MODULE III

VECTOR CALCULUS

CONTENTS:

Introduction:

Basically vector is a quantity having both magnitude and direction. Vector quantities like force, velocity, acceleration etc. have lot of reference in physical and engineering problems. We are familiar with vector algebra which gives an exposure to all the basic concepts related to vectors.

Differentiation and Integration are well acquainted topics in calculus. In the background of all these we discuss this chapter vector calculus comprising vector Differentiation. Many concepts are highly significant in various branches of engineering.

Basic Concepts – Vector function of a single variable and the derivative of a vector

Let the position vector of a point p(x, y, z) in space be

$$\vec{r} = xi + yi + zk$$

If x, y, z are all functions of a single parameter t, then $\vec{r}$ is said to be a vector function of t which is also referred to as a vector point function usually denoted as $\vec{r} = \vec{r}$ (t). As the parameter t varies, the point P traces in space. Therefore

$\vec{r}$ = x(t) i+y (t) j+z(t) k

is called as the vector equation of the curve.

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

Is a vector along the tangent to the curve at P.

If t is the time variable,

$\vec{\upsilon} = \frac{d\vec{r}}{dt}$ gives the velocity of the particle at time t.

Further $\vec{a} = \frac{d\vec{\upsilon}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity $\vec{\upsilon}$ and is called the acceleration of the particle at time t.

1. $\frac{d}{dt}\ c_1\vec{r_1}(t) \pm c_2\vec{r_2}(t) = c_1\vec{r_1}'(t) \pm c_2\vec{r_2}(t)$ where c_1 , c_2 are constants.

2. $\frac{d}{dt}\left(\vec{F}.\vec{G}\right) = \vec{F}.\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}.\vec{G}$

3. $\frac{d}{dt}\left(\vec{F}\times\vec{G}\right)=\vec{F}.\frac{d\vec{G}}{dt}+\frac{d\vec{F}}{dt}\times\vec{G}$

$Where \quad \vec{F}=\vec{F}(t) \; and \; \vec{G}=\vec{G}(t).$

Gradient, Divergence, Curl and Laplacian:

If ϕ is scalar function $\vec{A}$ is vector function $\vec{A}=a_1 i+a_2 j+a_3 k$ then

1. ie., grad $\phi=\nabla\phi=\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$

2. If $\vec{A}=a_1 i+a_2 j+a_3 k,$

$$div\,\vec{A}=\nabla.\vec{A}=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right).\; a_1 i+a_2 j+a_3 k$$

$$div\,\vec{A}=\nabla.\vec{A}=\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}$$

3. If $\vec{A}=a_1 i+a_2 j+a_3 k,$

$$div\,\vec{A}=\nabla\times\vec{A}=\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{pmatrix}.$$

$$=i\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)-j\left(\frac{\partial a_3}{\partial x}-\frac{\partial a_1}{\partial z}\right)+k\left(\frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial y}\right)$$

4. Laplacian of $\phi=\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

5. Laplacian of $\vec{A}=\nabla^2\vec{A}=\frac{\partial^2\vec{A}}{\partial x^2}+\frac{\partial^2\vec{A}}{\partial y^2}+\frac{\partial^2\vec{A}}{\partial z^2}$

Important points:

1. If $\vec{r}=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}, then\, \frac{d\vec{r}}{dt}$ is velocity and $\frac{d^2\vec{r}}{dt^2}$ is acceleration.

2. The unit tangent vector $\hat{T} = \dfrac{\left(\dfrac{d\vec{r}}{dt}\right)}{\left|\dfrac{d\vec{r}}{dt}\right|}$ and unit normal vector is $\hat{n} = \dfrac{\dfrac{d\vec{T}}{ds}}{|\hat{T}|}$ where

$\hat{T} = \dfrac{ds}{dt}$.

3. If $\vec{A}$ and $\vec{B}$ are any two vector and θ is angle between two vectors, then

$$\theta = \cos^{-1}\left(\frac{\vec{A}\,.\,\vec{B}}{|\vec{A}|\,.\,|\vec{B}|}\right).$$

4. Component of a vector (velocity or acceleration) $\vec{F}$ along a given vector $\vec{C}$ is the resolved part of $\vec{F}$ given by $\vec{F}\,.\,\hat{n}$ where $\hat{n} = \dfrac{\vec{c}}{|\vec{c}|}$.

5. Component of a vector $\vec{F}$ along normal to the $\vec{c}$ is given by

$$\left|\vec{F} - \text{resolved part of acceleration along } \vec{c}\right| = \left|F - \left(\vec{F}\,.\,\vec{c}\right).\hat{c}\right|.$$

1. Find the unit tangent vector to the curve $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$.

Soln: Given the space curve $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$

$$\therefore \vec{T} = \frac{d\vec{r}}{dt} = -\sin t\hat{i} + \cos t\hat{j} + \hat{k}$$

$$|\vec{T}| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

Therefore, the unit tangent vector to the given curve at any point is

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{-\sin t\hat{i} + \cos t\hat{j} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\ -\sin t\hat{i} + \cos t\hat{j} + \hat{k}$$

