MODULE II

DIFFERENTIAL CALCULUS-II

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Taylor's Mean Value Theorem:

(Generalized Mean Value Theorem):

(English Mathematician Brook Taylor 1685-1731)

Statement:

Suppose a function $f(x)$ satisfies the following two conditions:

(i) $f(x)$ and it's first (n-1) derivatives are continuous in a closed interval $[a,b]$

(ii) $f^{(n-1)}(x)$ is differentiable in the open interval (a,b)

Then there exists at least one point c in the open interval (a,b) such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \frac{(b-a)^3}{\lfloor 3} f'''(a) + \ldots..$$

$$\ldots.. + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(b-a)^n}{\lfloor n} f^{(n)}(c) \to (1)$$

Taking $b = a + h$ and for $0 < \theta < 1$, the above expression (1) can be rewritten as

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \frac{h^3}{\lfloor 3} f'''(a) + \ldots. + \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{h^n}{\lfloor n} f^{(n)}(a+\theta h) \to (2)$$

Taking b=x in (1) we may write

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \frac{(x-a)^3}{\lfloor 3} f'''(a) + \ldots + \frac{(x-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + R_n \to (3)$$

$$Where \quad R_n = \frac{(x-a)^n}{\lfloor n} f^{(n)}(c) \to \text{Remainder term after } n \text{ terms}$$

When $n \to \infty$, we can show that $|R_n| \to 0$, thus we can write the Taylor's series as

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(x-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \ldots.$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) \to (4)$$

Using (4) we can write a Taylor's series expansion for the given function f(x) in powers of (x-a) or about the point 'a'.

Maclaurin's series:

(Scottish Mathematician Colin Maclaurin's 1698-1746)

When a=0, expression (4) reduces to a Maclaurin's expansion given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \ldots.$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{x^n}{\lfloor n} f^{(n)}(0) \to (5)$$

Example 1: Obtain a Taylor's expansion for $f(x) = sin\, x$ in the ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to the fourth degree term.

The Taylor's expansion for $f(x)$ about $\frac{\pi}{4}$ is

$$f(x) = f(\frac{\pi}{4}) + (x - \frac{\pi}{4}) f'(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^2}{\lfloor 2} f''(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^3}{\lfloor 3} f'''(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^4}{\lfloor 4} f^{(4)}(\frac{\pi}{4}) \ldots. \to (1)$$

$$f(x) = sin\, x \Rightarrow f\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \qquad f'(x) = cos\, x \Rightarrow f'\left(\frac{\pi}{4}\right) = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(x) = -\,sin\, x \Rightarrow f''\left(\frac{\pi}{4}\right) = -sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -\,cos\, x \Rightarrow f'''\left(\frac{\pi}{4}\right) = -cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f^{(4)}(x) = sin\, x \Rightarrow f^{(4)}\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Substituting these in (1) we obtain the required Taylor's series in the form

$$f(x) = \frac{1}{\sqrt{2}} + (x - \frac{\pi}{4})(\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^2}{\lfloor 2}(-\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^3}{\lfloor 3}(-\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^4}{\lfloor 4}(\frac{1}{\sqrt{2}}) \ldots.$$

$$f(x) = \frac{1}{\sqrt{2}}\left[1 + (x - \frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^2}{\lfloor 2} + \frac{(x-\frac{\pi}{4})^3}{\lfloor 3} - \frac{(x-\frac{\pi}{4})^4}{\lfloor 4} + \ldots\right]$$

Example 2………………………….: Obtain a Taylor's expansion for $f(x) = \log_e x$ up to the term containing $x-1^{\;4}$ and hence find $\log_e(1.1)$.

The Taylor's series for $f(x)$ about the point 1 is

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{\lfloor 2} f''(1) + \frac{(x-1)^3}{\lfloor 3} f'''(1) + \frac{(x-1)^4}{\lfloor 4} f^{(4)}(1) .... \to (1)$$

Here $f(x) = \log_e x \Rightarrow f(1) = \log 1 = 0$; $\quad f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1; \quad f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \Rightarrow f^{(4)}(1) = -6 \text{ etc.,}$$

Using all these values in (1) we get

$$f(x) = \log_e x = 0 + (x-1)(1) + \frac{(x-1)^2}{\lfloor 2}(-1) + \frac{(x-1)^3}{\lfloor 3}(2) + \frac{(x-1)^4}{\lfloor 4}(-6) ....$$

$$\Rightarrow \log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} ....$$

Taking x=1.1 in the above expansion we get

$$\Rightarrow \log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} .... = 0.0953$$

Example 18: Using Taylor's theorem Show that

$$\log_e(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \quad \textit{for } 0 < \theta < 1, x > 0$$

Taking n=3 in the statement of Taylor's theorem, we can write

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{\lfloor 2} f''(a) + \frac{x^3}{\lfloor 3} f'''(a+\theta x) \to (1)$$

Consider $f(x) = \log_e x \Rightarrow f'(x) = \frac{1}{x}$; $f''(x) = -\frac{1}{x^2}$ *and* $f'''(x) = \frac{2}{x^3}$

Using these in (1), we can write,

$$\log(a+x) = \log a + x\left(\frac{1}{a}\right) + \frac{x^2}{\lfloor 2}\left(-\frac{1}{a^2}\right) + \frac{x^3}{\lfloor 3}\left(\frac{2}{(a+\theta x)^3}\right) \to (2)$$

For a=1 in (2) we write,

$$\log(1+x) = \log 1 + x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3} = x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3}$$

Since $x > 0$ *and* $\theta > 0, (1+\theta x)^3 > 1$ *and therefore* $\frac{1}{(1+\theta x)^3} < 1$

$$\therefore \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Example 19: Obtain a Maclaurin's series for $f(x) = \sin x$ up to the term containing x^5.

The Maclaurin's series for f(x) is

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(4)}(0) + \frac{x^5}{\lfloor 5} f^{(5)}(0) \ldots. \rightarrow (1)$$

Here $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$ $\quad f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$ $\quad f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = \sin 0 = 0$ $\quad f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = \cos 0 = 1$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = \sin x$ as

$$f(x) = \sin x = 0 + x\,(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(1) \ldots.$$

$$\Rightarrow \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots.$$

Indeterminate Forms:

While evaluating certain limits, we come across expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0$ *and* 1^∞ which do not represent any value. Such expressions are called Indeterminate Forms.

We can evaluate such limits that lead to indeterminate forms by using L'Hospital's Rule (French Mathematician 1661-1704).

