Integral Transforms

Transforms that involve integration in its process is called Integral Transforms.

General form:

$$T[f(t)] = \int_a^b K(s,t) f(t)\,dt$$

$K(s,t)$ is kernel of transformation

LAPLACE TRANSFORM

$f(t)$ be function with $t \geq 0$,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

* FORMULAS + PROPERTY

1) $L\{e^{at}\} = \dfrac{1}{s-a}$

2) $L\{e^{-at}\} = \dfrac{1}{s+a}$

Prop:

$L\{c_1 f(t) + c_2 g(t)\}$
$= c_1 F(s) + c_2 G(s)$

3) $L\{\cos at\} = \dfrac{s}{s^2+a^2}$

4) $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

To obtain both, compute $L\{e^{iat}\}$ using above formula, simplify e^{iat} as $\cos at + i\sin at$, equat real + imaginary parts.

* $\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$

$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$

$\Downarrow$

$\cosh x = (e^{x} + e^{-x})/2$

$\sinh x = (e^{x} - e^{-x})/2$

5) $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

6) $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$

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7) $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$ & $\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$.

$\Gamma(n+1) = n!$ $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(1) = 1$

8) $L\{1\} = \frac{1}{s}$ | $L\{t\} = \frac{1}{s^2}$ | $L\{t^3\} = \frac{6}{s^4}$

$L\{K\} = \frac{K}{s}$ | $L\{t^2\} = \frac{2}{s^3}$ | $L\{t^4\} = \frac{24}{s^5}$

* $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos 2A = 2\cos^2 A - 1$

$1 = 2\sin^2 A + \cos 2A$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$4\sin^3 x = 3\sin x - \sin 3x$

$4\cos^3 x = \cos 3x + 3\cos x$

Inverse Laplace

$L^{-1}[F(s)] = f(t)$

$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$ $\quad L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$ $\quad L^{-1}\left[\frac{k}{s}\right] =$

$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$ $\quad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$ $\quad L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma(n)}$

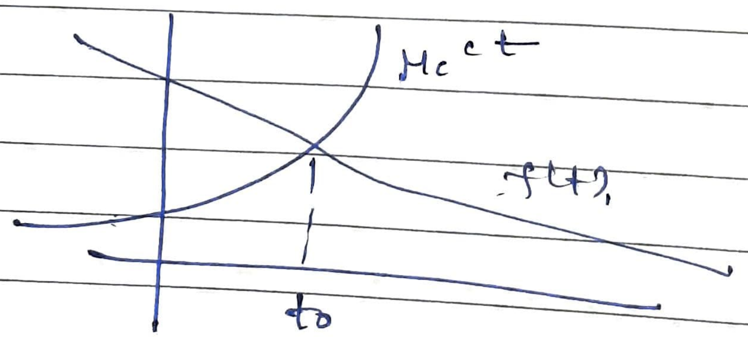
Existence of Laplace

$L\{f(t)\}$ exist iff $\boxed{\int_0^\infty e^{-st} f(t)\, dt < \infty}$

Based on characteristics of $f(t)$, the existence of Laplace can be determined in advance.

* **Exponential order**

↳ $f(t)$ is such if there exist constant c & positive constant t_0 & M such that $|f(t)| < Me^{ct}$ $\forall\ t > t_0$



Exponential order:

t^2, $\sin t$, $\cos t$, $\frac{1}{t}$, e^{2t}

Not Exponential:

e^{t^2}, $\tan t$, $\log t$

* Piecewise continuous

$f(t)$ is PC on $t \in [a, b]$ if interval has finite no: of partitions / points. $a = t_0 < t_1 \dots < t_n = b$ such that

i) f is cts on each open subinterval (t_{i-1}, t_i)

ii) f approaches finite limit as endpts of each subinterval are approached from within the subinterval.

Thus, f is cts on $[a, b]$ except for finite no: of jump discontinuities.

$f(t)$ is PC on $[0, \infty)$ if $f(t)$ is PC on $[0, N]$ $\forall N > 0$

* PC on every finite interval on semi axis $t \geq 0$; exponential order $\Rightarrow$ Laplace exists

$\not\Leftarrow$

(only sufficient, not necessary condition)

$f(t) = \frac{1}{\sqrt{t}}$ $\Rightarrow$ not piecewise but Laplace exist.

