CS 480

Introduction to Artificial Intelligence

October 25, 2022

Announcements / Reminders

- **Follow Week 09 TO DO List**

- **Grading TA assignment:**

  https://docs.google.com/spreadsheets/d/1ExS0bKnGt_fdf4LHa3YS1qRA7-Iq4xqXVjfSAPMaGVk/edit?usp=sharing

- **UPDATED Final Exam date:**
  - **December 1st, 2022 (last week of classes!)**
    - **Ignore the date provided by the Registrar**

Plan for Today

- **Predicate / First-Order Logic**

Predicate Logic Syntax: Summary

Predicate calculus symbols include:

- truth symbols: true and false
- terms represent specific objects in the world
  - constants, variables and functions
- predicate symbols refer to a particular relation between objects or represent facts
- function symbols refer to objects indirectly (via some relationship)
- quantifiers ($\forall$ and $\exists$) and variables refer to collections of objects without explicitly naming each object

Proof by Resolution

The process of proving by resolution is as follows:

A. Formalize the problem: "English to Predicate Logic"

B. Negate the input statement/claim C to obtain $\neg C$

C. Convert $\neg C$ into CNF ("standardized") form

D. Apply resolution rule to resulting clauses. New clauses will be generated (add them to the set if not already present)

E. Repeat (C) until:

   a. no new clause can be added (C is false)

   b. last two clauses resolve to yield the empty clause (C is true)

Problem: Argument Mismatch

The goal of the resolution process is to find a contradiction. When two contradicting literals "cancel" each other out, we can end up with an empty clause:

$$\frac{(w), (\neg w)}{()}$$

It is not so easy in predicate logic. This

$$\frac{(\text{setting}(\text{sun})), (\neg\text{setting}(\text{sun}))}{()}$$

will work (predicate arguments match). This

$$\frac{(\text{beautiful}(\text{day})), (\neg\text{beautiful}(\text{night}))}{???????}$$

will not, because predicate arguments don't match.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Universal Quantifier: Conjuctions

Universal quantifier ("for all") indicates that a sentence is true for all possible values of the variable. For example:

$$\forall x \; likes(x, cake)$$

is true if $likes(x, cake)$ **is** true **for all interpretations of variable** x**. Assuming that**

$$x \in \{x_1, x_2, \ldots, x_n\}$$

we can rewrite $\forall x \; likes(x, cake)$ **as:**

$$likes(x_1, cake) \wedge likes(x_2, cake) \wedge \ldots \wedge likes(x_n, cake)$$

Eliminating Universal Quantifiers

In general universal quantifiers can also be eliminated through the use of **Universal Instantiation**.

For any sentence S, variable x, and constant symbol g (that does not appear anywhere in the knowledge base), we can use the following rule:

$$\frac{\forall x\ S}{\mathrm{SUBST}(\{x / g\}, S)}$$

Where is a result of applying substitution $\{x / g\}$ to the sentence S.

Eliminating Universal Quantifiers

For example, from the sentence:

$$\forall x\ (king(x) \land greedy(x) \Rightarrow evil(x))$$

we can infer the sentence

$$king(John) \land greedy(John) \Rightarrow evil(John)$$

using the substitution $\{x\ /\ John\}$**.**

Eliminating Existential Quantifiers

In general: existential quantifiers can also be eliminated through the use of **Existential Instantiation.**

For any sentence S, variable x, and constant symbol k (that does not appear anywhere in the knowledge base), we can use the following rule:

$$\frac{\exists x\ S}{\mathrm{SUBST}(\{x / k\}, S)}$$

Where is a result of applying substitution $\{x / k\}$ to the sentence S.

