Functional Data Regression

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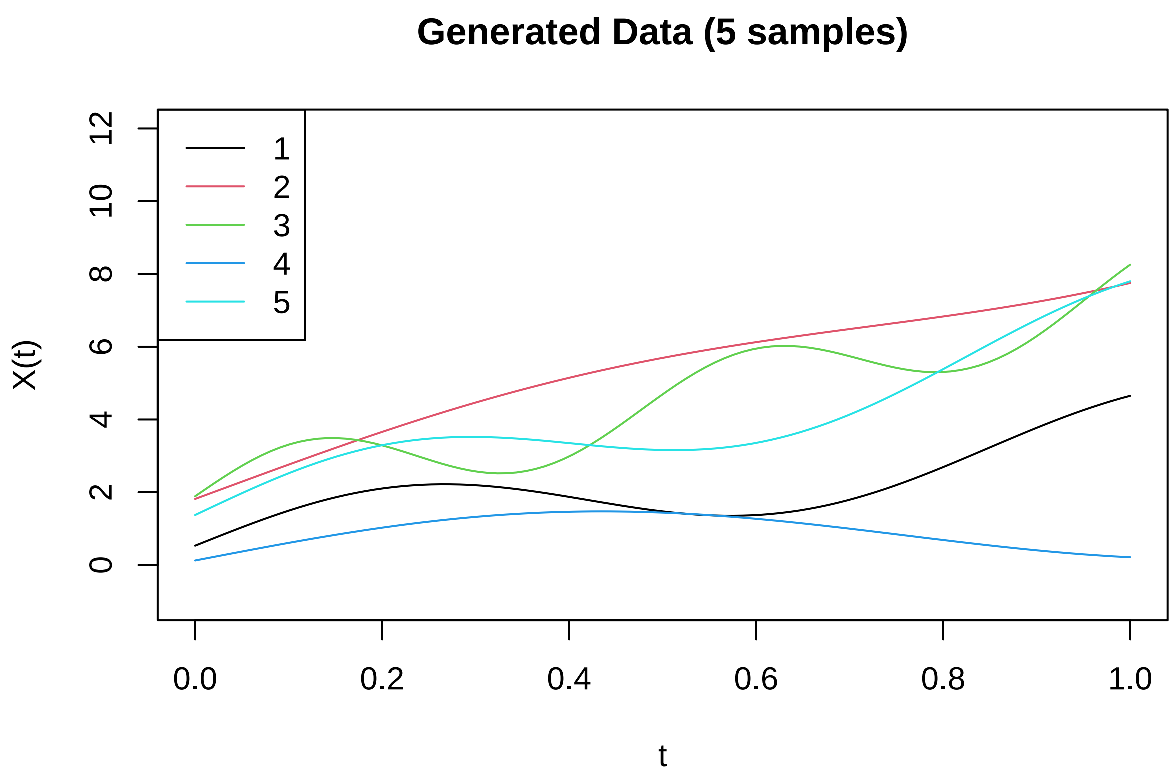
Simulating the data

We construct a random variable $X_{(t)}$ from $L^2_{[0,1]}$ space using the following:

$$X_{(t)} = (c_1 * e^t) + sin(c_2 * 10t) + (c_3 * 2t)$$

where c_1, c_2, c_3 are random variables drawn from $uniform(0, 2)$. We generate (n = 300) samples of $X_{(t)}$ as our given data. Here is the plot of 5 samples from the generated data.



We simulate real Y using the following:

$$Y = m(X_{(t)}) + \epsilon$$

where $m(X_{(t)}) = \int_0^1 X^2_{(t)}(sin(t) + cos(t))dt$ and $\epsilon \sim N(0, 1)$

Here are the real values of y_i for above $X_{(t)}^i$ in respective order:

```
## [1]  6.970074 43.629690 31.696077  1.947331 24.870695 22.454314
```

Functional Linear Regression

Initially I predict the response variable using in-built functional linear regression function *fdRegress(.)*. The only reason to predict using the Linear Regression model is to get a baseline for model comparison. It tries to find β such that:

$$Y_i = \beta_0 + \int_0^1 X_i(t)\beta(t)dt + \epsilon_i$$

Both $X_i(t)$ and $\beta(t)$ can be expanded using orthonormal basis functions in L_2 space, say K:

$$X_i(t) \approx \sum_{k=1}^{K} c_{ik}\phi_k, \beta(t) \approx \sum_{k=1}^{K} b_k\phi_k$$

