MODULE V

LINEAR ALGEBRA

CONTENTS:

Definition: A system of mn nos. arranged in a rectangular formation along m-rows & n-columns & bounded by the brackets or is called as m by n matrix or mxn matrix Matrix is denoted by a single capital letters A,B,C etc.

$$A=\begin{bmatrix} a_{11} & a_{12}\,.......a_{1n} \\ a_{21} & a_{22}\,.......a_{2n} \\ a_{m1} & a_{m2}\,......a_{mn} \end{bmatrix}_{m\times n}$$

<u>Elementary operations on Matrices:</u>

The following 3 operations are said to be elementary operations

1. Interchange of any two rows or columns.
2. Multiplication of each element of a row or column by a non-Zero scalar or constant.
3. Addition of a scalar multiple of one row or column to anpother row or column.

$$A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{R_3\to kR_1+R_3} \qquad B=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ (ka_1+c_1) & (ka_2+c_1) & (ka_3+c_3) \end{bmatrix}$$

If a matrix A gets transferred into another matrix B by any of these transformations then A is said to be equivalent to B written as $A\square\, B.$

Echelon form or Row reduced Echelon form.

A matrix A of order mxn is said to be in a row reduced echelon form if

1. The leading element (the first non-Zero entry) of each row is unity.
2. All the entries below this leading entry is Zero.
3. The no of Zeros appearing before the leading entry in each row is greater than that appears in its previous row.
4. The Zero rows must appear below the non-zero rows.

$$A=\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Normal form of a matrix

The given matrix A is reduced to an echelon form first by applying a series of elementary row transformations.

Later column transformations row performed to reduce the matrix to one of the following four forms, called the normal form of A.

$i)\,Ir \qquad ii)\;\; Ir,o \qquad iii)\begin{bmatrix} \dfrac{Ir}{o} \end{bmatrix} \qquad iv)\begin{bmatrix} Ir & o \\ o & o \end{bmatrix}$ where Ir is the identity matrix of order r.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix} \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$$I_3 \qquad I_3, 0 \qquad \begin{bmatrix}I_2\\0\end{bmatrix} \quad \begin{bmatrix}I_3&0\\0&0\end{bmatrix}$$

Rank of a matrix: The number of non-zero rows in echelon or normal form. It's in denoted by f(A)

1. Reduce the matrix to the row reduced echelon form

$$A=\begin{bmatrix}1&2&-2&3\\0&0&1&2\\-1&-3&2&-1\\2&4&-1&3\end{bmatrix}$$

$Sol: R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 + R_1 \quad R_4 \to R_4 - 2R_1$

$$A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&-1&0&2\\0&0&3&-3\end{bmatrix} \quad A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&8&3\\0&0&3&-3\end{bmatrix}$$

$R_3 \to R_3/8, \quad R_4 \to R_4/3 \quad R_4 \to R_4 - R_3$

$$A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&1&3/8\\0&0&1&-3\end{bmatrix} \quad A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&1&3/8\\0&0&0&-11/8\end{bmatrix}$$

$R_4 \to \frac{-8}{11}R_4$

$$A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&1&3/8\\0&0&0&1\end{bmatrix}$$

2) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$ find rank of a matrix

$Sol: R_3 \to R_3 - 3R_1, \quad R_3 \to R_3 - 13R_2$

$$A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 13 & -2 & -8 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -28 & -34 \end{bmatrix}$$

$\rho(A) = 3$

3) $\begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 5 & 1 & 1 & 4 \end{bmatrix}$ find the rank

$Sol: R_2 \to 4R_2 - R_1, \quad R_3 \to 8R_3 - 3R_1 \quad R_4 \to R_4 + R_2$

$$A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & -6 & 5 & 6 \\ 0 & -2 & 3 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

4) Using the elementary transformation reduce the matrix $A=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}$ to echelon form

$Sol: R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\to A=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix} = \begin{bmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & -14 & 16\end{bmatrix}$$

$R_3 \to R_3 - 2R_2, \quad R_2 \to {-R_2}/{7}$

$$=\begin{bmatrix}1 & 2 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{bmatrix} \quad A=\begin{bmatrix}1 & 2 & -2\\ 0 & 1 & -8/7\\ 0 & 0 & 0\end{bmatrix}$$

5) Applying elementary transformations reduce the following matrix to the normal form & hence find rank of matrix given $\begin{bmatrix}3 & 2 & 5 & 7 & 12\\ 1 & 1 & 2 & 3 & 5\\ 3 & 3 & 6 & 9 & 15\end{bmatrix}$

$Sol: R_1 \leftrightarrow R_2$

$$A=\begin{bmatrix}3 & 2 & 5 & 7 & 12\\ 1 & 1 & 2 & 3 & 5\\ 3 & 3 & 6 & 9 & 15\end{bmatrix} = \begin{bmatrix}1 & 1 & 2 & 3 & 5\\ 3 & 2 & 5 & 7 & 12\\ 3 & 3 & 6 & 9 & 15\end{bmatrix}$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 3R_1$

$$A=\begin{bmatrix}1 & 1 & 2 & 3 & 5\\ 0 & -1 & -1 & -2 & -3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

This echelon form, now we have to perform column trans to reduce to the normal form.

$c_2 \to c_2 - c_1, \quad c_3 \to c_3 - 2c_1 \quad c_4 \to c_4 - 3c_1 \quad c_5 \to c_5 - 5c_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_3 \to c_3 - c_2, \quad c_4 \to c_4 - 2c_2 \quad c_3 \to c_5 - 3c_2,$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to -R_2} A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

6) By performing elementary row & column transformations, reduce the following matrix to the normal form $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

Sol: $R_2 \leftrightarrow R_2$

$$A = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & -4 & 3 & 1 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \quad R_4 \to R_4 - 4R_2 \qquad R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

$c_2 \to c_2 + 2c_1, \quad c_3 \to c_3 - c_1 \quad c_4 \to c_4 + 4c_1, \quad c_5 \to c_5 - 2c_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_3 \to c_3 + c_2, \quad c_4 \to c_4 - 3c_2, \quad c_5 \to c_5 - c_2 \quad c_4 \to c_4 - 9c_3 \quad c_5 \to c_5 - 4c_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

7) Reducing the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$ into normal form and find the rank

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 3R_1 \quad R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - 2R_2 \quad R_3 \to \frac{-1}{3} R_3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_2 \to -R_2 \quad c_3 \to 2c_3 + 2c_2 \quad c_4 \to 2c_4 - c_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_3 \to {c_2}/{2} \quad c_4 \to c_4 - 3c_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$== \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 3.$$

8) Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

$Sol: R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1 \quad R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_3 \leftrightarrow R_4 \quad R_3 = \frac{1}{3} R_3$

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c_2 \to c_2 - 2c_1, \quad c_3 \to c_3 - 2c_1 \quad c_4 \to c_4 - 3c_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rho(A) = 4$$

9) Find the rank of the matrix by reducing to the normal form

1) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$$

$Sol: R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 2R_1 \quad R_4 \to R_4 - 3R_1$

2) $$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - 7R_2 \quad R_4 \to R_4 - 6R_3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

$c_2 \to c_2 - c_1, \quad c_3 \to c_3 - c_1 \quad c_4 \to c_4 - c_1 \quad c_3 \to c_3 - 2c_2 \quad c_4 \to c_4 - 5c_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

$c_4 \to c_4 - 2c_3 \quad c_4 \to \frac{1}{18} c_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 4.$

10) Find the value of K such that the following matrix A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix}$ may have rank equal to a) 3 b) 2.

