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Sensor fusion - III

Detailed Feedback

Task 1: *Bayesian estimation of an existence variable*

We are back at tracking the number of boats in a region. However, you now know that there is at most one boat in the region, and that it is there with probability $r_k \in (0,1)$ at time step k. The boat will stay in the region with probability P_S and leave with probability $1 - P_S$ if it is there. Otherwise, it may enter with probability P_E and not enter with probability $1 - P_E$ if it is not in the region. You have an unreliable measurement coming from a radar that detects a present boat with probability P_D, and may not detect a present boat with probability $1 - P_D$. If it did not detect a present boat, it may declare that there is a boat present anyway, termed a false alarm, with probability P_{FA}. The task is to apply a Bayesian filter to this problem.

> *Hint:* It might ease proper book keeping if you start by denoting the events with variables and use their (possibly conditional) pmfs before you insert the given probabilities.

(a) Apply the Bayes prediction step to get the predicted probability $r_{k+1|k}$ in terms of r_k, P_S and P_E.

> *Hint:* Introducing the events at time step k as "in region" R_k, "staying" S_k and "entering" E_k, where $S_{k+1} = R_{k+1}|R_k$ and $E_{k+1} = R_{k+1}|\neg R_k$ with $\neg$ denoting negation, can be helpful.

Boat in region $\Rightarrow P(R_k) = r_k$

Staying $\Rightarrow P(S_k | R_k) = P_S$

Entering $\Rightarrow P(E_k | \neg R_k) = P_E$

Using Bayes prediction step,

$$r_{k+1|k} = \sum_{r=0}^{1} P(R_{k+1} | R_k = r_k)\, P(R_k = r_k)$$

$$= P(R_{k+1} | R_k = 0)\, P(R_k = 0) + P(R_{k+1} | R_k = 1)\, P(R_k = 1)$$

$$= P_E \cdot P(\neg R_k) + P_S \cdot P(R_k)$$

$$= P_E \cdot (1 - r_k) + P_S\, r_k$$

(b) Apply the Bayes update step to get posterior probability for the boat being in the region, r_{k+1}, in terms of $r_{k+1|k}$, P_D and P_{FA}. That is, condition the probability on the measurement. There are two cases that needs to be considered; receiving a detection and not receiving a detection.

> *Hint:* Introduce the events $M_{k+1} = D_{k+1} \cup F_{k+1}$ to denote the sensor declaring a present boat, with D_{k+1} denoting the event of detecting a present boat and F_{k+1} denoting the event of a false alarm. We then for instance have $\Pr(D_{k+1}|\neg R_{k+1}) = 0$ and $\Pr(F_{k+1}|D_{k+1}) = 0$ from the problem setup.

Using Bayes update step,

$$P(R_{k+1} \mid M_{k+1}) = \frac{P(M_{k+1} \mid R_{k+1}) \cdot P(R_{k+1})}{P(M_{k+1})}$$

$$P(R_{k+1}) = \gamma_{k+1|k}$$

$$P(D_{k+1} \mid R_{k+1}) = P_D$$

$$P(F_{k+1} \mid \neg R_{k+1}) = P_{FA}$$

M_{k+1} can be 0 or 1

$$P(R_{k+1} \mid M_{k+1} = 1)$$

$$= P(M_{k+1} = 1 \mid R_{k+1})\, P(R_{k+1}) + P(M_{k+1} = 1 \mid \neg R_{k+1}) \cdot P(\neg R_{k+1})$$

$$= P_D \cdot \gamma_{k+1|k} + P_{FA}(1 - \gamma_{k+1|k})$$

$$P(R_{k+1} \mid M_{k+1} = 1) = \frac{P_D \cdot \gamma_{k+1|k}}{P_D \gamma_{k+1|k} + P_{FA}(1 - \gamma_{k+1|k})}$$

$$P(M_{k+1} = 0) = P(M_{k+1} = 0 \mid R_{k+1})\, P(R_{k+1}) + P(M_{k+1} = 0 \mid \neg R_{k+1})\, P(\neg R_{k+1})$$

$$= (1 - P_D)\gamma_{k+1|k} + (1 - P_{FA})(1 - \gamma_{k+1|k})$$

$$P(R_{k+1} \mid M_{k+1} = 0) = \frac{(1 - P_D)\gamma_{k+1|k}}{(1 - P_D)\gamma_{k+1|k} + (1 - P_{FA})(1 - \gamma_{k+1|k})}$$

Task 2: *KF initialization of CV model without a prior*

The KF typically uses a prior for initializing the filter. However, in target tracking we often have no specific prior and would like to infer the initialization of the filter from the data. For the CV model (see chapter 4) with positional measurements, the position is observable with a single measurement, while the velocity needs two measurements to be observable (observable is here used in a statistical sense to mean that there is information about the state from the measurements).

