Optimal solution for transportation problem

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IBFS by

(1) North West Corner

(2) Least Cost Method

	L_1	L_2	L_3	L_4	supply
a_1	10	2 $\boxed{15}$	20	11	15
a_2	12	7	9	20	25
a_3	4	14	16	18	10
Demand	5	15	15	15	50

Step 1:

Find the least cost cell and allocate.

When supply = demand

Cross out either row or Column and make other one as 0.

	L_1	L_2	L_3	L_4	
a_1	10	2 $\boxed{15}$	20	11	~~15~~ 0
a_2	12	7	9	20	25
a_3	4	14	16	18	10
	5	~~15~~	15	15	

Step 2

In remaining find least cost & repeat the process

	L_1	L_2	L_3	L_4	
a_1		$\boxed{15}$			~~15~~ 0

a_1	~~10~~	~~2~~	20	11	
a_2	~~12~~	~~7~~	9	20	25
a_3	~~4~~ $\boxed{5}$	~~14~~	16	18	~~15~~ 5
	~~5~~	~~15~~	15	15	

10 − 5

Step 3

~~10~~	~~2~~ $\boxed{15}$	~~20~~	11	~~15~~ 0
~~12~~	~~7~~	~~9~~ $\boxed{15}$	20	~~25~~ 10
~~4~~ $\boxed{5}$	~~14~~	~~16~~	18	~~15~~ 5
~~5~~	~~15~~	~~15~~	15	

25 − 15

Step 4.

~~10~~	~~2~~ $\boxed{15}$	~~20~~	~~11~~ $\boxed{0}$	~~15~~ 0
~~12~~	~~7~~	~~9~~ $\boxed{15}$	20 $\boxed{10}$	~~25~~ ~~10~~
~~4~~ $\boxed{5}$	~~14~~	~~16~~	~~18~~ $\boxed{5}$	~~15~~ ~~5~~
~~5~~	~~15~~	~~15~~	~~15~~	

~~10~~ → 15 − 5

$x_{12} = 15$ ✓ $\quad x_{23} = 15$ ✓

$x_{14} = 0$ ✓ $\quad x_{24} = 10$ ✓

$x_{31} = 5$ ✓ $\quad x_{34} = 5$ ✓

Initial Basic feasible soln is

$$Z = (15 \times 2) + (0 \times 11) + (15 \times 9) + (10 \times 20) + (5 \times 4) + (5 \times 18)$$

$$= \$475$$

IBFS: $Z = 475$

$x_{12} = \ldots\ldots$

3) Vogel's approximation.

Step 1

				Supply	Row Penalty
~~10~~	2	20	11	15	$10-2=(8)$
~~12~~	7	9	20	25	$9-7=(2)$
4 (5)	14	16	18	~~10~~ 5	$14-4=(10)$ ← Largest
~~5~~	15	15	15		
Column Penalty: $10-4=(6)$	$7-2=(5)$	$16-9=(7)$	$18-11=(7)$		

next smallest, smallest

Largest of (6, 5, 7, 7, 8, 2, 10) = 10

Choose largest Penalty (row/col)

In that row/col

Choose min & allocate

Repeat.

Step 2

			Supply	Row Penalty
~~2~~ (15)	20	11	~~15~~ 0	(9) ←
~~7~~	9	20	25	(2)
~~14~~	16	18	5	(2)
Demand: ~~15~~	15	15		
Col Penalty: (5)	(7)	(7)		

Step 3

		Supply	Row Penalty
~~20~~	11	0	(9)
~~9~~ (15)	20	~~25~~ 10	(11) ←
~~16~~	18	5	(2)
Demand: ~~15~~	15		
Col Penalty: (7)	(7)		

(Lar Penalty) Row/Col

In that Row/col smallest.

Step 4

Supply

~~11~~ [0]	~~0~~
~~20~~ [10]	~~10~~
18 [5]	~~5~~

Least cost 11

Demand ~~15~~ ~~5~~

Find Allocation matrix.

10	2 [15]	20	11 [0]
12	7	9 [15]	20 [10]
4 [5]	14	16	18 [5]

Initial Cost is

$= (2\times15) + (15\times9) + (11\times0) + (20\times10) + (4\times5) + (18\times5)$

$= 30 + 135 + 0 + 200 + 20 + 90$

$$\begin{array}{r} 135 \\ 30 \\ 200 \\ 20 \\ 90 \\ \hline 475 \end{array}$$

$z = \$475$

$x_{12} = 15$, $x_{14} = 0$

$x_{23} = 15$, $x_{24} = 10$

$x_{31} = 5$, $x_{34} = 5$

IBFS

IBFS — VAM Better choice

Requirements of Initial Basic Feasible soln.