L'Hospital's Rule:

If f(x) and g(x) are two functions such that

(i) $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$

(ii) $f'(x)$ $and g'(x)$ $exist$ and $g'(a) \neq 0$

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

The above rule can be extended, i.e, if

$f'(a)=0$ and $g'(a)=0$ $then$ $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = ...$

Note:

1. We apply L'Hospital's Rule only to evaluate the limits that in $\frac{0}{0}, \frac{\infty}{\infty}$ forms. Here we differentiate the numerator and denominator separately to write $\frac{f'(x)}{g'(x)}$ and apply the limit to see whether it is a finite value. If it is still in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form we continue to differentiate the numerator and denominator and write further $\frac{f''(x)}{g''(x)}$ and apply the limit to see whether it is a finite value. We can continue the above procedure till we get a definite value of the limit.

2. To evaluate the indeterminate forms of the form $0\times\infty, \infty-\infty,$ we rewrite the functions involved or take L.C.M. to arrange the expression in either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply L'Hospital's Rule.
3. To evaluate the limits of the form $0^0, \infty^0$ and 1^∞ i.e, where function to the power of function exists, call such an expression as some constant, then take logarithm on both sides and rewrite the expressions to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.

4. We can use the values of the standard limits like

$\lim_{x\to 0}\frac{\sin x}{x}=1;$ $\lim_{x\to 0}\frac{\tan x}{x}=1;$ $\lim_{x\to 0}\frac{x}{\sin x}=1;$ $\lim_{x\to 0}\frac{x}{\tan x}=1;$ $\lim_{x\to 0}\cos x=1;$ etc

Evaluate the following limits:

Example 1: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x - x}{\tan^3 x}$

$$\lim_{x\to 0}\frac{\sin x - x}{\tan^3 x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\cos x - 1}{3\tan^2 x\sec^2 x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-\sin x}{6\tan x\sec^4 x + 6\tan^3 x\sec^2 x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-\cos x}{6\sec^6 x + 24\tan^2 x\sec^4 x + 18\tan^2 x\sec^4 x + 12\tan^4 x\sec^2 x} = -\frac{1}{6}$$

Method 2:

$$\lim_{x\to 0}\frac{\sin x - x}{\tan^3 x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\dfrac{\sin x - x}{x^3}}{\left(\dfrac{\tan x}{x}\right)^3}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\sin x - x}{x^3}\left(\frac{0}{0}\right) \quad \because \lim_{x\to 0}\frac{\tan x}{x} = 1$$

$$= \lim_{x\to 0}\frac{\cos x - 1}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-\sin x}{6x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-\cos x}{6} = -\frac{1}{6}$$

Example 2: Evaluate $\lim\limits_{x\to 0}\dfrac{a^x - b^x}{x}$

$$\lim_{x\to 0}\frac{a^x - b^x}{x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{a^x\log a - b^x\log b}{1}$$

$$= \log a - \log b = \log\frac{a}{b}$$

Example 3: Evaluate $\lim\limits_{x\to 0}\dfrac{x\sin x}{(e^x - 1)^2}$

$$\lim_{x\to 0}\frac{x\sin x}{(e^x - 1)^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\sin x + x\cos x}{2(e^x - 1)e^x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\cos x + \cos x - x\sin x}{2\left[e^x.e^x + (e^x - 1)e^x\right]} = \frac{1+1-0}{2[1+0]} = \frac{2}{2} = 1$$

Example 4: Evaluate $\lim_{x\to 0}\frac{x\,e^x-\log(1+x)}{x^2}$

$$\lim_{x\to 0}\frac{x\,e^x-\log(1+x)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+x\,e^x-\frac{1}{1+x}}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+e^x+x\,e^x+\frac{1}{1+x^{\ 2}}}{2}=\frac{1+1+0+1}{2}=\frac{3}{2}$$

Example 5: Evaluate $\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}$

$$\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sinh x+\sin x}{\sin x+x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cosh x+\cos x}{\cos x+\cos x-x\sin x}=\frac{1+1}{1+1-0}=\frac{2}{2}=1$$

Example 6: Evaluate $\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}$

$$\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x-\frac{1}{1+x}+1}{\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x+\frac{1}{(1+x)^2}}{2\cos 2x}=\frac{-1+1}{2}=0$$

Example 7: Evaluate $\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}$

$$\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)-1}{1-\frac{1}{x}}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)^2+x^{x-1}}{\frac{1}{x^2}}=\frac{1+1}{1}=2$$

$$\textit{since}\;\; y=x^x\Rightarrow\log y=x\log x\Rightarrow\frac{1}{y}y'$$

$$=1+\log x\Rightarrow y'=y(1+\log x)$$

$$\textit{then}\;\frac{d}{dx}(x^x)=x^x(1+\log x)$$

Example 8: Evaluate $\lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{4}}\frac{2\sec^2 x\tan x-2\sec^2 x}{-4\sin 4x}\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{4}}\frac{2\sec^4 x+4\sec^2 x\tan^2 x-4\sec^2 x\tan x}{-16\cos 4x}$$

$$= \frac{2\left(\sqrt{2}\right)^4 + 4\left(\sqrt{2}\right)^2 (1)^2 - 4\left(\sqrt{2}\right)^2}{16} = \frac{8}{16} = \frac{1}{2}$$

Example 9: Evaluate $\lim_{x \to a} \frac{\log(\sin x.\operatorname{cosec} a)}{\log(\cos a.\sec x)}$

$$\lim_{x \to a} \frac{\log(\sin x.\operatorname{cosec} a)}{\log(\cos a.\sec x)}\left(\frac{0}{0}\right) = \lim_{x \to a} \frac{\left[\frac{\cos x \operatorname{cosec} a}{\sin x.\operatorname{cosec} a}\right]}{\left[\frac{\sec x \tan x.\cos a}{\cos a.\sec x}\right]} = \lim_{x \to a} \frac{\cot x}{\tan x} = \cot^2 a$$

Example 10: Evaluate $\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x}$

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^x - e^{-x} + 2\sin x}{\sin x + x\cos x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x}{\cos x + \cos x - x\sin x} = \frac{1+1+2}{1+1-0} = 2$$

Example 11: Evaluate $\lim_{x \to 0} \frac{x\cos x - \log(1+x)}{x^2}$

$$\lim_{x \to 0} \frac{x\cos x - \log(1+x)}{x^2}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\cos x - x\sin x - \frac{1}{1+x}}{2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x - \sin x - x\cos x + \frac{1}{(1+x)^2}}{2} = \frac{-0-0-0+1}{2} = \frac{1}{2}$$

