CS 480

Introduction to Artificial Intelligence

November 1, 2022

Announcements / Reminders

- **Follow Week 10 TO DO List**
- **Written Assignment #03 due on ~~Sunday (11/06/22)~~ Thursday (11/10) at 11:00 PM CST**
- **Grading TA assignment:**

  https://docs.google.com/spreadsheets/d/1ExS0bKnGt_fdf4LHa3YS1qRA7-Iq4xqXVjfSAPMaGVk/edit?usp=sharing

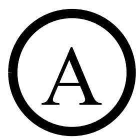
- **UPDATED Final Exam date:**
  - **December 1st, 2022 (last week of classes!)**
    - **Ignore the date provided by the Registrar**

Plan for Today

- **Quantifying and dealing with uncertainty**

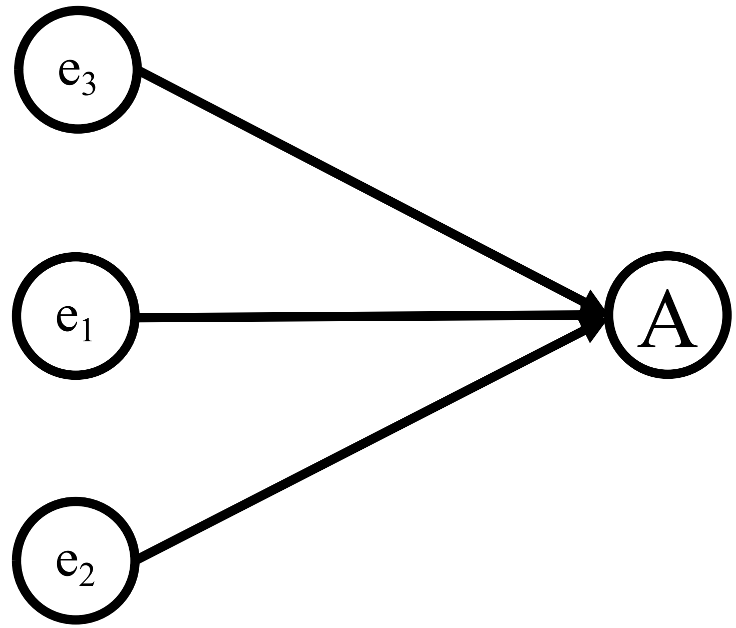
Prior vs. Posterior Probabilities

Prior Probability



$$P(A)$$

Posterior Probability



$$P(A \mid \text{parents}(A))$$

Marginal Probability

Marginal probability: the probability of an event occurring $P(A)$.

It may be thought of as an unconditional probability.

It is not conditioned on another event.

Joint Probability

The probability of event A and event B occurring (or more than two events). It is the probability of the intersection of two or more events.

$$P(A \wedge B)$$

For any propositions $f_1, f_2, \ldots, f_n$:

$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n)$$

Chain Rule

Conditional probabilities can be used to decompose conjunctions using the chain rule. For any propositions $f_1, f_2, \ldots, f_n$:

$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) =$$

$$P(f_1) *$$

$$P(f_2 \mid f_1) *$$

$$P(f_3 \mid f_1 \wedge f_2) *$$

$$\ldots$$

$$P(f_n \mid f_1 \wedge \ldots \wedge f_{n-1}) =$$

$$= \prod_{i=1}^{n} P(f_i \mid f_1 \wedge \ldots \wedge f_{i-1})$$

Chain Rule

Conditional probabilities can be used to decompose conjunctions using the chain rule. For any propositions $f_1, f_2, \ldots, f_n$:

$$P(f_1 = x_1 \wedge f_2 = x_2 \wedge \ldots \wedge f_n = x_n) =$$

$$P(f_1 = x_1) *$$

$$P(f_2 \mid f_1 = x_1) *$$

$$P(f_3 \mid f_1 = x_1 \wedge f_2 = x_2) *$$

$$\ldots$$

$$P(f_n = x_n \mid f_1 = x_1 \wedge \ldots \wedge f_{n-1} = x_{n-1}) =$$

$$= \prod_{i=1}^{n} P(f_i = x_i \mid f_1 = x_1 \wedge \ldots \wedge f_{i-1} = x_{i-1})$$

Bayes' Rule

$$\boldsymbol{P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}}$$

Bayes' Rule

$$P(cause \mid effect) = \frac{P(effect \mid cause) * P(cause)}{P(effect)}$$

Bayes' Rule

$P(cause \mid effect)$ **diagnostic direction relation**

$$P(cause \mid effect) = \frac{P(effect \mid cause) * P(cause)}{P(effect)}$$

$P(effect \mid cause)$ **causal direction relation**

Bayes' Rule

$P(disease \mid symptoms)$ **diagnostic direction relation**

$$P(disease \mid symptoms) = \frac{P(symptoms \mid disease) * P(disease)}{P(symptoms)}$$

$P(symptoms \mid disease)$ **causal direction relation**

Bayes' Rule

$$P(cause \mid effect) = \frac{P(effect \mid cause) * P(cause)}{P(effect)}$$

Problem: a single card is drawn from a standard deck of cards. What is the probability that we drew a queen if we know that a face card (J, Q, K) was drawn?

$$P(queen \mid face) = \frac{P(face \mid queen) * P(queen)}{P(face)}$$

$$P(queen \mid face) = \frac{1 * 4 / 52}{12 / 52} = \frac{1}{3}$$

Bayes' Rule

$$P(cause \mid effect) = \frac{P(effect \mid cause) * P(cause)}{P(effect)}$$

Problem: Calculate probability that a patient has meningitis if a patient has stiff neck. Meningitis is a cause of neck stiffness in 70% of cases, probability of having meningitis is 1/50000. Stiff neck happens to 1% of patients.

$$P(m \mid s) = \frac{P(s \mid m) * P(m)}{P(s)}$$

$$P(m \mid s) = \frac{0.7 * 1/50000}{0.01} = 0.0014$$

Independence

Assume that the knowledge of the truth of one proposition Y, does not affect the agent's belief in another proposition, X, in the context of other propositions Z. We say that X is independent of Y given Z.

Conditional Independence

Random variable X is conditionally independent of random variable Y given Z if for all $x \in Dx$, for all $y \in Dy$, and for all $z \in Dz$, such that

$$P(Y = y \wedge Z = z) > 0 \text{ and } P(Y = y' \wedge Z = z) > 0$$

$$P(X = x \mid Y = y \wedge Z = z) = P(X = x \mid Y = y' \wedge Z = z)$$

In other words, given a value of Z, knowing Y's value DOES NOT affect your belief in the value of X.

Conditional Independence

The following four statements are equivalent as long as conditional probabilities:

1. X **is conditionally independent of** Y **given** Z
2. Y **is conditionally independent of** X **given** Z
3. $P(X \mid Y, Z) = P(X \mid Z)$
4. $P(X, Y \mid Z) = P(X \mid Z) * P(Y \mid Z)$

Bayes’ Rule: Another Interpretation

Another way to think about Baye’s rule: it allows us to update the hypothesis H in light of some new data/evidence e.

$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)}$$

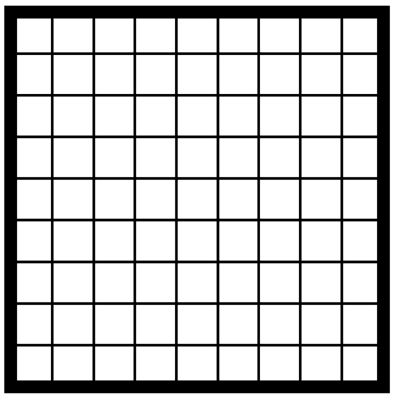
$$P(Hypothesis \mid evidence) = \frac{P(evidence \mid Hypothesis) * P(Hypothesis)}{P(evidence)}$$

where:

- P(H) **- probability of the** Hypothesis H **being true BEFORE we see new data/**evidence e **(prior probability)**
- P(H | e) **- probability of the** Hypothesis H **being true AFTER we see new data/**evidence e **(posterior probability)**
- P(e | H) **- probability of new data/**evidence e **being true under the** Hypothesis H **(likelihood)**
- P(e) **- probability of new data/**evidence e **being true under ANY hypothesis (normalizing constant)**

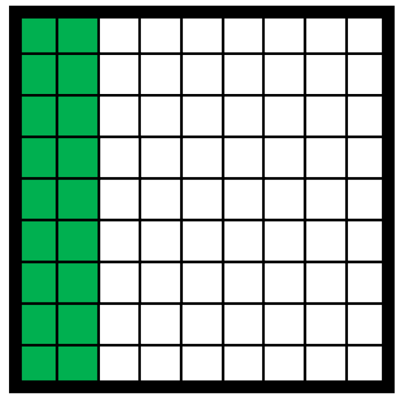
Bayes' Rule: Visual Interpretation

All possible cases



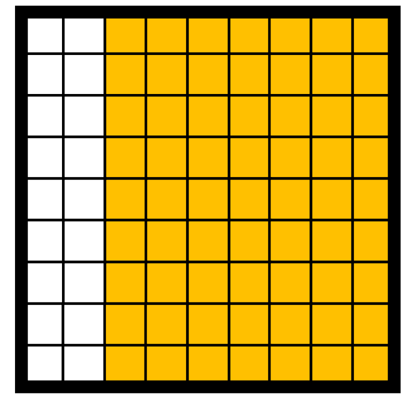
Cases where Hypothesis H **is true**

P(H)