Propositionalization

The idea:

- **Replace an existentially quantified sentence with ONE instantiation (Skolemization)**
- **Replace an universally quantified sentence with ALL POSSIBLE instantiations**

For example, from the sentence:

$$\forall x\ (king(x) \wedge greedy(x) \Rightarrow evil(x))$$

Assume: there are TWO possible values/objects for x**:** {John, Richard}**. We obtain:**

$$(king(John) \wedge greedy(John) \Rightarrow evil(John))$$

$$(king(Richard) \wedge greedy(Richard) \Rightarrow evil(Richard))$$

Propositionalization

Now, we can continue the conversion of:

$$(king(John) \land greedy(John) \Rightarrow evil(John))$$
$$(king(Richard) \land greedy(Richard) \Rightarrow evil(Richard))$$

by replacing each atomic predicate logic symbol with a propositional logic symbol

$$(JohnIsKing \land JohnIsGreedy \Rightarrow JohnIsEvil)$$
$$(RichardIsKing \land RichardIsGreedy \Rightarrow RichardIsEvil)$$

Can you see potential problems?

Propositionalization

What if, in addition to:

$$(king(John) \wedge greedy(John) \Rightarrow evil(John))$$

$$(king(Richard) \wedge greedy(Richard) \Rightarrow evil(Richard))$$

we also had a function Father(..)**?**

You can easily end up with infinite nesting of the following nature:

$$Father(Father(Father(Father(John)))$$

That leads to an infinite number of clauses!

Unification

Predicate logic inference rules require finding substitutions that make two different logical expressions look identical.

The process is called unification. A UNIFY **algorithm takes two sentences** p **and** q **and returns a unifier** θ **for them (a substitution)** <u>**if one exists**</u>**:**

$$\text{UNIFY}(p, q) = \theta, \text{ where SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Unification: Examples

UNIFY(sentenceA, sentenceB) = {unifier for sentenceA and sentenceB}

UNIFY(p, q) = {θ}

UNIFY(p, q) = {variable / unifying value}

Examples:

UNIFY(Knows(John, x), Knows(John, Jane)) = {x/Jane}

UNIFY(Knows(John, x), Knows(y, Bill)) = {x/Bill, y/John}

UNIFY(Knows(John, x), Knows(y, Mother(y))) = {x/Mother(John), y/John}

UNIFY(Knows(John, x), Knows(x, Elizabeth) = failure

({x/John, x/Elizabeth} is not possible)

Most General Unifier (MGU)

But.... ther can be multiple unifiers for a pair of sentences. Which one to choose?

Every UNIFIABLE pair of sentences has a SINGLE most general unifier that is unique.

UNIFY algorithm will find MGU.

Unification

function UNIFY(x, y, θ=*empty*) **returns** a substitution to make x and y identical, or *failure*
  if θ = *failure* **then return** *failure*
  else if $x = y$ **then return** θ
  else if VARIABLE?(x) **then return** UNIFY-VAR(x, y, θ)
  else if VARIABLE?(y) **then return** UNIFY-VAR(y, x, θ)
  else if COMPOUND?(x) **and** COMPOUND?(y) **then**
    return UNIFY(ARGS(x), ARGS(y), UNIFY(OP(x), OP(y), θ))
  else if LIST?(x) **and** LIST?(y) **then**
    return UNIFY(REST(x), REST(y), UNIFY(FIRST(x), FIRST(y), θ))
  else return *failure*

function UNIFY-VAR(var, x, θ) **returns** a substitution
  if $\{var/val\} \in \theta$ for some val **then return** UNIFY(val, x, θ)
  else if $\{x/val\} \in \theta$ for some val **then return** UNIFY(var, val, θ)
  else if OCCUR-CHECK?(var, x) **then return** *failure*
  else return add $\{var/x\}$ to θ

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
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By Implication Law ($p \Rightarrow q \equiv \neg p \vee q$):

$$\forall w\ ([\underline{\mathbf{P_1(w) \vee P_2(w) \Rightarrow P_3(w)}}] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x))]]) \wedge [\forall w\ (P_5(w))]$$

becomes:

$$\forall w\ ([\underline{\mathbf{\neg(P_1(w) \vee P_2(w)) \vee P_3(w)}}] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x))]]) \wedge [\forall w\ (P_5(w))]$$

