Dynamic Programming

29 March 2021 14:25

- decompose the problem into sub problems.

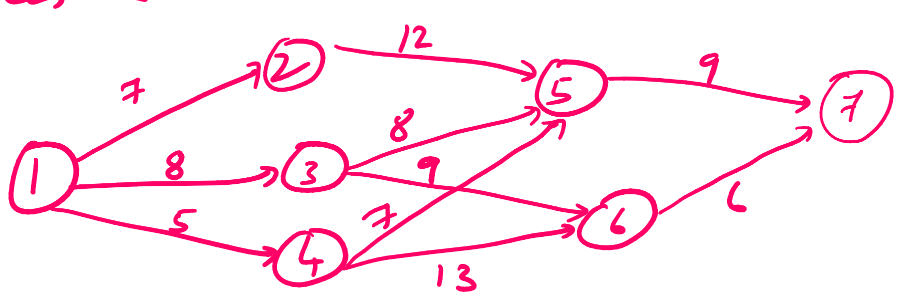
- output of one sub problem is input to another sub problem

- Determines optimization of a multi variable problem by decomposing it into stages, each stage is a single variable problem.

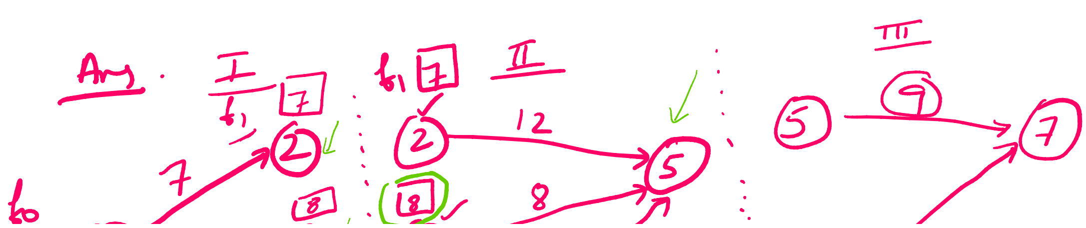
- Recursive in nature.

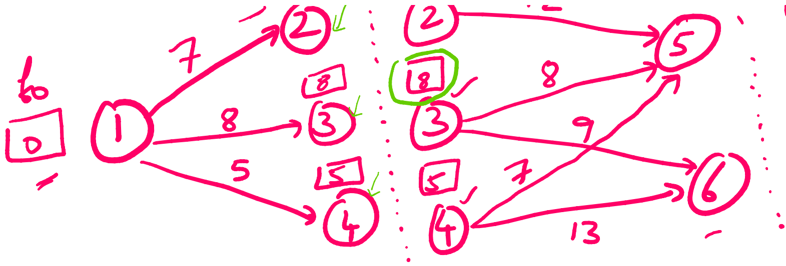
- Example:

To find shortest route between two cities. The following network provides the possible routes between starting city at node 1 and destination city at node 7. The routes pass through the intermediate cities designated by the nodes 2 to 6.



Ans.





Stage I

Shortest distance from node 1 to node 2 = 7 ✓

" " " to node 3 = 8 ✓

" " " node 4 = 5 ✓ (from node 1)

Stage II

Stage II has two end nodes 5 & 6

$$\left.\begin{array}{l}\text{Shortest}\\\text{distance}\\\text{to node 5}\end{array}\right\} = \min_{i=2,3,4}\left\{\left(\begin{array}{l}\text{shortest}\\\text{distance}\\\text{to node } i\end{array}\right) + \begin{array}{l}\text{distance}\\\text{from}\\\text{node } i\\\text{to node 5}\end{array}\right\}$$

$$= \min\{\overbrace{7+12}^{i=2},\ \overbrace{8+8}^{i=2},\ 5+7\}$$

$$= \min\{19, 16, 12\} = \underline{12} \text{ (from node 4)}$$

$$\left.\begin{array}{l}\text{Shortest}\\\text{distance}\\\text{to node 6}\end{array}\right\} = \min_{i=3,4}\left\{\begin{array}{l}\text{Shortest}\\\text{dist}\\\text{to node } i\end{array} + \begin{array}{l}\text{distance}\\\text{from}\\\text{node } i\\\text{to node 6}\end{array}\right\}$$

$$= \min\{8+9,\ 5+13\}$$

$$= \min\{17, 18\} = \underline{17} \text{ (from node 3)}$$

Stage 3

The destination 7 can be reached

from 5 & 6.

