MODULE IV

INTEGRAL CALCULUS

CONTENTS:

Reduction formula:

1. Reduction formula for $\int \sin^n x \, dx$ and $\int_0^{\pi/2} \sin^n x \, dx$, n is a positive integer.

Let
$$I_n = \int \sin^n x \, dx$$
$$= \int \sin^{n-1} x . \sin x \, dx = \int u \; v \; dx \, (say)$$

We have the rule of integration by parts,

$$\int u \; v \; dx = u \int v \, dx - \int \int v \, dx . u' \; dx$$

$$\therefore I_n = \sin^{n-1} x(-\cos x) - \int \; -\cos x \quad n-1 \; \sin^{n-2} x . \cos x \; dx$$

$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x . \cos^2 x \; dx$$

$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \; dx$$

$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x \; dx - (n-1) \int \sin^n x \; dx$$

$$i.e., \; I_n = \sin^{n-1} x . \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$i.e., \; I_n = \; 1 + (n-1) \; = -\sin^{n-1} x . \cos x + (n-1) I_{n-2}$$

$$\therefore \; I_n = \int \sin^n x \; dx = \frac{-\sin^{n-1} x . \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

This is the required reduction formula.

2. . Reduction formula for $\int \cos^n x$ and $\int_0^{\pi/2} \cos^n x \, dx$,

Where n is a positive integer.

Let
$$I_n = \int \cos^n x \, dx$$
$$= \int \cos^{n-1} x . \cos x \, dx$$

$$\therefore I_n = \cos^{n-1} x.\sin x - \int \sin x \;\; n-1 \;\; \cos^{n-2} x(-\sin x)\; dx$$

$$= -\cos^{n-1} x.\cos x + (n-1)\int \cos^{n-2} x.\sin^2 x\; dx$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\; dx$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x\; dx - (n-1)\int \cos^n x\; dx$$

$$I_n = \cos^{n-1} x.\sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$i.e.,\;\; I_n = \left[1+(n-1)\right] = \cos^{n-1} x.\sin x + (n-1) I_{n-2}$$

$$\therefore\;\; I_n = \int \cos^n x\; dx = \frac{\cos^{n-1} x.\sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Next, let $I_n = \int_0^{\pi/2} \cos^n x dx$

$$\therefore\;\; from (1),\; I_n = \left[\frac{\cos^{n-1} x.\sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

$But \;\; \cos\left(\pi/2\right) = 0 = \sin 0.$

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$Thus \;\; I_n = \frac{n-1}{n} I_{n-2}$

1. Reduction formula for $\int \sin^m x \cos^n x\, dx$ and $\int_0^{\pi/2} \sin^m \cos^n x\, dx$ where m and n are positive integers.

$$I_{m,n} = \int \sin^m x \;\; \cos^n x dx$$

$$= \int \sin^{m-1} x \;\; \sin x \cos^n x \;\; dx = \int u \;\; v \;\; dx\, (say)$$

we have $\int u\,v\;dx = u\int v\,dx - \iint v\,dx.u'\;dx$

$Here\;\; \int u\,dx = \int \sin x \cos^n dx$

$Put\quad \cos x = t \quad \therefore -\sin x dx = dt$

$$Hence\;\; \int v\;dx = \int -t^n dt = -\frac{t^{n+1}}{n+1} = -\frac{\cos^{n+1} x}{n+1}$$

$$Now\, I_{m,n} = \sin^{m-1} x\left(\frac{-\cos^{n+1} x}{n+1}\right) - \int \frac{-\cos^{n+1} x}{n+1}(m-1)\sin^{m-2} x \cos x dx$$

$$i.e., \; = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x.\cos^{n} x.\cos^2 x dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n} x(1-\sin^2 x) dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n} x dx - \frac{m-1}{n+1}\int \sin^{m} x \cos^{n} x dx$$

$$I_{m,n} = -\frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1}\; I_{m,n}$$

$$i.e.,\; I_{m,n}\left[1 + \frac{m-1}{n+1}\right] = \frac{1}{n+1}\left[-\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}\right]$$

$$I_{m,n}\left[\frac{m-1}{n+1}\right] = \frac{1}{n+1}\left[-\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}\right]$$

$$\therefore\; I_{m,n} = \int \sin^m x \cos^n x dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad ......(1)$$

