PROPERTIES OF MEMBERSHIP FUNCTIONS, FUZZIFICATION, AND DEFUZZIFICATION

BY SOUROAB MANDAL

WHY FUZZIFICATION AND DEFUZZIFICATION?

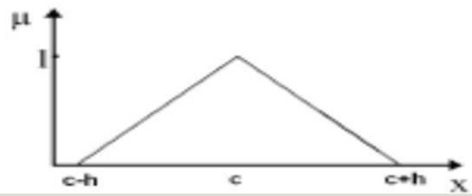
- The bulk of the information we assimilate every day is fuzzy but most of the actions or decisions implemented by humans or machines are crisp or binary.
- Example
  - In giving instructions to an aircraft autopilot, it is not possible to turn the plane "slightly to the west"; an autopilot device does not understand the natural language of a human. We have to turn the plane by 15◦, for example, a crisp number.
  - An electrical circuit typically is either on or off, not partially on.
- If one thinks of a fuzzy set as a collection of membership values, or a vector of values on the unit interval, defuzzification reduces this vector to a single scalar quantity – presumably to the most typical (prototype) or representative value.

MEMBERSHIP FUNCTION OF A FUZZY SET

- All information contained in a fuzzy set is described by its membership function, it is useful to develop a lexicon of terms to describe various special features of this function.
- For the purpose of simplicity, all functions taken here will be continuous but the terms apply equally for both discrete and continuous fuzzy sets.

Continuous example:

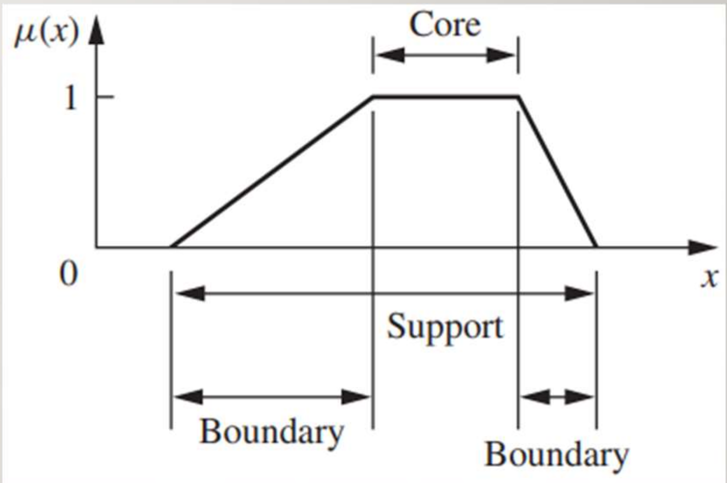
$$\mu_A(x) = \begin{cases} 1 + \dfrac{x-c}{h}, & x \in [c-h, c] \\ 1 - \dfrac{x-c}{h}, & x \in [c, c+h] \\ 0, & \text{otherwise} \end{cases}$$



PROPERTIES OF MEMBERSHIP FUNCTION

- Core: The core of a membership function for some fuzzy set $\underline{A}$ is defined as that region of the universe that is characterized by complete and full membership in the set $\underline{A}$. That is, the core comprises those elements x of the universe such that $\mu_{\underline{A}}(x) = 1$.

$$\mathrm{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$$



.. PROPERTIES OF MEMBERSHIP FUNCTION

- The support of a membership function for some fuzzy set A is defined as that region of the universe that is characterized by nonzero membership in the set A. That is, the support comprises those elements x of the universe such that $\mu_{\underline{A}}(x) > 0$.

$$\mathrm{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$$

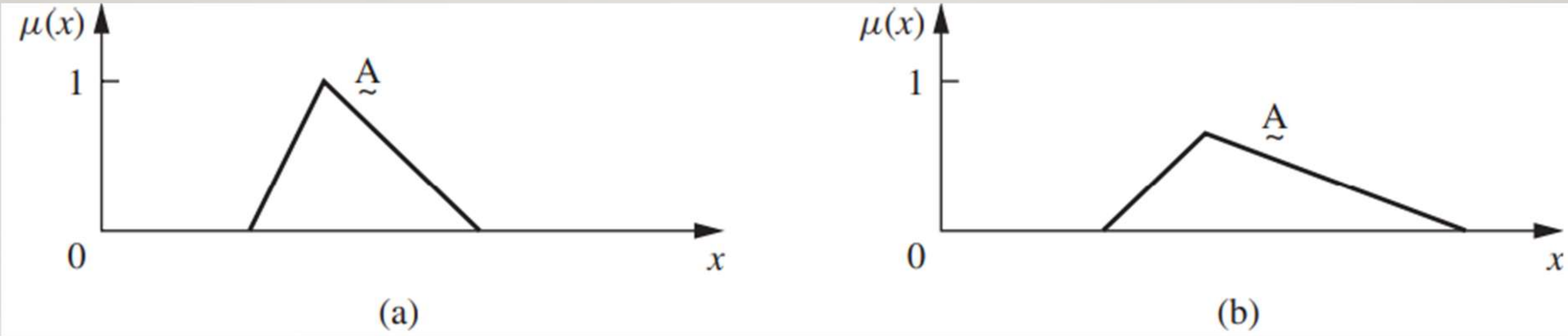
.. PROPERTIES OF MEMBERSHIP FUNCTION

- The boundaries of a membership function for some fuzzy set A are defined as that region of the universe containing elements that have a nonzero membership but not complete membership. That is, the boundaries comprise those elements x of the universe such that $0 < \mu_{\underline{A}}(x) < 1$.
- These elements of the universe are those with some degree of fuzziness, or only partial membership in the fuzzy set A.

$$\mathbf{bnd}(A) = \{x \in X \mid 0 < \mu_{A}(x) < 1\}$$

.. PROPERTIES OF MEMBERSHIP FUNCTION

- Normal Fuzzy Set: A normal fuzzy set is one whose membership function has at least one element x in the universe whose membership value is unity.
- In fuzzy sets, where one and only one element has a membership equal to one, the element is typically referred to as the prototype of the set, or the prototypical element.



