Classical Logic & Fuzzy Logic

Classical Predicate Logic

- Classical logic deals with bivalued logic (TRUE or FALSE).

- Fuzzy logic deals with multivalued logic (partial truth or approximate reasoning).

- A simple proposition *P* is a linguistic statement contained within a universe of elements (*X*).
    - ➢ Hence P can be identified as being a collection of elements (set) in *X* that are either strictly TRUE or strictly FALSE.

Classical Predicate Logic

- The truth values of an element in the proposition P, are either all TRUE or all FALSE.
- This binary truth value $T(P)$, has a value of 1 (truth) or 0 (falsity), i.e.,

$$T : u \in U \rightarrow [0,1]$$

- All elements u in universe U, that are true for proposition P is called the truth set of P, $T(P)$ and those that are false form the falsity set of P, $F(P)$.
  - ➢The boundary condition of the truth values are: $T(U)=1;\ T(\varphi)=0$

Classical Logical Connectives

- Let P and Q be two simple Propositions, connected by logical connectives to form logical expressions.

- Five main connectives are :
  - ✓Disjunction ($\vee$) [Logical *OR*/Inclusive *OR*]
  - ✓Conjunction ($\wedge$) [Logical *AND*]
  - ✓Negation (–) [Logical *NOT*]
  - ✓Implication ($\rightarrow$) [hypothesis $\rightarrow$ conclusion]
  - ✓Equivalence ($\leftrightarrow$) [*IF* $P \rightarrow Q$ *AND* $Q \rightarrow P$, *THEN* $P \leftrightarrow Q$].
- These connectives can be used to from new propositions.

Classical Logical Connectives

- For a set A, if $x \in A$ generates a proposition P which is TRUE and $x \in B$ generates a proposition Q which is TRUE, then

$$P: \quad truth \quad that \quad x \in A; \quad Q: \quad truth \quad that \quad x \in B$$

- If truth is measured in terms of truth value, then

$$If \quad x \in A, T(P) = 1; \quad otherwise, \quad T(P) = 0$$
$$If \quad x \in B, T(Q) = 1; \quad otherwise, \quad T(Q) = 0$$

- Using the characteristic function to represent truth (1) and falsity (0), the following notation results.

$$\chi_{\mathrm{A}}(x) = \begin{cases} 1, & x \in \mathrm{A} \\ 0, & x \notin \mathrm{A} \end{cases}$$

Classical Logical Connectives

- For a proposition, $P : x \in A, \overline{P} : x \notin A$ the following classical logical connectives result.
  - ✓Disjunction

$$P \vee Q : \quad x \in A \quad or \quad x \in B \quad Hence, \quad T(P \vee Q) = \max(T(P), T(Q))$$

  - ✓Conjunction

$$P \wedge Q : \quad x \in A \quad or \quad x \in B \quad Hence, \quad T(P \wedge Q) = \min(T(P), T(Q))$$

  - ✓Negation

$$If \quad T(P) = 1, \quad then \quad T(\overline{P}) = 0; \quad if \quad T(P) = 0, \quad then \quad T(\overline{P}) = 1$$

  - ✓Implication

$$(P \to Q) : x \notin A \quad or \quad x \in B \quad Hence, \quad T(P \to Q) = T(\overline{P} \cup Q)$$

Classical Logical Connectives

✓Equivalence

$$(P \leftrightarrow Q) : T(P \leftrightarrow Q) = \begin{cases} 1, & for \quad T(P) = T(Q) \\ 0, & for \quad T(P) \neq T(Q) \end{cases}$$

- IF P is a proposition on set A in universe X and Q is a proposition on set B in universe Y, then $P \rightarrow Q$ can be represented by a relation R,

$$R = (A \times B) \cup (\overline{A} \times Y) \equiv If \quad A, \quad THEN \quad B$$

$$i.e., \quad \begin{array}{cccccc} IF & x \in A & where & x \in X & and & A \subset X \\ THEN & y \in B & where & y \in Y & and & B \subset Y \end{array}$$

Compound Proposition

- A compound proposition in linguistic rule form can also be expressed in terms of predicate logic, e.g.,

$$IF \quad A, \quad THEN \quad B, \quad ELSE \quad C \Rightarrow IF \quad A, \quad THEN \quad B, \quad or \quad IF \quad \overline{A}, \quad THEN \quad C$$

- In predicate logic, this rule has the form,

$$(P \to Q) \vee (\overline{P} \to S) \quad where \quad \begin{array}{l} P : x \in A, A \subset X \\ Q : y \in B, B \subset Y \\ S : y \in C, C \subset Y \end{array}$$

- In set-theoretic form,

$$(A \times B) \cup (\overline{A} \times C) = R = relation \quad on \quad X \times Y$$

Tautologies

- Classical logical compound propositions that are always true irrespective of the veracity of the individual simple propositions are referred to as ***tautologies***.
  - ✓ ***Modus Ponens*** deduction concludes that, given two propositions, P and $P \rightarrow Q$, both of which are true, then the truth of the simple proposition, Q is automatically inferred.
  - ✓ In ***modus tollens***, an implication between two propositions is combined with a second proposition and both are used to imply a third proposition.