2. Find the unit normal vector to the curve $\vec{r} = 4\sin t\hat{i} + 4\cos t\hat{j} + 3t\hat{k}$.

Soln: Given $\vec{r} = 4\sin t\hat{i} + 4\cos t\hat{j} + 3t\hat{k}$

$$\therefore \vec{T} = \frac{d\vec{r}}{dt} = 4\cos t\hat{i} - 4\sin t\hat{j} + 3\hat{k}$$

$$\left|\vec{T}\right| = \sqrt{16\ \cos^2 t + \sin^2 t\ \ + 9} = \sqrt{25} = 5$$

Therefore, the unit tangent vector to the given curve at any point t is

$$\hat{T} = \frac{\vec{T}}{\left|\vec{T}\right|} = \frac{4\cos t\hat{i} - 4\sin t\hat{j} + 3\hat{k}}{5} = \frac{1}{5}\ 4\cos t\hat{i} - 4\sin t\hat{j} + 3\hat{k}$$

$$\frac{d\hat{T}}{ds} = \left(\frac{\frac{d\hat{T}}{dt}}{\frac{ds}{dt}}\right) = \left(\frac{\frac{d\hat{T}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}\right) = \frac{1}{5}\left[\frac{1}{5}\ -4\sin t\hat{i} - 4\cos t\hat{j}\ \right]$$

$$\frac{d\hat{T}}{ds} = -\frac{4}{25}\ \sin t\hat{i} + \cos tj$$

$$and\ \left|\frac{d\hat{T}}{ds}\right| = \sqrt{\left(\frac{4}{25}\right)^2\ \sin^2 t + \cos^2 t}\ = \frac{4}{25}$$

The unit normal vector to the given curve is

$$\hat{n} = \frac{\left(\frac{d\hat{T}}{ds}\right)}{\left|\frac{d\hat{T}}{ds}\right|} = \frac{(-4/25)\ \sin t\hat{i} + \cos t\hat{j}}{(4/25)} = -\sin t\hat{i} - \cos t\hat{j}$$

3. Find the angle between tangents to the curve $x = t^2, y = t^3, z = t^4$ at t=2 and t=3.

Soln: Define the position vector $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{r} = r^2\hat{i} + t^3\hat{j} + t^4\hat{k}$$

i.e., $$\vec{T} = \frac{d\vec{r}}{dt} = 2t\hat{i} + 3t^2\hat{j} + 4t^3\hat{k}$$

$$\vec{A} = \vec{T}\Big|_{t=2} = 4\hat{i} + 12\hat{j} + 32\hat{k} = 4(\hat{i} + 3\hat{j} + 8\hat{k})$$

$$\therefore \left|\vec{A}\right| = \sqrt{16(1+9+64)} = 4\sqrt{74}$$

$$\vec{B} = \vec{T}\Big|_{t=3} = 6\hat{i} + 27\hat{j} + 108\hat{k} = 3(2\hat{i} + 9\hat{j} + 36\hat{k})$$

$$\therefore \left|\vec{A}\right| = \sqrt{9(4+81+1296)} = 3\sqrt{1381}$$

Let θ be the angle between two vectors $\vec{A}$ and $\vec{B}$, then

$$\theta = \cos^{-1}\left(\frac{\vec{A}\,.\,\vec{B}}{\left|\vec{A}\right|.\left|\vec{B}\right|}\right) = \cos^{-1}\left(\frac{4\ \hat{i}+3\hat{j}+8\hat{k}\ .3\ \ 2\hat{i}+9\hat{j}+36\hat{k}}{4\sqrt{74}\times 3\sqrt{1381}}\right).$$

$$= \cos^{-1}\left(\frac{2+27+248}{\sqrt{74}\times\sqrt{1381}}\right) = \cos^{-1}\ 0.8665$$

$$\theta = \ 30^{\circ}$$

4. A particle moves along the curve $x = 1 - t^3, y = 1 + t^2, z = 2t - 5.$ Determine its velocity and acceleration. Find the component of velocity and acceleration at t=1 in the direction $2\hat{i} + \hat{j} + 2\hat{k}$

Soln: Given the position vector

$$\vec{r} = (1-t^3)\hat{i} + (1+t^2)\hat{j} + (2t-5)\hat{k}$$

$$the\,velocity,\ \vec{v} = \frac{d\vec{r}}{dt} = (-3t^2)\hat{i} + (2t)\hat{j} + (2)\hat{k}$$

$$and\ \ acceleration\ \vec{a} = \frac{d^2\vec{r}}{dt^2} = (-6t^3)\hat{i} + 2\hat{j}$$

$$at\,t = 1, \qquad \vec{v} = -3\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{a} = -6\hat{i} + 2\hat{j}$$

Therefore, the component of velocity vector in the given direction

$$\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}.$$

$$\vec{v}.\hat{n} = (-3\hat{i} + 2\hat{j} + 2\hat{k}).\frac{(2\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{4+1+4}} = \frac{-6+2+4}{3} = 0$$

The normal component of acceleration in the given direction

$$\vec{a}.\hat{n} = (-6\hat{i} + 2\hat{j}).\frac{(2\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{4+1+4}} = \frac{-12+2}{3} = -\frac{10}{3}.$$

5. A particle moves on the curve $x = 2t^2, y = t^2 - 4t, z = 3t - 5,$ where t is the time. Find the components of velocity and acceleration at time t=1 in the direction $\hat{i} + 3\hat{j} + 2\hat{k}.$

Soln: The position vector at any point (x,y,z) is given $\vec{r} = xi + yj + zk,$ but

$\vec{r} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$

Therefore, the velocity and acceleration are

$$\vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\hat{i} + 2\hat{j}$$

$$at\ t = 1, \quad \vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{a} = 4\hat{i} + 2\hat{j}$$

Therefore the component of velocity in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\vec{v}.\hat{n} = (4\hat{i} - 2\hat{j} + 3\hat{k}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}} = \frac{16}{\sqrt{14}}$$

Since dot product of two vector is a scalar.