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becomes:

$$\forall w\ ([\neg(P_1(w) \vee P_2(w)) \vee P_3(w)] \vee [\exists x\ (\exists y\ (\underline{\mathbf{\neg P_6(x, y) \vee P_4(w, x)}}))]]) \wedge [\forall w\ (P_5(w))]$$

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By De Morgan's Law ($\neg(p \vee q) \equiv \neg p \wedge \neg q$):

$$\forall w\ ([\underline{\neg(\mathbf{P_1}(\mathbf{w}) \vee \mathbf{P_2}(\mathbf{w}))} \vee P_3(w)] \vee [\exists x\ (\exists y\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\forall w\ (P_5(w))]$$

becomes:

$$\forall w\ ([\underline{(\neg\mathbf{P_1}(\mathbf{w}) \wedge \neg\mathbf{P_2}(\mathbf{w}))} \vee P_3(w)] \vee [\exists x\ (\exists y\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\forall w\ (P_5(w))]$$

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Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Variable w (w and w) is bound to two different quantifiers:

$$\forall w\ ([\neg(P_1(w) \vee P_2(w)) \vee P_3(w)] \vee [\exists x\ (\exists y\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\forall w\ (P_5(w))]$$

Replace w with z and the sentence S becomes:

$$\forall w\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [\exists x\ (\exists y\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\forall z\ (P_5(z))]$$

Predicate (First-Order) Logic to CNF

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$$\forall w\, ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\, (\exists y\, (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\, (P_5(z))]$$

Quantified variables unique, move quantifiers left (order!):

$$\forall w\, ([\neg(P_1(w) \vee P_2(w)) \vee P_3(w)] \vee [\underline{\exists \mathbf{x}\, (\exists \mathbf{y}}\, (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\underline{\forall \mathbf{z}}\, (P_5(z))]$$

becomes:

$$\forall w\, \underline{\exists \mathbf{x}\, \exists \mathbf{y}\, \forall \mathbf{z}}\, ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\neg P_6(x, y) \vee P_4(w, x))]) \wedge [(P_5(z))]$$

Predicate (First-Order) Logic to CNF

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We have two existential quantifiers to remove here:

$$\forall w\ \underline{\exists x}\ \exists y\ \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg \mathbf{P_6(x, y)}}) \vee \underline{\mathbf{P_4(w, x)}})]) \wedge [(P_5(z))]$$

and:

$$\forall w\ \exists x\ \underline{\exists y}\ \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg \mathbf{P_6(x, y)}}) \vee P_4(w, x))]) \wedge [(P_5(z))]$$

Both $\exists x$ and $\exists y$ are inside the scope of the universal quantifier $\forall w$. We need to use Skolem function substitution (Skolemization).

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall z\ (P_5(z))]$$

Let's start with $\exists x$ and replace x with a Skolem function:

$$\forall w\ \underline{\exists x}\ \exists y\ \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg P_6(x, y)} \lor \underline{P_4(w, x)})]) \land [(P_5(z))]$$

becomes:

$$\forall w \exists y \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg P_6(f(w), y)} \lor \underline{P_4(w, f(w))})]) \land [(P_5(z))]$$

Quantified variable x was replaced with Skolem function $f(w)$. Existential quantifier $\exists x$ was removed.

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall z\ (P_5(z))]$$

Now: remove $\exists y$ and replace y with a Skolem function:

$$\forall w \exists y \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg \mathbf{P_6(f(w), y)}}) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

becomes:

$$\forall w \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg \mathbf{P_6(f(w), g(w))}}) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

Quantified variable y was replaced with Skolem function $g(w)$. Existential quantifier $\exists y$ was removed.

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Original sentence S:

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Remaining quantified variables are universally quantified:

$$\forall w \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

We can simply "drop" universal quantifiers:

$$\forall w \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

becomes:

$$([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

We are "dropping" universal quantifiers for inferential purposes only. Equivalence is lost, but we can still use the remaining sentence to infer.