PROBLEMS:

1. $Let\;\; I = \int_0^{\pi} \sin^4 x\, dx$

$f(x) = \sin^4 x\, and \;\; 2a = \pi \;\; or \;\; a = \pi/2$

$f(2a-x) = \sin^4(\pi - x) = \sin^4 x = f(x) \quad ie., \;\; f(2a-x) = f(x)$

Thus by the property $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \;\; we\;\; have,$

$I = 2\int_0^{\pi/2} \sin^4 x\, dx = 2.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$ by reduction formula.

Thus $\;$ I=$3\pi/8$

2. $Let \;\; I = \int_0^{\pi} x \sin^8 x \, dx$

We have the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi} (\pi - x)\sin^8(\pi - x)dx = \int_0^{\pi} (\pi - x)\sin^8 x dx$$

$$= \pi \int_0^{\pi} \sin^8 x dx - \int_0^{\pi} x \sin^8 x dx$$

$$I = \pi \int_0^{\pi} \sin^8 x dx - I$$

$$or \;\; 2I = \pi . 2 \int_0^{\pi/2} \sin^8 x dx$$

$$Hence\, I = \pi . \frac{7}{8} . \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}, \; .$$

Thus by reduction formula

$$I = \frac{35\pi^2}{256}$$

3. $Let \;\; I = \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

$$I = \int_0^{\pi} (\pi - x)\sin^2(\pi - x)\cos^4(\pi - x)dx, \text{ by a property.}$$

$$= \int_0^{\pi} (\pi - x)\sin^2 x \cos^4 x dx$$

$$= \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx - \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx - I$$

$$2I = \pi . 2 \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

$$I = \pi . \frac{(1).(3).(1)}{6 \times 4 \times 2} . \frac{\pi}{2} \;\; \text{by reduction formula.}$$

Thus $I = \pi^2 / 32$

2. *Evaluate* $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

>> Let $I = \int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

Put $x = \sin^2\theta$, $dx = 2\sin\theta\,\cos\theta\,d\theta$ and θ varies from 0 to $\pi/2$.

Also $(1-x)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$

$\therefore$ $I = \int_{\theta=0}^{\pi/2} \sin^3\theta\cos^3\theta \cdot 2\sin\theta\cos\theta\,d\theta$

i.e., $I = 2\int_0^{\pi/2} \sin^4\theta\cos^4\theta\,d\theta$

Hence $I = 2\cdot\dfrac{[(3)(1)][(3)(1)]}{8\times6\times4\times2}\cdot\dfrac{\pi}{2}$ *by reduction formula.*

Thus $I = 3\pi/128$

Introduction:

Many problems in all branches of science and engineering when analysed for putting in a mathematical form assumes the form of a differential equation.

An engineer or an applied mathematician will be mostly interested in obtaining a solution for the associated equation without bothering much on the rigorous aspects. Accordingly the study of differential equations at various levels is focused on the methods of solving the equations.

Preliminaries:

Ordinary Differential Equation (O.D.E)

If $y = f(x)$ is an unknown function, an equation which involves atleast one derivative of y, *w.r.t.* x is called an **ordinary differential equation** which in future will be simply referred to as **Differential Equation (D.E)**.

The order of D.E is the order of the highest derivative present in the equation and the degree of the D.E. is the degree of the highest order derivative after clearing the fractional powers.

Finding y as a function of x explicitly $[y = f(x)]$ or a relationship in x and y satisfying the D.E. $[f(x, y) = c]$ constitutes the solution of the D.E.

Observe the following equations along with their order and degree.

1 $\frac{dy}{dx}=2x$ [order = 1, degree = 1]

2 $\left(\frac{dy}{dx}\right)^2+3\frac{dy}{dx}+2=0$ [order = 1, degree = 2]

General solution and particular solution:

A solution of a D.E. is a relation between the dependent and independent variables satisfying the given equation identically.

The general solution will involve arbitrary constants equal to the order of the D.E.

If the arbitrary constants present in the solution are evaluated by using a set of given conditions then the solution so obtained is called a **particular solution**. In many physical problems these conditions can be formulated from the problem itself.

