^2(2y)y'}{y^4}\right],\ hence\ at\ \ (a,a),\ y''=-\left[\frac{2a^3+2a^3}{a^4}\right]=-\frac{4}{a}$$

$$\therefore\ \rho=\frac{\left[1+(y')^2\right]^{3/2}}{y''}=\frac{\left[1+(-1)^2\right]^{3/2}}{-4/a}\ \ i.e.,\ |\rho|=\frac{a}{4}\cdot 2\sqrt{2}\ \ =\ \frac{a}{\sqrt{2}}$$

Example 10: Find the radius of curvature for $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point where it meets the line y=x.

On the line y = x, $\sqrt{x}+\sqrt{x}=\sqrt{a}\ \ i.e\ 2\sqrt{x}=\sqrt{a}\ \ \ or\ \ \ x=\dfrac{a}{4}$

i.e., We need to find ρ at $\left(\dfrac{a}{4},\dfrac{a}{4}\right)$

$$\sqrt{x}+\sqrt{y}=\sqrt{a}\Rightarrow\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0\ \ i.e\ \ y'=-\sqrt{\frac{y}{x}},\ hence\ at\left(\frac{a}{4},\frac{a}{4}\right),y'=-1$$

$$Also,\ y''=-\left[\frac{\sqrt{x}\dfrac{1}{2\sqrt{y}}\cdot y'-\sqrt{y}\dfrac{1}{2\sqrt{x}}}{x}\right]$$

$$\therefore at\left(\frac{a}{4},\frac{a}{4}\right),y''=-\left[\frac{\sqrt{\dfrac{a}{4}}\dfrac{1}{2\sqrt{\dfrac{a}{4}}}\cdot(-1)-\sqrt{\dfrac{a}{4}}\dfrac{1}{2\sqrt{\dfrac{a}{4}}}}{\dfrac{a}{4}}\right]=-\frac{(-\frac{1}{2}-\frac{1}{2})}{\dfrac{a}{4}}=-\frac{(-1)}{\dfrac{a}{4}}=\frac{4}{a}$$

$$\therefore \rho = \frac{\left[1+ (y')^2\right]^{3/2}}{y''} = \frac{\left[1+ (-1)^2\right]^{3/2}}{4/a} = \frac{a}{4}2\sqrt{2} = \frac{a}{\sqrt{2}}$$

Example 11: Show that the radius of curvature for the curve y = 4 Sin x - Sin 2x at x = $\pi/2$ is $5\sqrt{5}/4$

y = 4 sin x - sin 2x $\Rightarrow y' = 4\cos x - 2\cos 2x$

$\therefore$ when x $= \pi/2$, $y' = 4\cos \pi/2 - 2\cos\pi = 0 - 2(-1) = 2$

Also, $y'' = -4 \sin x + 4 \sin 2x$ and when x $= \pi/2$, $y'' = -4 \sin \pi/2 + 4 \sin \pi = -4$

$$\therefore \rho = \frac{\left[1+ (y')^2\right]^{3/2}}{y''} = \frac{\left[1+2^2\right]^{3/2}}{-4} \Rightarrow |\rho| = \frac{5\sqrt{5}}{4}$$

Example 12: Find the radius of curvature for $xy^2 = a^3 - x^3$ at (a, 0).

$$xy^2 = a^3 - x^3 \Rightarrow y^2 + 2xy\, y' = -3x^2$$

$$\therefore y' = \frac{-3x^2 - y^2}{2xy} \; and \; at \; (a,0), \;\; y' = \infty$$

In such cases we write $\dfrac{dx}{dy} = \dfrac{2xy}{-3x^2 - y^2}$ *and at*$(a, 0)$, $\dfrac{dx}{dy} = 0$

Also $$\frac{dx}{dy} = \frac{-2xy}{3x^2 + y^2} \Rightarrow \frac{d^2x}{dy^2} = \left[\frac{(3x^2 + y^2)\left(2\frac{dx}{dy}y + 2x\right) - 2xy\left(6x\frac{dx}{dy} + 2y\right)}{(3x^2 + y^2)^2}\right]$$

$$\therefore At\; (a, 0),\; \frac{d^2x}{dy^2} = \left[\frac{(3a^2 + 0)(0 + 2a) - 0}{(3a^2 + 0)^2}\right] = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

$$\therefore \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{\left[1+0^2\right]^{3/2}}{-2/3a} \; or \; |\rho| = \frac{3a}{2}$$

An expression for the radius of curvature in the case of a parametric curve $x = x(t),\ y = y(t)$

$$\rho = \frac{\left\{(\dot{x})^2 + (\dot{y})^2\right\}^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