3) $Sol: R_2 \to R_2 - R_1 \quad R_3 \to R_3 + R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & k-1 \\ 0 & 3 & 9 & k^2-1 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & k-1 \\ 0 & 0 & 0 & k^2-3k+2 \end{bmatrix}$$

a) Rank of A can be 3 if the equivalent form of A has 3 non-Zero rows.

This is possible if $k^2 - 3k + 2 \neq 0$

$i.e, \ (k-1)(k-2) \neq 0$

$\rho(A) = 3 \quad if \ \ k \neq 1 \ \& \ \ k \neq 2$

4) b) Rank of A can be 2 if the equivalent form of A has 2 non-zero rows.

5) This is possible if $k^2 - 3k + 2 = 0$

$i.e, \ (k-1)(k-2) = 0 \Rightarrow k = 1 \ or \ \ k - 2$

6) $\iota(A) = 2 \quad if \ \ k = 1 \ \& \ \ k = 2$

$$A:B = \begin{bmatrix} a_{11}a_{12}.........a_{1n} : b_1 \\ a_{21}a_{22}.........a_{2n} : b_2 \\ a_{m1}a_{m2}.........a_{mn} : b_m \end{bmatrix}$$

The given sys of equation is consistent & will have unique soln.

Let us convert $A:B$ into a set of equation as follows

x + y + z = 6		
-2y + z = 7	-2y+3=-1	x+y+z=6
-3z = -9	-2y=-4	x=6-2-3=1
$\Rightarrow$ z = 3	y=2	x=1

x=1, y=2, z=3is the unique soln.

7) Solve the system of equations: x+2y=3z=0
2x+3y+z=0
4x+5y+4z=0
x+y-2z=0

$$Sol:\ A:B=\begin{bmatrix}1 & 2 & 3 & :0\\ 2 & 3 & 1 & :0\\ 4 & 5 & 4 & :0\\ 1 & 1 & -2 & :0\end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 4R_1 \quad R_4 \to R_4 - R_1 \quad R_3 \to R_3 - 3R_2 \quad R_4 \to R_4 - R_2$

$$=\begin{bmatrix}1 & 2 & 3:0\\ 0 & -1 & -5:0\\ 0 & -3 & -8:0\\ 0 & -1 & -5:0\end{bmatrix}=\begin{bmatrix}1 & 2 & 3:0\\ 0 & -1 & -5:0\\ 0 & 0 & 7:0\\ 0 & 0 & 0:0\end{bmatrix}$$

$\rho(A)=3$

$\rho\ A:B\ =3 \Rightarrow n=3$

Hence the system is consistent & will have trivial soln x=0 y=0 z=0

8) Does the following system of homogenous equations possess a non-trivial solutions? If so find them

$x_1+x_2-x_3+x_4=0$

$x_1-x_2+2x_3+x_4=0$

$3x_1+x_2+x_4=0$

$$A:B=\begin{bmatrix}1 & 1 & -1 & 1 & :0\\ 1 & -1 & 2 & -1 & :0\\ 3 & 1 & 0 & 1 & :0\end{bmatrix}$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 3R_1 \quad R_3 - R_2$

$$=\begin{bmatrix}1 & 1 & -1 & 1 & :0\\ 0 & -2 & 3 & -2 & :0\\ 0 & -2 & 3 & -2 & :0\end{bmatrix}=\begin{bmatrix}1 & 1 & -1 & 1 & :0\\ 0 & -2 & 3 & -2 & :0\\ 0 & 0 & 0 & 0 & :0\end{bmatrix}$$

Gauss elimination method:

The simplest method of solving systems of the form (1) of section 5.2 is the elimination method.

The Working Rule for the method is as given below.

Working rule:

Step1: Reduce the augmented matrix (A:B) to the form where A is in echelon form or in upper triangular form, by employing appropriate elementary row operations.

Step2: Write the linear equations associated with the reduce form obtained in Step 1. Let the number of equations in this reduced system be equal to r, If r=n, then the reduced system yields the unique solution the given system. If r<n, then n-r unknowns in the reduce system can be chosen arbitrarily and the reduced system yields infinitely many solutions of the given system.

1. : Solve the following system of linear equations by the Gauss elimination method

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + x_2 - x_3 = 1$$
$$x_1 - x_2 + 2x_3 = 2$$

Sol: For the given system, the coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

And the augmented matrix is $A:B = \begin{bmatrix} 1 & 1 & 1:4 \\ 2 & 1 & -1:1 \\ 1 & -1 & 2:2 \end{bmatrix}$

We reduce this matrix $A:B$ to the upper triangular form by using elementary operations

Using the row operation $R_2 \to R_2 - 2R_1 \ and\ R_3 \to R_3 - R_1, We\ get$

$$A:B \sim \begin{bmatrix} 1 & 1 & 1:4 \\ 0 & -1 & -3:-7 \\ 0 & -2 & 1:-2 \end{bmatrix}$$

Now, Using the row operation $\mathrm{R}_3 \to \mathrm{R}_3 - 2\,\mathrm{R}_2$ in this, we get

$$A:B \sim \begin{bmatrix} 1 & 1 & 1:\ 4 \\ 0 & -1 & -3:-7 \\ 0 & 0 & 7:12 \end{bmatrix}$$

We note that A is now reduced to the upper triangular form. The linear equations which correspond to this reduced form of $A:B$ are

$x_1 + x_2 + x_3 = 4$

$-x_2 - 3x_3 = -7$

$7x_3 = 12$

From equation (iii), wew find that $x_3 = 12/7$.

