TTK4250 Sensor Fusion
Assignment 2

Hand in: *Friday 13. September* on Blackboard.

Read Python setup guide on BB, it can be found under *Course work*.
This assignment should be handed in on Blackboard, as a PDF pluss **the zip created by running *create_handin.py***, before the deadline.
You are supposed to show how you got to each answer unless told otherwise.
If you struggle, we encourage you to ask for help from a classmate or come to the exercise class on Friday.

Task 1: *Transformation of Gaussian random variables*

Let $x \in \mathbb{R}^n$ be $\mathcal{N}(\mu, \Sigma)$. Find the distribution and see if you recognize it:

Hint: they are all given in the book.

(a) $z = \Sigma^{-\frac{1}{2}}(x - \mu)$, where $\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^T = \Sigma$

> *Hint:* If you are using theorem 2.4.1, you might need $\det(A^{\frac{1}{2}}) = \det(A)^{\frac{1}{2}}$, $(A^{-1})^T = (A^T)^{-1}$, and $\det(A^T) = \det(A)$ whenever A has full rank.

(b) Use transformation of random variables to find $y_i = z_i^2$, where z_i is the i'th variable in the vector z.

(c) $y = (x - \mu)^T \Sigma^{-1}(x - \mu) = z^T z = \sum z_i^2 = \sum y_i$.

> *Hint:* The MGF of y_i is given in the book through example 2.8 and 2.10. Example 2.6 might also be handy.

Task 2: *Sensor fusion*

In this task we want to find out if a boat is above the line $x_2 = x_1 + 5$. In order to do this we will fuse measurements from two sensors with our prior belief: A drone-mounted camera, and a maritime surveillance radar. You have some prior knowledge of the state of the boat. You get 1 measurement from each sensor that are processed so that you know them to be (approximately) Gaussian conditioned on the position.

To be more specific, let us denote the state by x and our prior Gaussian by $\mathcal{N}(x; \bar{x}, P)$. The measurement from the camera is given by $z^c = H^c x + v^c$ and the measurement from the radar by $z^r = H^r x + v^r$, where v^c, v^r denotes the measurement noise and is distributed according to $\mathcal{N}(0, R^c)$ and $\mathcal{N}(0, R^r)$, respectively.

Only insert the numbers when asked to. The needed values are given by

$$\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \qquad P = 25 I_2, \qquad H^c = H^r = I_2,$$
$$R^c = \begin{bmatrix} 79 & 36 \\ 36 & 36 \end{bmatrix}, \qquad R^r = \begin{bmatrix} 28 & 4 \\ 4 & 22 \end{bmatrix}, \qquad z^c = \begin{bmatrix} 2 & 14 \end{bmatrix}^T, \qquad z^r = \begin{bmatrix} -4 & 6 \end{bmatrix}^T$$

(a) What is $p(z^c|x)$?

(b) Show that the joint $p(x, z^c)$ can be written as a Gaussian distribution.

> *Hint:* Use conditional probability and the proof of theorem 3.3.1.

(c) Find the marginal $p(z^c)$ and the conditional $p(x|z^c)$, using the above and either theorems from the book or calculations.

(d) Given what you found above, what is the marginal $p(z^r)$ and the conditional $p(x|z^r)$?

(e) What is the MMSE and MAP estimate of x given z^c? You do not need to do calculations to find the answer, but briefly state what you would do if you had to.

(f) Finish the `sensor_model.LinearSensorModel2d.get_pred_meas` method that can be used to calculate marginal probabilities $p(z)$.

(g) Finish the `conditioning.get_cond_state` function that can be used to calculate conditional probabilities $p(x|z)$.

(h) Finish the `task2.get_conds` function that is used to calculate the conditional probabilities $p(x|z^c)$ and $p(x|z^r)$.

(i) Finish the `task2.get_double_conds` function that is used to calculate the conditional probabilities $p(x|z^c, z^r)$, i.e. the posterior of x conditioned on z^c then z^r, and $p(x|z^r, z^c)$, i.e. the posterior of x conditioned on z^r then z^c. Does it matter which order we condition?

