1. Consider the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. What does it remind you of? What does it denote? Where and why do we use a matrix? Can you enlist a few applications of matrices? Keep revisiting this question as you cruise through the questions coming ahead, remember to sit UTT as you solve the questions.

2. Define a function. What is a surjective, injective and bijective function?

3. Given an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$.

4. Given an example of a very nice function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Make extra efforts to make this function really nice. Explain what is so nice about your function? Why should one study such functions?

5. Define a function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies the following property: The point $\phi(2,3) = (7,4)$. Note that this function should be defined at all points on $\mathbb{R}^2$. What are the properties of your function? Observe it microscopically.

6. Given a function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ which stisfies the following two conditions:

   (a) $\phi(u + v) = \phi(u) + \phi(v)$

   (b) $\phi(\alpha v) = \alpha v$

   What can one say of such functions? Observe this function closely, such functions form the crus of several discplines, mainly of sciences and engineering. Spend at least 20 minutes dissecting them to the best possible extent.

7. Consider $\mathbb{R}^2$ what are all the properties of this set? $\mathbb{R}^2$ is called a space of all vectors, aka a vector space. Lookup for the definition of a vector space.

8. A subset of a vector space which in itself is a vector space is caled a subspace. Given an example of a subspace of $\mathbb{R}^2$.

9. Given a vector $(1, 7)$ what does the set $\{\alpha(1, 7)|\alpha \in \mathbb{R}\}$ represent? Is it a subspace of $\mathbb{R}^2$?

10. Is $\mathbb{R}^3$ a vector space?

11. Conside the two points $(1, 2, 3)$ and $(4, 5, 7) \in \mathbb{R}^3$. What does the following set denote:
$$\{\alpha(1, 2, 3) + \beta(4, 5, 7)|\alpha, \beta \in \mathbb{R}\}$$
. Is this a subspace?

12. Consider a straight line $y = 2x + 1$ in $\mathbb{R}^2$, does it form a subspace of $\mathbb{R}^2$?

13. Consider a unit circle in $\mathbb{R}^2$, centered at origin, is it a subspace of $\mathbb{R}^2$?

14. What are all the subspaces of $\mathbb{R}^2$?

15. What are all the subspaces of $\mathbb{R}^3$?

16. Given $\mathbb{R}^3$, pick any two points $u, v \in \mathbb{R}^3$. Note that $\{\alpha u + \beta v | \alpha, \beta \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. Generalize this idea!

17. The set $\{\alpha u + \beta v | \alpha, \beta \in \mathbb{R}\}$ is called the linear combination of vectors u and v. We can generalize this to k vectors. Observe what this set is all about.

18. We say that a vector w is manufactured by u and v if $w \in \{\alpha u + \beta v | \alpha, \beta \in \mathbb{R}\}$.

19. Show that $(1, 2, 3)$ and $(4, 5, 6)$ can manufacture $(7, 8, 9)$. Also $(4, 5, 6)$ and $(7, 8, 9)$ can manufacture $(1, 2, 3)$. Finally $(4, 5, 6)$ can be manufactured by the other two vectors.

20. Can $(2, 1, 0)$ and $(3, 0, 8)$ manufacture $(1, 1, 1)$?

21. Can $(0, 0, 1)$ and $(0, 1, 0)$ manufacture $(1, 0, 0)$?

22. When can two vectors in $\mathbb{R}^3$ manufacture a given third vector?

23. When can two vectors in $\mathbb{R}^3$ fail to manufacture a given third vector?

24. If $\{u, v, w\}$ are such that a vector in this set can be manufactured by some vectors in the same set, then such a set is called a linearly dependent set.

25. Give examples of a linearly dependent set in $\mathbb{R}^3$ and get conversant with the definition.

26. If $\{u, v, w\}$ are such that no vector in this set can be manufactured by any comination of vectors in the same set, then such a set is called a linearly independent set.

27. Give examples of a linearly dependent set in $\mathbb{R}^3$ and $\mathbb{R}^2$ and get conversant with the definition.

28. Are $(1, 2), (3, 4)$ linearly independent in $\mathbb{R}^2$? Prove!

29. Are $(1, 1), (2, 3), (7, 17)$ Linearly Independent or Dependent?

30. Construct a set of 3 vectors that are Linearly Independent in $\mathbb{R}^2$. Can you?

    Construct a set of 3 vectors that are Linearly Independent in $\mathbb{R}^3$. Can you?

