Kalman Filter

- Bayes filter with **Gaussians**
- Developed in the late 1950's
- Most relevant Bayes filter variant in practice
- Applications range from economics, weather forecasting, satellite navigation to robotics and many more.

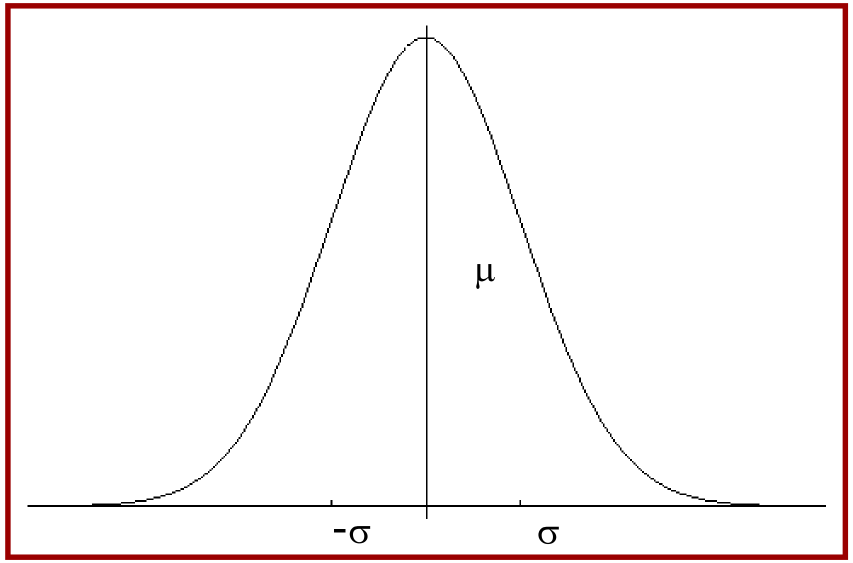
- The Kalman filter “algorithm” is “just” a couple of **matrix multiplications**!

Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

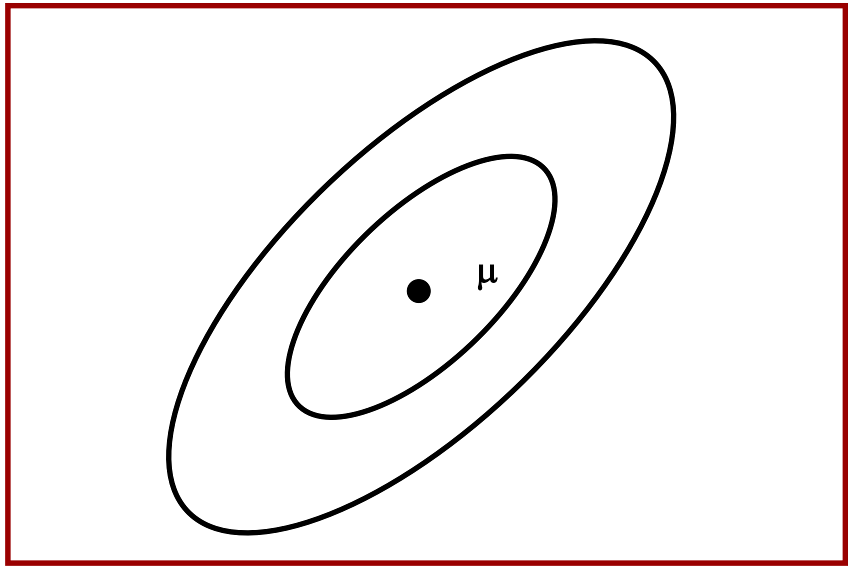
Univariate



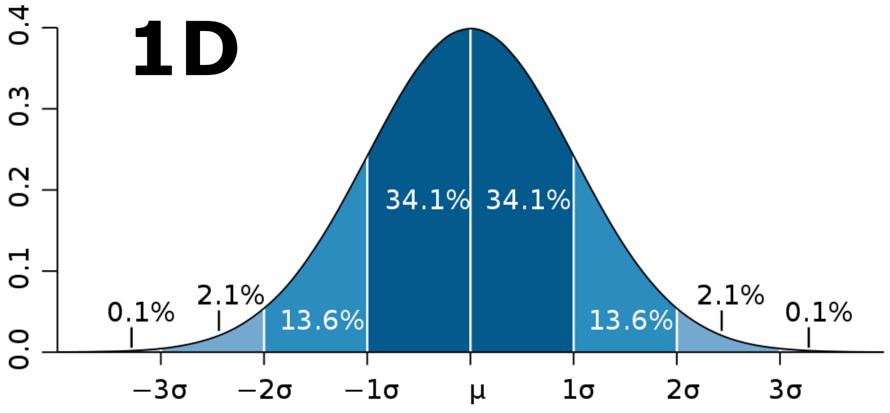
$$p(\mathbf{x}) \sim N(\mu, \Sigma):$$

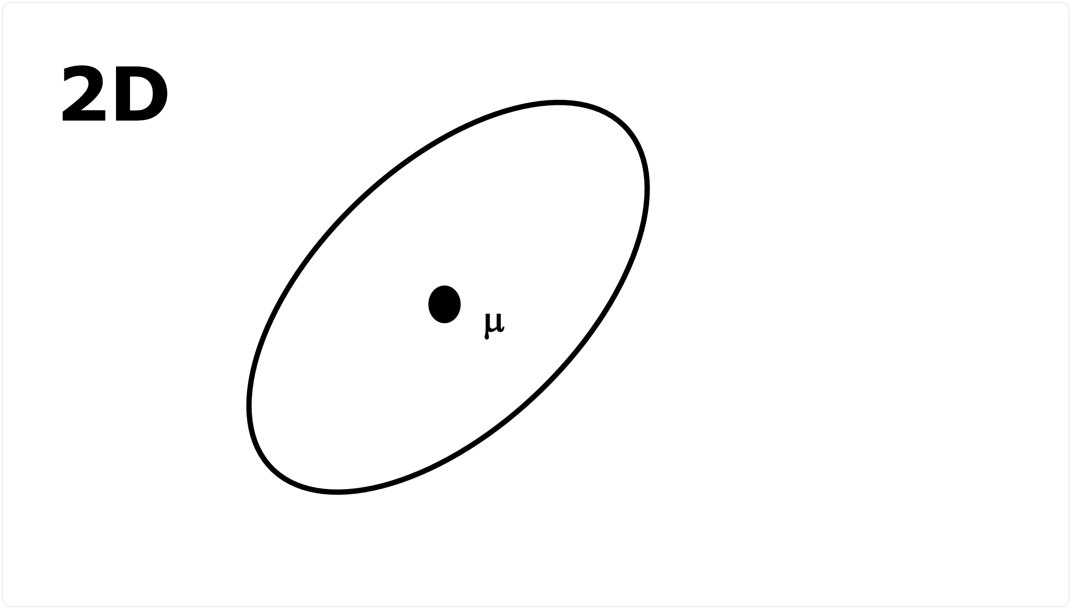
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^t \Sigma^{-1} (\mathbf{x}-\mu)}$$

