TRANSFORM OF DERIVATIVES (HW)

1) $y'' + 7y = 10e^{2t}$ $\quad y(0) = 0 \quad y'(0) = 3$

$s^2 L(y) - sf(0) - f'(0) + 7L(y) = L(10e^{2t})$

$(s^2+7)L(y) - 3 = 10/s-2$

$(s^2+7)L(y) = \frac{10}{s-2} + 3$

$L(y) = \frac{10}{(s-2)(s^2+7)} + \frac{3}{s^2+7}$

$\frac{10}{(s-2)(s^2+7)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+7}$

$10 = A(s^2+7) + Bs+C(s-2)$

At $s=2$; $\quad 10 = A(11) \longrightarrow \boxed{A = 10/11}$

At $s=0$;

$10 = 7A + C(-2) \rightarrow -2C = \frac{110-70}{11}$

$C = \frac{-40}{11 \times 2} \rightarrow \boxed{C = -20/11}$

At $s=1$;

$10 = 8A + -(B+C) \rightarrow$ $\quad 8A - B - C = 10$

$\frac{80}{11} + \frac{20}{11} - 10 = B$

$\Rightarrow \boxed{B = -10/11}$

$\therefore y = L^{-1}\left[\frac{10}{11}\left(\frac{1}{s-2}\right) - \frac{10}{11}\left(\frac{s}{s^2+1}\right) - \frac{20}{11}\left(\frac{1}{s^2+7}\right)\right].$

$+ \frac{3}{s^2+7}.$

$$= \frac{10}{11} e^{2t} - \frac{10}{11} \cos\sqrt{7}t - \frac{20}{11\sqrt{7}} \sin\sqrt{7}t + \frac{3}{\sqrt{7}} \sin\sqrt{7}t$$

$$= \boxed{\frac{10}{11} e^{2t} - \frac{10}{11} \cos\sqrt{7}t - \frac{13}{11\sqrt{7}} \sin\sqrt{7}t.}$$

2) $y'' - 3y' - 10y = 2 \quad y(0) = 1 \quad y'(0) = 2$

$$L(y'') - 3L(y') - 10L(y) = L(2)$$

$$s^2 L(y) - s y(0) - y'(0) - 3 L(y') - 10 L(y) = L(2)$$

$$\left. \begin{array}{l} s^2 L(y) - s y(0) - y'(0) - 3 s L(y) + 3 y(0) \\ \quad - 10 L(y) \end{array} \right\} = 2/s$$

$$(s^2 - 3s - 10) L(y) - s - 2 + 3 = \frac{2}{s}$$

$$(s^2 - 3s - 10) L(y) = \frac{2}{s} + s - 1$$

$$L(y) = \frac{2}{s(s-5)(s+2)} + \frac{s-1}{(s-5)(s+2)}$$

$$= \frac{2 + s^2 - s}{s(s-5)(s+2)}$$

$$\frac{2 + s^2 - s}{s(s-5)(s-2)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s-2}$$

$$2 + s^2 - s = A(s-5)(s+2) + Bs(s+2) + Cs(s-5)$$

at $s = 0$; $\quad 2 = A(-10) \rightarrow \boxed{A = -1/5}$

$\boxed{C = 4/7}$

at $s = -2$; $\quad 2 + 4 + 2 = -2C(-7) \rightarrow \boxed{\cancel{C = 4/7}}$

at $s = 5$; $\quad 2 + 25 - 5 = 5B(7) \rightarrow \boxed{B = 22/35}$

$$\therefore y = L^{-1}\left[\frac{-1}{5s} + \frac{4}{7(s+2)} + \frac{22}{35(s+5)}\right]$$

$$= \cancel{-\frac{1}{5} + \frac{5}{7}e^{-5t} + \frac{22}{35}e}$$

$$y = \boxed{-\frac{1}{5} + \frac{22}{35}e^{-5t} + \frac{4}{7}e^{-2t}}$$

3) $y'' - y' - 2y = e^{2t}$, $y(0) = 0$ $y'(0) = 1$.

$L(y'') - L(y') - 2L(y) = L(e^{2t})$

$$\left.\begin{array}{l} s^2L(y) - \cancel{sf+y}\ sy(0) - y'(0) \\ -2L(y) - sL(y) + y(0) \end{array}\right\} = \frac{1}{s-2}$$

$$(s^2 - s - 2)L(y) - 1 = \frac{1}{s-2}$$

$$(s^2 - s - 2)L(y) = \frac{1}{s-2} + 1$$

$$L(y) = \frac{1}{(s-2)^2(s+1)} + \frac{1}{(s-2)(s+1)} = \frac{s-1}{(s-2)^2(s+1)}$$

$$\frac{s-1}{(s-2)^2(s+1)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{(s+1)}$$

$$s-1 = A(s-2)(s+1) + B(s+1) + C(s-2)^2$$

at $s = -1$; $-2 = C(-3)^2 \rightarrow \boxed{C = -2/9}$

at $s = 2$; $1 = 3B \rightarrow \boxed{B = 1/3}$

at $s = 0$; $-1 = A(-2)(1) + B + 4C$.

