Dimensionality Reduction

Question 1: Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7,3/7,6/7], and another is [6/7, 2/7, -3/7]. Let the third column be [x,y,z]. Since the length of the vector [x,y,z] must be 1, there is a constraint that $x^2+y^2+z^2 = 1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

Let C1 be $[\frac{2}{7} \quad \frac{3}{7} \quad \frac{6}{7}]$, C2 be $[\frac{6}{7} \quad \frac{2}{7} \quad \frac{-3}{7}]$ and C3 be $[x \quad y \quad z]$

The dot product of any two columns must be zero

C1.C2 = $(\frac{2}{7} \times \frac{6}{7}) + (\frac{3}{7} \times \frac{2}{7}) + (\frac{6}{7} \times \frac{-3}{7}) = 0$

C2.C3 = $(\frac{6}{7} \times x) + (\frac{2}{7} \times y) + (\frac{-3}{7} \times z) = 0 \Rightarrow 6x + 2y - 3z = 0 \rightarrow$ eq1

C3.C1 = $(x \times \frac{2}{7}) + (y \times \frac{3}{7}) + (z \times \frac{6}{7}) = 0 \Rightarrow 2x + 3y + 6z = 0 \rightarrow$ eq2

2×eq1 + eq2 $\Rightarrow 12x + 4y - 6z + 2x + 3y + 6z = 0 \Rightarrow 14x + 7y = 0 \rightarrow y = -2x$

3×eq2 - eq1 $\Rightarrow 6x + 9y + 18z - 6x - 2y + 3z = 0 \Rightarrow 7y + 21z = 0 \rightarrow y = -3x$

X : Y : Z = -2 : 1 : -3

Question 2: Find the eigenvalues and eigenvectors of the following matrix:

2	3
3	10

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.

The given matrix be A = $\begin{matrix} 2 & 3 \\ 3 & 10 \end{matrix}$ and the eigen vector be of the form $\begin{matrix} 1 \\ e \end{matrix}$

$Ax = \lambda x = \begin{matrix} 2 & 3 \\ 3 & 10 \end{matrix} \times \begin{matrix} 1 \\ e \end{matrix} = \lambda \times \begin{matrix} 1 \\ e \end{matrix} \Rightarrow 2 + 3e = \lambda$ and $3 + 10e = \lambda e$

$3 + 10e = (2 + 3e)e$

$3e^2 - 8e + 3 = 0$ ➔ $e = 3, \frac{-1}{3}$

The eigen vectors are $\begin{matrix} 1 \\ 3 \end{matrix}$ *and* $\begin{matrix} 1 \\ \frac{-1}{3} \end{matrix}$

The eigen values are $2 + 3e = \lambda \rightarrow \ \lambda = 2 + 3 * 3 = 11$ and $\lambda = 2 + 3 * \left(\frac{-1}{3}\right) = 1$

Question 3: Suppose [1, 3, 4, 5, 7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

Given the eigen vector of some matrix be M = [1, 3, 4, 5, 7]

To get the unit eigen vector of given matrix, we need to divide each component by square root of sum of squares in the same direction.

Sum of squares = $1^2 + \ 3^2 \ + \ 4^2 + 5^2 + \ 7^2 = 100$ and its square root is 10

Unit Eigen vector = [$\frac{1}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{7}{10}$]

Question 4: Suppose we have three points in a two-dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

The given three points in a 2- D space are (1,1), (2,2), and (3,4). We should construct a matrix whose rows correspond to points and columns correspond to dimensions of the space.

The matrix will be M = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$

$M^TM = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 17 \\ 17 & 21 \end{bmatrix}$

Question 5: Consider the diagonal matrix M = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Compute its Moore-Penrose pseudoinverse.

Moore-Penrose pseudoinverse means the matrix having diagonal elements replaced by 1 and divided by corresponding elements of given matrix and the other elements will be zero. Moore-Penrose pseudoinverse of given matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Question 6: When we perform a CUR decomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

1	2	3
4	5	6
7	8	9
10	11	12

Calculate the probability distribution for the rows.

Probability with which we choose now = $\frac{\text{sum of squares of elements in the rows}}{\text{sum of squares of elements in the matrix}}$

Sum of squares of elements in the matrix = $\frac{12\times13\times25}{6} = \frac{3900}{6} = 650$

P(R1) = $\frac{1^2 + 2^2 + 3^2}{650} = \frac{14}{650} = 0.02$

P(R2) = $\frac{4^2 + 5^2 + 6^2}{650} = \frac{77}{650} = 0.12$

P(R3) = $\frac{7^2 + 8^2 + 9^2}{650} = \frac{194}{650} = 0.298$

P(R1) = $\frac{10^2 + 11^2 + 12^2}{650} = \frac{365}{650} = 0.56$