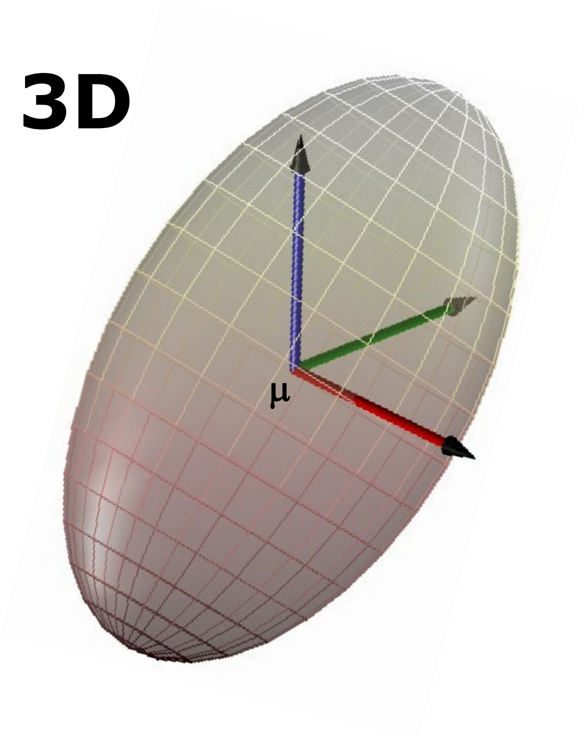
Multivariate



Gaussians


1D
0.0
0.1
0.2
0.3
0.4
0.1%
2.1%
13.6%
34.1%
34.1%
13.6%
2.1%
0.1%
−3σ
−2σ
−1σ
μ
1σ
2σ
3σ


2D
μ


3D
μ

Properties of Gaussians

- Univariate case

$$\left.\begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y = aX + b \end{array}\right\} \Rightarrow \quad Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^{\ 2}) \\ X_2 \sim N(\mu_2, \sigma_2^{\ 2}) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2^{\ 2}}{\sigma_1^{\ 2} + \sigma_2^{\ 2}}\mu_1 + \frac{\sigma_1^{\ 2}}{\sigma_1^{\ 2} + \sigma_2^{\ 2}}\mu_2, \quad \frac{1}{\sigma_1^{\ -2} + \sigma_2^{\ -2}}\right)$$

Properties of Gaussians

- Multivariate case

$$\left.\begin{array}{l} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2}\mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2}\mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}}\right)$$

(where division (fractions) correspond to matrix inversion)

- We **stay Gaussian** as long as we start with Gaussians and perform only **linear transformations**

Discrete Kalman Filter

Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t}$$

with a measurement

$$\boxed{z_t = C_t x_t + \delta_t}$$

Components of a Kalman Filter

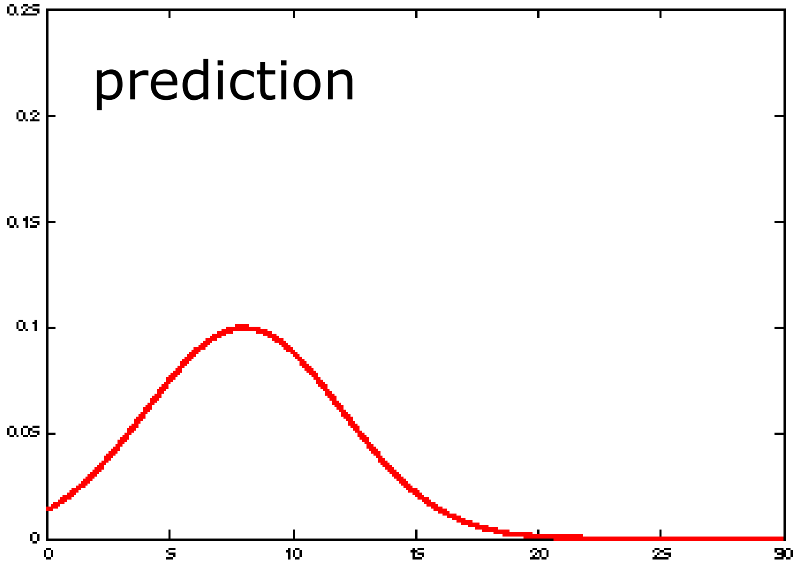
$\boxed{A_t}$ Matrix $(n \times n)$ that describes how the state evolves from t-1 to t without controls or noise.