Some common Tautologies

$$\overline{B} \cup B \leftrightarrow X$$

$$A \cup X; \quad \overline{A} \cup X \leftrightarrow X$$

$$A \leftrightarrow B$$

$$(A \wedge (A \rightarrow B)) \rightarrow B \quad (\text{mod}us \quad ponens)$$

$$(\overline{B} \wedge (A \rightarrow B)) \rightarrow \overline{A} \quad (\text{mod}us \quad tollens)$$

Proof of Modus Ponens

$$(A \wedge (A \to B)) \to B$$
$$(A \wedge (\overline{A} \cup B)) \to B \quad \text{Implication}$$
$$((A \wedge \overline{A}) \cup (A \wedge B)) \to B \quad Distributivity$$
$$(\phi \cup (A \wedge B)) \to B \quad Excluded \quad middle \quad laws$$
$$(A \wedge B) \to B \quad Identity$$
$$(\overline{A \wedge B}) \cup B \quad \text{Implication}$$
$$(\overline{A} \vee \overline{B}) \cup B \quad De \quad Morgan's \quad laws$$
$$\overline{A} \vee (\overline{B} \cup B) \to B \quad Associativity$$
$$\overline{A} \cup X \quad Excluded \quad middle \quad laws$$
$$X \Rightarrow T(X) = 1 \quad Identity$$

Proof of Modus Tollens

$$(\overline{B}\wedge(A\to B))\to\overline{A}$$

$$(\overline{B}\wedge(\overline{A}\cup B))\to\overline{A}\quad \text{Implication}$$

$$((\overline{B}\wedge\overline{A})\cup(\overline{B}\wedge B))\to\overline{A}\quad Distributivity$$

$$((\overline{B}\wedge\overline{A})\cup\phi)\to\overline{A}\quad Excluded\quad middle\quad laws$$

$$(\overline{B}\wedge\overline{A})\to\overline{A}\quad Identity$$

$$\overline{(\overline{B}\wedge\overline{A})}\cup\overline{A}\quad \text{Implication}$$

$$(\overline{\overline{B}}\vee\overline{\overline{A}})\cup\overline{A}\quad De\quad Morgan's\quad laws$$

$$B\cup(A\cup\overline{A})\quad Associativity$$

$$B\cup X\quad Excluded\quad middle\quad laws$$

$$X\Rightarrow T(X)=1\quad Identity$$

Contradictions

- Compound propositions that are always false irrespective of the truth values of the individual simple propositions are referred to as *contradictions*.

- Is A is set of all prime numbers, then the proposition "A_i is a multiple of 2" is a contradiction.
  - e.g.,

$$\overline{B} \cap B; A \cap \phi; \overline{A} \cap \phi$$

Equivalence

- P and Q are equivalent i.e.,

  $P \leftrightarrow Q$ is true only when both P and Q are true or when both are false.

- If $P \to Q$ is a relation R, then:
  - ✓ $P \to Q$ is the *inverse* relation of R.
  - ✓ $\overline{Q} \to P$ is called the *contrapositive* of R.
  - ✓ $Q \to P$ is called the *converse* of R.

- Every proposition has a dual proposition.
- For two propositions,
  - P defined on set A in universe X and
  - Q defined on set B in universe Y

  the set theoretic form is given by $(P \to Q) = R = (A \times B) \cup (\overline{A} \times Y)$

  and the function-theoretic form is

$$\chi_R(x, y) = \max[(\chi_A(x) \wedge \chi_B(y)), ((1 - \chi_A(x)) \wedge 1)]$$

Example

- Consider X (universe of temperatures) = {1,2,3,4} and Y (universe of pressures) = {1,2,3,4,5,6}. A $\subseteq$ X={2,3} and B $\subseteq$ Y={3,4} then find the deductive inference *IF A, THEN B.*

  Find A X B

  Find $\overline{A}\times Y$

  If A, Then B is $R=(A\times B)\cup(\overline{A}\times Y)$

- Similarly the Compound Rule IF A, THEN B, ELSE C

  Can be defined as $R=(A\times B)\cup(\overline{A}\times C)$ $\Rightarrow$ $(P\rightarrow Q)\vee(\overline{P}\rightarrow S)$