$x_2 = 7 - 3x_3 = 7 - \frac{36}{7} = \frac{13}{7}$

Substituting for $x_3$and x_2 found above in (i), we get

$x_1 = 4 - x_2 - x_3 = 4 - \frac{13}{7} - \frac{12}{7} = \frac{3}{7}.$

Thus, x_1=3/7, x_2=13/7, x_3=12/7 constitute the solutionof the given system.

2. Solve the following system of equations by Gauss's elimination method:

$4x_1 + x_2 + x_3 = 4$

$x_1 + 4x_2 - 2x_3 = 4$

$3x_1 + 2x_2 - 4x_3 = 6$

Sol: The augmented matrix is $A:B = \begin{bmatrix} 4 & 1 & 1 : 4 \\ 1 & 4 & -2 : 4 \\ 3 & 2 & -4 : 6 \end{bmatrix}$

$$A:B \sim \begin{bmatrix} 4 & 1 & 1 & :4 \\ 0 & 15/4 & -9/4 & :3 \\ 0 & -10 & 2 & :-6 \end{bmatrix}$$

Using $R_2 \to R_2 - (1/4)R_1, R_3 \to R_3 - 3R_2$

$$A:B \sim \begin{bmatrix} 4 & 1 & 1 & :4 \\ 0 & 5 & -3 & :4 \\ 0 & -5 & 1 & :-3 \end{bmatrix}$$

Using $R_2 \to (4/3)R_2, R_3 \to (1/2)R_3$

$$A:B \sim \begin{bmatrix} 4 & 1 & 1 & :4 \\ 0 & 5 & -3 & :4 \\ 0 & 0 & -2 & :1 \end{bmatrix}$$

Using $R_3 \to R_3 - R_2$,

We note that A is now reduced to the upper triangular form. The equations that correspond to (i) are

$4x_1 + x_2 + x_3 = 4$

$5x_2 - 3x_3 = 4$

$-2x_3 = 1.$

These yield.

$x_3 = -\frac{1}{2},\ x_2 = \frac{1}{5}(4+3x_3) = \frac{1}{2},\ x_1 = \frac{1}{4}(4 - x_2 - x_3) = 1.$

Thus, x_1=1, x_2=1/2, x_3=-1/2 constitute the solution of the given system.

3._ Solve the following system of equations by Gauss's elimination method:

$x + 2y + 2y = 1$

$2x + y + z = 2$

$3x + 2y + 2z = 3$

$x + z = 0$

Sol: The augmented matrix is $A:B = \begin{bmatrix} 1 & 2 & 2:1 \\ 2 & 1 & 1:2 \\ 3 & 2 & 2:3 \\ 0 & 1 & 1:0 \end{bmatrix}$

$$A:B \sim \begin{bmatrix} 1 & 2 & 2:1 \\ 0 & -3 & -3:0 \\ 0 & -4 & -4:0 \\ 0 & 1 & 1:0 \end{bmatrix}$$

Using $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$A:B \sim \begin{bmatrix} 1 & 2 & 2:1 \\ 0 & 1 & 1:0 \\ 0 & 1 & 1:0 \\ 0 & 1 & 1:0 \end{bmatrix}$$

Using $R_2 \to (-1/3)R_2, R_3 \to (-1/4)R_3$

$$A:B \sim \begin{bmatrix} 1 & 2 & 2:1 \\ 0 & 1 & 1:0 \\ 0 & 0 & 0:0 \\ 0 & 0 & 0:0 \end{bmatrix}$$

Using $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2,$

We note that A is now reduced to the echelon form. The system correspond to (i) is

$x + 2y = 2z = 1$

$y + z = 0$

These are two equations for three unknowns. Therefore, we can choose one of the un-known arbitrarily. Taking z=k, we get y=-k and x=1

Thus x=1,y=-k, z=k, Where k is arbitrary, is a solution of the given system.

5) Solve the following system of equations by Gauss's elimination method:

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + x_4 = -6$$

Sol : Consider augmented matrix $A:B$ *by* $R_1 \leftrightarrow R_4$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 5 & 1 & 1 & 1 & : & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1, R_4 \to R_4 - 5R_1$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 6 & 0 & -3 & : & 18 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix}$$

$R_4 \to 5R_4 + 4R_3$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & 0 & -117 & : & 234 \end{bmatrix}$$

Hence we have $x_1 + x_2 + x_3 + 4x_4 = -6$

$2x_2 - x_4 = 6$

$5x_3 - 3x_4 = 1$

$-117x_4 = 234$

$\therefore x_4 = -2, x_3 = -1, x_2 = 2$ *and* $x_1 = 1$ *is the reqd. soln.*

Gauss -Jordan method:

This method can be regarded as the modification of Gauss – elimination method.

This method aims in reducing the coefficient matrix A to a diagonal matrix.

1) Applying Gauss Jordan method solve $2x+3y-z=5$, $4x+4y-3z=3$, $2x-3y+2z=2$

Soln: $A:B = \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 4 & 4 & -3 & : & 3 \\ 2 & -3 & 2 & : & 2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A:B = \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 0 & -2 & -1 & : & -7 \\ 0 & -6 & 3 & : & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 0 & 2 & 1 & : & 7 \\ 0 & -2 & 1 & : & -1 \end{bmatrix}$$

$R_1 \to 2R_1 - 3R_2, R_3 \to R_3 + R_2$

$$A:B = \begin{bmatrix} 4 & 0 & -5 & : & -11 \\ 0 & 2 & 1 & : & 7 \\ 0 & 0 & 2 & : & 6 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & -5 & : & -11 \\ 0 & 2 & 1 & : & 7 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$R_1 \to R_1 + 5R_3, R_2 \to R_2 - R_3$

$$A:B = \begin{bmatrix} 4 & 0 & 0 & : & 4 \\ 0 & 2 & 0 & : & 4 \\ 0 & 0 & 2 & : & 6 \end{bmatrix}$$

Hence $4x = 4, 2y = 4, 2z = 6$

$\therefore x = 1, y = 2, z = 3$

2. Apply Gauss - Jordan method to solve the system of equations

$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$
$$x + y + z = 9$$

>> As it is convenient to have the leading coefficient as 1 we shall interchange the first and third equations. The augmented matrix will be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2, \; R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_1 \to R_2 + R_1, \; R_3 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$-1/4 \cdot R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \to 2R_3 + R_1, \; R_2 \to 3R_3 + R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Hence we have $x = 1, \; -y = -3, \; z = 5$

Thus $\boldsymbol{x = 1, \; y = 3, \; z = 5}$ is the required solution.

Iterative methods of solution of a system of algebraic equations

In this article we discuss two numerical iterative methods for solving a system of algebraic equations.

These two methods cannot be applied to any system of equations. It is applicable only when the numerically large coefficients are along the leading / principal diagonal of the coefficient matrix A associated with the system of equations usually represented in the form $AX = B$. Such a system is called a *diagonally dominant system*.

