

CS 480

Introduction to Artificial Intelligence

October 18, 2022

Announcements / Reminders

- **Grading TA assignment:**

https://docs.google.com/spreadsheets/d/1ExS0bKnGt_fdf4LHa3YS1qRA7-lq4xqXVjfSAPMaGVk/edit?usp=sharing

Plan for Today

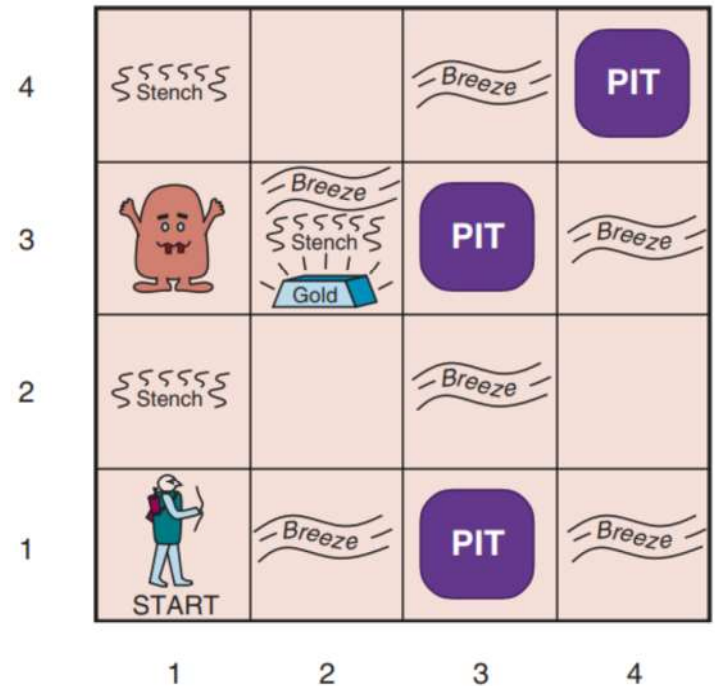
- **Predicate / First-Order Logic**

Propositional Logic Weaknesses

Wumpus World is represented by several variables in propositional logic:

- $B_{x,y}$: there is a breeze in square (x,y)
- $P_{x,y}$: there is a pit in square (x,y)
- $S_{x,y}$: there is a stench in square (x,y)
- $W_{x,y}$: there is a wumpus in square (x,y)
- $G_{x,y}$: there is gold in square (x,y)
- $L_{x,y}$: agent is located in square (x,y)

Either one of those can be either **true** or **false**. There is 4 x 4 variables for every type.



What if this Wumpus World was a $N \times N$ grid and N was a really large number. How many variables we would have?

Propositional Logic Weaknesses

Similarly, consider the following English sentence:

“robot R514 is standing at coordinates (123, 4501)”

It could be represented by a following propositional logic variable:

robot_R514_is_standing_at_coordinates_(123_4501)

and a **true** / **false** value. What's the problem?

Propositional Logic Weaknesses

How about relationships such as this one:

“robot R514 is to the left of robot R89”

It could be represented by a following propositional logic variable:

robot_R514_is_to_the_left_of_R89

and a **true** / **false** value. What's the problem?

Propositional Logic Weaknesses

How about relationships such as this one:

“robot R514 is to the left of robot R89”

It would be much easier to represent it in a form similar to this one:

toTheLeftOf(R514, R89)

Propositional Logic Weaknesses

Propositional logic:

- CAN represent facts
- CANNOT represent objects
- CANNOT represent relationships between objects
- is overall not very expressive, for example it is impossible to transform this sentence into propositional logic:

“Some people live in Illinois.”

A More Expressive Formal Language

A more expressive formal language could be obtained by augmenting propositional logic with:

- **objects:**
 - people, houses, humbers, theories, colors, basketball games, wars, etc.
- **relations:**
 - unary relations (properties): red, round, bogus, etc.
 - n-ary relations: brother of, bigger than, inside, part of, etc.
- **functions:**
 - relations in which there is only one “value” for a given “input”
 - father of, best friend, one more than, beginning of, etc.

Predicate / First-Order Logic

Predicate Logic is an extension of Propositional Logic. It is a formal language in which propositions are expressed in terms of **predicates**, **variables**, and **quantifiers**.