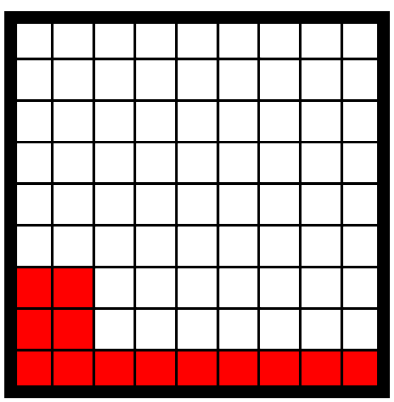
Cases where Hypothesis H **is false**

P(¬H)

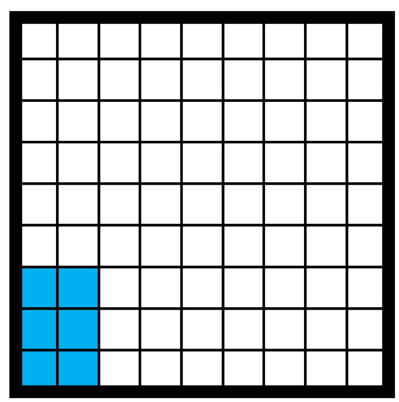


Cases where evidence e **is true**

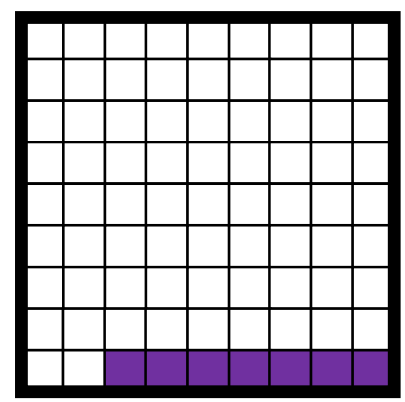
P(e)



Cases where evidence e **is true given** Hypothesis H **true** P(e | H)



Cases where evidence e **is true given** Hypothesis H **false** P(e | ¬H)



Bayes’ Rule: Visual Interpretation

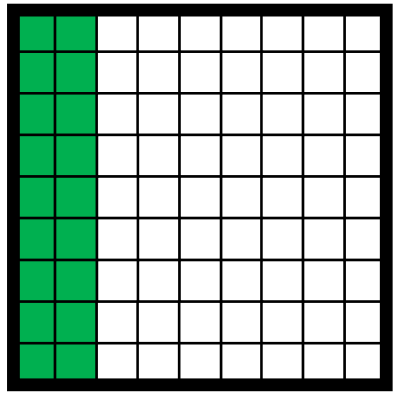
Bayes’ Rule:

$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)}$$

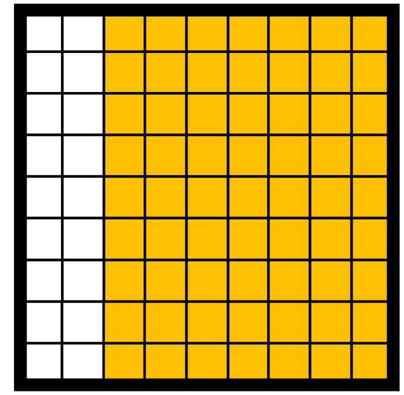
$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)}$$

$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(H) * P(e \mid H) + P(\neg H) * P(e \mid \neg H)}$$

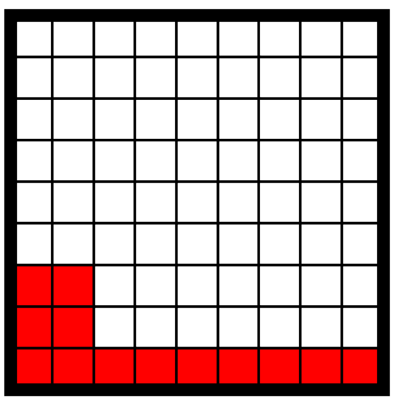
Cases where Hypothesis H is true
$P(H)$



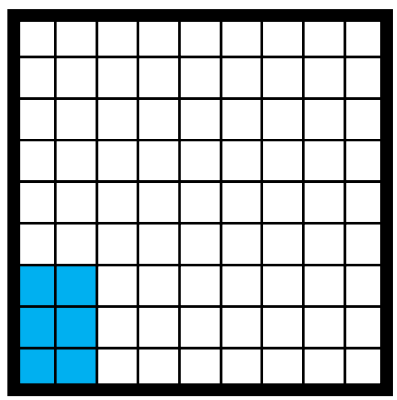
Cases where Hypothesis H is false
$P(\neg H)$



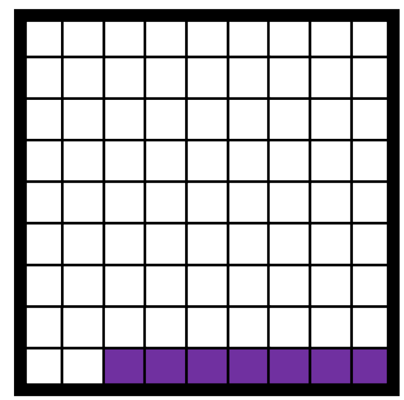
Cases where evidence e is true
$P(e)$



Cases where evidence e is true given Hypothesis H true $P(e \mid H)$



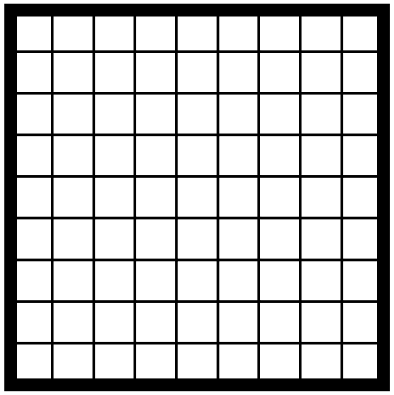
Cases where evidence e is true given Hypothesis H false $P(e \mid \neg H)$



Bayes' Rule: Visual Interpretation

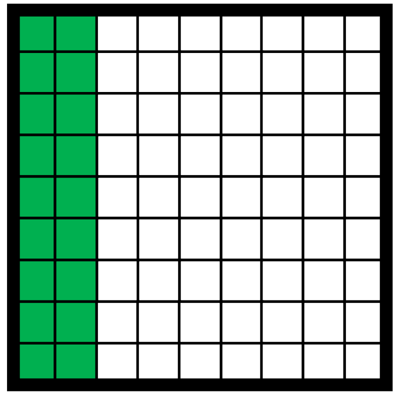
All CS 480 Students

Hypothesis H: **graduate student**



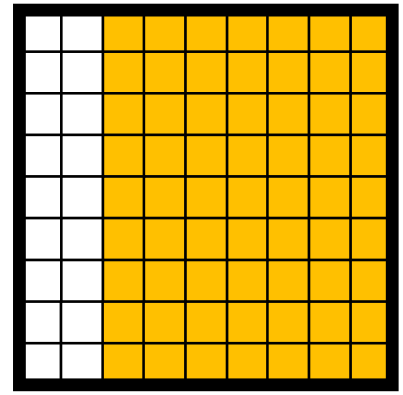
Cases where Hypothesis H **is true**

$P(H) = P(\text{grad} = \text{true})$



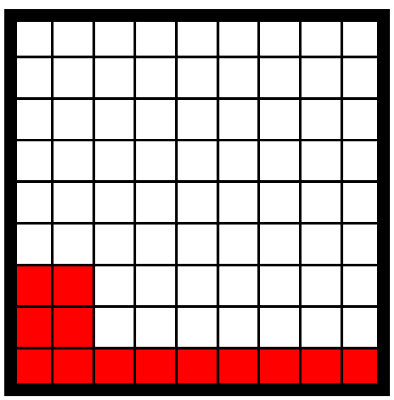
Cases where Hypothesis H **is false**

$P(\neg H) = P(\text{grad} = \text{false})$



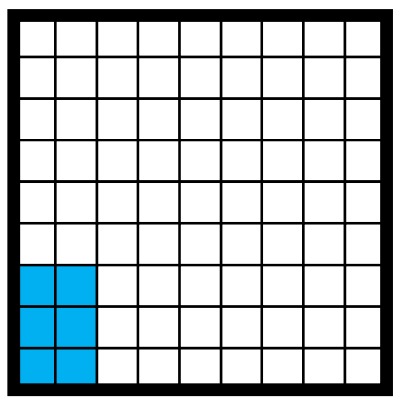
Cases where evidence e **is true**

$P(e) = P(\text{female} = \text{true})$



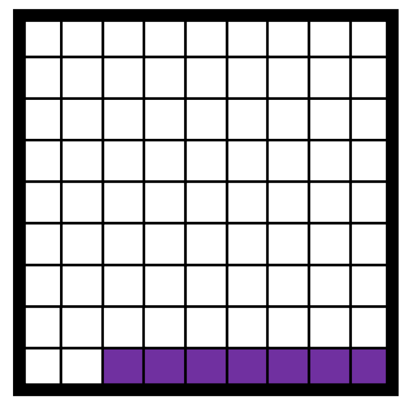
Cases where e **true given** H **true**

$P(e \mid H)=P(\text{female} = \text{true} \mid \text{grad} = \text{true})$



Cases where e **true given** H **false**

$P(e \mid \neg H)=P(\text{female} = \text{true} \mid \text{grad} = \text{false})$



