Assignment 1: Modelling and Simulation

Varun Hegde
Arshad N

10 october 2019

1 Introduction

Our goal is to solve the "Hovering mass in Helicopter" problem using generalized coordinate system to compute the Euler Lagrange form for the model and then solve the same problem using constrained Euler Lagrange approach. The complexities of the two methods are then compared to find the simpler model. The constrained Euler Lagrange approach are then expressed in implicit and explicit form in order to compare the advantages and disadvantages of the models. These models are developed in Matlab to compute the problem.

2 Hovering mass

2.1 (a)

Given that generalized coordinates will have the form

$$q = \begin{pmatrix} p1 \\ \theta \\ \phi \end{pmatrix} \tag{1}$$

The kinetic and potential energy associated to the hanging mass will then be read as:

$$T_2 = 1/2 m_2 \, \dot{p}_2 \, (q, \dot{q})^T p_2 \, (q, \dot{q}) \tag{2}$$

$$V_2 = m_2 \, g[001] p_2 \, (q) \tag{3}$$

Let p1 and p2 be the position of Helicopter and Hanging mass,

$$p_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4}$$

Then the new generalized coordinates are

$$q = \begin{pmatrix} x \\ y \\ z \\ \theta \\ \phi \end{pmatrix} \tag{5}$$

The position of the Hanging mass is described by two angles $\theta and \phi$

$$p_2 = \begin{pmatrix} x + L\, \cos(\theta)\, \sin(\phi) \\ y + L\, \sin(\phi)\, \sin(\theta) \\ z - L\, \cos(\phi) \end{pmatrix} \tag{6}$$

where L is the length of the non-elastic link connecting the helicopter and the mass. The potential energy for the 2 respective masses for the system is $V_1\, V_2$

$$V_1 = m_1\, g\, [001] p_1(q)\, ; V_2 = m_2\, g[001] p_2\, (q) \tag{7}$$

$$T_1 = 1/2 m_1\, \dot{p}_1\, (q, \dot{q})^T \dot{p}_1\, (q, \dot{q}); T_2 = 1/2 m_2\, \dot{p}_2\, (q, \dot{q})^T \dot{p}_2\, (q, \dot{q}) \tag{8}$$

The kinetic (T) and potential energy(V) associated to the whole system is:

$$V = V_1 + V_2 = m_1\, g\, [001] p_1(q) + m_2\, g[001] p_2\, (q) \tag{9}$$

$$T = T1 + T2 = 1/2 m_1\, \dot{p}_1\, (q, \dot{q})^T \dot{p}_1\, (q, \dot{q}) + 1/2 m_2\, \dot{p}_2\, (q, \dot{q})^T \dot{p}_2\, (q, \dot{q}) \tag{10}$$

The Lagrange Equation L will be,

$$L = T - V \tag{11}$$

Matrix W is obtained by evaluating equation 12;

$$W(q) = \frac{\partial^2 T(q, \dot{q})}{\partial q^2} \tag{12}$$

Thus we obtain the following symmetric matrix W(q) -Equation 13 (complete matrix is detailed in the MATLAB script as its exceeding the page margins).

$$\left(\begin{array}{ccccc} m_1 + m_2 & 0 & 0 & -L\, m_2\, \sin(\phi)\, \sin(\theta) \\ 0 & m_1 + m_2 & 0 & L\, m_2\, \cos(\theta)\, \sin(\phi) \\ 0 & 0 & m_1 + m_2 & 0 \\ -L\, m_2\, \sin(\phi)\, \sin(\theta) & L\, m_2\, \cos(\theta)\, \sin(\phi) & 0 & m_2\, L^2 \cos(\theta)^2 \sin(\phi)^2 + m_2\, L^2 \sin(\phi)^2 \sin(\\ L\, m_2\, \cos(\phi)\, \cos(\theta) & L\, m_2\, \cos(\phi)\, \sin(\theta) & -L\, m_2\, \sin(\phi) & 0 \end{array} \right. \tag{13}$$

