Revised simplex method

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The LPP can be written in Matrix form

Max/Min $\boxed{Z = cX}$

s.t.

$AX = b$

$X \geq 0$

where c - Vector representing objective fn coefficient

X - vector representing variables

A - $m \times n$ matrix representing constraint coefficients

b - vector representing RHS of constraint

We can write $Ax = b$ as,

$$\sum_{j=1}^{n} P_j x_j = b.$$

Where P_j j^{th} column vector of matrix A.

A subset of 'm' vectors of P_j is called basis B:

eg:

Max $Z = 2x_1 + x_2$

s.t.

$3x_1 + 4x_2 \leq 6$

$6x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

Ans:

$Z = 2x_1 + x_2 + 0x_3 + 0x_4$

s.t.

$3x_1 + 4x_2 + x_3 = 6$

$6x_1 + x_2 + x_4 = 3$

$x_1, x_2, x_3, x_4 \geq 0$

writing in matrix form → C - cost/Profit coefficient matrix

$$Z = \begin{array}{cccc} (2 & 1 & 0 & 0) \\ c_1 & c_2 & c_3 & c_4 \end{array} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

→ X Variable matrix

s.t.

A - Constraint Coeff matrix | X | b RHS

$$\begin{pmatrix} 3 & 4 & 1 & 0 \\ 6 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 & 1 & 0 \\ 6 & 1 & 0 & 1 \\ P_1 & P_2 & P_3 & P_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

Iteration 0

Basic Variable : x_3, x_4

Non Basic Variable : x_1, x_2

Basic Vector X_{B_0}

$$X_{B_0} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Corresponding Cost/Profit Vector (c_3, c_4)

$$C_{B_0} = (0 \quad 0)$$

[∴ (P_3, P_4) corresponding (x_3, x_4)]

Basis $B_0 = (P_3, P_4)$

$$\boxed{B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\boxed{X_{B_0} = B_0^{-1} b} \qquad B_0^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\boxed{X_{B_0} = I \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}}$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \implies x_3 = 6, \quad x_4 = 3$$

$$Z = C_{B_0} X_{B_0}$$

$$\boxed{\begin{aligned} X_{B_0} &= B_0^{-1} b \\ Z &= C_{B_0} X_{B_0} \end{aligned}}$$

$$= (0 \quad 0) \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 0$$

$$\boxed{Z = 0} \qquad \boxed{x_3 = 6, \quad x_4 = 3}$$

Initial Basic feasible solution.

Optimality Condition

Compute for non basic Variable x_1, x_2

$j = 1, 2$

$$\boxed{\begin{aligned} & z_j - c_j \\ & = C_{B_0} B_0^{-1} P_j - c_j \end{aligned}}$$

$$\{z_j - c_j\}_{j=1,2} = C_{B_0} B_0^{-1} (P_1, P_2) - c_1, c_2$$

$[j = 1, 2]$

$$C_{B_0} B_0^{-1} = (0 \quad 0)\, I = (0 \quad 0)$$

$$z_j - c_j = (0 \quad 0) \begin{array}{cc} P_1 & P_2 \end{array} \begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix} - (2 \quad 1)$$

$$= (0 \quad 0) - (2 \quad 1)$$

$$= (-2 \quad -1)$$

Most negative -2 corresponding variable x_1 enters basis.

Feasibility Condition

$$X_{B_0} = \begin{pmatrix} \boxed{6} \\ \boxed{3} \end{pmatrix}$$

$$\boxed{\frac{X_{B_0}}{B_0^{-1} P}}$$

Since x_1 enters the basis corresponding p vector is P_1.

$$\therefore B_0^{-1} P_1 = I \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$B_0^{-1} P_1 = \begin{pmatrix} \boxed{3} \\ \boxed{6} \end{pmatrix}$$

$$\text{min ratio} = \left\{ \boxed{6/3}, \boxed{3/6} \right\}$$

$$= \{ \underset{x_3}{2}, \underset{x_4}{1/2} \} = 1/2$$

$\therefore$ The corresponding variable x_4 leaves the basis.