Note : Basic integration and integration methods are essential prerequisites for this chapter.

Solution of differential equations of first order and first degree

Recollecting the definition of the order and the degree of a D.E., a first order and first degree equation will be the form

$\frac{dy}{dx}=f(x,y)$ or M(x,y)dx+N(x,y)dy=0

We discuss mainly classified four types of differential equations of first order and first degree. They as are as follows:

- Variables separable equations
- Homogenous equations
- Exact equations
- Linear equations

Variables separable Equations:

If the given D.E. can be put in the form such that the coefficient of *dx* is a function of the variable *x* only and the coefficient of *dy* is a function of *y* only then the given equation is said to be in the separable form.

The modified form of such an equation will be,

$P(x)\ dx + Q(y)\ dy = 0$

This is the general solution of the equation.

Example 1: $Solve\ \frac{dy}{dx} = xe^{y-x^2}$ given that y(0)=0

Soln: $\frac{dy}{dx} = xe^{y-x^2} \quad or \quad \frac{dy}{dx} = xe^{y}e^{-x^2}$

put $-x^2 = t \quad \therefore -2xdx = dt \, or -xdx = dt$

Hence we have, $-e^{-y} + \int e^{t}\, dt = c$

$$i.e.\ \frac{dy}{e^{y}} = xe^{-x^2}dx \ by\ separating\ the Variables$$

$$\Rightarrow \int e^{-y}dy - \int xe^{-x^2}dx = 0$$

$$i.e \quad -e^{-y} - \int xe^{-x^2}dx = c$$

The general solution becomes

$$i.e. \quad -e^{-y} + \frac{e^{t}}{2} = c$$

$or \quad \frac{e^{-x^2}}{2} - e^{-y} = c$ is the general solution.

Now we consider $y(0)$ =0 That is y=0 when x=0,

$\frac{1}{2} - 1 = c$ or $c = -\frac{1}{2}$

Now the general solution becomes

$$\frac{e^{-x^2}}{2} - e^{-y} = -\frac{1}{2}$$

This is the required solution.

Example –2 $solve: xy\dfrac{dy}{dx} = 1 + x + y + xy$

□ $xy\dfrac{dy}{dx} = 1 + x + y + xy$

$i.e., \quad xy\dfrac{dy}{dx} = (1+x) + y(1+x)$

$i.e., \quad xy\dfrac{dy}{dx} = (1+x)(1+y)$

$or \quad \dfrac{ydy}{1+y} = \dfrac{1+x}{x}dx$ by separating the variables.

$\Rightarrow \quad \int \dfrac{y}{1+y}dy - \int \dfrac{1+x}{x}dx = c$

$or \quad \int \dfrac{(1+y)-1}{1+y}dy - \int \dfrac{1}{x}dx - \int 1dx = c$

$ie., \quad \int 1dy - \int \dfrac{1}{1+y}dy - \log x - x = c$

$i.e \quad y - \log y - \log x - x = c$

$or \quad y - x \;\; - \log yx = c \;\; is\, the\, required \;\; solution$

Example – 4 : Solve : $y - x \;\; \dfrac{dy}{dx} = y^2 + \dfrac{dy}{dx}$

>> Rearranging the given equation we have,

$y - y^2 = \dfrac{dy}{dx} \;\; x+1$

$or \;\; \int \dfrac{dx}{x+1} = \int \dfrac{dy}{y - y^2}$

$i.e \; \log \;\; x+1 \;\; = \int \dfrac{dy}{y \;\; 1-y} ----------(1)$

We have to employ the method of partial fractions for the second term of the above.