The components of acceleration in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\vec{a}.\hat{n} = (4\hat{i} + 2\hat{j}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}} = \frac{-2}{\sqrt{14}}$$

Unit Normal Vector:

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{where } \hat{n} = \nabla\phi = \text{Normal vector}$$

Directional Derivative (D.D.):

If $\vec{a}$ is any vector and ϕ is any scalar point function then Directional Derivative

(D.D.)= $\nabla\phi.\vec{a} = \dfrac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Maximum directional derivative (Normal derivative)

The directional derivative will be the maximum in the direction $\nabla\phi (i.e., \vec{a} = \nabla\phi)$

and the maximum value of directional derivative = $\nabla\phi. = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{|\nabla\phi|^2}{|\nabla\phi|} = |\nabla\phi|.$

Note : The maximum directional derivative is also called normal derivative i.e., Normal derivative $=|\nabla\phi|$

Equations of Tangent plane and Normal Line:

Let $\phi(x,y,z)=c$ be any given surface and (x_1, y_1, z_1) be a point on it, then

(i) Equation of tangent plane to $\phi(x,y,z)=c \ at \ P(x_1,y_1,z_1)$ is

$$(x-x_1)\left(\frac{\partial\phi}{\partial x}\right)_p+(y-y_1)\left(\frac{\partial\phi}{\partial y}\right)_p+(z-z_1)\left(\frac{\partial\phi}{\partial z}\right)_p=0$$

(ii) Equation of normal line to $\phi(x,y,z)=c \ at \ P(x_1,y_1,z_1)$ is

$$\frac{x-x_1}{\left(\frac{\partial\phi}{\partial x}\right)_p}=\frac{y-y_1}{\left(\frac{\partial\phi}{\partial y}\right)_p}=\frac{z-z_1}{\left(\frac{\partial\phi}{\partial z}\right)_p}$$

Ex. 1: If $\phi=x^3y^3z^3$, find $\nabla\phi$ at (1,2,1) along $\hat{i}+2\hat{j}+2\hat{k}$

Soln: Given $\phi=x^3y^3z^3$, then

$$\nabla\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$=(3x^2y^3z^3)\hat{i}+(3x^3y^2z^3)\hat{j}+(3x^3y^3z^2)\hat{k}$$

$$\nabla\phi|_{(1,2,1)}=24\hat{i}+12\hat{j}+24\hat{k}$$

$$Let\ \vec{a}=\hat{i}+2\hat{j}+2\hat{k}\Rightarrow\left|\vec{a}\right|$$

$$=\sqrt{1+4+4}=3$$

$$\hat{a}=\frac{\vec{a}}{\left|\vec{a}\right|}=\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

$\nabla\phi$ at (1,2,1) along the vectors $\vec{a}$ is

$$\nabla\phi.\vec{a} = 24\hat{i}+12\hat{j}+24\hat{k} \ .\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

$$=\frac{1}{3}(24+24+48)=\frac{96}{3}$$

$$\nabla\phi.\vec{a}=32$$

Ex. 2: Find the unit normal vector to the surface $xy^3z^2 = 4$ at the point (-1,-1,2).

$$\phi = xy^3z^2$$

Soln: Let $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = y^3z^2)\hat{i} + 3x^2z^2\hat{j} + 2xy^3z\hat{k}$

$$and\ \nabla\phi\,\Big|_{(-1,-1,2)} = -4\hat{i} - 12\hat{j} + 4\hat{k}$$

Therefore, the normal vector to the given surface is

$$\vec{n}\ \nabla\phi\,\Big|_{(-1,-1,2)} = -4\hat{i} - 12\hat{j} + 4\hat{k}$$

$$\left|\vec{n}\right| = \sqrt{16+144+16} = \sqrt{176} = 4\sqrt{11}$$

$$\therefore \vec{n} = \frac{\vec{n}}{\left|\vec{n}\right|} = \frac{1}{\sqrt{11}}\ -\hat{i} - 3\hat{j} + \hat{k}$$

Ex. 3: Find the angle between the normals to the surface $xy = z^2$ at the points (1, 4, 2) and (-3,-3, 3).

Soln: Let $\phi\ \ x, y, z\ = xy - z^2$ be the surface

The normal to the surface is

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy\text{-}z^2) + \hat{j}\frac{\partial}{\partial y}(xy\text{-}z^2) + \hat{k}\frac{\partial}{\partial z}(xy\text{-}z^2)$$

$$\nabla\phi = y\hat{i} - x\hat{j} + 2z\hat{k}$$

$$\nabla\phi\,\Big|_{(1,4,2)} = 4\hat{i} + \hat{j} - 4\hat{k}$$

$$\nabla\phi\,\Big|_{(-3,-3,3)} = -3\hat{i} - 3\hat{j} + 6\hat{k}$$

Are normals to the surface at (1, 4, 2) and (-3,-3,3).

Let θ be the angle between the normals

$$\therefore \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|} = \frac{4\hat{i} + \hat{j} - 4\hat{k}\ \cdot\ -3\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{16+1+16}\ \ \sqrt{9+9+36}}$$

$$= \frac{-12-3-24}{\sqrt{33}\sqrt{54}}$$

$$= \frac{-39}{\sqrt{33}\sqrt{54}}$$

$$\theta = \cos^{-1}\left(-\frac{-39}{\sqrt{33}\sqrt{54}}\right)$$

Ex.4: Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point (-1,-2,-1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

Soln: Given

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2yz+4xz^2)\hat{i} + \frac{\partial}{\partial y}(x^2yz+4xz^2)\hat{j} + \frac{\partial}{\partial z}(x^2yz+4xz^2)\hat{k}$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y+8xz)\hat{k}$$

$$\nabla\phi\big|_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

The directional derivative of ϕ at the point (1,-2,-1) in the direction of vector $2\hat{i} - \hat{j} - 2\hat{k}$. is

$$= (8\hat{i} - \hat{j} - 10\hat{k}).\left(\frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}}\right) = \frac{1}{3}\ 16+1+20\ = \frac{37}{3}$$

Ex.5: Find the directional derivative of $\phi = xy^2 + yz^3$ at the point (1,-2,-1) in the direction of the normal to the surface x log $z - y^2 = -4 at (-1,2,1)$.

