1) $y'' + 9y = r(t)$, $r(t) = 8\sin t$ if $0 < t < \pi$ and 0 if $t > \pi$, $y(0) = 0$ $y'(0) = 4$.

$$r(t) = \begin{cases} 0 < t < \pi & 8\sin t \\ t > \pi & 0 \end{cases}$$

$y(0) = 0$; $y'(0) = 4$, $r(t) = 8\sin t\,[1 - u(t-\pi)]$

$$L(y'') + 9L(y) = L[r(t)]$$

$$s^2 L(y) - sy(0) - y'(0) + 9L(y)$$

$$= \frac{8}{s^2+1} - e^{-\pi s} L(8\sin(t+\pi))$$

$$= \frac{8}{s^2+1} - 8e^{-\pi s} L(-\sin t)$$

$$= \frac{8}{s^2+1} + \frac{8e^{-\pi s}}{s^2+1}$$

$$(s^2+9)L(y) - 4 = \frac{8}{s^2+1} + \frac{8e^{-\pi s}}{s^2+1}$$

$$L(y) = \frac{8}{(s^2+9)(s^2+1)} + \frac{8e^{-\pi s}}{(s^2+9)(s^2+1)} + \frac{4}{s^2+9}$$

$$= \left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) + \frac{4}{s^2+9}$$

$$y = L^{-1}\left\{\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) + \frac{4}{s^2+9}\right\}$$

$= \sin t - \frac{\sin 3t}{3} + u(t-\pi)\sin(t-\pi)$.

$+ \frac{4}{3}\sin 3t \quad - \frac{1}{3}\sin 3(t-\pi)$

$= \sin t + \sin 3t + u(t-\pi)\left[\sin(t-\pi) - \frac{1}{3}\sin 3(t-\pi)\right]$

$\therefore y = (\sin t + \sin 3t) +$

$u(t-\pi)\left[\sin(t-\pi) - \frac{1}{3}\sin 3(t-\pi)\right]$

$$g(t) = \begin{cases} \sin t + \sin 3t & 0 < t < \pi \\ \frac{4}{3}\sin 3t & t > \pi \end{cases}$$

2) $y'' + y = r(t)$ $\quad r(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 0 & \text{if } t > 1 \end{cases}$

$y(0) = y'(0) = 0$.

$\therefore r(t) = -u(t-1) + t$.

$L(y'') + L(y) = L\{r(t)\} = L\{t - u(t-1)\}$

$s^2 L(y) - sy(0) - y'(0) + L(y) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} - \frac{1}{s}\right)$

$(s^2+1)L(y) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$

$L(y) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}$

$L(y) = \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$

$y = L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1} - e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\}$

$= t - \sin t - u(t-1)(t-1) +$

$u(t-1)\sin(t-1) - u(t-1) +$

$u(t-1)\cos(t-1)$

$= t - \sin t - u(t-1)\,[t - \sin(t-1) - \cos(t-1)]$

$$\therefore g(t) = \begin{cases} t - \sin t & 0 < t < 1 \\ -\sin t + \sin(t-1) + \cos(t-1) & t > 1 \end{cases}$$

3) $y'' + y = \delta(t - 2\pi) \quad y(0) = 10 \quad y'(0) = 0.$

$L\{y''\} + L\{y\} = L\{\delta(t-2\pi)\}$

$s^2 L\{y\} - s y(0) - y'(0) + L\{y\} = e^{-2\pi s}$

$(s^2 + 1)\, L\{y\} = e^{-2\pi s} + 10 s$

$$L\{y\} = \frac{e^{-2\pi s}}{s^2+1} + \frac{10 s}{s^2+1}$$

$$y = L^{-1}\left[\frac{e^{-2\pi s}}{s^2+1}\right] + L^{-1}\left[\frac{10 s}{s^2+1}\right]$$

$= u(t-2\pi)\sin(t-2\pi) + 10\cos t$

$y = 10\cos t - u(t-2\pi)[-\sin t]$

$$\therefore g(t) = \begin{cases} 10\cos t & 0 < t < 2\pi \\ 10\cos t + \sin t & t > 2\pi \end{cases}$$

4)

$$y'' + 2y' + 5y = 25t - 100\,\delta(t-\pi),\quad y(0) = -2,\ y'(0) = 5$$

$$\left.\begin{array}{l} s^2 L\{y\} - s y(0) - y'(0) + 2sL(y) - 2 \\ 5L(y) \end{array}\right] = \frac{25}{s^2} - 100e^{-\pi s}$$

$$L(y)\left[s^2 + 2s + 5\right] - s(-2) - 5 - 2(-2) = \frac{25}{s^2} - 100e^{-\pi s}$$

$$L(y) = \frac{25}{s^2(s^2+2s+5)} - \frac{2s}{s^2+2s+5} - \frac{100e^{-\pi s}}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

$$y = L^{-1}\left(\frac{25}{s^2(s^2+2s+5)}\right) + L^{-1}\left(\frac{1}{s^2+2s+5}\right) - 2L^{-1}\left(\frac{s}{s^2+2s+5}\right) - 100L^{-1}\left(\frac{e^{-\pi s}}{s^2+2s+5}\right).$$

$$\frac{25}{s^2(s^2+2s+5)} = \frac{-2}{s} + \frac{5}{s^2} + \frac{2(s+1) - Q}{(s-1)^2 + 4}$$

$$L^{-1}\left(\frac{25}{s^2(s^2+2s+5)}\right) = 5t - 2 + e^{-t}\left(2\cos 2t - \frac{Q}{2}\sin 2t\right) \quad — (1)$$

$$100L^{-1}\left[\frac{e^{-\pi s}}{s^2+2s+5}\right] = 50U(t-\pi)e^{-(t-\pi)}\sin 2t \quad — (2)$$

$$L^{-1}\left[\frac{1}{s^2+2s+5}\right] = L^{-1}\left[\frac{1}{(s+1)^2+4}\right] = \frac{e^{-t}}{2}\sin 2t \quad — (3)$$

$$L^{-1}\left[\frac{2s}{s^2+2s+5}\right] = L^{-1}\left[\frac{2(s+1)-2}{(s+1)^2+4}\right] = e^{-t}(2\cos 2t - \sin 2t) \quad — (4)$$

$\Rightarrow$ (1) − (4) + (2) + (3) = (1) − (2) + (3) − (4)

$$\boxed{= 5t - 2 - 50U(t-\pi)e^{-t+\pi}\sin 2t}$$