

# CS 480

## *Introduction to Artificial Intelligence*

September 29, 2022

# Announcements / Reminders

- **Midterm Exam: October 13th!**
  - **Online section:** please make arrangements. Contact Mr. Charles Scott (scott@iit.edu) if in doubt
- **Written Assignment #02:**
  - will be posted next week
- **Quiz #02:**
  - due: October 9th, 11:00 PM CST
- **Programming Assignment #01:**
  - due: October 15th, 11:00 PM CST
- **Please follow the Week 06 To Do List instructions**
- **Spring Semester midterm course evaluation is UP.**
  - Please fill it out if you can. It means a lot to me.
- **Grading TA assignment:**

[https://docs.google.com/spreadsheets/d/1ExS0bKnGt\\_fdf4LHa3YS1qRA7-lq4xqXVjfSAPMaGVk/edit?usp=sharing](https://docs.google.com/spreadsheets/d/1ExS0bKnGt_fdf4LHa3YS1qRA7-lq4xqXVjfSAPMaGVk/edit?usp=sharing)

# Plan for Today

- **Propositional logic**

# Propositional Logic and KB-Agents

**Propositional  
Logic:  
Syntax**

**Propositional  
Logic:  
Semantics**

**Propositional  
Logic:  
Inference and  
Proof Systems**

**KB-Agents:  
Inference  
algorithms**

# Propositional Logic

Propositional logic, also known as **sentential logic** and **statement logic**, is the branch of logic that studies ways of joining and/or modifying entire propositions, statements or sentences to form more complicated propositions, statements or sentences, as well as the logical relationships and properties that are derived from these methods of combining or altering statements.

# Language: Syntax and Semantics

- **Syntax:**

- defines a set of rules for producing legal (well formed) sentences in a given language

- **Semantics:**

- defines the “meaning” of a sentence → it has semantic value
- NOT all legal sentences will have semantic value:

**Example: Colorless green ideas sleep furiously**

# Proposition / Sentence

A proposition / **sentence** (also called a logical expression) is an assertion about the world in a mathematically defined knowledge representation language. It can be true or false.

Examples:

John is sick

When it thunders, there is also lightning

# Propositional Logic: Syntax

- Logical constants: **true**, **false**
- Propositional symbols / variables:
  - atomic sentences:  $p, q, r$
  - compound / complex sentences:  $P, Q, R$
- Wrapping parentheses:  $(...)$
- Sentences are combined by logical connectives:  
 $\neg \wedge \vee \Leftrightarrow \Rightarrow$
- Literals:
  - atomic sentence  $p$  or negated atomic sequence  $\neg p$



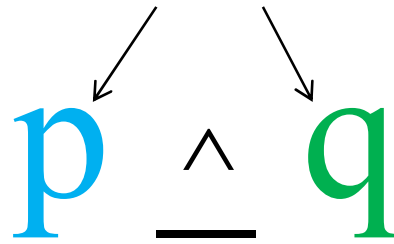
# Symbols: Refresher

Symbol	Name	Alternative symbols*	Should be read
$\neg$	Negation	$\sim, !$	not
$\wedge$	(Logical) conjunction	$\bullet, \&$	and
$\vee$	(Logical) disjunction	$+,   $	or
$\Rightarrow$	(Material) implication	$\rightarrow, \supset$	implies
$\Leftrightarrow$	(Material) equivalence	$\leftrightarrow, \equiv, \text{iff}$	if and only if
$\top$	Tautology	$T, 1, \blacksquare$	truth
$\perp$	Contradiction	$F, 0, \square$	falsum empty clause
$\therefore$	Therefore		therefore

\* you can encounter it elsewhere in literature

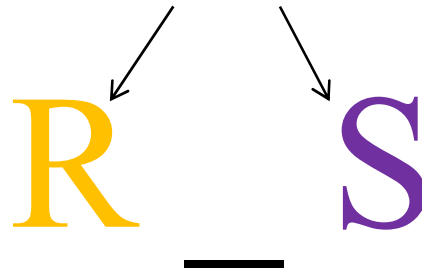
# Creating Complex Sentences

atomic sentences



logical connective

complex sentences

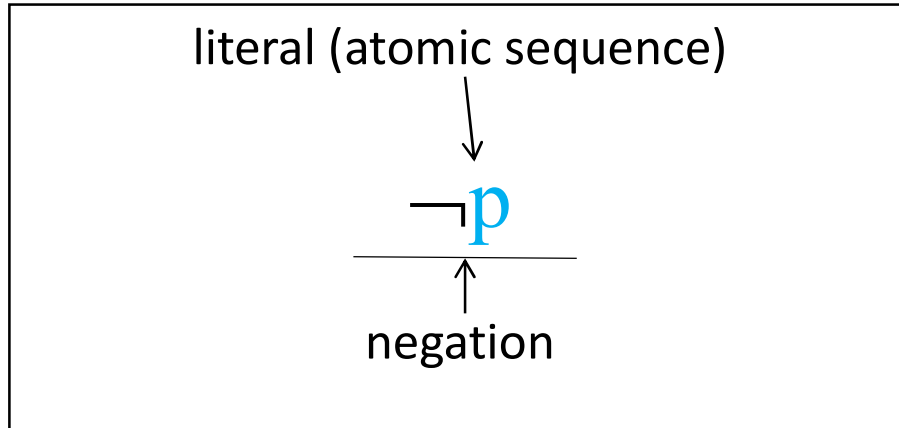


logical connective

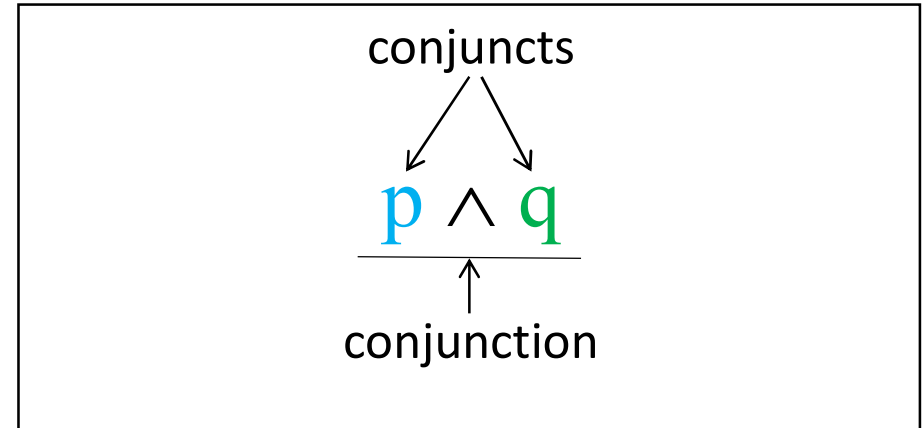
$p, q, R, S$  - proposition (sentence) symbols / variables | \_ logical connective:  $\neg \wedge \vee \Leftrightarrow \Rightarrow$

