CS 480

Introduction to Artificial Intelligence

October 20, 2022

Announcements / Reminders

- **Grading TA assignment:**

  https://docs.google.com/spreadsheets/d/1ExS0bKnGt_fdf4LHa3YS1qRA7-Iq4xqXVjfSAPMaGVk/edit?usp=sharing

- **UPDATED Final Exam date:**
  - **November 30th, 2022 (last week of classes!)**

Plan for Today

- **Predicate / First-Order Logic**

Predicate Logic Syntax: Summary

Predicate calculus symbols include:

- **truth symbols: true and false**
- **terms represent specific objects in the world**
  - **constants, variables and functions**
- **predicate symbols refer to a particular relation between objects or represent facts**
- **function symbols refer to objects indirectly (via some relationship)**
- **quantifiers ($\forall$ and $\exists$) and variables refer to collections of objects without explicitly naming each object**

Universal Quantifier: Conjuctions

Universal quantifier ("for all") indicates that a sentence is true for all possible values of the variable. For example:

$$\forall x \; likes(x, cake)$$

is true if $likes(x, cake)$ **is** true **for all interpretations of variable** x**. Assuming that**

$$x \in \{x_1, x_2, \ldots, x_n\}$$

we can rewrite $\forall x \; likes(x, cake)$ **as:**

$$likes(x_1, cake) \wedge likes(x_2, cake) \wedge \ldots \wedge likes(x_n, cake)$$

Existential Quantifier: Disjunctions

Existential quantifier ("there exists") indicates that a sentence is true for at least one value of the the variable. For example:

$$\exists x \ likes(x, cake)$$

is true if $likes(x, cake)$ is true for at least one interpretation of variable x. Assuming that

$$x \in \{x_1, x_2, \ldots, x_n\}$$

we can rewrite $\exists x \ likes(x, cake)$ as:

$$likes(x_1, cake) \lor likes(x_2, cake) \lor \ldots \lor likes(x_n, cake)$$

Universal/Existential Quantifiers

We assumed that $x \in \{x_1, x_2, \ldots, x_n\}$ **and then we rewrote** $\forall x \text{ likes}(x, \text{cake})$ **as:**

$$\text{likes}(x_1, \text{cake}) \wedge \text{likes}(x_2, \text{cake}) \wedge \ldots \wedge \text{likes}(x_n, \text{cake})$$

and $\exists x \text{ likes}(x, \text{cake})$ **as:**

$$\text{likes}(x_1, \text{cake}) \vee \text{likes}(x_2, \text{cake}) \vee \ldots \vee \text{likes}(x_n, \text{cake})$$

From De Morgan's rules we can obtain the following equivalence:

$$\forall x \text{ likes}(x, \text{cake}) \equiv \neg \exists x \neg \text{likes}(x, \text{cake})$$

"Everyone likes cake" ≡ **"Nobody dislikes cake"**

Universal/Existential Q. Equivalences

Selected equivalences:

$$\forall x\ (P(x) \wedge Q(x)) \equiv \forall x\ (P(x)) \wedge \forall x\ (Q(x))$$

$$\exists x\ (P(x) \vee Q(x)) \equiv \exists x\ (P(x)) \vee \exists x\ (Q(x))$$

$$\neg[\exists x\ (N(x))] \equiv \forall x\ (\neg N(x))$$

$$\neg[\forall x\ (N(x))] \equiv \exists x\ (\neg N(x))$$

Quantifiers: Scope of Quantification

Consider the following sentence:

$$\forall x\ \underbrace{(P(x) \wedge Q(x))}_{\text{Scope of quantification for variable } x}$$

Variable x is universally quantified in both $P(x)$ and $Q(x)$.

In this sentence:

$$\exists x\ \underbrace{(P(x) \vee Q(y) \Rightarrow R(x))}_{\text{Scope of quantification for variable } x}$$

Variable x is existentionally quantified in both $P(x)$ and $R(x)$.