With $x_k = \begin{bmatrix} p_k^T & u_k^T \end{bmatrix}^T$, where p_k is the position and u_k is the velocity at time step k, you should recognize the CV model as

$$x_{k+1} = \begin{bmatrix} p_{k+1} \\ u_{k+1} \end{bmatrix} = F x_k + v_k,$$

with $v_k \sim \mathcal{N}(0, Q)$ and F and Q as defined in (4.64) in the book. The measurement model is given by $z_k = \begin{bmatrix} I_2 & 0_2 \end{bmatrix} x_k + w_k = p_k + w_k$ and $w_k \sim \mathcal{N}(0, R) = \mathcal{N}(0, \sigma_r^2 I_2)$.

Since the KF is linear, we would like to use a linear initialization scheme that uses two measurements and the model parameters. That is

$$\hat{x}_1 = \begin{bmatrix} \hat{p}_1 \\ \hat{u}_1 \end{bmatrix} = \begin{bmatrix} K_{p_1} & K_{p_0} \\ K_{u_1} & K_{u_0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix}. \tag{1}$$

(a) Write z_1 and z_0 as a function of the noises, true position and speed, p_1 and u_1, using the CV model with positional measurements. Use v_k to denote the process disturbance and w_k to denote the measurement noise at time step k, and T for the sampling time between $k-1$ and k.

> *Hint:* A discrete time transition matrix is always invertible, and it is easy to find for the CV model: remove the process noise and write out the transition as a system of linear equations, solve the system for the inverse and rewrite it as a matrix equation again.

$$\begin{bmatrix} p_{x_{k+1}} \\ p_{y_{k+1}} \\ u_{x_{k+1}} \\ u_{y_{k+1}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \vdots & \ddots & & \vdots \\ a_{41} & \cdots & & a_{44} \end{bmatrix}$$

General eqn of motion,

$$S = S_0 + ut + \frac{1}{2} at^2 \qquad v = u + at$$

if $a = 0$, $S = S_0 + ut$, $v = u$

$$\begin{bmatrix} p_{x_{k+1}} \\ p_{y_{k+1}} \\ u_{x_{k+1}} \\ u_{y_{k+1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{x_k} \\ p_{y_k} \\ u_{x_k} \\ u_{y_k} \end{bmatrix}$$

$$p_{x_1} = p_{x_0} + T u_{x_0} \Rightarrow p_{x_0} = p_{x_1} - T u_{x_0}$$
$$p_{y_1} = p_{y_0} + T u_{y_0} \Rightarrow p_{y_0} = p_{y_1} - T u_{y_0}$$

$$u_{x_1} = u_{x_0}$$
$$u_{y_1} = u_{y_0}$$

$$F^{-1} = \begin{bmatrix} 1 & 0 & -T & 0 \\ 0 & 1 & 0 & -T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad x_0 = F^{-1} x_1$$

$$z_0 = H x_0 + \omega_0 = H F^{-1} x_1 + \omega_0$$
$$z_1 = H x_1 + \omega_0$$

$$z_0 = \begin{bmatrix} I_2 & 0_2 \end{bmatrix} \begin{bmatrix} I_2 & -T I_2 \\ 0_2 & I_2 \end{bmatrix} \begin{bmatrix} p_1 \\ u_1 \end{bmatrix} + \omega_0 = p_1 - T u_1 + \omega_0$$

$$z_1 = \begin{bmatrix} I_2 & 0_2 \end{bmatrix} \begin{bmatrix} p_1 \\ u_1 \end{bmatrix} + \omega_1 = p_1 + \omega_1$$

$$\therefore \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} p_1 - T u_1 + \omega_0 \\ p_1 + \omega_1 \end{bmatrix}$$

(b) Show that to get an unbiased initial estimate, the initialization gain matrix must satisfy $K_{p_1} = I_2$, $K_{p_0} = 0_2$, $K_{u_1} = \frac{1}{T} I_2$ and $K_{u_0} = -\frac{1}{T} I_2$, where T is the sampling time. That is, find the $K_\times$ so that $E[\hat{x}_1] = x_1 = \begin{bmatrix} p_1 & u_1 \end{bmatrix}^T$

Note: To find estimator biases, one fixes the values to be estimated and do not treat them as random variables.