# Logical Connectives: $\neg \wedge \vee \Leftrightarrow \Rightarrow$

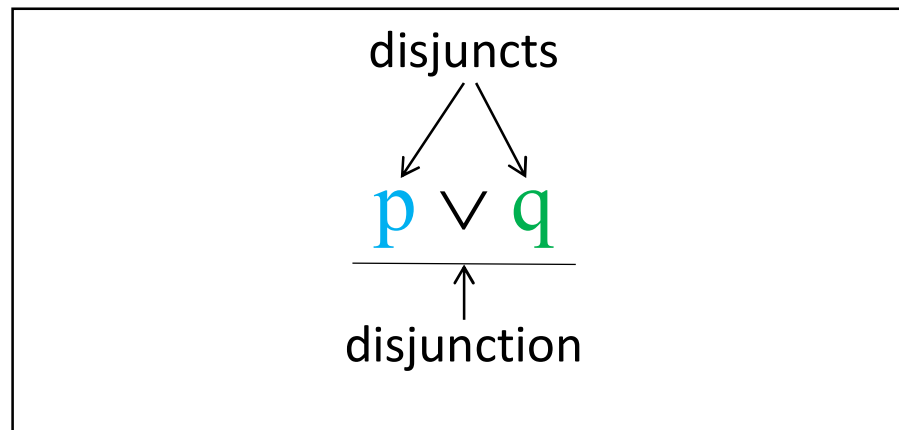
## Negation (not)



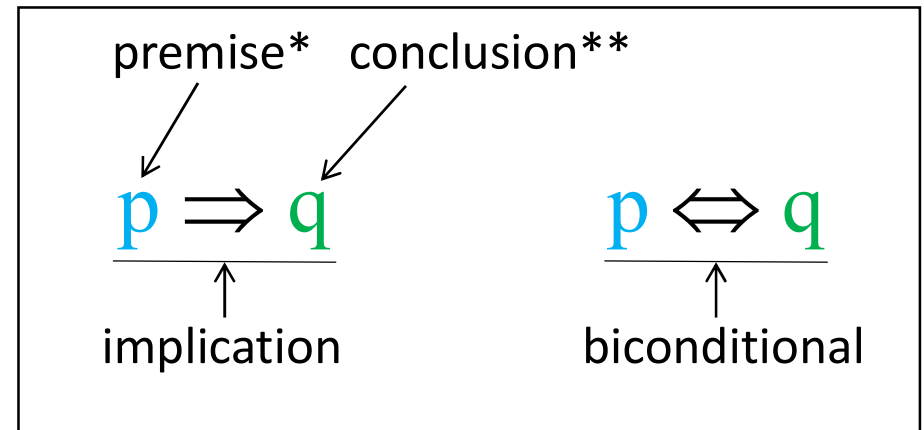
## Logical conjunction (and)



## Logical disjunction (or)



## Material implication and equivalence



\* also called antecedent | \*\* also called consequent

# Operator Precedence

## Operator Precedence

Higher precedence

$()$

$\neg$

$\wedge$

$\vee$

$\Rightarrow \Leftrightarrow$

Lower precedence

## Precedence in Sentences

If in doubt: left can be rewritten as right

$$\neg p \wedge q \quad ((\neg p) \wedge q)$$

$$p \wedge \neg q \quad (p \wedge (\neg q))$$

$$p \wedge q \vee r \quad ((p \wedge q) \vee r)$$

$$p \vee q \wedge r \quad (p \vee (q \wedge r))$$

$$p \Rightarrow q \Rightarrow r \quad (p \Rightarrow (q \Rightarrow r))$$

$$p \Rightarrow q \Leftrightarrow r \quad (p \Rightarrow (q \Leftrightarrow r))$$

# Well-formed Sentences

**A well-formed sentence is a finite sequence of symbols from a given alphabet that is part of a formal language (grammatically correct)**

- **well-formed propositional logic sentence:**

$$(((p \Rightarrow q) \wedge (r \Rightarrow s)) \vee (\neg q \wedge \neg s))$$

- **NOT well-formed propositional logic sentence:**

$$((p \Rightarrow q) \Rightarrow (qq))p))$$

# BNF (Backus-Naur Form) Grammar

$$\begin{aligned} \textit{Sentence} &\rightarrow \textit{AtomicSentence} \mid \textit{ComplexSentence} \\ \textit{AtomicSentence} &\rightarrow \textit{True} \mid \textit{False} \mid P \mid Q \mid R \mid \dots \quad * \\ \textit{ComplexSentence} &\rightarrow (\textit{Sentence}) \\ &\mid \neg \textit{Sentence} \\ &\mid \textit{Sentence} \wedge \textit{Sentence} \\ &\mid \textit{Sentence} \vee \textit{Sentence} \\ &\mid \textit{Sentence} \Rightarrow \textit{Sentence} \\ &\mid \textit{Sentence} \Leftrightarrow \textit{Sentence} \end{aligned}$$

OPERATOR PRECEDENCE :  $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

\* I will:

- be using true and false instead of True and False
- use lowercase p, q for atomic and uppercase P, Q for complex

# English $\rightarrow$ Propositional Logic

Consider three atomic sentences in propositional logic: cool, funny, and popular. Each can be assigned a truth value of true or false.

Natural language encoded using propositional logic examples:

IF a person is cool OR funny, THEN she is popular.

$$(\text{cool} \vee \text{funny}) \Rightarrow \text{popular}$$

A person is popular ONLY IF she is EITHER cool OR funny.

$$\text{popular} \Rightarrow (\text{cool} \vee \text{funny})$$

A person is popular IF AND ONLY IF she is EITHER cool OR funny.

$$\text{popular} \Leftrightarrow (\text{cool} \vee \text{funny})$$

There is NO one who is both cool AND funny.

$$\neg(\text{cool} \wedge \text{funny})$$

# Propositional Logic: Laws/Theorems

Equivalence	Law / Theorems
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutative laws
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associative laws
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distributive laws
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \wedge (p \vee q) \Leftrightarrow p$ $p \vee (p \wedge q) \Leftrightarrow p$	Absorption laws
$\neg(\neg p) \Leftrightarrow p$	Double Negation law (involution)
$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$	Idempotent laws
$p \vee \neg p \Leftrightarrow T$	Law of Excluded Middle (Negation law)
$p \wedge \neg p \Leftrightarrow \perp$	Contradiction (Negation law)
$p \wedge T \Leftrightarrow p$ $p \vee \perp \Leftrightarrow p$	Identity laws
$p \wedge \perp \Leftrightarrow \perp$ $p \vee T \Leftrightarrow T$	Domination laws
$\neg p \vee q \Leftrightarrow p \Rightarrow q$	Implication law
$p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$	Contraposition law
$(p \wedge q) \vee (\neg q \wedge \neg p) \Leftrightarrow (p \Leftrightarrow q)$ $(p \Rightarrow q) \wedge (q \Rightarrow p) \Leftrightarrow (p \Leftrightarrow q)$	Equivalence law