The methods are illustrated for the following system of three independent equations in three unknowns.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of equations is said to be diagonally dominant if

$$|a_{11}| > |a_{12}| + |a_{13}|, \ |a_{22}| > |a_{21}| + |a_{23}|, \ |a_{33}| > |a_{31}| + |a_{32}|$$

Sometimes we may have to rearrange the given system of equations to meet this requirement. If this condition is satisfied, the solution exists as the iteration process will converge.

Gauss - Seidel iterative method

We write the system of equations in the form

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \quad \ldots (1)$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \quad \ldots (2)$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \quad \ldots (3)$$

We start with the trial solution (*initial approximation*)

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$$

The first approximation are as follows.

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}\cdot 0 - a_{13}\cdot 0\right] = \frac{b_1}{a_{11}}$$

This approximation is immediately used in (2) so that we have

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\,x_1^{(1)} - a_{23}\cdot 0\right]$$

ie., $$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right)\right]$$

Finally, we use both these approximations in (3), so that we have

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}\,x_1^{(1)} - a_{32}\,x_2^{(1)}\right]$$

This completes first iteration.

The process is continued till we get the solution to the desired degree of accuracy.

Problem 1

Solve the following system of equations by Gauss - Seidel method

$$10x + y + z = 12$$

$$x + 10y + z = 12$$

$$x + y + 10z = 12$$

The given system of equations are diagonally dominant and the equations are put in the form

$$x = \frac{1}{10}\left[12 - y - z\right] \qquad \ldots (1)$$

$$y = \frac{1}{10}\left[12 - x - z\right] \qquad \ldots (2)$$

$$z = \frac{1}{10}\left[12 - x - y\right] \qquad \ldots (3)$$

Let us start with the trial solution $x = 0,\ y = 0,\ z = 0.$

First iteration :

$$x^{(1)} = \frac{1}{10}\left[12 - 0 - 0\right] = 1.2$$

$$y^{(1)} = \frac{1}{10}\left[12 - 1.2 - 0\right] = 1.08 \qquad z^{(1)} = \frac{1}{10}\left[12 - 1.2 - 1.08\right] = 0.972$$

Second iteration :

$$x^{(2)} = \frac{1}{10}\left[12 - 1.08 - 0.972\right] = 0.9948$$

$$y^{(2)} = \frac{1}{10}\left[12 - 0.9948 - 0.972\right] = 1.00332$$

$$z^{(2)} = \frac{1}{10}\left[12 - 0.9948 - 1.00332\right] = 1.000188$$

Third iteration :

$$x^{(3)} = \frac{1}{10}\left[12 - 1.00332 - 1.000188\right] = 0.99965$$

$$y^{(3)} = \frac{1}{10}\left[12 - 0.99965 - 1.000188\right] = 1.00002$$

$$z^{(3)} = \frac{1}{10}\left[12 - 0.99965 - 1.00002\right] = 1.00003$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}\left[12 - 1.00002 - 1.00003\right] = 0.999995 \approx 1$$

$$y^{(4)} = \frac{1}{10}\left[12 - 1 - 1.00003\right] = 0.999997 \approx 1$$

$$z^{(4)} = \frac{1}{10}\left[12 - 1 - 1\right] = 1$$

Thus $\boldsymbol{x = 1,\ y = 1,\ z = 1}$

Problem 2

Solve the following system of equations by Gauss - Seidel method.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

>> The equations are diagonally dominant and hence we first write them in the following form.

$$x = \frac{1}{20}\left[17 - y + 2z\right] \qquad y = \frac{1}{20}\left[-18 - 3x + z\right] \qquad z = \frac{1}{20}\left[25 - 2x + 3y\right]$$

We start with the trial solution $x = 0,\ y = 0,\ z = 0$

First iteration:

$$x^{(1)} = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20}\left[-18 - 3(0.85)\right] = -1.0275$$

$$z^{(1)} = \frac{1}{20}\left[25 - 2(0.85) + 3(-1.0275)\right] = 1.0109$$

Second iteration:

$$x^{(2)} = \frac{1}{20}\left[17 - (-1.0275) + 2(1.0109)\right] = 1.0025$$

$$y^{(2)} = \frac{1}{20}\left[-18 - 3(1.0025) + 1.0109\right] = -0.9998$$

$$z^{(2)} = \frac{1}{20}\left[(25 - 2(1.0025) + 3(-0.9998)\right] = 0.9998$$

Third iteration:

$$x^{(3)} = \frac{1}{20}\left[17 - (-0.9998) + 2(0.9998)\right] = 0.99997$$

$$y^{(3)} = \frac{1}{20}\left[-18 - 3(0.99997) + 0.9998\right] = -1.0000055$$

$$z^{(3)} = \frac{1}{20}\left[(25 - 2(0.99997) + 3(-1.0000055)\right] = 1.0000022$$

Thus $x = 1$, $y = -1$, $z = 1$ is the required solution.

Problem 3

Employ Gauss - Seidel iteration method to solve

$$5x + 2y + z = 12$$

$$x + 4y + 2z = .15$$

$$x + 2y + 5z = 20$$

Carryout 4 *iterations taking the initial approximation to the solution as* $(1, 0, 3)$

Soln:

>> The given system of equations are diagonally dominant and we put them in the following form.

$$x = \frac{1}{5}\,[\,12 - 2y - z\,]$$

$$y = \frac{1}{4}\,[\,15 - x - 2z\,]$$

$$z = \frac{1}{5}\,[\,20 - x - 2y\,]$$

By data, $x^{(0)} = 1\,,\ \ y^{(0)} = 0\,,\ \ z^{(0)} = 3$

First iteration :

$$x^{(1)} = \frac{1}{5}\,[\,12 - 2\,(\,0\,) - 3\,] = 1.8$$

$$y^{(1)} = \frac{1}{4}\,[\,15 - 1.8 - 2\,(\,3\,)\,] = 1.8$$

$$z^{(1)} = \frac{1}{5}\,[\,20 - 1.8 - 2\,(\,1.8\,)\,] = 2.92$$

Second iteration :

$$x^{(2)} = \frac{1}{5}\,[\,12 - 2\,(\,1.8\,) - 2.92\,] = 1.096$$

$$y^{(2)} = \frac{1}{4}\,[\,15 - 1.096 - 2\,(\,2.92\,)\,] = 2.016$$

$$z^{(2)} = \frac{1}{5}\,[\,20 - 1.096 - 2\,(\,2.016\,)\,] = 2.9744$$

Third iteration :