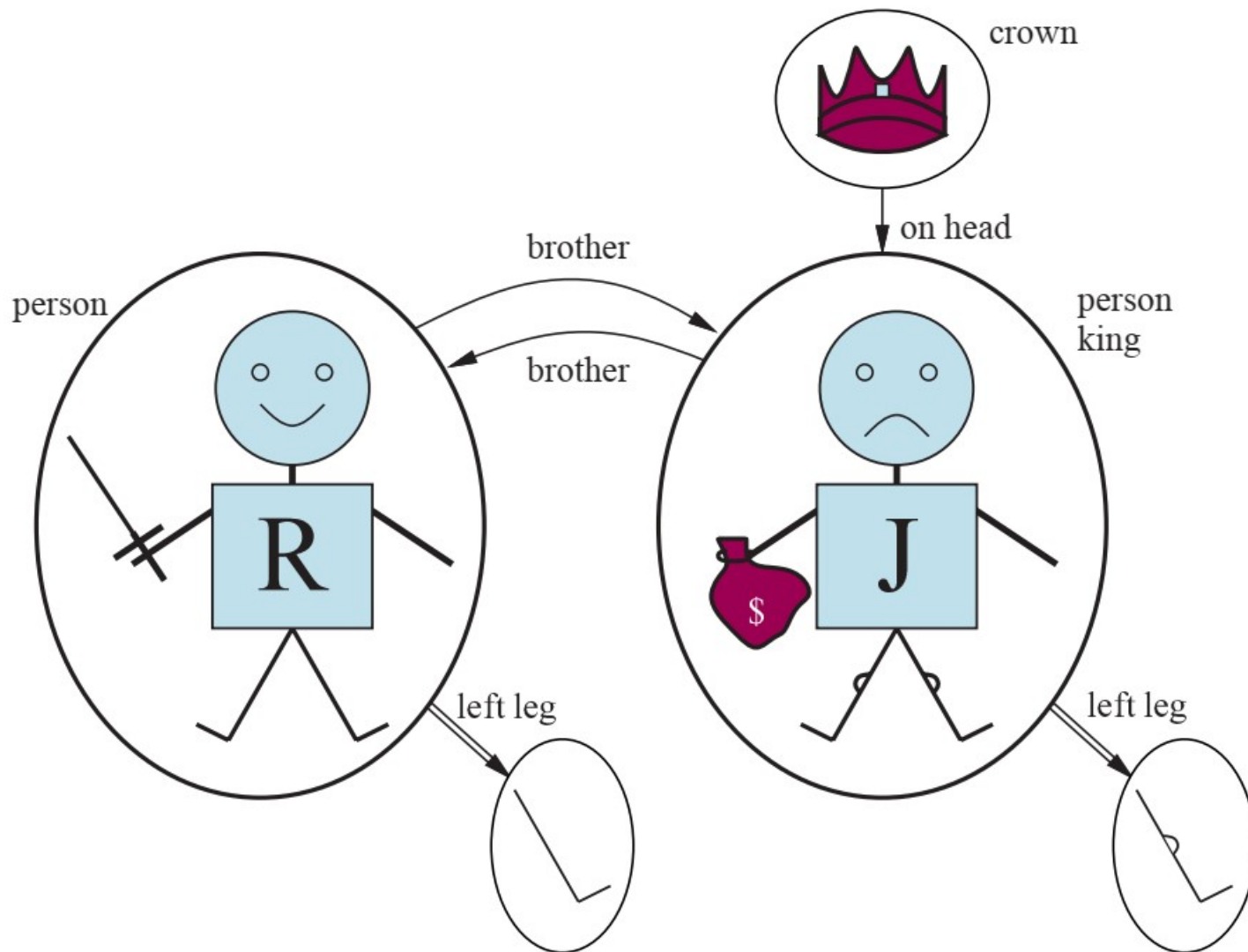
Predicates

In logic, a **predicate** is a symbol which represents a property or a relation. For example:

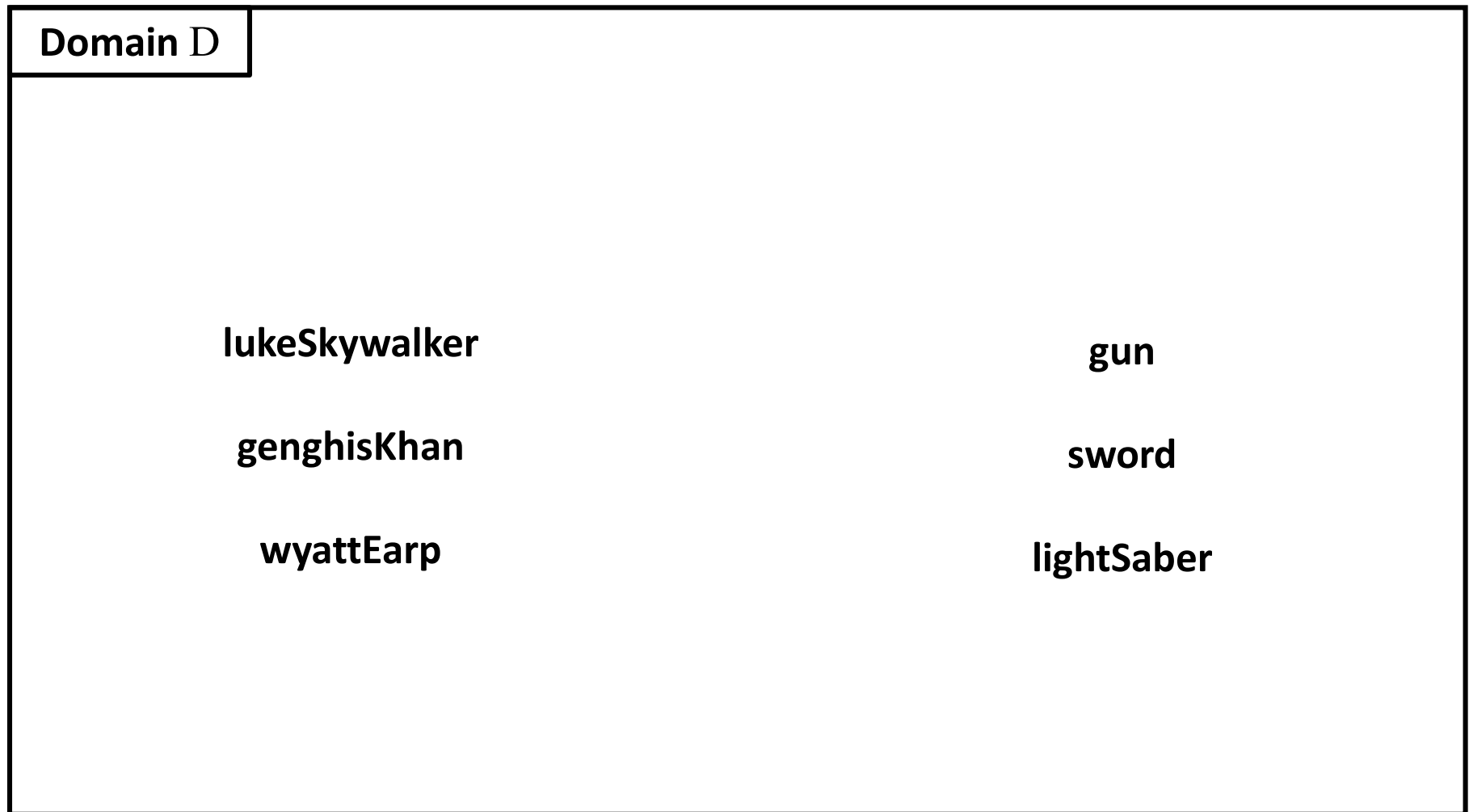
- in the sentence $P(a)$, the symbol P is a predicate which applies to the individual constant a ,
- in the formula $R(a,b)$ the symbol R is a predicate which applies to the individual constants a and b .

In the semantics of logic, **predicates are interpreted as relations.**

Predicate Logic: Model of the World



Predicate Logic: Model and Objects



The domain of a model D is a **nonempty** set of domain elements it contains. Objects may be related.