# Deduction

Laws/theorems in propositional logic can be used to prove additional theorems through a process known as **deduction**:

**Prove that  $((\neg m \vee n) \wedge \neg n) \Rightarrow \neg m$  is a tautology:**

$$((\neg m \wedge \neg n) \vee (n \wedge \neg n)) \Rightarrow \neg m$$

$$((\neg m \wedge \neg n) \vee \perp) \Rightarrow \neg m$$

$$(\neg m \wedge \neg n) \Rightarrow \neg m$$

$$\neg(\neg m \wedge \neg n) \vee \neg m$$

$$(\neg\neg m \vee \neg\neg n) \vee \neg m$$

$$(m \vee n) \vee \neg m$$

$$m \vee (n \vee \neg m)$$

$$m \vee (\neg m \vee n)$$

$$(m \vee \neg m) \vee n$$

$$T \vee n$$

$$n \vee T$$

$$T$$

by Distributive law  $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

by Negation law (contradiction)  $p \wedge \neg p \Leftrightarrow \perp$

by Identity law  $p \vee \perp \Leftrightarrow p$

by Implication law  $\neg p \vee q \Leftrightarrow p \Rightarrow q$

by De Morgan's law  $\neg(p \wedge q) \Leftrightarrow \neg q \vee \neg p$

by Double Negation law  $\neg(\neg p) \Leftrightarrow p$

by Associative law  $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

by Commutative law  $p \vee q \Leftrightarrow q \vee p$

by Associative law  $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

by Law of Excluded Middle  $p \vee \neg p \Leftrightarrow T$

by Commutative law  $p \vee q \Leftrightarrow q \vee p$

by Domination Law  $p \vee T \Leftrightarrow T$

# Deduction

Laws/theorems in propositional logic can be used to prove additional theorems through a process known as **deduction**:

Note that we only manipulated symbols at the syntactic level!

**Prove that**  $((\neg m \vee n) \wedge \neg n) \Rightarrow \neg m$  **is a tautology:**

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$$\neg(\neg m \wedge \neg n) \vee \neg m$$

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$$(m \vee \neg m) \vee n$$

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by Domination Law  $p \vee T \Leftrightarrow T$

# Propositional Logic and KB-Agents

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# Interpretation

The truth value assignment to propositional sentences is called an **interpretation** (an assertion about their truth **in some possible world / model**).

**Definition:** A mapping  $I : \Sigma \rightarrow \{\text{true}, \text{false}\}$ , which assigns a truth value to **every proposition variable**, is called an **interpretation**.

**Sentence:**  $(p \vee q) \wedge (\neg q \vee r)$

**Interpretation i:**  $p^i = \text{true}$ ,  $q^i = \text{false}$ ,  $r^i = \text{true}$

# Truth Values and Truth Tables

Propositional logic sentences can have a **truth value** assigned to them:

- Atomic sentences:
  - either **true** or **false**
- Compound / complex sentence truth value can be established using a truth table:

<b>p</b>	<b>q</b>	<b><math>\neg p</math></b>	<b><math>p \wedge q</math></b>	<b><math>p \vee q</math></b>	<b><math>p \Rightarrow q</math></b>	<b><math>p \Leftrightarrow q</math></b>
true	true	false	true	true	true	true
true	false	false	false	true	false	false
false	true	true	false	true	true	false
false	false	true	false	false	true	true

# Evaluation

Evaluation is the process of **determining the truth values of compound/complex sentences** given a **truth assignment for the truth values of proposition constants/atomic sentences**. Consider the following truth assignment i:

$p^i = \text{true}, q^i = \text{false}, r^i = \text{true}$  Assignment

Let's evaluate the following complex sentence  $(p \vee q) \wedge (\neg q \vee r)$ :

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$(p \vee q) \wedge (\neg q \vee r) \rightarrow (\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$  Subsitute

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$(\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$  Disjunction



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$(\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$  Disjunction

$\text{true} \wedge (\neg \text{false} \vee \text{true})$  Negation

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$(\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$  Disjunction

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$(\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$  Disjunction

$\text{true} \wedge (\neg \text{false} \vee \text{true})$  Negation

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$\text{true} \wedge \text{true}$  Conjunction

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$p^i = \text{true}, q^i = \text{false}, r^i = \text{true}$  Assignment

Let's evaluate the following complex sentence  $(p \vee q) \wedge (\neg q \vee r)$ :

$(p \vee q) \wedge (\neg q \vee r) \rightarrow (\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$  Subsitute

$(\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$  Disjunction

$\text{true} \wedge (\neg \text{false} \vee \text{true})$  Negation

$\text{true} \wedge (\text{true} \vee \text{true})$  Disjunction

$\text{true} \wedge \text{true}$  Conjunction

$\text{true}$  Interpretation

# Complex Sentence: Truth Table

Consider a complex sentence **R** built with **N** propositional variables  $p_1, p_2, p_3, \dots, p_{N-1}, p_N$  and logical connectives ( $\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$ ). Here is a corresponding truth table for sentence **R**.

		<b>N Propositional Variables</b>						Complex sentence <b>R</b>		
		$p_1$	$p_2$	$p_3$	...	$p_{N-1}$	$p_N$			
<b><math>2^N</math> Truth Assignments</b>		true	true	true	...	true	true	false	<b><math>2^N</math> Interpretations of <b>R</b></b>	
		true	true	true	...	true	false	true		
		true	true	false	...	false	true	false		
		...	...	...	...	...	...	...		
		false	false	true	...	true	false	true		
		false	false	true	...	false	true	true		
		false	false	false	...	false	false	false		

# Complex Sentence: Truth Table

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<b>N Propositional Variables</b>							Complex sentence <b>R</b>
$p_1$	$p_2$	$p_3$	...	$p_{N-1}$	$p_N$		
true	true	true	...	true	true		false
true	true	true	...	true	false		true
true	true	false	...	false	true		false
...	...	...	...	...	...		...
false	false	true	...	true	false		true
false	false	true	...	false	true		true
false	false	false	...	false	false		false

**$2^N$  Possible Worlds (Models)**

**$2^N$  Interpretations of **R****

# Sentence: Syntactic / Semantic Levels

Each propositional logic “exists” on two levels:

- Syntactic: where a sentence is just a legal arrangement of language symbols. Sentence

$$(p \vee q) \wedge (\neg q \vee r)$$

**WITHOUT** interpretation **HAS NO MEANING**

- we can manipulate symbols, but we CANNOT reason
- Semantic: where a sentence has a truth value is assigned to it (interpretation):

$$(p \vee q) \wedge (\neg q \vee r) \text{ where } p^i = \text{true}, q^i = \text{false}, r^i = \text{true}$$

**HAS MEANING (through interpretation) → it is true**

# Sentence: Syntactic / Semantic Levels

Each propositional logic “exists” on two levels:

- Syntactic: where a sentence is just a legal arrangement of language symbols. Sentence

$$(\text{cool} \vee \text{funny}) \Rightarrow \text{popular}$$

**WITHOUT** interpretation **HAS NO MEANING**

- we can't tell if a given person is popular here
- Semantic: where a sentence has a truth value is assigned to it (interpretation):

$(\text{cool} \vee \text{funny}) \Rightarrow \text{popular}$  where  $\text{cool} = \text{true}$ ,  $\text{funny} = \text{false}$   
**HAS MEANING** → we can deduce that a person is popular



# Sentence Semantical Equivalence

Two propositional logic sentences  $F$  and  $G$  are called **semantically equivalent** if they take on the **same interpretation** for all truth value assignments. If that is the case  $F \equiv G$ .