Bayes' Rule: Visual Interpretation

Given (made up roster data):

%of G students: $P(H)$

%of UG students: $P(\neg H)$

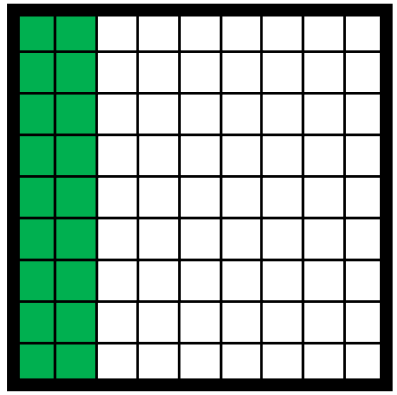
%of female students: $P(e)$

%of female G students: $P(e \mid H)$

%of female UG students: $P(e \mid \neg H)$

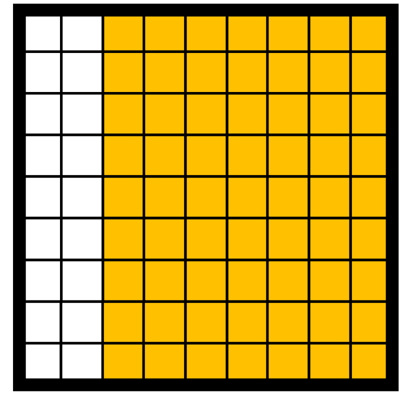
Cases where Hypothesis H is true

$P(H) = 18 / 81$



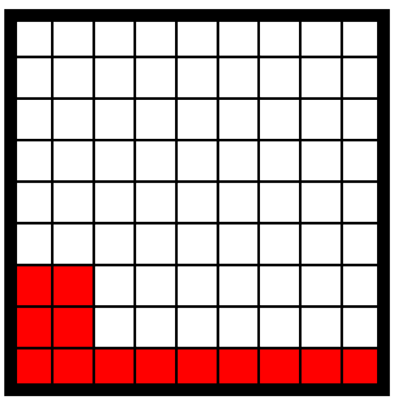
Cases where Hypothesis H is false

$P(\neg H) = 63 / 81$



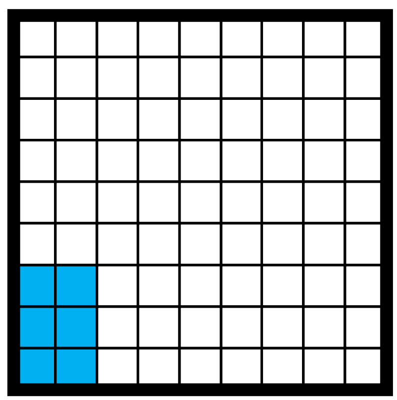
Cases where evidence e is true

$P(e) = 13 / 81$



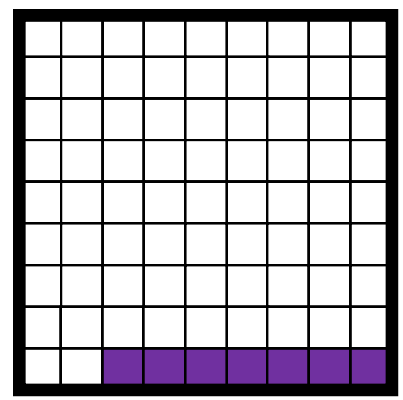
Cases where e true given H true

$P(e \mid H) = 6 / 18$



Cases where e true given H false

$P(e \mid \neg H) = 7 / 63$



Bayes' Rule: Visual Interpretation

Bayes' Rule:

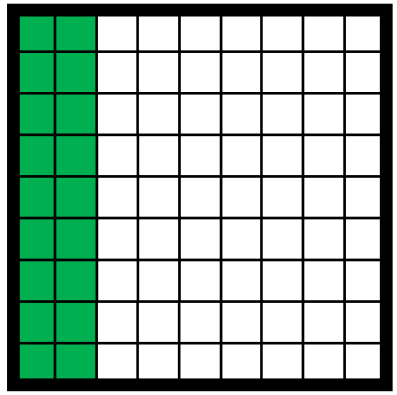
$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)}$$

$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)}$$

$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(H) * P(e \mid H) + P(\neg H) * P(e \mid \neg H)}$$

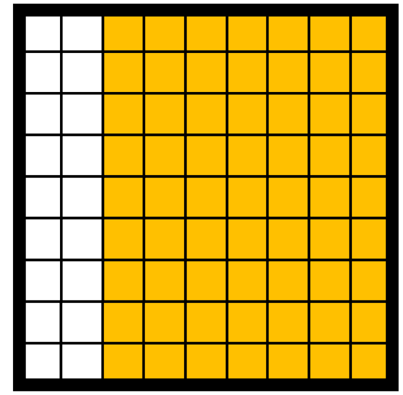
Cases where Hypothesis H is true

P(H) = 18 / 81



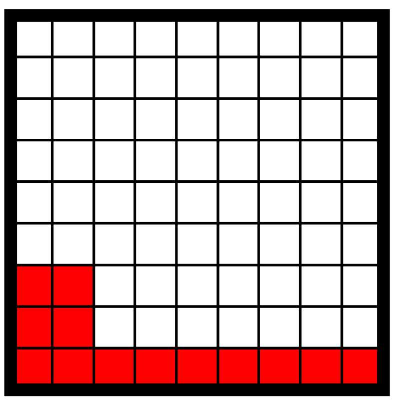
Cases where Hypothesis H is false

P(¬H) = 63 / 81



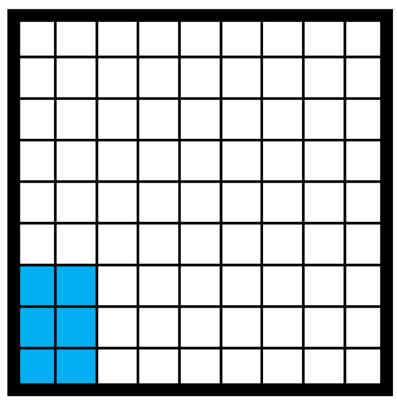
Cases where evidence e is true

P(e) = 13 / 81



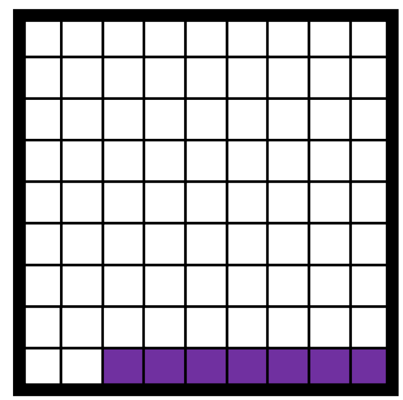
Cases where e true given H true

P(e | H) = 6 / 18



Cases where e true given H false

P(e | ¬H) = 7 / 63



Bayes' Rule: Visual Interpretation

Bayes' Rule:

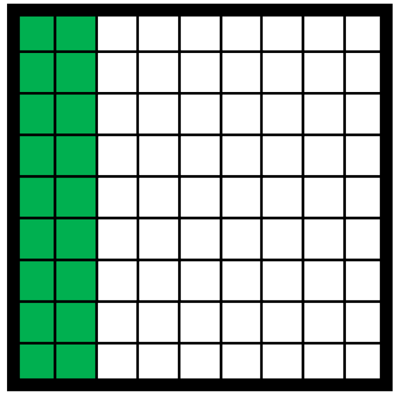
$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)}$$