Soln: Given

$$\phi = xy^2 + yz^3$$

$$\therefore \nabla\phi = y^2\hat{i} + \ 2xy + z^3\ \hat{j} + \ 3yz^2\ \hat{k}$$

$$\therefore \nabla\phi\big|_{(1,-2,-1)} = 4\hat{i} - 5\hat{j} - 5\hat{k}$$

$$Let\, \phi_1 = x\log z - y^2$$

Therefore the normal vector to the surface x log $z - y^2 = -4 is \nabla\phi_1$

$$\therefore \nabla\phi_1 = \log z\hat{i} - +(-2y)\hat{j} + \left(\frac{x}{z}\right)\hat{k}$$

and $\vec{n} = \nabla\phi_1\big|_{(-1,2,1)} = \log(1)\hat{i} - (2\times 2)\hat{j} + \left(-\frac{1}{1}\right)\hat{k} = -4\hat{j} - \hat{k}$

$$\left|\vec{n}\right| = \sqrt{16+1} = \sqrt{17}$$

$$\hat{n} = \frac{\vec{n}}{\left|\vec{n}\right|} = \frac{1}{\sqrt{17}}\ -4\hat{j} - \hat{k}$$

Therefore, the directional derivative of the surface $\phi = xy^2 + yz^3$ at (1,-2,-1) in the direction normal to the surface x log $z - y^2 = -4at\,(-1,2,1)$. is

$$\therefore \nabla\phi.\hat{n} = \ 4\hat{i} - 5\hat{j} - 5\hat{k}\ .\frac{1}{\sqrt{17}}\ -4\hat{j} - \hat{k}\ = \frac{1}{\sqrt{17}}\ 20+5\ = \frac{25}{\sqrt{17}}.$$

Ex.6: Find the equation of tangent plane and normal to line to the surface x log $z - y^2 = -4$at the point $(-1, 2, 1)$.

Soln: Let $\phi = x\log z - y^2$.

$$\frac{\partial\phi}{\partial x}\bigg| = \log z \Rightarrow \frac{\partial\phi}{\partial x}\bigg|_{(-1,2,1)} = 0$$

Therefore
$$\frac{\partial\phi}{\partial y}\bigg| = -2y \Rightarrow \frac{\partial\phi}{\partial y}\bigg|_{(-1,2,1)} = -4$$

$$\frac{\partial\phi}{\partial z}\bigg| = \frac{x}{z} \Rightarrow \frac{\partial\phi}{\partial z}\bigg|_{(-1,2,1)} = -1$$

Thertefore, equation of the tangent plane is

$$(x - x_1)\left(\frac{\partial\phi}{\partial x}\right)_p + (y - y_1)\left(\frac{\partial\phi}{\partial y}\right)_p + (z - z_1)\left(\frac{\partial\phi}{\partial z}\right)_p = 0$$

$Hence\,(x_1, y_1, z_1) = (-1, 2, 1),\ then$

$$(x+1)0 + (y-2)(-4) + (z-1)(-1) = 0$$

$$-4y + 8 - z + 1 = 0$$

$$4y + z + 9 = 0$$

And the equation of the normal line is

$$\frac{x - x_1}{\left(\frac{\partial\phi}{\partial x}\right)} = \frac{y - y_1}{\left(\frac{\partial\phi}{\partial y}\right)} = \frac{z - z_1}{\left(\frac{\partial\phi}{\partial z}\right)} \Rightarrow \frac{x+1}{0} = \frac{y-2}{-4} = \frac{z-1}{-1}$$

<u>Properties of divergence</u>:

1.Prove that $div\left(\vec{A} + \vec{B}\right) = div\,\vec{A} + div\,\vec{B}$

$Or\ \nabla.\left(\vec{A} + \vec{B}\right) = \nabla.\vec{A} + \nabla.\vec{B}$

1. Proof: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k},\quad \vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$$\therefore\left(\vec{A} + \vec{B}\right) = \left(A_1 + B_1\right)\hat{i} + \left(A_2 + B_2\right)j + \left(A_3 + B_3\right)\hat{k}$$

$$\nabla.\left(\vec{A} + \vec{B}\right) = \frac{\partial}{\partial x}\left(A_1 + B_1\right) + \frac{\partial}{\partial y}\left(A_2 + B_2\right) + \frac{\partial}{\partial z}\left(A_3 + B_3\right)$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$\nabla.\left(\vec{A} + \vec{B}\right) = \nabla.\vec{A} + \nabla.\vec{B}$$

2. **Prove that**

$$div\left(\phi\vec{A}\right) = \left(grad\phi\right).\vec{A} + \phi\left(div\vec{A}\right)$$

$$Or\ \nabla.\left(\phi\vec{A}\right) = \left(\nabla\phi\right).\vec{A} + \phi\left(\nabla.\vec{A}\right)$$

$$\text{Proof: Let } \vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$then,\ \phi\vec{A} = \phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}$$

$$\nabla.\left(\phi\vec{A}\right) = \frac{\partial}{\partial x}\left(\phi A_1\right) + \frac{\partial}{\partial y}\left(\phi A_2\right) + \frac{\partial}{\partial z}\left(\phi A_3\right) \text{ by the property,}$$

$$= \phi\frac{\partial A_1}{\partial x} + A_1\frac{\partial\phi}{\partial x} + \phi\frac{\partial A_2}{\partial y} + A_2\frac{\partial\phi}{\partial y} + \phi\frac{\partial A_3}{\partial z} + A_3\frac{\partial\phi}{\partial z}$$

$$= \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + A_1\frac{\partial\phi}{\partial x} + A_2\frac{\partial\phi}{\partial y} + A_3\frac{\partial\phi}{\partial z}$$

$$= \phi\left(\nabla\vec{A}\right) + \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right).\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial y}\hat{k}\right)$$

$$\nabla.\left(\phi\vec{A}\right) = \left(\nabla\phi\right).\vec{A} + \phi\left(\nabla.\vec{A}\right)$$

Properties of curl :