Proof by Resolution

The process of proving by resolution is as follows:

A. Formalize the problem: "English to Predicate Logic"

B. Derive $KB \wedge \neg Q$

C. Convert $KB \wedge \neg Q$ into CNF ("standardized") form

D. Apply resolution rule to resulting clauses. New clauses will be generated (add them to the set if not already present)

E. Repeat (C) until:

   a. no new clause can be added (KB does NOT entail Q)

   b. last two clauses resolve to yield the empty clause (KB entails Q)

Proof by Resolution

The process of proving by resolution is as follows:

A. **Formalize the problem: “English to Predicate Logic”**

B. **Negate the input statement/claim C to obtain $\neg C$**

C. **Convert $\neg C$ into CNF (“standardized”) form**

D. **Apply resolution rule to resulting clauses. New clauses will be generated (add them to the set if not already present)**

E. **Repeat (C) until:**

   a. **no new clause can be added (C is false)**

   b. **last two clauses resolve to yield the empty clause (C is true)**

Problem: Argument Mismatch

The goal of the resolution process is to find a contradiction. When two contradicting literals "cancel" each other out, we can end up with an empty clause:

$$\frac{(w), (\neg w)}{()}$$

It is not so easy in predicate logic. This

$$\frac{(\text{setting}(\text{sun})), (\neg\text{setting}(\text{sun}))}{()}$$

will work (predicate arguments match). This

$$\frac{(\text{beautiful}(\text{day})), (\neg\text{beautiful}(\text{night}))}{???????}$$

will not, because predicate arguments don't match.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

1. Remove Equivalences/Implications

Use propositional logic laws to do it where possible.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

2. Reduce the Scope of All $\neg$

Consider a predicate $N(x)$ asserting the fact that x is non-vegetarian. Now, let's create the following sentence:

$$\forall x\ (\neg N(x))$$

Which roughly translates to "No one is a non-vegetarian." Let's try a slightly different sentence:

$$\neg[\forall x\ (N(x))]$$

Which roughly translates to "It is not true that everyone is a non-vegetarian". This also means "At least one person is NOT a non-vegetarian" and we could rewrite it as:

$$\exists x\ (\neg N(x))\text{, so } \neg[\forall x\ (N(x))] \equiv \exists x\ (\neg N(x))$$

Using the following logic, we can establish this equivalence:

$$\forall x\ (\neg N(x)) \equiv \neg[\exists x\ (N(x))]$$

2. Reduce the Scope of All $\neg$

Recall that for a domain of objects $\{a, b, c\}$, the sentence

$\forall x\ (N(x))$ **is equivalent to** $N(a) \wedge N(b) \wedge N(c)$

Similarly, the sentence equivalence:

$\exists x\ (N(x))$ **is equivalent to** $N(a) \vee N(b) \vee N(c)$

Now, if we apply De Morgan's rules, we get the following:

$$\neg[N(a) \vee N(b) \vee N(c)] \equiv [\neg N(a) \wedge \neg N(b) \wedge \neg N(c)]$$

and:

$$\neg[N(a) \wedge N(b) \wedge N(c)] \equiv [\neg N(a) \vee \neg N(b) \vee \neg N(c)]$$

2. Reduce the Scope of All $\neg$

Recall that for a domain of objects $\{a, b, c\}$, the sentence

$\forall x\ (N(x))$ **is equivalent to** $N(a) \wedge N(b) \wedge N(c)$

Similarly, the sentence equivalence:

$\exists x\ (N(x))$ **is equivalent to** $N(a) \vee N(b) \vee N(c)$

Now, if we apply De Morgan's rules, we get the following:

$$\neg\underbrace{[N(a) \vee N(b) \vee N(c)]}_{\neg[\exists x\ (N(x))]} \equiv \underbrace{[\neg N(a) \wedge \neg N(b) \wedge \neg N(c)]}_{\forall x\ (\neg N(x))}$$

and:

$$\neg\underbrace{[N(a) \wedge N(b) \wedge N(c)]}_{\neg[\forall x\ (N(x))]} \equiv \underbrace{[\neg N(a) \vee \neg N(b) \vee \neg N(c)]}_{\exists x\ (\neg N(x))}$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