Expanding the respective terms, we get:

$$Y_i = \beta_0 + \sum_{k=1}^{K} b_k c_{ik}$$

Proposed Estimator (Based on Nadaraya-Watson Estimator)

I am using something similar to Nadaraya-Watson estimator for estimating m. The proposed estimator is as follows:

$$\hat{m}(X_{(t)}) = \frac{\Sigma_{i=1}^n K(||X - X_{(t)}^i||_{L_2}/h)y_i}{\Sigma_{i=1}^n K(||X - X_{(t)}^i||_{L_2}/h)}$$

where:

$K(.)$ is a valid kernel. I have used standard Normal $N(0,1)$

h is the band-width

$||.||$ is the norm on $L_{2[0,1]}$ space

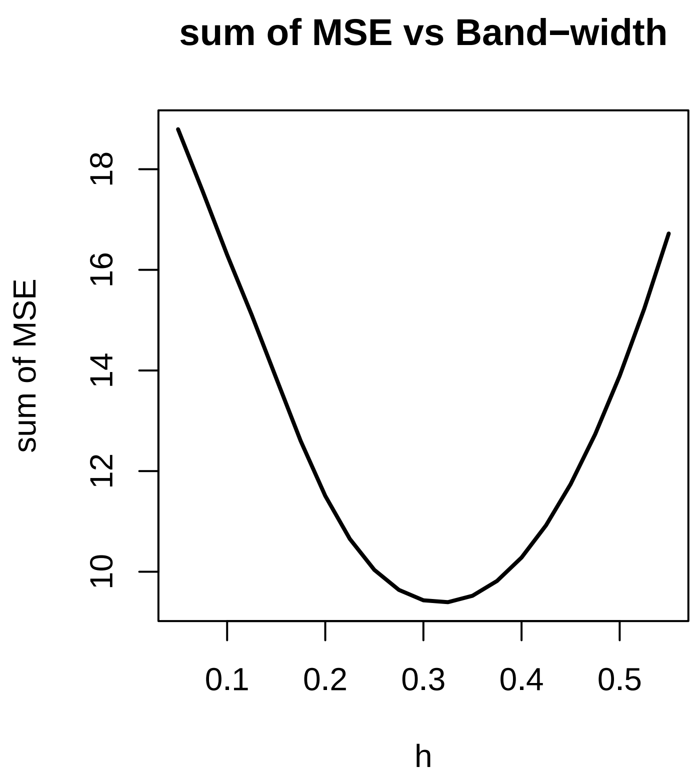
Choosing the Bandwidth (h)

In order to choose the appropriate band-width for the simulated sample data, I am using K-fold cross validation.

Here are the steps I followed:

1. I generate a grid of band-width containing 21 values - $h = \{0.05, 0.075, ...0.55\}$
2. Then I perform K-fold cross validation (with $K = 4$), and calculate the sum of the MSE obtained from cross-validation, for every h in the grid.

3. Chose the h which gives the lowest sum of MSE from the K cross validation results.




```
## [1] "Chosen Band-width: 0.325"
```

Estimation for new data

Now, we generate 100 new $(X_{(t)}, y)$ in order to evaluate the performance of our estimator.

Here are some of the estimates made for the newly generated data:

Estimation using Linear Regression:

```
##         Real y Estimated y     Error
## [1,] 19.382151  21.5130396 -2.130888
## [2,]  1.664833   0.3223428  1.342490
## [3,]  1.566500   3.3098935 -1.743394
## [4,]  6.425769   5.4234245  1.002345
## [5,] 13.571157  22.6553290 -9.084173
## [6,] 19.895903  21.7855517 -1.889649
```

```
## [1] "MSE =  9.8529470947704"
```

Estimation using proposed estimator

```
##         Real y Estimated y       Error
## [1,] 19.382151   19.679719 -0.29756778
## [2,]  1.664833    1.526623  0.13821023
## [3,]  1.566500    1.922560 -0.35606053
## [4,]  6.425769    6.391972  0.03379736
## [5,] 13.571157   13.480416  0.09074011
## [6,] 19.895903   19.457724  0.43817868
```

```
## [1] "MSE =  0.673709591846734"
```

Conclusion

Both estimators successfully predicted the dependent variable; however, the Nadaraya-Watson estimator outperformed the linear model slightly. This indicates that the linear model may not be the most appropriate fit for the data.