* # Z-TRANSFORM

$$Z\{x(n)\} = \sum_{n=0} x(n) z^{-n}$$
$$= X(z)$$

$Z\{a^n\} = \frac{z}{z-a}$ $\quad |z|>|a|$

$Z\{n\} = \frac{z}{(z-1)^2}$ $\quad |z|>1$

$Z\{\frac{1}{n}\} = \log\left(\frac{z}{z-1}\right)$ $\quad |z|>1$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3}$

$|x|<1$

* ## Properties

1. Linear - $Z\{c_1 x(n) + c_2 y(n)\} = c_1 X(z) + c_2 Y(z)$

$$= c_1 \sum_{n=0}^{\infty} x(n) z^{-n} + c_2 \sum_{n=0}^{\infty} y(n) z^{-n}$$

2. change of scale:

$Z\{x(n)\} = X(z)$ $\quad Z\{a^n x(n)\} = X\left(\frac{z}{a}\right)$

$Z\{a^n n\} = \frac{az}{(z-a)^2}$

3. Shifting property:

$Z\{x(n-k)\} = z^{-k} Z\{x(n)\}$

4. Right shifting:

$Z\{x(n+1)\} = z Z\{x(n)\} - z x(0)$

$Z\{x(n+2)\} = z^2 Z\{x(n)\} - z^2 x(0) - z x(1)$

$Z\{x(n+3)\} = z^3 Z\{x(n)\} - z^3 x(0) - z^2 x(1) - z x(2)$

5. derivative of Transform:

$$Z\{n\,x(n)\} = -z\frac{d}{dz}X(z)$$

FORMULAS

$$Z\{a^n\} = z/z-a$$

$$Z\left\{\frac{1}{n}\right\} = \log\left(\frac{z}{z-1}\right)$$

$$Z\left\{\frac{1}{n-1}\right\} = \frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

$$Z\left\{\frac{1}{n+1}\right\} = z\log\left(\frac{z}{z-1}\right)$$

$$Z\{n^2\} = \frac{z(z+1)}{(z-1)^3}$$

$$Z\{n(n+1)\} = \frac{2z^2}{(z-1)^3}$$

$$Z\left\{\frac{1}{n(n-1)}\right\} = \frac{1-z}{z}\log\frac{z}{z-1}$$

$$Z\left\{\frac{1}{n(n+1)}\right\} = (1-z)\log\frac{z}{z-1}$$

$$Z\{r^n\cos n\theta\} = \frac{z^2 - zr\cos\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$Z\{r^n\sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$Z\left\{r^n\cos\frac{n\pi}{2}\right\} = \frac{z^2}{z^2+r^2}$$

$$Z\left\{r^n\sin\frac{n\pi}{2}\right\} = \frac{zr}{z^2+r^2}$$

$$Z\{\cos n\pi\} = \frac{z}{z+1}$$

$$Z\{n\} = \frac{z}{(z-1)^2}$$

$$Z\{n+1\} = \frac{z^2}{(z-1)^2}$$

$$Z\{1\} = \frac{z}{z-1}$$

$$Z\{k\} = \frac{kz}{z-1}$$

$$Z\{n-1\} = \frac{2z-z^2}{(z-1)^2}$$

$$Z\left\{\sin\frac{n\pi}{2}\right\} = \frac{z}{z^2+1}$$

$$Z\left\{\cos\frac{n\pi}{2}\right\} = \frac{z^2}{z^2+1}$$

$$Z\{\delta(n)\} = 1$$

$$Z\{u(n)\} = \frac{z}{z-1}$$

$\forall\ |z|>1$

* Inverse z - Transforms

$$z^{-1}\left[\frac{z}{z-a}\right] = a^n \qquad z^{-1}\left[\frac{az}{(z-a)^2}\right] = a^n n$$

$$z^{-1}\left[\frac{z}{(z-1)^2}\right] = n \qquad z^{-1}\left[\frac{z^2}{(z-1)^2}\right] = n+1$$

$$z^{-1}\left[\frac{z^2}{z^2+1}\right] = \cos\frac{n\pi}{2}$$

$$z^{-1}\left[\log\left(\frac{z}{z-1}\right)\right] = \frac{1}{n}$$

$$z^{-1}\left[\frac{z}{z^2+1}\right] = \sin\frac{n\pi}{2}$$

$$z^{-1}\left[\frac{z}{z-1}\right] = 1$$

$$z^{-1}\left[\frac{kz}{z-1}\right] = k$$

* Example:

$$z^{-1}\left[\frac{3}{3z-1}\right] \Rightarrow \frac{3}{3z-1} = z^{-1}\,\frac{3z}{3z-1}$$

$$z^{-1}\left[z^{-1}\,\frac{3z}{z-1}\right] = \left(\frac{1}{3}\right)^n \quad n \to n-1$$

$$= \left(\frac{1}{3}\right)^{n-1}$$

* Always preserve z in numerator.

i.e $\dfrac{X(z)}{z} = \dfrac{1}{\text{Denominator}}$

* Difference equation

Equation involving differences between successive values of a function of a discrete variable.

* Formulation of Diff equations:

fibnoci: $F_n = F_{n-1} + F_{n-2}$ $\quad a_0 = 3 \quad a_3 = 10$

Hanoi: $H_n = 2H_{n-1} + 1$ $\quad H_1 = 1 \quad H_2 = 3$

No: of bit strings of length n that has no 2 consecutive 0's $\Big\}$ $a_{n-1} + a_{n-2}$

$a_2 = 3$ $\quad n \geq 3$

$a_3 = 5$

$a_1 = 2$

length n, ends with 0 : $a_n = a_{n-1} + a_{n-2}$

- Refer MFOC RR-1 slides, FDE (TT) slides.

* Z-Transform convolution

$Z\{f(n)\} = F(z)$

$Z\{g(n)\} = G(z)$

$Z\{f(n) * g(n)\} = F(z)\,G(z)$

$\bar{f}(z)\,\bar{g}(z)$

$\{f(n) * g(n)\} = \sum_{r=0}^{n} f(r)\,g(n-r)$

$Z \rightarrow$ $\boxed{\text{Non linear operator, exist over entire plane}}$ — FALSE

* Dirac Delta Z-Transform

$\delta(n)$ - unit impulse function. $\begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$

$$Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1$$

* unit-step fn

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad n = 0, 1, 2 \ldots$$

$$Z\{u(n)\} = Z(1) = \frac{z}{z-1}, \text{ if } |z| > 1$$

* check Z-Transform convolution 3rd sum.

* G-P formulas:

$$S_n = \frac{a(r^n - 1)}{r-1} \qquad r > 1$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r < 1$$

$$S_n = na \qquad r = 1$$

* i power values

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i^4 \cdot i = i$

$i^6 = i^4 \cdot i^2 = -1$

$i^7 = i^4 \times i^3 = -i$

* Z(n) doesnt exist over entire complex plane

$$\frac{n}{2}\left[2a + (n-1)d\right]$$