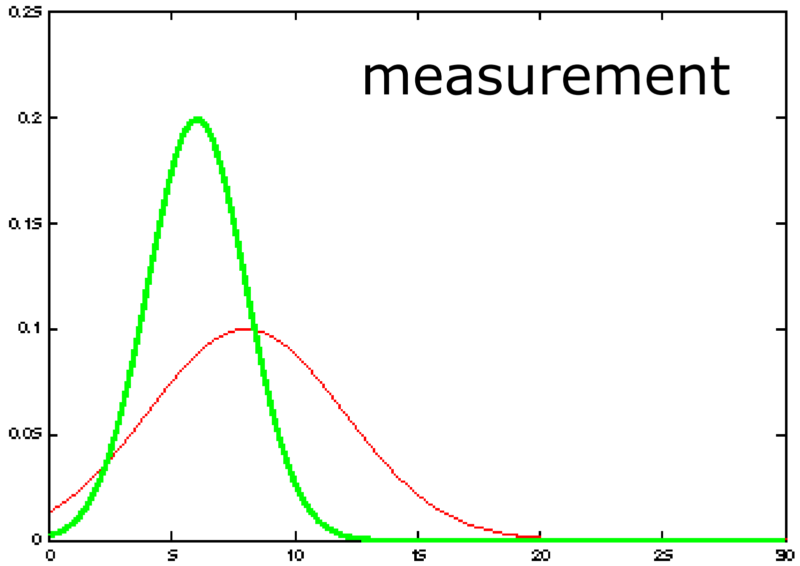
$\boxed{B_t}$ Matrix $(n \times l)$ that describes how the control u_t changes the state from t-1 to t.

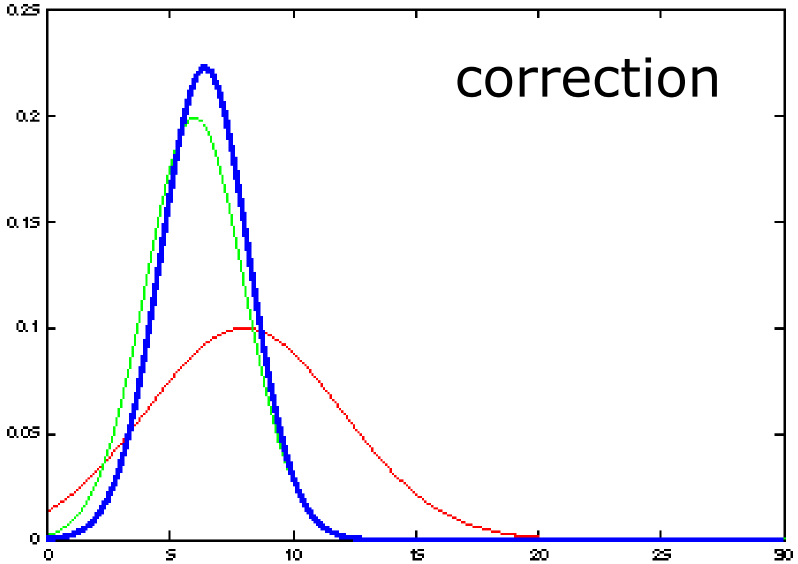
$\boxed{C_t}$ Matrix $(k \times n)$ that describes how to map the state x_t to an observation z_t.

$\boxed{\varepsilon_t}$ $\boxed{\delta_t}$ Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance Q_t and R_t respectively.

Kalman Filter Updates in 1D

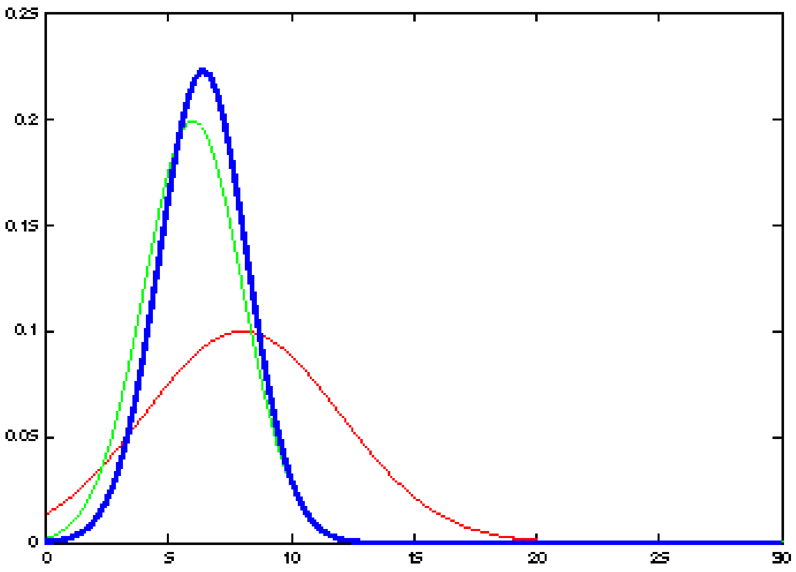






It's a weighted mean!