    Construct a set of k vectors that are Linearly Independent in $\mathbb{R}^k$. Can you?

    Construct a set of k+1 vectors that are Linearly Independent in $\mathbb{R}^k$. Can you?

31. Show that in $\mathbb{R}^2$ we can at most have 2 Linearly Independent vectors.

    Show that in $\mathbb{R}^3$ we can at most have 3 Linearly Independent vectors.

    Show that in $\mathbb{R}^n$ we can at most have n Linearly Independent vectors.

32. Set of any two linearly independent vectors in $\mathbb{R}^2$ is called a basis. Set of any three linearly independent vectors in $\mathbb{R}^3$ is called a basis. Similarly, for n.

33. Show that a basis can manufacture any and every vector of the vector space.

34. Show that any set of linearly independent vectors form a subset of some basis. In other words, one can include a few more elements to a linearly independent set and make it a basis.

35. Show that the number of elements in any basis for a given vector space is always constant.

36. Try taking a few vectors in $\mathbb{R}^3$ and discuss its linear independence or linear dependence. Try atleast 10 examples and familiarize yourself.

37. Show that any 3 vectors on a plane passing through the origin in $\mathbb{R}^3$ cannot be linearly independent. Prove.

38. What are all the subspaces of R^3.

39. Consider a subspace of R^3 and write down its basis. Do this for 3 to 4 different subspaces.

40. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of the function is the straight line $y = 10x$.

41. As an example consider the following basis set of $\mathbb{R}^2 : \{(1,2),(2,2)\}$. Construct a function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\phi(u+v) = \phi(u) + \phi(v)$$
$$\phi(\alpha u) = \alpha\phi(u)$$

    Such a function is called a LT function,

    (a) Where is (1,2) and (2,2) mapped?
    (b) Where is (1,0) and (0,1) mapped?
    (c) What is the range of this function?

42. Consider a function ϕ which is LT which maps

$$\phi(1,2) = (3,4)$$
$$\phi(6,6) = (2,0)$$

    (a) How many such functions can you construct?
    (b) What is the range of this function?

43. Discuss the range of the following functions, given their values at few points

$$\phi(1,2) = (3,4) \ \& \ \phi(1,1) = (6,8)$$
$$\phi(1,1) = (2,2) \ \& \ \phi(3,3) = (4,4)$$
$$\phi(1,2) = (3,5) \ \& \ \phi(0,8) = (3,0)$$
$$\phi(-2,2) = (0,0) \ \& \ \phi(8,2) = (1,1)$$
$$\phi(1,1) = (2,2) \ \& \ \phi(1,2) = (3,3)$$

44. Give an example of an LT function which maps a L.I. set of vectors to a L.I. set of vectors i.e. $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, where $\{u, v\}$ are L.I. & $\{\phi(u), \phi(v)\}$ are also L.I.

45. Same as the previous question, but $\{u, v\}$ is L.I. and $\{\phi(u), \phi(v)\}$ is L.D. What can one say about the range?

46. Note that range of a L.I function is always a subspace.

47. Note that the set N=$\{v \| \phi(v) = 0\}$ is a subspace of R^2 where $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is L.I .Take atleast 5 different examples of L.I functions and see if this is true.The set N⊂=Domain is called the null space of ϕ

48. Construct a Linear Transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ $\phi(0,1) = (3,4) and \phi(1,0) = (1,2)$. Construct an inverse of linear transformation and observe it carefully.Is it a L.T function too?

49. Do you realize the importance of inverting a matrix ?

50. L.T stands for Linear Transformation .It is aka a matrix.

51. Define a L.T. $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(0,1) = (2,3)$ and $\phi(1,0) = (7,4)$.This function ϕ is same as the matrix $\begin{bmatrix} 7 & 2 \\ 4 & 3 \end{bmatrix}$. $\phi(x,y)$ is same as $\begin{bmatrix} 7 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, Verify and make your share of observations. What is ϕ^{-1} ? Is this the traditional matrix inverse of $\phi(x,y)$ is same as inverse of $\begin{bmatrix} 7 & 2 \\ 4 & 3 \end{bmatrix}$? Think !