Example: sentence  $\neg a \vee b$  is equivalent to sentence  $a \Rightarrow b$ . Proof with a truth table:

$a$	$b$	$\neg a$	$\neg a \vee b$	$\Leftrightarrow$	$a \Rightarrow b$
true	true	false	true	$\equiv$	true
true	false	false	false		false
false	true	true	true		true
false	false	true	true		true

# Sentence Classes

## SATISFIABLE

A sentence is **satisfiable** if it is **true for AT LEAST ONE interpretation.**

In plain English:

“You can find **AT LEAST one assignment** of logical values of true and false to individual propositional variables that will make this sentence **true.**”

Example:

$$p \Rightarrow q$$

p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

## (LOGICALLY) VALID/TAUTOLOGY

A sentence is (logically) **valid** if it is **true for ALL interpretations.**  
Also called a **tautology.**

In plain English:

“This sentence is **ALWAYS true** regardless of value assignment to individual propositional variables.”

Example:

$$p \vee \neg p$$

p	$\neg p$	$p \vee \neg p$
true	false	true
true	false	true
false	true	true
false	true	true

## UNSATISFIABLE/CONTRADICTION

A sentence is **unsatisfiable** if it is **NOT true for ANY interpretation.**  
Also called a **contradiction.**

In plain English:

“This sentence is **ALWAYS false** regardless of value assignment to individual propositional variables.”

Example:

$$p \wedge \neg p$$

p	$\neg p$	$p \wedge \neg p$
true	false	false
true	false	false
false	true	false
false	true	false

# Propositional Logic and KB-Agents

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# Inference: The idea

## The idea:

**Given everything (expressed as sentences) that we know, can we infer if some query (another sentence) is true or not (satisfied or not)?**

# **(Automated) Proof System**

**In AI we are interested in taking existing knowledge (sentences in KB) and from that:**

- **deriving new knowledge (new sentences)**
- **answering questions (query sentences)**

**In Propositional Logic this means showing that some sentence  $Q$  follows from a Knowledge Base KB**

**where:**

- **$Q$  - some query sentence**
- **KB - knowledge base (a sentence made of sentences)**

# **Inference: Real-life Example**

**If it is raining, I will need an umbrella. It is raining. Therefore, I will need an umbrella.**

# Inference: Real-life Example

**If** it is raining, **then** I will need an umbrella. It is raining. **Therefore**, I will need an umbrella.

# Inference: Real-life Example

If it is raining, then I will need an umbrella. It is raining. Therefore, I will need an umbrella.



# Propositional Logic: An Argument

An argument  $A$  in propositional logic has the following form:

A:	P1	PREMISES
	P2	
	...	
	PN	
	<hr/>	
	$\therefore C$	CONCLUSION

An argument  $A$  is said to be **valid** if the implication formed by taking the conjunction of the premiseses (antecedent) and the conclusion  $C$  (consequent),

$(P1 \wedge P2 \wedge P3 \wedge \dots \wedge PN) \Rightarrow C$  is a **tautology**.

# Propositional Logic: An Argument

An argument  $A$  in propositional logic has the following form:

A:	P1	PREMISES
	P2	
	...	
	PN	
	<hr/>	
	$\therefore C$	CONCLUSION

Premises are taken for granted (assumed to be **true**).

# Inference: Real-life Example

If it is raining, then I will need an umbrella.

It is raining.

---

Therefore, I will need an umbrella.

# Inference: Real-life Example

If it is raining, then I will need an umbrella.

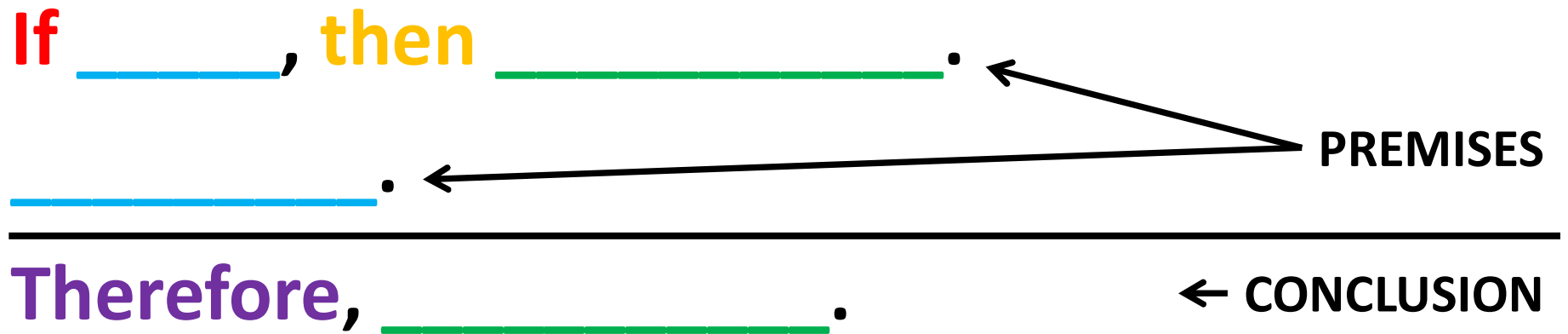
It is raining.