$Let \;\; \dfrac{1}{y \;\; 1-y} = \dfrac{A}{y} + \dfrac{B}{y-1}$

$\Rightarrow 1 = A \;\; y-1 \;\; + By$

Put y=0,1 A=-1 and B=1

$$\therefore \int \frac{dy}{y\ 1-y} = -\int \frac{dy}{y} + \int \frac{dy}{1-y}$$

$$\int \frac{dy}{y\ 1-y} = -\log y - \log\ 1-y\ = \log \frac{1-y}{y}$$

Using this result in (1) we get,

$$\log\ x+1\ + \log\left(\frac{1-y}{y}\right) = c$$

$$or\ \log\left[\frac{x+1\ \ 1-y}{y}\right] = \log k$$

$\therefore\ \ x+1\ \ 1-y\ = ky$ *is the required solution.*

Example – 5 :

Solve:

$$\tan y \frac{dy}{dx} = \cos(x+y) + \cos(x-y)$$

 The given equation on expanding terms in the R.H.S. becomes

$$\tan \mathrm{y} \frac{\mathrm{dy}}{\mathrm{dx}} = \cos x \cos y - \sin x \sin y + \cos x \cos y + \sin x \sin y$$

$$ie.,\ \tan \mathrm{y} \frac{\mathrm{dy}}{\mathrm{dx}} = 2\cos x \cos y$$

$or\ \frac{\mathrm{tany}}{\mathrm{cosy}} dy = 2\cos x\, dx$ by separating the variables.*

$$\Rightarrow \int \mathrm{tany.}\ \sec y\, dy - \int 2\cos x\, dx = c$$

$\therefore \sec y - 2\sin x = c$ is the required solution.

Exact Differential Equations:

The differential equation M(x , y) dx + N(x , y)dy=0 to be an exact equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Further the solution of the exact equation is given by

$$\int M\, dx + \int N(y)\, dy = c$$

Where, in the first term we integrate M(x,y) w.r.t x keeping y fixed and N(y) indicate the terms in N with out x
(not containing x)

1. Solve: $5x^4 + 3x^2y^2 - 2xy^3\ \ dx + \ 2x^3y - 3x^2y^2 - 5y^4\ \ dy = 0$

(Though it is evident that the equation is a homogeneous one, before solving by putting y=vx we should check for exactness)

 $Let\ M = 5x^4 + 3x^2y^2 - 2xy^3 and\ \ N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 and\ \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

$ie., \int 5x^4 + 3x^2y^2 - 2xy^3\ dx + \int -5y^4 dy = c$

$Thus\ \ x^5 + x^3y^2 - x^2y^3 - y^5 = c$, is the required solution.

2. Solve: $\cos x \tan y + \cos(x+y)\ dx + \left[\sin x \sec^2 y + \cos(x+y)\right] dy = 0$

 $Let\ M = \cos x \tan y + \cos(x+y) : N = \sin x \sec^2 y + \cos(x+y)$

$\therefore\ \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y);\ \frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

$ie., \int \cos x \tan y + \cos(x+y)\ dx + \int 0\ dy = c$

Thus $\sin x \tan y + \sin(x+y) = c$, is the required solution.

3. Solve: $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

>> The given equation is put in the form,

$y\cos x + \sin y + y\ \ dx + \ \sin x + x\cos y + x\ \ dy = 0.$

$Let\ \ M = y\cos x + \sin y + y\ \ and\ \ N = \sin x + x\cos y + x$

$\therefore\ \frac{\partial M}{\partial y} = \cos x + \cos y + 1 and\ \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$Since\ \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

$ie., \int\ y\cos x + \sin y + y\ \ dx + \int 0\,dy = c$

$Thus\ \ y\sin x + x\sin y + xy = c$, is the required solution.

4. Solve: $ye^{xy}dx + \left(xe^{xy} + 2y\right)dy = 0$

 Let $M = ye^{xy}, N = xe^{xy} + 2y$

$\frac{\partial M}{\partial y} = ye^{xy}x + e^{xy}; \frac{\partial N}{\partial x} = xe^{xy}y + e^{xy}$

$Since \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

solution is given by $\int M\, dx + \int N(y)\, dy = c$

$ie., \int ye^{xy}dx + \int 2y\, dy = c$

$ie., y\frac{e^{xy}}{y} + y^2 = c$

Thus $e^{xy} + y^2 = c$, is the required solution.

5. Solve: $\left(y(1+1/x) + \cos y\right)dx + \left(x + \log x - x\sin y\right)dy = 0$

 $Let\ M = y(1+1/x) + \cos y\ \ and\ \ N = x + \log x - x\sin y$

$\therefore \frac{\partial M}{\partial y} = 1 + 1/x - \sin y\ \ and\ \ \frac{\partial N}{\partial x} = 1 + 1/x - \sin y$

$Since\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\, dx + \int N(y)\, dy = c$

$ie., \int \left(y(1+1/x) + \cos y\right)dx + \int 0\, dy = c$

$Thus\ y(x + \log x) + x\cos y = c$, is the required solution.

