FOURIER TRANSFORMS

- 19PW13

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1. Find fourier cosine transform of $f(x)$ if $f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \\ 0 & x>2 \end{cases}$

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{x \sin sx}{s} + \frac{\cos sx}{s^2} \right)_0^1 + \left(\frac{(2-x)\sin sx}{s} - \frac{\cos sx}{s^2} \right)_1^2$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right)_0^1 + \left(-\frac{\cos s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} \times \frac{1}{s^2} \right)$$

$$F_c\{f(x)\} = \boxed{\sqrt{\frac{2}{\pi}} \left(\frac{1}{s^2} \right) (2\cos s - \cos 2s - 1)}$$

2. Fourier cosine Transform of

$$f(x) = \begin{cases} 1 & 0<x<a \\ 0 & \text{otherwise} \end{cases}$$

$$F_c\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt = F_c(s)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos st \, dt = \sqrt{\frac{2}{\pi}} \left(\frac{\sin st}{s} \right)_0^a$$

$$\boxed{F_c(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s} \right)}$$

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3. Fourier Sine Transform of $f(x)' = \frac{x}{x^2+1}$

W.K.T $F_s\{x f(x)\} = \frac{-d}{ds} F_c\{f(x)\}$

$f(x) = \frac{1}{x^2+1}$

$I(s) = F_c\left[\frac{1}{x^2+1}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos sx}{1+x^2}\, dx \quad — (1)$

Diff w.r.t s on both sides

$\frac{dI(s)}{ds} = \sqrt{\frac{2}{\pi}} (-\sin sx) \cdot x \int_0^\infty \frac{dx}{1+x^2}$

Multiply and divide by x,

$\frac{dI(s)}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{-x^2 \sin sx\, dx}{x(x^2+1)}$

$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(x^2+1-1)}{x^2+1} \frac{\sin sx}{x}\, dx$

$= -\sqrt{\frac{2}{\pi}} \int_0^\infty 1 - \frac{1}{x^2+1} \left(\frac{\sin sx}{x}\right) dx$

$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x}\, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x(x^2+1)}\, dx.$

We have $\int_0^\infty \frac{\sin t}{t} = \pi/2$

$= -\sqrt{\frac{2}{\pi}} * \frac{\pi}{2} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x} \left(\frac{1}{1+x^2}\right) dx.$

$\frac{dI(s)}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{\frac{\sin sx}{x}\right\} \frac{1}{1+x^2}\, dx - \sqrt{\frac{\pi}{2}}$

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differentiate w.r.t s

$$\frac{d^2 I(s)}{ds^2} = \sqrt{\frac{2}{\pi}} \frac{(\cos sx).x}{x} \int_0^\infty \frac{dx}{1+x^2}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos sx}{1+x^2} dx = I(s) \quad \text{--- (3)}$$

From (3) $\quad \dfrac{d^2 I(s)}{ds^2} = I(s)$

$$\frac{d^2 I(s)}{ds^2} - I = 0$$

$$(D^2 - 1) I = 0 \quad \Rightarrow m^2 - 1 = 0$$

$$m = \pm 1$$

$$\boxed{I(s) = c_1 e^{s} + c_2 e^{-s}}$$

Put $s = 0$ in eqn (1) & (2)

(1) $\Rightarrow$ $I(0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos(0)}{1+x^2} dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{dx}{1+x^2} = \sqrt{\frac{2}{\pi}} \left(\tan^{-1} x\right)_0^\infty$$

$$= \sqrt{\frac{\pi}{2}}$$

(2) $\Rightarrow$ $I'(0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(0) - \frac{x^2}{x(x^2+1)} dx - \sqrt{\frac{\pi}{2}}$

$$= -\sqrt{\frac{\pi}{2}}$$

$I(0) = c_1 + c_2 = \sqrt{\frac{\pi}{2}} \quad \text{--- (4)}$

$I'(0) = c_1 - c_2 = -\sqrt{\frac{\pi}{2}} \quad \text{--- (5)}$

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4. Find fourier Transform of $f(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & \text{otherwise} \end{cases}$

Hence P.T $\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1-|x|)(\cos sx - i\sin sx)\,dx$$

$\because |x|>1$ $f(x)=0$ in $-\infty<x<-1$

$1<x<\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx - i\sin sx)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\cos sx - \int_{-1}^{1}(1-|x|)\, i\sin sx\,dx$$

$= 0$

(odd fn)

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-|x|)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \cos sx - x\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin sx}{s} - \frac{x\sin sx}{s} - \frac{\cos sx}{s} \right)_0$$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{\cos s}{s^2} + \frac{1}{s^2} \right) = \boxed{\sqrt{\frac{2}{\pi}} \frac{1}{s^2}(1-\cos s)}$$

using Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\int_{-\infty}^{\infty} (1-|x|)^2\,dx = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\,ds$$

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$$2\int_0^1 (1-x)^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4} ds$$

$$2\int_0^1 1+x^2-2x\, dx = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4} ds$$

$$\cancel{2}\left(x+\frac{x^3}{3}-\frac{2x^2}{2}\right)_0^1 = \frac{\cancel{2}}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4} ds$$

$$\frac{\pi}{3} = \int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4} ds$$

put $s=2t$ $\quad s^4 = 16t^4$ $\quad ds = 2dt$

$$\frac{\pi}{3} = \frac{2}{16}\int_{-\infty}^{\infty} \frac{(1-\cos 2t)^2}{t^4} dt$$

$$\frac{8\pi}{3} = 8\int_0^{\infty} \frac{\sin^4 t}{t^4} dt$$

$$\boxed{\int_0^{\infty} \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}}$$

5. Fourier inverse Transform of

$$F(s) = \begin{cases} a-|s| & |s|<a \\ 0 & |s|>a \end{cases}$$

Hence find $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$

$$F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist} F(s)\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ist}(a-s)\, ds$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos st\,(a-s)\,ds + \int_{-a}^{a} i \sin st\,(a-s)\,ds$$

$= 0$ (odd fn)

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a-s) \cos st\, ds$$

$$= \sqrt{\frac{2}{\pi}} \left[(a-s) \frac{\sin st}{t} - \frac{\cos st}{t^2} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{\cos at}{t^2} + \frac{1}{t^2} \right) = \sqrt{\frac{2}{\pi}} \left(\frac{1}{t^2} \right) (1 - \cos at)$$

$$F^{-1}[F(s)] = \boxed{\sqrt{\frac{2}{\pi}} \frac{1}{t^2} (1 - \cos at)}$$

$$F\left\{ \sqrt{\frac{2}{\pi}} \frac{1}{t^2} (1-\cos at) \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} \sqrt{\frac{2}{\pi}} \times \frac{(1-\cos at)}{t} dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} (\cos st - i \sin st) \left[\frac{1 - \cos at}{t^2} \right] dt$$

Put $a = 2$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \cos st - i \sin st \left(\frac{2 \sin^2 t}{t^2} \right) dt$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \cos st \left(\frac{\sin t}{t} \right)^2 \quad [\because \text{odd func}]$$

$$= \frac{4}{\pi} \int_{0}^{\infty} \cos st \left(\frac{\sin t}{t} \right)^2 dt$$

Put $s = 0$; LHS $= \left\{ \begin{array}{l} a - |s| \\ \Rightarrow 2 \end{array} \right.$ where $a = 2$ $s = 0$

$$2 = \frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt \Rightarrow \boxed{\int_{0}^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}}$$

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