- Should have $m+n-1$ basic variables
- If not then the transportation will have degenerate solution.

Suppose $m=2, n=3$

			supply
x_1	x_2	x_3	a_1
x_4	x_5	x_6	a_2
demand b_1	b_2	b_3	

$$\begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{matrix}$$

Total supply = Total demand
$b_1+b_2+b_3 = a_1+a_2$

Constraint eqns

$$\left.\begin{aligned}
&① \quad \boxed{x_1 + x_2 + x_3 = a_1} \\
&② \quad x_4 + x_5 + x_6 = a_2 \\
&③ \quad x_1 + x_4 = b_1 \\
&④ \quad x_2 + x_5 = b_2 \\
&⑤ \quad x_3 + x_6 = b_3
\end{aligned}\right\}$$

$m+n$
$m=2, n=3$
$=5$
eqns

Add ③, ④ & ⑤.

$$x_1+x_2+x_3+x_4+x_5+x_6 = b_1+b_2+b_3$$

Subtracting ②,

$$(x_1+x_2+x_3+x_4+x_5+x_6) - (x_4+x_5+x_6) = (b_1+b_2+b_3) - a_2$$

$b_1+b_2+b_3 = a_1+a_2$

$$x_1+x_2+x_3 = (a_1+a_2) - a_2$$

$$x_1+x_2+x_3 = a_1$$

$x_1+x_2+x_3 = a_1$ is redundant.

$m+n-1$ = basic variables

Optimum Solution

Modified distribution Method (MODI method)

(or) $u - v$ - method

eg Given:

				Supply
4	6	8	8	40
6	8	6	7	60
5	7	6	8	50
Demand: 20	30	50	50	150

$\therefore$ Supply = Demand

Balanced transportation problem.

Step 1

Initial Basic feasible Solution

VAM is,

4 [20]	6 [20]	8	8	40
6	8	6 [50]	7 [10]	60
5	7 [10]	6	8 [40]	50
20	30	50	50	

$\boxed{Z = 960.}$

Step 2:

Computation of u_i & v_j

For each current basic variable x_{ij}, compute u_i & v_j using

$$\boxed{u_i + v_j = c_{ij}}$$

	$v_1=4$	$v_2=6$	$v_3=6$	$v_4=7$
$u_1=0$	[20] 4	[20] 6	8	8
			[50]	[10]

1) $u_1 + v_1 = 4 \Rightarrow v_1 = 4$

$u_2 = 0$	6	8	5	7
$u_3 = 1$	5	$\boxed{20}$ 7	6	$\boxed{40}$ 8

$u_1 + v_2 = 6 \Rightarrow v_2 = 6$

$\cancel{u_2} + v_3 = 6 \Rightarrow v_3 = 6$

$u_2 + \cancel{v_4} = 7 \Rightarrow u_2 = 0$

$u_3 + \cancel{v_4} = 7 \Rightarrow u_3 = 1$

$1\ \cancel{u_3} + v_4 = 8 \Rightarrow v_4 = 7$

For each non basic Variable x_{ij}

Compute $\boxed{C_{ij} - (u_i + v_j)}$

Non basic Variables are

$x_{13}, x_{14}, x_{21}, x_{22}, x_{31}, x_{33}$

① for x_{13} $i=1, j=3.$

$$c_{13} - (u_1 + v_3) = 8 - (0+6) = 2$$

② for x_{14} $i=1, j=4$

$$c_{14} - (u_1 + v_4) = 8 - (0+7) = 1$$

③ for x_{21} $i=2, j=1$

$$c_{21} - (u_2 + v_1) = 6 - (0+4) = 2$$

④ for x_{22} $i=2, j=2$

$$c_{22} - (u_2 + v_2) = 8 - (0+6) = 2$$

⑤ for x_{31} $i=3, j=1$

$$c_{31} - (u_3 + v_1) = 5 - (1+4) = 0$$

6) for x_{33}

$i=3, j=3$

$$c_{33} - (u_3 + v_3) = 6 - (1+6) = -1$$

The matrix is represented as,

$\boxed{28}$ 4	$\boxed{28}$ 6	(2) 8	(1) 8
(2)	(2)	$\boxed{50}$	$\boxed{10}$

(2) 6	8	6	7
(0) 5	[10] 7	(−1) 6	[40] 8

negative

Most negative is −1 associated with variable x_{33}.