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Original sentence S:

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By Associative Law ($(p \lor q) \lor r \Leftrightarrow p \lor (q \lor r)$):

$([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$

becomes:

$([P_3(w) \lor (\neg P_1(w) \land \neg P_2(w))] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$

Converting FOL to CNF: Example 1

Original sentence S:

$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$

By Associative Law ($(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$):

$([\underline{\mathbf{P_3(w) \vee (\neg P_1(w) \wedge \neg P_2(w))}}] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$

becomes:

$([\underline{\mathbf{P_3(w) \vee (\neg P_1(w) \wedge \neg P_2(w))}}] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

By Distributive Law ($p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$):

$$([\underline{\mathbf{P_3(w) \vee (\neg P_1(w) \wedge \neg P_2(w))}}] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

becomes:

$$([\underline{\mathbf{(P_3(w) \vee \neg P_1(w)) \wedge (P_3(w) \vee \neg P_2(w))}}] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Let's make some substitutions:

$$([(P_3(w) \vee \neg P_1(w)) \wedge (P_3(w) \vee \neg P_2(w))] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

$A \equiv (P_3(w) \vee \neg P_1(w))$

$B \equiv (P_3(w) \vee \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$

so the sentence becomes:

$$([A \wedge B] \vee [C]) \wedge [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

By Distributive Law ($p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$):

$$([A \wedge B] \vee [C]) \wedge [(P_5(z))]$$

becomes:

$$((A \vee C) \wedge (B \vee C)) \wedge [(P_5(z))]$$

where:

$A \equiv (P_3(w) \vee \neg P_1(w))$

$B \equiv (P_3(w) \vee \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Remove substitutions:

$$((A \vee C) \wedge (B \vee C)) \wedge [(P_5(z))]$$

becomes:

$$(((P_3(w) \vee \neg P_1(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))) \wedge ((P_3(w) \vee \neg P_2(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w))))) \wedge [(P_5(z))]$$

where:

$A \equiv (P_3(w) \vee \neg P_1(w))$

$B \equiv (P_3(w) \vee \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$

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We can remove some parentheses:

$$(((P_3(w) \vee \neg P_1(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))) \wedge ((P_3(w) \vee \neg P_2(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w))))) \wedge [(P_5(z))]$$

becomes:

$$(P_3(w) \vee \neg P_1(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$$
$$\wedge\ (P_3(w) \vee \neg P_2(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$$
$$\wedge\ (P_5(z))$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

We obtained sentence S in CNF form:

$$(P_3(w) \vee \neg P_1(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$$
$$\wedge\ (P_3(w) \vee \neg P_2(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$$
$$\wedge\ (P_5(z))$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts (CNF)**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Let's number all clauses:

$$(P_3(w) \vee \neg P_1(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))_1$$
$$\wedge\ (P_3(w) \vee \neg P_2(w) \vee \neg P_6(f(w), g(w)) \vee P_4(w, f(w)))_2$$
$$\wedge\ (P_5(z))_3$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

“Everyone who loves all animals is loved by someone”

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
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6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Implication Law ($p \Rightarrow q \equiv \neg p \lor q$):

$$\forall x\ \underline{[\forall y\ (\mathbf{Animal}(y) \Rightarrow \mathbf{Loves}(x, y))] \Rightarrow [\exists y\ \mathbf{Loves}(y, x)]}$$

becomes:

$$\forall x\ \underline{\neg[\forall y\ (\mathbf{Animal}(y) \Rightarrow \mathbf{Loves}(x, y))] \lor [\exists y\ \mathbf{Loves}(y, x)]}$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (\text{Animal}(y) \Rightarrow \text{Loves}(x, y))] \Rightarrow [\exists y\ \text{Loves}(y, x)]$$