1. Find the radius of curvature of the curve

$x = a\log(\text{sect}+\text{tant}),\ \ \text{y} = \text{asect}$

$\Rightarrow x = a\log(\text{sect}+\text{tant})$

$$\frac{dx}{dt} = \frac{a}{\sec t + \tan t}\sec t\tan t + \sec^2 t = \frac{a\sec t(\text{seet}+\text{tant})}{(\text{seet}+\text{tant})}$$

$$\therefore \frac{dx}{dt} = a\sec t$$

Also $y = a\sec t$ *gives*

$$\frac{dy}{dt} = a\sec t\tan t$$

$$Now\ ,\ y_1 = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a\sec t\tan t}{a\sec t}$$

$$y_1 = \tan t$$

Differentiating w.r.t x we get

$$y_2 = \sec^2 t\frac{dt}{dx}$$

$$\therefore y_2 = \frac{\sec t}{a}$$

$$we\,have\ \rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2}$$

$$\rho = \frac{a\left(1 + \tan^2 t\right)^{3/2}}{\sec t}$$

$$\rho = a\sec^2 t$$

2. *Show that the radius of curvature at any point* θ *on the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ *is* $4a\cos(\theta/2)$

\>> $x = a(\theta + \sin\theta)$; $y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1+\cos\theta) \quad ; \quad \frac{dy}{d\theta} = a\sin\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$\therefore \quad y_1 = \tan(\theta/2)$

Differentiating w.r.t. x we get,

$$y_2 = \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{d\theta}{dx}$$

$$= \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{1}{a(1+\cos\theta)} = \frac{\sec^2(\theta/2)}{4a\cos^2(\theta/2)}$$

$\therefore \quad y_2 = \frac{1}{4a}\sec^4(\theta/2)$

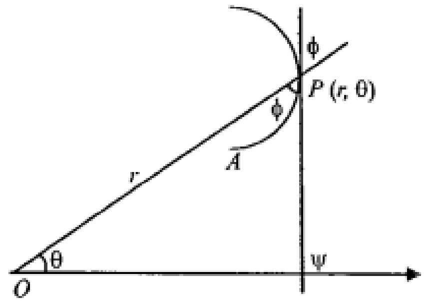
We have $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+\tan^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)}$$

$$= \frac{[\sec^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)} = \frac{4a\sec^3(\theta/2)}{\sec^4(\theta/2)}$$

Thus $\rho = 4a\cos(\theta/2)$

An expression for the radius of curvature in the case of a polar curve $r = f(\theta)$



Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at $P(r, \theta)$.

Let ψ be the angle made by the tangent at P with the initial line.

Let A be a fixed point on the curve and let $\overset{\frown}{AP} = s$.

We have $\psi = \theta + \phi$

$$\therefore \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} \qquad ie., \ \frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\text{or} \qquad \rho = \frac{\left(\dfrac{ds}{d\theta}\right)}{1 + \dfrac{d\phi}{d\theta}} \qquad \ldots (1)$$

We know that $\tan\phi = r\,\dfrac{d\theta}{dr} = r \Big/ \left(\dfrac{dr}{d\theta}\right)$

$$ie., \quad \tan\phi = \frac{r}{r_1} \quad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating w.r.t θ we get,

$$\sec^2\phi \, \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r \cdot r_2}{r_1^2} \quad \text{where } r_2 = \frac{d^2 r}{d\theta^2}$$

$$\text{or} \qquad \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 \sec^2\phi} = \frac{r_1^2 - r r_2}{r_1^2 (1 + \tan^2\phi)}$$

$$ie., \qquad \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 [1 + (r^2/r_1^2)]} = \frac{r_1^2 - r r_2}{r_1^2 + r^2}$$

Hence $1+\dfrac{d\phi}{d\theta} = 1+\dfrac{r_1^2-rr_2}{r^2+r_1^2} = \dfrac{r^2+r_1^2+r_1^2-rr_2}{r^2+r_1^2}$

ie., $$1+\frac{d\phi}{d\theta} = \frac{r^2+2r_1^2-rr_2}{r^2+r_1^2} \qquad \ldots(2)$$

Also, we know that $$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2+r_1^2} \qquad \ldots(3)$$

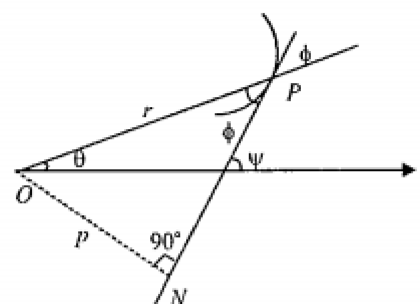
Using (2) and (3) in (1) we get

$$\rho = \sqrt{r^2+r_1^2}\cdot\frac{(r^2+r_1^2)}{r^2+2r_1^2-rr_2}$$

Thus in the polar form, $\boxed{\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}}$

An expression for the radius of curvature in the case of a pedal curve

Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at P. Let ψ be the angle made by the tangent at P with the initial line. Draw $ON = p$, a perpendicular from the pole to the tangent.



