Decision Analysis

- Choosing the best from the given set of alternatives

I) Decision making in Certainty

Using Analytic Hiravchy Process (AHP)

Eg

Martin have a Choice of doing higher studies in Institutions A, B, C.

His choice is based on two criteria.

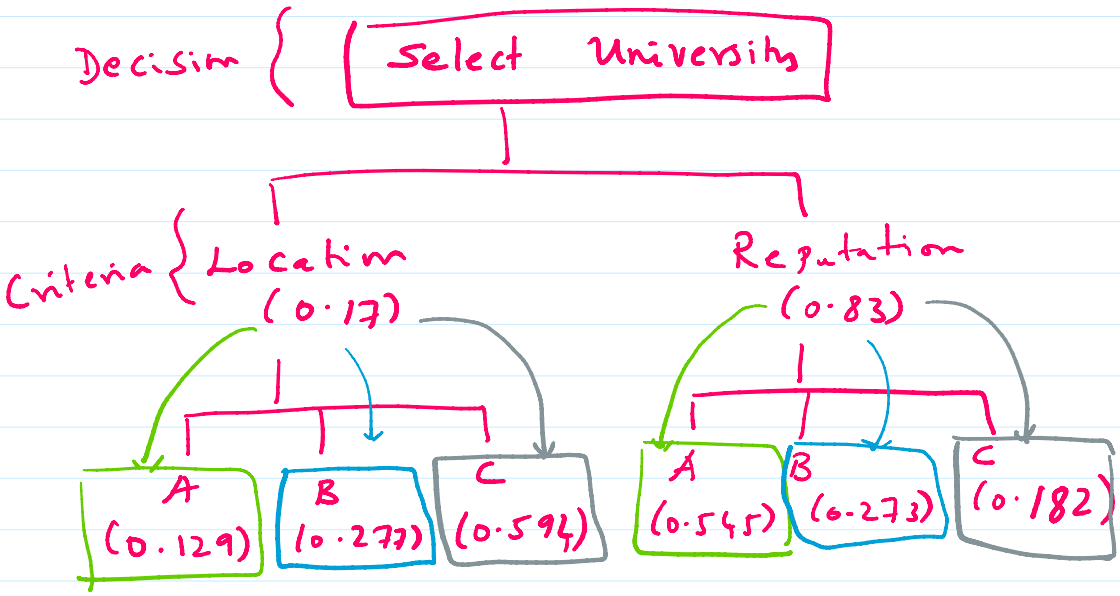
(i) Reputation
(ii) Location

The weights assign to
reputation is 83%
Location is 17%

A systematic Procedure is used to find weights and given as follows.

Criteria	University A	B	C
Location	12.9	27.7	59.4
Reputation	54.5	27.3	18.2

Find Martin's decision



To find Ranks

Weightage of university A $= (0.17 \times 0.129 + 0.83 \times 0.545$

$= 0.474$

$\therefore$ B $= (0.17 \times 0.277) + (0.83 \times 0.273)$

$= 0.273$

C $= (0.17 \times 0.594) + (0.83 \times 0.182)$

$= 0.250$

Largest weightage is 0.474

$\therefore$ Corresponding University is A

Martin will select university A

To determine weights

Comparison matrix A $\rightarrow n \times n$

No. of Criteria.

Comparison matrix A → n×n

where 'n' – number of Criteria.

a_{ij} – element of A

→ Scale of numerical Value from 1 to 9

a_{ij} = 1 i, j are equal

= 5 i is strongly more important than j.

= 9 i is extremely more important than j.

Other Values are intreprected Correspondingly.

Suppose Martin give a rating Reputation is strongly more important than Location (or) 5. [R–L is 5]

i) For Criteria: Reputation (R)
Location (L)

Comparism matrix

$$A = \begin{array}{c} \\ L \\ R \end{array} \begin{pmatrix} L & R \\ 1 & 1/5 \\ 5 & 1 \end{pmatrix}$$

L–L, L–R, R–L, R R

L – R is inverse of R–L

$$L\,R = \frac{1}{RL} = \frac{1}{5}$$

Normalization Matrix

Normalization matrix

$$N = \begin{pmatrix} 1/6 & 1/5 \\ 5/6 & 1 \end{pmatrix} \quad \begin{matrix} 1/5 \\ 1.2 \end{matrix} \quad \begin{pmatrix} 1 & 1/5 \\ 5 & 1 \end{pmatrix}$$