* Theorems

1. Shifting : $L\{e^{at} f(t)\} = F(s-a)$

$L^{-1}[F(s-a)] = e^{at} L^{-1}[F(s)]$
$L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]$

2. Transform of derivative

$L\{f'(t)\} = sL\{f(t)\} - f(0)$
$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$
$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$

3. Derivative of Transform

$L\{t f(t)\} = -\frac{d F(s)}{ds} = -F'(s)$

$L\{t^2 f(t)\} = \frac{d^2}{ds^2} L\{f(t)\}$

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}$

$$\boxed{L^{-1}[F'(s)] = -t L^{-1}[F(s)]}$$

(x) $L\{t^5 e^{2t}\}$ => use shifting

$= \left\{\frac{5!}{s^6}\right\}_{s \to (s-2)} = \frac{120}{(s-2)^6}$.

(x) $\ln\left(1+\frac{a^2}{s^2}\right)$

$\ln(s^2+a^2) - 2\ln s = F(s)$

$\frac{2s}{s^2+a^2} - \frac{2}{s} = F'(s)$

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4. <u>Transform of integrals</u>

$L\{f(t)\} = F(s)$, $L\{\int_0^t f(t)\,dt\} = \frac{F(s)}{s}$

$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

$$\boxed{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt}$$

$L^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t \int_0^t L^{-1}[F(s)]\,dt.$

5. <u>Integration of Transform</u>

If $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists , then $L\{\frac{f(t)}{t}\} = \int_s^\infty F(s)\,ds$

Ex: $\left.\begin{array}{l} L\{\frac{\cos t}{t}\} \\ L\{\frac{e^t}{t}\} \end{array}\right\} \Rightarrow$ do not exists

6. <u>Second-shifting Theorem / t-shifting</u>

shifted function $F(s) = f(t-a)u(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t > a \end{cases}$
has the transform $e^{-as}F(s)$

$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s) = e^{-as}L\{f(t)\}$

$L\{f(t)\,u(t-a)\} = e^{-as}L\{f(t+a)\}$

$L^{-1}[e^{-as}F(s)] = u(t-a)\{L^{-1}[F(s)]\}_{t \to (t-a)}$

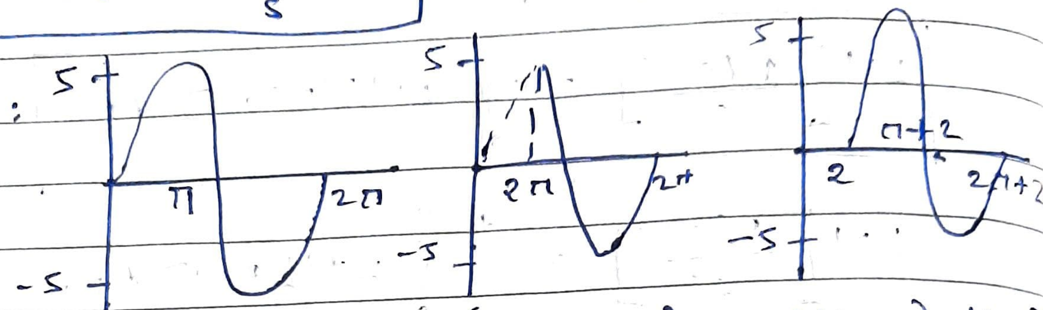
Transform of special functions

* unit-step / heaviside function $= \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

(switch on light in dark room)

$$\boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}}$$

Effect :



$f(t) = 5\sin t$ $\quad$ $f(t)u(t-2)$ $\quad$ $f(t-2)u(t-2)$

Example :

$$f(t) = \begin{cases} 4 & 0 \le t < 8 \\ 6 & t \ge 8 \end{cases} \rightarrow 4 - 4u(t-8) + 6u(t-8)$$