Equations reducible to the exact form:

Integrating factor: Type-1:

Suppose that, for the equation M dx + N dy =0

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then we take their difference.

The difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ should be close to the expression of M or N.

If it is so, then we compute $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ or $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ or $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$

Then $e^{\int f(x)dx}$ or $e^{-\int g(y)dy}$ is an integrating factor.

The following basic results will be useful :

(i) $e^{\log x} = x$ $(ii)\, e^{n\log x} = x^n$

1. Solve: $4xy+3y^2-x\ dx+x\ x+2y\ dy=0$

 $Let\, M = 4xy+3y^2-x\ and\ N = x\ x+2y\ = x^2+2xy$

$$\frac{\partial M}{\partial y} = 4x+6y\ and\ \frac{\partial N}{\partial x} = 2x+2y.\ \text{ The equation is not exact}$$

$$Consider\ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x+4y = 2(x+2y)....close\ to\ N.$$

$$Now\ \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) - \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$$

$Hence\ e^{\int f(x)dx}$ is an integrating factor.

ie., $e^{\int f(x)dx} = e^{\int f\frac{2}{x}dx} = e^{2\log x} = e^{\log(x^2)} = x^2$

Multiplying the given equation by x^2 we now have,

$M = 4x^3y+3x^2y^2-x^3\ \ and\ \ N = x^4+2x^3y$

$$\frac{\partial M}{\partial y} = 4x^3+6x^2y\ and\ \frac{\partial N}{\partial x} = 4x^3+6x^2y$$

Solution of the exact equation is $\int M\, dx + \int N(y)\, dy = c$

$ie.,\ \int\ 4x^3y+3x^2y^2-x^3\ dx + \int 0\, dy = c$

$Thus\ x^4y+x^3y^2-\frac{x^4}{4} = c$, is the required solution.

2. Solve: y(2x-y+1)dx+x(3x-4y+3)dy=0

 $Let\, M = y\ 2x-y+1\ \ and\ N = x\ 3x-4y+3$

$ie.,\ M = 2xy-y^2+y\ \ and\ \ N = 3x^2-4xy+3x$

$$\frac{\partial M}{\partial y} = 2x-2y+1,\ \frac{\partial N}{\partial x} = 6x-4y+3$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4x+2y-2 = -2(2x-y+1)....near\ to\ M.$$

$$Now\ \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2(2x-y+1)}{y(2x-y+1)} = -\frac{2}{y} = g(y)$$

$$Hence\, I.F = e^{-\int g(y)dy} = e^{\int\frac{2}{y}dy} = e^{2\log y} = e^{\log(y^2)} = y^2$$

Multiplying the given equation by y^2 we now have,

$$M = 2xy^3 - y^4 + y^3 \;\; and \;\; N = 3x^2y^2 - 4xy^3 + 3xy^2$$

$$\frac{\partial M}{\partial y} = 6xy^2 - 4y^3 + 3y^2 \;\; and \;\; \frac{\partial N}{\partial x} = 6xy^2 - 4y^3 + 3y^2$$

The Solution is $\int M\,dx + \int N(y)\,dy = c$

$$ie., \int \; 2xy^3 - y^4 + y^3 \;\; dx + \int 0\,dy = c$$

Thus $x^2y^3 - xy^4 + xy^3 = c$, is the required solution.