1. Prove that curl:

$$\left(\vec{A}_1 + \vec{B}_1\right) = curl\,\vec{A} + curl\,\vec{B}$$

$$Or\ \nabla\times\left(\vec{A}_1 + \vec{B}_1\right) = \nabla\times\vec{A} + \nabla\times\vec{B}$$

$$\text{Proof: Let } \vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k},\quad \vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$$

$$Therefore, \vec{A} + \vec{B} = \left(A_1 + B_1\right)\hat{i} + \left(A_2 + B_2\right)\hat{j} + \left(A_3 + B_3\right)\hat{k}$$

$$\therefore \nabla\times\left(\vec{A} + \vec{B}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\therefore \nabla\times\left(\vec{A}_1 + \vec{B}_1\right) = \nabla\times\vec{A} + \nabla\times\vec{B}$$

2. If $\vec{A}$ is a vector function and ϕ is a scalar function then

$$curl\left(\phi\vec{A}\right)=\phi\left(curl\,\vec{A}\right)+grad\phi\times\vec{A}$$

$$Or\ \nabla\times\left(\phi\vec{A}\right)=\phi\left(\nabla\times\vec{A}\right)+\nabla\phi\times\vec{A}$$

$$\text{Proof: Let } \vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$$

$$then\ ,\ \phi\vec{A}=\phi A_1\hat{i}+\phi A_2\hat{j}+\phi A_3\hat{k}$$

$$\nabla\times\phi\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \phi A_1 & \phi A_2 & \phi A_3\end{vmatrix}=\sum\left[\frac{\partial}{\partial y}\ \phi A_3\ -\frac{\partial}{\partial z}\ \phi A_3\ \right]\hat{i}$$

$$=\sum\left[\phi\frac{\partial A_3}{\partial y}+A_3\frac{\partial\phi}{\partial y}-\phi\frac{\partial A_2}{\partial z}-A_2\frac{\partial\phi}{\partial z}\right]\hat{i}$$

$$=\sum\left[\phi\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+A_3\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial z}\right]\hat{i}$$

$$=\phi\sum\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\sum\left(A_3\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial z}\right)\hat{i}$$

$$=\phi\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

$$\nabla\times\phi\vec{A}=\phi\left(\nabla\times\vec{A}\right)+\nabla\phi\times\vec{A}$$

<u>Laplacian:</u> The Laplacian operator ∇^2 is defined by

$$\nabla\cdot\nabla=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

<u>Irrotational Vector (or Conservative Force Field):</u> A vector field $\vec{F}$ is said to be irrotational vector or conservative force field or curl free vector if $\nabla\times\vec{F}=0$ or curl $\vec{F}=0$.

<u>Scalar Potential</u>: A vector field $\vec{F}$ which can be derived from the scalar field ϕ such that $\vec{F}=\nabla\phi$ is called conservative force field and ϕ is called scalar potential.

<u>Solenoidal Vector Function:</u> A vector $\vec{A}$ is said to be solenoidal vector or divergence free vector if div $\vec{A} = \nabla.\ \vec{A} = 0$.

<u>Curl of a vector function:</u> If $\vec{A}$ is any vector function differentiable at each point (x,y,z) then curl of $\vec{A}$ is denoted by curl $\vec{A}$ *or* $\nabla \times \vec{A}$ and it is defined by

$$curl\ \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$curl\ \vec{A} = \nabla \times \vec{A} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

Hence, curl of a vector function is a vector.

Ex.1: If $\vec{F} = grad\ \ x^3y + y^3z + z^3x - x^2y^2z^2$ then find div $\vec{F}$ *at* $(1,2,3)$.

Soln: Given

$$\vec{F} = grad\ \ x^3y + y^3z + z^3x - x^2y^2z^2$$

$$= \nabla\ \ x^3y + y^3z + z^3x - x^2y^2z^2$$

$$\vec{F} = \ 3x^3y + z^3 - 2xy^2z^2\ \ \hat{i} + \ \ x^3 + 3y^2z - 2x^2yz^2\ \ \hat{j} + \ \ y^3 + 3z^2x - 2x^2y^2z\ \ \hat{k}$$

$$div\ \vec{F} = 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6zx - 2x^2y^2$$

$$\therefore \nabla\ \vec{F}\Big|_{1,2,3} = 12 - 72 + 36 - 18 + 18 - 8 = -32.$$

Ex.2 : If $\vec{F}$ =(x+y+1) $\hat{i} + \hat{j} - (x+y)\hat{k},$ then show that $\vec{F}$ .curl $\vec{F}$ =0.

Soln: Given $\vec{F}$ =(x+y+1) $\hat{i} + \hat{j} - (x+y)\hat{k},$

$$curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$=(-1-0)\hat{i}-(-1-0)\hat{j}+(0-1)\hat{k}$$

$$curl\ \vec{F}=-\hat{i}+\hat{j}-\hat{k}$$

$$\therefore\ \vec{F}.curl\ \vec{F}=\ \ \mathrm{x+y+1}\ \hat{i}+\hat{j}-(x+y)\hat{k}\ .\ -\hat{i}+\hat{j}-\hat{k}$$

$$=-\ \ \mathrm{x+y+1}\ +1+(x+y)$$

$$=\ 0$$

Ex.3: If $\vec{F}=(ax+3y+4z)\hat{i}+(x-2y+3z)\hat{j}+(3x+2y-z)\hat{k}$ is solenoidal, find ;'a'.
Soln: Given

$$\vec{F}=(ax+3y+4z)\hat{i}+(x-2y+3z)\hat{j}+(3x+2y-z)\hat{k},\ then$$

$$\nabla.\vec{F}=\frac{\partial}{\partial x}(ax+3y+4z)+\frac{\partial}{\partial y}(x-2y+3z)+\frac{\partial}{\partial z}(3x+2y-z)=a-2-1$$

$$\nabla.\vec{F}=a-3$$

Since the vector field is solenoidal therefore $\nabla.\vec{F}=0,\ then$

$$a-3=0\Rightarrow a=3$$

Ex.4: Find the constants a,b,c such that the vector

$\vec{F}=(x+y+az)\hat{i}+(x+cy+2z)\hat{k}+(bx+2y-z)\hat{j}$ is irrotational.