* $\boxed{L\{\delta(t-a)\} = e^{-as}}$ $\quad$ $g(t-a) = \begin{cases} \frac{1}{h} & a < t < a+h \\ 0 & \text{otherwise} \end{cases}$

$$\lim_{h \to 0} g(t-a) = \delta(t-a)$$

CONVOLUTION

↓↓ $\quad$ $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

Blending of 2 functions,

Area of overlapping, Amount of overlapping

* convolution is
i) commutative
ii) distributive
iii) Associative

i) ⇒ $f(t) * g(t) = g(t) * f(t)$

Theorem:

$$L\{f(t) * g(t)\} = L\{f(t)\} L\{g(t)\} = F(s)G(s)$$

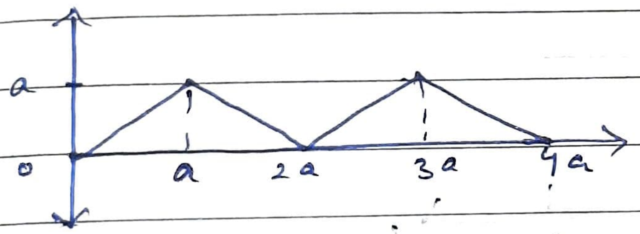
$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[g(s)]$$

PERIODIC FUNCTIONS

→ $f(t)$ → periodic fn with period P if $f(t+p) = f(t)$

→ If $f(t)$ is periodic with period p, then $f(t)$ is of period np $n \in Z^+$

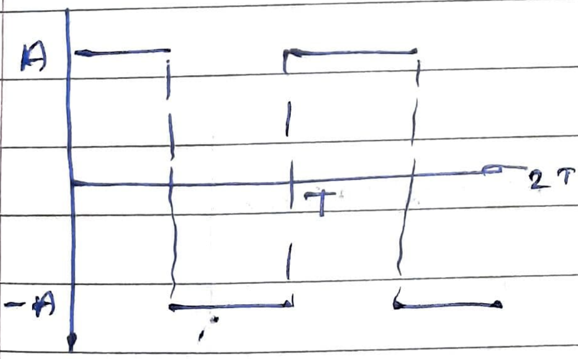
1. TRIANGULAR WAVE FN ⇒ period = 2a;



$$f(t) = \begin{cases} t & 0 \le t \le a \\ 2a - t & a \le t \le 2a \end{cases}$$

$$f(t+2a) = f(t)$$

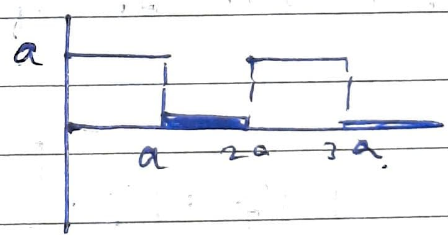
2. RECTANGULAR (T)



$$f(t) = \begin{cases} A & 0 \le t \le T/2 \\ -A & \frac{T}{2} < t < T \end{cases}$$

$$f(t+T) = f(t)$$

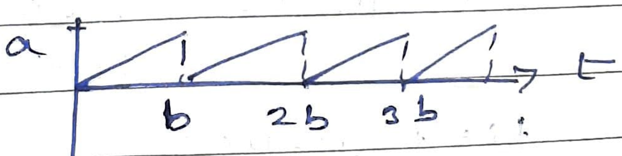
3. SQUARE (2a)



$$f(t) = \begin{cases} a & 0 \le t \le a \\ 0 & a < t < 2a \end{cases}$$

$$f(t+2a) = f(t)$$

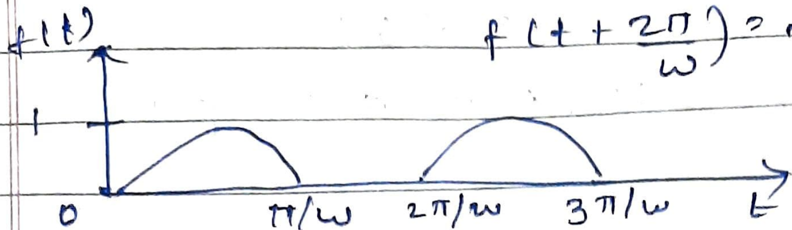
4. SAWTOOTH (b)