3. Make All Variable Names Unique

Given a qualified sentence:

$$\forall x\ (crown(x) \lor (\exists x\ brother(Richard, x))$$

change variables to avoid duplicates:

$$\forall x\ (crown(x) \lor (\exists z\ brother(Richard, z))$$

3. Make All Variable Names Unique

Given a qualified sentence:

$$\forall x\ (P(x) \Rightarrow Q(x)) \land \exists x\ (Q(x)) \land \exists z\ (P(z)) \land \exists x\ (Q(z) \Rightarrow R(z))$$

change variables to avoid duplicates:

$$\forall y\ (P(y) \Rightarrow Q(y)) \land \exists u\ (Q(u)) \land \exists w\ (P(w)) \land \exists z\ (Q(z) \Rightarrow R(z))$$

Also called: "standardizing variables apart"

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

4. Move Quantifiers Left

Consider the following sentence in predicate logic:

$$\exists x\ (A(x) \lor \forall x\ (B(x)))$$

The two occurences of x **(**x **and** x**) do not refer to the same variable. Let's make all variables unique (standardize) first:**

$$\exists x\ (A(x) \lor \forall y\ (B(y)))$$

Because variable y **bound by** $\forall y$ **does not interact with the variable** x **bound by** $\exists x$**, we can extend the scope of the universal quantifier** $\forall y$ **to entire sentence:**

$$\exists x\ \forall y\ (A(x) \lor (B(y)))$$

Now, $A(x) \lor (B(y))$ **is almost a propositional logic sentence.**

4. Move Quantifiers Left | PNF

A predicate logic formula φ is in prenex normal form (PNF) if it holds that:

- $\varphi = Q_1 x_1 \ldots Q_n x_n \, \psi$
- ψ **is a quantifierless sentence**
- $Q_i \in \{\forall, \exists\}$ **for** $i = 1, \ldots, n$

For example this sentence is NOT in PNF:

$$\exists x \, (A(x) \vee \forall y \, (B(y)))$$

This sentence is in PNF:

$$\exists x \, \forall y \, (A(x) \wedge (B(y)))$$

4. Move Quantifiers Left | PNF

Every predicate logic sentence can be transformed into an equivalent sentence in prenex normal form.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

5. Eliminating Existential Quantifiers

In order to convert a predicate logic to a propositional logic CNF form we need to remove quantifiers.

Existential quantifiers can appear in sentences:

- **in isolation ($\exists$ is OUTSIDE the scope of some $\forall$):**

$$\exists x\ (A(x))$$

This can be resolved using Skolem constant(s).

- **in relation ($\exists$ is INSIDE the scope of some $\forall$)**

$$\forall y\ (\exists x\ (A(x,\ y)))$$

This can be resolved using Skolem function(s).

The process is known as skolemization.

5. Eliminating Existential Quantifiers

Any object variable that is existentially quantified <u>outside</u> <u>of</u> the scope of a <u>universal</u> quantifier, such as

$$\exists x\ (A(x))$$

can be replaced by a single new constant expression $A(t)$, where t is a Skolem constant:

$$\exists x\ (A(t))$$

and the existential quantifier can be dropped to obtain

$$A(t)$$

5. Eliminating Existential Quantifiers

With multiple variables that are existentially quantified outside of the scope of a universal quantifier, such as

$$\exists x, y\ (A(x, y))$$

multiple corresponding Skolem constants (t, u) will be needed to create a new constant expression $A(t, u)$,

$$\exists x, y\ (A(t, u))$$

and the existential quantifier can be dropped to obtain

$$A(t, u)$$

5. Eliminating Existential Quantifiers

With multiple variables bound to different existential quantifiers outside of the scope of a universal quantifier:

$$[\exists x\ (B(x))] \vee [\exists y\ (C(y))]$$

multiple Skolem constants t, u **will be needed to create new constant expressions** $B(t)$ **and** $C(u)$**:**

$$[\exists x\ (B(t))] \vee [\exists y\ (C(u))]$$

and existential quantifiers can be dropped to obtain

$$[B(t)] \vee [C(u)]$$

5. Eliminating Existential Quantifiers

Any object variable that is existentially quantified outside of the scope of a universal quantifier, such as

$$[\forall x\ (B(x))] \vee [\exists y\ (C(y))]$$

can be replaced by a single new constant expression $C(t)$, where t is a Skolem constant:

$$[\forall x\ (B(x))] \vee [\exists y\ (C(t))]$$

and the existential quantifier can be dropped to obtain

$$[\forall x\ (B(x))] \vee [(C(t))]$$

5. Eliminating Existential Quantifiers

We can say that following predicate logic sentences:

$$\exists x\, (A(x)) \equiv_I A(t)$$

$$\exists x, y\, (A(x, y)) \equiv_I A(t, u)$$

$$[\exists x\, (B(x))] \vee [\exists y\, (C(y))] \equiv_I [B(t)] \vee [C(u)]$$

$$[\forall x\, (B(x))] \vee [\exists y\, (C(y))] \equiv_I [\forall x\, (B(x))] \vee [(C(t))]$$

are <u>inferentially</u> equivalent ($\equiv_I$). Skolemization leads to sentences that are not completely equivalent, but this is good enough for proofs and inference.

5. Eliminating Existential Quantifiers

Inferentially equivalent sentences are not completely equivalent, but this is good enough for proofs. Why?

Consider following two predicate logic sentences:

$\exists$x (studies(x)): **there exist at least one** x **who studies**

studies(t): t **studies (just one, specific object** t**)**

Constant t **is assumed to be a possible value for variable** x**.**
If for some object t, studies(t) **is true, then** $\exists$x (studies(x)) **also must be true (**t **and <u>possibly</u> other objects** study**).**

5. Eliminating Existential Quantifiers

Note: when choosing a Skolem constant for a existentially quantified expressions such as:

$$\exists x\,(A(x))$$

DON'T choose EXISTING constants as Skolem constants to create a new constant expression $A(t)$, where t is a Skolem constant:

So: $\exists x\,(A(t))$ **YES,**

but $\exists x\,(A(lukeSkywalker))$ **NO**

Assuming that $lukeSkywalker$ **is an existing object.**

5. Eliminating Existential Quantifiers

Any object variable that is existentially quantified <u>inside of</u> the scope of a <u>universal</u> quantifier, such as:

$$\forall y\ (\exists x\ (A(x,\ y)))$$

can be replaced by with a Skolem function of the universal variable $f(y)$:

$$\forall y\ (\exists x\ (A(f(y),\ y)))$$

and the existential quantifier can be dropped to obtain

$$\forall y\ (A(f(y),\ y))$$

5. Eliminating Existential Quantifiers

An existential quantifier inside of the scope of MORE THAN ONE universal quantifier, such as:

$$\forall y\ \forall z\ (\exists x\ (B(x, y, z)))$$

can be replaced by with a multivariable Skolem function of $g(y, z)$**:**

$$\forall y\ \forall z\ (\exists x\ (B(g(y, z), y, z)))$$

and the existential quantifier can be dropped to obtain

$$\forall y\ \forall z\ (B(g(y, z), y, z))$$

5. Eliminating Existential Quantifiers

Consider the following example:

$$\forall x\ [\exists y\ (A(x) \Rightarrow B(y)) \lor \forall w\ (\exists z\ (D(x) \land E(w) \land F(z) \Rightarrow C(z))]$$

can be modified using skolemization to obtain:

$$\forall x\ [(A(x) \Rightarrow B(\underline{f(x)})) \lor \forall w\ ((D(x) \land E(w) \land F(\underline{g(x, w)}) \Rightarrow C(\underline{g(x, w)})))]$$

Skolem functions $f()$ **and** $g()$.