$2A = \frac{1}{3} - \frac{8}{9} + 1. \rightarrow \boxed{A = 2/9}$

$$= \frac{2}{9}\left(\frac{1}{s-2}\right) + \frac{1}{3}\left(\frac{1}{(s-2)^2}\right) + \frac{-2}{9}\left(\frac{1}{s+1}\right)$$

$$y = L^{-1}\left[\frac{2}{9}\left(\frac{1}{s-2}\right)\right] + \frac{1}{3} L^{-1}\left[\frac{1}{(s-2)^2}\right] - \frac{2}{9} L^{-1}\left[\frac{1}{s+1}\right]$$

$$= \frac{2}{9}e^{2t} + \frac{e^{2t}}{3} L^{-1}\left(\frac{1}{s^2}\right) - \frac{2}{9}e^{-t}$$

$$= \boxed{\frac{2}{9}e^{2t} + \frac{e^{2t}}{3}t - \frac{2}{9}e^{-t}}$$

4) $y'' - 2y' + 2y = e^{-t}$ $\quad y(0) = 0 \quad y'(0) = 1$

$$L(y'') - 2L(y') + 2L(y) = L(e^{-t})$$

$$\left.\begin{array}{l} s^2L(y) - sy(0) - y'(0) - \\ 2sL(y) + 2y(0) \end{array}\right\} = L(e^{-t})$$

$$\cancel{s^2 - 2s + 2} \quad (s^2 - 2s + 2)L(y) - (s+2)y(0) - y'(0) = \frac{1}{s+1}$$

$$L(y) = \left(\frac{1}{s+1} + 1\right)\frac{1}{s^2 - 2s + 2} = \frac{s+2}{(s+1)s^2 - 2s + 2}$$

$$\frac{s+2}{(s+1)(s^2 - 2s + 2)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 2}$$

$$s + 2 = A(s^2 - 2s + 2) + Bs + C(s+1)$$

at $s = -1$; $\quad 1 = A(1 + 2 + 2) \rightarrow \boxed{A = 1/5}$

at $s = 0$; $\quad 2 = 2A + C \rightarrow C = 2 - \frac{2}{5}$ $\quad \boxed{C = 8/5}$

at $s = 1$; $\quad 3 = A(1) + 2B + 2C$

$$\frac{1}{5} + 2B + \frac{16}{5} = 3 \rightarrow B = \frac{3 - 17}{2 \cdot 10} = \frac{15 - 17}{10}$$

$$\boxed{B = -1/5}$$

$$\therefore y = L^{-1}\left[\frac{1}{5(s+1)} + \frac{(-)s}{5(s^2-2s+2)} + \frac{8}{5(s^2-2s+2)}\right]$$

$$y = L^{-1}\left(\frac{1}{5(s+1)}\right) - \frac{1}{5}L^{-1}\left(\frac{s}{s^2-2s+2}\right) + \frac{8}{5}L^{-1}\left(\frac{1}{s^2-2s+2}\right)$$

$$= \frac{e^{-t}}{5} - \frac{1}{5}L^{-1}\left[\frac{s-1+1}{(s-1)^2-1+2}\right] + \frac{8}{5}L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= \frac{e^{-t}}{5} - \frac{1}{5}L^{-1}\left[\frac{(s-1)}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}\right] + \frac{8}{5}e^{t}\sin t$$

$$= \frac{e^{-t}}{5} - \frac{e^{t}\cos t}{5} - \frac{e^{t}\sin t}{5} + \frac{8}{5}e^{t}\sin t.$$

$$= \boxed{\frac{e^{-t}}{5} - \frac{e^{t}\cos t}{5} + \frac{7}{5}e^{t}\sin t.}$$

5) $y'' + y = \cos 2t \qquad y(0) = 2,\ y'(0) = 1$

$$L(y'') + L(y) = L(\cos 2t)$$

$$s^2L(y) - sy(0) - y'(0) + L(y) = \frac{s}{s^2+4}$$

$$(s^2+1)L(y) - 2s - 1 = \frac{s}{s^2+4}$$

$$(s^2+1)L(y) = \frac{s}{s^2+4} + 2s + 1$$

$$L(y) = \frac{s}{(s^2+4)(s^2+1)} + \frac{2s+1}{(s^2+1)}$$

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$s = \cancel{A}\,s(s^2+1) + Bs + c(s^2+4)$$

~~at s = -4;~~

~~$-4 = A(16+1) \rightarrow \boxed{A = -4/17}$~~

~~at $s = 0$; $A + 4C = 0$~~

$s = As + B(s^2+1) + (Cs+D)\, s^2+4$

$A + C = 0 \rightarrow A = -C.$

$A - 4C = 1 \rightarrow 3C = 1 \quad \boxed{C = 1/3}$

$\boxed{A = -1/3} \quad B + D = 0, \; B + 4D = 0 \rightarrow \boxed{B = 0} \quad \boxed{D = 0}$

$\therefore \dfrac{s}{(s^2+4)(s^2+1)} = \dfrac{-1}{3}\dfrac{s}{s^2+4} + \dfrac{1}{3}\dfrac{s}{s^2+1}$

$y = L^{-1}\left(\dfrac{-1}{3}\dfrac{s}{s^2+4} + \dfrac{1}{3}\dfrac{s}{s^2+1} + 2\dfrac{s}{s^2+1} + \dfrac{1}{s^2+1} \right)$

$= \boxed{-\frac{1}{3}\cos 2t + \frac{1}{3}\cos t + 2\cos t + \sin t}$