$$P(H \mid e) = \frac{6 / 18 * 18 / 81}{13 / 81}$$

$$P(H \mid e) = \frac{6 / 18 * 18 / 81}{18 / 81 * 6 / 18 + 63 / 81 * 7 / 63}$$

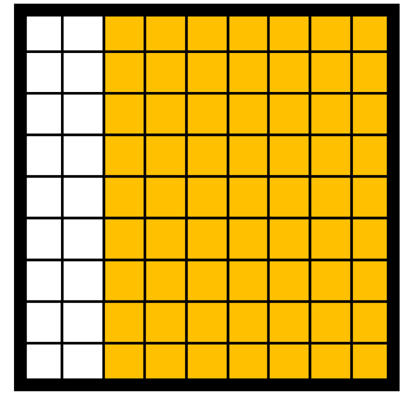
Cases where Hypothesis H is true

P(H) = 18 / 81



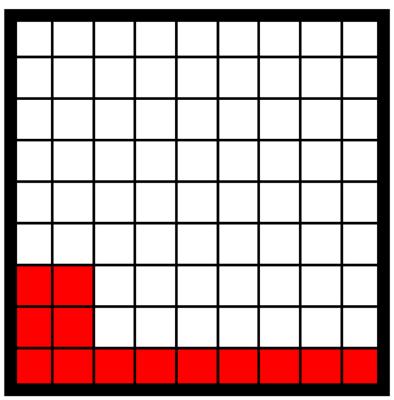
Cases where Hypothesis H is false

P(¬H) = 63 / 81



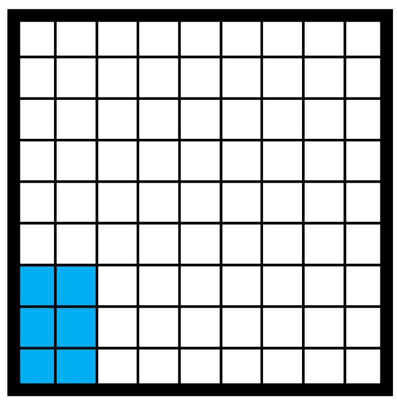
Cases where evidence e is true

P(e) = 13 / 81



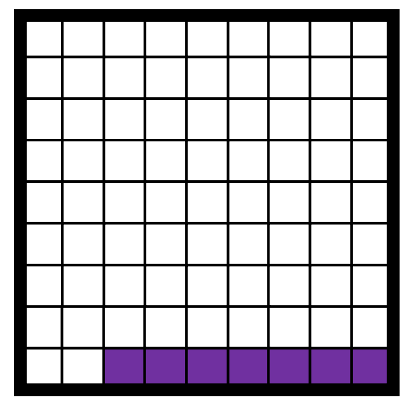
Cases where e true given H true

P(e | H) = 6 / 18



Cases where e true given H false

P(e | ¬H) = 7 / 63



Bayes' Rule: Visual Interpretation

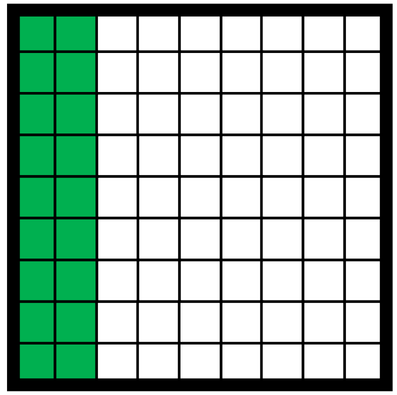
Bayes' Rule:

$$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)}$$

$$P(H \mid e) \approx 0.462$$

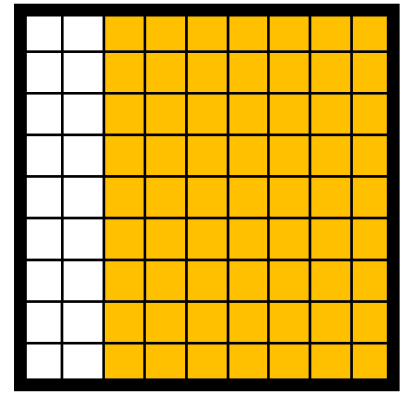
Cases where Hypothesis H is true

P(H) = 18 / 81



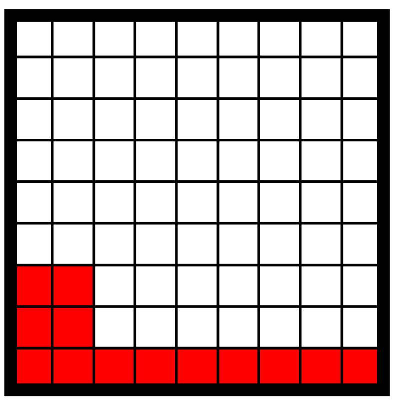
Cases where Hypothesis H is false

P(¬H) = 63 / 81



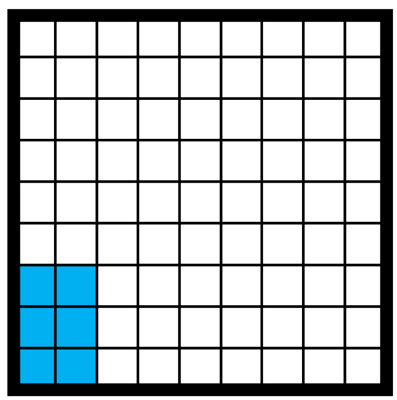
Cases where evidence e is true

P(e) = 13 / 81



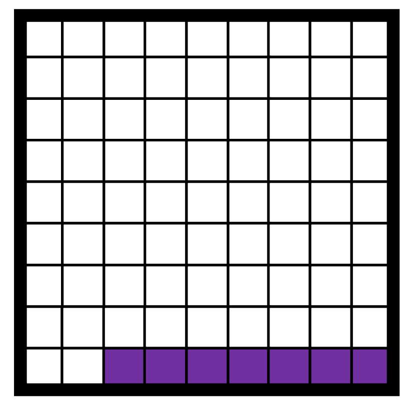
Cases where e true given H true

P(e | H) = 6 / 18



Cases where e true given H false

P(e | ¬H) = 7 / 63



Bayes' Rule: Visual Interpretation

Prior probability:

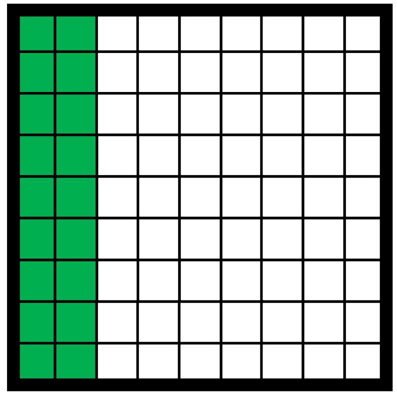
$$P(H) = 18 / 81 \approx 0.222$$

Posterior probability:

$$P(H \mid e) \approx 0.462$$

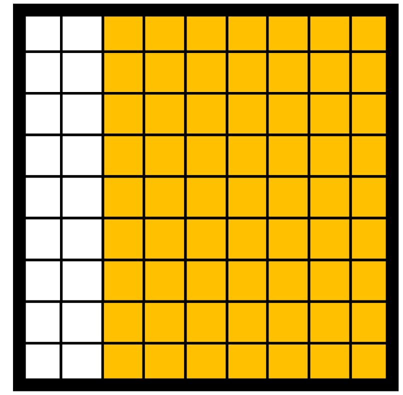
Cases where Hypothesis H **is true**

P(H) = 18 / 81



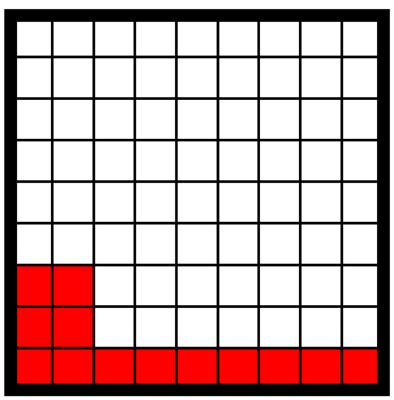
Cases where Hypothesis H **is false**

P(¬H) = 63 / 81



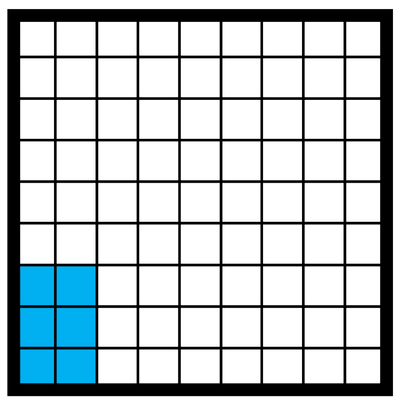
Cases where evidence e **is true**

P(e) = 13 / 81



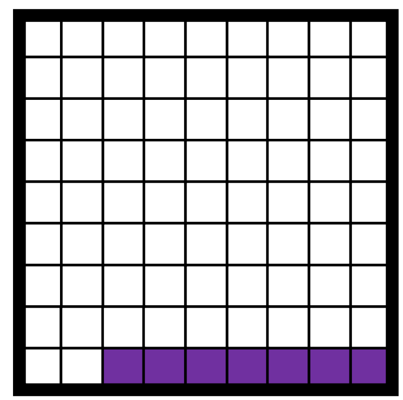
Cases where e **true given** H **true**

P(e | H) = 6 / 18



Cases where e **true given** H **false**

P(e | ¬H) = 7 / 63



Bayes' Rule: Belief/Probability Update

A student approaches the podium. Without looking I create a hypothesis H:

this is a grad student ($\mathrm{grad} = \mathrm{true}$)

My <u>belief</u> in H being true is based on prior probability:

$$P(H) = 18\,/\,81 \approx 0.222$$

I look up and see a female student, which is <u>new data / evidence</u> e ($\mathrm{female} = \mathrm{true}$). Bayes' Rule helps me update my <u>belief</u> in H being true with <u>posterior</u> probability:

$$P(H \mid e) = \frac{6\,/\,18 * 18\,/\,81}{18\,/\,81 * 6\,/\,18 + 63\,/\,81 * 7\,/\,63} \approx 0.462$$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)	Conditional probabilities
true	true	$P(H \mid e)*P(e) \approx 0.074$	$P(H \mid e) = \frac{P(e \mid H) * P(H)}{P(e)} = \frac{6/18 * 18/81}{13/81} \approx 0.462$
true	false	$P(H \mid \neg e)*P(\neg e) \approx 0.148$	$P(H \mid \neg e) = \frac{P(\neg e \mid H) * P(H)}{P(\neg e)} = \frac{12/18 * 18/81}{68/81} \approx 0.176$
false	true	$P(\neg H \mid e)*P(e) \approx 0.086$	$P(\neg H \mid e) = \frac{P(e \mid \neg H) * P(\neg H)}{P(e)} = \frac{7/63 * 63/81}{13/81} \approx 0.538$
false	false	$P(\neg H \mid \neg e)*P(\neg e) \approx 0.691$	$P(\neg H \mid \neg e) = \frac{P(\neg e \mid \neg H) * P(\neg H)}{P(\neg e)} = \frac{56/63 * 63/81}{68/81} \approx 0.824$
		SUM = 1	