$\therefore x_{33}$ enters the basis.

To decide the leaving Variable

Assign 'θ' to entering variable x_{33}.

Construct a closed loop that starts & ends at x_{33}.

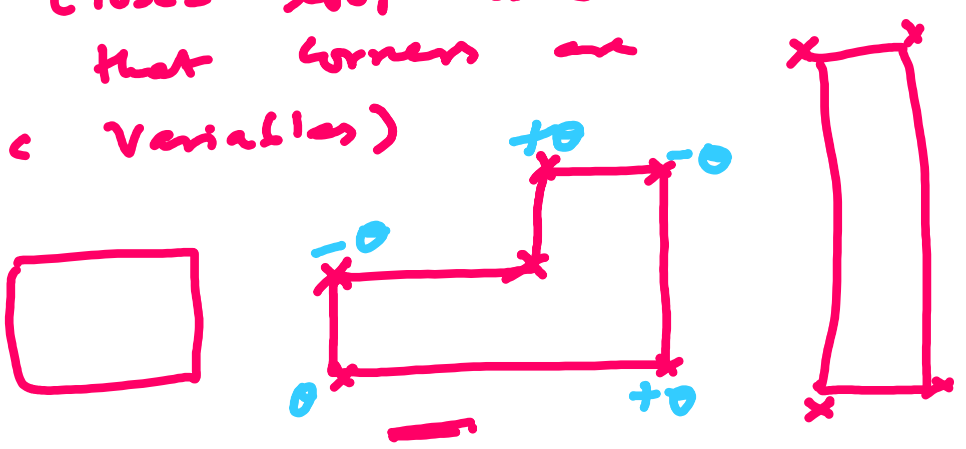
(Horizontal & vertical lines connecting basic variables).

– only one loop exists.

	a	b	c	d	Supply
1	[20] 4	[20] 6	(2) 8	(1) 8	40
2	(2) 6	(2) 8	−θ [50]	[10] +θ 7	60
3	(0) 5	[10] 7	(−1) θ 6	[40] −θ 8	50
Demand	20	30	50	50	

x_{11}
x_{12}
x_{23}
x_{24}
x_{32}
x_{34}
x_{33}

(The closed loop is drawn such that corners are basic variables)



Consider min of allocated values in −θ cells.

(iv) min (40, 50) = 40.

Corresponding Variable x_{34} is leaving Variable.

Assign $\theta = 40$,

the reallocated matrix is,

$\boxed{20}$ 4	$\boxed{20}$ 6	(2) 8	(1) 8	40
(2) 6	(2) 8	$\boxed{10}$ 6	$\boxed{50}$ 7	60
(0) 5	$\boxed{10}$ 7	$\boxed{40}$ 6	8	50
20	30	50	50	

50 − 40 = 10

10 + 40 = 50

40 − 40 Remove allocation

New Basic feasible soln is

$x_{11} = 20$, $x_{12} = 20$, $x_{23} = 10$,
$x_{24} = 50$ $x_{32} = 10$ $x_{33} = 40$

$Z = (20 \times 4) + (20 \times 6) + (10 \times 6) + (50 \times 7) + (7 \times 10) + (6 \times 40)$

$= 920$

Iteration 2

Recomputing u & v

	$v_1 = 4$	$v_2 = 6$	$v_3 = 5$	$v_4 = 6$
$u_1 = 0$	$\boxed{20}$ 4	$\boxed{20}$ 6	(3) 8	(2) 8
$u_2 = 1$	(1) 6	(1) 8	$\boxed{10}$ 6	$\boxed{50}$ 7
$u_3 = 1$	$\boxed{(0)}$ 5	$\boxed{10}$ 7	$\boxed{40}$ 6	(1) 8

0 at optimum level

$u_1 + v_1 = 4$
$u_1 + v_2 = 6$
$u_2 + v_3 = 6$
$u_2 + v_4 = 7$
$u_3 + v_2 = 7$
$u_3 + v_3 = 6$