Symbols: Refresher

Symbol	Name	Alternative symbols*	Should be read
\neg	Negation	$\sim, !$	not
\wedge	(Logical) conjunction	$\bullet, \&$	and
\vee	(Logical) disjunction	$+, $	or
\Rightarrow	(Material) implication	\rightarrow, \supset	implies
\Leftrightarrow	(Material) equivalence	$\leftrightarrow, \equiv, \text{iff}$	if and only if
\top	Tautology	$T, 1, \blacksquare$	truth
\perp	Contradiction	$F, 0, \square$	falsum empty clause
\forall	Universal quantification		for all; for any; for each
\exists	Existential quantification		there exist

* you can encounter it elsewhere in literature

Predicate Logic Syntax: Symbols

Predicate calculus symbols include:

- truth symbols: true and false
- **constant** symbols
- **variable** symbols
- **predicate** symbols
- **function** symbols

Syntax: Constant

A **constant** *c* (*c* - symbol) represent objects

▪ **for example:**

- lightSaber
- kingJohn
- lukeSkywalker
- box
- table
- crown

Predicate Logic: Model and Objects



Existing constants in this representation are

$D = \{\text{lukeSkywalker}, \text{genghisKhan}, \text{wyattEarp}, \text{gun}, \text{sword}, \text{lightSaber}\}$

Syntax: Variable

A **variable** v (v - symbol) can be used to represent classes of objects or properties in the world.

- **for example:**

- weapon
- character
- person
- height
- distance

Predicate Logic: Model and Objects



Possible variables for this model: person and weapon

$D_{\text{person}} = \{\text{lukeSkywalker}, \text{genghisKhan}, \text{wyattEarp}\}$ $D_{\text{weapon}} = \{\text{gun}, \text{sword}, \text{lightSaber}\}$

Syntax: Predicate Symbols

A **predicate** p (p - symbol) **maps object(s) to a boolean value.**

$p(\text{argument}_1, \text{argument}_2, \dots, \text{argument}_N)$ or just p

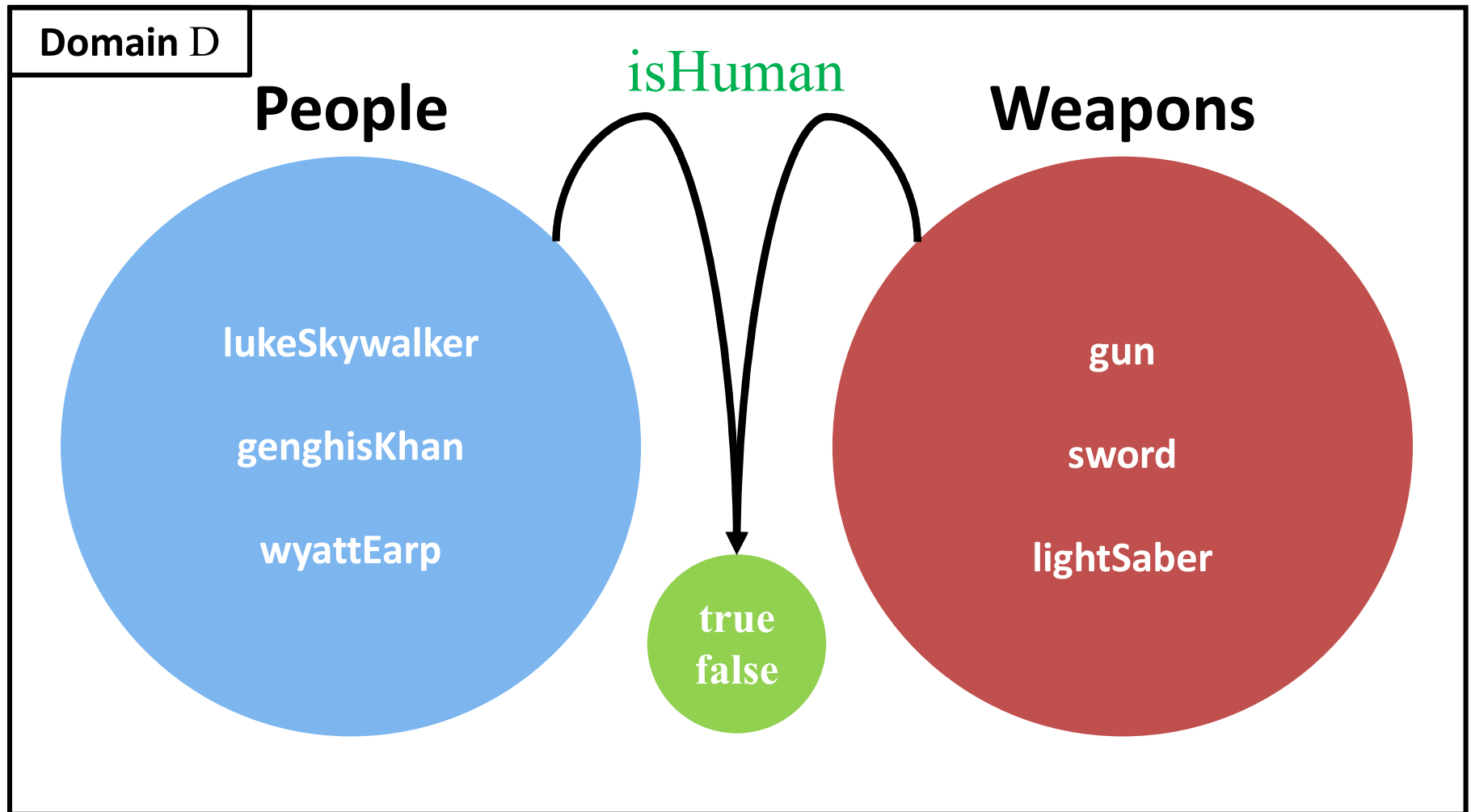
Predicates:

- take **objects** as **arguments**
- output **boolean values** true / false
- for example:

$\text{happy}(\text{student})$ would output value false or true

$\text{wasRaining}(\text{today})$ would output value true

Predicates: Example



$\text{isHuman}(\text{lukeSkywalker}) = \text{true}$

$\text{isHuman}(\text{sword}) = \text{false}$

Syntax: Function Symbols

A **function** f (f - symbol) **maps object(s) to an object.**

$f(\text{argument}_1, \text{argument}_2, \dots, \text{argument}_N)$

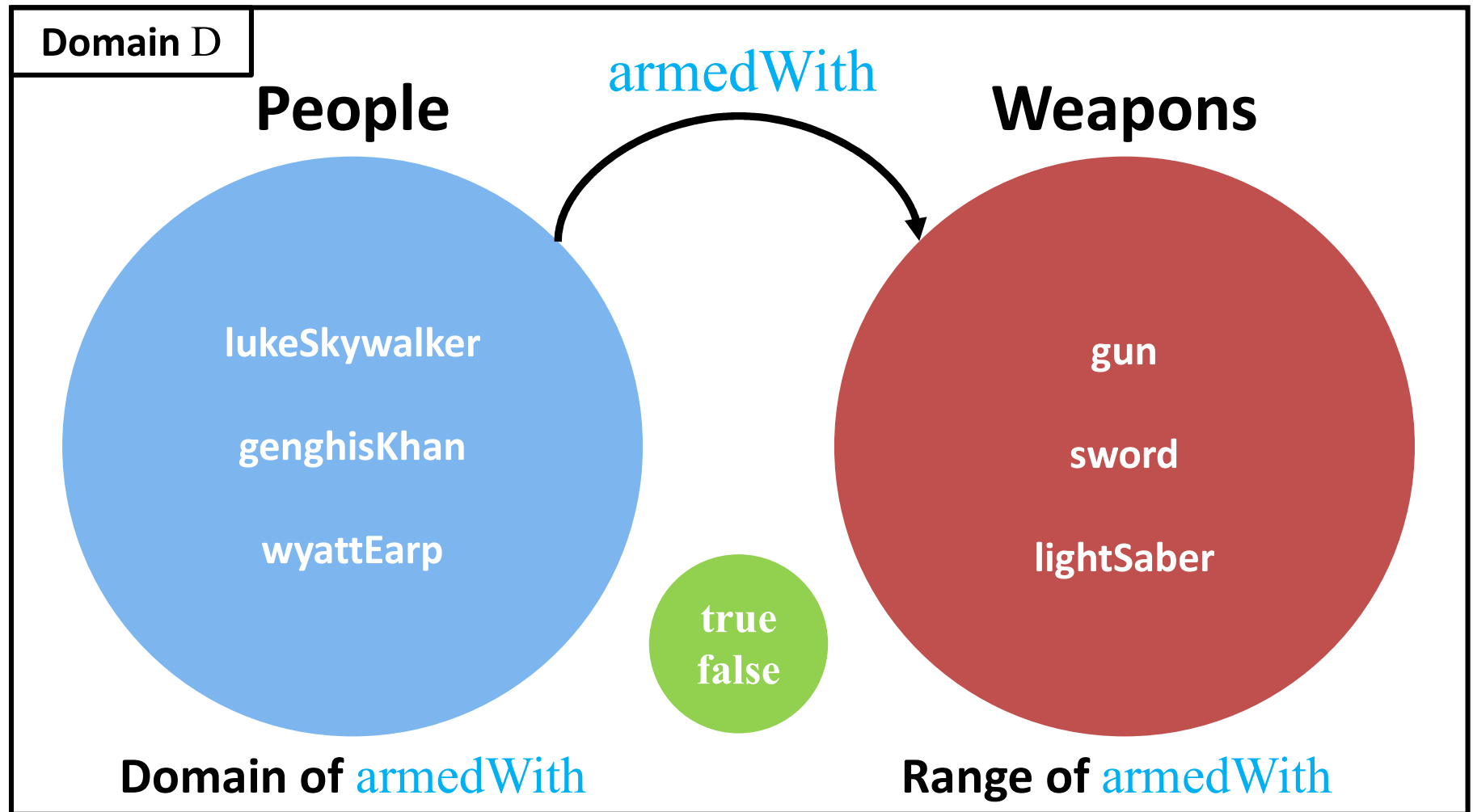
Unlike predicates, functions:

- take **objects** as **arguments**
- output **objects**
- for example:

