Fourier Transform

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)\,dt$$

$$F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} F(s)\,ds$$

* Existence conditions

$f(t)$ – absolutely integrable, $\rightarrow$ $\boxed{(a,b) \quad \int_a^b |f(t)|\,dt < \infty}$

piecewise continuous

* $\cancel{\dfrac{e^{x}-e^{-x}}{2x}}$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin ax}{x} = \frac{\pi}{2}$$

$e^{ist} = \cos st + i\sin st$

$e^{-ist} = \cos st - i\sin st$

unit step

$$u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

$$F\{u(t-a)\} = \boxed{\frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-ias}}{is} \right\}}$$

Dirac delta

$$\delta(t-a) = \lim_{h\to 0} g(t-a)$$

$$g(t-a) = \begin{cases} 1/h & a < t < a+h \\ 0 & \text{otherwise} \end{cases}$$

$$F\{g(t-a)\} = \frac{1}{2\pi} \int_a^{a+h} e^{-ist} g(t-a)\,dt$$

$$= \frac{e^{-ias}}{is\sqrt{2\pi}} \left[\frac{1 - e^{-ish}}{h}\right]$$

$$F\{\delta(t-a)\} = \frac{e^{-ias}(is)}{is(\sqrt{2\pi})}$$

$$\boxed{= \frac{e^{-ias}}{\sqrt{2\pi}}}$$

* Find fourier Transform of ques, to compute given integral, apply inverse formula to $F(s)$ and substitute value of t according to the question

* Bernoulli's: $\int u\,dx = uv - u'v_1 + u''v_2 - u'''v_3$

* A function $f(t) \Rightarrow$ is self reciprocal under FT if $F\{f(t)\} = f(s)$

* $\int_0^\infty e^{-u} u^{n-1}\,du = \Gamma(n)$

* $F\{e^{-a^2x^2}\} = \dfrac{e^{-s^2/4a^2}}{a\sqrt{2}}$ | $F\{e^{-\frac{x^2}{2}}\} = e^{-\frac{s^2}{2}}$ self-reciprocal,

* Fourier inverse transform of $F(s) = \begin{cases} a-|s| & |s|<a \\ 0 & |s|>a \end{cases}$

$= \sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{t^2}(1-\cos at)$

$\int_0^\infty \left(\dfrac{\sin x}{x}\right)^2 dx = \dfrac{\pi}{2}$

$\dfrac{1-\cos 2t}{2} = \sin^2 t$

$\dfrac{1+\cos 2t}{2} = \cos^2 t$.

*

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

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Properties :

① Linear property -

$$F\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s)$$

② Change of scale -

$$F\{f(at)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

③ s-shifting -

$$F\{e^{-iat} f(t)\} = F(s+a)$$

④ t-shifting -

$$F\{f(t-a)\} = e^{-ias} F(s)$$

⑤ Modulation:

$$F\{f(t)\cos at\} = \frac{1}{2}[F(s-a) + F(s+a)]$$

⑥ sine cosine modulation:

i) $F_c\{f(t)\cos at\} = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

ii) $F_c\{f(t)\sin at\} = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

iii) $F_s\{f(t)\cos at\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

iv) $F_s\{f(t)\sin at\} = \frac{-1}{2}[F_s(s+a) - F_c(s-a)]$

⑦ Transform of derivative:

$$\boxed{F\{f'(t)\} = (is) \cdot F(s)}$$

$f(t)$ - continuous

$f'(t)$ - piecewise continuously differentiable

$f(t), f'(t)$ are absolutely integrable in $(-\infty, \infty)$

$$\lim_{t \to \pm\infty} f(t) = 0$$

⑧ Derivative of Transform

$$F\{f(t)\} = F(s) \; ; \; \boxed{F\{t f(t)\} = i \frac{d}{ds} F(s)}$$

⑨ Faultang theorem

convolution : $f(t) * g(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$.

$$F\{f(t) * g(t)\} = F(s)\, G(s)$$

⑩ Parseval's identity / Energy theorem

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

i) $\int_0^{\infty} f(t)\, g(t)\, dt = \int_0^{\infty} F_c(s)\, G_c(s)\, ds = \int_0^{\infty} F_s(s)\, G_s(s)\, ds$

ii) $\int_0^{\infty} |f(t)|^2\, dt = \int_0^{\infty} |F_s(s)|^2\, ds = \int_0^{\infty} |F_c(s)|^2\, ds$

* $\boxed{F\{e^{iat} f(t)\} = F(s+a)}$

$\boxed{F^{-1}\{F(s+a)\} = e^{-iat} f(t)}$

*

$$F\{x^2 f(x)\} = i \frac{d}{ds} F\{x f(x)\}$$

$$F\{x f(x)\} = i \frac{d}{ds} F\{f(x)\}$$

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* $f(t) = \begin{cases} 1 & |t| < a \\ 0 & \text{otherwise.} \end{cases}$ | $F\{f(t)(1+\cos \frac{\pi t}{a})\} = ?$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin as}{s}\right] \left\{\frac{\pi^2/a^2}{\pi^2/a^2 - s^2}\right\}$$