Joint probabilities calculated using the Product Rule:

$$P(A \wedge B) = P(A \mid B) * P(B)$$

Conditional probabilities calculated using Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}$$

Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(grad = true \wedge female = true) = P(H, e) = P(H \wedge e) = P(H \mid e) * P(e) \approx 0.074$
true	false	$P(grad = true \wedge female = false) = P(H, \neg e) = P(H \mid \neg e) * P(\neg e) \approx 0.148$
false	true	$P(grad = false \wedge female = true) = P(\neg H, e) = P(\neg H \mid e) * P(e) \approx 0.086$
false	false	$P(grad = false \wedge female = false) = P(\neg H, \neg e) = P(\neg H \mid \neg e) * P(\neg e) \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Product Rule:

$$P(A \wedge B) = P(A \mid B) * P(B)$$

Conditional probabilities calculated using Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}$$

Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	0.074
true	false	0.148
false	true	0.086
false	false	0.691
		SUM = 1

If we know the joint probability distribution, we can infer:

- **marginal probabilities** P(H), P(¬H), P(e), **and** P(¬e)
- **conditional probabilities** P(H | e), P(H | ¬e), P(¬H | e), **and** P(¬H | ¬e)

Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Probability P(H)**:**

$$P(H) = P(grad = true) = 0.074 + 0.148 \approx 18 / 81$$

Probability P(H)**: "sum of all probabilities where** H **true"**

Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	0.074
true	false	0.148
false	true	0.086
false	false	0.691
		SUM = 1

Probability P(e)**:**

$$P(e) = P(female = true) = 0.074 + 0.086 \approx 13 / 81$$

Probability P(e)**: "sum of all probabilities where** e **true"**

Marginal Probability

Marginal probability: the probability of an event occurring $P(A)$.

It may be thought of as an unconditional probability.

It is not conditioned on another event.

Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Joint probabilities calculated using the Product Rule:

$$P(A \wedge B) = P(A \mid B) * P(B)$$

Conditional probabilities calculated using Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}$$

Joint Probability: Conditionals

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

From product rule:

$$P(H \wedge e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \wedge e)}{P(e)}$$

Joint Probability: Conditionals

H: grad	e: female	$P(H, e) = P(H \wedge e)$: $P(grad \wedge female)$
true	true	0.074
true	false	0.148
false	true	0.086
false	false	0.691
		SUM = 1

From product rule:

$$P(H \wedge e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \wedge e)}{P(e)} = \frac{0.074}{0.074 + 0.086} \approx 0.462$$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

Joint probabilities calculated using the Product Rule:

$$P(A \wedge B) = P(A \mid B) * P(B)$$

Conditional probabilities calculated using Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A) * P(A)}{P(B)}$$

Full Joint Probability Distribution

	Toothache		¬Toothache	
	Catch	¬Catch	Catch	¬Catch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Random variables:

Toothache **- Boolean**

Cavity **- Boolean**

Catch **(dentist's probe catches tooth) - Boolean**

Full Joint Probability Distribution

	Toothache		¬Toothache	
	Catch	¬Catch	Catch	¬Catch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Probability P(Cavity ∨ Toothache)**:**

$$P(Cavity = true \vee Toothache = true) =$$
$$= 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064$$
$$= 0.28$$

Full Joint Probability Distribution

	Toothache		¬Toothache	
	Catch	¬Catch	Catch	¬Catch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Marginal probability P(Cavity)**:**

$$P(Cavity = true) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$$

Full Joint Probability Distribution

	Toothache		¬Toothache	
	Catch	¬Catch	Catch	¬Catch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Conditional probability P(Cavity | Toothache)**:**

$$P(Cavity = true \mid Toothache = true) =$$

$$= \frac{P(Cavity = true \wedge Toothache = true)}{P(Toothache = true)} =$$

$$= \frac{0.108 + 0.012}{0.108 + 0.012 + 0.016 + 0.064} = 0.6$$

Full Joint Probability Distribution

	Toothache		¬Toothache	
	Catch	¬Catch	Catch	¬Catch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Conditional probability P(¬Cavity | Toothache)**:**

$$P(\neg Cavity \ = true \mid Toothache = \ true) =$$

$$= \frac{P(\neg Cavity \ = true \wedge Toothache = \ true)}{P(Toothache = \ true)} =$$

$$= \frac{0.016 + 0.164}{0.108 + 0.012 + 0.016 + 0.064} = \ 0.4$$

Full Joint Probability Distribution

	Toothache		¬Toothache	
	Catch	¬Catch	Catch	¬Catch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Note that:

$$P(Cavity \mid Toothache) = \frac{P(Cavity \wedge Toothache)}{P(Toothache)} = 0.6$$

$$P(\neg Cavity \mid Toothache) = \frac{P(\neg Cavity \wedge Toothache)}{P(Toothache)} = 0.4$$

add up to 1 and the same denominator is involved.

Full Joint Probability Distribution

	Toothache		¬Toothache	
	Catch	¬Catch	Catch	¬Catch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Note that $\mathrm{P}()$ is the distribution, NOT individual probability:

$$\boldsymbol{P}(Cavity \mid Toothache) = \alpha * \boldsymbol{P}(Cavity, Toothache) =$$

$$= \alpha * [\boldsymbol{P}(Cavity, Toothache, Catch) + \boldsymbol{P}(Cavity, Toothache, \neg Catch)] =$$

$$= \alpha * [\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle] =$$

$$= \alpha * \langle 0.12, 0.08\rangle =$$

$$= \langle 0.6, 0.4\rangle$$

General Inference Procedure

Given:

- **a query involving a single variable X (in our example: Cavity),**
- **a list of evidence variables $\mathbf{E}$ (in our example: just Toothache),**
- **a list of observed values e for $\mathbf{E}$,**
- **a list of remaining unobserved variables $\mathbf{Y}$ (in our example: just Catch),**

where X, $\mathbf{E}$, and $\mathbf{Y}$ together are a COMPLETE set of variables for the domain, the probability $\mathrm{P}(\mathrm{X} \mid \mathbf{E})$ can be evaluated as:

$$\boldsymbol{P}(X \mid \boldsymbol{e}) = \alpha * \boldsymbol{P}(X, \boldsymbol{e}) = \alpha * \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$$

where ys are all possible values for $\mathbf{Y}$s, α - normalization constant.

$\mathbf{P}(\mathrm{X}, \mathrm{e}, \mathrm{y})$ is a subset of probabilities from the joint distribution

Complex Joint Distributions

Consider a complex joint probability distribution involving N random variables P_1, P_2, P_3, …, P_{N-1}, Pp$_N$.

	N Random Variables						**Joint Probability**	
	P_1	P_2	P_3	…	P_{N-1}	P_N		
2^N Possible Worlds (Models)	true	true	true	…	true	true	false	**2^N values**
	true	true	true	…	true	false	true	
	true	true	false	…	false	true	false	
	…	…	…	…	…	…	…	
	false	false	true	…	true	false	true	
	false	false	true	…	false	true	true	
	false	false	false	…	false	false	false	

Non-binary / Non-Boolean RVs

Some Random Variables are going to have more than two possible, discrete, values:

- height **-> short, average, tall**
- size **-> S, M, L, XL**
- streetLight **-> green, orange, red**
- vision **-> 20/20, 15/15, etc.**
- continent **-> Africa, Antarctica, Asia, Australia, Europe, North America, South America**

Non-binary RVs increase the complexity.

This May Be Impossible to Manage!

	N Random Variables						
	P_1	P_2	P_3	…	P_{N-1}	P_N	Joint Probability
2^N Possible Worlds (Models)	true	true	true	…	true	true	false
	true	true	true	…	true	false	true
	true	true	false	…	false	true	false
	…	…	…	…	…	…	…
	false	false	true	…	true	false	true
	false	false	true	…	false	true	true
	false	false	false	…	false	false	false

2^N values

Independent Variable

		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
¬Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576
		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
Cloudy	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

Let's introduce another random variable Cloudy **representing some weather conditions. It is difficult to imagine the other random variables here (**Toothache**,** Cavity**,** Catch**) being dependent on** Cloudy **and vice versa.**

Independent Variable

		Toothache		¬Toothache	
¬Cloudy		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576
Cloudy		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

Let's try to calculate the following probability:

$$P(Toothache, Catch, Cavity, Cloudy)$$

using the Product Rule:

$$P(Toothache, Catch, Cavity, Cloudy) =$$
$$= P(Cloudy \mid Toothache, Catch, Cavity) * P(Toothache, Catch, Cavity)$$

Independent Variable

¬Cloudy		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

Cloudy		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

It's hard to imagine Cloudy influencing other variables, so:

$$P(Cloudy \mid Toothache, Catch, Cavity) = P(Cloudy)$$

and then:

$$P(Toothache, Catch, Cavity, Cloudy) =$$
$$= P(Cloudy) * P(Toothache, Catch, Cavity)$$