PREMISES

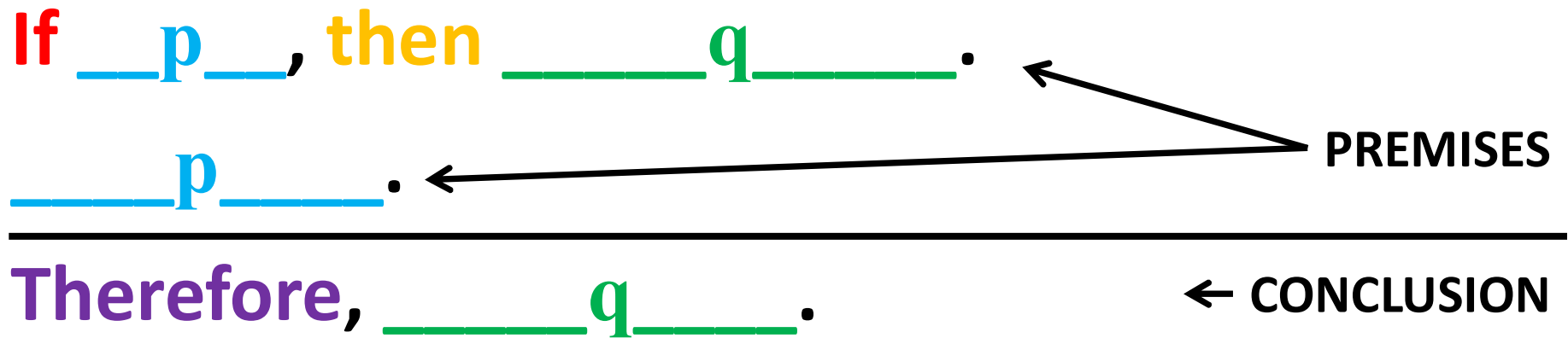
---

Therefore, I will need an umbrella. ← CONCLUSION

# Inference: Real-life Example



# Inference: Real-life Example



$p$  = "It is raining."

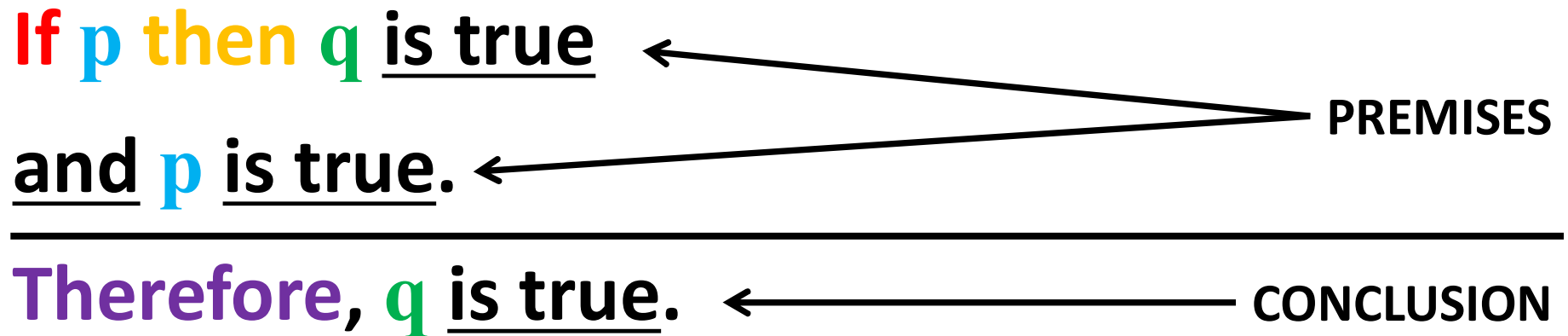
$q$  = "I will need an umbrella."

PREMISE1 = "If it is raining, then I will need an umbrella."

PREMISE2 = "It is raining."

CONCLUSION = "I will need an umbrella."

# Inference: Real-life Example



**p** = "It is raining."

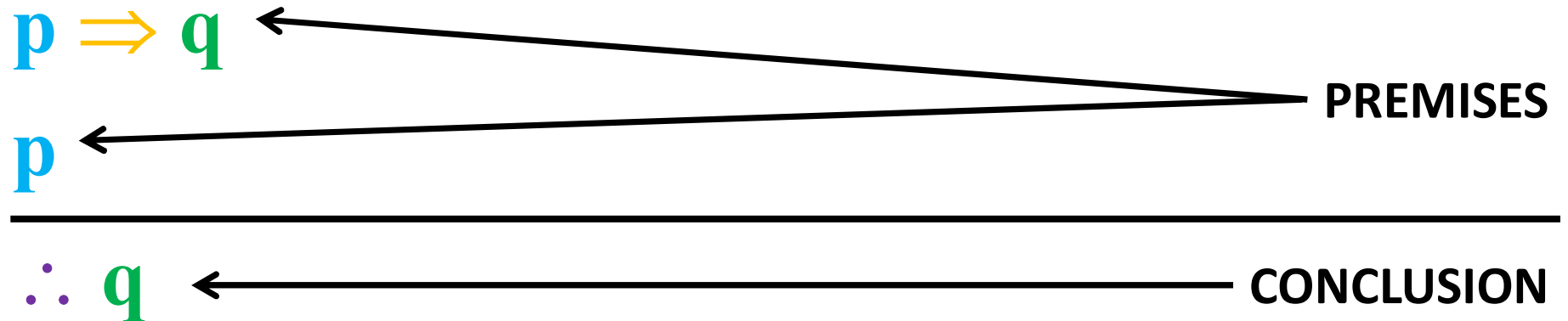
**q** = "I will need an umbrella."

PREMISE1 = "If it is raining, then I will need an umbrella."

PREMISE2 = "It is raining."

CONCLUSION = "I will need an umbrella."

# Inference Rules: Modus Ponens



$p$  = "It is raining."

$q$  = "I will need an umbrella."

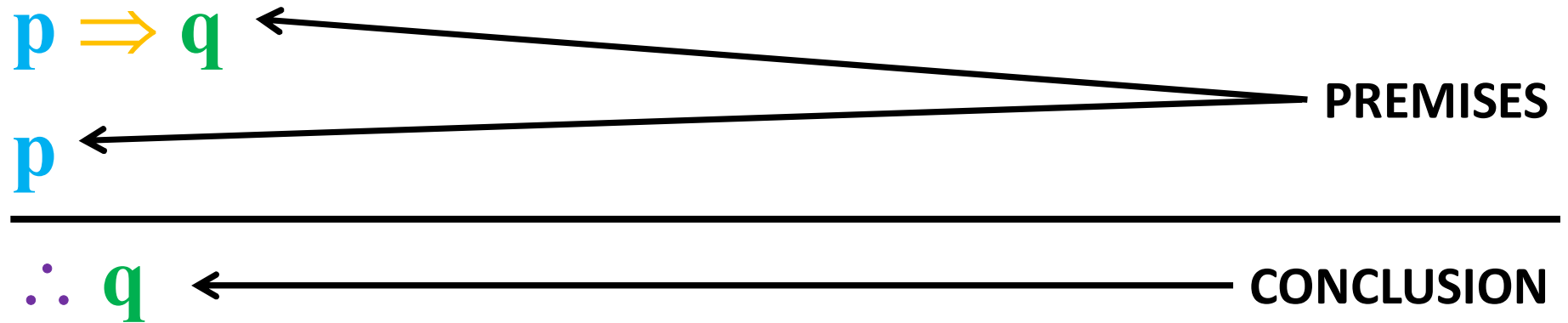
PREMISE1 =  $p \Rightarrow q$

PREMISE2 =  $p$

CONCLUSION =  $q$



# Inference Rules: Modus Ponens



$p$  = "It is raining."