$$x^{(3)} = \frac{1}{5}\,[\,12 - 2\,(\,2.016\,) - 2.9744\,] = 0.99872$$

$$y^{(3)} = \frac{1}{4}\,[\,15 - 0.99872 - 2\,(\,2.9744\,)\,] = 2.01312$$

$$z^{(3)} = \frac{1}{5}\,[\,20 - 0.99872 - 2\,(\,2.01312\,)\,] = 2.995$$

Fourth iteration :

$$x^{(4)} = \frac{1}{5}\,[\,12 - 2\,(\,2.01312\,) - 2.995\,] = 0.995752$$

$$y^{(4)} = \frac{1}{4}\,[\,15 - 0.995752 - 2\,(\,2.995\,)\,] = 2.003562$$

$$z^{(4)} = \frac{1}{5}\,[\,20 - 0.995752 - 2\,(\,2.003562\,)\,] = 2.9994248$$

Thus the solution after four iterations correct to four decimal places is given by

$$\boldsymbol{x = 0.9958\,,\ \ y = 2.0036\,,\ \ z = 2.9994}$$

LINEAR TRANSFORMATION:

A Linear transformation in two dimensions is given by

$y_1 = a_1x_1 + a_2x_2$

$y_2 = b_1x_1 + b_2x_2$

This can be represented in the matrix form as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =)\ Y = AX$$

Similarly a linear transformation in 3 dimensions along with its matrix form is as,

$y_1 = a_1x_1 + a_2x_2 + a_3x_3$

$y_2 = b_1x_1 + b_2x_2 + b_3x_3$

$y_3 = c_1x_1 + c_2x_2 + c_3x_3$

A is called transformation matrix

If $|A| \neq 0$ Then y=AX is called non-Singular transformation or regular transformation.

If $|A| = 0$ Then y=AX is called Singular transformation

X= $A^{-1}y$ is called the inverse transformation.

Let z=By =B(AX)=(BA)X=CX Where C=BA Z=CX is called a composite linear transformation .

1. Show that the transformation

   $y_1 = 2x_1 + x_2 + x_3;$ $y_2 = x_1 + x_2 + 2x_3$ $y_3 = x_1 - 2x_3$ is regular. Write down the inverse transformation.

Sol: The given transformation may be written as

Y=AX

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} = 2(-2) - 1(-2-2) + 1(-1) = -4 + 4 - 1 = -1 \neq 0$$

$|A| \neq 0 \Rightarrow$ A is a non-singular matrix

The transformation is regular

The inverse transformation is X= $A^{-1}Y$

$$A^{-1}=\frac{adjA}{|A|}=\begin{bmatrix}2 & -2 & -1\\ -4 & 5 & 3\\ 1 & -1 & -1\end{bmatrix}$$

$$X = A^{-1}Y$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}2 & -2 & -1\\ -4 & 5 & 3\\ 1 & -1 & -1\end{bmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$$

$x_1 = 2y_1 + 2y_2 + y_3$

$x_2 = 4y_1 + 5y_2 + 3y_3$ is the inverse transformation.

$x_3 = y_1 - y_2 - y_3$

2. Prove that the following matrix is orthogonal

$$A\begin{bmatrix}-2/3 & 1/3 & 2/3\\ 2/3 & 2/3 & 1/3\\ 1/3 & -2/3 & 2/3\end{bmatrix}$$

$$\text{Sol:Consider}\quad AA' = I = \begin{bmatrix}-2/3 & 1/3 & 2/3\\ 2/3 & 2/3 & 1/3\\ 1/3 & -2/3 & 2/3\end{bmatrix}\begin{bmatrix}-2/3 & 2/3 & 1/3\\ 1/3 & 2/3 & -2/3\\ 2/3 & 1/3 & 2/3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

3. If A=1/3 $\begin{bmatrix}1 & 2 & a\\ 2 & 1 & b\\ 2 & -2 & c\end{bmatrix}$ is orthogonal. Find a,b,c & A^{-1} A is orthogonal $AA' = I$

$$Sol:\frac{1}{3}\begin{bmatrix}1 & 2 & a\\ 2 & 1 & b\\ 2 & -2 & c\end{bmatrix}\ \frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & b & c\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1+4+a^2 & 2+2+ab & 2-4+ac\\ 2+2+ab & 4+1+b^2 & 4-2+bc\\ 2-4+ac & 4-2+bc & 4+4+c^2\end{bmatrix}=\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$5+a^2=9 \qquad 5+b^2=9 \qquad 8+c^2=9$

$a^2=4 \qquad b^2=4 \qquad c^2=1$

$a=2 \qquad b=2 \qquad c=1$

$$AA' = I \Rightarrow A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

4. Find the inverse transformation of the following linear transformation

$$y_1 = x_1 + 2x_2 + 5x_3$$
$$y_2 = 2x_1 + 4x_2 + 11x_3$$
$$y_3 = -x_1 + 2x_3$$

Sol: Y=AX

$$A^{-1}Y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$x_1 = 19y_1 - 9y_2 + 2y_3$

$x_2 = -4y_1 + 2y_2 - y_3$ *is the inverse transformation*

$x_3 = -2y_1 + y_2$

5. Represents each of the transformation $y_1 = z_1 - 2z_2$ & $x_2 = -y_1 - 4y_2$, $y_2 = 3z$ by the use of matrix & find the composite transformation which express x_1, x_2 in terms of z_1, z_2

$Sol: x_1 = 3y_1 + 2y_2$

$x_2 = -y_1 + 4y_2$

$$\Rightarrow x = AY \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_1 = z_1 + 2z_2$

$$y_2 = 3z_1 \quad \Rightarrow y = BZ \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$X = AY = A(BZ) = ABZ = AB\ Z$$

$$AB = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$x_1 = 9z_1 + 6z_2$

$x_2 = 11z_1 + 2z_2$ is the required composite transformation

6.` Given the linear transformation

$y_1 = 5x_1 + 3x_2 + 3x_3$

$y_2 = 3x_1 + 2x_2 + 2x_3$

$y_3 = 2x_1 - x_2 + 2x_3$

$Sol: z_1 = 4x_1 + 2x_3$

$z_2 = x_2 + 4x_3$

$z_3 = 5y_3$

Express y_1, y_2, y_3

$\text{int erms of } z_1, z_2, z_3$

Given : Y=AX

$$Z = BX \Rightarrow X = B^{-1}Z \qquad B^{-1}\begin{bmatrix} 1/4 & 0 & -1/10 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$Y = (AB^{-1})Z$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} 5/4 & 3 & -23/10 \\ 3/4 & 1 & -23/10 \\ 1/2 & -1 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

<u>Eigen values and Eigen vectors of a square matrix:</u>

<u>Definition:</u> Let A be a given square matrix of order n. suppose I a non-zero Column vector X of order n and real or complex no. λ Such that AX= λ x

Then X is called an Eigen vector of A.

