

Functional Data Regression

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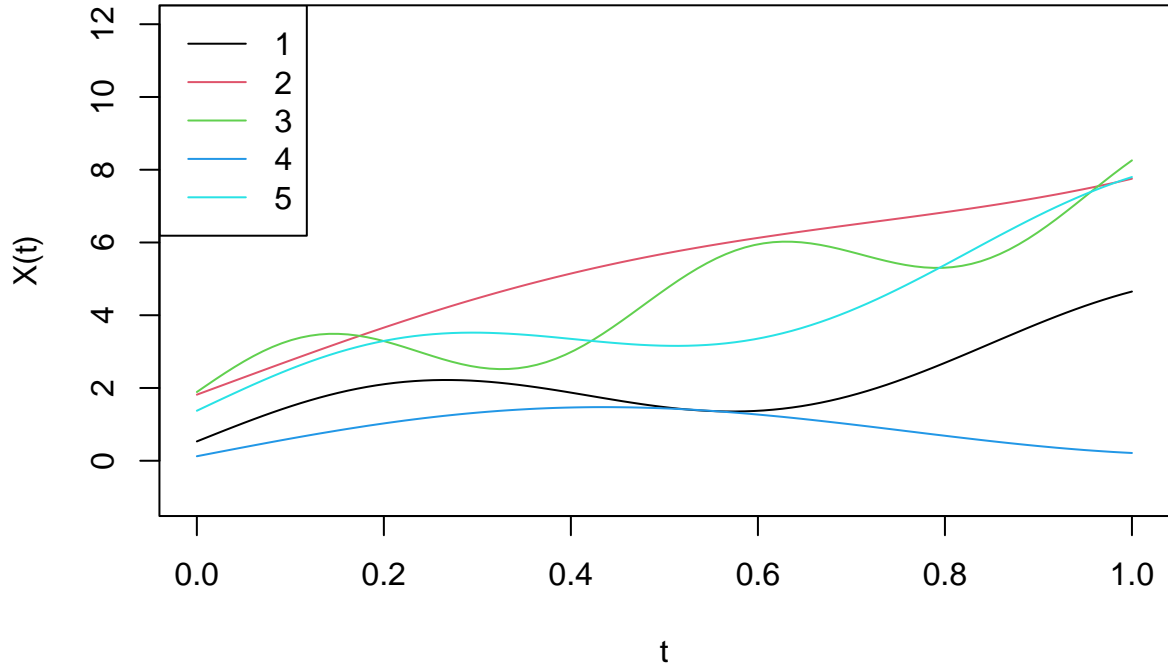
Simulating the data

We construct a random variable $X_{(t)}$ from $L^2_{[0,1]}$ space using the following:

$$X_{(t)} = (c_1 * e^t) + \sin(c_2 * 10t) + (c_3 * 2t)$$

where c_1, c_2, c_3 are random variables drawn from $uniform(0, 2)$. We generate ($n = 100$) samples of $X_{(t)}$ as our given data. Here is the plot of 5 samples from the generated data.

Generated Data (5 samples)



We simulate real Y using the following:

$$Y = m(X_{(t)}) + \epsilon$$

where $m(X_{(t)}) = \int_0^1 X_{(t)}^2 (\sin(t) + \cos(t)) dt$ and $\epsilon \sim N(0, 1)$

Here are the real values of y_i for above $X_{(t)}^i$ in respective order:

```
## [1] 8.204814 41.079967 33.139107 1.021508 26.661513 25.103260
```

Proposed Estimator (Based on Nadaraya-Watson Estimator)

I am using something similar to Nadaraya-Watson estimator for estimating m . The proposed estimator is as follows:

$$\hat{m}(X_{(t)}) = \frac{\sum_{i=1}^n K(\|X - X_{(t)}^i\|_{L_2}/h) y_i}{\sum_{i=1}^n K(\|X - X_{(t)}^i\|_{L_2}/h)}$$

where:

$K(\cdot)$ is a valid kernel. I have used standard Normal $N(0, 1)$

h is the band-width

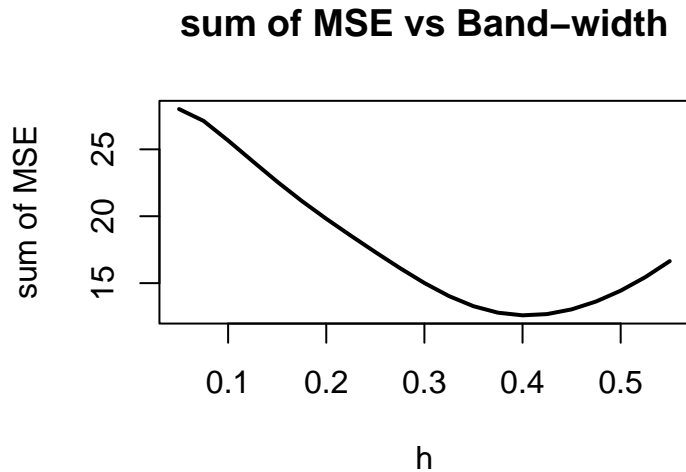
$\|\cdot\|$ is the norm on $L_{2[0,1]}$ space

Choosing the Bandwidth (h)

In order to choose the appropriate band-width for the simulated sample data, I am using K-fold cross validation.

Here are the steps I followed:

1. I generate a grid of band-width containing 21 values - $h = \{0.05, 0.075, \dots, 0.55\}$
2. Then I perform K-fold cross validation (with $K = 4$), and calculate the sum of the MSE obtained from cross-validation, for every h in the grid.
3. Chose the h which gives the lowest sum of MSE from the K cross validation results.



```
## [1] "Chosen Band-width: 0.4"
```

Estimation for new data

Now, we generate 100 new $(X_{(t)}, y)$ in order to evaluate the performance of our estimator.

Here are some of the estimates made for the newly generated data:

##		Real y	Estimated y	Error
##	[1,]	25.039944	25.519573	-0.47962918
##	[2,]	21.278169	20.465545	0.81262407
##	[3,]	1.686397	2.611027	-0.92463012
##	[4,]	8.952435	9.773870	-0.82143463
##	[5,]	18.974028	18.906697	0.06733104
##	[6,]	14.837879	14.011128	0.82675184

```
## [1] "MSE = 1.08949042918219"
```

```
## [1] "R2 = 0.989799998756462"
```

Conclusion

The implementation of the Nadaraya-Watson estimator for functional regression was successful, with the model achieving a Mean Squared Error (MSE) of 1.09 and an R-squared (R^2) score of 0.98. These results show that the model performs very well in terms of accuracy and explains almost all of the variability in the data.

Outlier Detection

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The objective is to generate simulated data along with some outliers. Then we have to propose a method to detect the outliers and estimate the proportion of outliers in the data.

Simulating Data:

We generate a random $X_{(t)}$ from $L^2_{[0,1]}$ space using the following:

$$X_{(t)} = 0.5(c_1 \sin(10.c_1.t) + c_2 \sin(10.c_2.t) + c_3 \sin(10.c_3.t)) + c_4$$

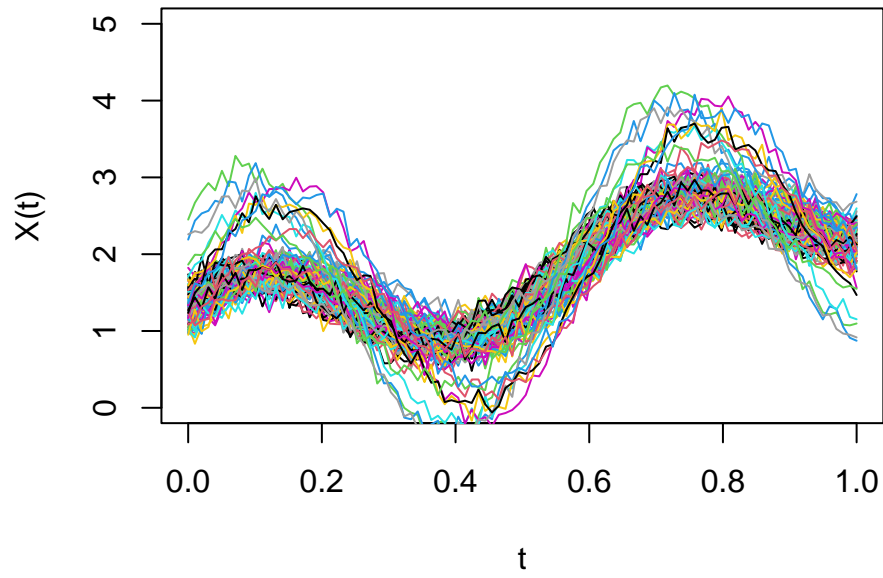
where c_i are random constants from $Unif(0, 1)$.