- a: Normal Fuzzy Set; b: Subnormal Fuzzy Set

.. PROPERTIES OF MEMBERSHIP FUNCTION

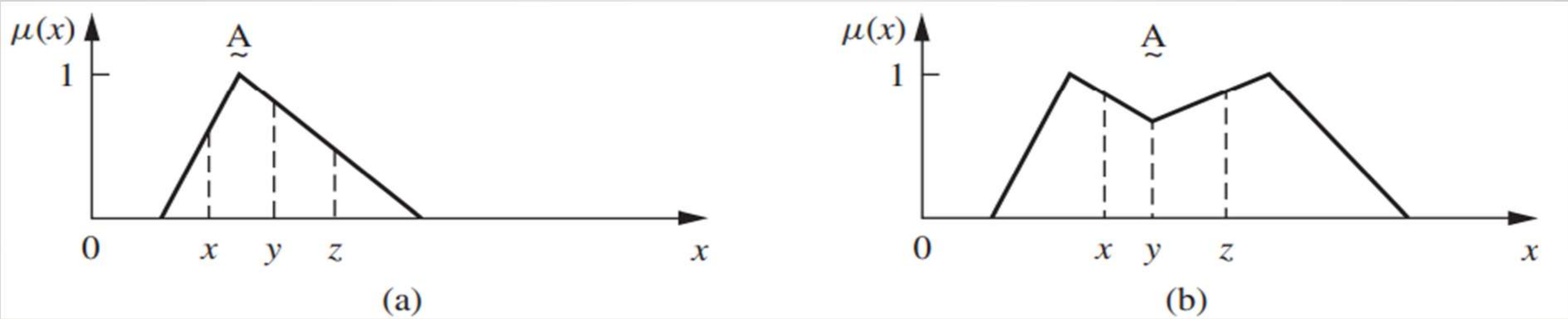
- Convex Fuzzy Set: A convex fuzzy set is described by a membership function whose membership values are:
  - strictly monotonically increasing,
  - or whose membership values are strictly monotonically decreasing,
  - or whose membership values are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe.
- For any values of x, y, z in fuzzy set A, the relation x<y<z implies that,

$$\mu_{\underline{A}}(y) \geq \min[\mu_{\underline{A}}(x), \mu_{\underline{A}}(z)]$$

- Then the set $\underline{A}$ is convex set.
- Also if A and B are convex $\underline{A} \cap \underline{B}$ then is also convex.

.. PROPERTIES OF MEMBERSHIP FUNCTION

- a: Convex, normal Fuzzy Set; b: Non-convex, normal Fuzzy Set



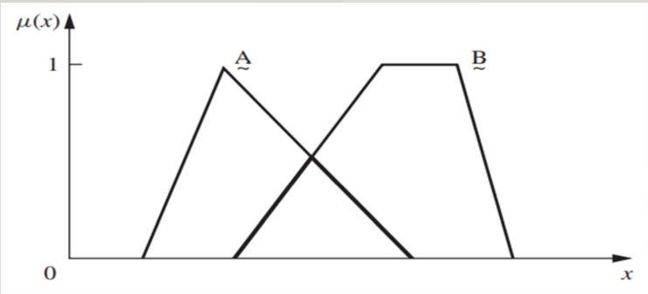


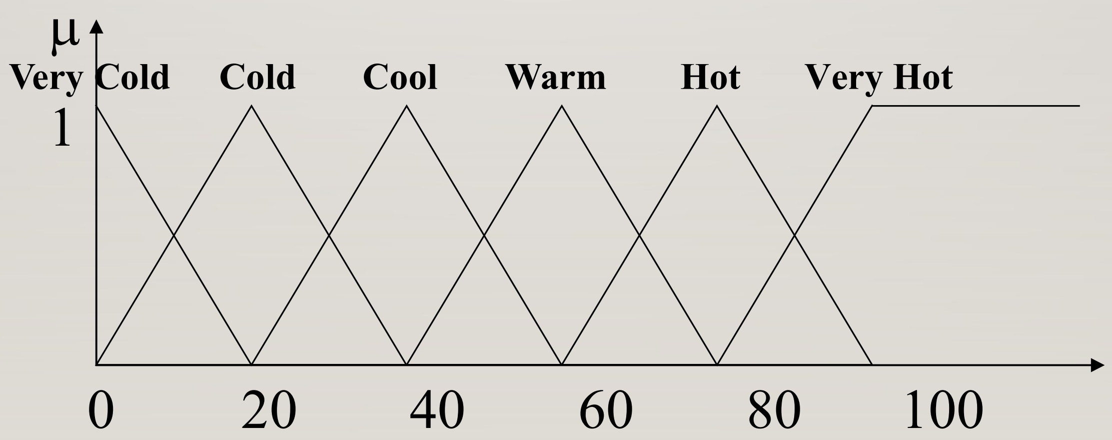
Figure: $\underline{A} \cap \underline{B}$ is also a Convex, normal Fuzzy Set

FUZZIFICATION

- Process of making a crisp quantity fuzzy.
- Following are some of the methods of assigning membership values or functions to fuzzy variables.
  - Intuition
  - Inference
  - Rank Ordering
  - Angular fuzzy sets
  - Neural Networks
  - Genetic Algorithms
  - Inductive reasoning
  - Soft Partitioning
  - Meta rules
  - Fuzzy statistics.