Independent Variable / Factoring

¬Cloudy		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

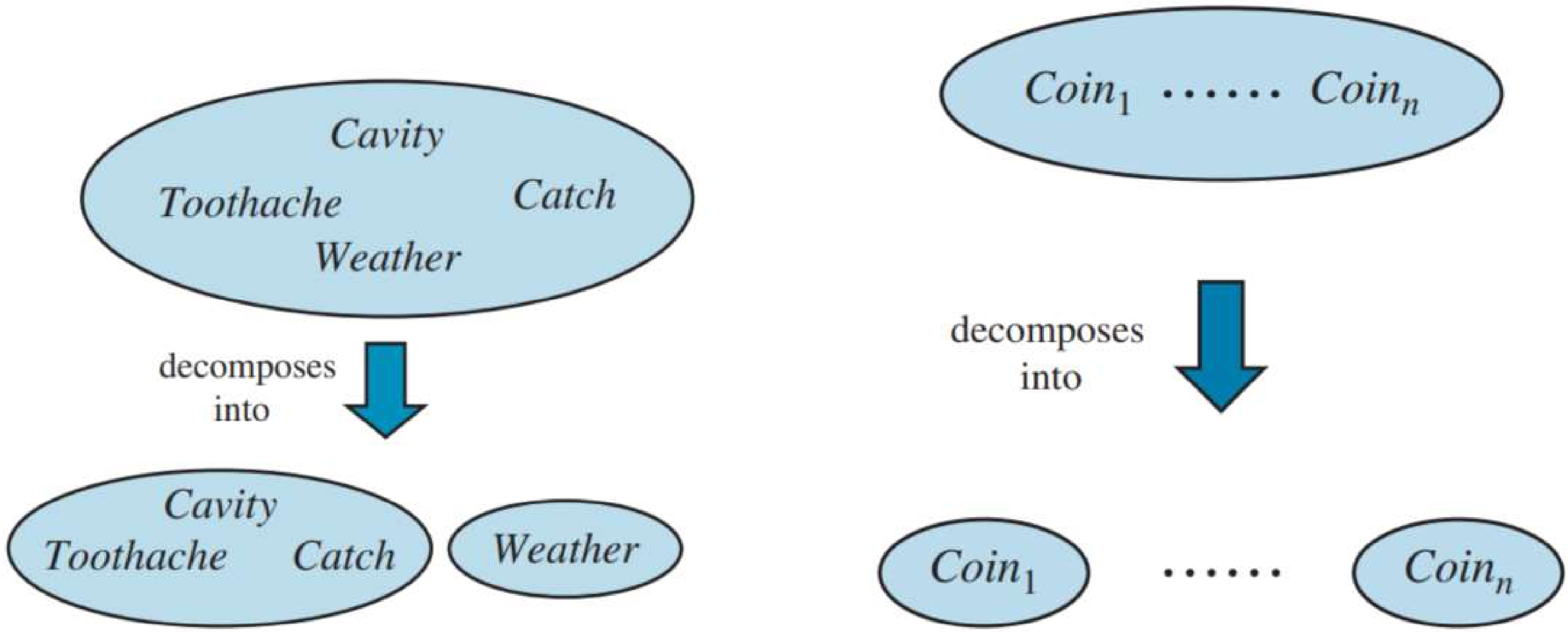
Cloudy		Toothache		¬Toothache	
		Catch	¬Catch	Catch	¬Catch
	Cavity	0.108	0.012	0.072	0.008
	¬Cavity	0.016	0.064	0.144	0.576

It's hard to imagine Cloudy influencing other variables, so:

$$P(Toothache, Catch, Cavity, Cloudy) =$$
$$= P(Cloudy) * P(Toothache, Catch, Cavity)$$

This shows that Cloudy **is INDEPENDENT of other variables and factoring can be applied.**

Factoring / Decomposition



Use Chain Rule To Decompose

N Random Variables						Joint Probability
P_1	P_2	P_3	…	P_{N-1}	P_N	
true	true	true	…	true	true	false
true	true	true	…	true	false	true
true	true	false	…	false	true	false
…	…	…	…	…	…	…
false	false	true	…	true	false	true
false	false	true	…	false	true	true
false	false	false	…	false	false	false

Chain Rule

Conditional probabilities can be used to decompose conjunctions using the chain rule. For any propositions $f_1, f_2, \ldots, f_n$ **:**

$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) = \prod_{i=1}^{n} P(f_i \mid f_1 \wedge \ldots \wedge f_{i-1})$$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(grad = true \wedge female = true) = P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) \approx 0.074$
true	false	$P(grad = true \wedge female = false) = P(H, \neg e) = P(H) * P(\neg e \mid H) \approx 0.148$
false	true	$P(grad = false \wedge female = true) = P(\neg H, e) = P(\neg H) * P(e \mid \neg H) \approx 0.086$
false	false	$P(grad = false \wedge female = false) = P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Chain Rule:

$$P(f_1 \wedge f_2) = \prod_{i=1}^{2} P(f_i \mid f_1 \wedge \ldots \wedge f_{i-1})$$

$$P(f_1 \wedge f_2) = P(f_1) * P(f_2 \mid f_1)$$

so: $P(grad \wedge female) = P(H \wedge e) = P(H) * P(e \mid H)$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) = 18 / 81 * 6 / 18 \approx 0.074$
true	false	$P(H, \neg e) = P(H) * P(\neg e \mid H) = 18 / 81 * 12 / 18 \approx 0.148$
false	true	$P(\neg H, e) = P(\neg H) * P(e \mid \neg H) = 63 / 81 * 7 / 63 \approx 0.086$
false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63 / 81 * 56 / 63 \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Chain Rule:

$$P(f_1 \wedge f_2) = \prod_{i=1}^{2} P(f_i \mid f_1 \wedge \ldots \wedge f_{i-1})$$

$$P(f_1 \wedge f_2) = P(f_1) * P(f_2 \mid f_1)$$

so: $P(grad \wedge female) = P(H \wedge e) = P(H) * P(e \mid H)$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) = 18/81 * 6/18 \approx 0.074$
true	false	$P(H, \neg e) = P(H) * P(\neg e \mid H) = 18/81 * 12/18 \approx 0.148$
false	true	$P(\neg H, e) = P(\neg H) * P(e \mid \neg H) = 63/81 * 7/63 \approx 0.086$
false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63/81 * 56/63 \approx 0.691$
		SUM = 1

Joint probabilities calculated using the Chain Rule:

$$P(f_1 \wedge f_2) = \prod_{i=1}^{2} P(f_i \mid parents(f_i))$$

$$P(f_1 \wedge f_2) = P(f_1) * P(f_2 \mid parents(f_i))$$

so: $P(H \wedge e) = P(H) * P(e \mid parents(e)) = P(H) * P(e \mid H)$

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) = 18/81 * 6/18 \approx 0.074$
true	false	$P(H, \neg e) = P(H) * P(\neg e \mid H) = 18/81 * 12/18 \approx 0.148$
false	true	$P(\neg H, e) = P(\neg H) * P(e \mid \neg H) = 63/81 * 7/63 \approx 0.086$
false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63/81 * 56/63 \approx 0.691$
		SUM = 1

$$P(H \wedge e) = P(H) * P(e \mid parents(e)) = \boxed{P(H)} * \boxed{P(e \mid H)}$$

H: grad	¬H: ¬grad
18 / 81 ≈ 0.22	63 / 81 ≈ 0.78

H: grad	e: female	P(e | H)
true	true	6 / 18 ≈ 0.333
true	false	12 / 18 ≈ 0.667
false	true	7 / 63 ≈ 0.111
false	false	56 / 63 ≈ 0.889

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) = 18 / 81 * 6 / 18 \approx 0.074$
true	false	$P(H, \neg e) = P(H) * P(\neg e \mid H) = 18 / 81 * 12 / 18 \approx 0.148$
false	true	$P(\neg H, e) = P(\neg H) * P(e \mid \neg H) = 63 / 81 * 7 / 63 \approx 0.086$
false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63 / 81 * 56 / 63 \approx 0.691$
		SUM = 1

$$P(H \wedge e) = P(H) * P(e \mid parents(e)) = \boxed{P(H)} * \boxed{P(e \mid H)}$$

H: grad	¬H: ¬grad
18 / 81 ≈ 0.22	63 / 81 ≈ 0.78

H: grad	e: female	P(e | H)
true	true	6 / 18 ≈ 0.333
true	false	12 / 18 ≈ 0.667
false	true	7 / 63 ≈ 0.111
false	false	56 / 63 ≈ 0.889

Conditional Probability Table (CPT)

Bayesian (Belief) Network

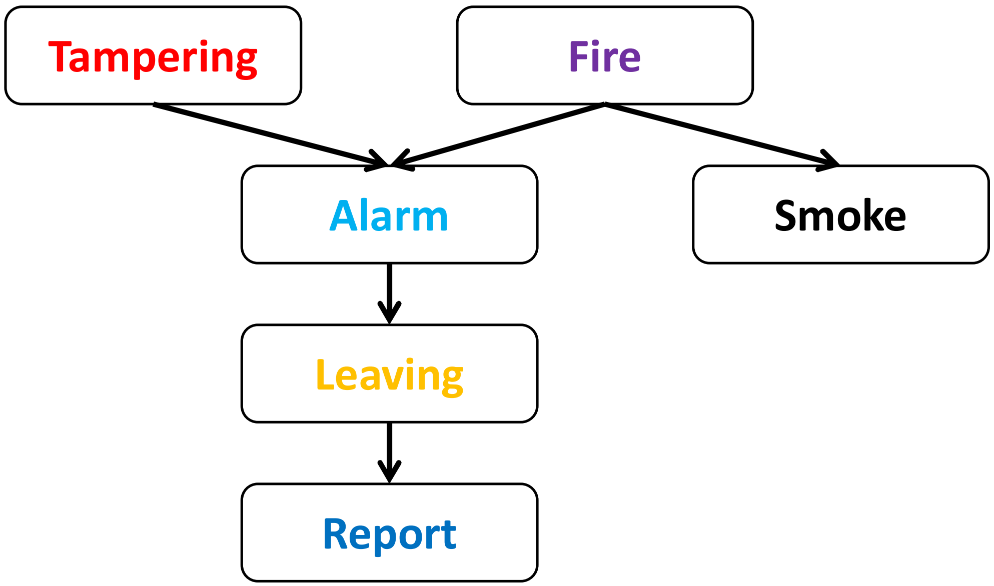
A Bayesian belief network describes the joint probability distribution for a set of variables.