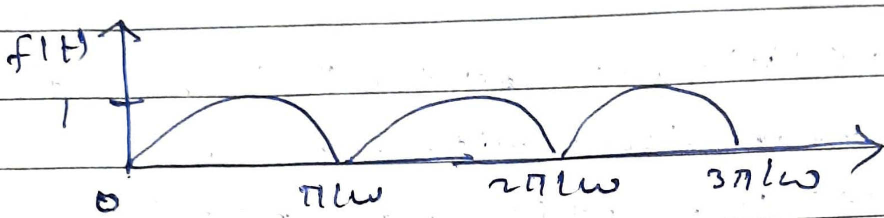
$$f(t) = \frac{a}{b} t \quad \Big\} \quad f(t+b) = f(t)$$

HALF WAVE: $f(t) = \begin{cases} \sin\omega t & 0 \le t \le \pi/\omega \\ 0 & \pi/\omega \le t \le 2\pi/\omega \end{cases}$

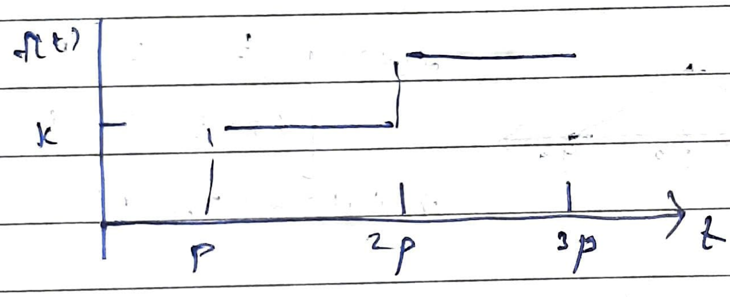
$$f\left(t + \frac{2\pi}{\omega}\right) = f(t)$$



FULL WAVE: $f(t) = \sin\omega t$, $f(t+\pi/\omega) = f(t)$



STAIRCASE



periodic fn. formula

$$L\{f(t)\} = \boxed{\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt}$$

Triangular wave : $\frac{1}{s^2}\left(\tanh\frac{as}{2}\right)$

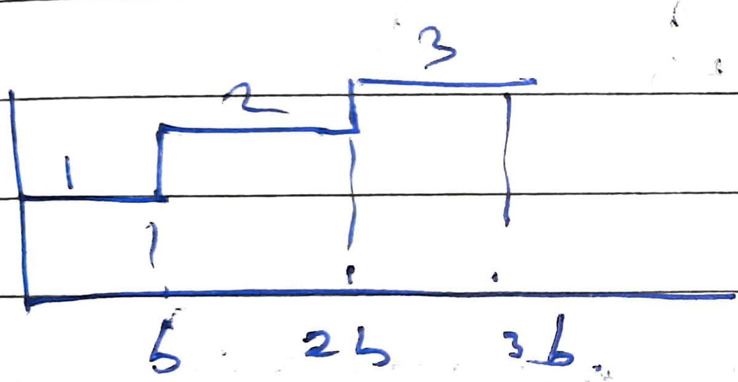
Saw-tooth : $\frac{a/b}{1-e^{-bs}}\left[\frac{1}{s^2} - \frac{be^{-bs}}{s} - \frac{e^{-bs}}{s^2}\right]$

Half wave : $\frac{\omega}{(s^2+\omega^2)(1-e^{-\frac{s\pi}{\omega}})}$

Full-wave: $\frac{\omega}{s^2+\omega^2} \coth\left(\frac{s\pi}{2\omega}\right) \Rightarrow \frac{\omega}{s^2+\omega^2}\left[\frac{e^{\frac{s\pi}{2\omega}}+e^{-\frac{s\pi}{2\omega}}}{e^{\frac{s\pi}{2\omega}}-e^{-\frac{s\pi}{2\omega}}}\right]$

staircase: $x(s) = \int_0^\infty x(t)e^{-st}dt$

$$= \int_0^b e^{-st}dt + \int_b^{2b} 2e^{-st}dt + \int_{2b}^{3b} 3e^{-st}dt$$



$$= \boxed{\frac{1}{s(1-e^{-sb})}}$$

$$\left[\frac{K}{s} \cdot \frac{e^{-ps}}{1-e^{-ps}}\right]$$

square-wave

$$\frac{a}{s(1+e^{-as})}$$

rectangular-wave

$$\frac{A}{s}\tanh\left(\frac{sT}{4}\right)$$