$$\left.\begin{array}{l}\text{Shortest distance} \\ \text{to node } 7\end{array}\right\} = \min_{i=5,6} \left\{ \begin{array}{l}\text{shor dis} \\ \text{to} \\ \text{node } i\end{array} + \begin{array}{l}\text{dis from} \\ \text{node } i \\ \text{to} \\ \text{node } 7\end{array} \right\}$$

$$= \min \{ 12 + 9, \; 17 + 6 \}$$

$$= \min \{ 21, 23 \} = 21$$

(from node $\underline{\underline{5}}$)

$\therefore$ Shortest path is $1 \rightarrow 4 \leftarrow 5 \rightarrow 7$

& distance is

$5 + 7 + 9 = 21$

Stage 3 : $5 \rightarrow 7$

Stage 2 : $4 \rightarrow 5$

Stage 1 : $1 \rightarrow 4$

Recursive Equation

The recursive definition is given as

$\underline{f_i(x_i)}$ – shortest distance to node x_i in i^{th} stage

x_i – $\underline{\text{state of the system at stage } i}$.

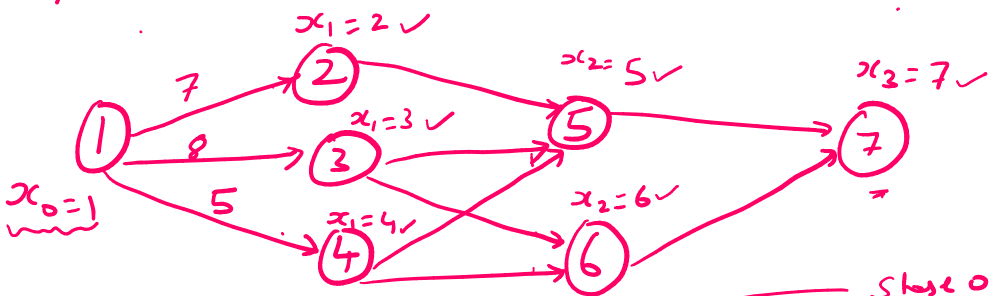
$d(x_i, x_{i-1})$ – $\underline{\text{distance from node } x_i \text{ to } x_{i-1}}$

previous

$\therefore f_i(x_i) = \min_{i \; [\ldots]} \{ d(x_i, x_{i-1}) + f(x_{i-1}) \}$

$$\therefore \; f_i(x_i) = \min_{\substack{\text{all feasible} \\ \text{route from} \\ i \text{ to } i-1}} \left\{ d(x_i, x_{i-1}) + f_{i-1}(x_{i-1}) \right\}$$

For the shortest route,

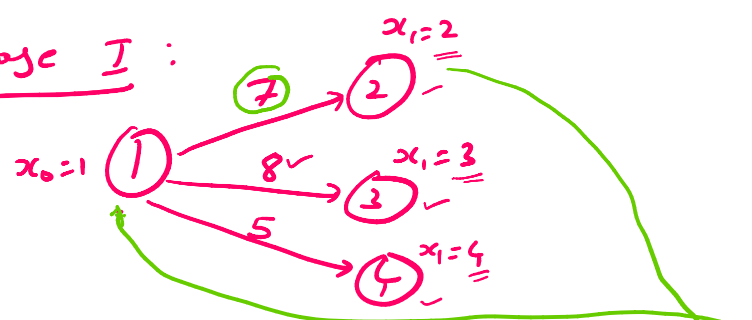


Stage 0

Initial Condition

$$f_0(x_0 = 1) = 0$$

Stage I :



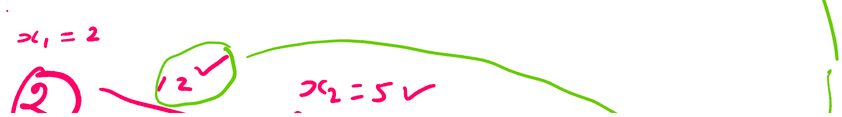
$$f_1(x_1 = 2) = \min_{\substack{\text{all feasible} \\ \text{from } x_0=1 \\ \text{to } x_1=2}} \left\{ d(x_1, x_0) + f_0(x_0) \right\}$$