INTUITIVE APPROACH

- From Human experience.
- From knowledge base.
- From a detailed study of the system in hand.



INFERENCE APPROACH

- We use knowledge to perform deductive reasoning.
- We deduce or infer a conclusion.
- Family of Triangles Example
  - Let
    - $\underline{I}$ is a set of fuzzy isosceles triangle
    - $\underline{R}$ is a set of fuzzy right triangle
    - $\underline{IR}$ is a set of fuzzy isosceles and right triangle
    - $\underline{E}$ is a set of fuzzy equilateral triangle
    - $\underline{T}$ is a set of other fuzzy triangles
  - The universe of discourse is defined as $U = \{(A, B, C) \mid A \geq B \geq C \geq 0;\ A + B + C = 180^0\}$

.. INFERENCE APPROACH

$$\mu_{\underline{I}}(A,B,C) = 1 - \frac{1}{60^0}\min(A-B,B-C)$$

$$\mu_{\underline{R}}(A,B,C) = 1 - \frac{1}{90^0}|A-90^0|$$

$$\mu_{\underline{E}}(A,B,C) = 1 - \frac{1}{180^0}(A-C)$$

$$\underline{IR} = \underline{I} \cap \underline{R}; \quad \underline{T} = (\overline{\underline{I} \cup \underline{R} \cup \underline{E}}) = \overline{\underline{I}} \cap \overline{\underline{R}} \cap \overline{\underline{E}}$$

$$If \quad \{X : A = 85^0 \geq B = 50^0 \geq C = 45^0; A+B+C = 180^0\}$$

$$then \quad \mu_{\underline{R}}(x) = 0.94, \mu_{\underline{I}}(x) = 0.916, \mu_{\underline{IR}}(x) = 0.916, \mu_{\underline{E}}(x) = 0.7, \mu_{\underline{T}}(x) = 0.05$$