Kalman Filter Updates in 1D



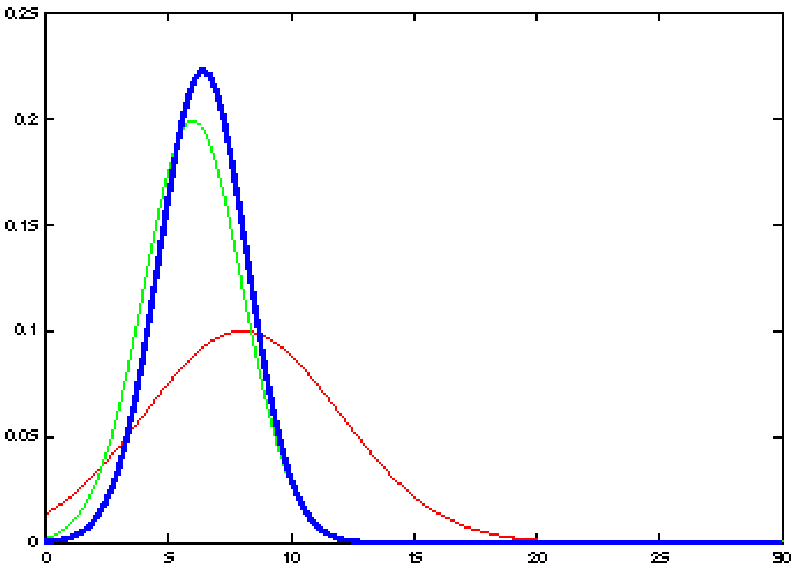
How to get the blue one?