In this analysis, I have generated a total of 100 samples, with exactly 10 of them as outliers. I explore two different cases: Frequency Difference and Scale Difference

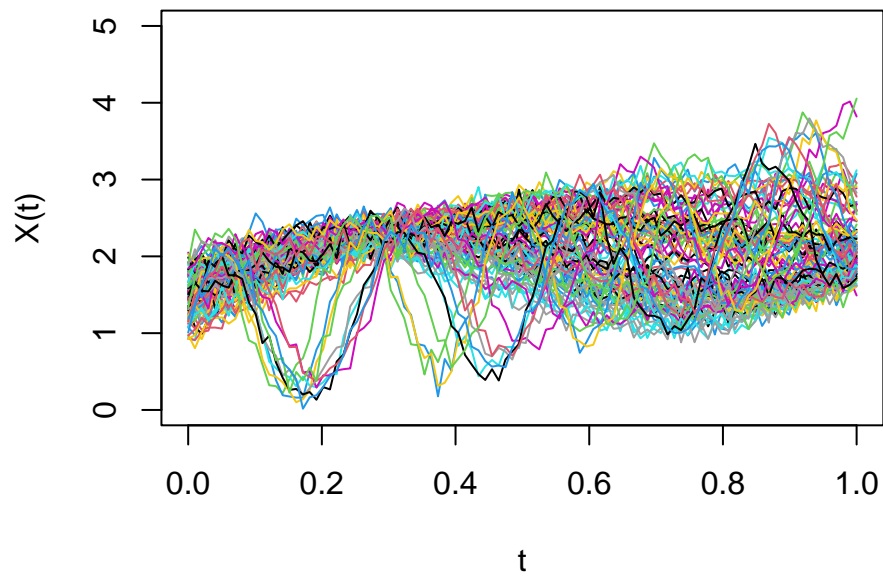
- In case 1, the outliers have similar shape to the data but attain more extreme values compared to the data. To introduce this difference, I multiply a random generated value uniform distribution $Unif(1, 2)$ to the original function.
- In case 2, the outliers have a different frequency compared to the rest of the data. To achieve this, I multiply the frequency of sinusoidal function with $10*c$ where $c \sim \text{runif}(1, 1.5)$

Here are the plots of the datasets for the two cases:

Case 2 (More Extreme Values)



Case 1 (Different Frequency)



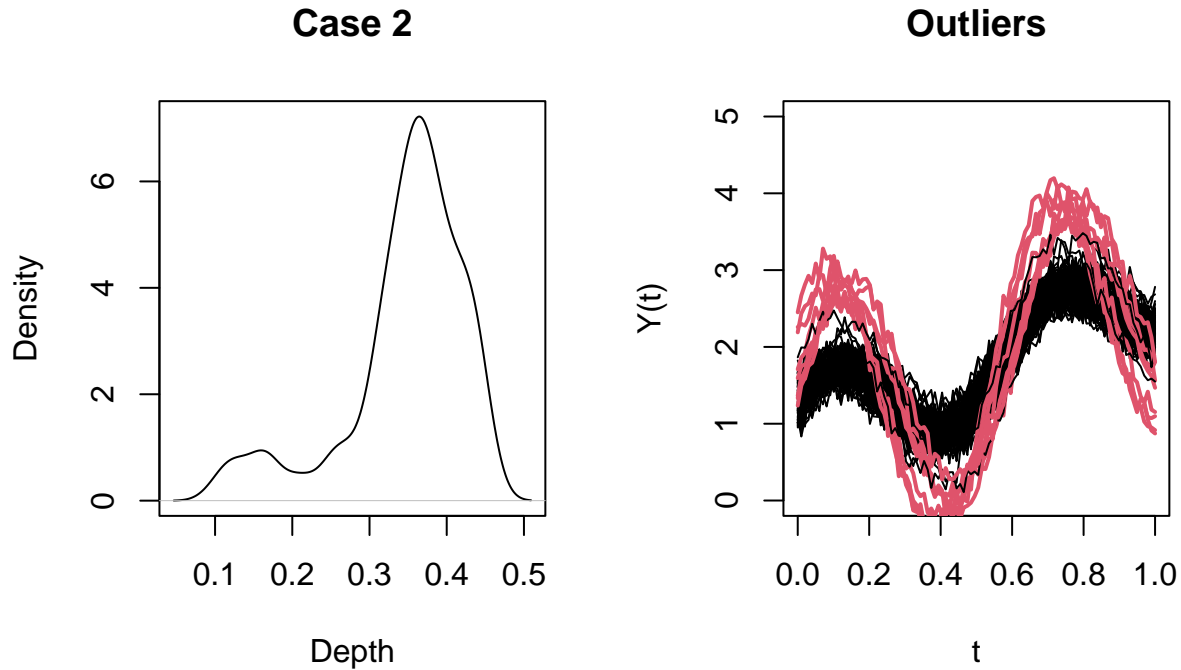
Proposed method:

To detect the outliers in the data, I calculate the depth of the $X_{(t)}$'s in the dataset. Subsequently, then assuming the depth comes from a normal distribution, a cut-off value based on the 95% region for $N(\mu, \sigma^2)$ where $\mu = \text{mean}(\text{depth})$ and $\sigma^2 = \text{Var}(\text{depth})$. Any samples that lie outside the 95% region, are labelled as outliers.

Note: As a measure of depth, we are using Modified Band-Depth (MBD) presented by Sara Lopez-Pintado and Juan Romo. This is the link to the original paper: <https://www.jstor.org/stable/40592217>

Case 1

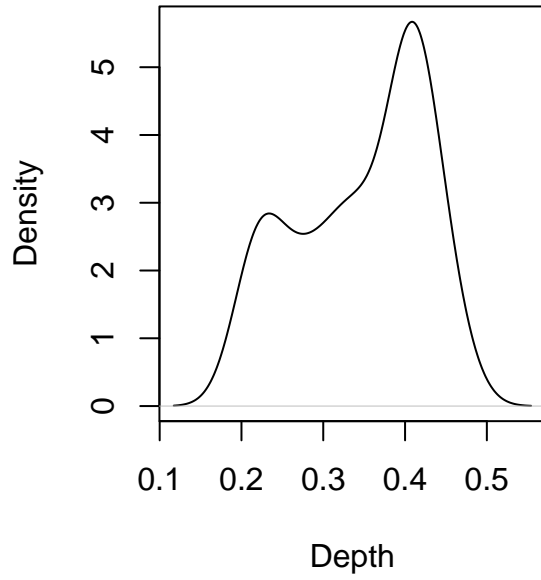
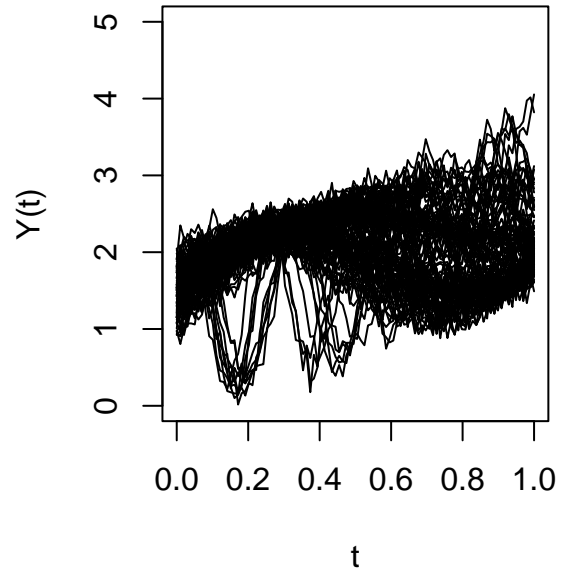
The method works well in this case. It is able to identify most of the outliers correctly.



```
##           predicted_outlier
## is_outlier  0  1
##           0 90  0
##           1  2  8
```

Case 2

In this case, the method was unable to accurately predict the outliers. Due to higher frequency of the random function, the function spends most of the time inside the band. Consequently, these functions, which should be identified as outliers, are instead ranked with a greater depth than non-outliers

Case 2**Outliers**

```
##           predicted_outlier
## is_outlier  0
##           0 90
##           1 10
```

Conclusion

The method was effective in detecting most outliers with more extreme values. However, it struggled to accurately identify outliers that had higher frequency but were on the same scale. To improve the performance, additional features that account for frequency variations can be included while detecting outliers.