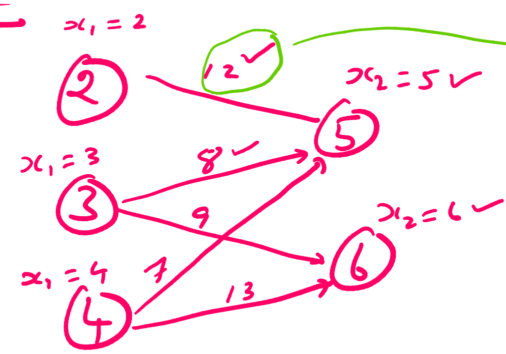
$$= \min\{7+0\} = 7$$

$$f_1(x_1 = 3) = \min\{8+0\} = 8$$

$$f_1(x_1 = 4) = \min\{5+0\} = 5$$

Stage 2





$$f_2(x_2 = 5) = \min_{\substack{\text{all feasible} \\ x_1 = 2,\,3,\,4 \\ \text{to } x_2 = 5}} \left\{ d(x_2, x_1) + f_1(x_1) \right\}$$

$$= \min\{ \underbrace{12}_{x_2=5,\ x_1=2} + \underbrace{7}_{f_1(x_1=2)},\ \underbrace{8}_{x_2=5} + \underbrace{8}_{x_1=3},\ \underbrace{7}_{\substack{x_2=5 \\ x_1=4 \\ d(x_2=5,\ x_1=4)}} + \underbrace{5}_{f_1(x_1=4)} \}$$

$$= \min\{19,\ 16,\ 12\}$$

$$= 12 \quad (x_1 = 4)$$

$$f_2(x_2 = 6) = \min_{\substack{\text{all feasible route} \\ x_1 = 3,\,4 \\ \text{to } x_2 = 6}} \left\{ d(x_1, x_2) + f_1(x_1) \right\}$$

$$= \{ \underbrace{8}_{f_1(x_1=3)} + \underbrace{9}_{d(x_1=3,\ x_2=6)},\ \underbrace{5}_{f_1(x_1=4)} + \underbrace{13}_{d(x_1=4,\ x_2=6)} \}$$

$$= \{17,\ 18\}$$

$$= 17$$

Stage III

$$f_3(x_3 = 7) = \min_{\substack{\text{all feasible} \\ x_2 = 5,\,6 \\ \text{to } x_3 = 7}} \left\{ d(x_2, x_3) + f_2(x_2) \right\}$$

$$= \min \{ d(x_2=5, x_3=7) + f_2(x_2=5),$$
$$d(x_2=6, x_3=7) + f_2(x_2=6) \}$$

$$= \min \{ 9+12, \quad 6+17 \}$$

$$= \min \{ 21, \quad 23 \}$$

$$= 21$$

Path

$$\overset{x_0}{1} \rightarrow \overset{x_1}{4} \rightarrow \overset{x_2}{5} \rightarrow \overset{x_3}{7}.$$

This is called. Forward Recursion

Backward Recursion

$$f_i(x_i) = \min_{x_i \text{ to } x_{i+1}} \{ d(x_i, x_{i+1}), f_{i+1}(x_{i+1}) \}$$

Initial $f_4(x_4 = 7) = 0$

Initial $\boxed{f_4, f_3, f_2, f_1}$ Backward

initial $\boxed{f_0, f_1, f_2, f_3}$ Forward

3 stage pattern

Principle of optimality

Future decisions for the remaining stages will constitute an optimal Policy regardless ...

remaining ... an optimal policy regardless of the policy adopted in previous stages. 'Bellman'

Knapsack / fly away / Cargo loading

A 4 ton vessel can be loaded with one or more of three items. The following table gives the unit weight W_i in tons and the unit revenue in thousands of dollars r_i for item i. How should the vessel be loaded to maximize the total return?

Item i	W_i	r_i
1	2	31
2	3	47
3	1	14

Ans

Backward recursion.

