CLASSICAL AND FUZZY RELATIONS

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CARTESIAN PRODUCT

- An ordered sequence of r elements in the form $(a_1, a_2, a_3, \ldots, a_r)$ is called an ordered tuple.
- If $r = 2$, it forms an ordered pair.
- For crisp sets $A_1, A_2, A_3, \ldots, A_r$
  - $(a_1, a_2, a_3, \ldots, a_r)$, where $a_1 \in A_1$, $a_2 \in A_2$ etc. is called the cartesian product $A_1 \times A_2 \times A_3 \ldots \times A_r$.
- Cartesian product is not the same as arithmetic product.
- When all A_r are identical, $A_1 \times A_2 \times A_3 \ldots \times A_r$ can be denoted as A^r.

EXAMPLE

- The elements in two sets A and B are given as $A = \{0, 1\}$ and $B = \{a,b,c\}$
- Various Cartesian products of these two sets can be written as shown:
  - $A \times B = \{(0, a),(0, b),(0, c), (1,a), (1,b), (1,c)\}$.
  - $B \times A = \{(a, 0),(a, 1),(b, 0),(b, 1),(c, 0), (c, 1)\}$.
  - $A \times A = A^2 = \{(0, 0),(0, 1),(1, 0),(1, 1)\}$.
  - $B \times B = B^2 = \{(a,a), (a,b), (a,c), (b, a),(b, b),(b, c), (c, a),(c, b), (c, c)\}$.

CLASSICAL/CRISP RELATIONS

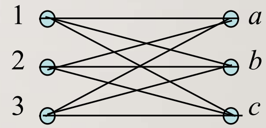
- A subset of the cartesian product of *r* sets is called a *r-ary relation* over $A_1, A_2, A_3, \ldots, A_r$.
- If $r = 2$, it is a binary relation from A_1 into A_2 $(A_1 \times A_2)$.
- For a cartesian product of two universe X and Y, the *strength* of the relationship between the pairs of elements is measured by a characteristic function χ. If $\chi = 1$, it implies complete relationship and $\chi = 0$ implies no relationship, i.e.,

$$\mathrm{X} \times \mathrm{Y} = \{(x, y) \mid x \in \mathrm{X}, y \in \mathrm{Y}\}$$

$$\chi_{\mathrm{X} \times \mathrm{Y}}(x, y) = \begin{cases} 1, & (x, y) \in \mathrm{X} \times \mathrm{Y} \\ 0, & (x, y) \notin \mathrm{X} \times \mathrm{Y} \end{cases}$$

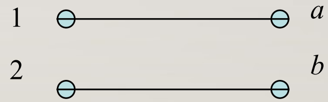
CLASSICAL/CRISP RELATIONS

- A binary relationship can be represented by a two dimensional matrix, called a relation matrix.
- This is an example of unconstrained or universal relation



$$R = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{array}$$

- This is an example of constrained or identity relation



$$R = \begin{array}{c|cc} & a & b \\ \hline 1 & 1 & 0 \\ 2 & 0 & 1 \end{array}$$

CARDINALITY OF CRISP RELATIONS

- If n elements of set X are related to m elements of set Y, then the cardinality of the relation R between these two sets is the product of the cardinalities of the two sets, i.e., $n_X * m_Y$.

COMPOSITION OF CRISP RELATIONS

- Given a relation *R* between *X* & *Y* and Given a relation *S* between *Y* & *Z*, find a relation *T* between *X* & *Z*.
- Two forms of composition operations are there.
  - Max-min composition, T = R O S

$$\chi_T(x, z) = \bigvee_{y \in Y} (\chi_R(x, y) \wedge \chi_S(y, z))$$

  - Max-product (sometimes called max–dot) composition, T = R O S

$$\chi_T(x, z) = \bigvee_{y \in Y} (\chi_R(x, y) \bullet \chi_S(y, z))$$

Here, the symbol "•" is arithmetic product.