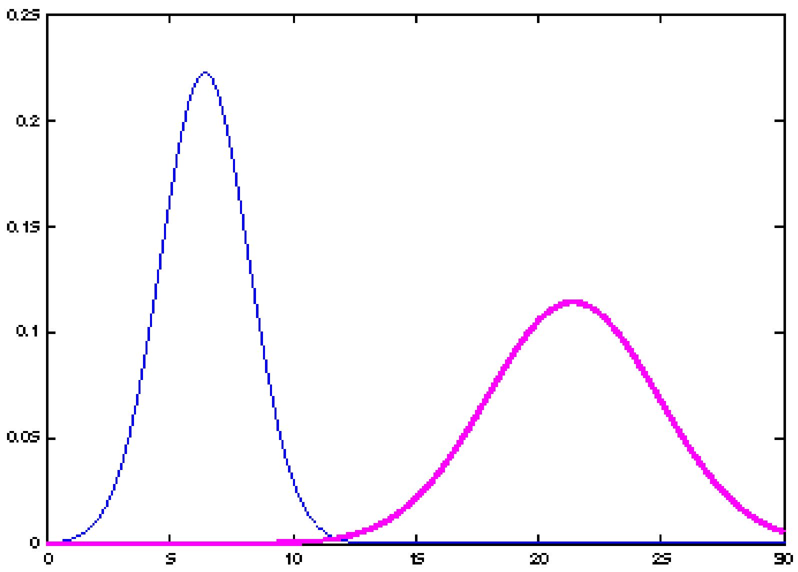
Kalman correction step

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Kalman Filter Updates in 1D





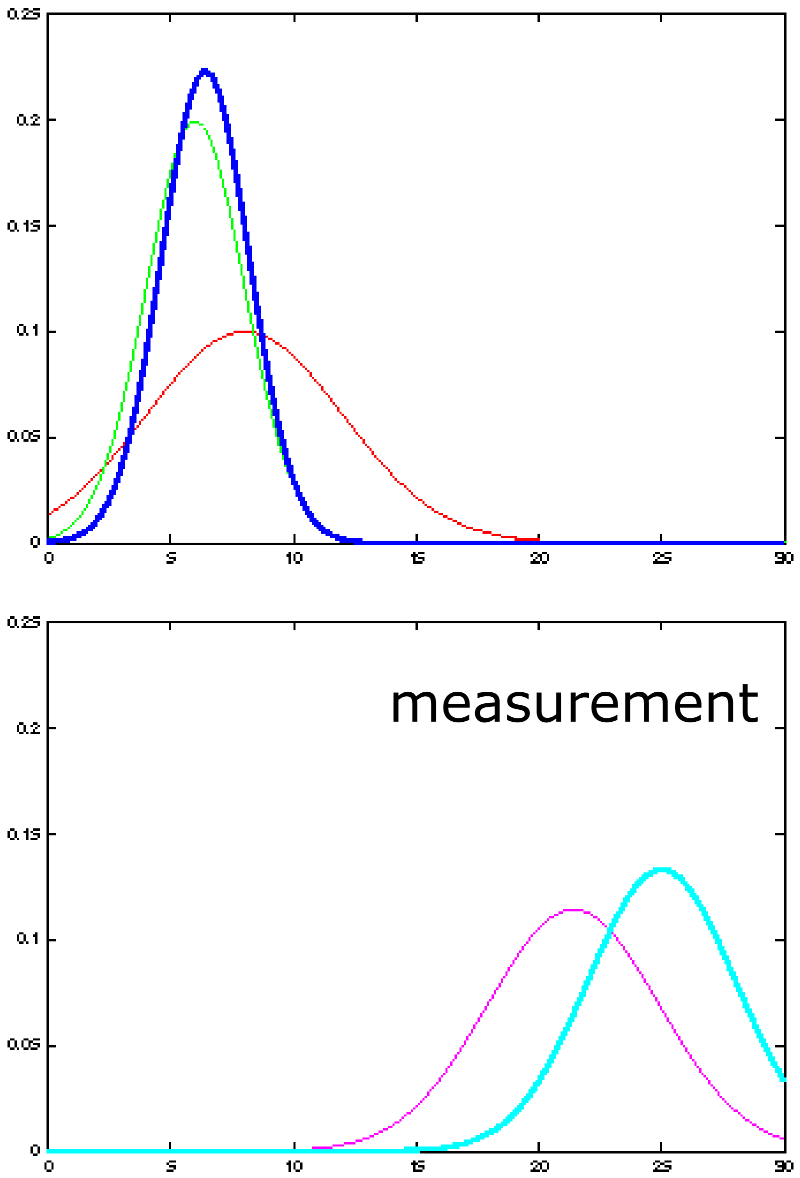
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

How to get the magenta one?

State prediction step

Kalman Filter Updates



Linear Gaussian Systems: Initialization

Initial belief is normally distributed:

$$\boxed{bel(x_0) = N(x_0; \mu_0, \Sigma_0)}$$

Linear Gaussian Systems: Dynamics

Dynamics are linear functions of the state and the control plus additive noise:

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t}$$

$$\boxed{p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right)}$$

$$\boxed{\begin{array}{ccc} \overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) & & bel(x_{t-1})\, dx_{t-1} \\ \Downarrow & & \Downarrow \\ \sim N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right) & & \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right) \end{array}}$$

Linear Gaussian Systems: Dynamics

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1})}_{\sim N(x_t; A_t x_{t-1} + B_t u_t, Q_t)} \quad \underbrace{bel(x_{t-1})}_{\sim N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1})} \, dx_{t-1}$$

$$\Downarrow$$

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T Q_t^{-1}(x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

Linear Gaussian Systems: Observations

Observations are a linear function of the state plus additive noise:

$$\boxed{z_t = C_t x_t + \delta_t}$$

$$\boxed{p(z_t \mid x_t) = N(z_t; C_t x_t, R_t)}$$

$$\boxed{\begin{array}{cccc} bel(x_t) = & \eta & p(z_t \mid x_t) & \overline{bel}(x_t) \\ & & \Downarrow & \Downarrow \\ & & \sim N(z_t; C_t x_t, R_t) & \sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t) \end{array}}$$

Linear Gaussian Systems: Observations

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad\qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\sim N(z_t; C_t x_t, R_t) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T R_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \qquad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