Variable y **is inside the scope of** $\forall x$**, hence:** $f(x)$

Variable z **is inside the scope of** $\forall x$ **and** $\forall w$**, hence:** $g(x, w)$

5. Eliminating Existential Quantifiers

In general: existential quantifiers can also be eliminated through the use of Existential Instantiation.

For any sentence S, variable x, and constant symbol k (that does not appear anywhere in the knowledge base), we can use the following rule:

$$\frac{\exists x\ S}{\mathrm{SUBST}(\{x / k\}, S)}$$

Where is a result of applying substitution $\{x / k\}$ to the sentence S.

5. Eliminating Existential Quantifiers

For example, from the sentence:

$$\exists x\ (crown(x) \wedge onHead(x, John))$$

we can infer the sentence

$$crown(C_1) \wedge onHead(C_1, John)$$

using the substitution $\{x / C_1\}$ **as long as** C_1 **does not exist in the knowledge base.**

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

6. Eliminating Universal Quantifiers

In general universal quantifiers can also be eliminated through the use of Universal Instantiation.

For any sentence S, variable x, and constant symbol g (that <u>does not appear anywhere in the knowledge base</u>), we can use the following rule:

$$\frac{\forall x\ S}{\mathrm{SUBST}(\{x / g\}, S)}$$

Where is a result of applying substitution $\{x / g\}$ to the sentence S.

6. Eliminating Universal Quantifiers

For example, from the sentence:

$$\forall x\ (king(x) \land greedy(x) \Rightarrow evil(x))$$

we can infer the sentence

$$king(John) \land greedy(John) \Rightarrow evil(John)$$

using the substitution $\{x\ /\ John\}$**.**

Problem: Argument Mismatch

The goal of the resolution process is to find a contradiction. When two contradicting literals “cancel” each other out, we can end up with an empty clause:

$$\frac{(w), (\neg w)}{()}$$

It is not so easy in predicate logic. This

$$\frac{(\text{setting}(\text{sun})), (\neg\text{setting}(\text{sun}))}{()}$$

will work (predicate arguments match). This

$$\frac{(\text{beautiful}(\text{day})), (\neg\text{beautiful}(\text{night}))}{???????}$$

will not, because predicate arguments don’t match.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Universal Quantifier: Conjuctions

Universal quantifier ("for all") indicates that a sentence is true for all possible values of the variable. For example:

$$\forall x \; likes(x, cake)$$

is true if $likes(x, cake)$ **is** true **for all interpretations of variable** x**. Assuming that**

$$x \in \{x_1, x_2, \ldots, x_n\}$$

we can rewrite $\forall x \; likes(x, cake)$ **as:**

$$likes(x_1, cake) \wedge likes(x_2, cake) \wedge \ldots \wedge likes(x_n, cake)$$

Eliminating Universal Quantifiers

In general universal quantifiers can also be eliminated through the use of Universal Instantiation.

For any sentence S, variable x, and constant symbol g (that <u>does not appear anywhere in the knowledge base</u>), we can use the following rule:

$$\frac{\forall x\ S}{\text{SUBST}(\{x / g\}, S)}$$

Where is a result of applying substitution $\{x / g\}$ to the sentence S.

Eliminating Universal Quantifiers

For example, from the sentence:

$$\forall x\ (king(x) \wedge greedy(x) \Rightarrow evil(x))$$

we can infer the sentence

$$king(John) \wedge greedy(John) \Rightarrow evil(John)$$

using the substitution $\{x\ /\ John\}$**.**

Eliminating Existential Quantifiers

In general: existential quantifiers can also be eliminated through the use of **Existential Instantiation.**

For any sentence S, variable x, and constant symbol k (that <u>does not appear anywhere in the knowledge base</u>), we can use the following rule:

$$\frac{\exists x\ S}{\text{SUBST}(\{x / k\}, S)}$$

Where is a result of applying substitution $\{x / k\}$ to the sentence S.