Example 12: Evaluate $\lim_{x \to 0} \frac{\log(1-x^2)}{\log\cos x}$

$$\lim_{x \to 0} \frac{\log(1-x^2)}{\log\cos x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\left[\frac{-2x}{1-x^2}\right]}{\left(\frac{-\sin x}{\cos x}\right)} = \lim_{x \to 0} \frac{2x\cos x}{(1-x^2)\sin x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{2\cos x - 2x\sin x}{(1-x^2)\cos x - 2x\sin x} = \frac{2-0}{1-0} = 2$$

Example 13: Evaluate $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\sec^2 x - \cos x}{3\sin^2 x\cos x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{2\sec^2 x\tan x + \sin x}{6\sin x\cos^2 x - 3\sin^3 x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x + \cos x}{6\cos^3 x - 12\sin^2 x \cos x - 9\sin^2 x \cos x} = \frac{0+2+1}{6-0-0} = \frac{3}{6} = \frac{1}{2}$$

Method 2:

$$\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x\to 0} \frac{\dfrac{\tan x - \sin}{x^3}}{\left(\dfrac{\sin x}{x}\right)^3} = \lim_{x\to 0} \frac{\tan x - \sin x}{x^3}\left(\frac{0}{0}\right) \quad \because \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \lim_{x\to 0} \frac{\sec^2 x - \cos x}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2\sec^2 x \tan x + \sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x + \cos x}{6} = \frac{0+2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

Example 14: Evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$

$$\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x} = \lim_{x\to 0} \frac{\dfrac{\tan x - x}{x^3}}{\left(\dfrac{\tan x}{x}\right)} = \lim_{x\to 0} \frac{\tan x - x}{x^3}\left(\frac{0}{0}\right) \qquad \because \lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \frac{0+2}{6} = \frac{1}{3}$$

Example 15: Evaluate $\lim_{x\to 0} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}$

$$\lim_{x\to 0} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{ae^{ax} + ae^{-ax}}{b/(1+bx)}$$

$$= \frac{a+a}{b} = \frac{2a}{b}$$

Example 16: Evaluate $\lim_{x\to 0} \frac{a^x - 1 - x\log a}{x^2}$

$$\lim_{x\to 0} \frac{a^x - 1 - x\log a}{x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{a^x \log a - \log a}{2x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{a^x (\log a)^2}{2} = \frac{1}{2}(\log a)^2$$

Example 17: Evaluate $\lim_{x\to 0} \frac{e^x - \log(e+ex)}{x^2}$

$$\lim_{x\to 0}\frac{e^x-\log(e+ex)}{x^2}=\lim_{x\to 0}\frac{e^x-\log e(1+x)}{x^2}=\lim_{x\to 0}\frac{e^x-\log e-\log(1+x)}{x^2}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{e^x-\frac{1}{1+x}}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+\frac{1}{(1+x)^2}}{2}=\frac{1+1}{2}=1$$

Limits of the form $\left(\frac{\infty}{\infty}\right)$:

Example 18: Evaluate $\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}$

$$\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{(2\cos 2x/\sin 2x)}{(\cos x/\sin x)}=\lim_{x\to 0}\frac{2\cot 2x}{\cot x}=\lim_{x\to 0}\frac{2\tan x}{\tan 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x}{2\sec^2 2x}=\frac{2}{2}=1$$

Example 19: Evaluate $\lim_{x\to 0}\frac{\log x}{\cos ecx}$

$$\lim_{x\to 0}\frac{\log x}{\cos ecx}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{1\backslash x}{-\cos ec\, x.\cot x}=\lim_{x\to 0}\frac{-\sin^2 x}{x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-2\sin 2x}{\cos x-x\sin x}=\frac{0}{1-0}=0$$

Example 20: Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}$

$$\lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=-\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin x\cos x}{1}=\frac{-0}{1}=0$$

Example 21: Evaluate $\lim_{x\to 1}\frac{\log(1-x)}{\cot\pi x}$

$$\lim_{x\to 1}\frac{\log(1-x)}{\cot\pi x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 1}\frac{-1/(1-x)}{-\pi\cos ec^2\pi x}=\lim_{x\to 1}\frac{\sin^2\pi x}{\pi(1-x)}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{2\pi\sin\pi x\cos\pi x}{-\pi}=\frac{0}{-\pi}=0$$

Example 22: Evaluate $\lim_{x\to 0}\log_{\tan 2x}\tan 3x$

$$\lim_{x\to 0}\log_{\tan 2x}\tan 3x=\lim_{x\to 0}\left(\frac{\log\tan 3x}{\log\tan 2x}\right)\left(\frac{\infty}{\infty}\right)\qquad \because \log_b a=\frac{\log_e a}{\log_e b}$$

$$= \lim_{x\to 0}\left(\frac{3\sec^2 3x/\tan 3x}{2\sec^2 2x/\tan 2x}\right) = \lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right) = \lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{6/\sin 6x}{4/\sin 4x}\right) = \lim_{x\to 0}\left(\frac{6\sin 4x}{4\sin 6x}\right)\left(\frac{0}{0}\right) = \lim_{x\to 0}\left(\frac{24\cos 4x}{24\cos 6x}\right) = \frac{24}{24} = 1$$

Example 23: Evaluate $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}$

$$\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}\left(\frac{\infty}{\infty}\right) = \lim_{x\to a}\frac{1/(x-a)}{e^x/(e^x-e^a)} = \lim_{x\to a}\frac{(e^x-e^a)}{e^x(x-a)}\left(\frac{0}{0}\right) = \lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x} = \frac{e^a}{e^a} = 1$$

Limits of the form $0\times\infty$ **:** To evaluate the limits of the form $0\times\infty$ **,** we rewrite the given expression to obtain either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 24: Evaluate $\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x$

$$\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x \quad 0\times\infty \;\; form \;\; = \lim_{x\to\infty}\frac{(a^{\frac{1}{x}}-1)}{\left(\frac{1}{x}\right)}\left(\frac{0}{0}\right) = \lim_{x\to\infty}\frac{a^{\frac{1}{x}}(\log a)\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to\infty} a^{\frac{1}{x}}(\log a) = a^0\log a = \log a$$

Example 25: Evaluate $\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

$$\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x \quad 0\times\infty \;\; form \;\; = \lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)}{\cot x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\operatorname{cos}ec^2 x} = \frac{0}{1} = 0$$