$\text{currentCourse}(\text{student})$ would output object **cs480**

$\text{instructor}(\text{cs480})$ would output object **Jacek**

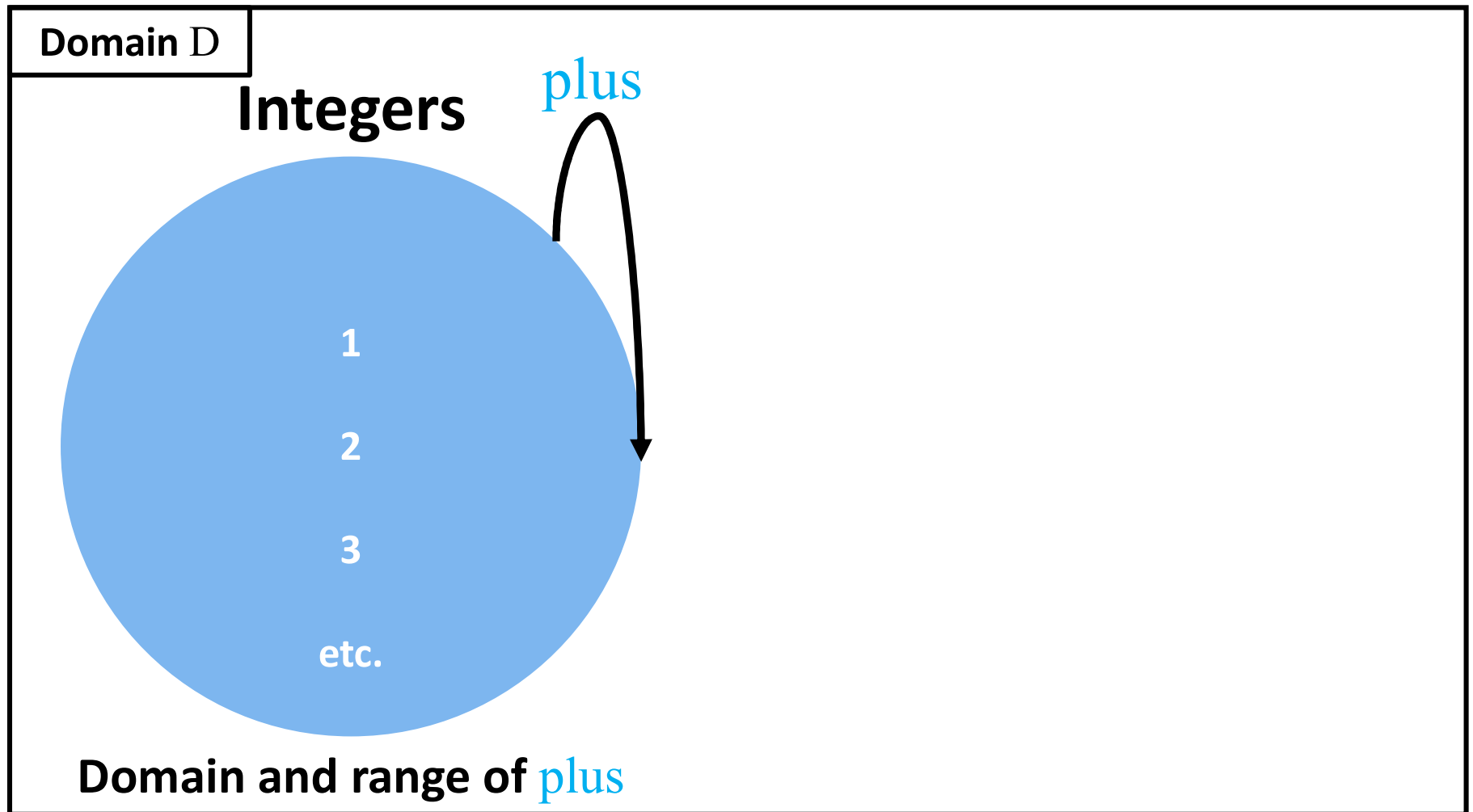
Functions: Example



`armedWith`(lukeSkywalker) = lightSaber

`armedWith`(genghisKhan) = sword

Functions: Example



$$\text{plus}(2,3) = 5$$

Syntax: Terms

A term is a **logical expression that refers to an object**. Terms can be:

- **constants:** lukeSkywalker
- **variables:** son
- **complex term** - a **function symbol** followed by a **parenthesized list of terms** as arguments:

father(lukeSkywalker) or
father(brother(princessLeia))

Both output an OBJECT: darthVader

Syntax: Complex Sentences

Complex sentences can be constructed using logical connectives (just like in propositional logic). For example:

$\neg \text{brother}(\text{leftLeg}(\text{Richard}), \text{John})$

$\text{brother}(\text{Richard}, \text{John}) \wedge \text{brother}(\text{John}, \text{Richard})$

$\text{king}(\text{Richard}) \vee \text{king}(\text{John})$

$\neg \text{king}(\text{Richard}) \Rightarrow \text{king}(\text{John})$

Syntax: Universal Quantifier

Quantifiers allow expressing properties of collections of objects.

Universal quantifier (“for all”) indicates that a sentence is true for **all possible values** of the variable. For example:

$$\forall x \text{ likes}(x, \text{cake})$$

- \forall is the universal quantifier symbol
- x is a variable
- $\text{likes}(x, \text{cake})$ is a sentence

Syntax: Existential Quantifier

Quantifiers allow expressing properties of collections of objects.

Existential quantifier (“there exists”) indicates that a sentence is true for at least one value of the the variable. For example:

$$\exists x \text{ likes}(x, \text{cake})$$

- \exists is the existential quantifier symbol
- x is a variable
- $\text{likes}(x, \text{cake})$ is a sentence

Syntax: Equality

In predicate logic we can use the **equality symbol** to indicate that two terms refer to the same object.

For example:

$$\text{father}(\text{John}) = \text{Henry}$$

This sentence means that the output of function **father**(John) is the same as object (constant) Henry.

Syntax: Summary

Sentence \rightarrow *AtomicSentence* | *ComplexSentence*

AtomicSentence \rightarrow *Predicate* | *Predicate*(*Term*,...) | *Term* = *Term*

ComplexSentence \rightarrow (*Sentence*)
| \neg *Sentence*
| *Sentence* \wedge *Sentence*
| *Sentence* \vee *Sentence*
| *Sentence* \Rightarrow *Sentence*
| *Sentence* \Leftrightarrow *Sentence*
| *Quantifier* *Variable*,... *Sentence*

Term \rightarrow *Function*(*Term*,...)
| *Constant*
| *Variable*

Quantifier \rightarrow \forall | \exists

Constant \rightarrow *A* | *X*₁ | *John* | ...

Variable \rightarrow *a* | *x* | *s* | ...

Predicate \rightarrow *True* | *False* | *After* | *Loves* | *Raining* | ...

Function \rightarrow *Mother* | *LeftLeg* | ...

OPERATOR PRECEDENCE : $\neg, =, \wedge, \vee, \Rightarrow, \Leftrightarrow$

Predicate Calculus: Interpretation

Let the domain D be a non-empty set. An *interpretation* I over D is an assignment of the entities in D to each of the constant, variable, **predicate**, and **function** symbols of a predicate calculus expression, such that:

- each **constant** is assigned an element of D ,
- each **variable** is assigned to non-empty subset of D ; these are the allowable substitutions for that variable,
- each **function** f of arity m is defined on m arguments of D and defines mapping from D^m into D ,
- each **predicate** p of arity n is defined on n arguments from D and defines a mapping from D^n into $\{\text{true}, \text{false}\}$.

Predicate Calculus: Sentences

Predicate calculus sentences are created according to following rules:

- every atomic sentence s is a sentence
- if s a sentence, so it its negation $\neg s$,
- if s_1 and s_2 are sentences, then so is their conjunction $s_1 \wedge s_2$,
- if s_1 and s_2 are sentences, then so is their disjunction $s_1 \vee s_2$,
- if s_1 and s_2 are sentences, then so is their implication $s_1 \Rightarrow s_2$,
- if s_1 and s_2 are sentences, then so is their equivalence $s_1 \Leftrightarrow s_2$,
- if x is a variable s a sentence, then $\forall x$ is a sentence,
- if x is a variable s a sentence, then $\exists x$ is a sentence.