Propositionalization

The idea:

- **Replace an existentially quantified sentence with ONE instantiation (Skolemization)**
- **Replace an universally quantified sentence with ALL POSSIBLE instantiations**

For example, from the sentence:

$$\forall x\ (king(x) \wedge greedy(x) \Rightarrow evil(x))$$

Assume: there are TWO possible values/objects for x**:** $\{John, Richard\}$**. We obtain:**

$$(king(John) \wedge greedy(John) \Rightarrow evil(John))$$

$$(king(Richard) \wedge greedy(Richard) \Rightarrow evil(Richard))$$

Propositionalization

Now, we can continue the conversion of:

$$(\text{king}(\text{John}) \wedge \text{greedy}(\text{John}) \Rightarrow \text{evil}(\text{John}))$$
$$(\text{king}(\text{Richard}) \wedge \text{greedy}(\text{Richard}) \Rightarrow \text{evil}(\text{Richard}))$$

by replacing each atomic predicate logic symbol with a propositional logic symbol

$$(\text{JohnIsKing} \wedge \text{JohnIsGreedy} \Rightarrow \text{JohnIsEvil})$$
$$(\text{RichardIsKing} \wedge \text{RichardIsGreedy} \Rightarrow \text{RichardIsEvil})$$

Can you see potential problems?

Propositionalization

What if, in addition to:

$$(king(John) \wedge greedy(John) \Rightarrow evil(John))$$

$$(king(Richard) \wedge greedy(Richard) \Rightarrow evil(Richard))$$

we also had a function Father(..)**?**

You can easily end up with infinite nesting of the following nature:

$$Father(Father(Father(Father(John)))$$

That leads to an infinite number of clauses!

Unification

Predicate logic inference rules require finding substitutions that make two different logical expressions look identical.

The process is called unification. A UNIFY **algorithm takes two sentences** p **and** q **and returns a unifier** θ **for them (a substitution) if one exists:**

$$\text{UNIFY}(p, q) = \theta, \text{ where } \text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Unification: Examples

UNIFY(sentenceA, sentenceB) = {unifier for sentenceA and sentenceB}

UNIFY(p, q) = {θ}

UNIFY(p, q) = {variable / unifying value}

Examples:

UNIFY(Knows(John, x), Knows(John, Jane)) = {x/Jane}

UNIFY(Knows(John, x), Knows(y, Bill)) = {x/Bill, y/John}

UNIFY(Knows(John, x), Knows(y, Mother(y))) = {x/Mother(John), y/John}

UNIFY(Knows(John, x), Knows(x, Elizabeth) = failure

({x/John, x/Elizabeth} is not possible)

Most General Unifier (MGU)

But.... ther can be multiple unifiers for a pair of sentences. Which one to choose?

Every UNIFIABLE pair of sentences has a SINGLE most general unifier that is unique.

UNIFY algorithm will find MGU.

Unification

function UNIFY(x, y, θ=*empty*) **returns** a substitution to make x and y identical, or *failure*
  if θ = *failure* **then return** *failure*
  else if $x = y$ **then return** θ
  else if VARIABLE?(x) **then return** UNIFY-VAR(x, y, θ)
  else if VARIABLE?(y) **then return** UNIFY-VAR(y, x, θ)
  else if COMPOUND?(x) **and** COMPOUND?(y) **then**
    return UNIFY(ARGS(x), ARGS(y), UNIFY(OP(x), OP(y), θ))
  else if LIST?(x) **and** LIST?(y) **then**
    return UNIFY(REST(x), REST(y), UNIFY(FIRST(x), FIRST(y), θ))
  else return *failure*

function UNIFY-VAR(var, x, θ) **returns** a substitution
  if $\{var/val\} \in \theta$ for some val **then return** UNIFY(val, x, θ)
  else if $\{x/val\} \in \theta$ for some val **then return** UNIFY(var, val, θ)
  else if OCCUR-CHECK?(var, x) **then return** *failure*
  else return add $\{var/x\}$ to θ