EXAMPLE

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$\chi_T(x_1, z_1) = \max[\min(1,0), \min(0,0), \min(1,0), \min(0,0)] = 0$

$\chi_T(x_1, z_2) = \max[\min(1,1), \min(0,0), \min(1,1), \min(0,0)] = 1$

$$\therefore\ T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

FUZZY RELATIONS

- It is a mapping $\underline{R}$ from the cartesian product $X \times Y$ to the interval [0, 1], where the strength of $\underline{R}$ is expressed by the membership function of the relation, $\mu_{\underline{R}}(x, y)$.
- Let $\underline{R}$ be a fuzzy relation between two sets X = {Kolkata, Bhuvaneshwar} and Y={Mumbai, Pune, Kolkata} representing "very far". Then R can be represented by
- $\underline{R}$ = 0.8/Kolkata, Mumbai + 1.0/Kolkata, Pune + 0.0/Kolkata, Kolkata + 0.7/ Bhuvaneshwar, Mumbai + 0.9/ Bhuvaneshwar, Pune+ 0.4/ Bhuvaneshwar, Kolkata

	Kolkata	**Bhubaneshwar**
Mumbai	0.8	0.7
Pune	1.0	0.9
Kolkata	0.0	0.4

2-D Matrix

OPERATIONS ON FUZZY RELATIONS

- Let R and S be fuzzy relations for X x Y, the following operations can be applied:

Union	$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)).$
Intersection	$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)).$
Complement	$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y).$
Containment	$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y).$

FUZZY CARTESIAN PRODUCT

- For two fuzzy sets $\underset{\sim}{A}$ on universe X and $\underset{\sim}{B}$ on universe Y, the cartesian product (or relation) $\underset{\sim}{R}$ between $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is given by

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y,$$

where the fuzzy relation $\underset{\sim}{R}$ has membership function

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x, y) = \min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)).$$

EXAMPLE

- If $\underset{\sim}{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}$ and $\underset{\sim}{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}$

- Then $\underline{A} \times \underline{B} = \underline{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$

DOMAIN OF A FUZZY RELATION

- We define the domain of a binary fuzzy relation R(X,Y) as the fuzzy set:

$$dom\underline{R}(x) = \max_{y \in Y} R(x, y)$$

- Each element of X belongs to domain of R to the degree equal to strength of its strongest relation to any member of set Y.

RANGE OF A FUZZY RELATION

- The range of a binary fuzzy relation is defined as the fuzzy set:

$$ran\underline{R}(y) = \max_{x \in X} R(x, y)$$

- This is the strength of strongest relation which each element of Y has to an element of X.

HEIGHT OF A FUZZY RELATION

- The height of a fuzzy relation R is a number h(R) and is given by

$$h(\underline{R}) = \max_{y \in Y} \max_{x \in X} R(x, y)$$

- It is the largest membership grade attended by any pair (x,y) in R.

INVERSE OF A FUZZY RELATION

The Inverse of a fuzzy relation R is given by

$$R^{-1}(y, x) = R(x, y)$$

Is a relation on Y x X.

COMPOSITION OF FUZZY RELATION

- Given two relations R(X,Y) and S(Y, Z) with a common set (Y), we define their standard composition as:
  - Two forms of composition operations are there
    - Max-min composition, $\underline{T} = \underline{R} \circ \underline{S}$

$$\mu_{\underset{\sim}{T}}(x, z) = \bigvee_{y \in Y} (\mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{\underset{\sim}{S}}(y, z))$$

    - Max-product (Max-dot) composition, $\underline{T} = \underline{R} \circ \underline{S}$

$$\mu_{\underset{\sim}{T}}(x, z) = \bigvee_{y \in Y} (\mu_{\underset{\sim}{R}}(x, y) \bullet \mu_{\underset{\sim}{S}}(y, z))$$