By Implication Law ($p \Rightarrow q \equiv \neg p \vee q$):

$$\forall x\ \neg[\forall y\ \underline{\mathbf{(Animal(y) \Rightarrow Loves(x, y))}}] \vee [\exists y\ \text{Loves}(y, x)]$$

becomes:

$$\forall x\ \neg[(\forall y\ (\neg\mathbf{Animal(y)}) \vee \mathbf{Loves(x, y)}))] \vee [\exists y\ \text{Loves}(y, x)]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

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7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (\mathrm{Animal}(y) \Rightarrow \mathrm{Loves}(x, y))] \Rightarrow [\exists y\ \mathrm{Loves}(y, x)]$$

By the equivalence ($\neg\forall x\ (p) \equiv \exists x\ (\neg p)$):

$$\forall x\ \underline{\neg[(\forall y\ (\neg\mathbf{Animal}(y) \lor \mathbf{Loves}(x, y))]} \lor [\exists y\ \mathrm{Loves}(y, x)]$$

becomes:

$$\forall x\ \underline{[(\exists y\ \neg(\neg\mathbf{Animal}(y) \lor \mathbf{Loves}(x, y))]} \lor [\exists y\ \mathrm{Loves}(y, x)]$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By De Morgan's Law ($\neg(p \lor q) \equiv \neg p \land \neg q$):

$$\forall x\ [(\exists y\ \underline{\neg(\neg \mathbf{Animal}(y) \lor \mathbf{Loves}(x, y))})] \lor [\exists y\ Loves(y, x)]$$

becomes:

$$\forall x\ [(\exists y\ \underline{(\neg\neg \mathbf{Animal}(y) \land \neg \mathbf{Loves}(x, y))})] \lor [\exists y\ Loves(y, x)]$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Double Negation Law ($\neg(\neg p) \equiv p$):

$$\forall x\ [(\exists y\ \underline{\mathbf{(\neg\neg Animal(y)}} \land \neg Loves(x, y))] \lor [\exists y\ Loves(y, x)]$$

becomes:

$$\forall x\ [(\exists y\ \underline{\mathbf{(Animal(y)}} \land \neg Loves(x, y))] \lor [\exists y\ Loves(y, x)]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

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6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Variable y (y and y) is bound to two different quantifiers:

$$\forall x\ [(\exists y\ \underline{(\neg\neg \mathbf{Animal(}y\mathbf{)}} \wedge \neg Loves(x, y))] \vee [\exists y\ Loves(y, x)]$$

Replace y with z and the sentence S becomes:

$$\forall x\ [(\exists y\ \underline{(\mathbf{Animal(}y\mathbf{)}} \wedge \neg Loves(x, y))] \vee [\exists z\ Loves(z, x)]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

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2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
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6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

We <u>CAN'T</u> move $\exists z$ left, as it is on the same "level" as $\exists y$:

$$\forall x\ [(\exists y\ (Animal(y) \land \neg Loves(x, y))] \lor [\exists z\ Loves(z, x)]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

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2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
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5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\, [\forall y\, (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\, Loves(y, x)]$$

We have two existential quantifiers to remove ($\exists y$, $\exists z$) :

$$\forall x\, [(\underline{\exists y}\, (Animal(y) \wedge \neg Loves(x, y))] \vee [\exists z\, Loves(z, x)]$$

and:

$$\forall x\, [(\exists y\, (Animal(y) \wedge \neg Loves(x, y))] \vee [\underline{\exists z}\, Loves(z, x)]$$

Both $\exists y$ and $\exists z$ are inside the scope of the universal quantifier $\forall x$. We need to use Skolem function substitution (Skolemization).

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Let's start with $\exists y$ and replace y with a Skolem function:

$$\forall x\ [(\underline{\exists y}\ (Animal(y) \wedge \neg Loves(x, y))] \vee [\exists z\ Loves(z, x)]$$

becomes:

$$\forall x\ [(Animal(F(x)) \wedge \neg Loves(x, F(x)))] \vee [\exists z\ Loves(z, x)]$$

Quantified variable y was replaced with Skolem function $F(x)$. Existential quantifier $\exists y$ was removed.