We have from the $\Delta\ ONP$, $\sin\phi = \dfrac{p}{r}$

ie., $$p = r\sin\phi \qquad \ldots(1)$$

Differentiating (1) w.r.t r we get,

$$\frac{dp}{dr} = r\cos\phi\,\frac{d\phi}{dr}+1\cdot\sin\phi$$

But we know that, $\sin\phi = r\dfrac{d\theta}{ds}$ and $\cos\phi = \dfrac{dr}{ds}$

$$\therefore \quad \frac{dp}{dr} = r\frac{d\phi}{dr}\,\frac{dr}{ds}+r\,\frac{d\theta}{ds} = r\left[\frac{d\phi}{ds}+\frac{d\theta}{ds}\right] = r\frac{d}{ds}(\phi+\theta)$$

But $\phi+\theta = \psi$

$$\therefore \quad \frac{dp}{dr} = r\frac{d\psi}{ds} \quad \text{or} \quad \frac{ds}{d\psi} = r\frac{dr}{dp}$$

Thus $\boxed{\rho = r\dfrac{dr}{dp}}$

1. *Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ varies inversely as r^{n-1}*

>> $r^n = a^n \cos n\theta$

$\Rightarrow$ $n \log r = n \log a + \log(\cos n\theta)$

Differentiating w.r.t. θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n \sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$\therefore$ $r_1 = -r \tan n\theta$

Hence $r_2 = \dfrac{d^2 r}{d\theta^2} = -r_1 \tan n\theta - n\, r \sec^2 n\theta$

We have $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\, r_2}$

$$\therefore \quad \rho = \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r(-r_1 \tan n\theta - nr \sec^2 n\theta)}$$

$$= \frac{(r^2)^{3/2} (\sec^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{r^2(1 + \tan^2 n\theta + n \sec^2 n\theta)}$$

$$= \frac{r \sec^3 n\theta}{\sec^2 n\theta (1+n)} = \frac{r \sec n\theta}{(1+n)}$$

Thus $\rho = \dfrac{r}{1+n} \sec n\theta$

But $a^n/r^n = \sec n\theta$ by data.

$$\therefore \quad \rho = \frac{r}{1+n} \cdot \frac{a^n}{r^n} = \left[\frac{a^n}{1+n}\right] \frac{1}{r^{n-1}}$$

ie., $\rho = \text{const} \cdot \dfrac{1}{r^{n-1}}$

Thus $\rho \propto 1/r^{n-1}$

2. *Find the radius of curvature of the curve* $r = a\sin n\theta$ *at the pole.*

>> $r = a\sin n\theta$

$\therefore \quad r_1 = an\cos n\theta, \; r_2 = -an^2\sin n\theta$

At the pole we have $\theta = 0$. When $\theta = 0$: $r = 0, \; r_1 = an, \; r_2 = 0$

We have $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

$$\therefore \quad \rho = \frac{(a^2 n^2)^{3/2}}{2a^2 n^2} = \frac{a^3 n^3}{2a^2 n^2} = \frac{an}{2}$$

Thus $\rho = an/2$ at the pole.

3. *Find the radius of curvature of the curve* $\theta = \dfrac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}(a/r)$ *at any point on it.*

>> Differentiating the given equation w.r.t. r we have,

$$\frac{d\theta}{dr} = \frac{1}{a}\cdot\frac{2r}{2\sqrt{r^2 - a^2}} - \left\{\frac{-1}{\sqrt{1-(a/r)^2}}\cdot\frac{-a}{r^2}\right\}$$

$$= \frac{r}{a\sqrt{r^2 - a^2}} - \frac{r}{\sqrt{r^2 - a^2}}\cdot\frac{a}{r^2}$$

$$= \frac{1}{\sqrt{r^2 - a^2}}\left(\frac{r}{a} - \frac{a}{r}\right) = \frac{r^2 - a^2}{\sqrt{r^2 - a^2}\cdot ar}$$

ie., $$\frac{d\theta}{dr} = \frac{\sqrt{r^2 - a^2}}{ar} \qquad \ldots (1)$$

We prefer to find the pedal equation of the given curve and then apply the formula for ρ in the pedal form.

From (1) $\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{r^2 - a^2}}$ *ie.,* $\cot\phi = \dfrac{a}{\sqrt{r^2 - a^2}}$

Consider $p = r\sin\phi$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad ie., \ \frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{a^2}{r^2 - a^2}\right]$$

$$ie., \quad \frac{1}{p^2} = \frac{1}{r^2}\left[\frac{r^2}{r^2 - a^2}\right] \quad ie., \ \frac{1}{p} = \frac{1}{\sqrt{r^2 - a^2}}$$

$\therefore \quad p = \sqrt{r^2 - a^2}$ is the pedal equation of the curve.

Differentiating w.r.t. p we get,

$$1 = \frac{2r}{2\sqrt{r^2 - a^2}}\frac{dr}{dp} \quad ie., \ \sqrt{r^2 - a^2} = r\frac{dr}{dp} = \rho$$

Thus $\rho = \sqrt{r^2 - a^2}$