* $F\{x^2 e^{-ax} u(x)\} = \dfrac{2}{\sqrt{2\pi}\,(is+a)^3}$

$F\{x e^{-ax} u(x)\} = \dfrac{1}{\sqrt{2\pi}\,(is+a)^2}$

* $F\left\{\dfrac{\sin ax}{x}\right\} = \begin{cases} \sqrt{\frac{\pi}{2}} & |s| < a \\ 0 & \text{otherwise} \end{cases} \longrightarrow \begin{matrix} s+a>0 \\ s-a<0 \end{matrix}$

(parseval's)

$$\int_0^\infty \left(\frac{\sin ax}{x}\right)^2 dt = \boxed{a\pi}$$

* Fourier Transform of $f(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & \text{other} \end{cases}$

$\downarrow$

$$\sqrt{\frac{2}{\pi}}\,\frac{1}{s^2}(1-\cos s)$$

$$\left.\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \boxed{\frac{\pi}{3}}\right\}$$ | use $s = 2t$; use parseval's identity

* **FOURIER SINE, COSINE Transform**

$$F_s\{f(t)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt = F_s(s)$$

$$F_c\{f(t)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt = F_c(s)$$

Inverse formula :

$$F_S^{-1}[F_S(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S(s) \sin st \, ds$$

$$F_C^{-1}[F_C(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(s) \cos st \, ds$$

Formula :

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{(a^2+b^2)} (a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$$

* $F_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$
  $F_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$ } can also be proved using modulation property.

* If $f(x)$ is even function, $F\{f(x)\} = F_C\{f(x)\}$

* $F\{1\}$ = unit impulse function = $u(x)$

* NOTE : To compute integrals related to parseval's identity ques, square ques in LHS, Fourier transform of ques in RHS & reduce to desired format.

* $$\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}$$

$$\int_0^\infty \frac{x^2 \, dx}{(a^2+x^2)^2} = \frac{\pi}{4a}.$$

$$*\ \int_0^\infty \frac{dx}{(1+x^2)(4+x^2)}$$

i) check if even [$F_c = F$]
ii) compute its $f(x), g(x)$
+ use parseval's 2nd form

Here $\frac{a}{a^2+x^2} \Rightarrow e^{-ax}$

so $\frac{1}{1+x^2} = e^{-x}$; $\frac{1}{4+x^2} = e^{-2x}$

& always use cosine (F_c) for comparison

$$*\ \int_0^\infty \frac{dx}{(1+x^2)(4+x^2)} = \frac{\pi}{12}$$

$$\int_0^\infty \frac{x^2}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}$$

$-\frac{d}{ds}$

$$*\ F_s\{x f(x)\} = -\frac{d}{ds} F_c\{f(x)\}$$

$$F_c\{x f(x)\} = \frac{d}{ds} F_s\{f(x)\}$$

$*\ \sin n\pi = 0$

$\cos n\pi = 1$ if n is even

$\cos n\pi = -1$ if n is odd.

$$*\ F_c\left\{\frac{1}{x^2+1}\right\} = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

$$F_s\left\{\frac{x}{x^2+1}\right\} = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

sol : $\frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t \, dt \, d\lambda$

$\boxed{w_n = \frac{n\pi}{L}}$

* FOURIER INTEGRAL FORMULA

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x)$$

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx] \, dw$$

fourier integral rep.

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv \, dv$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv \, dv$

* Fourier cosine integral

$f(x)$ - even function in $(-\infty, \infty)$

then,

$$f(x) = \int_0^\infty A(w)\cos wx \, dw$$

* where $A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv$

* Fourier sine Integral

$f(x)$ - odd function in $(-\infty, \infty)$

then,

$$f(x) = \int_0^\infty B(w)\sin wx \, dw$$

where

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv \, dv.$$

* Fourier Transform is derived from fourier series with condition that period of the fn is infinite

Imp

* $F_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$

$F_c\{e^{-|x|}\cos 2x\} = \frac{1}{2}[F(s+a) + F(s-a)]$

$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}\right]_{s\to s+a} + \left[\sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}\right]_{s\to s-a}$

$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \frac{1}{1+(s+2)^2} + \sqrt{\frac{2}{\pi}} \frac{1}{1+(s-2)^2}\right]$

* $f(t) = \int_{-\infty}^{\infty} e^{-ist} f(t)\,dt = F(s)$

then, Inverse fourier transform of $F(s)$ is

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} F(s)\,ds}$$

* $F\{e^{-|x|}\cos x\} = \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \frac{1}{1+(s+1)^2} + \sqrt{\frac{2}{\pi}} \frac{1}{1+(s-1)^2}\right]$

then,

$F_s\{e^{-|x|}\cos x\} =$