λ is called the corresponding Eigen value of A.

Working Rule:

1. Given square matrix A write down the characteristic equation $|A-\lambda I| = 0$
2. Solve the characteristic equation for Eigen values $\lambda_1, \lambda_2, \lambda_3, ...$
3. To find Eigen vector, write down the matrix equation as

$$A - \lambda I \quad X = 0 \quad where\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$$

4. We set $\lambda = \lambda_1$ in the matrix equation & solve it for Eigen vector x_1. Similarly we obtain Eigen vector $x_2, x_3.....$ for corresponding Eigen value $\lambda_2, \lambda_3.....$

1. Find the Eigen values & corresponding eigen vector of the following matrix

$A=\begin{bmatrix}-3 & 8\\ -2 & 7\end{bmatrix}$

$Sol: |A-\lambda I| = \begin{vmatrix}-3-\lambda & 8\\ -2 & 7-\lambda\end{vmatrix} = (-3-\lambda)(7-\lambda)+16$

$= \lambda^2 - 4\lambda - 5$

$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$

The roots of this equation are λ_1=5 & λ_2=-1. These are the two Eigen value of the given matrix A. Let X= x, y^T, then the matrix equation $(A-\lambda I)X = 0$

$\Rightarrow \begin{bmatrix}-3-\lambda & 8\\ -2 & 7-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$\lambda = \lambda_1 = 5$

$\begin{bmatrix}-8 & 8\\ -2 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

-8x+8y=0, -2x+2y=0. Both of these reduces to some equation x-y=0 =) x=y. If we choose x=a, then y=a These, when $\lambda = \lambda_1 = 5, x_1 = \begin{bmatrix}a\\ a\end{bmatrix}$ is the solution of (1)

$\lambda_0 = \lambda_2 = -1$ equation (1) becomes

$\begin{bmatrix}-2 & 8\\ -2 & 8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

For $-2x+8y=0,\ x-4y=0$. Hence if we choose y=b, then x=4b

Thus when $\lambda = \lambda_2 = -1,\ x_2 = \begin{bmatrix}Ab\\ b\end{bmatrix}$ is the soln of (1)

2. Find the Eigen values & the Eigen vectors of the matrix

$A=\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$

Sol: For the given matrix, the characteristic polynomial is

$|A-\lambda I| = \begin{bmatrix}2-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 1 & 0 & 2-\lambda\end{bmatrix}$

$= (2-x)^3 - (2-\lambda)$

$= (\lambda-3)(2-\lambda)(\lambda-1)$

The characteristic equation of the given matrix is
$(2-\lambda)(\lambda-3)(\lambda-1)=0$

The roots are $\lambda_1=1$ $\lambda_2=2$ & $\lambda_3=3$ these are the Eigen value of the given matrix.

$x, y, z^{\ T}=X$, then the matrix equation (A-λI)x=0

$$\begin{bmatrix} 2\text{-}\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\rightarrow(1)$$

$For\ \lambda=\lambda_1=1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x+z=0\ \&\ y=0$

If we choose x=a, then z=-a; $x_1 = a_1\ \ 0_1\ -a^{\ T}$

$If\ \ a=1$ $x_1=\left[1\ \ 0\ -1\right]^T$

$$For\ \lambda=\lambda_2=2\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x=0,\ z=0.$ we take y=b; $x_2=\left[0\ \ b\ \ 0\right]^T$

$If\ \ b=2$ $x_2=\left[0\ \ 2\ \ 0\right]^T$

Let

$$For\ \lambda=\lambda_3=3\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x+z=0,\ y=0.$ we take x=c; then z=c $x_3=\left[c\ \ o\ \ c\right]^T$

$If\ \ c=3$ $x_3=\left[3\ \ 0\ \ 3\right]^T$

3. Find the matrix P which reduces the matrix $A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form

Hence find A^4

$$\left|A-\lambda I\right|=\begin{bmatrix} 1\text{-}\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}=0$$

$\lambda_1=-2$ $\lambda_2=3$ $\lambda_3=6$

$$|A-\lambda I| = \begin{bmatrix} 1\text{-}\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = 0$$

$\lambda_1 = -2 \quad \lambda_2 = 3 \quad \lambda_3 = 6$

Case(1): $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x + y + 3z = 0$

$x + 7y + z = 0$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x}{-20} = \frac{-y}{0} = \frac{z}{20}$$

$Case ii): \lambda_2 = 3$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + 3z = 0$

$x + 2y + z = 0$

$3x + y - 2z = 0$

$$\frac{x}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5}$$

$Case iii): \lambda_3 = 6$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x + y + 3z = 0$

$x - y + z = 0$

$3x + y - 5z = 0$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 1 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{4}=\frac{-y}{-8}=\frac{z}{4}$$

$$p = \; x_1\,x_2\,x_3 \; = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$p^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$p^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$$

$$A^4 = PD^4p^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$$

5) Diagonalizable the matrix $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ *and find* A^5

$$(A-\lambda I)=0$$

$$\begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{bmatrix} = 0$$

$$-\lambda^3+3\lambda^3-2\lambda=0$$

$$-\lambda\left[\lambda^2-3\lambda+2\right.=0$$

$$x=0 \quad \lambda=1 \quad \lambda=2$$

$Case i : \lambda = 0$

$$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$11x - 4y - 7z = 0$

$7x - 2y - 5z = 0$

$10x - 4y - 6z = 0$

$$\frac{x}{\begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 7 & -5 \\ 10 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 7 & -2 \\ 10 & -4 \end{vmatrix}}$$

$$\frac{x}{-8} = \frac{-y}{8} = \frac{z}{-8}$$

$Case ii : \lambda = 1$

$$\begin{bmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$10x - 4y - 7z = 0$

$7x - 3y - 5z = 0$

$10x - 4y - 7z = 0$

$$\frac{x}{\begin{vmatrix} -4 & -7 \\ -3 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 10 & -7 \\ 7 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 10 & -4 \\ 7 & -3 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-2}$$

$Case iii : \lambda = 2$

$$\begin{bmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$9x - 4y - 7z = 0$

$7x - 4y - 5z = 0$

$10x - 4y - 8z = 0$

$$\frac{x}{\begin{vmatrix}-4 & -5\\ -4 & -8\end{vmatrix}} = \frac{-y}{\begin{vmatrix}7 & -5\\ 10 & -8\end{vmatrix}} = \frac{z}{\begin{vmatrix}7 & -4\\ 10 & -4\end{vmatrix}}$$

$$\frac{x}{12} = \frac{-y}{-6} = \frac{z}{12}$$

$$p = \; x_1\, x_2\, x_3 \; = \begin{bmatrix}1 & 1 & 2\\ 0 & -1 & 1\\ 1 & 2 & 2\end{bmatrix}$$

$$p^{-1} = \begin{bmatrix}-4 & 2 & 3\\ -1 & 0 & 1\\ 3 & -1 & -2\end{bmatrix}$$

$$p^{-1}AP = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix} = D$$

$$A^5 = PD^5 p^{-1} = \begin{bmatrix}191 & -64 & -127\\ 97 & -32 & -65\\ 190 & -64 & -126\end{bmatrix}$$

Quadratic forms:

A homogeneous expression of the second degree in any number of variables is called a quadratic form (Q.F).