52. Consider a L.T. ϕ such that : $\phi(1,2) = (3,5)$ and $\phi(3,1) = (8,2)$. What is the matrix equivalent of ϕ ?

53. Solve the following :

    (i) 3x-2y=15

    x+4y=19

    (ii) Isn't this the same as $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \end{bmatrix}$

    (iii) You are trying to find if there is an element in the domain which maps to (15,19) .

    (iv) Say all that you can , on what exactly is happening here ?

54. Given V_1 , V_2 & V_3 are L.I. , S.T.

    (i) $\{V_2 - V_3, V_1 - V_3, V_1 - V_2\}$ are L.D.

    (ii) S.T. $\{V_1 + V_2, V_1 + V_3, V_2 + V_3\}$ are L.I.

55. Given vectors $\{V_1, V_2, V_3, V_4\}$ their sum is $0 = V_1 + V_2 + V_3 + V_4$. Is this set L.I or L.D. ?

56. Show that the following are equivalent:

    (a) The vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent

    (b) $\forall \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\ (\sum_{i=1}^{n} \alpha_i v_i = 0 \implies \alpha_j = 0\ \forall 1 \leq j \leq n)$

57. In the following six matrices find out the following:

    (a) Rank of the matrix

    (b) Dimension of the range

        i. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

        ii. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

        iii. $\begin{bmatrix} 7 & -7 \\ 2 & -2 \end{bmatrix}$

        iv. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

        v. $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

        vi. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

    What is happening here? Describe in detail.

58. The dimension of the range of the matrix M and M^T is always the same. Why?

59. Take three linearly independent vectors in $\mathbb{R}^3$. Show that they form a basis of $\mathbb{R}^3$.

60. Consider the matrix $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$. The dimension of the range is 2. The range of this linear transformation is obviously a linear combination of three vectors. Do you see which are those three vectors?

61. ***** The questions is wrong**** Consider any $2X2$ matrix. Do you observe the following?

    Dimension of the Null space = Dimension of the range

62. S.T. this doesn't hold good in $3X3$ matrix.

63. What else do you observe in a $3X3$ matrix? How is the dimension of null space and the dimension of the range of a L.T. related?

64. When is a L.T. function is one-one and when is it onto?

65. A $2X2$ matrix A can be seen as two vectors placed as columns. For example $(1,2)$ and $(3,4)$ when place as columns give rise to the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

    Note that $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \{\alpha(1,2) + \beta(3,4)/\alpha, \beta \in \mathbb{R}\}$.

    Note that matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ simply means that x(a,c)+y(b,d).

    Note that a matrix takes a vector (x,y) to a linear combination of the column vector.

    It is now clear that the range of the matrix, say $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix}$ is nothing but the linear combination of the vectors $(1,0,1), (1,2,3), (4,4,4)$.

66. Consider a $10X10$ matrix A, defined as following:

    $A[i,j] = 0$ if $i + j \cong 0(mod2)$

    $A[i,j] = 1$ if $i + j \cong 1(mod2)$

    What is the dimension of range?

67. Consider the sub-space $S : (y = 13x)$ of $\mathbb{R}^2$

    (a) Give an example of a ϕ such that S is its null space.

    (b) Give an example of a ϕ such that S is the dimension of range of ϕ

68. Let $A =$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

    (a) Show that null space of $A \subset$ Null space of A^2

    (b) Is this true for any $2X2$ matrix.

    (c) Is this true for any $3X3$ matrix.

    (d) Is this true for any nXn matrix.

69. Define column space. S.T. Column space of $A^2 \subset$ Column space of A.

70. Solve the following simultaneous equation:

    (a) $x + 4y + 7z = 9$
    $2x + 5y + 8z = 9$
    $3x + 6y + 9z = 9$

    (b) What is the null space of

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

    Question (a) and (b) are strongly related and make a big theory. Do you observe what is happening there?

71. What does one mean by linear combination of vectors and what is a linear span?

72. Show that, if a matrix A has linearly independent column vectors, then the columns of the matrix A^2 are also linearly independent.

73. $Ax = Ay \iff x = y$. Is this true? If False, when is the statement False and what leads to the falsity of the statement?

74. Let M be a 3×3 matrix $\ni$ dim(Range) =3. Show that, $M^2 = M \iff M$ is an identity matrix.

75. Consider a 10×10 matrix with all its entries to be 1. What is the dimension of its range?