Soln: Given $\vec{F}=(x+y+az)\hat{i}+(x+cy+2z)\hat{k}+(bx+2y-z)\hat{j}$

Since the vector field is irrotational, therefore $\nabla\times\vec{F}=0$.

$$curl\ \vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y+az & bx+2y-z & x+cy+2z\end{vmatrix}$$

$$=(c+1)\hat{i}-(1-a)\hat{j}+(b-1)\hat{k}$$

$i.e.,\ (c+1)\hat{i}-(1-a)\hat{j}+(b-1)\hat{k}=0$

This is possible only when,

c-1=0, 1-a=0, b-1=0 $\Rightarrow$ a=1, b=1, c=1.

Ex.5: Prove that 1. $\nabla r^n = nr^{n-2}\vec{r}$,

Soln: We have by the relation x=rcosθ, y=r sinθ.

By definition

$$\nabla r^n = \frac{\partial}{\partial x} r^n \; \hat{i} + \frac{\partial}{\partial y} r^n \; \hat{j} + \frac{\partial}{\partial z} r^n \; \hat{k}$$

$$= nr^{n-1}\frac{\partial r}{\partial x}\hat{i} + nr^{n-1}\frac{\partial r}{\partial y}\hat{j} + nr^{n-1}\frac{\partial r}{\partial z}\hat{k}$$

$$But \;\; r^2 = x^2 + y^2$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \therefore \frac{\partial r}{\partial x} = \frac{x}{r} \;\; lll^{ly} \;\; \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= nr^{n-1}\left(\frac{x}{r}\right)\hat{i} + nr^{n-1}\left(\frac{y}{r}\right)\hat{j} + nr^{n-1}\left(\frac{z}{r}\right)\hat{k}$$

$$= nr^{n-2}x\hat{i} + nr^{n-2}y\hat{j} + nr^{n-2}xz\hat{k}$$

$$= nr^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= nr^{n-2}\vec{r}$$

Ex.6: Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r),\ where\, r^2 = x^2 + y^2 + z^2.$

Soln: Given $r^2 = x^2 + y^2 + z^2.$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$Let\, \phi = f(x) \Rightarrow \nabla^2\phi = \sum\frac{\partial^2\phi}{\partial x^2} = \sum\frac{\partial^2}{\partial x^2} f(x) = \sum\frac{\partial}{\partial x}\left[f'(r)\frac{\partial r}{\partial x}\right]$$

$$Where \sum\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \sum\frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] = \sum\frac{\partial}{\partial x}\left[\frac{xf'(r)}{r}\right]$$

$$= \sum\frac{1}{r^2}\left[r\frac{\partial}{\partial x}\, xf'(r) \; - xf'(r)\frac{\partial r}{\partial x}\right]$$

$$= \sum\frac{1}{r^2}\left[r\left\{xf''(r)\frac{\partial r}{\partial x} + 1.f'(r)\right\} - xf'(r)\frac{x}{r}\right]$$

$$= \sum\frac{1}{r^2}\left[r\left\{xf''(r)\frac{x}{r} + f'(r)\right\} - \frac{x^2}{r} f'(r)\right]$$

$$=\sum \frac{1}{r^2}\left[x^2 f''(r)+rf'(r)-\frac{x^2}{r}f'(r)\right]$$

$$=\sum \frac{1}{r^2}\left[x^2 f''(r)+\left(r-\frac{x^2}{r}\right)f'(r)\right]$$

$$=\frac{1}{r^2}\left[x^2 f''(r)+\left(r-\frac{x^2}{r}\right)f'(r)\right]+\frac{1}{r^2}\left[x^2 f''(r)+\left(r-\frac{y^2}{r}\right)f'(r)\right]$$

$$+\frac{1}{r^2}\left[x^2 f''(r)+\left(r-\frac{z^2}{r}\right)f'(r)\right]$$

$$=\frac{1}{r^2}\left[x^2+y^2+z^2\right]f''(r)+\frac{1}{r^2}\left[r-\frac{x^2}{r}+r-\frac{y^2}{r}+r-\frac{z^2}{r}\right]f'(r)$$

$$=\frac{1}{r^2}r^2 f''(r)+\frac{1}{r^2}\left[3r-\frac{1}{r}\left(x^2+y^2+z^2\right)\right]f'(r)$$

$$=\frac{1}{r^2}\left[r^2 f''(r)\right]+\frac{1}{r^2}\left[3r-\frac{1}{r}\left(r^2\right)\right]f'(r)$$

$$=\frac{1}{r^2}\left[r^2 f''(r)\right]+\frac{1}{r^2}\left[3r-r\right]f'(r)$$

$$\therefore \nabla^2 f(r)= f''(r)+\frac{3}{r}f'(r)$$

Ex.7: Find the constants 'a' and 'b' so that $\vec{F} = axy+z^3\ \hat{i} + 3x^2-z\ \hat{j} + bxz^2-y\ \hat{k}$ irrotational and find ϕ such that $\vec{F}=\nabla\phi$.

Soln: Given $\vec{F}=(axy+z^3)\hat{i}+(3x^2-z)\hat{j}+(bxz^2-y)\hat{k}$

Since $\vec{F}$ is irrotational i.e., $\nabla\times\vec{F}=0$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ axy+z^3 & 3x^2-z & bxz^2-y \end{vmatrix}=0$$

$$i.e.,\ (-1+1)\hat{i}-(bz^2-3z^2)\hat{j}+(6x-ax)\hat{k}=0$$

$$i.e.,\ -z^2(b-3)\hat{j}+(6-a)=0$$

which holds good if any only if b-3=0 and 6-a=0 $\Rightarrow a$=6 and b=3.