Given that q̇ =v and v can be written as

$$v = \begin{pmatrix} \dot{\mathrm{x}}_1 \\ \dot{\mathrm{y}}_1 \\ \dot{\mathrm{z}}_1 \\ \dot{\mathrm{x}}_2 \\ \dot{\mathrm{y}}_2 \\ \dot{\mathrm{z}}_2 \end{pmatrix} \tag{14}$$

In order to get our model in desired form, we use simplify the Euler Lagrange equation to the form below.

$$W(q)\ddot{q} = -\frac{\partial}{\partial q}(W(q)\dot{q})\dot{q} + \nabla qT - \nabla qV \tag{15}$$

Here, W(q) is equivalent to M(q). The computation to obtain both LHS (Equation 17) and RHS (Equation 18) of Equation 16 are detailed in the Matlab script.

$$M(q)\ddot{v} = b(q, \dot{q}, u) \tag{16}$$

$$M(q)\ddot{v} = MA = \begin{pmatrix} \text{xddot } (m_1 + m_2) + L\, m_2\, \text{phiddot } \cos(\phi)\, \cos(\theta) - \\ \text{yddot } (m_1 + m_2) + L\, m_2\, \text{phiddot } \cos(\phi)\, \sin(\theta) + \\ \text{zddot } (m_1 + m_2) - L\, m_2\, \text{phidd} \\ \text{thetaddot } \bigl(m_2\, L^2 \cos(\theta)^2 \sin(\phi)^2 + m_2\, L^2 \sin(\phi)^2 \sin(\theta)^2\bigr) + L\, m_2\, \text{y} \\ \text{phiddot } \bigl(m_2\, L^2 \cos(\phi)^2 \cos(\theta)^2 + m_2\, L^2 \cos(\phi)^2 \sin(\theta)^2 + m_2\, L^2 \sin(\phi)^2\bigr) - L\, m_2\, \text{zdd} \end{pmatrix} \tag{17}$$

$$b(q, \dot{q}, u) = \begin{pmatrix} \\ \\ \\ \text{thetadot } (L\, m_2\, \text{xdot } \cos(\theta)\, \sin(\phi) + L \\ \text{phidot } \bigl(\text{phidot } \bigl(2\, m_2\, \cos(\phi)\, \sin(\phi)\, L^2 \cos(\theta)^2 + 2\, m_2\, \cos(\phi)\, \sin(\phi)\, L^2 \sin(\theta)^2 - 2\, m_2\, \cos \end{pmatrix} \tag{18}$$

The b(q,q̇,u) and M(q)v̈ are too huge and are out of bounds of the page but the matrix is well defined in the MATLAB code.

2.2 (b)

In this part, we are going to express the model equations using the constrained Lagrange approach.
The generalized coordinates will be

$$q = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{19}$$

Scalar constraint will be of the form:

$$C = 1/2\left(\ e^T e - L^2\ \right) \tag{20}$$

Where e is defined as e=p1-p2

$$e = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix} \tag{21}$$

$$then, C = \frac{(x_1 - x_2)^2}{2} + \frac{(y_1 - y_2)^2}{2} + \frac{(z_1 - z_2)^2}{2} - \frac{L^2}{2} \tag{22}$$

The potential energy V = V1 + V2 and Kinetic energy T= T1 + T2 similarly as in 1.a).Then the Lagrange equation considering constraint will be as follows,

$$L = T - V - Z^T * C \tag{23}$$

The simplified form of the Euler Lagrange equation (equation-25) considering the constraint is given by Equation (27) where W(q) is obtained through a similar process followed in 1.a)

$$W(q) = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{pmatrix} \tag{24}$$

$$W(q)\ddot{q} = Q - \frac{\partial}{\partial q}(W(q)\dot{q})\dot{q} + \nabla qT - \nabla qV - \nabla qc.z \tag{25}$$

$$M(q)\dot{v} = b(q, \dot{q}, u) \tag{26}$$

$$C(q) = 0 = 1/2\left(\ e^T e - L^2\ \right) \tag{27}$$

Thus by comparing the equation 26 to 25 the following can be inferred ;

$$M(q) = W(q) \tag{28}$$

$$b = Q - \frac{\partial}{\partial q}(W(q)\dot{q})\dot{q} + \nabla qT - \nabla qV - \nabla qc.z \tag{29}$$