Col sum → 6 → $\frac{1}{5}+1 = 1.2$

[Each element is divided by corresponding col. sum]

Consistent matrix (All cols are equal)

$$N = \begin{matrix} \\ L \\ R \end{matrix} \begin{matrix} \begin{matrix} L & R \end{matrix} \\ \begin{pmatrix} 0.17 & 0.17 \\ 0.83 & 0.83 \end{pmatrix} \end{matrix} \quad \begin{matrix} \text{Row average} \\ W_L = \frac{0.17+0.17}{2} = 0.17 \\ W_R = \frac{0.83+0.83}{2} = 0.83 \end{matrix}$$

Weight $W_L = 0.17$ $W_R = 0.83$ for Criteria

To find weights for alternatives

$$A_L = \begin{matrix} \\ A \\ B \\ C \end{matrix} \begin{matrix} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} 1 & 1/2 & 1/5 \\ 2 & 1 & 1/2 \\ 5 & 2 & 1 \end{pmatrix} \end{matrix}$$

(University A Location criteria)

C to A is 5

B - A = 2

C - A = 5

C - B = 2

A - B = 1/2

A - C = 1/5

B - C = 1/2

Similarly.

$$A_R = \begin{matrix} \\ A \\ B \\ C \end{matrix} \begin{matrix} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 3/2 \\ 1/3 & 2/3 & 1 \end{pmatrix} \end{matrix}$$

To find normalized matrix

$$A_L = \begin{pmatrix} 1 & 1/2 & 1/5 \\ 2 & 1 & 1/2 \\ 5 & 2 & 1 \end{pmatrix}$$

Col sum $= 8 \quad 7/2 \quad 17/10$

Dividing by Col Sum

Normalized matrix is

Row average

$$N_{A_L} = \begin{pmatrix} 0.125 & 0.142 & 0.117 \\ 0.25 & 0.285 & 0.294 \\ 0.625 & 0.571 & 0.588 \end{pmatrix} \begin{matrix} W_{A_L} = 0.128 \checkmark \\ W_{B_L} = 0.276 \checkmark \\ W_{C_L} = 0.594 \checkmark \end{matrix}$$

Inconsistent (not all cols are equal)

$$N_{A_R} = \begin{pmatrix} 0.545 & 0.545 & 0.545 \\ 0.272 & 0.272 & 0.272 \\ 0.181 & 0.181 & 0.181 \end{pmatrix} \begin{matrix} W_{A_R} = 0.545 \\ W_{B_R} = 0.272 \\ W_{C_R} = 0.181 \end{matrix}$$

Consistent.

Weight are (Location)

(wrt A) $W_{A_L} = 0.128$

(wrt B) $W_{B_L} = 0.276$

(wrt C) $W_{C_L} = 0.594$

Reputation

wrt A - $W_{A_R} = 0.545$

B - $W_{B_R} = 0.272$

C - $W_{C_R} = 0.181$

To check level of Consistency is reasonable

Consistency Ratio (CR) = $\dfrac{\text{Consistency Index (CI)}}{\text{Random Consistency (RI) Index}}$

Random [illegible] Index

$$CI = \frac{n_{max} - n}{n-1}$$

$$RI = \frac{1.98\,(n-2)}{n}$$

$$n_{max} = A \times \bar{w}$$

where A - matrix to be tested ✓

$\bar{w}$ - column vector of average weight ✓

If $C_R \leq (0.1)$ → threshold then the matrix can be accepted.

Checking N_L for Consistency

The corresponding matrix is A_L

$$n_{max} = A \times \bar{w} \quad [\because A \text{ is } A_L]$$

we have $A_L = \begin{pmatrix} 1 & 1/2 & 1/5 \\ 2 & 1 & 1/2 \\ 5 & 2 & 1 \end{pmatrix}$

$$\bar{w} = \begin{pmatrix} 0.128 \\ 0.276 \\ 0.594 \end{pmatrix}$$

$$n_{max} = A_L \times \bar{w}$$

$$= \begin{pmatrix} 1 & 1/2 & 1/5 \\ 2 & 1 & 1/2 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0.128 \\ 0.276 \\ 0.594 \end{pmatrix}$$

$$n_{max} = \begin{pmatrix} 0.3863 \\ 0.832 \\ 1.793 \end{pmatrix}$$

n_{max} → Sum of all elements in resultant matrix.