(j) Finish the `gaussian.MultiVarGauss2d.get_transformed` method that is used to calculate the probability $p(Tx)$, where T is a linear transformation.

(k) You now want to know the probability that the boat is above the line, $x_2 = x_1 + 5$.
Finish `task2.get_prob_over_line` using the appropriate linear transform and the CDF.

> *Hint:* This is the same as finding $\Pr(x_2 - x_1 > 5) = \Pr(\begin{bmatrix} -1 & 1 \end{bmatrix} x > 5)$
>
> To get the cdf you can use `from scipy.stats import norm`, and then use `norm.cdf(value, mean, std)`. Note that it takes the standard deviation (std) and not the variance as input.

Task 3: *Working with the canonical form*

In Section 3.3 the fundamental product identity was studied using a moment-based parametrization. Clearly, it must also be possible to establish an equivalent result using the canonical representation. In this exercise we shall therefore consider the product

$$\mathcal{N}^{-1}(\mathbf{x}\,;\,\mathbf{a},\mathbf{B})\mathcal{N}^{-1}(\mathbf{y}\,;\,\mathbf{C}\mathbf{x},\mathbf{D}). \tag{6}$$

(a) Show that (6) is identical to the Gaussian

$$\mathcal{N}^{-1}\left(\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix};\begin{bmatrix}\mathbf{a}\\ \mathbf{0}\end{bmatrix},\begin{bmatrix}\mathbf{B}+\mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C} & -\mathbf{C}^\mathsf{T}\\ -\mathbf{C} & \mathbf{D}\end{bmatrix}\right) \tag{7}$$

> *Hint:*
>
> Taking the logarithm of the form (3.17) in the book with (3.20) inserted give a relatively simple way to the goal, after the terms constant in $\mathbf{x}$ and $\mathbf{y}$ are subtracted. Also
>
> $$\mathbf{a}^\mathsf{T}\mathbf{A}\mathbf{a}+\mathbf{b}^\mathsf{T}\mathbf{B}\mathbf{b}+2\mathbf{a}^\mathsf{T}\mathbf{C}\mathbf{b}=\mathbf{a}^\mathsf{T}\mathbf{A}\mathbf{a}+\mathbf{b}^\mathsf{T}\mathbf{B}\mathbf{b}+\mathbf{a}^\mathsf{T}\mathbf{C}\mathbf{b}+\mathbf{b}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{a}=\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}^\mathsf{T}\begin{bmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{C}^\mathsf{T} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix},$$
>
> is handy, and valid for any vectors $\mathbf{a}$ and $\mathbf{b}$ and matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of interest (variable names are not related to the task).

(b) Show that the marginal distribution of $\mathbf{y}$, from the joint density (7), is

$$\mathcal{N}^{-1}\left(\mathbf{y}\,;\,\mathbf{C}^\mathsf{T}(\mathbf{B}+\mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{a},\mathbf{D}-\mathbf{C}(\mathbf{B}+\mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{C}^\mathsf{T}\right). \tag{8}$$

> *Hint:* Theorem 3.4.1

(c) Show that the conditional distribution of $\mathbf{x}$ given $\mathbf{y}$ is

$$\mathcal{N}^{-1}\left(\mathbf{x}\,;\,\mathbf{a}+\mathbf{C}^\mathsf{T}\mathbf{y},\mathbf{B}+\mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C}\right). \tag{9}$$

> *Hint:* Theorem 3.4.1

(d) Let us now return to the original formulation of the product identity in Theorem 3.3.1. Use the result from c) to show that

$$\widehat{\mathbf{P}}^{-1}=\mathbf{P}^{-1}+\mathbf{H}^\mathsf{T}\mathbf{R}^{-1}\mathbf{H}. \tag{10}$$

> *Hint:* Match variables in (3.21) with (3.10) in the book.