Iteration 1

Basic Variable : $x_3 \quad \boxed{x_1}$

Non basic variable : $x_2 \quad x_4$

Basic vector $X_{B_1} = \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}$

Cost vector $C_{B_1} = \begin{pmatrix} c_3 & c_1 \\ 0 & 2 \end{pmatrix}$

Cost vector $C_{B_1} = (0 \quad 2)$

Basis $B_1 = \underline{P_3 \quad P_1} = \begin{array}{cc} P_3 & P_1 \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix}} \end{array}$

$$\underline{B_1^{-1} = \frac{1}{6}\begin{pmatrix} 6 & -3 \\ 0 & 1 \end{pmatrix}}$$

$$B_1^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/6 \end{pmatrix}$$

$$X_{B_1} = B_1^{-1} b$$

$$= \begin{pmatrix} 1 & -1/2 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$X_{B_1} = \begin{pmatrix} 9/2 \\ 1/2 \end{pmatrix}$$

$$Z = \underline{C_{B_1}} X_{B_1}$$

$$= \underline{(0 \quad 2)} \begin{pmatrix} 9/2 \\ 1/2 \end{pmatrix}$$

$$\boxed{Z = 1}$$

<u>Optimality</u> <u>Condition</u>

Compute for non basic Variable

$$\{Z_j - C_j\}_{\underline{\underline{j=2,4}}} = C_{B_1} B_1^{-1} (\underline{P_2} \quad \underline{P_4}) - (\underline{C_2} \quad \underline{C_4})$$

nonbasic

$$C_{B_1} B_1^{-1} = (0 \quad 2) \begin{pmatrix} 1 & -1/2 \\ 0 & 1/6 \end{pmatrix} = (0 \quad 1/3)$$

$\underline{C_{B_1} B_1^{-1}}\ \underline{(P_2\ P_4)} - (c_2\ c_4) = (0\ \ 1/3)\begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} - (1\ \ 0)$

$$= \begin{pmatrix} 1/3 & 1/3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$z_j - c_j = \begin{array}{cc} \begin{pmatrix} -2/3 & 1/3 \end{pmatrix} \\ \quad x_2 \quad\quad x_4 \end{array}$$

Most negative is $-2/3$. x_2 enters Basis

<u>feasibility Condition</u>

$$X_{B_1} = \begin{pmatrix} 9/2 \\ 1/2 \end{pmatrix}$$

Since x_2 enters, corresponding vector is P_2.

$$B_1^{-1} P_2 = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/6 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$B_1^{-1} P_2 = \begin{pmatrix} 7/2 \\ 1/6 \end{pmatrix}$$

$$\text{Min ratio} = \left\{ \frac{9/2}{7/2}, \frac{1/2}{1/6} \right\}$$

$$= \left\{ 9/7, 3 \right\}$$

$$= 9/7$$

$= 9/7$

$\therefore$ Corresponding vector x_3 leaves the basis.

<u>Iteration 2</u>

Basic Variable $\underline{\underline{x_2}}$ $\underline{\underline{x_1}}$

Non Basic x_3 x_4

Basic vector

$$X_{B_2} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

Cost vector

$$C_{B_2} = \begin{array}{cc} c_2 & c_1 \\ \end{array} \begin{pmatrix} 1 & 2 \end{pmatrix}$$

Basis

$$B_2 = (P_2 \quad P_1) = \begin{pmatrix} 4 & 3 \\ 1 & 6 \end{pmatrix}$$

$$B_2^{-1} = \begin{pmatrix} 2/7 & -1/7 \\ -1/21 & 4/21 \end{pmatrix}$$

$$X_{B_2} = B_2^{-1} b$$

$$= \begin{pmatrix} 2/7 & -1/7 \\ -1/21 & 4/21 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9/7 \\ 2/7 \end{pmatrix}$$

$$Z = C_{B_2} X_{B_2}$$

$$= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 9/7 \\ 2/7 \end{pmatrix} = \frac{13}{7}$$

$$\boxed{Z = \frac{13}{7}}$$

<u>Optimality Condition</u>

Compute for basic variable x_3, x_4

$$\{z_j - c_j\}_{j=3,4} = c_{B_2} B_2^{-1} \underline{P_j} - \underline{c_j}$$

$$= \underline{c_{B_2} B_2^{-1}} (P_3 \; P_4) - (c_3 \; c_4)$$

$$c_{B_2} B_2^{-1} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2/7 & -1/7 \\ -1/21 & 4/21 \end{pmatrix}$$

$$= \begin{pmatrix} 4/21, & 5/21 \end{pmatrix}$$

$$\{z_j - c_j\}_{j=3,4} = \begin{pmatrix} 4/21 & 5/21 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4/21 & 5/21 \end{pmatrix}$$

Since all $z_j - c_j$ are +ve,

optimum achieved.

Soln $Z = \frac{13}{7}$, $x_1 = 2/7$

Soln $Z = \frac{13}{7}$, $x_1 = 2/7$

$x_2 = 9/7$