A Bayesian network is an acyclic, directed graph (DAG), where the nodes are random variables (propositions). There is an edge (arc) from each elements of $\text{parents}(X_i)$ into X_i. Associated with the Bayesian network is a set of conditional probability distributions - the conditional probability of each variable given its parents (which includes the prior probabilities of those variables with no parents).

Consists of:

- **a graph (DAG) with nodes corresponding to random variables**
- **a domain for each random variable**
- **a set of conditional probability distributions $P(X_i \mid \text{parents}(X_i))$**

Bayesian (Belief) Network: Example



Random Variables (Propositions):

- Tampering: true if the alarm is tampered with
- Fire: true if there is a fire
- Alarm: true if the alarm sounds
- Smoke: true if there is smoke
- Leaving: true if people leaving the building at once
- Report: true if someone who left the building reports fire

Domain for all variables: $\{\text{true}, \text{false}\}$

NOTE: RVs don't have to be Boolean

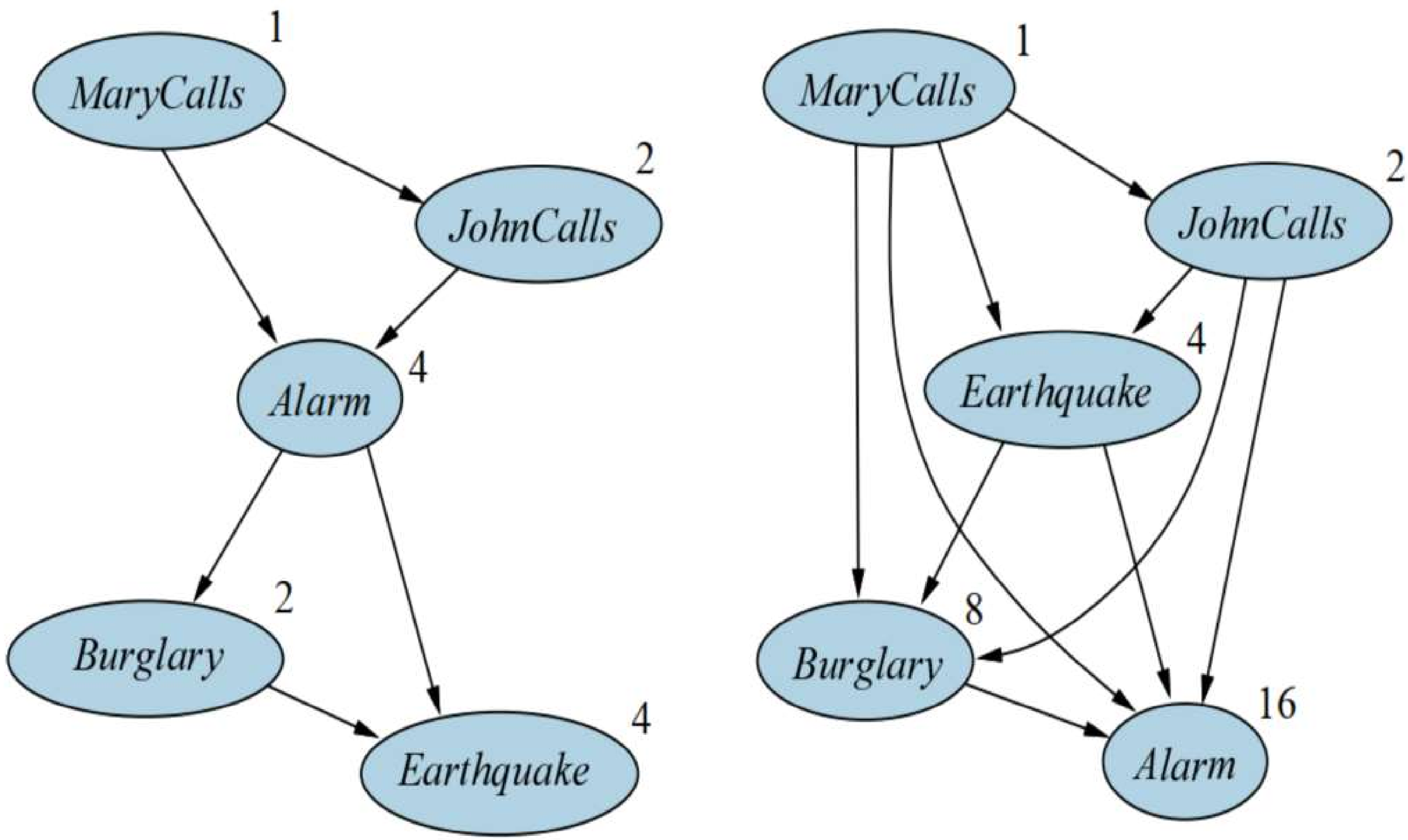
Building Bayesian (Belief) Network

1. Order Random Variables (ordering matters!)
2. Create network nodes for each Random Variable
3. Add edges between parent nodes and children nodes
   - For every node node X_i:
     - choose a minimal set S of parents for X_i
     - for each parent node Y in S add an edge from Y to X_i

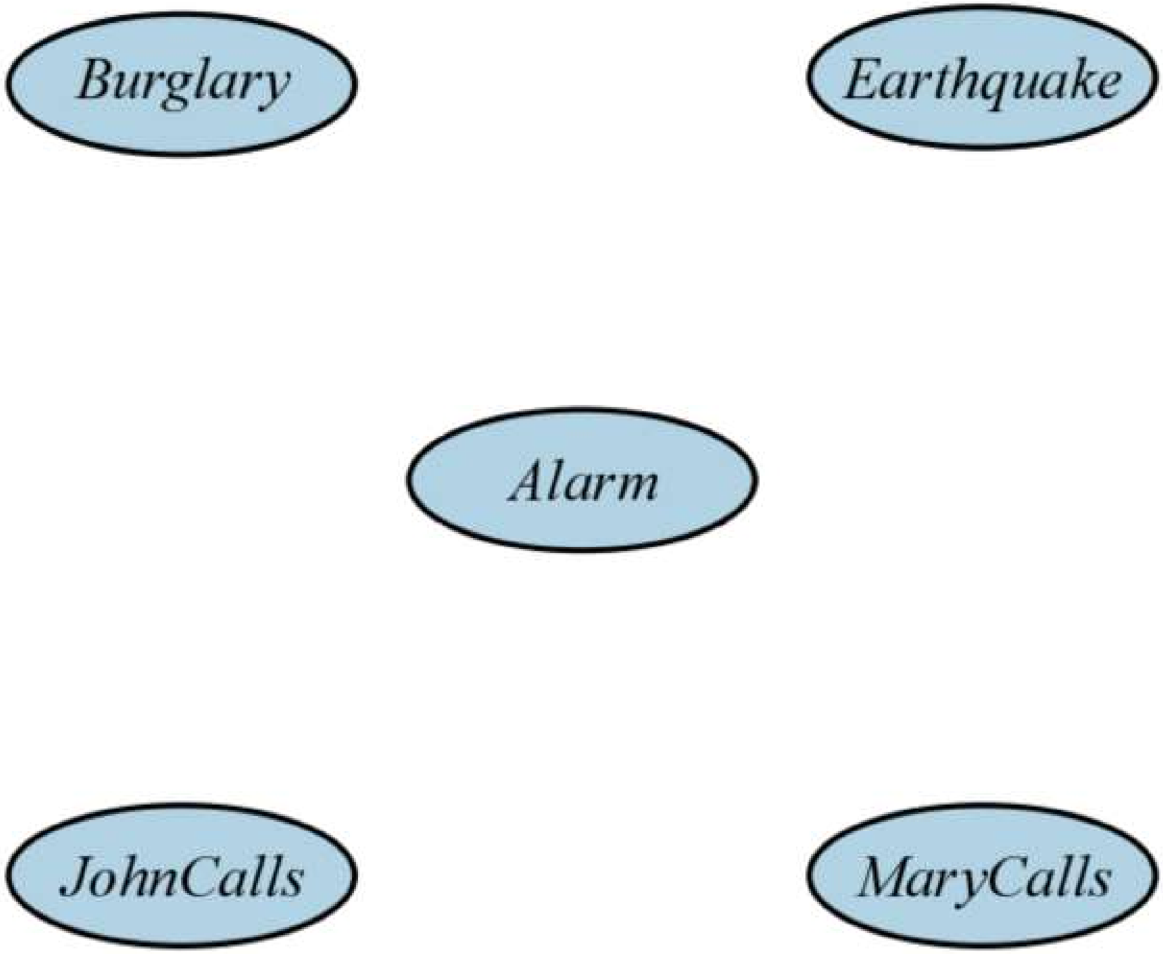
4. Add Conditional Probability Tables

Make it compact / sparse: choose your Random Variable ordering wisely.