  OR

$$X_R(x,y)=\max[(X_A(x)\wedge X_B(y)),((1-X_A(x)\wedge X_C(y))]$$

- If C={5,6}, find *IF A, THEN B, ELSE C*

Fuzzy Logic

- Fuzzy logic deals with multivalued logic
- A fuzzy logic proposition $\underline{P}$ is a statement concept without clearly defined boundaries.
- The truth value assigned to $\underline{P}$ can be any value on the interval [0, 1], i.e.,

$$T : u \in U \rightarrow \{0,1\}$$

- Fuzzy propositions are assigned to fuzzy sets.

$$IF \quad \underline{P} : x \in \underline{A} \quad THEN, \quad T(\underline{P}) = \mu_{\underline{A}}(x); \quad 0 \le \mu_{\underline{A}}(x) \le 1$$

Fuzzy Logical Connectives

- For a proposition, $\underset{\sim}{P}: x \in \underset{\sim}{A}, \overline{\underset{\sim}{P}}: x \notin \underset{\sim}{A}$ the following fuzzy logical connectives result.
  - ✓Disjunction

$$\underset{\sim}{P} \vee \underset{\sim}{Q}: \quad x \in \underset{\sim}{A} \quad or \quad x \in \underset{\sim}{B} \quad Hence, \quad T(\underset{\sim}{P} \vee \underset{\sim}{Q}) = \max(T(\underset{\sim}{P}), T(\underset{\sim}{Q}))$$

  - ✓Conjunction

$$\underset{\sim}{P} \wedge \underset{\sim}{Q}: \quad x \in \underset{\sim}{A} \quad or \quad x \in \underset{\sim}{B} \quad Hence, \quad T(\underset{\sim}{P} \wedge \underset{\sim}{Q}) = \min(T(\underset{\sim}{P}), T(\underset{\sim}{Q}))$$

  - ✓Negation

$$T(\overline{\underset{\sim}{P}}) = 1 - T(\underset{\sim}{P})$$

  - ✓Implication

$$(\underset{\sim}{P} \rightarrow \underset{\sim}{Q}): x \in \underset{\sim}{A}, \quad then \quad x \in \underset{\sim}{B} \quad Hence, \quad T(\underset{\sim}{P} \rightarrow \underset{\sim}{Q}) = T(\overline{\underset{\sim}{P}} \vee \underset{\sim}{Q})$$

$$= \max(T(\overline{\underset{\sim}{P}}), T(\underset{\sim}{Q}))$$

Fuzzy Logical Connectives

- A simple or a compound rule form can be represented in terms of fuzzy logical expression, e.g.,
  - ✓Simple rule form

$$IF \quad x \quad is \quad \underline{A}, \quad THEN \quad y \quad is \quad \underline{B}$$

$$\underline{R} = (\underline{A} \times \underline{B}) \cup (\overline{\underline{A}} \times Y)$$

$$\mu_{\underline{R}}(x, y) = \max[(\mu_{\underline{A}}(x) \wedge \mu_{\underline{B}}(y)), (1 - \mu_{\underline{A}}(x))]$$

  - ✓Compound rule form

$$IF \quad x \quad is \quad \underline{A}, \quad THEN \quad y \quad is \quad \underline{B}, ELSE \quad y \quad is \quad \underline{C}$$

$$\underline{R} = (\underline{A} \times \underline{B}) \cup (\overline{\underline{A}} \times \overline{\underline{C}})$$

$$\mu_{\underline{R}}(x, y) = \max[(\mu_{\underline{A}}(x) \wedge \mu_{\underline{B}}(y)), ((1 - \mu_{\underline{A}}(x) \wedge \mu_{\underline{C}}(y))]$$

Approximate Reasoning

- Fuzzy logic is meant for reasoning about imprecise propositions.
- Approximate reasoning is analogous to predicate logic for reasoning with precise propositions, it is an extension of the propositional calculus that deals with partial truths.
- Let

  IF x is $\underline{A}$, *THEN* y is $\underline{B}$ and *IF* x is $\underline{A}'$, *THEN* y is $\underline{B}'$
- How to find B'?
  - By Using the fuzzy composition, $B' = \underline{A} \circ \underline{R}$

Example

‡ Let $\underline{A}$ = "*temperature is high*"= $\underline{A} = \{ {0.1}/{50} + {.5}/{75} + {.7}/{100} + {.9}/{125} + {1}/{150} \}$

"*Voltage is low*"= $\underline{B} = \{ {1}/{40} + {.8}/{4.25} + {.5}/{4.5} + {.2}/{4.75} + {01}/{5.0} \}$

Then find, IF "*the temperature is high*", THEN "*voltage will be low*". OR $\underline{A} \rightarrow \underline{B}$

Let there is another temperature $\underline{A}' = \{ {0}/{50} + {.2}/{75} + {.4}/{100} + {.6}/{125} + {1}/{150} \}$

Then find, IF "*the temperature is high*", THEN "*voltage will be low*", IF " *temperature is* $\underline{A}$*'* " THEN " *fuzzy voltage is* $\underline{V}$*'* ".