DIFFERENT MEMBERSHIP FUNCTIONS

- Triangular MF

$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

- Trapezoidal MF

$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

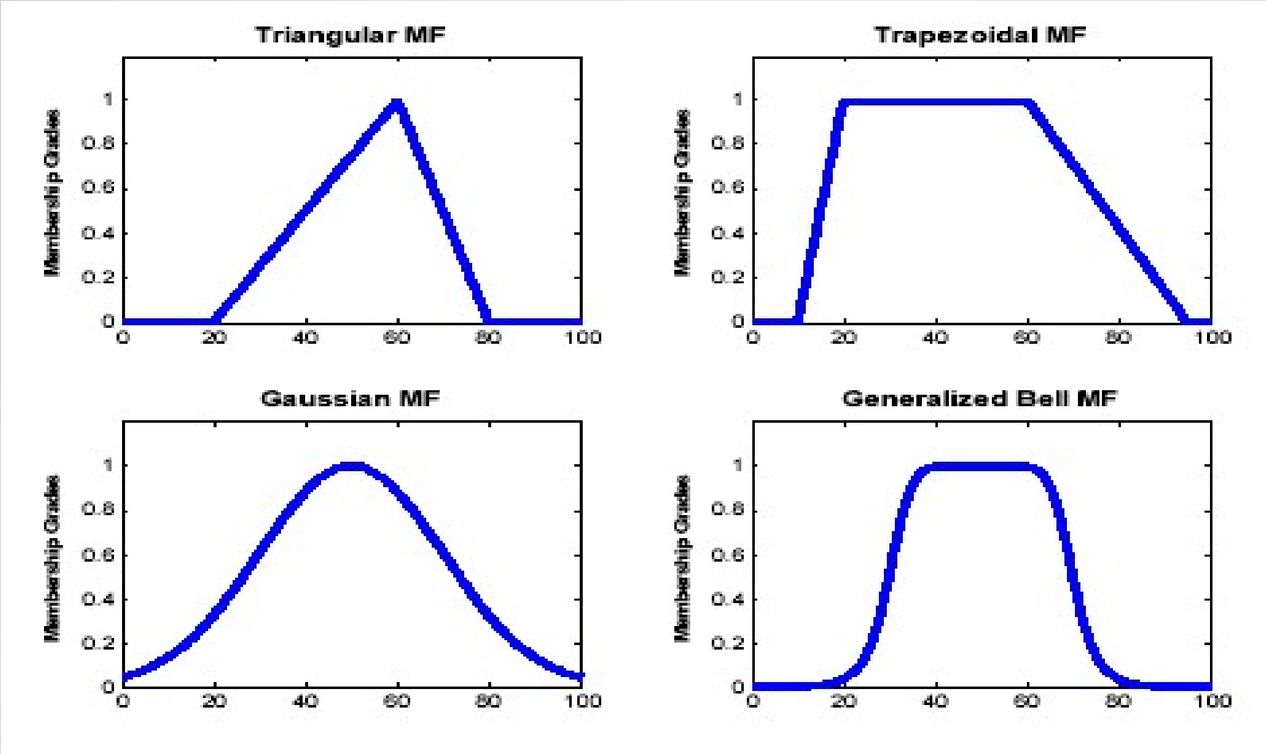
- Gaussian MF

$$gaussmf(x;\sigma,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

- Generalized bell MF

$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

..DIFFERENT MEMBERSHIP FUNCTIONS

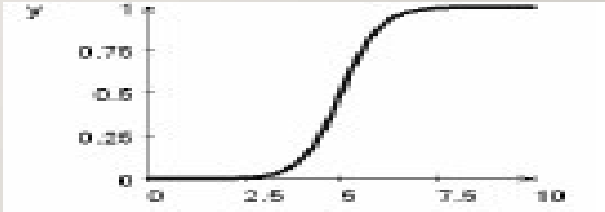


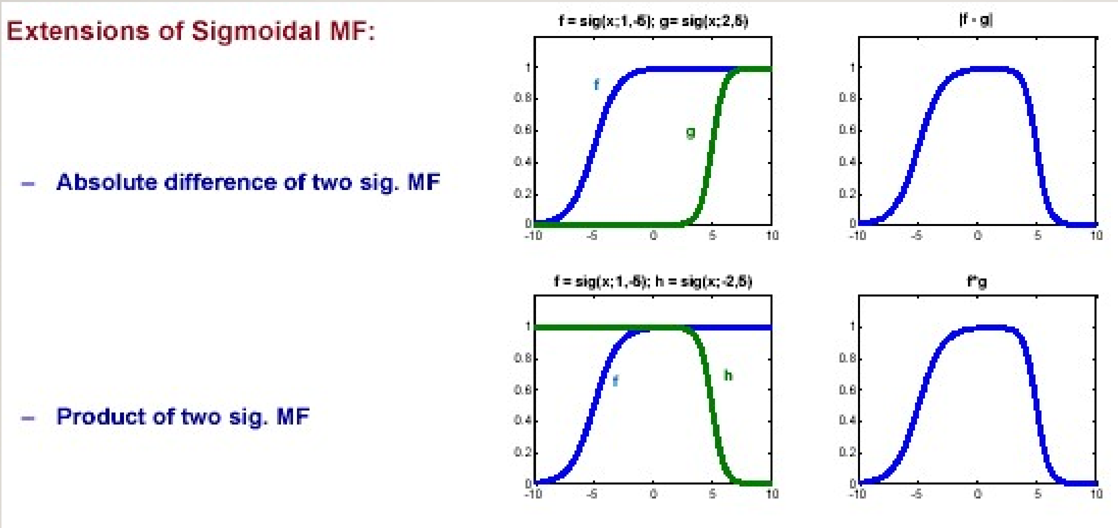
..DIFFERENT MEMBERSHIP FUNCTIONS

- Sigmoidal MF

$$sigmf(x;a,c) = \frac{1}{1+e^{-a(x-c)}}$$

$$\frac{1}{1+e^{-2.5(x-6)}}$$





DEFUZZIFICATION OF FUZZY SETS AND RELATIONS

λ-CUTS OR α CUTS FOR FUZZY SETS

- $[1 \geq \lambda \geq 0]$
- A λ-cut set is a fuzzy set containing elements with membership values greater than λ.

$$A_{\lambda} = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \lambda\}$$

- Any element $x \in A_{\lambda}$ belongs to $\underset{\sim}{A}$ with $\mu \geq \lambda$

$$\underset{\sim}{A} = \{\frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f}\} \qquad \lambda = 1, 0.9, 0.6, 0.3, 0^{+}, 0$$

$$A_1 = \{a\}; \quad A_{0.9} = \{a, b\}; \quad A_{0.6} = \{a, b, c\}; \quad A_{0.3} = \{a, b, c, d\}$$

$$A_{0^+} = \{a, b, c, d, e\}; \quad A_0 = \{a, b, c, d, e, f\}$$

λ-CUTS FOR FUZZY RELATIONS

$$\underline{R} = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

$\lambda = 1$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If $\underline{R}$ is a two-dimensional array defined on the universe X and Y, then any pair $(x,y) \in R_\lambda$ belongs to $\underline{R}$ with a strength of relation gerater than or equal to λ.

λ-CUTS FOR FUZZY SETS & RELATIONS

- Any λ-cut set of a fuzzy set is referred to as the nearest ordinary set to the fuzzy set.
- Any λ-cut relation of a fuzzy relation is referred to as the nearest ordinary relation to the fuzzy relation.
- The distance *d* between a fuzzy set *F* and its nearest ordinary set $\underline{F}$ is referred to as the index of fuzziness. It is given as

$$v(F) = \frac{2}{n^k} d(F, \underline{F})$$

where *k* depends on the type of distance used.