In general for two variables x_1, x_2 $i.e, n = 1, 2$ $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ is called QF in two variables.

The matrix A of the above Q.F is $A = \begin{bmatrix} coeff\ of\ x_1^2 & \frac{1}{2}\ coeff\ of\ \ x_1x_2 \\ \frac{1}{2}\ coeff\ of\ \ x_1x_2 & coeff\ of\ x_1^2 \end{bmatrix}$

$Eg : x^2 + y^2 + xy$ $\qquad A = \begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}$

2) $2x^2 + 3y^2 + 6xy$ $\qquad A = \begin{bmatrix}2 & 3\\ 3 & 3\end{bmatrix}$

3) $5x_1^2 + 7x_2^2 + 12\, x_1x_2$ $\qquad A = \begin{bmatrix}5 & 6\\ 6 & 7\end{bmatrix}$

Similarly Q.F in 3 variables is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_2 + 2a_{23}x_1x_2$$

$$A = \begin{bmatrix} coeff\ of\ x_1^2 & \frac{1}{2}coeff\quad x_1x_2 & \frac{1}{2}coeff\quad x_1x_3 \\ \frac{1}{2}Coeff\ of\quad x_1x_2 & coeff\ of\ x_1^2 & \frac{1}{2}coeff\quad x_2x_3 \\ \frac{1}{2}Coeff\ of\quad x_1x_2 & \frac{1}{2}coeff\quad x_2x_3 & coeff\ of\ x_3^2 \end{bmatrix}$$

$$Examples: 1)\ QF: x^2 + y^2 + z^2 + xy + 2yz = 4zx \qquad A = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ \frac{1}{2} & 1 & \\ 2 & 1 & 1 \end{bmatrix}$$

$$2)\ 5x^2 + 2yz + 6y^2 + 9z^2 + 4xy \qquad A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 9 \end{bmatrix}$$

$$3)\ xy + yz + zx \quad A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Canonical Form (sum of squares):

$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$ *where* $\lambda_1, \lambda_2, ...\lambda_n$ *are* Eigen values called canonical form.

Rank Index & Signature of canonical form

The number of non-zero terms present in a canonical form of Q is called rank of Q it, (r)

$Ex: 2y_1^2 + y_2^2 - y_3^2 \Rightarrow r = 3 \qquad y_1^2 + y_3^2 \Rightarrow r = 2$

1. The number of the terms present in a canonical form is called index of Q. (p)

$Ex: 2y_1^2 + 3y_2^2 - 5y_3^2 \qquad p = 2$

2. The difference between the negative terms in a canonical form is called signature of Q (s)

$Ex: y_1^2 - 3y_2^2 + y_3^2 \quad s = 2-1=1$

Nature of Quadratic Form:

r=rank, p=index n=number of variables

Condition	Meaning	Nature of Q.F	Eg;
r=n,p=n	All n Co efficient are positive	+ve definite	$2y_1^2 + y_2^2 + 8y_3^2 +$
r=n,p=0	All n Co efficient are -ve	-ve definite	$-y_1^2 - y_2^2 - 6y_3^2$
r=p,p<n for r=2=p 2<3	At least one of the Co-efficient Zero & all other Co-efficientp are +ve	+ve Semi definite	$y_2^2 + 5y_3^2$
r<n,p=0	At least one of the Co-efficient Zero & all other Co-efficient are -ve	-ve Semi definite	$-y_2^2 - 10y_3^2$

Note: Q.F is indefinite if some of the Co-efficient are +ve and some are -ve

Eg: $y_1^2 - y_2^2 + 3y_3^2$

Working rule to reduce Q.F to Canonical (sum of squares) form by orthogonal transformation.

1. Write down the matrix A to Q.F
2. Find the Eigen values & the corresponding eigen vectors of matrixA.
3. Normalise the Eigen vector x_1, x_2, x_3

$$i.e, x_1^1 = \frac{x_1}{\|x_1\|}$$

$$If \ \ x_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \|x_{11}\| = \sqrt{a^2 + b^2 + c^2}$$

$$111^{iy} \ x_2^1 = \frac{x_2}{\|x_2\|} \qquad x_3^1 = \frac{x_3}{\|x_3\|}$$

4. Write down the associated orthogonal model matrix $Q = \begin{bmatrix} x_1^1 & x_2^1 & x_3^1 \end{bmatrix}$
5. Since $pp^1 = I \Rightarrow p^{-1} = p^1$

   Then $p^{-1}AP = p^1AP =$ diagonal matrix $\lambda_1, \lambda_2, \lambda_3$

6. The associated canonical form is $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

7. x=py where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \& Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

   will give us the orthogonal linear transformation.

1) Obtain the canonical form of the quadratic form

   $Sol: 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$

   $$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

   $(A - \lambda I) = 0$

   $$\Rightarrow \begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix} = 0$$

   $$(2-\lambda)\ \ (2-\lambda)^2 - 1\ \ +1\ \ -(2-\lambda) - 1\ \ -1\ \ 1 + (2-\lambda)\ \ = 0$$

   $$(2-\lambda)\ \ 4 + \lambda^2 - 4\lambda - 1\ \ -\ \ 2 - \lambda + 1\ \ -\ \ 3 - \lambda\ \ = 0$$

   $$2-\lambda\ \ \ \lambda^2 - 4\lambda + 3\ \ -\ \ 3-\lambda\ \ -3 + \lambda = 0$$

   $$2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda - 3 + \lambda - 3 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 9) = 0 \quad \lambda(\lambda - 3)^3 = 0$$

$\lambda = 0, 3, 3$ *i.e*, $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0$ are roots and are the Eigen values of A.

2) The canonical form of the given Q.P that we get by an orthogonal transformation is $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 3y_2^2 + 3y_3^2$

   Sol: Since one Co- efficient in this canonical form is zero & the other two are +ve, the Q.F is +ve Semi-definite

   Rank, r=2 Index, p=2 & Signature, s=2.