$$\hat{x}_1 = \begin{bmatrix} \hat{p}_1 \\ \hat{u}_1 \end{bmatrix} = \begin{bmatrix} K_{p_1} & K_{p_0} \\ K_{u_1} & K_{u_0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{p}_1 \\ \hat{u}_1 \end{bmatrix} = \begin{bmatrix} z_1 K_{p_1} + z_0 K_{p_0} \\ z_1 K_{u_1} + z_0 K_{u_0} \end{bmatrix}$$

$$= \begin{bmatrix} (p_1 + w_1) K_{p_1} + (p_1 - T u_1 + w_0) K_{p_0} \\ (p_1 + w_1) K_{u_1} + (p_1 - T u_1 + w_0) K_{u_0} \end{bmatrix}$$

$$= \begin{bmatrix} p_1 (K_{p_1} + K_{p_0}) + w_1 K_{p_1} + w_0 K_{p_0} - T u_1 K_{p_0} \\ p_1 (K_{u_1} + K_{u_0}) + w_1 K_{u_1} + w_0 K_{u_0} - T u_1 K_{u_0} \end{bmatrix}$$

putting $K_{p_1} = I_2$, $K_{p_0} = 0_2$; $K_{u_1} = \frac{1}{T} I_2$, $K_{u_0} = -\frac{1}{T} I_2$ in R.H.S

$$p_1 (I_2 + 0_2) + w_1 I_2 + w_0 0_2 - T u_1 0_2 = P_1 + w_1$$

$$\frac{p_1}{T} (I_2 - I_2) + \frac{w_1}{T} - \frac{w_0}{T} + T u_1 \frac{I_2}{T} = u_1 + \frac{1}{T} (w_1 - w_0)$$

Since $w_K = \mathcal{N}(0, R_K)$
$v_K = \mathcal{N}(0, \theta_u)$

Expectation of RHS $\Rightarrow$

$$E[p_1 + w_1] = p_1 \quad ; \quad E\left[u_1 + \frac{1}{T}(w_1 - w_0)\right] = u_1$$

$\therefore$ we get the expectation of $\begin{bmatrix} \hat{p} \\ \hat{u} \end{bmatrix}$ as $\begin{bmatrix} p \\ u \end{bmatrix}$ after putting the values of given const.

(c) What is the covariance of this estimate?

we know, $\hat{x} = Kz$

from linearity, $E[\hat{x}] = K E[z]$

$$\mathrm{Cov}(\hat{x}) = K\,\mathrm{Cov}(z)\,K^T$$

$$z = \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} = \begin{bmatrix} P_1 + \omega_1 \\ P_1 - Tu_1 + \omega_0 \end{bmatrix} \qquad \begin{matrix} \omega_1 \sim \mathcal{N}(0, R_1) \\ \omega_0 \sim \mathcal{N}(0, R_0) \end{matrix}$$

$$\mathrm{Cov}(z) = \begin{bmatrix} R_1 & O_2 \\ O_2 & R_0 \end{bmatrix}$$

$$\mathrm{Cov}(\hat{x}) = \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T} I_2 & -\frac{1}{T} I_2 \end{bmatrix} \begin{bmatrix} R_1 & O_2 \\ O_2 & R_0 \end{bmatrix} \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T} I_2 & -\frac{1}{T} I_2 \end{bmatrix}^T$$

$$= \begin{bmatrix} R_1 & O_2 \\ \frac{R_1}{T} & -\frac{R_0}{T} \end{bmatrix} \begin{bmatrix} I_2 & \frac{1}{T} I_2 \\ O_2 & -\frac{1}{T} I_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 & R_1/T \\ R_1/T & (R_1 + R_0)/T^2 \end{bmatrix}$$

taking $R_1 = R_0 = R$

$$\mathrm{Cov}(\hat{x}) = \frac{1}{T^2} \begin{bmatrix} T^2 R & RT \\ RT & 2R \end{bmatrix}$$

(d) You have used this initialization scheme for your estimator and found a mean and covariance. What distribution does the true state have after this initialization? What are its parameters.

Hint: From equations you have already used, you can write x_1 in terms of $\hat{x}$ and disturbances and noises. You should be able to see the result as a linear transformation of some random variables. Note that $\hat{x}$ is given since the measurements are given and thus can be treated as a constant. x_1 is now treated as a random variable as opposed to when finding the mean and variance of the estimator.

we know that

$$x_{k+1} = \begin{bmatrix} p_{k+1} \\ u_{k+1} \end{bmatrix} = F x_k + v_k$$

$$x_1 = F\hat{x} + v_1 \qquad v_1 \sim N(0, Q)$$

$$E[x_1] = F \cdot E[\hat{x}] = \begin{bmatrix} I_2 & TI_2 \\ O_2 & I_2 \end{bmatrix} \begin{bmatrix} \hat{p} \\ \hat{u} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{p} + \hat{u}T \\ \hat{u} \end{bmatrix} = \mu_{x_1}$$

$$Cov(x_1) = F \cdot \frac{1}{T^2} \begin{bmatrix} T^2R & RT \\ RT & 2R \end{bmatrix} F^T \cdot Q$$

$$= \frac{1}{T^2} \begin{bmatrix} I_2 & TI_2 \\ O_2 & I_2 \end{bmatrix} \begin{bmatrix} T^2R & RT \\ RT & 2R \end{bmatrix} \begin{bmatrix} I_2 & TI_2 \\ O_2 & I_2 \end{bmatrix}^T + Q$$