Predicate Calculus: Evaluation

Assume a sentence E and an interpretation I for E over a non-empty domain D . The truth value for E is determined by:

- the value of each **constant** is the element of D it is assigned to by I ,
- the value of a **variable** is the set of elements of D it is assigned to by I ,
- the value of a **function** expression is that element of D obtained by evaluating the **function** for the parameter values I assigned ,
- the value of “true” is true and “false” is false,
- The value of an atomic sentence is either true or false, as determined by the interpretation I .

Predicate Calculus: Evaluation

- the value of negation of a sentence is true if the value of the sentence is false (is false if the value of the sentence is true),
- the value of the conjunction of two sentences is true if the value of both sentences is true and is false otherwise
- similarly, the truth value of expressions using \vee , \Rightarrow , and \Leftrightarrow is determined using related truth tables,
- The value of $\forall x \, s$ is true if s is true **for all assignments** to x under I , and it is false otherwise,
- The value of $\exists x \, s$ is true if s is true if **there is an assignment** to x under I , and it is false otherwise,

First-Order Logic Sentences: Examples

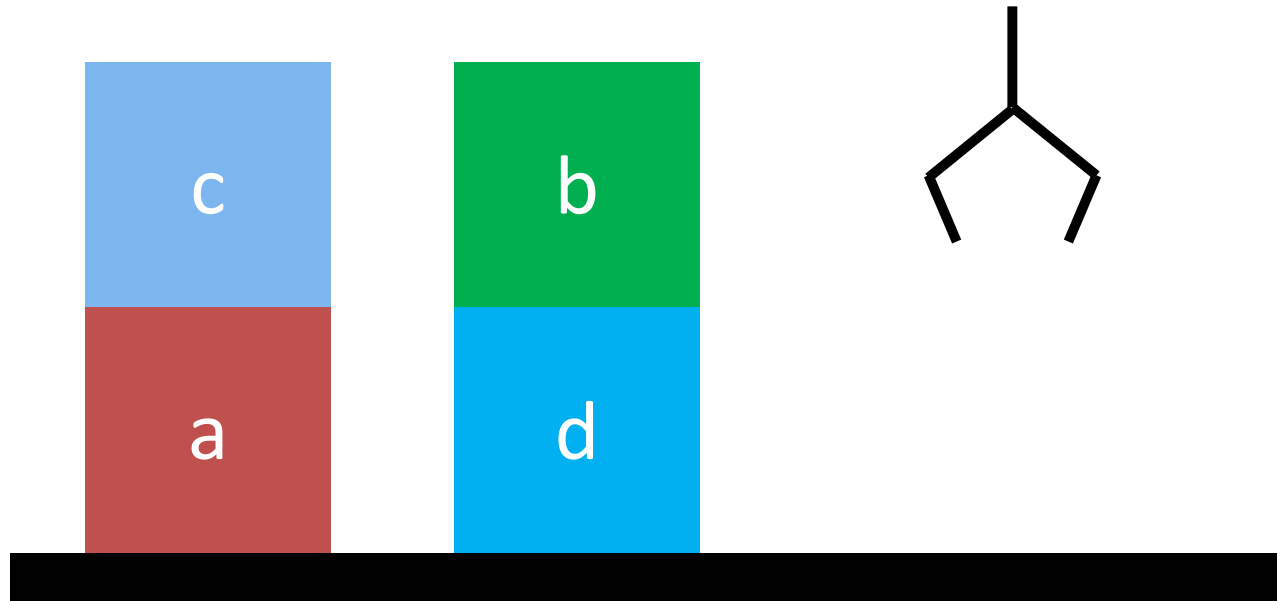
Sentence in First-Order Logic	Sentence in English
$\forall x \text{ frog}(x) \Rightarrow \text{green}(x)$	All frogs are green. (For all x s being a frog implies being green)
$\forall x \text{ frog}(x) \wedge \text{brown}(x) \Rightarrow \text{big}(x)$	All brown frogs are big. (For all x s being a frog and brown implies being big)
$\forall x \text{ likes}(x, \text{cake})$	Everyone likes cake. (All x s like cake.)
$\neg \forall x \text{ likes}(x, \text{cake})$	Not everyone likes cake. (Not all x s like cake.)
$\exists x \forall y \text{ likes}(y, x)$	There is something that everyone likes. (There exists something x that all y s like.)
$\exists x \forall y \text{ likes}(x, y)$	There is someone that likes everyone. (There exists an x that is likes all y s.)
$\forall x \exists y \text{ likes}(y, x)$	Everything is liked by someone. (For all x s there exists a y that likes them.)
$\forall x \exists y \text{ likes}(x, y)$	Everyone likes something. (For all x s there exists a y that is being liked by x .)

First-Order Logic Sentences: Examples

Sentence in First-Order Logic	Sentence in English
$\forall x \text{ customer}(x) \Rightarrow \text{likes}(\text{bob}, x)$	Bob likes every customer. (For all x s being a customer implies being liked by Bob.)
$\exists x \text{ customer}(x) \wedge \text{likes}(x, \text{bob})$	There is a customer who likes Bob. (There exist an x who is a customer and likes Bob.)
$\exists x \text{ baker}(x) \wedge \forall y \text{ customer}(y) \Rightarrow \text{likes}(x, y)$	There is a baker who likes all customers. (There exist an x who is a baker such that for all y s being a customer, implies that y is liked by x .)
$\forall x \text{ older}(\text{mother}(x), x) \quad *$	Every mother is older than her child. (For all x s, mother of x is older than x .)
$\forall x \text{ older}(\text{mother}(\text{mother}(x)), x) \quad *$	Every grandmother is older than her daughter's child. (For all x s, mother of mother of x is older than x .)
$\text{brother}(x, \text{bob}) \Rightarrow \exists y \text{ parent}(y, x) \wedge \text{parent}(y, \text{bob})$	If x is Bob's brother, x and Bob must have the same parent.
$\forall x \forall y \forall z f(x, y) \wedge f(y, z) \Rightarrow f(x, z)$	f is a transitive relation.