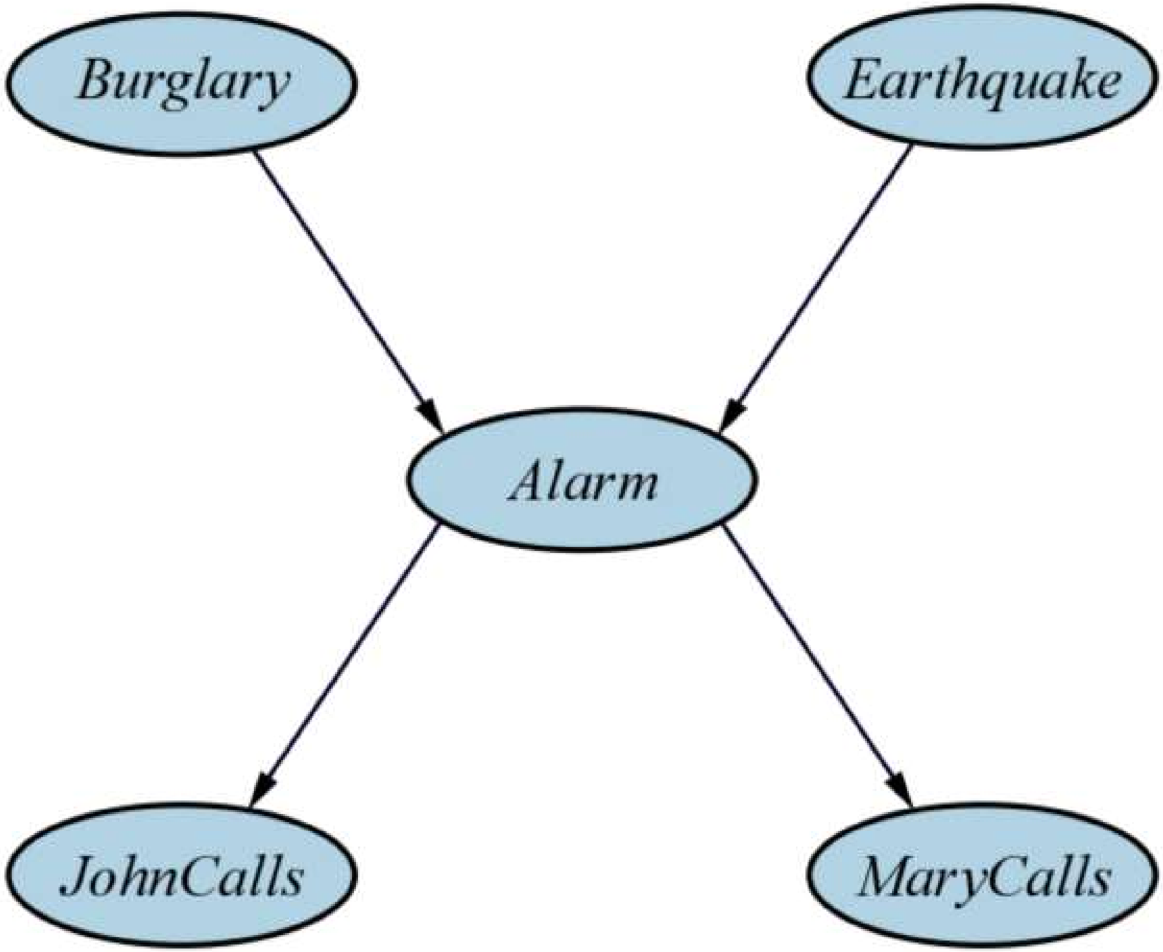
Ordering Matters!



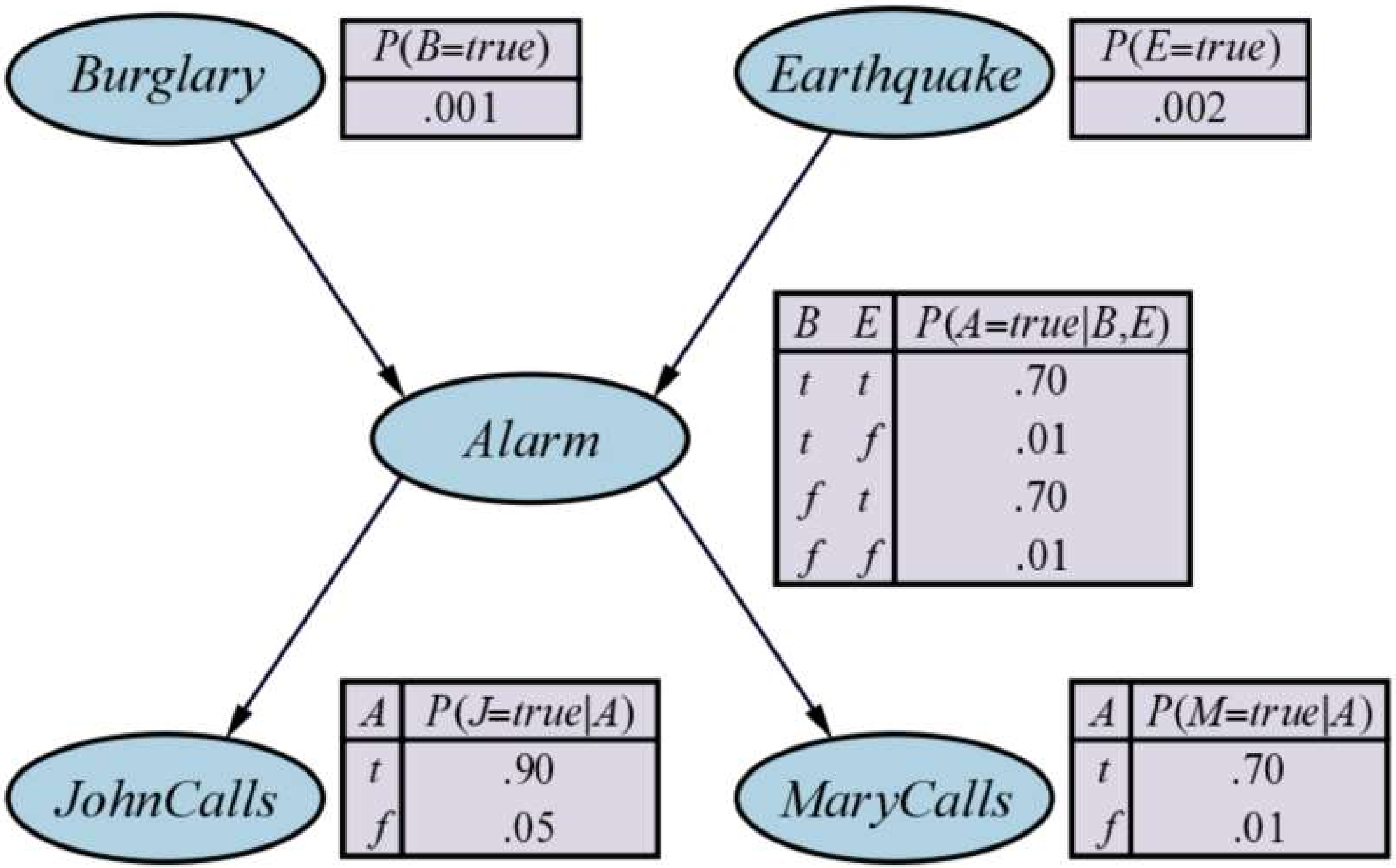
Create Vertices / Node / Random Vars



Add Edges



Add Conditional Probability Tables



P(B=true)
.001

P(E=true)
.002

B	E	P(A=true|B,E)
t	t	.70
t	f	.01
f	t	.70
f	f	.01

A	P(J=true|A)
t	.90
f	.05

A	P(M=true|A)
t	.70
f	.01

Full Joint Probability Distribution

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	$P(H, e) = P(H \wedge e) = P(H) * P(e \mid H) = 18 / 81 * 6 / 18 \approx 0.074$
true	false	$P(H, \neg e) = P(H) * P(\neg e \mid H) = 18 / 81 * 12 / 18 \approx 0.148$
false	true	$P(\neg H, e) = P(\neg H) * P(e \mid \neg H) = 63 / 81 * 7 / 63 \approx 0.086$
false	false	$P(\neg H, \neg e) = P(\neg H) * P(\neg e \mid \neg H) = 63 / 81 * 56 / 63 \approx 0.691$
		SUM = 1

$$P(H \wedge e) = P(H) * P(e \mid parents(e)) = \boxed{P(H)} * \boxed{P(e \mid H)}$$

H: grad	¬H: ¬grad
18 / 81 ≈ 0.22	63 / 81 ≈ 0.78

H: grad	e: female	P(e | H)
true	true	6 / 18 ≈ 0.333
true	false	12 / 18 ≈ 0.667
false	true	7 / 63 ≈ 0.111
false	false	56 / 63 ≈ 0.889

Conditional Probability Table (CPT)

Create Vertices / Node / Random Vars

Create Vertices / Node / Random Vars

Add Edges

Add Conditional Probability Tables

H: grad	¬H: ¬grad
18 / 81 ≈ 0.22	63 / 81 ≈ 0.78

H: grad	e: female	P(e | H)
true	true	6 / 18 ≈ 0.333
true	false	12 / 18 ≈ 0.667
false	true	7 / 63 ≈ 0.111
false	false	56 / 63 ≈ 0.889

Bayesian Network: Car Insurance