Also given that $\nabla\phi = \vec{F}$

$$\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right) = 6xy+z^3\ \hat{i}++(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy+z^3 \Rightarrow \phi = 3x^2y+xz^3+f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = 3x^2-z \Rightarrow \phi = 3x^2y-yz+f_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2-y \Rightarrow \phi = xz^3-yz+f_2(x,y)$$

Hence $\phi = 3x^2y+xz^3-yz.$

Vector Identities–

These are some properties relating to various meaningful combinations of *gradient, divergence, curl and laplacian*. These are established by taking a general scalar point function or a vector point function.

V.I - 1 curl (grad ϕ) = $\vec{0}$. or $\nabla \times (\nabla\phi) = \vec{0}$

Proof : Let ϕ be a scalar point function of x, y, z. grad $\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$

$$\text{curl (grad}\,\phi) = \nabla\times(\nabla\phi) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}$$

ie., $$= \Sigma\left\{\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right\} i = \Sigma\left(\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right) i = \vec{0}$$

Thus **curl (grad ϕ) = 0**, for any scalar function ϕ

V.I - 2 div (curl $\vec{A}$) = 0. or $\nabla\cdot(\nabla\times\vec{A}) = 0$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z

$$\text{curl}\,\vec{A} = \nabla\times\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3\end{vmatrix} = \Sigma\, i\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)$$

Now div (curl $\vec{A}$) = $\nabla\cdot(\nabla\times\vec{A})$

$$= \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) = \Sigma \left(\frac{\partial^2 a_3}{\partial x\, \partial y} - \frac{\partial^2 a_2}{\partial x\, \partial z} \right)$$

On expanding we get,

$$\frac{\partial^2 a_3}{\partial x\, \partial y} - \frac{\partial^2 a_2}{\partial x\, \partial z} + \frac{\partial^2 a_1}{\partial y\, \partial z} - \frac{\partial^2 a_3}{\partial y\, \partial x} + \frac{\partial^2 a_2}{\partial z\, \partial x} - \frac{\partial^2 a_1}{\partial z\, \partial y} = 0$$

Thus **div (curl $\vec{A}$) = 0**, for any vector function $\vec{A}$

V.I - 3 curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$ or $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Proof: Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z

$$\therefore \quad \text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

Now curl (curl $\vec{A}$) = $\nabla \times (\nabla \times \vec{A})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) & \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) & \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) \end{vmatrix}$$

$$= \Sigma\, i \left\{ \frac{\partial}{\partial y} \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) \right\}$$

$$= \Sigma\, i \left(\frac{\partial^2 a_2}{\partial y\, \partial x} + \frac{\partial^2 a_3}{\partial z\, \partial x} \right) - \Sigma\, i \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} \right), \text{ by rearranging.}$$

Adding and subtracting $\Sigma\, i \frac{\partial^2 a_1}{\partial x^2}$ we get

$$\Sigma\, i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_2}{\partial y\, \partial x} + \frac{\partial^2 a_3}{\partial z\, \partial x} \right) - \Sigma\, i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} \right)$$

$$= \Sigma\, i \frac{\partial}{\partial x} \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \right) - \Sigma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) a_1 i$$

$$= \Sigma \frac{\partial}{\partial x}(\text{div } \vec{A})\, i - \nabla^2 \Sigma a_1 i = \text{grad}(\text{div } \vec{A}) - \nabla^2 \vec{A}$$

Thus $\quad \textbf{curl}(\textbf{curl } \vec{A}) = \textbf{grad}(\textbf{div } \vec{A}) - \nabla^2 \vec{A}$

V.I - 4 div $\left(\phi \vec{A}\right) = \phi\,(\text{div } \vec{A}) + \text{grad } \phi \cdot \vec{A}$ or $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z and ϕ be a scalar point function of x, y, z

$$\therefore \quad \phi \vec{A} = \phi(a_1 i + a_2 j + a_3 k) = \Sigma(\phi a_1)\, i$$

Now div $(\phi \vec{A}) = \nabla \cdot (\phi \vec{A})$

$$= \left(\Sigma \frac{\partial}{\partial x} i\right) \cdot \Sigma(\phi a_1)\, i$$

$$= \Sigma \frac{\partial}{\partial x}(\phi a_1) = \Sigma \left(\phi \frac{\partial a_1}{\partial x} + \frac{\partial \phi}{\partial x} a_1\right)$$

ie., $\quad$ div $(\phi \vec{A}) = \phi\, \Sigma \frac{\partial a_1}{\partial x} + \Sigma \frac{\partial \phi}{\partial x} i \cdot \Sigma a_1 i$

Thus $\quad \textbf{div}(\phi \vec{A}) = \phi\,(\textbf{div } \vec{A}) + \textbf{grad } \phi \cdot \vec{A}$

V.I - 5 curl $\left(\phi \vec{A}\right) = \phi\,(\text{curl } \vec{A}) + \text{grad } \phi \times \vec{A}$ or

$\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$

Proof : Let ϕ and $\vec{A} = a_1 i + a_2 j + a_3 k$ be respectively scalar and vector point functions of x, y, z

$$\therefore \quad \phi \vec{A} = (\phi a_1)\, i + (\phi a_2)\, j + (\phi a_3)\, k$$

$$\text{Now} \quad \text{curl}(\phi \vec{A}) = \nabla \times (\phi \vec{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix}$$

ie., $$= \Sigma\, i \left\{\frac{\partial}{\partial y}(\phi a_3) - \frac{\partial}{\partial z}(\phi a_2)\right\}$$

$$= \Sigma\, i \left\{\left(\phi \frac{\partial a_3}{\partial y} + \frac{\partial \phi}{\partial y} a_3\right) - \left(\phi \frac{\partial a_2}{\partial z} + \frac{\partial \phi}{\partial z} a_2\right)\right\}$$

$$= \phi\, \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) i + \Sigma \left(\frac{\partial \phi}{\partial y} a_3 - \frac{\partial \phi}{\partial z} a_2\right) i$$

$$= \phi \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \phi\,(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$$