$$= \begin{bmatrix} 2R & 3R/T \\ R/T & 2R/T^2 \end{bmatrix} \begin{bmatrix} I_2 & O_2 \\ TI_2 & I_2 \end{bmatrix} + Q$$

$$= \begin{bmatrix} 5R & 3R/T \\ 3R/T & 2R/T^2 \end{bmatrix} + Q = P_{x_1}$$

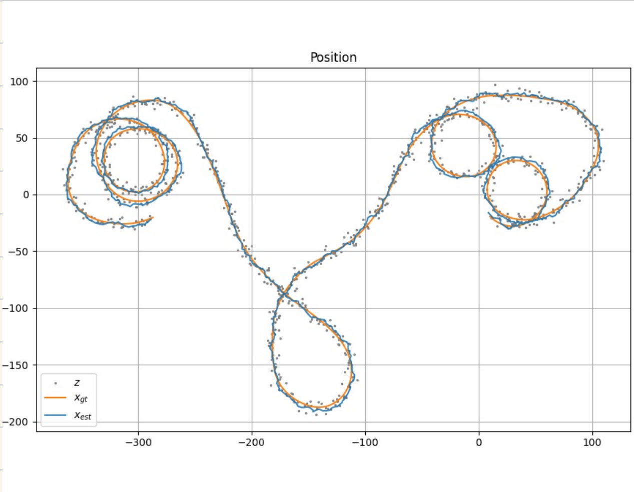
$\therefore$ The true state is in normal distribution.

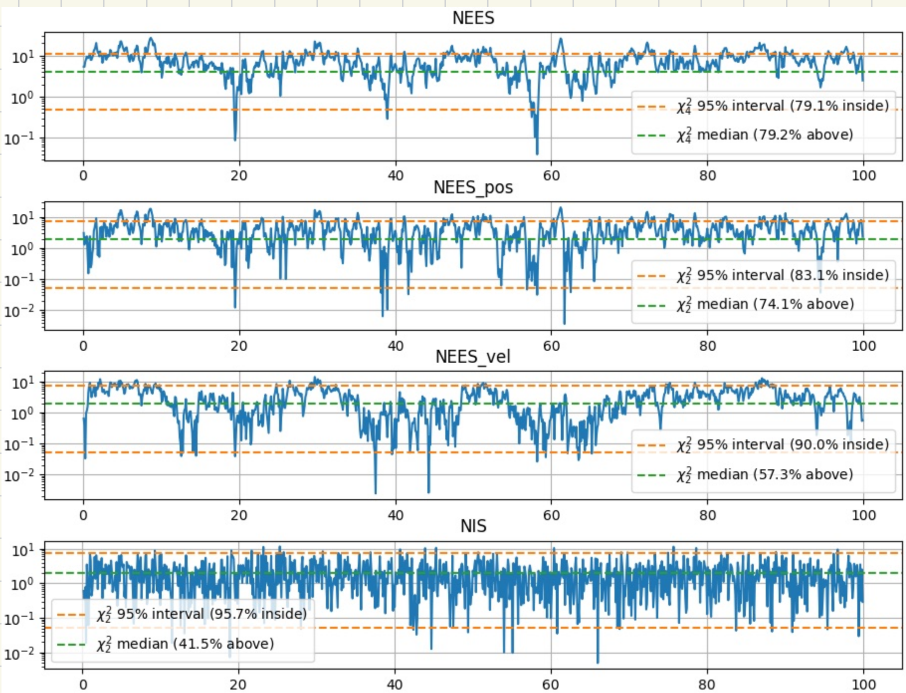
$$x_1 \sim N(\mu_{x_1}, P_{x_1})$$

Task 5: *Analyse the outputs of the KF*

Finish the implementation of `analysis.get_nis`

(b) Finish the implementation of `analysis.get_nees`





(d) For each of the five requirements under (4.6.1) in the book, give a brief explanation of why the requirement is important and how one can check if a filter satisfies it.

The 5 requirements are:

i) The state errors should be accepted as zero mean so that the filter doesn't over/under-estimate the state.

ii) The state error should have magnitude commensurate with the state covariance yielded by the filter to ensure the filter's prediction is realistic.

iii) The innovations should be acceptable as zero mean to ensure the filter is not biased in its state updates.

iv) The innovation error should have magnitude commensurate with the innovation covariance yielded by the filter to ensure that the filter correctly predicts the uncertainity in its measurement updates

v) The innovations should be acceptable as white to ensure there is no pattern or bias in the measurement errors over time.

For criteria (ii) and (iv), we use NEES and NIS and check whether the average NEES or NIS follows a χ^2 distribution.