$$n_{max} = 0.3863 + 0.832 + 1.793$$

$$= 3.04$$

$$n = 3$$

$$CI = \frac{n_{max} - n}{n-1} = \frac{3.04 - 3}{3-1}$$

$$= \underline{0.02}$$

$$RI = \frac{1.98\,(n-2)}{n}$$

$$= \frac{1.98\,(3-2)}{3} = \underline{0.66}$$

$$CR = \frac{CI}{RI} = \frac{0.02}{0.66} = \underline{0.03}$$

$CR \leq 0.1$ is true

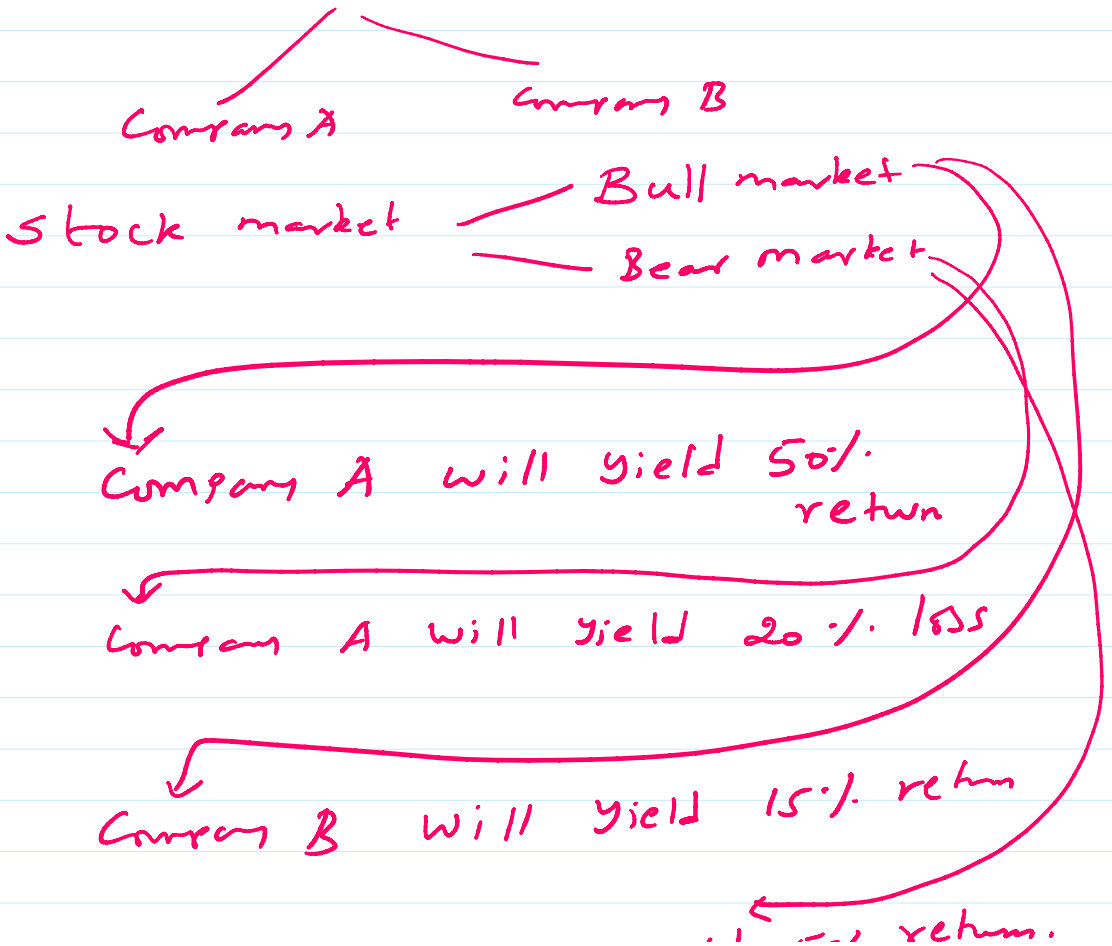
Accept the matrix for Consistency

II) Decision making Under risk

- Decision alternatives are represented by Probability distributions.
- based on expected Value Criterion
- Maximization of expected Profit
  Minimization of expected cost.