Stage 3

Exact weight to be allocated in stage 3 is not known in advance, it takes one of the values (1, - 2)

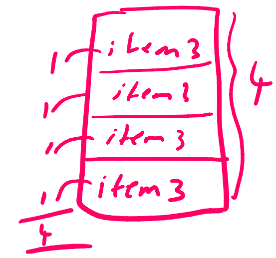
Stage 3 is ...

it takes one of the values
0, 1, 2, 3, 4 (because $W = 4$)

Given $w_3 = 1$, total weight $W = 4$.

The maximum number of units of item 3 that can be loaded is,

$$\frac{W}{w_3} = \frac{4}{1} = 4.$$



$\therefore$ m_3 – no. of units of item 3 = 0, 1, 2, 3, 4

Optimum value at stage 3 is given by,

$$f_3(x_3) = r_3 m_3$$
$$= 14 m_3$$

Following table gives feasible alternatives for each value of x_3

Stage 3 — $14 m_3$

x_3	$m_3 = 0$	$m_3 = 1$	$m_3 = 2$	$m_3 = 3$	$m_3 = 4$	$f_3(x_3)$	m_3^*
0	0	–	–	–	–	–	0
1	0	14	–	–	–	14 ✓	1
2	0	14	28 ✓	–	–	28 ✓	2
3	0	14	28	42	–	42 ✓	3
4	0	14	28	42	56	56 ✓	4

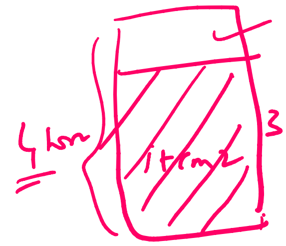
$$f_i(x_i) = \max_{m_i = 0, 1, 2, \ldots \left[\frac{W}{w_i}\right]} \left\{ v_i m_i + f_{\cdot}(x_i - w_i m_i) \right\}$$

$$f_i(x_i) = \max_{\substack{m_i = 0,1,2,\dots\left[\frac{W}{w_i}\right] \\ x_i \le W}} \left\{ r_i m_i + f_{i+1}(x_i - w_i m_i) \right\}$$

$$f_4(x_i) = 0$$

$$f_3(x_3) = r_3 m_3 + 0$$

Stage 2



$$m_2 = \frac{W}{w_2} = \frac{4}{3} = 1$$

$3 \times 2 = 6$

$$\therefore m_2 = 0, \text{ or } 1$$

$$f_2(x_2) = r_2 m_2 + f_3(x_2 - 3m_2)$$

x_2	$47 m_2 + f_3(x_2 - 3m_2)$, $m_2 = 0$	$m_2 = 1$	$f_2(x_2)$	m_2^*
0	0 + 0	–	0	0
1	0 + 14	–	14	0
2	0 + 28	–	28	0
3	0 + 42	47 + 0	47	1
4	0 + 56	47 + 14	61	1

47×1, $f_3(0)$, 47×1

$$f_3(4 - 3 \times 1) = f_3(4-3) = f_3(1)$$

Stage 1

$$m_1 = \frac{4}{2} = 2, \quad \therefore m_1 = 0, 1, 2$$

$$f_1(x_1) = 31 m_1 + f_2(x_1 - 2m_1)$$

$$f_1(x_1) = 31 m_1 + f_2(x_1 - 2m_1)$$

x_1	$m_1=0$	$m_1=1$	$m_1=2$	opt $f_1(x_1)$	m_1^*
0	0	–	–	0	0
1	0+14	–	–	14	0
2	0+28	31+0	–	31	1
3	0+47	31+14	–	47	0
4	0+6	31+28	62+0	62	2

Given $W=4$

Stage 1 optimum $x_1 = 4$, $m_1 = 2$

$\therefore x_2 = x_1 - w_1 m_1$

$= 4 - 2\times 2$

$= 4 - 4 = 0$

$x_2 = 0$ Refer stage 2
$m_2 = 0$

$x_3 = x_2 - w_2 m_2$

$= 0 - 0 = 0$

Refer stage 3 $\longrightarrow m_3 = 0$

opti sol is

$$\boxed{\begin{array}{c} m_1 = 2,\ m_2 = 0.\ m_3 = 0 \\ f_1(x_1 = 4) = 62. \\ \text{(ie) } \$62{,}000 \end{array}}$$

Item1 — 2 units

Item2 — 0

Item 3 — 0

Revenue $62,000