For each non basic Variable x_{ij} compute $C_{ij} - (u_i + v_j)$

Non basic variables : $x_{13}, x_{14}, x_{21}, x_{22}$
x_{31}, x_{34}

① $i=1, j=3$

$c_{13} - (u_1 + v_3)$
$= 8 - (0+5) = 3$ ✓

② $i=1, j=4$: $c_{14} - (u_1 + v_4)$
$= 8 - (0+6) = 2$ ✓

③ $i=2, j=1$: $c_{21} - (u_2 + v_1)$
$= 6 - (1+4) = 1$

④ $i=2, j=2$: $c_{22} - (u_2 + v_2)$
$= 8 - (1+6) = 1$

⑤ $i=3, j=1$: $c_{31} - (u_3 + v_1)$
$= 5 - (1+4) = 0$

⑥ $i=3, j=4$: $c_{34} - (u_3 + v_4)$
$= 8 - (1+6)$
$= 1$

All $c_{ij} - (u_i + v_j)$ $\boxed{\geq 0}$

$\therefore$ optimal is reached.

Soln $x_{11} = 20$, $x_{12} = 20$
$x_{23} = 10$, $x_{24} = 50$
$x_{32} = 10$, $x_{33} = 40$

Min cost $Z = 920$

Note:

one non basic variable is 0 at optimum level.
There exists alternative optimum exists.

Allocate 'θ' to non basic variable with zero. Form closed loop with starting cell with θ.

-θ [20] 4	+θ [20] 6	(3) 8	(2) 8	40
(1) 6	(1) 8	[10] 6	[50] 7	60
(0) θ 5	-θ [10] 7	[40] 6	(1) 8	50
20	30	50	50	

Min. of '-θ' allocated cells = (20, 10) = 10

∴ θ = 10.

New allocation is,

20-10　　20+10

[10] 4	[30] 6			40
		[10] 6	[50] 7	60
[10] 5	–	[40] 6		50

10-10 =0 deallocate

The min cost = (10×4) + (30×6) + (10×6) + (50×7) + (10×5) + (40×6)

$x_{11} = 10$　$x_{31} = 10$
$x_{12} = 30$　$x_{33} = 40$
$x_{23} = 10$
$x_{24} = 50$

= 40 + 180 + 60 + 350 + 50 + 240

= 920

Degeneracy:

1) obtain IBFS

2) If the number of Positive allocation (>0) is not equal to

m+n−1, where

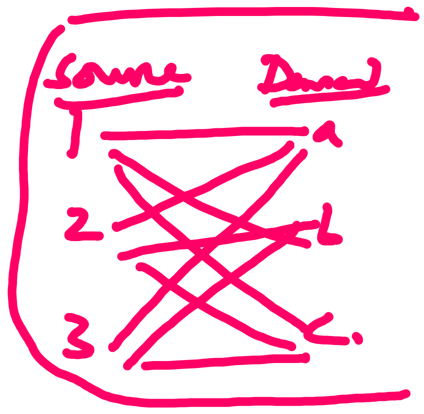
m – no. of supply variables

n – no. of demand "

then the solution is said to be degenerate.

Example :

	a	b	c	Supply
1	8	7	3	60
2	3	8	9	70
3	11	3	5	80
Demand	50	80	80	210



Step 1 Find IBFS by VAM

(i)

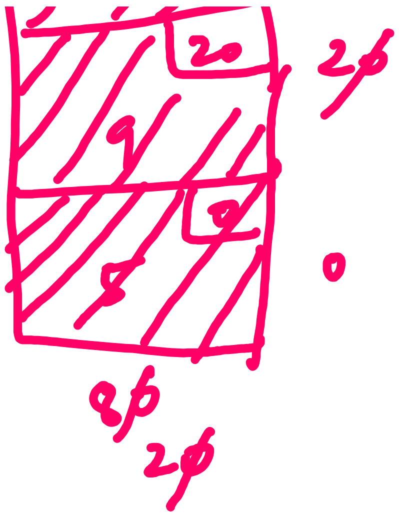
				Row Penalty
8	7	3	60	(4) — 7−3
3 [50]	8	9	~~70~~ 20	(5)
11	3	5	80	(2)
(5)	(4)	(2)		
~~50~~	80	80		

(ii)

7	3	60	(4)
8	9	20	(1)
3 [80]	5	~~80~~ 0	(2)
~~80~~	80		
(4)	(2)		