Thus:

$$b = \begin{pmatrix} \text{FX} + Z\ (x_1 - x_2) \\ \text{FY} + Z\ (y_1 - y_2) \\ \text{FZ} +\ m_1 g + Z\ (z_1 - z_2) \\ -Z\ (x_1 - x_2) \\ -Z\ (y_1 - y_2) \\ m_2 g - Z\ (z_1 - z_2) \end{pmatrix} \tag{30}$$

Comparison: By comparing M(q) for 1.a)(eqn(17)) and 1.b) (eqn(24)) for the LHS and b for 1.a) (eqn 18) and 1.b) (eqn 30) for the RHS , we clearly observe that constrained Euler Lagrange have much simpler models in comparison with regular Euler Lagrange with minimal/generalized coordinates.

3 Explicit vs. Implicit model

3.1 (a)

Expressing the 1.b) model in the form mentioned below (equation-31), by considering W(q) and a(q), relations are mentioned below which are obtained from 1.b) portion of the problem where equation 34 represents the LHS and 33 represents the RHS of the equation. physics

$$\begin{pmatrix} M & a\left(q\right) \\ a\left(q\right)^{T} & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ z \end{pmatrix} = c(q, \dot{q}, u) \tag{31}$$

c(q,q̇,u) is obtained by the solving for the elements in the below expression to realize the matrix. Its first element is obtained by evaluating the equation 25 by solving for W(q)q̈. Its second element is obtained by solving for $(\partial c/\partial q)\ddot{q}$.

$$c(q, \dot{q}, u) = \begin{pmatrix} Q - \frac{\partial}{\partial q}(W(q)\dot{q})\dot{q} + \nabla qT - \nabla qV - \nabla qc.z \\ -\frac{\partial}{\partial q}(\frac{\partial c}{\partial q}\dot{q})\dot{q} \end{pmatrix} \tag{32}$$

$$c(q, \dot{q}, u) = \begin{pmatrix} \mathrm{FX} \\ \mathrm{FY} \\ \mathrm{FZ} - m_1\, g \\ 0 \\ 0 \\ -m_2\, g \\ \dot{\mathrm{x}}_2\ (\dot{\mathrm{x}}_1 - \dot{\mathrm{x}}_2) - \dot{\mathrm{x}}_1\ (\dot{\mathrm{x}}_1 - \dot{\mathrm{x}}_2) - \dot{\mathrm{y}}_1\ (\dot{\mathrm{y}}_1 \\ -\dot{\mathrm{y}}_2 + \dot{\mathrm{y}}_2\ (\dot{\mathrm{y}}_1 - \dot{\mathrm{y}}_2) - \dot{\mathrm{z}}_1\ (\dot{\mathrm{z}}_1 - \dot{\mathrm{z}}_2) + \dot{\mathrm{z}}_2\ (\dot{\mathrm{z}}_1 - \dot{\mathrm{z}}_2) \end{pmatrix} \tag{33}$$

The above equation is a 6*1 matrix but the last element is extended to a new line due to space constraints

$$\begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & x_1 - x_2 \\ 0 & m_1 & 0 & 0 & 0 & 0 & y_1 - y_2 \\ 0 & 0 & m_1 & 0 & 0 & 0 & z_1 - z_2 \\ 0 & 0 & 0 & m_2 & 0 & 0 & x_2 - x_1 \\ 0 & 0 & 0 & 0 & m_2 & 0 & y_2 - y_1 \\ 0 & 0 & 0 & 0 & 0 & m_2 & z_2 - z_1 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 & x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ z \end{pmatrix} \tag{34}$$

Function a(q) is as mentioned below

$$a(q) = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \tag{35}$$

3.2 (b)