Integrating Factor: Type-2:

If the given equation M dx +N dy=0 is of the form
yf (xy) dx+xg(xy)dy=0

then $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$

1. Solve: $y \;\; 1 + xy + x^2y^2 \;\; dx + x \;\; 1 - xy + x^2y^2 \;\; dy = 0$

>> The equation is of the form yf(xy)dx+xg(xy)dy=0 where,

$$M = yf(xy) = y + xy^2 + x^2y^3 \;\; and$$

$$N = xg(xy) = x - x^2y + x^3y^2$$

$$Now\; Mx - Ny = \;\; xy + x^2y^2 + x^3y^3 \;\; - \;\; xy - x^2y^2 + x^3y^3 \;\; = 2x^2y^2$$

$$\therefore \; \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2} \;\; is\; the\; I.F.$$

Multiplying the given equation with $1/2x^2y^2$ it becomes an exact equation where we now have,

$$M = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \;\; and \;\; N = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

The solution is given by $\int M\,dx + \int N(y)dy = c$

$$ie., \int \left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \right) dx + \int -\frac{1}{2y}\,dy = c$$

$$ie., \frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

2. Solve: $y \;\; xy + 2x^2y^2 \;\; dx + x \;\; xy - x^2y^2 \;\; dy = 0$

>> The equation is of the form yf(xy)dx+xg(xy)dy=0 where,

$M = xy^2 + 2x^2y^3$ *and*

$N = x^2y - x^3y^2$

Now $Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3$

Thus $1/3x^3y^3$ is the I.F. Multiplying the given equation by this I.F we have an exact equation where we now have,

$$M = \frac{1}{3x^2y} + \frac{1}{3x} \text{ and } N = \frac{1}{3xy^2} - \frac{1}{3y}$$

The solution is $\int M\,dx + \int N(y)dy = c$

$$ie., \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$$

$$ie., \quad -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

Integrating factor: Type-3:

If the given equation Mdx+Ndy =0 is of the form

$$x^{k_1}y^{k_2}(c_1ydx + c_2x\,dy) + x^{k_3}y^{k_4}(c_3ydx + c_4x\,dy) = 0$$

Where k_i and c_i (i=1 to 4) are constants then $x^a y^b$ is an integrating factor. The constants a and b are determined such that the condition for an exact equation is satisfied.

1. Solve: x(4y dx+2x dy)+ y^3(3y dx+5x dy) = 0

>> We have $(4xy + 3y^4)dx + (2x^2 + 5xy^3)dy = 0$

Multiplying the equation by $x^a y^b$ we have,

$\Rightarrow 4(b+1) = 2(a+2)$ *and* $3(b+4) = 5(a+1)$

ie., $a = 2b$ *and* $5a - 3b = 7$

By solving we get a=2and b=1

We now have, $M = 4x^3y^2 + 3x^2y^5$ *and* $N = 2x^4y + 5x^3y^4$

The solution is $\int Mdx + \int N(y)dy = c$

$M = 4x^{a+1}y^{b+1} + 3x^{a}y^{b+4}$ *and*

$N = 2x^{a+2}y^{b} + 5x^{a+1}y^{b+3}$

$\frac{\partial M}{\partial y} = 4(b+1)x^{a+1}y^{b} + 3(b+4)x^{a}y^{b+3}$

$\frac{\partial N}{\partial x} = 2(a+2)x^{a+1}y^{b} + 5(a+1)x^{a}y^{b+3}$

We have to find a and b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

ie., $\int 4x^{3}y^{2} + 3x^{2}y^{5}\ dx + \int 0\ dy = c$

Thus $x^{4}y^{2} + 3x^{2}y^{5}$=c, is the required solution.

2. Solve: $(y^{2} + 2x^{2}y)dx + (2x^{3} - xy)\,dy = 0$

>>Multiplying the given equation by $x^{a}y^{b}$ we have,

$M = x^{a}y^{b+2} + 2x^{a+2}y^{b+1}$ *and*

$N = 2x^{a+3}y^{b} - x^{a+1}y^{b+1}$

$\frac{\partial M}{\partial y} = (b+2)x^{a}y^{b+1} + 2(b+1)x^{a+2}y^{b}$

$\frac{\partial N}{\partial x} = 2(a+3)x^{a+2}y^{b} - (a+1)x^{a}y^{b+1}$

Let us find a and b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow (b+2) = -(a+1)$ *and* $2(b+1) = 2(a+3)$

ie., $a + b = -3$ *and* $a - b = -2$

By solving we get a = -5/2 and b=-1/2.