.. λ-CUTS FOR FUZZY SETS & RELATIONS

- k = 1 is used for generalized Hamming distance.
- The corresponding index of fuzziness is referred to as the linear index of fuzziness. It is given by

$$v_l(F) = \frac{2}{n}\sum_{i=1}^{n} |\mu_F(x_i) - \mu_{\underline{F}}(x_i)|$$
$$= \frac{2}{n}\sum_{i=1}^{n} [\min\{\mu_F(x_i), (1 - \mu_F(x_i))\}]$$

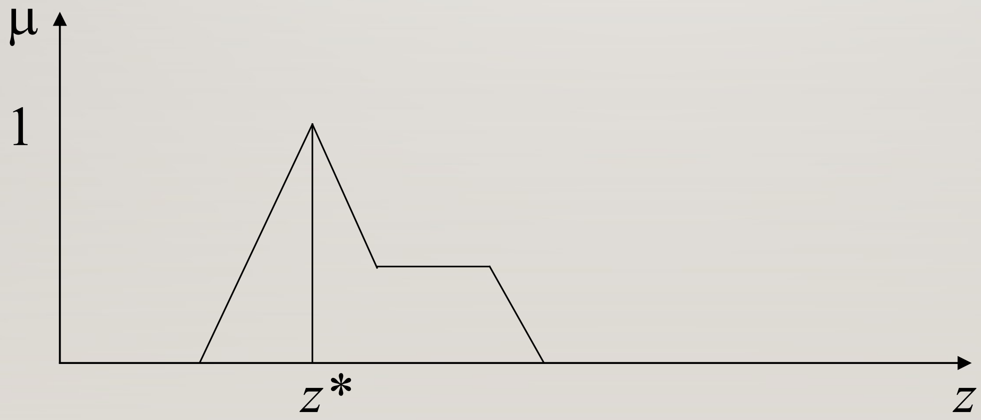
- Higher values of k gives higher orders of the index of fuzziness.

METHODS FOR DEFUZZIFICATION TO SCALARS

- The process of obtaining back the crisp values from the fuzzified state.
- It involves the conversion of a fuzzy quantity to a precise quantity.
- Several methods out of recently proposed ones are shown here:
    - Max-membership principle
    - Centroid method
    - Weighted average method
    - Mean-max membership
    - Centre of sums
    - Center of highest area
    - First (or last) of maxima

MAX-MEMBERSHIP PRINCIPLE

- Also referred to as height method.
- Limited to peaked output functions.
- This method is given by the algebraic expression $\mu_{\underset{\sim}{C}}(z^*) \geq \mu_{\underset{\sim}{C}}(z), \qquad \text{for all } z \in Z$

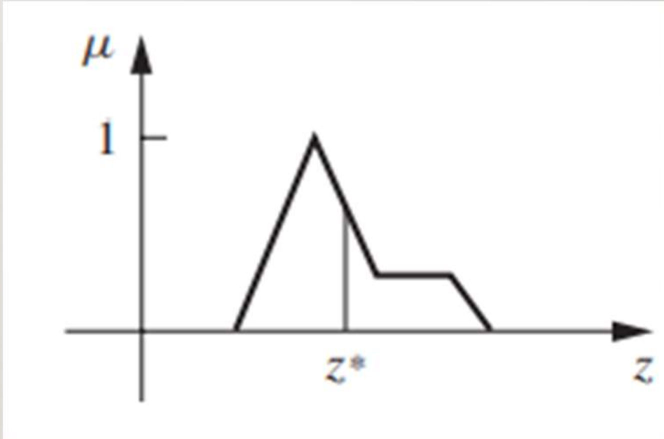


z* is the defuzzified value

CENTROID METHOD

- This procedure (also called center of area or center of gravity) is the most prevalent and physically appealing of all the defuzzification methods.
- It is given by the algebraic expression

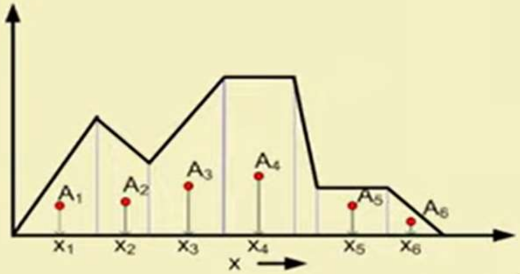
$$z^* = \frac{\int \mu_{\underset{\sim}{C}}(z) \cdot z \, dz}{\int \mu_{\underset{\sim}{C}}(z) \, dz},$$



..CENTROID METHOD: A GEOMETRICAL CALCULATION

Steps:

1) Divide the entire region into a number of small regular regions (e.g. triangles, trapezoid, etc.)



2) Let A_i and x_i denotes the area and $c.g.$ of the i^{th} portion.
3) Then x^* according to CoG is

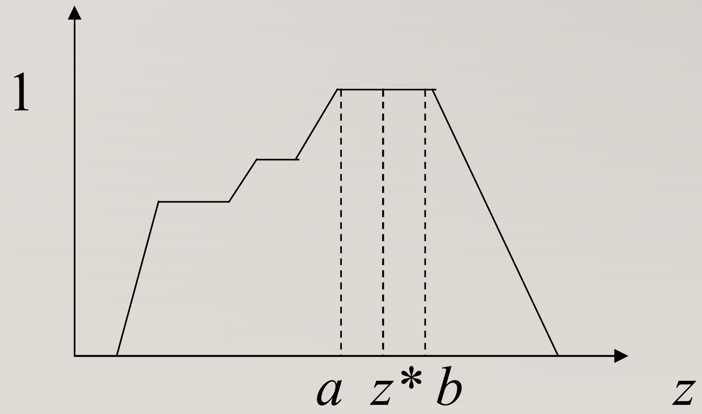
$$x^* = \frac{\sum_{i=1}^{n} x_i.(A_i)}{\sum_{i=1}^{n} A_i}$$

where n is the number of smaller geometrical components.