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

By Implication Law ($p \Rightarrow q \equiv \neg p \vee q$):

$$\forall w\ ([\underline{\mathbf{P_1(w) \vee P_2(w) \Rightarrow P_3(w)}}] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

becomes:

$$\forall w\ ([\underline{\mathbf{\neg(P_1(w) \vee P_2(w)) \vee P_3(w)}}] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

By Implication Law ($p \Rightarrow q \equiv \neg p \vee q$):

$$\forall w\ ([\neg(P_1(w) \vee P_2(w)) \vee P_3(w)] \vee [\exists x\ (\exists y\ (\underline{\mathbf{P_6(x, y) \Rightarrow P_4(w, x)}}))]) \wedge [\forall w\ (P_5(w))]$$

becomes:

$$\forall w\ ([\neg(P_1(w) \vee P_2(w)) \vee P_3(w)] \vee [\exists x\ (\exists y\ (\underline{\mathbf{\neg P_6(x, y) \vee P_4(w, x)}}))]) \wedge [\forall w\ (P_5(w))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

By De Morgan's Law ($\neg(p \vee q) \equiv \neg p \wedge \neg q$):

$$\forall w\ ([\underline{\neg(\mathbf{P_1(w)} \vee \mathbf{P_2(w)})} \vee P_3(w)] \vee [\exists x\ (\exists y\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\forall w\ (P_5(w))]$$

becomes:

$$\forall w\ ([\underline{(\neg \mathbf{P_1(w)} \wedge \neg \mathbf{P_2(w)})} \vee P_3(w)] \vee [\exists x\ (\exists y\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\forall w\ (P_5(w))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

Variable w **(**w **and** w**) is bound to two different quantifiers:**

$$\forall w\ ([\neg(P_1(w) \lor P_2(w)) \lor P_3(w)] \lor [\exists x\ (\exists y\ (\neg P_6(x, y) \lor P_4(w, x))]) \land [\forall w\ (P_5(w))]$$

Replace w **with** z **and the sentence S becomes:**

$$\forall w\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [\exists x\ (\exists y\ (\neg P_6(x, y) \lor P_4(w, x))]) \land [\forall z\ (P_5(z))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

Quantified variables unique, move quantifiers left (order!):

$$\forall w\ ([\neg(P_1(w) \vee P_2(w)) \vee P_3(w)] \vee [\underline{\exists \mathbf{x}\ (\exists \mathbf{y}}\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\underline{\forall \mathbf{z}}\ (P_5(z))]$$

becomes:

$$\forall w\ \underline{\exists \mathbf{x}\ \exists \mathbf{y}\ \forall \mathbf{z}}\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\neg P_6(x, y) \vee P_4(w, x))]) \wedge [(P_5(z))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

We have two existential quantifiers to remove here:

$$\forall w\ \underline{\exists x}\ \exists y\ \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg \mathbf{P_6}(\mathbf{x}, \mathbf{y})} \lor \underline{\mathbf{P_4}(\mathbf{w}, \mathbf{x})})]) \land [(P_5(z))]$$

and:

$$\forall w\ \exists x\ \underline{\exists y}\ \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg \mathbf{P_6}(\mathbf{x}, \mathbf{y})} \lor P_4(w, x))]) \land [(P_5(z))]$$

Both $\exists x$ and $\exists y$ are inside the scope of the universal quantifier $\forall w$. We need to use Skolem function substitution (Skolemization).

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

Let's start with $\exists x$ and replace x with a Skolem function:

$$\forall w\ \underline{\exists x}\ \exists y\ \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg P_6(x, y)} \vee \underline{P_4(w, x)})]) \wedge [(P_5(z))]$$

becomes:

$$\forall w \exists y \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg P_6(f(w), y)} \vee \underline{P_4(w, f(w))})]) \wedge [(P_5(z))]$$

Quantified variable x was replaced with Skolem function $f(w)$. Existential quantifier $\exists x$ was removed.