* mother is function symbol here

Predicate Calculus: Model Example



Predicate calculus description of the world:

`on(c, a)`

`on(b, d)`

`onTable(a)`

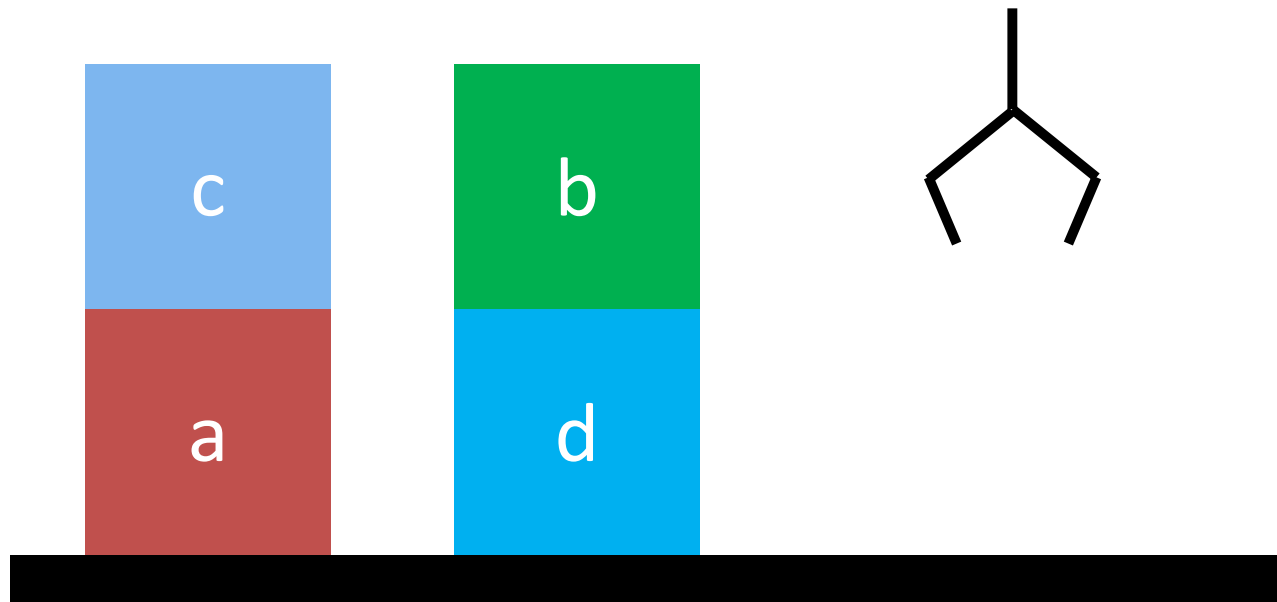
`onTable(d)`

`clear(b)`

`clear(c)`

`manipulatorEmpty`

Predicate Calculus: Objects



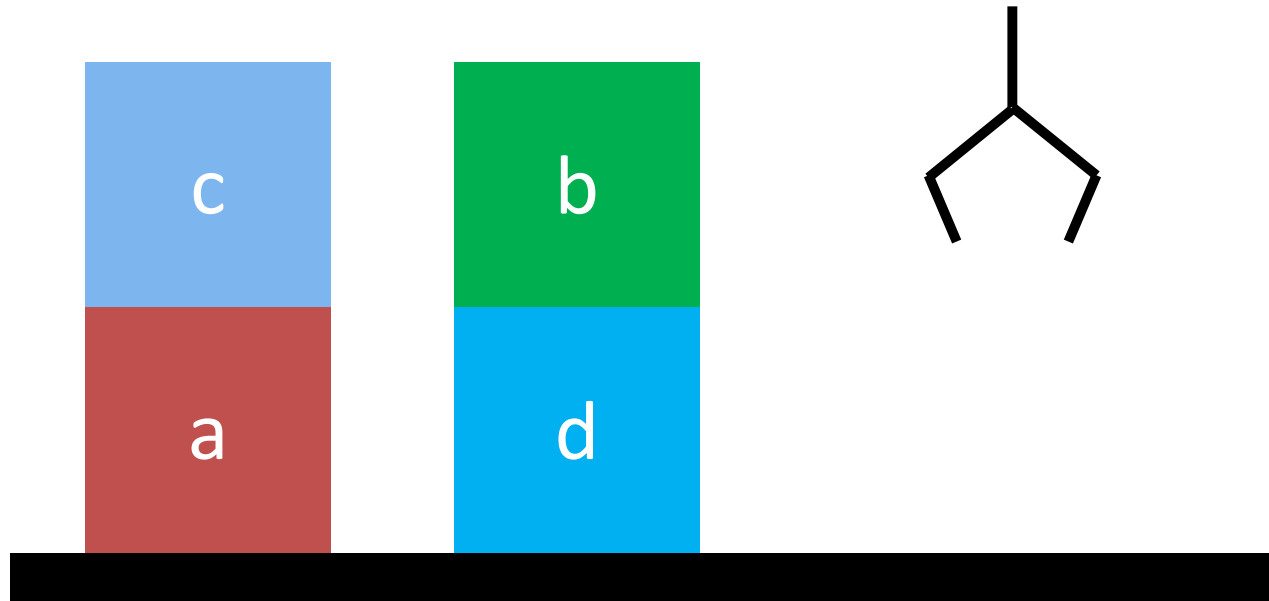
Objects:

table

blocks a, b, c, d

manipulator

Predicate Calculus: Predicates



Predicates:

`on(x, y)`

asserts that object `x` is on top of `y`

`onTable(x)`

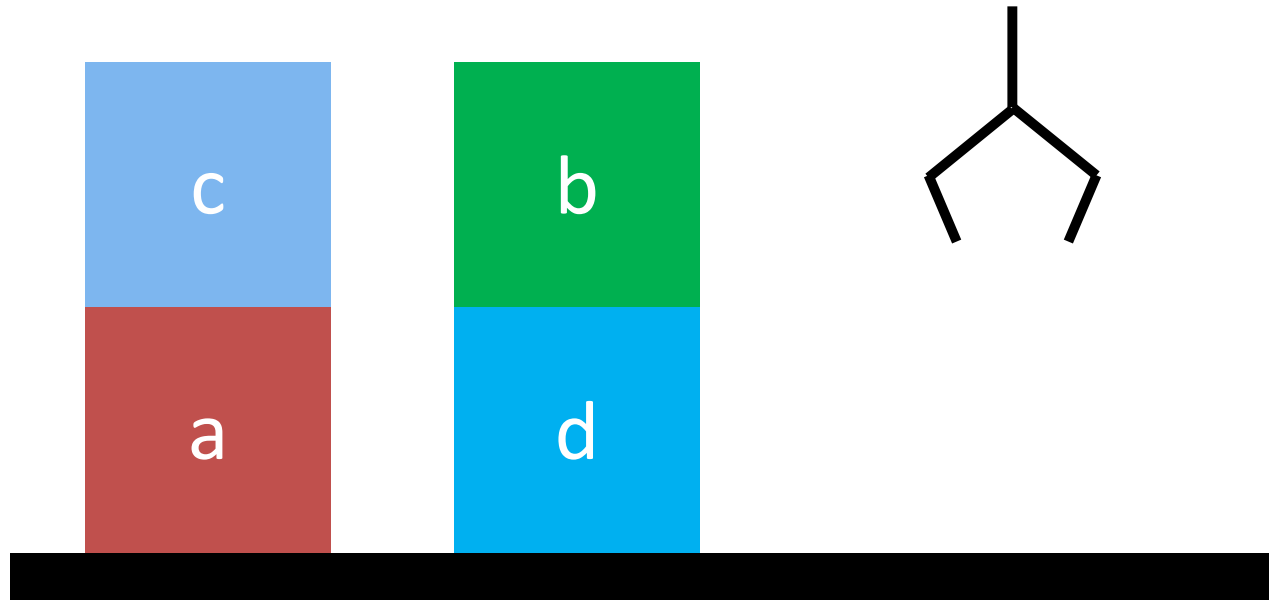
asserts that object `x` is on the table

`clear(x)`

asserts that no objects is on top of `x`

`manipulatorEmpty` asserts the status of the manipulator

Model of the World



Predicate calculus description of the world (model):

`on(c, a)`

`on(b, d)`

`onTable(a)`

`onTable(d)`

`clear(b)`

`clear(c)`

`manipulatorEmpty`

Quantifier Nesting

Quantifiers can be nested to obtain more complex expressions. For example:

$$\forall x \forall y \text{ brother}(x, y) \Rightarrow \text{sibling}(x, y)$$

means “Brothers are siblings”. Here

$$\forall x \forall y \text{ sibling}(y, x) \Leftrightarrow \text{sibling}(x, y)$$

a symmetric relationship is expressed.

Quantifier Nesting: Ordering

When quantifiers are nested it is important to pay attention to ordering. For example:

$$\forall x \exists y \text{ loves}(x, y)$$

means “Everybody loves somebody”. Here

$$\exists x \forall y \text{ loves}(y, x)$$

we have “There exists someone who is loved by everyone”.

Quantifier Nesting (Use Parentheses)

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Quantifier Nesting (Use Parentheses)

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Quantifier Nesting: Variable Names

Certain quantified sentences may be confusing:

$$\forall x (\text{crown}(x) \vee (\exists x \text{ brother}(\text{Richard}, x)))$$

Variable x is used twice:

- universally quantified x in $\forall x (\text{crown}(x) \vee$
- existentially quantified x in $\exists x \text{ brother}(\text{Richard}, x)$
- x and x are NOT the same (different “context”)

Rule:

variable belongs to innermost quantifier that mentions it.

Universal Quantifier: Conjunctions

Universal quantifier (“for all”) indicates that a sentence is true for **all possible values of the variable**. For example:

$$\forall x \text{ likes}(x, \text{cake})$$

is true if $\text{likes}(x, \text{cake})$ is true **for all interpretations** of variable x . Assuming that

$$x \in \{x_1, x_2, \dots, x_n\}$$

we can rewrite $\forall x \text{ likes}(x, \text{cake})$ as:

$$\text{likes}(x_1, \text{cake}) \wedge \text{likes}(x_2, \text{cake}) \wedge \dots \wedge \text{likes}(x_n, \text{cake})$$

Existential Quantifier: Disjunctions

Existential quantifier (“there exists”) indicates that a sentence is true for at least one value of the the variable. For example:

$$\exists x \text{ likes}(x, \text{cake})$$

is true if $\text{likes}(x, \text{cake})$ is true for **at least one** interpretation of variable x . Assuming that

$$x \in \{x_1, x_2, \dots, x_n\}$$

we can rewrite $\exists x \text{ likes}(x, \text{cake})$ as:

$$\text{likes}(x_1, \text{cake}) \vee \text{likes}(x_2, \text{cake}) \vee \dots \vee \text{likes}(x_n, \text{cake})$$

Universal/Existential Quantifiers

We assumed that $x \in \{x_1, x_2, \dots, x_n\}$ and then we rewrote $\forall x \text{ likes}(x, \text{cake})$ as:

$$\text{likes}(x_1, \text{cake}) \wedge \text{likes}(x_2, \text{cake}) \wedge \dots \wedge \text{likes}(x_n, \text{cake})$$

and $\exists x \text{ likes}(x, \text{cake})$ as:

$$\text{likes}(x_1, \text{cake}) \vee \text{likes}(x_2, \text{cake}) \vee \dots \vee \text{likes}(x_n, \text{cake})$$

From De Morgan's rules we can obtain the following equivalence:

$$\forall x \text{ likes}(x, \text{cake}) \equiv \neg \exists x \neg \text{likes}(x, \text{cake})$$

“Everyone likes cake” \equiv “Nobody dislikes cake”

Universal/Existential Q. Equivalences

Selected equivalences:

$$\forall x (P(x) \wedge Q(x)) \equiv \forall x (P(x)) \wedge \forall x (Q(x))$$

$$\exists x (P(x) \vee Q(x)) \equiv \exists x (P(x)) \vee \exists x (Q(x))$$

$$\neg[\exists x (N(x))] \equiv \forall x (\neg N(x))$$

$$\neg[\forall x (N(x))] \equiv \exists x (\neg N(x))$$

Quantifiers: Scope of Quantification

Consider the following sentence:

$$\forall x (P(x) \wedge Q(x))$$

Scope of quantification
for variable x

Variable x is universally quantified in both $P(x)$ and $Q(x)$.

In this sentence:

$$\exists x (P(x) \vee Q(y) \Rightarrow R(x))$$

Scope of quantification
for variable x

Variable x is existentially quantified in both $P(x)$ and $R(x)$.