MEAN-MAX MEMBERSHIP METHOD

- Also referred to as the middle-of-maxima method.
- Similar to max-membership method, but the locations of the maximum membership can be non-unique.
- The maximum membership can be a plateau rather than a single point.
- This method is given by the expression

$$z^* = \frac{a + b}{2}$$

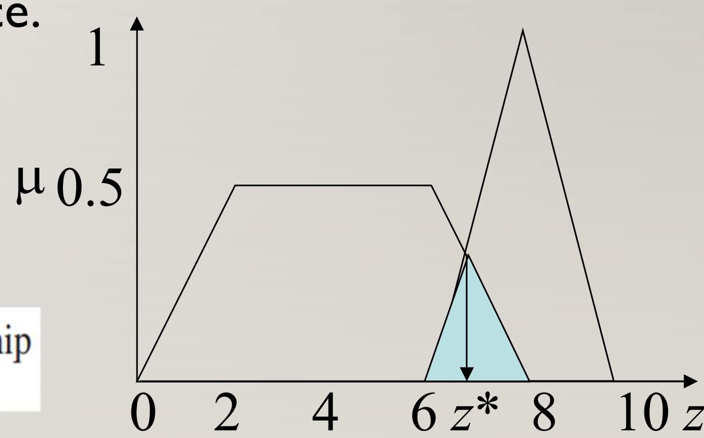


CENTER OF SUMS METHOD

- A faster method.
- Involves algebraic sum of individual output fuzzy sets instead of their union.
- The drawback is that the intersecting areas are added twice.

$$z^* = \frac{\sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z) \int_z \bar{z}\, \mathrm{d}z}{\sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z) \int_z \mathrm{d}z},$$

where the symbol $\bar{z}$ is the distance to the centroid of each of the respective membership functions.



FIRST (OR LAST) OF MAXIMA METHOD

- Uses the overall output or union of all the individual output fuzzy sets $\underset{\sim}{C}_k$ to determine the smallest value of the domain with maximized membership degree in $\underset{\sim}{C}_k$.

First, the largest height in the union (denoted $\mathrm{hgt}(\underset{\sim}{C}_k)$) is determined,

$$\mathrm{hgt}(\underset{\sim}{C}_k) = \sup_{z \in Z} \mu_{\underset{\sim}{C}_k}(z).$$

Then, the first of the maxima is found,

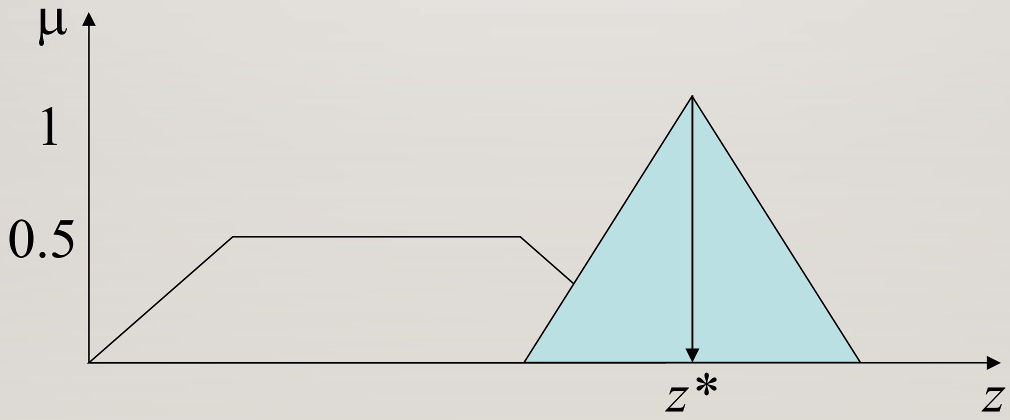
$$z^* = \inf_{z \in Z}\{z \in Z | \mu_{\underset{\sim}{C}_k}(z) = \mathrm{hgt}(\underset{\sim}{C}_k)\}.$$

An alternative to this method is called the *last of maxima*, and it is given as

$$z^* = \sup_{z \in Z}\{z \in Z | \mu_{\underset{\sim}{C}_k}(z) = \mathrm{hgt}(\underset{\sim}{C}_k)\}.$$

.. FIRST (OR LAST) OF MAXIMA METHOD

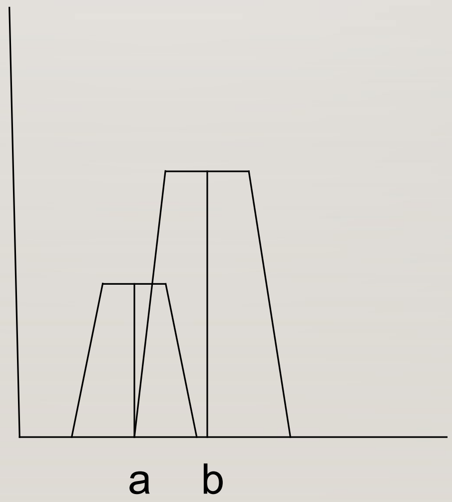
- Sup: Supremum is the least upper bound.
- Inf: Infimum is the greatest lower bound.