Example 26: Evaluate $\lim_{x\to 1}\sec\frac{\pi}{2x}.\log x$

$$\lim_{x\to 1}\sec\frac{\pi}{2x}.\log x \quad \infty\times 0 \;\; form \;\; = \lim_{x\to 1}\frac{\log x}{\cos\frac{\pi}{2x}}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{1/x}{-\frac{\pi}{2}\left(\sin\frac{\pi}{2x}\right)\left(\frac{-1}{x^2}\right)}=\lim_{x\to 1}\frac{2x}{\pi\sin\frac{\pi}{2x}}=\frac{2}{\pi}$$

Example 27: Evaluate $\lim_{x\to 0} x\log\tan x$

$$\lim_{x\to 0} x\log\tan x \quad 0\times\infty \;\; form \;\; =\lim_{x\to 0}\frac{\log\tan x}{(1/x)}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{\sec^2 x/\tan x}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to 0}\frac{-x^2}{\sin x.\cos x}$$

$$=\lim_{x\to 0}\frac{-2x^2}{\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-4x}{2\cos 2x}=\frac{0}{2}=0$$

Example 28: Evaluate $\lim_{x\to 1}(1-x^2)\tan\frac{\pi x}{2}$

$$\lim_{x\to 1}(1-x^2)\tan\frac{\pi x}{2} \quad 0\times\infty \;\; form$$

$$=\lim_{x\to 1}\frac{1-x^2}{\cot\frac{\pi x}{2}}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{-2x}{-\frac{\pi}{2}\cos ec^2\frac{\pi x}{2}}$$

$$=\frac{2}{\left(\frac{\pi}{2}\right)}=\frac{4}{\pi}$$

Example 29: Evaluate $\lim_{x\to 0}\tan x.\log x$

$$\lim_{x\to 0}\tan x.\log x \;\; 0\times\infty \;\; form \;\; =\lim_{x\to 0}\frac{\log x}{\cot x}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{1/x}{-\cos ec^2 x}=\lim_{x\to 0}\frac{-\sin^2 x}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin 2x}{1}=\frac{0}{1}=0$$

Limits of the form $\infty-\infty$ **:** To evaluate the limits of the form $\infty-\infty$ **,** we take L.C.M. and rewrite the given expression to obtain either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 30: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x}-\cot x\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]=\lim_{x\to 0}\left[\frac{1}{x}-\frac{\cos x}{\sin x}\right]\ \infty-\infty\ \textit{form}$$

$$=\lim_{x\to 0}\left[\frac{\sin x-x\cos x}{x\sin x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right]$$

$$=\lim_{x\to 0}\left[\frac{x\sin x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}\right]$$

$$=\frac{0+0}{1+1-0}=0$$

Example 31: Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\sec x-\tan x$

$$\lim_{x\to\frac{\pi}{2}}\sec x-\tan x\ =\lim_{x\to\frac{\pi}{2}}\left[\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right]\ \infty-\infty\ \textit{form}$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\cos x}{-\sin x}\right]=\frac{0}{1}=0$$

Example 32: Evaluate $\lim\limits_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]$

$$\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]\ \infty-\infty\ \textit{form}\ =\lim_{x\to 1}\left[\frac{(x-1)-x\log x}{(x-1)\log x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\left[\frac{1-1-\log x}{\frac{x-1}{x}+\log x}\right]=\lim_{x\to 1}\left[\frac{-\log x}{1-\frac{1}{x}+\log x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 1}\left[\frac{-1/x}{\frac{1}{x^2}+\frac{1}{x}}\right]=\frac{-1}{1+1}=\frac{-1}{2}$$

Example 33: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]\ \infty-\infty\ \textit{form}\ =\lim_{x\to 0}\left[\frac{(e^x-1)-x}{x(e^x-1)}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{e^x-1}{(e^x-1)+xe^x}\right]\left(\frac{-}{0}\right)=\lim_{x\to 0}\left[\frac{e^x}{e^x+e^x+xe^x}\right]$$

$$=\frac{1}{1+1+0}=\frac{1}{2}$$

Example 34: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]$

$$\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]\ \infty-\infty\ \textit{form}\ =\lim_{x\to 0}\left[\frac{x-\sin x}{x\sin x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{1-\cos x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)==\lim_{x\to 0}\left[\frac{\sin x}{\cos x+\cos x-x\sin x}\right]=\frac{0}{1+1}=0$$

Example 35: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{1-\frac{1}{1+x}}{2x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\frac{1}{(1+x)^2}}{2}\right]=\frac{1}{2}$$

Example 36: Evaluate $\lim\limits_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]$

$$\lim_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]=\lim_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right]\ \infty-\infty\ \textit{form}\ =\lim_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\left[\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{x\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to0}\left[\frac{a.\frac{1}{a}\cos\frac{x}{a}-\cos\frac{x}{a}+\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]$$

$$=\lim_{x\to0}\left[\frac{\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to0}\left[\frac{\frac{1}{a}\sin\frac{x}{a}+\frac{x}{a}.\frac{1}{a}.\cos\frac{x}{a}}{\frac{1}{a}\cos\frac{x}{a}+\frac{1}{a}\cos\frac{x}{a}-\frac{x}{a^2}\sin\frac{x}{a}}\right]=\frac{0+0}{\frac{1}{a}+\frac{1}{a}-0}=0$$

Example 37: Find the value of 'a' such that $\lim_{x\to0}\frac{\sin 2x+a\sin x}{x^3}$ is finite. Also find the value of the limit.

Let $A=\lim_{x\to0}\frac{\sin 2x+a\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to0}\frac{2\cos 2x+a\cos x}{3x^2}=\frac{2+a}{0}\neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if 2+a=0 i.e., a= -2 .

For $a=-2$,

$$A=\lim_{x\to0}\frac{2\cos 2x-2\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to0}\frac{-4\sin 2x+2\sin x}{6x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\frac{-8\cos 2x+2\cos x}{6}=\frac{-8+2}{6}=-1$$

$\therefore$ *The given* lim*it will have a finite value when* $a=-2$ *and it is* -1.

Example 38: Find the values of 'a' and 'b' such that $\lim_{x\to0}\frac{x(1-a\cos x)+b\sin x}{x^3}=\frac{1}{3}$.

Let $A=\lim_{x\to0}\frac{x(1-a\cos x)+b\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}=\frac{1-a+b}{0}\neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if 1-a+b=0 i.e., a-b = 1.