(iii)

3	[60]	~~60~~



Allocated matrix (without 0 allocation)

8	7	3 [60]
3 [50]	8	9 [20]
11	3 [80]	5

IBFS is,

$x_{13} = 60$ $\quad x_{21} = 50$

$x_{23} = 20$ $\quad x_{32} = 80$

$x_{33} = 0$

Min cost $Z = (60 \times 3) + (50 \times 3) + (20 \times 9) + (80 \times 3) + (5 \times 0)$

$= 180 + 150 + 180 + 240$

$= 750$

No. of allocations $= 4$ (without zero allocation)

$m + n - 1 = 3 + 3 - 1 = 5$

No. of alloc $\neq m + n - 1$

$\therefore$ Solution degenerates.

Non allocated cells are

x_{11} ✓

x_{12} ✓

x_{22} ✓

x_{31} ✓

x_{33} ✓

Check whether closed loops can be drawn from the cell. If loop can't be drawn then the cell is called independent cell.

From the above non allocated cell, independent cells are

x_{11} – ✗

x_{12} – ✓

x_{22} – ✓

x_{31} – ✓

x_{33} ✓

loop can't be drawn

min cost variable $\{x_{12}, x_{22}, x_{31}, x_{33}\}$
$= \{7, 8, 11, 5\} = 5$

(e) min cost independent variable is $\underline{x_{33}}$

Allocate a small negligible quantity ϵ (>0, but for calculation purpose we can take $\epsilon \to 0$)

	$v_1=-3$	$v_2=1$	$v_3=3$	
$u_1=0$	8	7	3 [60]	60
$u_2=6$	3 [50]	8	7 [20]	70
$u_3=2$	11	3 [80]	5 [ϵ]	80
	50	80	80	

allocated in min cost ind. cell.

$\overset{0}{u_1} + v_3 = 3$

$\overset{6}{u_2} + v_1 = 3 \Rightarrow v_1 = 3-6=-3$

$u_2 + \cancel{v_3}^{3} = 9 \Rightarrow u_2 = 6$

$\cancel{u_3}^{2} + v_2 = 3 \Rightarrow v_2 = 1$

$u_3 + \cancel{v_3}^{3} = 5 \Rightarrow u_3 = 5-3=2$

Find $c_{ij} - (u_i + v_j)$ for non allocated cells.

i) x_{11} $c_{11} - (u_1 + v_1) = 11$

ii) x_{12} $c_{12} - (u_1 + v_2) = 6$

iii) x_{22} $c_{22} - (u_2 + v_2) = 1$

iv) x_{31} $c_{31} - (u_3 + v_1) = 12$

All $c_{ij} - (u_i + v_j) \geq 0$, so optimum is reached.

$\therefore$ Soln is,

$x_{13} = 60$ $x_{32} = 80$

$x_{21} = 50$ $\boxed{x_{33} = 6 \to 0 \\ (\text{or}) \; x_{33} = 0}$

$x_{23} = 20$

$\therefore Z = (60 \times 3) + (50 \times 3) + (20 \times 9) + (80 \times 3) + (0 \times 5)$

$= 180 + 150 + 180 + 240 + 0$

$= 750$

Un Balanced Transportation Problem

Total supply $\neq$ Total demand

Supply

Q

				Supply
	8	7	3	60
	3	8	9	70
	11	3	5	85
Demand	50	80	80	210 / 215

Total Supply = 215

" Demand = 210

Tot. Supply ≠ Tot. demand.

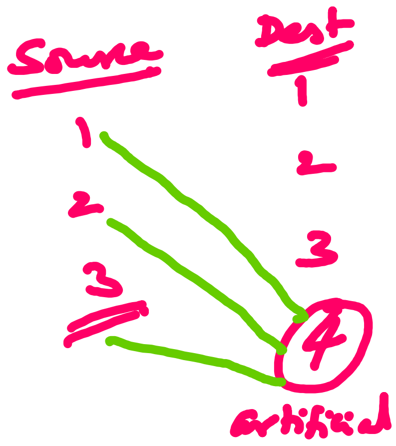
∴ Unbalanced Transportation

↓

Balanced

				Supply
8	7	3	0	60
3	8	9	0	70
11	3	5	0	85
50	80	80	5	215 / 215

Additional Demand



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Balanced transpn

Column to added } when Supply > Demand.

Row is added } when demand > supply