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Now, remove $\exists z$ and replace y with a Skolem function:

$$\forall x\ [(\exists y\ (Animal(y) \wedge \neg Loves(x, y))] \vee [\underline{\exists z}\ Loves(z, x)]$$

becomes:

$$\forall x\ [(Animal(F(x)) \wedge \neg Loves(x, F(x)))] \vee [Loves(G(x), x)]$$

Quantified variable z was replaced with Skolem function $G(x)$. Existential quantifier $\exists z$ was removed.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Remaining quantified variables are universally quantified:

$$\forall x\ [(Animal(F(x)) \land \neg Loves(x, F(x)))] \lor [Loves(G(x), x)]$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

We can simply “drop” universal quantifiers:

$$\forall x\ [(Animal(F(x)) \wedge \neg Loves(x, F(x)))] \vee [Loves(G(x), x)]$$

becomes:

$$[(Animal(F(x)) \wedge \neg Loves(x, F(x)))] \vee [Loves(G(x), x)]$$

We are “dropping” universal quantifiers for inferential purposes only. Equivalence is lost, but we can still use the remaining sentence to infer.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Commutative Law ($p \vee q \Leftrightarrow q \vee p$):

$$[(Animal(F(x)) \wedge \neg Loves(x, F(x)))] \vee [Loves(G(x), x)]$$

becomes:

$$[Loves(G(x), x)] \vee [(Animal(F(x)) \wedge \neg Loves(x, F(x)))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Distributive Law ($p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$):

$$[Loves(G(x), x)] \vee [(Animal(F(x)) \wedge \neg Loves(x, F(x)))]$$

becomes:

$$(Loves(G(x), x) \vee Animal(F(x))) \wedge (Loves(G(x), x) \vee \neg Loves(x, F(x)))$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Sentence S is now in CNF form:

$$(Loves(G(x), x) \lor Animal(F(x))) \land (Loves(G(x), x) \lor \neg Loves(x, F(x)))$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
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6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Let's number all clauses:

$$(Loves(G(x), x) \vee Animal(F(x))) \wedge (Loves(G(x), x) \vee \neg Loves(x, F(x)))$$

becomes:

$$(Loves(G(x), x) \vee Animal(F(x)))_1 \wedge (Loves(G(x), x) \vee \neg Loves(x, F(x)))_2$$

Predicate Logic Resolution: Example

Consider following sentences in English

A. **Everyone who loves all animals is loved by someone**
B. **Anyone who kills an animal is loved by no one**
C. **Jack Loves all animals**
D. **Either Jack or Curiosity killed the cat, who is named Tuna**

Q. Did Curiosity kill the cat?

FOL: The Resolution Inference Rule

Two clauses, which are assumed to be standardized apart, so that they share no variables, can be resolved if they contain complementary literals:

- **Propositional literals are complementary if one is the negation of the other**
- **Predicate logic literals are complimentary if one unifies with the negation of the other**

$$\frac{(l_1 \vee \ldots \vee l_k),\ (m_1 \vee \ldots \vee m_n)}{\text{SUBST}(\theta,\ l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1}\ \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1}\ \vee \ldots \vee m_n)}$$

where $\theta = \text{UNIFY}(l_{i-1}, m_j)$.

FOL: The Resolution Inference Rule

For example, the following two clauses:

$$[\mathrm{Animal}(F(x)) \vee \mathrm{Loves}(G(x), x)]$$ **and**

$$[\neg\mathrm{Loves}(u, v) \vee \neg\mathrm{Kills}(u, v)]$$

can be resolved by eliminating complementary literals

$\mathrm{Loves}(G(x), x)$ **and** $\neg\mathrm{Loves}(u, v)$

with the unifier

$$\theta = \{u/G(x), v/x\},$$

to produce the resolvent clause:

$$[\mathrm{Animal}(F(x)) \vee \neg\mathrm{Kills}(G(x), x)]$$

Predicate Logic Resolution: Example

Now, let's turn them into predicate logic sentences/KB:

A. $\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$

B. $\forall x\ [\exists z\ (Animal(z) \land Kills(x, z))] \Rightarrow [\forall y\ \neg Loves(y, x)]$

C. $\forall x\ [Animal(x) \Rightarrow Loves(Jack, x)]$

D. $Kills(Jack, Tuna) \lor Kills(Curiosity, Tuna)$

E. $Cat(Tuna)$

F. $\forall x\ [Cat(x) \Rightarrow Animal(x)]$

Q. $Kills(Curiosity, Tuna)$, **so** $\neg Q \equiv \neg Kills(Curiosity, Tuna)$

Predicate Logic Resolution: Example

Let's turn them into predicate logic CNF sentences/KB:

A1. $(Animal(F(x)) \lor Loves(G(x), x))$ **(A1 and A2 related)**

A2. $(\neg Loves(x, F(x)) \lor Loves(G(x), x))$

B. $(\neg Loves(y, x) \lor \neg Animal(z) \lor \neg Kills(x, z))$

C. $(\neg Animal(x) \lor Loves(Jack, x))$

D. $(Kills(Jack, Tuna) \lor Kills(Curiosity, Tuna))$

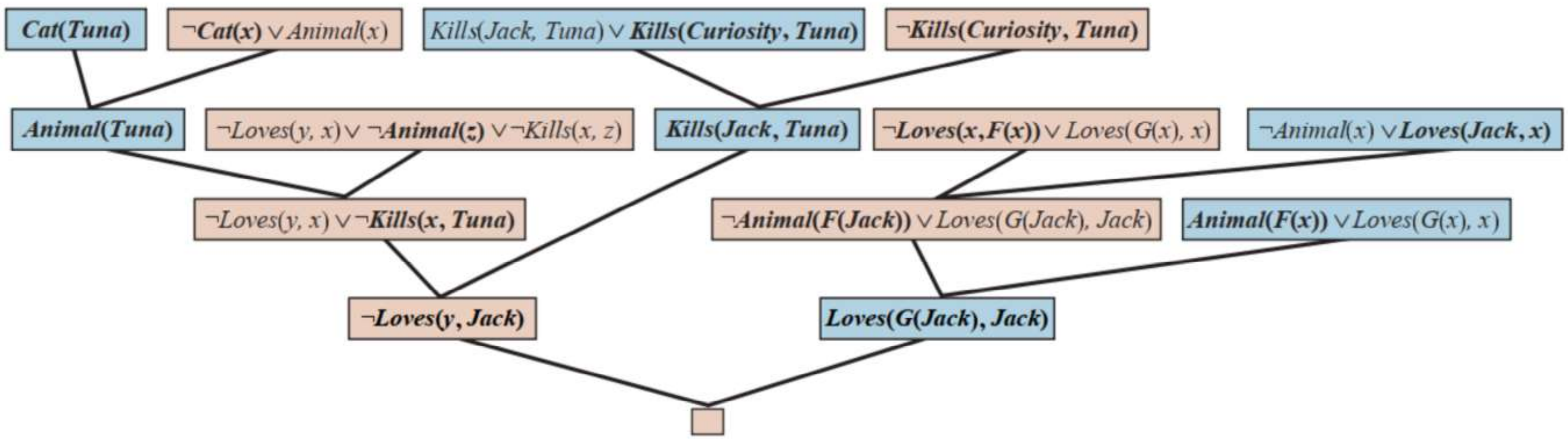
E. $(Cat(Tuna))$

F. $(\neg Cat(x) \lor Animal(x))$

Q. $Kills(Curiosity, Tuna)$**, so** $\neg Q \equiv (\neg Kills(Curiosity, Tuna))$

Predicate Logic Resolution: Example

Resolution process with substitutions:



Notice the use of factoring in derivation of the clause(Loves(G(Jack), Jack)**)**