For $a-b=1$,

$$A=\lim_{x\to0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to0}\frac{2a\sin x+ax\cos x-b\sin x}{6x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{3a\cos x - ax\sin x - b\cos x}{6} = \frac{3a-b}{6} = \textit{finite}$$

$$\textit{This finite value is given as } \frac{1}{3}.\ \textit{i.e.,} \frac{3a-b}{6} = \frac{1}{3} \Rightarrow 3a-b=2$$

$$\textit{Solving the equations } a-b=1 \textit{ and } 3a-b=2 \textit{ we obtain } a=\frac{1}{2} \textit{ and } b=-\frac{1}{2}.$$

Example 39: Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2} = 1$**.**

$$\textit{Let } A = \lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2} = \frac{a-b}{0} \neq \textit{finite}$$

We can continue to apply the L'Hospital's Rule, if a-b=0, since the denominator=0 .

For $a-b=0$,

$$A = \lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{a\sinh x + b\sin x}{2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{a\cosh x + b\cos x}{2} = \frac{a+b}{2}$$

But this is given as 1.

$\therefore a+b=2$

Solving the equations $a-b=0$ *and* $a+b=2$ *we obtain* $a=1$ *and* $b=1$.

Limits of the form $0^0, \infty^0$ *and* 1^∞ **:** To evaluate such limits, where function to the power of function exists, we call such an expression as some constant, then take logarithm on both sides and rewrite the expressions to get $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.

Example 40: Evaluate $\lim_{x\to 0} x^x$

$$\textit{Let } A = \lim_{x\to 0} x^x \ (0^0 \ \textit{form})$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0}\log x^x = \lim_{x\to 0} x.\log x\ (0\times\infty\ \ form) = \lim_{x\to 0}\frac{\log x}{1/x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{1/x}{(-1/x^2)} = \lim_{x\to 0}\frac{-x}{1} = \frac{0}{1} = 0$$

$$\log_e A = 0 \Rightarrow A = e^0 = 1 \quad \therefore \lim_{x\to 0} x^x = 1$$

Example 41: Evaluate $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

$$Let\ \ A = \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}\ \ (1^\infty\ \ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0}\log(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{x^2}\log\cos x\ (\infty\times 0\ \ form) = \lim_{x\to 0}\frac{\log\cos x}{x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-\tan x}{2x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-\sec^2 x}{2} = \frac{-1}{2}$$

$$\log_e A = -\frac{1}{2} \Rightarrow A = e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}} \quad \therefore \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$$

Example 42: Evaluate $\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$

$$Let\ \ A = \lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}\ \ (\infty^0\ \ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to\frac{\pi}{2}}\log(\tan x)^{\cos x} = \lim_{x\to\frac{\pi}{2}}\cos x\log(\tan x)\ (0\times\infty\ \ form)$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\log\tan x}{\sec x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x/\tan x}{\sec x.\tan x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$$

$$\log_e A = 0 \Rightarrow A = e^0 = 1 \quad \therefore \lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x} = 1$$

Example 43: Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

$$Let\ A=\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}\ (1^\infty\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=\lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)\ (\infty\times 0\ form)$$

$$=\lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\sin x.\cos x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{\frac{2}{\sin 2x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{2x-\sin 2x}{2x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2-2\cos 2x}{4x\sin 2x+4x^2\cos 2x}\left(\frac{0}{0}\right)==\lim_{x\to 0}\frac{-4\sin 2x}{4\sin 2x+16x\cos 2x-8x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-8\cos 2x}{24\cos 2x-48x\sin 2x-16x^2\cos 2x}=\frac{-8}{24}=\frac{-1}{3}$$

$$\log_e A=-\frac{1}{3}\Rightarrow A=e^{-\frac{1}{3}}\quad \therefore \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=e^{-\frac{1}{3}}$$

Example 44: Evaluate $\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}$

$$Let\ A=\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}\ (1^\infty\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log(a^x+x)^{\frac{1}{x}}=\lim_{x\to 0}\frac{1}{x}\log(a^x+x)\ (\infty\times 0\ form)$$

$$=\lim_{x\to 0}\frac{\log(a^x+x)}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(a^x\log a+1)/(a^x+x)}{1}$$

$$=\log a+1=\log a+\log e=\log ae$$

$$\therefore \log_e A=\log ea\Rightarrow A=ea\qquad Hence\ \lim_{x\to 0}(a^x+x)^{\frac{1}{x}}=ea.$$

Example 45: Evaluate $\lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}$

$$Let\ A=\lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}\ (1^{\infty}\ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to a}\log(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}=\lim_{x\to a}\tan\frac{\pi x}{2a}.\log(2-\frac{x}{a})\ \ \infty\times 0\ form$$

$$=\lim_{x\to a}\frac{\log(2-\frac{x}{a})}{\cot\frac{\pi x}{2a}}\left(\frac{0}{0}\right)=\lim_{x\to a}\left(\frac{\frac{(-1/a)}{2-\frac{x}{a}}}{-\frac{\pi}{2a}\cos ec^2\frac{\pi x}{2a}}\right)=\lim_{x\to a}\frac{2}{\pi}.\frac{\sin^2\frac{\pi x}{2a}}{2-\frac{x}{a}}=\frac{2}{\pi}$$

$$\therefore \log_e A=\frac{2}{\pi}\Rightarrow A=e^{\frac{2}{\pi}}\qquad Hence\ \lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}=e^{\frac{2}{\pi}}.$$

PARTIAL DIFFERENTIATION:

Introduction: We often come across qualities which depend on two or more variables. For e.g. the area of a rectangle of length x and breadth y is given by Area = A(x,y) = xy. The area A(x, y) is, obliviously, a function of two variables. Similarly, the distances of the point (x, y, z) from the origin in three-dimensional space is an example of a function of three variables x, y, z.

Partial derivatives: Let z = f(x, y) be a function of two variables x and y.

The first order partial derivative of z w.r.t. x, denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or z_x or f_x is defined as

$$\frac{\partial z}{\partial x}=\lim_{\delta x\to 0}\frac{f(x+\delta x, y)-f(x,y)}{\delta x}$$

From the above definition, we understand that $\left(\frac{\partial z}{\partial x}\right)$ is the ordinary derivative of z w.r.t x, treating y as constant.

The first order partial derivative of z w.r.t y, denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or z_y or f_y is defined as

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

From the above definition, we understand that $\left(\frac{\partial z}{\partial y}\right)$ is the ordinary derivative of z w.r.t y, treating x as constant

The partial derivatives $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or z_{xx} or f_{xx};

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } z_{yy} \text{ or } f_{yy};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } z_{yx} \text{ or } f_{yx}$$

$$\text{and } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } z_{xy} \text{ or } f_{xy}$$

are known as second order Partial derivatives.

In all ordinary cases, it can be verified that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

The third and higher order partial derivatives of f(x,y) are defined in an analogous way Also, the second and higher order partial derivatives of more than two independent variables are defined similarly.