3) Reduce the Q.F $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2 x_3$ to the canonical form by an orthogonal transformation

   $$Sol: A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$(A-\lambda I)=0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix}=0$$

$$(1-\lambda)\ \ (3-\lambda)^2-1\ \ -0=0$$

$$(1-\lambda)\ \ 9+\lambda^2-6\lambda-1\ \ =0$$

$$(1-\lambda)\ \ \lambda^2-6\lambda+8\ \ =0$$

$$\lambda^2-6\lambda+8-\lambda^3+6\lambda^2-8\lambda=0$$

$$-\lambda^3+7\lambda^2-14\lambda+8=0 \Rightarrow \lambda^3-7\lambda^2+14\lambda-8=0$$

$$(\lambda-1)\ \ (\lambda-2)\ \ (\lambda-4)=0$$

The eigen values of A are $\lambda_1=1 \quad \lambda_2=2 \quad \lambda_3=4$

$Case i: For\ \lambda_1=1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}}=\frac{z}{\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{3}=\frac{-y}{0}=\frac{z}{0}$$

$$x_1=\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$11x_1 11=\sqrt{1^2+0+0}=1$$

$$x_1^1=\frac{x_1}{\|x_1\|}=\frac{100^{\ T}}{1}=\ 1 \quad 0 \quad 0^{\ T}$$

$$\lambda_2=2$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}}=\frac{z}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{0}=\frac{-y}{-1}=\frac{z}{+1}$$

$$x_2=\begin{bmatrix}3\\1\\+1\end{bmatrix}$$

$$11x_2 11=\sqrt{0+1+1}=\sqrt{2}$$

$$x_2' = \frac{x_2}{\|x_2\|}=\frac{1}{\sqrt{2}}= \ 0 \quad 1 \quad +1^{\ T}=\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^T$$

$$\lambda_3=4$$

$$\begin{bmatrix}-3 & 0 & 0\\0 & -1 & -1\\0 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{x}{\begin{vmatrix}-1 & -1\\-1 & -1\end{vmatrix}}=\frac{-y}{\begin{vmatrix}-3 & 0\\0 & -1\end{vmatrix}}=\frac{z}{\begin{vmatrix}-3 & 0\\0 & -1\end{vmatrix}}$$

$$\frac{x}{0}=\frac{-y}{3}=\frac{z}{3}$$

$$x_1=\begin{bmatrix}0\\-3\\3\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$11x_3 11=\sqrt{0+1+1}=\sqrt{2}$$

$$x_3^1=\frac{x_3}{\|x_3\|}=\frac{0 \ -1 \ \ 1^{\ T}}{\sqrt{2}}=\begin{bmatrix}1 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^T$$

The orthogonal model matrix for A is

$$Q=\begin{bmatrix}x_1^1 & x_2^1 & x_3^1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

X=Q.F is the orthogonal transformation that reduces the given Q.F to the canonical form.
The canonical form is

$$\lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2=y_1^2+2y_2^2+4y_3^2$$

Rayleigh's power method

Rayleigh's power method is an iterative method to determine the numerically largest eigen value and the corresponding eigen vector of a square matrix.

Working procedure:

- Suppose A is the given square matrix, we assume initially an eigen vector X_0 in a simple for like $[1,0,0]'$ *or* $[0,1,0]'$ *or* $[0,0,1]'$ *or* $[1,1,1]'$ and find the matrix product AX_0 which will also be a column matrix.
- We take out the largest element as the common factor to obtain $AX_0 = \lambda^1 X^1$.
- We then find AX^1 and again put in the form $AX^1 = \lambda^2 X^2$ by normalization.
- The iterative process is continued till two consecutive iterative values of λ and X are same upto a desired degree of accuracy.
- The values so obtained are respectively the largest eigen value and the corresponding eigen vector of the given square matrix A.

Problems:

1) Using the Power method find the largest eigen value and the corresponding eigen vector starting with the given initial vector.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \text{ given } [\text{redacted}] \quad 0 \quad 0]^T$$

$$Solution: \quad AX^0 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^1 X^1$$

$$AX^1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^2 X^2$$

$$AX^2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda^3 X^3$$

$$AX^{3} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda^{4} X^{4}$$

$$AX^{4} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = 2.98 \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda^{5} X^{5}$$

$$AX^{5} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \begin{bmatrix} 2.997 \\ 0 \\ 2.994 \end{bmatrix} = 2.997 \begin{bmatrix} 1 \\ 0 \\ 0.999 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Thus the **largest eigen value** is approximately **3** and the corresponding **eigen vector** is [1 ,0, 1]’

2) Using the Power method find the largest eigen value and the corresponding eigen vector starting with the given initial vector.

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \textit{given } [1 \quad 0.8 \quad -0.8]^{T}$$

$$\textit{Solution}: \quad AX^{0} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6 \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \lambda^{1} X^{1}$$

$$AX^{1} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86 \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{2} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96 \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \lambda^{3} X^{3}$$

$$AX^{3} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98 \begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \lambda^{4} X^{4}$$

$$AX^{4} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \begin{bmatrix} 5.994 \\ 5.988 \\ -5.988 \end{bmatrix} = 5.994 \begin{bmatrix} 1 \\ 0.999 \\ -0.999 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

Thus after five iterations the numerically largest eigen value is **5.994** and corresponding eigen vector is **[1, 0.999, -0,999]'**

6) Using Rayleigh's power method to find the largest Eigen value and the corresponding Eigen vector of the matrix.
(Dec 2012)

Sol: $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ $\quad X^{(0)} = [1,0,0]'$

$$AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.333 \\ 0.3333 \end{bmatrix}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 0.3333 \end{bmatrix} = \begin{bmatrix} 7.3332 \\ -3.3332 \\ 3.3332 \end{bmatrix} = 7.3332\begin{bmatrix} 1 \\ -0.4545 \\ 0.4545 \end{bmatrix}$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.4545 \\ 0.4545 \end{bmatrix} = \begin{bmatrix} 7.818 \\ -3.818 \\ 3.818 \end{bmatrix} = 7.818\begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix}$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix} = \begin{bmatrix} 7.952 \\ -3.952 \\ 3.952 \end{bmatrix} = 7.952\begin{bmatrix} 1 \\ -0.4969 \\ 0.4969 \end{bmatrix}$$

The largest Eigen value is $\lambda = 7.952$ and the corresponding Eigen vector is

$$\begin{bmatrix} 1 & -0.4969 & 0.4969 \end{bmatrix}'$$