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

Now: remove $\exists y$ and replace y with a Skolem function:

$$\forall w \exists y \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg \mathbf{P_6}(\mathbf{f}(\mathbf{w}), \mathbf{y})} \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

becomes:

$$\forall w \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg \mathbf{P_6}(\mathbf{f}(\mathbf{w}), \mathbf{g}(\mathbf{w}))} \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

Quantified variable y was replaced with Skolem function $g(w)$. Existential quantifier $\exists y$ was removed.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall z\ (P_5(z))]$$

Remaining quantified variables are universally quantified:

$$\forall w \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall z\ (P_5(z))]$$

We can simply "drop" universal quantifiers:

$$\forall w \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

becomes:

$$([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

We are "dropping" universal quantifiers for inferential purposes only. Equivalence is lost, but we can still use the remaining sentence to infer.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts (CNF)**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

By Associative Law $((p \vee q) \vee r \Leftrightarrow p \vee (q \vee r))$**:**

$$([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

becomes:

$$([P_3(w) \vee (\neg P_1(w) \wedge \neg P_2(w))] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

By Associative Law $((p \lor q) \lor r \Leftrightarrow p \lor (q \lor r))$**:**

$$([\underline{\mathbf{P_3(w)} \lor (\neg \mathbf{P_1(w)} \land \neg \mathbf{P_2(w)})}] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

becomes:

$$([\underline{\mathbf{P_3(w)} \lor (\neg \mathbf{P_1(w)} \land \neg \mathbf{P_2(w)})}] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

By Distributive Law ($p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$):

$$([\underline{\mathbf{P_3(w) \lor (\neg P_1(w) \land \neg P_2(w))}}] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

becomes:

$$([\underline{\mathbf{(P_3(w) \lor \neg P_1(w)) \land (P_3(w) \lor \neg P_2(w))}}] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Let's make some substitutions:

$$([(P_3(w) \vee \neg P_1(w)) \wedge (P_3(w) \vee \neg P_2(w))] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

$A \equiv (P_3(w) \vee \neg P_1(w))$

$B \equiv (P_3(w) \vee \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$

so the sentence becomes:

$$([A \wedge B] \vee [C]) \wedge [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

By Distributive Law ($p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$):

$$([A \land B] \lor [C]) \land [(P_5(z))]$$

becomes:

$$((A \lor C) \land (B \lor C)) \land [(P_5(z))]$$

where:

$A \equiv (P_3(w) \lor \neg P_1(w))$

$B \equiv (P_3(w) \lor \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

Remove substitutions:

$$((A \lor C) \land (B \lor C)) \land [(P_5(z))]$$

becomes:

$$(((P_3(w) \lor \neg P_1(w)) \lor (\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))) \land ((P_3(w) \lor \neg P_2(w)) \lor (\neg P_6(f(w), g(w)) \lor P_4(w, f(w))))) \land [(P_5(z))]$$

where:

$A \equiv (P_3(w) \lor \neg P_1(w))$

$B \equiv (P_3(w) \lor \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

We can remove some parentheses:

$$(((P_3(w) \vee \neg P_1(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))) \wedge ((P_3(w) \vee \neg P_2(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w))))) \wedge [(P_5(z))]$$

becomes:

$$(P_3(w) \vee \neg P_1(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$$
$$\wedge\ (P_3(w) \vee \neg P_2(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$$
$$\wedge\ (P_5(z))$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

We obtained sentence S in CNF form:

$$(P_3(w) \lor \neg P_1(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$$
$$\land\ (P_3(w) \lor \neg P_2(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$$
$$\land\ (P_5(z))$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts (CNF)**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

Let's number all clauses:

$$(P_3(w) \lor \neg P_1(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))_1$$
$$\land\ (P_3(w) \lor \neg P_2(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))_2$$
$$\land\ (P_5(z))_3$$