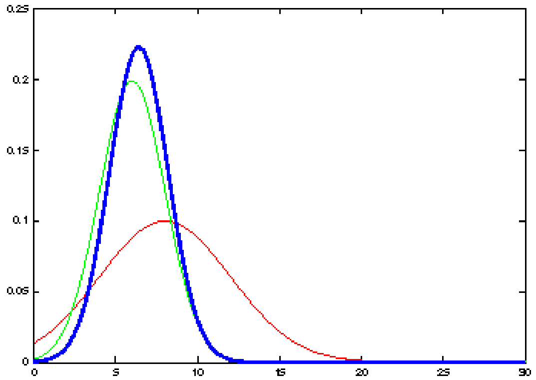
2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

The Prediction-Correction-Cycle

Prediction



$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

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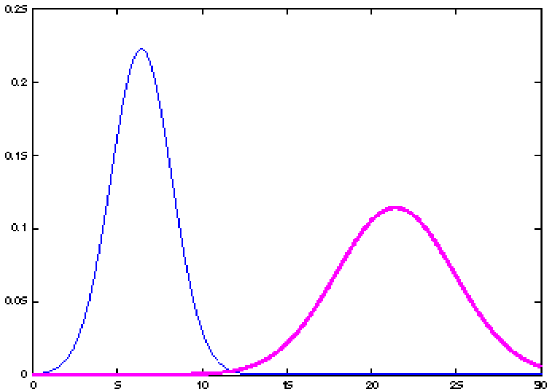
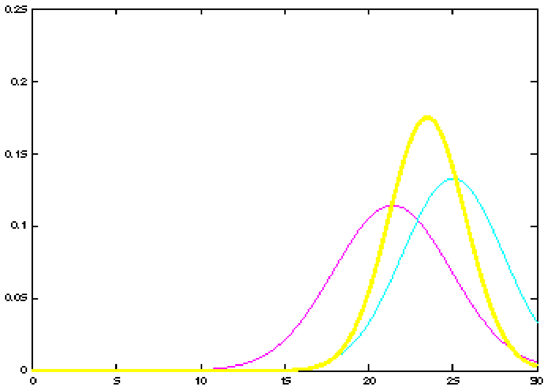
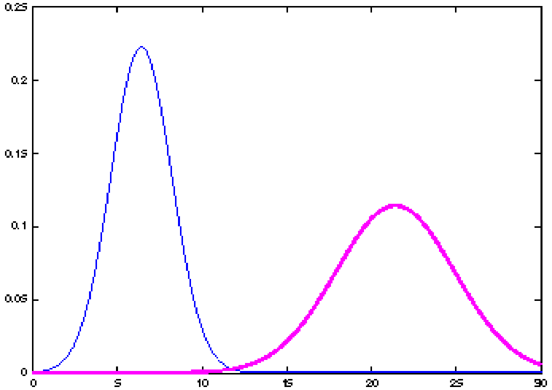


The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, \; K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$



Correction

The Prediction-Correction-Cycle

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

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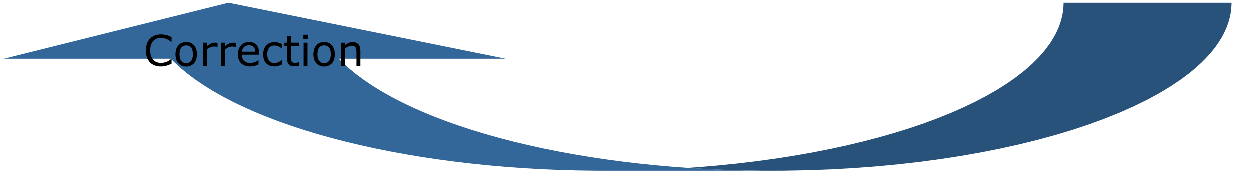

Kalman Filter Summary

- Only two parameters describe belief about the state of the system
- **Highly efficient:** Polynomial in the measurement dimensionality k and state dimensionality n:

$$O(k^{2.376} + n^2)$$

- **Optimal for linear Gaussian systems**
- However: Most robotics systems are **nonlinear**
- Can only model unimodal beliefs