We now have, $M = x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}$ *and*

$N = 2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}$

The solution is $\int M dx + \int N(y) dy = c$

ie., $\int \left(x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2} \right) dx + \int 0 \, dy = c$

ie., $\dfrac{x^{-3/2}}{-3/2} y^{3/2} + 2\dfrac{x^{1/2}}{1/2} y^{1/2} = c$

ie., $\dfrac{-2}{3} x^{-3/2} y^{3/2} + 4x^{1/2} y^{1/2} = c$

Thus $6\sqrt{xy} - \sqrt{y^3/x^3} = k$, is the required solution, where k=3c/2

Type-4 Exactness by inspection:

1. Solve : $\left(1 + y\tan(xy)\right) dx + \left(x\tan(xy)\right) dy = 0$

>> The given equation can be put in the form

$dx + \tan(xy)\left(y\,dx + x\,dy\right) = 0$

ie., $dx + \tan(xy)d(xy) = 0$

Integrating we get, $x + \log\sec(xy) = c$, being the required solution.

2. Solve: $\dfrac{y\,dx - x\,dy}{y^2} + (x\,dx + y\,dy) = 0$

>> The given equation is equivalent to the form,

$d\left(\dfrac{x}{y}\right) + x\,dx + y\,dy = 0$

$\Rightarrow \dfrac{x}{y} + \dfrac{x^2}{2} + \dfrac{y^2}{2} = c$, on integration.

Thus $\dfrac{x}{y} + \dfrac{1}{2}\left(x^2 + y^2\right) = c$, is the required solution.

Orthogonal trajectories

Definition: If two family of curves are such that every member of one family intersect every member of the other family at right angles then they are said to be orthogonal trajectories each other

Method of finding the orthogonal trajectories

Case - (i) Cartesian family $f(x, y, c) = 0$

We differentiate *w.r.t* x and eliminate the parameter c. The equation so obtained is called as the differential equation of the given family.

We know that if $\tan\psi = \frac{dy}{dx}$ is the slope of a given line then the slope of the line perpendicular to it is $\frac{-1}{\tan\psi} = -\frac{dx}{dy}$. Accordingly in the differential equation of the given family we shall replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to arrive at a new differential equation. Solving this new differential equation we get the orthogonal trajectories of the given family of curves.

Self orthogonal family : If the differential equation of the given family remains unaltered after replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ then the given family of curves is said to be *self orthogonal.*

Case - (ii) : Polar family $f(r, \theta, c) = 0$

We know that $\tan\phi = r\frac{d\theta}{dr}$ for a polar curve where ϕ is the angle between the radius vector and the tangent. $\phi_2 - \phi_1 = 90^0$ is the condition for two polar curves to be orthogonal.

$\therefore \quad \phi_2 = 90^0 + \phi_1 \Rightarrow \tan\phi_2 = \tan(90^0 + \phi_1)$

ie., $\quad \tan\phi_2 = -\cot\phi_1 \quad$ or $\quad \tan\phi_2 = \frac{-1}{\tan\phi_1}$

But $\tan\phi_1 = r\frac{d\theta}{dr}$ for the given curve and $\tan\phi_2 = r\frac{d\theta}{dr}$ for the orthogonal curve at the same point.

$\therefore \quad r\frac{d\theta}{dr}$ for the curve to be replaced by $\frac{-1}{r\frac{d\theta}{dr}}$

ie., $\quad -r^2\frac{d\theta}{dr}$ to be replaced by $\frac{dr}{d\theta}$ or vice - versa.

In otherwords, we have to differentiate $f(r, \theta, c) = 0$ *w.r.t* θ and eliminate c to obtain the D.E of the given family. We have to replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ to obtain the new D.E and solve the same to obtain the required orthogonal trajectories.

1. *Find the O.T of the family of parabolas* $y^2 = 4ax$.

>> Consider $\dfrac{y^2}{x} = 4a$ ... (1)

(If the parameter is on one side of the equation exclusively, then the same gets eliminated once we differentiate)

Now differentiating (1) $w.r.t\ x$ we have

$$\frac{x \cdot 2y\dfrac{dy}{dx} - y^2 \cdot 1}{x^2} = 0 \quad \text{or} \quad 2xy\frac{dy}{dx} - y^2 = 0$$

ie., $2x\dfrac{dy}{dx} - y = 0$, is the D.E of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy}\right) - y = 0 \quad \text{or} \quad 2x\,dx + y\,dy = 0$$

$\Rightarrow$ $\int 2x\,dx + \int y\,dy = c$

ie., $x^2 + \dfrac{y^2}{2} = c$ or $2x^2 + y^2 = 2c = k$ (say)

Thus $2x^2 + y^2 = k$, is the required O.T.