WEIGHTED AVERAGE METHOD

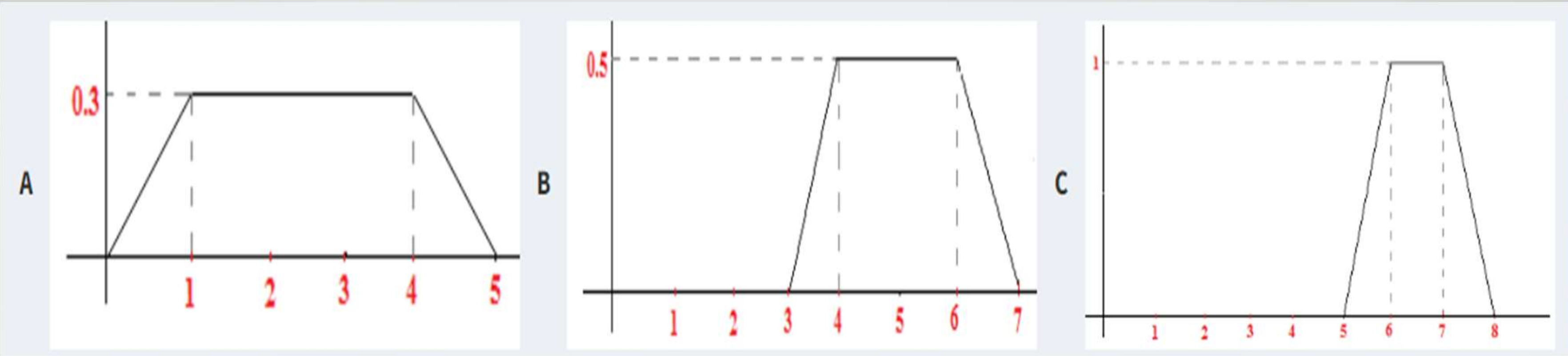
- The weighted average method is the most frequently used in fuzzy applications since it is one of the more computationally efficient methods. Usually restricted to symmetrical output membership functions.

$$z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\overline{z}) \cdot \overline{z}}{\sum \mu_{\underset{\sim}{C}}(\overline{z})}$$

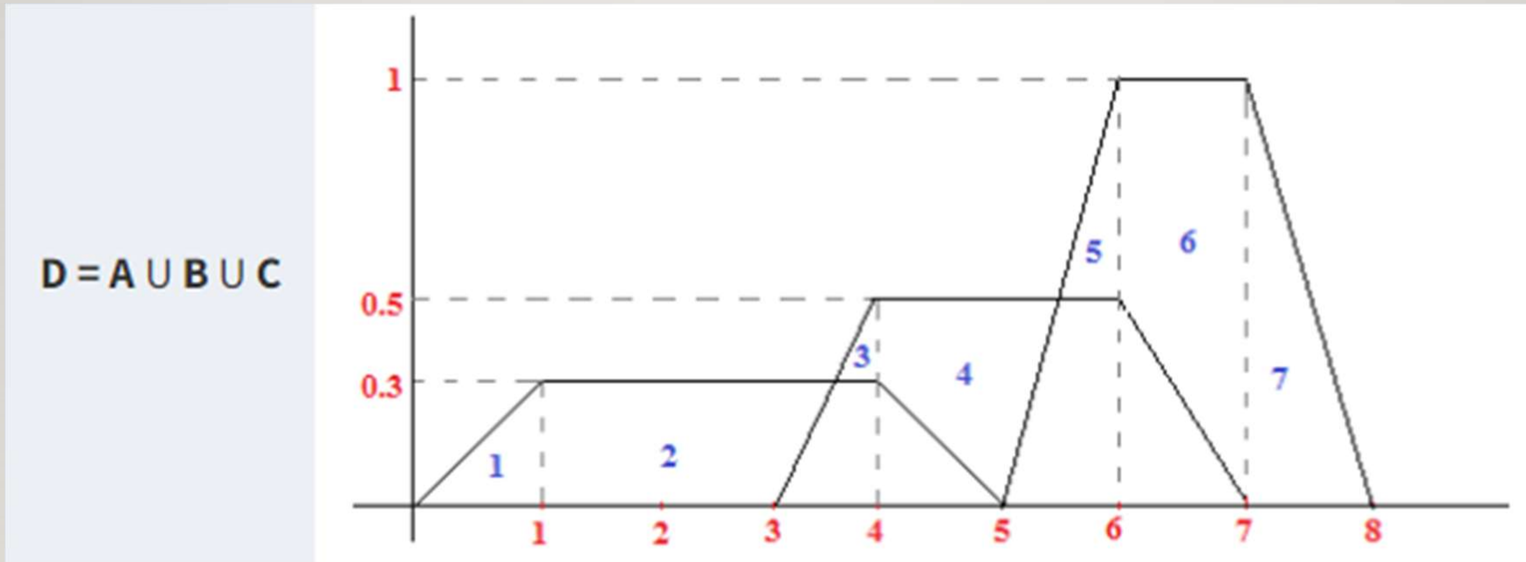


a and b are values of means of their respective shapes.