The explicit counter part for the above model is represented below where 37 represents the LHS of the model (equation 36) as mentioned in MATLAB and 38 represents the RHS.

$$\begin{pmatrix} \ddot{q} \\ z \end{pmatrix} = \begin{pmatrix} M & a\,(q) \\ a\,(q) & 0 \end{pmatrix}^{-1} c(q, \dot{q}, u) \tag{36}$$

$$\begin{pmatrix} \ddot{q} \\ z \end{pmatrix} = \begin{pmatrix} \ddot{\mathrm{x}}_1 \\ \ddot{\mathrm{y}}_1 \\ \ddot{\mathrm{z}}_1 \\ \ddot{\mathrm{x}}_2 \\ \ddot{\mathrm{y}}_2 \\ \ddot{\mathrm{z}}_2 \\ Z \end{pmatrix} \tag{37}$$

$$\begin{pmatrix} M & a\,(q) \\ a\,(q) & 0 \end{pmatrix}^{-1} c(q, \dot{q}, u) = \left(\begin{array}{c} \frac{\mathrm{FX}\left(m_1\,{x_1}^2+m_1\,{x_2}^2+m_1\,{y_1}^2+m_1\,{y_2}^2+m_2\,{y_1}^2+m_2\,\cdots\right.}{{m_1}^2\,{x_1}^2-2\,{m_1}^2\,x_1\,x_2+{m_1}^2\,{x_2}^2+{m_1}^2\,{y_1}^2-2\,{m_1}^2\,y_1\,y_2+{m_1}^2\,{y_2}^2+{m_1}^2\,{z_1}^2-2\,{m_1}^2\,z_1\,z\cdots} \\ \frac{\mathrm{FY}\left(m_1\,{x_1}^2+m_1\,{x_2}^2+m_2\,{x_1}^2+m_2\,{x_2}^2+m_1\,{y_1}^2+m_1\,\cdots\right.}{{m_1}^2\,{x_1}^2-2\,{m_1}^2\,x_1\,x_2+{m_1}^2\,{x_2}^2+{m_1}^2\,{y_1}^2-2\,{m_1}^2\,y_1\,y_2+{m_1}^2\,{y_2}^2+{m_1}^2\,{z_1}^2-2\,{m_1}^2\,z_1\,z\cdots} \\ \frac{(\mathrm{FZ}-m_1\,g)\left(m_1\,{x_1}^2+m_1\,{x_2}^2+m_2\,{x_1}^2+m_2\,{x_2}^2+m_1\,{y_1}^2+\cdots\right.}{{m_1}^2\,{x_1}^2-2\,{m_1}^2\,x_1\,x_2+{m_1}^2\,{x_2}^2+{m_1}^2\,{y_1}^2-2\,{m_1}^2\,y_1\,y_2+{m_1}^2\,{y_2}^2+{m_1}^2\,{z_1}^2-2\,{m_1}^2\,z_1\,z\cdots} \\ \frac{\cdots}{m_1\,{x_1}^2+m_1\,{x_2}^2+m_2\,{x_1}^2+m_2\,{x_2}^2+m_1\,{y_1}^2+\cdots} \\ \frac{\cdots}{m_1\,{x_1}^2+m_1\,{x_2}^2+m_2\,{x_1}^2+m_2\,{x_2}^2+m_1\,{y_1}^2+\cdots} \\ \frac{\cdots}{m_1\,{x_1}^2+m_1\,{x_2}^2+m_2\,{x_1}^2+m_2\,{x_2}^2+m_1\,{y_1}^2+\cdots} \\ \frac{\cdots}{m_1\,{x_1}^2+m_1\,{x_2}^2+m_2\,x\cdots} \end{array} \right. \tag{38}$$

Comparison: We can observe that the explicit model becomes too complicated with too many expressions on the RHS (equation 38) when we try to solve it for q̈ and the unknown z, in comparison to the model expressed through equation 33 and 34. Thus its advisable to carryout the formulation in implicit form than its explicit counterpart.