But $\underline{R} \circ \underline{S} \neq \underline{S} \circ \underline{R}$

EXAMPLE (MAX-MIN COMPOSITION)

$$Let \quad X = \{x_1, x_2\}; \quad Y = \{y_1, y_2\}; \quad Z = \{z_1, z_2, z_3\}$$

$$\underline{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \qquad \underline{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$\therefore \ \mu_{\underline{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

$$Thus, \quad \underline{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

EXAMPLE (MAX-DOT COMPOSITION)

$$Let \quad X=\{x_1, x_2\}; \quad Y=\{y_1, y_2\}; \quad Z=\{z_1, z_2, z_3\}$$

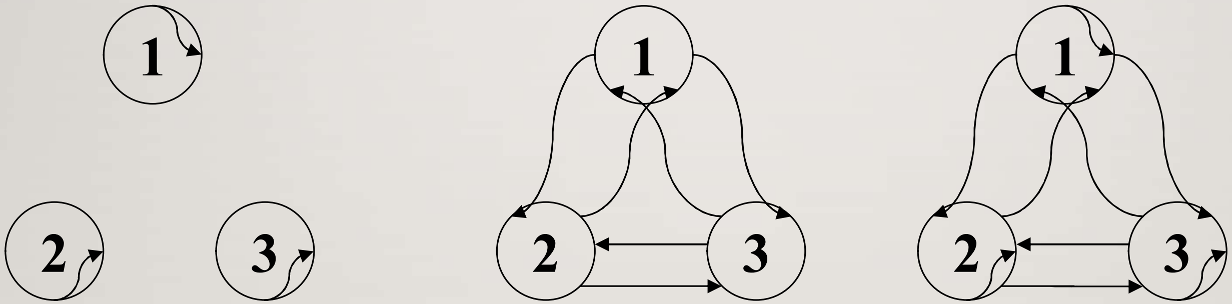
$$\underline{R}= \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \qquad \underline{S}= \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$\therefore \; \mu_{\underline{T}}(x_2, z_2) = \max\left[(0.8 \cdot 0.6), (0.4 \cdot 0.7)\right] = 0.48$$

$$Thus, \quad \underline{T}= \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \end{array}$$

PROPERTIES OF RELATIONS

- Any relation can have the properties of reflexivity, symmetry and transitivity.



Reflexivity **Symmetry** **Transitivity**

- There are some more antireflexive, asymmetric and antisymmetric, antitransitive.

CRISP EQUIVALENCE

- A relation R on a universe X can also be thought of as a relation from X to X. The relation R is an equivalence relation if it has the following three properties: (1) reflexivity, (2) symmetry, and (3) transitivity.

Reflexivity	$(x_i, x_i) \in \mathrm{R}$ or $\chi_{\mathrm{R}}(x_i, x_i) = 1.$
Symmetry	$(x_i, x_j) \in \mathrm{R} \longrightarrow (x_j, x_i) \in \mathrm{R}$
	or $\chi_{\mathrm{R}}(x_i, x_j) = \chi_{\mathrm{R}}(x_j, x_i).$
Transitivity	$(x_i, x_j) \in \mathrm{R}$ and $(x_j, x_k) \in \mathrm{R} \longrightarrow (x_i, x_k) \in \mathrm{R}$
	or $\chi_{\mathrm{R}}(x_i, x_j)$ and $\chi_{\mathrm{R}}(x_j, x_k) = 1 \longrightarrow \chi_{\mathrm{R}}(x_i, x_k) = 1$

FUZZY EQUIVALENCE OR SIMILARITY RELATION

- A fuzzy relation, R, on a single universe X is also a relation from X to X. It is a fuzzy equivalence relation if all three of the following properties for matrix relations define it:

Reflexivity $\mu_{\underset{\sim}{R}}(x_i, x_i) = 1.$

Symmetry $\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i).$

Transitivity $\mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1 \quad \text{and} \quad \mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2 \longrightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda,$

where $\lambda \geq \min[\lambda_1, \lambda_2].$

FUZZY TOLERANCE/PROXIMITY

- Exhibits only reflexivity and symmetry.

THANK YOU