<u>A note on rules of partial differentiation</u>:-

All the rules of differentiation applicable to functions of a single independent variable are applicable for partial differentiation also; the only difference is that while differentiating partially w.r.t one independent variable all other independent variables are treated as constants.

<u>Total derivatives, Differentiation of Composite and Implicit functions</u>

In this lesson we learn the concept of total derivatives of functions of two or more variables and, also rules for differentiation of composite and implicit functions.

a) <u>Total differential and Total derivative:-</u>

For a function $z = f(x, y)$ of two variables, x and y the **<u>total differential (or exact differential</u>) dz** is defined by:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy \text{ --------------------------(1)}$$

Further, if $z = f(x, y)$ where $x = x(t)$, $y = y(t)$ i.e. x and y are themselves functions of an independent variable t, then total **<u>derivative of z</u>** is given by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \text{-------------------------}(2)$$

Similarly, the total differential of a function $u = f(x, y, z)$ is defined by

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \text{-----------------}(3)$$

Further, if $u = f(x, y, z)$ and if $x = x(t)$, $y = y(t)$, $z = z(t)$, then the total derivative of u is given by

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \text{----------------}(4)$$

(b) <u>Differentiation of implicit functions:-</u>

An implicit function with x as an independent variable and y as the dependent variable is generally of the form $z = f(x, y) = 0$. This gives $\left(\frac{dz}{dx}\right) = \left(\frac{df}{dx}\right) = 0$. Then, by virtue of expression (2) above, we get

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad \text{or} \quad \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}, \text{ and hence}$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}, \text{ so that we get } \frac{dy}{dx} = -\frac{\left(\partial f/\partial x\right)}{\left(\partial f/\partial y\right)} \text{-----------}(5)$$

(c) <u>Differentiation of composite functions:-</u>

Let z be an function of x and y and that $x = \phi(u, v)$ and $y = \varphi(u, v)$ are functions of u and v then,

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \& \quad \frac{\partial z}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\} \text{-------------------}(6)$$

Similarly, if $z = f(u, v)$ are functions of u and v and if $u = \phi(x, y)$ and $v = \varphi(x, y)$ are functions of x and y then,

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ \& \ \frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\end{aligned}\right\} \quad \text{---------------------------(7)}$$

Note:-1) The above formulae can be extended to functions of three are more variables and formulas (6) and(7) are called **Chain rule** for partial differentiation.

2) The second and higher order partial derivatives of $z = f(x, y)$ can be obtained by repeated applications of the above formulas

Evaluate:

1. Find the total differential of

(i) $e^x [x \sin y + y \cos y]$ (ii) e^{xyz}

Sol:- (i) Let $z = f(x, y) = e^x [x \sin y + y \cos y]$ Then

$$\frac{\partial z}{\partial x} = e^x [(1 + x)\sin y + y \cos y]$$

and $\frac{\partial z}{\partial y} = e^x [(1 + x)\cos y - y \sin y]$ Hence, using formula (1), we get

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

i.e $dz = e^x [(1 + x)\sin y + y \cos y] dx + e^x [(1 + x)\cos y - y \sin y] dy$

(ii) Let $z = f(x, y, z) = e^{xyz}$ Then

$$\frac{\partial u}{\partial x} = (yz)e^{xyz}; \frac{\partial u}{\partial y} = (xz)e^{xyz}; \frac{\partial u}{\partial z} = (xy)e^{xyz}$$

$\therefore$ Total differential of $z = f(x, y, z)$ is (see formula (3) above)

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$= e^{xyz}(yzdx + zxdy + xydz)$$

2. Find $\frac{dz}{dt}$ if

(i) $z = xy^2 + x^2 y$,where $x = at^2$, $y = 2at$

(ii) $u = \tan^{-1}\left(y/x\right)$,where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ **(VTU-Jan 2003)**

Sol:- (i) Consider $z = xy^2 + x^2 y$

$$\frac{\partial z}{\partial x} = y^2 + 2xy \quad \& \frac{\partial z}{\partial y} = 2xy + x^2$$

Since $x = at^2$ & $y = 2at$, We have $\dfrac{dx}{dt} = 2at$, $\dfrac{dy}{dt} = 2a$

Hence, using formula (2), we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

$$= (y^2 + 2xy)y + 2a(2xy + x^2) \text{ Using } y = 2at$$

$$\frac{dz}{dt} = y^3 + 2xy^2 + 4axy + 2ax^2$$

To get $\left(\dfrac{dz}{dt}\right)$ explicitly in terms of t, we substitute

$x = at^2$ & $y = 2at$, to get

$$\left(\frac{dz}{dt}\right) = 2a^3(8t^3 + 5t^4)$$

(ii) Consider

$$u = \tan^{-1}\left(y/x\right)$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2 + y^2}, \frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2}$$

Since $x = e^t - e^{-t}$ & $y = e^t + e^{-t}$,we have

$$\frac{dx}{dt} = e^t + e^{-t} = y \quad \frac{dy}{dt} = e^t - e^{-t} = x$$

Hence $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$ (see eqn (2))

$$= \left(\frac{-y}{x^2 + y^2}\right)y + \left(\frac{x}{x^2 + y^2}\right)x = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

Substituting $x = e^t - e^{-t}$ & $y = e^t + e^{-t}$, we get

$$\frac{du}{dt} = \frac{-2}{e^{2t} + e^{-2t}}$$

3. Find $\left(\dfrac{dy}{dx}\right)$ if (i) $x^y + y^x$ =Constant

(ii) $x + e^y = 2xy$

Sol: - (i) Let $z = f(x, y) = x^y + y^x$ =Constant. Using formula (5)

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \text{-----------------(*)}$$

But $\dfrac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$ and $\dfrac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$ Putting those in(*), we get

$$\frac{dy}{dx} = -\left\{\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right\}$$

(ii) Let $z = f(x, y) = e^x + e^y - 2xy$ =Constant

Now, $\dfrac{\partial f}{\partial x} = e^x - 2y$; $\dfrac{\partial f}{\partial y} = e^y - 2x$ Using this in (8),

$$\frac{dy}{dx} = -\left\{\frac{\partial f/\partial x}{\partial f/\partial y}\right\} = -\left\{\frac{e^x - 2y}{e^y - 2x}\right\}$$

4. (i) If $z = f(x, y)$,where $x = r\cos\theta$, $y = r\sin\theta$ show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

(VTU July-2005)

(ii) If $z = f(x, y)$,where $x = e^u + e^{-v}$ & $y = e^{-u} - e^v$,Show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Sol: As $x = r\cos\theta$ and $y = r\sin\theta$, we have

$\dfrac{\partial x}{\partial r} = \cos\theta$, $\dfrac{\partial x}{\partial \theta} = -r\sin\theta$; $\dfrac{\partial y}{\partial r} = \sin\theta$ & $\dfrac{\partial y}{\partial \theta} = r\cos\theta$.