EXAMPLE



- Hence the union of all the three sets can be represented by the following figure



MAX-MEMBERSHIP METHOD

- From D we can say that the maximum membership value is 1. Hence the scalar value corresponding to the maximum membership value. Therefore, $x^* = 6 , 7$

CENTROID METHOD

- In this method, we have to find the area bounded by the union of the 3 sets and find its centroid which will be the defuzzified value.

Sub Area No.	Area	$\bar{x}$	Area × $\bar{x}$
1.	$\frac{1\times0.3}{2} = 0.150$	0.67	0.100
2.	$3\times0.3 = 0.90$	2.50	2.250
3.	$\frac{0.4\times0.2}{2} = 0.04$	3.73	0.149
4.	$2\times0.5 = 1.00$	5.00	5.000
5.	$\frac{0.5\times0.5}{2} = 0.125$	5.87	7.330
6.	$1\times1 = 1.00$	6.50	6.500
7.	$\frac{1\times1}{2} = 0.50$	7.33	3.660
	$\sum$ Area=3.715		$\sum$ Area × $\bar{x}$=24.989

$x^* = \sum \text{Area} \times \bar{x} \div \sum \text{Area}$

$x^* = 24.989 \div 3.715$

$x^* = 6.72$

WEIGHTED AVERAGE METHOD

- In this case for each fuzzy set the centre is calculated individually by multiplying the mean with its membership value and then the average of all sets is calculated.
- Membership value of A at 2.5 = 0.3
- Membership value of B at 5 = 0.5
- Membership value of C at 6.5 = 1
- $x^* = (2.5 \times 0.3 + 5 \times 0.5 + 6.5 \times 1) \div (0.3 + 0.5 + 1)$
- $x^* = 9.75 \div 1.8$ $x^* = 5.416$

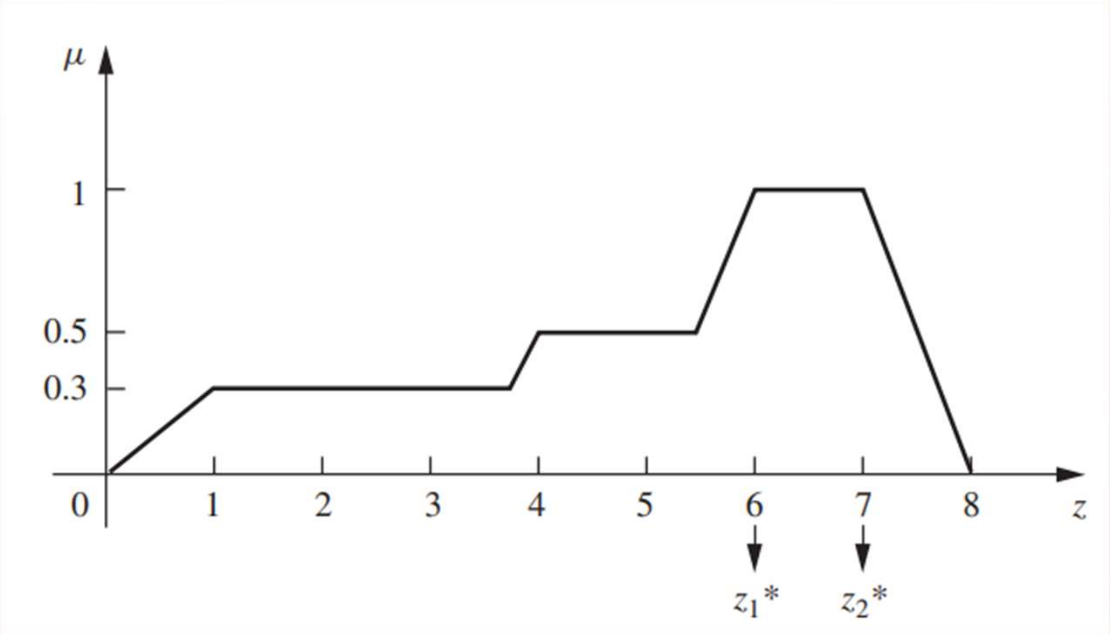
MEAN-MAX MEMBERSHIP

- Here we consider the mean of the range with maximum membership value.
- Here the maximum membership value is 1 which is between 6 and 7.
- Therefore, $x^* = (6 + 7) \div 2$
- $x^* = 13 \div 2 = 6.5$

CENTRE OF SUMS

- In this method, we calculate the sum of the areas of individual fuzzy sets and then find the centre of this area.
- $\text{Area}(A) = [(5 + 3) \times 0.3] \div 2 = 1.2$
- $\text{Area}(B) = [(4 + 2) \times 0.5] \div 2 = 1.5$
- $\text{Area}(C) = [(3 + 1) \times 1] \div 2 = 2$
- $x^* = [(1.2 \times 2.5) + (1.5 \times 5) + (2 \times 6.5)] \div (1.2 + 1.5 + 2)$
- $x^* = 5$

FIRST OF MAXIMA (OR LAST OF MAXIMA)



CENTRE OF LARGEST AREA

- In this method, the defuzzified value is the mean value of the fuzzy set having the maximum or the largest area.
- From method Center of Sums we can see that C has the largest area.
- Therefore, $x^* = (6 + 7) \div 2$
- $x^* = 6.5$

THANK YOU