Thus **curl** $(\phi \vec{A}) = \phi\,(\text{curl } \vec{A}) + \nabla\phi \times \vec{A}$

V.I - 6 div $(\vec{A} \times \vec{B}) = \vec{B} \cdot \text{curl } \vec{A} - \vec{A} \cdot \text{curl } \vec{B}$ or

$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ and $\vec{B} = b_1 i + b_2 j + b_3 k$, be two vector point functions of x, y, z

$$\therefore \quad \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \Sigma\, i\,(a_2 b_3 - a_3 b_2)$$

Now div $(\vec{A} \times \vec{B}) = \nabla \cdot (\vec{A} \times \vec{B})$

$$= \left(\Sigma \frac{\partial}{\partial x} i\right) \cdot \Sigma\, i\,(a_2 b_3 - a_3 b_2)$$

$$= \Sigma \frac{\partial}{\partial x}(a_2 b_3 - a_3 b_2)$$

$$= \Sigma \left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x}\right)$$

On expanding we get

$$\left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x}\right) + \left(a_3 \frac{\partial b_1}{\partial y} + b_1 \frac{\partial a_3}{\partial y} - a_1 \frac{\partial b_3}{\partial y} - b_3 \frac{\partial a_1}{\partial y}\right)$$

$$+ \left(a_1 \frac{\partial b_2}{\partial z} + b_2 \frac{\partial a_1}{\partial z} - a_2 \frac{\partial b_1}{\partial z} - b_1 \frac{\partial a_2}{\partial z}\right)$$

ie., $$= \Sigma\, b_1 \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - \Sigma\, a_1 \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right)$$

$$= (\Sigma\, b_1 i) \cdot \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) i - (\Sigma\, a_1 i) \cdot \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right) i$$

$(\because \quad \Sigma A_1 B_1 = \Sigma A_1 i \cdot \Sigma B_1 i)$

$$= (\Sigma\, b_1 i)\cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} - (\Sigma\, a_1 i)\cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

Thus $\;\textbf{div}\,(\vec{A}\times\vec{B}) = \vec{B}\cdot \textbf{curl}\,\vec{A} - \vec{A}\cdot\textbf{curl}\,\vec{B}$

1. *If $\vec{F_1}$ and $\vec{F_2}$ are irrotational, prove that $\vec{F_1}\times\vec{F_2}$ is solenoidal.*

>> $\vec{F_1}$ and $\vec{F_2}$ are irrotational by data.

$\Rightarrow$ $\quad$ curl $\vec{F_1} = \vec{0}$ and curl $\vec{F_2} = \vec{0}$ $\quad\ldots(1)$

We have to prove that div $(\vec{F_1}\times\vec{F_2}) = 0$

We have the vector identity (V.I - 6),

$$\text{div}\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\text{curl}\,\vec{A} - \vec{A}\cdot\text{curl}\,\vec{B} \quad \textit{(assumed)}$$

$\therefore\quad$ div $(\vec{F_1}\times\vec{F_2}) = \vec{F_2}\cdot$ curl $\vec{F_1} - \vec{F_1}\cdot$ curl $\vec{F_2}$

ie., $\quad$ div $(\vec{F_1}\times\vec{F_2}) = \vec{F_2}\cdot\vec{0} - \vec{F_1}\cdot\vec{0} = 0$, by using (1)

$\therefore\quad$ div $(\vec{F_1}\times\vec{F_2}) = 0 \Rightarrow$ $\vec{F_1}\times\vec{F_2}$ **is solenoidal.**

2. *If $u\,\vec{F} = \nabla v$, prove that $\vec{F}$ and curl $\vec{F}$ are at right angles.*

>> $\vec{F} = \dfrac{1}{u}\nabla v$ and we have to prove that $\vec{F}\cdot$ curl $\vec{F} = 0$

We shall first find curl $\vec{F}$ where $\vec{F}$ is of the form $\phi\vec{A}$
Let us consider the vector identity (V.I - 5)

$$\nabla\times(\phi\vec{A}) = \phi(\nabla\times\vec{A}) + \nabla\phi\times\vec{A}$$

$$\therefore\quad \nabla\times\left(\frac{1}{u}\nabla v\right) = \frac{1}{u}\{\nabla\times(\nabla v)\} + \nabla\left(\frac{1}{u}\right)\times\nabla v$$

The first term in the R.H.S of this equation is zero by a vector identity curl (grad ϕ) = 0 (V.I - 1)

$$\therefore\quad \nabla\times\left(\frac{1}{u}\nabla v\right) = \nabla\left(\frac{1}{u}\right)\times\nabla v \quad \textit{ie.,}\ \text{curl}\,\vec{F} = \nabla\left(\frac{1}{u}\right)\times\nabla v$$

Now $\quad \vec{F}\cdot\text{curl}\,\vec{F} = \left(\dfrac{1}{u}\nabla v\right)\cdot\left\{\nabla\left(\dfrac{1}{u}\right)\times\nabla v\right\}$

R.H.S of this equation is a scalar triple product or the box product of three vectors.

ie., $\vec{F} \cdot \text{curl } \vec{F} = \left[\frac{1}{u} \nabla v, \nabla \left(\frac{1}{u} \right), \nabla v \right]$

ie., $\vec{F} \cdot \text{curl } \vec{F} = \frac{1}{u} \left[\nabla v, \nabla \left(\frac{1}{u} \right), \nabla v \right] = 0$

Hence $\vec{F} \cdot \text{curl } \vec{F} = 0$, since two vectors are identical in the box product.

Thus $\vec{F}$ is perpendicular to curl $\vec{F}$