Using Chain rule (6) & (7) we have

$$\left(\frac{\partial z}{\partial r}\right) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(\cos\theta) + \frac{\partial z}{\partial y}(\sin\theta)$$

$$\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

Squaring on both sides, the above equations, we get

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\sin\theta\cos\theta$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta - 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\sin\theta\cos\theta$$

Adding the above equations , we get

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}(\cos^2\theta+\sin^2\theta)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 \text{ as desired.}$$

(ii) As $x=e^u+e^{-v}$ & $y=e^{-u}-e^v$, We have

$$\frac{\partial x}{\partial u}=e^u,\ \frac{\partial x}{\partial v}=-e^{-v},\ \frac{\partial y}{\partial u}=-e^{-u}\ \&\ \frac{\partial y}{\partial v}=-e^v$$

Using Chain rule (6) we get

$$\left(\frac{\partial z}{\partial u}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}(e^u)+\frac{\partial z}{\partial y}(-e^{-v})$$

$$\left(\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})$$

$$\therefore\ \left(\frac{\partial z}{\partial u}\right)-\left(\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial x}(e^u+e^{-v})+\frac{\partial z}{\partial y}(e^{-u}-e^v)$$

$$=\frac{\partial z}{\partial x}x-\frac{\partial z}{\partial y}y$$

5. (i) If $u=f(x,z,y/z)$ Then show that $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=0$ **(VTU-July-2004)**

(ii) If $H=f(x-y,y-z,z-x)$, show that

$$\frac{\partial H}{\partial x}+\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z}=0$$ **(VTU-July-2003)**

Sol: - (i) Let $u=f(v,w)$, where $v=xz$ and $w=y/z$

$$\frac{\partial v}{\partial x}=z,\ \frac{\partial v}{\partial y}=0,\ \frac{\partial v}{\partial z}=x\quad \&\quad \frac{\partial w}{\partial x}=0,\ \frac{\partial w}{\partial y}=1/z,\ \frac{\partial w}{\partial z}=-y/z^2$$

Using Chain rule,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial u}{\partial v}(z)+\frac{\partial u}{\partial w}(0)=z\frac{\partial u}{\partial v}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial u}{\partial v}(0)+\frac{\partial u}{\partial w}(1/z)=\frac{1}{z}\frac{\partial u}{\partial w}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial z}=\frac{\partial u}{\partial v}(x)+\frac{\partial u}{\partial w}\left(-y/z^2\right)=x\frac{\partial u}{\partial v}-\frac{y}{z^2}\frac{\partial u}{\partial w}$$

From these, we get

$$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=xz\frac{\partial u}{\partial v}-\frac{y}{z}\frac{\partial u}{\partial w}-z\left(x\frac{\partial u}{\partial v}-\frac{y}{z^2}\frac{\partial u}{\partial w}\right)$$

$$=0$$

(ii) Let $H=f(u,v,w)$ Where $u=x-y, v=y-z, w=z-x$

Now, $\frac{\partial u}{\partial x}=1, \frac{\partial u}{\partial y}=-1, \frac{\partial u}{\partial z}=0$

$\frac{\partial v}{\partial x}=0, \frac{\partial v}{\partial y}=1, \frac{\partial v}{\partial z}=-1$

$\frac{\partial w}{\partial x}=-1, \frac{\partial w}{\partial y}=0, \frac{\partial u}{\partial z}=1$ Using Chain rule,

$$\frac{\partial H}{\partial x}=\frac{\partial H}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial H}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial H}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial H}{\partial u}(1)+\frac{\partial H}{\partial v}(0)+\frac{\partial H}{\partial w}(-1)$$

$$\frac{\partial H}{\partial y}=\frac{\partial H}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial H}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial H}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial H}{\partial u}(-1)+\frac{\partial H}{\partial v}(1)+\frac{\partial v}{\partial w}(0)$$

$$\frac{\partial H}{\partial z}=\frac{\partial H}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial H}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial H}{\partial w}\frac{\partial w}{\partial z}=\frac{\partial H}{\partial u}(0)+\frac{\partial H}{\partial v}(-1)+\frac{\partial H}{\partial w}(1)$$

Adding the above equations, we get the required result $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=0$

Applications to Jacobians:

In this lesson, we study Jacobians, errors and approximations using the concept of partial differentiation.

Jacobians:-

Jacobians were invented by German mathematician C.G. Jacob Jacobi (1804-1851),who made significant contributions to mechanics, Partial differential equations and calculus of variations.

Definition:- Let u and v are functions of x and y, then Jacobian of u and v w.r.t x and denoted by

J or $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

is defined by

$$J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

Similarly, if u, v, w are functions of three independent variables of x, y, z, then

$$J = J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

<u>Remark:-</u> In a similar way, Jacobian of n functions in n-variables can be defined

Note:- (i) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ then the "inverse Jacobian" of the Jacobian J,

denoted by J',is defined as

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

(ii) Similarly, "inverse Jacobian" of $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ is defined as $J' = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

<u>Properties of Jacobians</u> :-

Property 1:- If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

Proof:- Consider

$$JJ^1 = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 1$$

<u>**Property 2:-**</u> (Chain rule for Jacobians):- If u and v are functions of r&s and r,s are functions x&y,then

$$J = \left(\frac{u,v}{x,y}\right) = J\left(\frac{u,v}{r,s}\right) \times J\left(\frac{r,s}{x,y}\right)$$

Proof:- Consider

$$J\left(\frac{u,v}{r,s}\right)\times J\left(\frac{r,s}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s}\end{vmatrix}\times\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}\\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=J\left(\frac{u,v}{x,y}\right)$$

Jacobians in various co-ordinate systems:-

1. In Polar co-ordinates, $x=r\cos\theta$, $y=r\sin\theta$

   we have $\frac{\partial(x,y)}{\partial(r,\theta)}=r$

2. In spherical coordinates, $x=\rho\cos\phi, y=\rho\sin\phi, z=z,$ we have

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}=\rho$$

3. In spherical polar co-ordinates, $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$

Proof of 1:- we have, $\frac{\partial x}{\partial r}=\cos\theta$ and $\frac{\partial y}{\partial r}=\sin\theta$