2. *Find the O.T of the family of astroids* $x^{2/3} + y^{2/3} = a^{2/3}$

>> Consider $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating $w.r.t\ x$, we have

$$\frac{2}{3} \cdot x^{-1/3} + \frac{2}{3} \cdot y^{-1/3}\frac{dy}{dx} = 0$$

ie., $x^{-1/3} + y^{-1/3}\dfrac{dy}{dx} = 0$, is the D.E of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0 \quad ie., \quad x^{-1/3}\,dy = y^{-1/3}\,dx$$

ie., $y^{1/3}\,dy = x^{1/3}\,dx$ by separating the variables.

$\Rightarrow$ $\int y^{1/3}\,dy - \int x^{1/3}\,dx = c$

ie., $\dfrac{y^{4/3}}{(4/3)} - \dfrac{x^{4/3}}{(4/3)} = c$ or $x^{4/3} - y^{4/3} = -\dfrac{4c}{3} = k$ (say)

Thus $x^{4/3} - y^{4/3} = k$ is the required O.T.

3) **Find the orthogonal trajectories of the family of curves**
$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ 'λ' being the parameter . **(July 2015)**

Soln: $we\,have\,\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ ....(1)

Differentiating the (1) equation we get,

$$\frac{2x}{a^2}+\frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx}=0$$

$$i.e \qquad \frac{x}{a^2}=\frac{-y}{b^2+\lambda}\cdot\frac{dy}{dx} \text{........(2)}$$

$$Also\ \ from(1) \qquad \frac{x^2}{a^2}-1=\frac{-y^2}{b^2+\lambda}$$

$$\Rightarrow \frac{x^2-a^2}{a^2}=\frac{-y^2}{b^2+\lambda} \text{.............(3)}$$

$Now, dividing$ (2) by(3) we get

$$\frac{x}{x^2-a^2}=\frac{y}{y^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{x}{x^2-a^2}=\frac{1}{y}\cdot\frac{dy}{dx}$$

$Now\ let\,us\,replace\ \frac{dy}{dx}\ by\,\frac{-dx}{dy}$

$$\therefore\ \frac{x}{x^2-a^2}=\frac{1}{y}\cdot\left(-\frac{dx}{dy}\right)$$

$$or\ \ ydy=-\frac{x^2-a^2}{x}dx\ by\,separating\,the\ \text{variables}$$

$$\Rightarrow \int ydy=-\int xdx+a^2\int\frac{dx}{x}+c$$

$$i.e\ \frac{y^2}{2}=\frac{-x^2}{2}+a^2\log x+c\ \ is\,the\,required\ orthogonal\,trajectories$$

4) **Find the orthogonal trajectory of the cardiods** $r=a\ 1-\cos\theta$ **, using the differential equation method .** **(Jan 2015 ,Jan2014)**

Soln:

Differentiating the given equation w.r.t θ,we get $\frac{dr}{d\theta}=a\sin\theta$.Substituting for a in the given equation,we get

$r=\left(\frac{1-\cos\theta}{\sin\theta}\right)\frac{dr}{d\theta}$.........DE of given equation

Changing $\frac{dr}{d\theta}$ *to* $-r^2\frac{d\theta}{dr}$ $\quad r=\left(\frac{1-\cos\theta}{\sin\theta}\right)\left(-r^2\frac{d\theta}{dr}\right)$

$\frac{dr}{r}+(\cos ec\theta-\cot\theta)\,d\theta=0$........DE of orthogonal trajectories

solving this equation,we get

$\log r+\log(\cos ec\theta-\cot\theta)-\log\sin\theta=\log c$

$r\frac{\cos ec\theta-\cot\theta}{\sin\theta}=c\Rightarrow r\frac{1-\cos\theta}{\sin^2\theta}=c$

$r=c(1+\cos\theta)$, this is requried orthogonal trajectories.