Example.

Invest $ 10000 in stock market



Company B will yield 5% return.

Predictions gave 60% chance for bull market & 40% bear market.

"How should you invest?"

10000 Invest

Given data	Bull Market	Bear Market
Company A	5000	−2000
Company B	1500	500
Probability	0.6	0.4

For Company A

Expected Value = $(5000 \times 0.6) + (-2000 \times 0.4)$

$= 3000 - 800 = \$2200$

For Company B

Expected Value = $(1500 \times 0.6) + (500 \times 0.4)$

$= 900 + 200$

$= \$1100$

Conclusion: Max Expected Value is 2200

∴ Investment in Company A is Preferd.

ii) Expected Value Criterion based on Posterior Probabilities
[(or) Bayes Probabilities]

(Previous Problem)

Additional

Investment recommendation based on a friend,

In bull market – 70% recommendation is "for"

In bear market – 50% recommendation is "for"

How does additional information affects the decision?

Any

Step 1) Let V_1 = "for"

V_2 = "against"

m_1 = "bull"

m_2 = "bear"

Conditional probabilities

$P(V_1/m_1) = 0.7 \rightarrow (1-0.7)$

Conditional probabilities:

$P(V_1/m_1) = 0.9$

$P(V_2/m_1) = 0.1$ → $(1-0.9)$

$P(V_1/m_2) = 0.5$

$P(V_2/m_2) = 0.5$

Conditional Probabilities are

	V_1	V_2
m_1	0.9	0.1
m_2	0.5	0.5

Step 2

Compute joint Probability

$P[m_i, V_j] = P(V_j/m_i)\, P(m_i)$ for all i, j

Given the Prior Probability

$P(m_1) = 0.6$ [Bull market] $P(m_2) = 0.4$ [Bear market]

$P(m_1, V_1) = P(V_1/m_1)\, P(m_1)$
$= 0.9 \times 0.6 = 0.54$

$P(m_1, V_2) = P(V_2/m_1) * P(m_1)$
$= 0.1 \times 0.6 = 0.06$

$P(m_2, V_1) = P(V_1/m_2) * P(m_2)$
$= 0.5 \times 0.4 = 0.20$

$$P(m_2, V_2) = P(V_2/m_2) * P(m_2)$$
$$= 0.5 \times 0.4 = 0.20$$

Step 3:

Compute absolute Probability

$$P(V_j) = \sum_{\forall i} P(m_i, V_j) \quad \forall j$$

$P(V_1)$	$P(V_2)$
0.54	0.06
0.20	0.20
= 0.54 + 0.20 = 0.74	0.06 + 0.20 = 0.26

} col sum

$$P(V_1) = 0.74$$
$$P(V_2) = 0.26$$

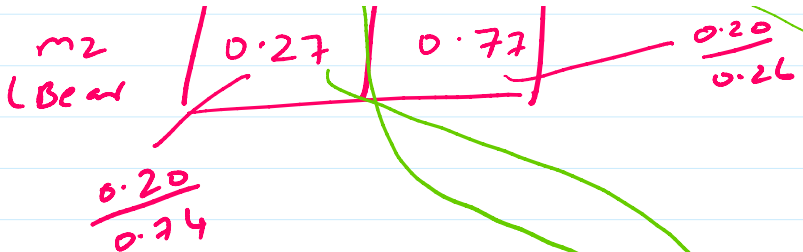
step 4:

Posterior probability.

$$P(m_i/V_j) = \frac{P(m_i, V_j)}{P(V_j)}$$

[i.e. Divide each col. in step 2 by corresponding col sum in step 3]

	for (V_1)	against (V_2)	
m_1 (Bull)	0.73	0.23	0.54/0.74 ; 0.06/0.26
m_2 (Bear)	0.27	0.77	0.20/0.26



Recommendation based on 'for'

Stock A:

Return = $5000 \times 0.73 + (-2000) \times 0.27$

$= \$3110$

Stock B

Return = $1500 \times 0.73 + 500 \times 0.27$

$= \$1230$

Stock A is recommended

(if the friend recommendation is "for" which stock will you invest?)