$\frac{\partial x}{\partial\theta}=-r\sin\theta$ and $\frac{\partial y}{\partial\theta}=r\cos\theta$

$$\therefore\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r$$

Proof of 2 :-we have $\frac{\partial x}{\partial\rho}=\cos\phi$, $\frac{\partial y}{\partial\rho}=\sin\phi$, $\frac{\partial z}{\partial\rho}=0$

$\frac{\partial x}{\partial\phi}=-\rho\sin\phi$, $\frac{\partial y}{\partial\phi}=\rho\cos\phi$, $\frac{\partial z}{\partial\phi}=0$

$$\frac{\partial z}{\partial \rho} = 0\,, \frac{\partial z}{\partial \phi} = 0\,, \frac{\partial z}{\partial z} = 0$$

$$\therefore \frac{\partial (x, y, z)}{\partial (\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

<u>Proof of 3:-</u> We have

$$\frac{\partial x}{\partial r} = \sin\theta\cos\theta\,, \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi\,, \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi\,, \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi\,, \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta\,, \frac{\partial z}{\partial \theta} = -r\sin\theta\,, \frac{\partial z}{\partial \phi} = 0$$

$$\therefore \frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\theta & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

<u>Evaluate</u>

1. If $u = x^2 - 2y^2\,, v = 2x^2 - y^2$, where
   $x = r\cos\theta\,, y = r\sin\theta$ show that
   $$\frac{\partial (u, v)}{\partial (r, \theta)} = 6r^3\sin 2\theta$$ **(VTU-Jan-2006)**

Consider $u = x^2 - 2y^2 = r^2\cos^2\theta - 2r^2\sin^2\theta$

$$v = 2x^2 - y^2 = 2r^2\cos^2\theta - r^2\sin^2\theta$$

$$\therefore \frac{\partial u}{\partial r} = 2r\cos^2\theta - 4r\sin^2\theta\,, \frac{\partial v}{\partial r} = 4r\cos^2\theta - 2r\sin^2\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta$$

$$\frac{\partial v}{\partial \theta} = -4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta$$

$$\frac{\partial (u, v)}{\partial (r, \theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos^2\theta - 4r\sin^2\theta & -2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta \\ 4r\cos^2\theta - 2r\sin^2\theta & -4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta \end{vmatrix}$$

$$= (2r\cos^2\theta - 4r\sin^2\theta)(4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta)$$
$$= (-2r\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta)(4r\cos^2\theta - 2r\sin^2\theta)$$
$$= 6r^3\sin 2\theta$$

2. If $x = u(1-v),\ y = uv$, Prove that $J\left(\frac{x,y}{u,v}\right) \times J^1\left(\frac{x,y}{u,v}\right) = 1$ **(VTU-2001)**

Consider $\frac{\partial x}{\partial u} = 1 - v,\ \frac{\partial x}{\partial v} = -u$

$\frac{\partial y}{\partial u} = v,\ \frac{\partial y}{\partial v} = u$

$$\therefore J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= (1-v)u - (-uv) = u - uv + uv = u$$

$$\therefore J\left(\frac{x,y}{u,v}\right) = u \quad ---(1)$$

Further, as $x = u(1-v),\ y = uv,$

$= u - uv$

We write, $x = u - y \quad \therefore u = x + y$ and

$v = \frac{y}{u} = \left(\frac{y}{x+y}\right) \qquad \therefore v = \left(\frac{y}{x+y}\right)$

$\therefore \frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 1$ and

$\frac{\partial v}{\partial x} = -\frac{y}{(x+y)^2},\ \frac{\partial v}{\partial y} = \frac{x}{(x+y)^2}$

$$\therefore J^1\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \left(\frac{1}{x+y}\right) = \frac{1}{u} \qquad \therefore JJ^1 = u\frac{1}{u} = 1$$

3. If $x = e^u\cos v,\ y = e^u\sin v$, Prove that

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$$

Consider $x = e^u\cos v \qquad y = e^u\sin v$

$$\frac{\partial x}{\partial u} = e^u \cos v \qquad \frac{\partial y}{\partial u} = e^u \sin v$$

$$\frac{\partial x}{\partial v} = -e^u \sin v \qquad \frac{\partial y}{\partial v} = e^u \cos v$$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$$i.e \frac{\partial(x, y)}{\partial(u, v)} = e^{2u} \; ------(1)$$

Again Consider $x = e^u \cos v,\ y = e^u \sin v,$

$$\therefore x^2 + y^2 = e^{2u} \quad \text{or} \quad u = \frac{1}{2}\log(x^2 + y^2)$$

$$\& \ \frac{y}{x} = \tan v \quad \text{or} \quad v = \tan^{-1}\left(y/x\right)$$

$$\text{Hence } \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2},;\ \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2},;$$

$$\& \ \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2};\ \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{x^2+y^2} = (x^2+y^2)^{-1}$$

$$\text{i.e } \frac{\partial(u, v)}{\partial(x, y)} = e^{-2u} \; --------(2)$$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = e^{2u} \times e^{-2u} = 1$$

4. If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$, Show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

$$\text{Now, } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

i.e $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\left(-\frac{yz}{x^2}\right)\left\{\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right)\right\}-\left(\frac{z}{x}\right)\left\{\left(\frac{z}{y}\right)\left(\frac{-xy}{z^2}\right)-\left(\frac{y}{z}\right)\left(\frac{x}{y}\right)\right\}$$
$$+\left(\frac{y}{x}\right)\left\{\left(\frac{z}{y}\right)\left(\frac{x}{z}\right)-\left(\frac{y}{z}\right)\left(\frac{-zx}{y^2}\right)\right\}$$

= 4, as desired.

5. If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$

Now, by definition

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}$$

i.e $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$=\sin\theta\cos\phi\,\{0-(r^2\sin^2\theta\cos\phi)\}$$
$$-r\cos\theta\cos\phi\,\{0-(r\sin\theta\cos\theta\cos\phi)\}$$
$$-r\sin\theta\sin\phi\,\{-r\sin^2\theta\sin\phi-r\cos^2\theta\sin\phi\}$$
$$=(r^2\sin^2\theta)\sin\theta\cos^2\phi+r^2\sin\theta\cos^2\theta\cos^2\phi$$
$$=r^2\sin\theta(\sin^2\theta+\cos^2\theta)\cos^2\phi+r^2\sin\theta\sin^2\phi$$
$$=r^2\sin\theta(\cos^2\phi+\sin^2\phi)$$
$$=r^2\sin\theta\text{, as required}$$