$q$  = "I will need an umbrella."

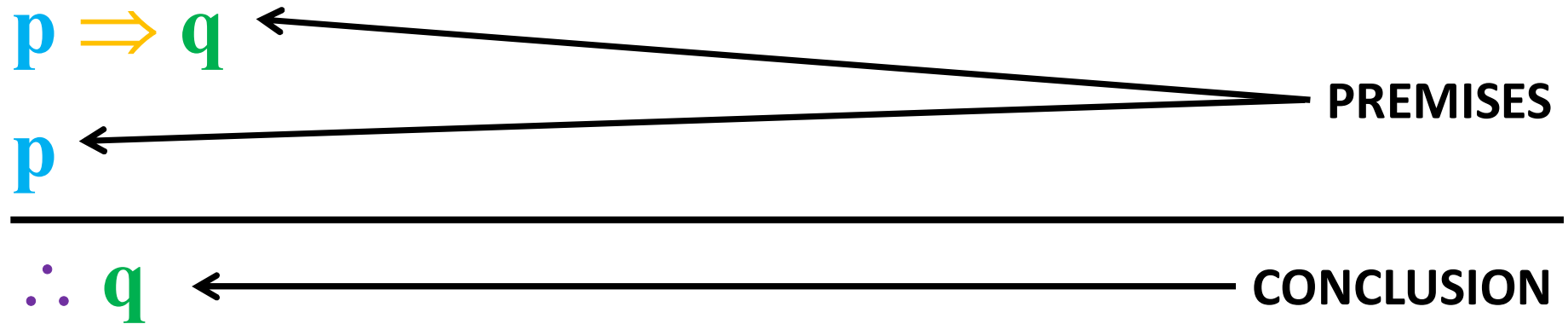
PREMISE1 =  $p \Rightarrow q$

PREMISE2 =  $p$

CONCLUSION =  $q$

IF PREMISES ARE TRUE,  
THEREFORE THE  
CONCLUSION MUST  
ALSO BE TRUE

# Inference: Modus Ponens



$p$  = "It is raining."

$q$  = "I will need an umbrella."

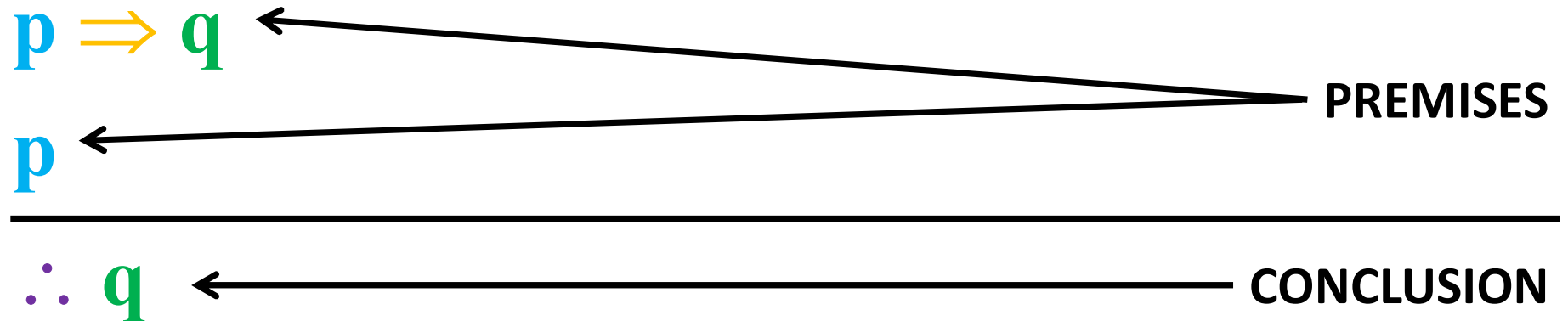
PREMISE1 =  $p \Rightarrow q$

PREMISE2 =  $p$

CONCLUSION =  $q$

PROPOSITIONAL VARIABLES		IMPLICATION
$p$	$q$	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

# Inference Rules: Modus Ponens



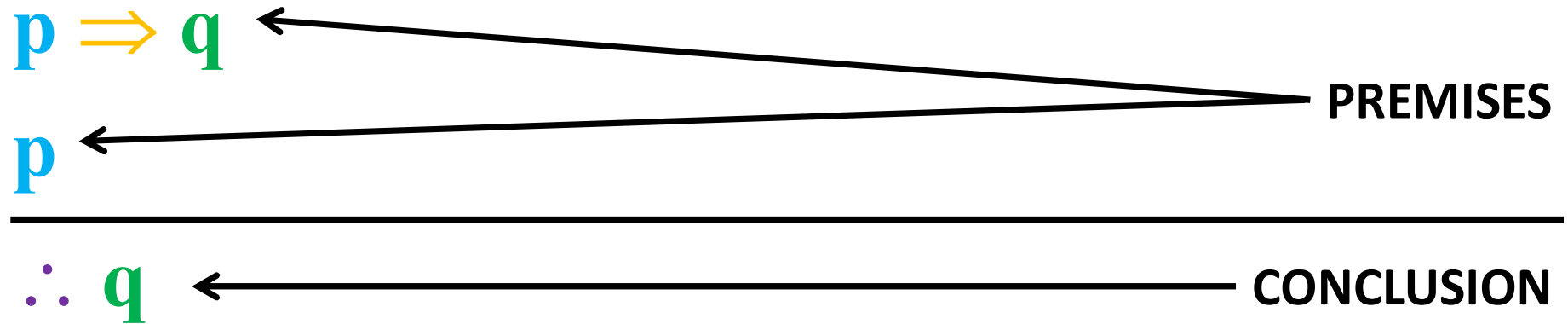
$p$  = "It is raining."

$q$  = "I will need an umbrella."