Recommendation Based on against

Stock A

return = $5000 \times 0.23 + (-2000 \times 0.77)$

$= -\$390$

Stock B

return = $1500 \times 0.23 + (500 \times 0.77)$

$= \$730$

Stock B is recommended

[if the friend recommendation is against in which stock will you invest?]

invest?]

III) Decision Under Uncertainty

— Involves alternative actions whose 'pay off' depends on the state of nature.

eg.

National outdoor school

↓

Summer Camp

↓

Attendance

4 categories

200 | 250 | 300 | 350

350
× 100
35000

200 × 100
20000

The cost of camp will be smallest when its size meets demand exactly.

— Deviation above or below will incur additional cost or loss.

— Let a_1 to a_4 represent size of camp (200, 250, 300, 350)

— Let S_1 to S_4 — level of attendance.

– Let S_1 to S_4 – level of attendance.

The following table summarize the cost matrix for the situation

(in 1000's dollars)

State	S_1	S_2	S_3	S_4 (att. level)
a_1	5	10	18	25
a_2	8	7	12	23
a_3	21	18	12	21
a_4	30	22	19	15

Recommend a course of action based on <u>four criteria of decision making under uncertainty</u>.

<u>Criteria I</u> : Laplace

(Principle of insufficient reason)

Since the probability is not known, all states are considered to have equal probabilities.

Probability of each state $= 1/4$ / no. of states.

$P(S_j) = 1/4$, $j = 1, 2, 3, 4$.

$$E(a_1) = \frac{1}{4}(5 + 10 + 18 + 25)$$

$$= \frac{58}{4} = 14.5 = \$14,500$$

$$E(a_2) = \frac{1}{4}(8 + 7 + 12 + 23)$$

$$E(a_2) = \frac{50}{4} = 12.5 = \$12,500$$ min

$$E(a_3) = \frac{1}{4}(21 + 18 + 12 + 21)$$

$$= \frac{72}{4} = 18 = \$18,000$$

$$E(a_4) = \frac{1}{4}(30 + 22 + 19 + 15)$$

$$= \frac{86}{4} = 21.5 = \$21,500$$

Action a_2 is Recommended.
Size is 250

Criteria II : Min Max

Cost matrix

	s_1	s_2	s_3	s_4	Row Max
a_1	5	10	18	25	25
a_2	8	7	12	23	23
→ a_3	21	18	12	21	21 ←
a_4	30	22	19	15	30

Min (Row max) = Min (25, 23, 21, 30)

= 21

Recommended action is a_3 (size is 300)

[best of the worst possible conditions]

Criteria III : Savage

Cost matrix

a_1	5	10	18	25
a_2	8	7	12	23
a_3	21	18	12	21
a_4	30	22	19	15

The Regret matrix is as follows:

(Find col min & subtract from every element in that col)

5 − 5 col min; 10 − 7 col min; 18 − 12 col min; 25 − 15 col min

					Row Max
a_1	0	3	6	10	10
a_2	3	0	0	8	8
a_3	16	11	0	6	16
a_4	25	15	7	0	25

Min row max = 8

∴ Corresponding row is a_2

Recommended action is a_2.

Recommended action is a_2,
Size is 250.

Note:

Min Max	s_1	s_2	
a_1	11000	90	11000
a_2	10000	10000	10000
		Min	10000

$\therefore$ select a_2

Regret matrix

	s_1	s_2	Max
a_1	1000	0	1000
a_2	0	9910	9910
		Min	1000

$\therefore$ Select a_1

IV) Hurwicz — α - threshold value set b/w 0 & 1

Alternatives	Row min	Row max	α (Row min) + (1−α) (Row max)
a_1	5	25	$5\alpha + 25 - 25\alpha = -20\alpha + 25$
a_2	7	23	$-16\alpha + 23$
a_3	12	21	$-9\alpha + 21$
a_4	15	30	$-15\alpha + 30$

$\alpha = 0.5$

$a_1 = 15$ } min

$a_2 = 15$ }

$a_3 = 16.5$

$a_4 = 22.5$

Either a_1 or a_2 is recommended

$\alpha = 0.25$

$a_1 = 20$

$a_2 = 19$

$a_3 = \boxed{18.75}$ min

$a_4 = 26.25$

a_3 is recommended