PREMISES = PREMISE1 AND PREMISE2 =  $(p \Rightarrow q) \wedge p$

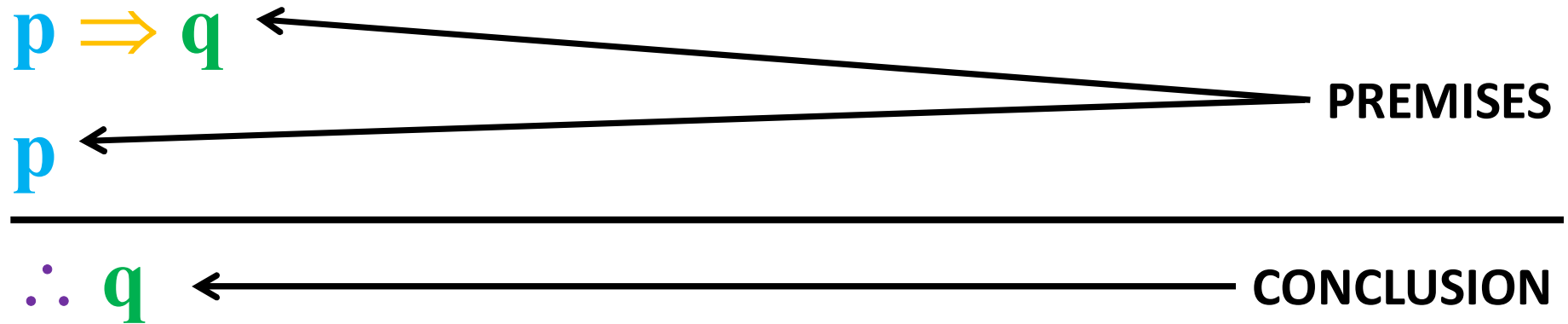
CONCLUSION =  $q$

# Inference Rules: Modus Ponens



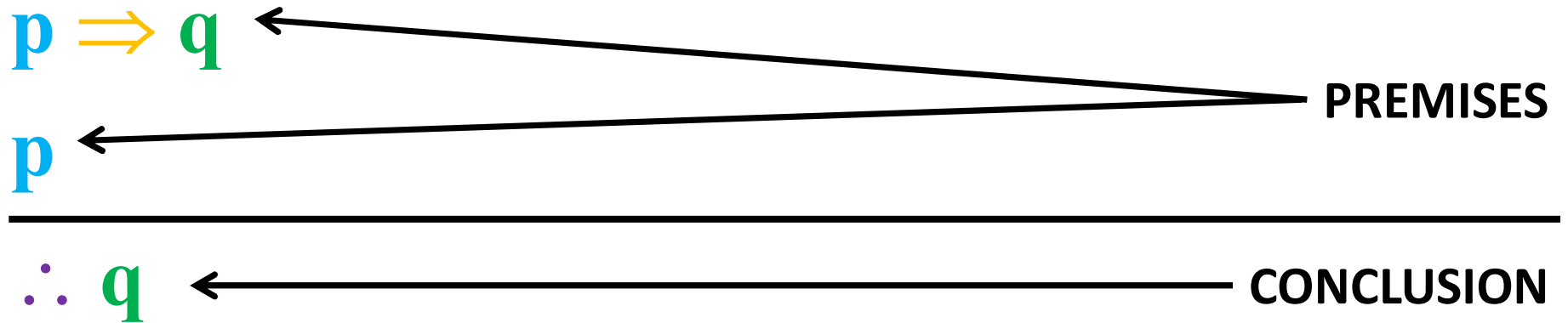
PROPOSITIONAL VARIABLES		INDIVIDUAL PREMISE		PREMISES	CONCLUSION
$p$	$q$	P1: $p \Rightarrow q$	P2: $p$	$P1 \wedge P2: (p \Rightarrow q) \wedge p$	$q$
true	true	true	true	true	true
true	false	false	true	false	false
false	true	true	false	false	true
false	false	true	false	false	false

# Inference Rules: Modus Ponens



PROPOSITIONAL VARIABLES		INDIVIDUAL PREMISE		PREMISES	CONCLUSION
$p$	$q$	P1: $p \Rightarrow q$	P2: $p$	$P1 \wedge P2: (p \Rightarrow q) \wedge p$	$q$
true	true	true	true	true	true
true	false	false	true	false	false
false	true	true	false	false	true
false	false	true	false	false	false

# Inference Rules: Modus Ponens



IF **PREMISES** ARE **TRUE**,  
THEREFORE THE  
**CONCLUSION MUST**  
**ALSO BE TRUE**

INDIVIDUAL PREMISE		PREMISES	CONCLUSION
P1: $p \Rightarrow q$	P2: $p$	$P1 \wedge P2: (p \Rightarrow q) \wedge p$	$q$
true	true	true	true
false	true	false	false
true	false	false	true
true	false	false	false

# Inference Rules: Summary

## Rules of Inference:

<b>Modus Ponens</b> $P \Rightarrow Q$ $P$ <hr/> $\therefore Q$	<b>Modus Tollens</b> $P \Rightarrow Q$ $\neg Q$ <hr/> $\therefore \neg P$	<b>Hypothetical Syllogism (Transitivity)</b> $P \Rightarrow Q$ $Q \Rightarrow R$ <hr/> $\therefore P \Rightarrow R$	<b>Conjunction</b> $P$ $Q$ <hr/> $\therefore P \wedge Q$
<b>Addition</b> $P$ <hr/> $\therefore P \vee Q$	<b>Simplification</b> $P \wedge Q$ <hr/> $\therefore P$	<b>Disjunctive Syllogism</b> $P \vee Q$ $\neg P$ <hr/> $\therefore Q$	<b>Resolution</b> $P \vee Q$ $\neg P \vee R$ <hr/> $\therefore Q \vee R$

## Tautological forms:

**Modus Ponens:**  $((P \Rightarrow Q) \wedge P) \Rightarrow Q$  | **Modus Tollens:**  $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$

**Hypothetical Syllogism:**  $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$

**Disjunctive Syllogism:**  $((P \vee Q) \wedge \neg P) \Rightarrow Q$

**Addition:**  $P \Rightarrow P \vee Q$  | **Simplification:**  $(P \wedge Q) \Rightarrow P$

**Conjunction:**  $(P) \wedge (Q) \Rightarrow (P \wedge Q)$  | **Resolution:**  $((P \vee Q) \wedge (\neg P \vee R)) \Rightarrow (Q \vee R)$

# Argument Validity: Truth Table Proof

$$p \Rightarrow q$$

$$q \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

---

$$\therefore \neg r$$

p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	$P1 \wedge P2 \wedge P3$	$(P1 \wedge P2 \wedge P3) \Rightarrow \neg r$
true	true	true	true	false	true	false	true
true	true	false	true	true	true	true	true
true	false	true	false	true	true	false	true
true	false	false	false	true	true	false	true
false	true	true	true	false	false	false	true
false	true	false	true	true	true	true	true
false	false	true	true	true	false	false	true
false	false	false	true	true	true	true	true



# Argument Validity: Truth Table Proof

$$p \Rightarrow q$$

$$A \Leftrightarrow ((p \Rightarrow q) \wedge (q \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \Rightarrow \neg r)$$

$$q \Rightarrow \neg r$$

An argument A is **valid** if it is a **tautology**.

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r$$

p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	$P1 \wedge P2 \wedge P3$	$(P1 \wedge P2 \wedge P3) \Rightarrow \neg r$
true	true	true	true	false	true	false	true
true	true	false	true	true	true	true	true
true	false	true	false	true	true	false	true
true	false	false	false	true	true	false	true
false	true	true	true	false	false	false	true
false	true	false	true	true	true	true	true
false	false	true	true	true	false	false	true
false	false	false	true	true	true	true	true

# Argument Validity: Truth Table Proof

$$p \Rightarrow q$$

$$A \Leftrightarrow ((P1) \wedge (P2) \wedge (P3) \Rightarrow \neg r)$$

$$q \Rightarrow \neg r$$

An argument A is **valid** if it is a **tautology**.

$$\neg p \Rightarrow \neg r$$

Argument A is valid, because it is a tautology

$$\therefore \neg r$$

(always true regardless of **p**, **q**, **r** truth assignments)

p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	P1 $\wedge$ P2 $\wedge$ P3	A
true	true	true	true	false	true	false	true
true	true	false	true	true	true	true	true
true	false	true	false	true	true	false	true
true	false	false	false	true	true	false	true
false	true	true	true	false	false	false	true
false	true	false	true	true	true	true	true
false	false	true	true	true	false	false	true
false	false	false	true	true	true	true	true

# Logical Entailment

A set of sentences (called **premises**) logically **entails** a sentence (called a **conclusion**) if and only if **every truth assignment that satisfies the premises also satisfies the conclusion.**

PREMISES  $\sqsubset$  CONCLUSION

# Logical Entailment

**Definition:** A sentence KB entails a sentence Q (or Q follows from KB) if every model of KB is also a model of Q. We write:

$$KB \models Q$$

**In other words:**

- For every interpretation in which KB is **true**, Q is also **true**
- “Whenever KB is **true**, Q is also **true**”

# Entailment: Deriving Conclusions

You can prove if:

$$KB \models Q$$

is **true** in a number of ways:

- Model checking (enumeration)
- Truth table proof (enumeration)
- Proof by resolution

You can also prove related sentences:

- prove that  $KB \wedge \neg Q$  is **unsatisfiable** (by contradiction)
- prove that  $KB \Rightarrow Q$  is a **tautology**

# Model / “Possible World”

A **model** (a “possible world”) is a single truth assignment / interpretation.

If a sentence  $U$  is **true** in model  $K$ ,  $K$  satisfies  $U$ .

$M(U)$ : set of **ALL** models of  $U$  (that satisfy  $U$ )

**Now:**

$KB \models Q$  if and only if  $M(KB) \subseteq M(Q)$

$KB \models Q$  is **true** if and only if **in EVERY model** in which  $KB$  is **true**,  $Q$  is also **true**.

# Logical Entailment with Truth Table

$$p \Rightarrow q \quad \text{KB}$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\text{KB} \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r)$$

$$Q \Leftrightarrow \neg r$$

$$\therefore \neg r \quad Q$$

Model	p	q	r	P1:p $\Rightarrow$ q	P2:q $\Rightarrow$ $\neg$ r	P3: $\neg$ p $\Rightarrow$ $\neg$ r	KB	Q
M1	true	true	true	true	false	true	false	false
M2	true	true	false	true	true	true	true	true
M3	true	false	true	false	true	true	false	false
M4	true	false	false	false	true	true	false	true
M5	false	true	true	true	false	false	false	false
M6	false	true	false	true	true	true	true	true
M7	false	false	true	true	true	false	false	false
M8	false	false	false	true	true	true	true	true

# Entailment: Model Checking

$$p \Rightarrow q \quad \text{KB}$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r \quad Q$$

$$\text{KB} \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \mid Q \Leftrightarrow \neg r$$

Models where KB is true:  $M(\text{KB}) = \{M2, M6, M8\}$

Models where Q is true:  $M(Q) = \{M2, M4, M6, M8\}$

Model	p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	KB	Q
M1	true	true	true	true	false	true	false	false
M2	true	true	false	true	true	true	true	true
M3	true	false	true	false	true	true	false	false
M4	true	false	false	false	true	true	false	true
M5	false	true	true	true	false	false	false	false
M6	false	true	false	true	true	true	true	true
M7	false	false	true	true	true	false	false	false
M8	false	false	false	true	true	true	true	true



# Entailment: Model Checking

$$p \Rightarrow q \quad \text{KB}$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r \quad Q$$

$$\text{KB} \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \mid Q \Leftrightarrow \neg r$$

Models where KB is true:  $M(\text{KB}) = \{M2, M6, M8\}$

Models where Q is true:  $M(Q) = \{M2, M4, M6, M8\}$

$M(\text{KB}) \subseteq M(Q)$  so Q follows KB

Model	p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	KB	Q
M1	true	true	true	true	false	true	false	false
M2	true	true	false	true	true	true	true	true
M3	true	false	true	false	true	true	false	false
M4	true	false	false	false	true	true	false	true
M5	false	true	true	true	false	false	false	false
M6	false	true	false	true	true	true	true	true
M7	false	false	true	true	true	false	false	false
M8	false	false	false	true	true	true	true	true

# KB $\Rightarrow$ Q is a **Tautology** Proof

$$p \Rightarrow q$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r$$

KB  $\Rightarrow$  Q is **true** for ALL models / interpretations

KB  $\Rightarrow$  Q is a **tautology**

p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	KB	KB $\Rightarrow$ Q
true	true	true	true	false	true	false	<b>true</b>
true	true	false	true	true	true	true	<b>true</b>
true	false	true	false	true	true	false	<b>true</b>
true	false	false	false	true	true	false	<b>true</b>
false	true	true	true	false	false	false	<b>true</b>
false	true	false	true	true	true	true	<b>true</b>
false	false	true	true	true	false	false	<b>true</b>
false	false	false	true	true	true	true	<b>true</b>

# Enumeration: Issues

Consider a complex sentence  $R$  built with  $N$  propositional variables  $p_1, p_2, p_3, \dots, p_{N-1}, p_N$  and logical connectives ( $\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$ ). Each truth assignment is a different possible world.

$N$ Propositional Variables							Complex sentence $R$
$p_1$	$p_2$	$p_3$	...	$p_{N-1}$	$p_N$		
true	true	true	...	true	true		false
true	true	true	...	true	false		true
true	true	false	...	false	true		false
...	...	...	...	...	...		...
false	false	true	...	true	false		true
false	false	true	...	false	true		true
false	false	false	...	false	false		false

$2^N$  Possible Worlds (Models)

$2^N$  Interpretations of  $R$

**Can we do better?**  
**Can we automate the process?**

# Conjunctive Normal Form (CNF)

A sentence is in conjunctive normal form (CNF if and only if consists of **conjunction**:

$$K_1 \wedge K_2 \wedge \dots \wedge K_m$$

of clauses. A clause  $K_i$  consists of a **disjunction**

$$(l_{i1} \vee l_{i2} \vee \dots \vee l_{ini})$$

of literals. Finally, a literal is propositional variable (positive literal) or a negated propositional variable (negative literal).

# Conjunctive Normal Form (CNF)

**Example:**

$$(a \vee b \vee \neg c) \wedge (a \vee b \vee \neg c) \wedge (\neg b \vee \neg c)$$

**where: a, b, c are literals.**