Extension Principle

Crisp Functions, and Relations

- A function of the form $y = f(x)$ is a relationship between a single-input and a single-output process where the transfer function f represents the mapping.
- Can we extend this mapping in the case where the input, x, is a fuzzy variable or a fuzzy set, and where the function itself could be fuzzy?
- How can be determine the fuzziness in the output, y, based on either a fuzzy input or a fuzzy function or both?

Crisp Functions, and Relations

- An extension principle developed by Zadeh [1975] and later elaborated by Yager [1986] enables us to extend the domain of a function on fuzzy sets.
- ***The mapping***
- The mapping $y = f(x)$ is referred to as the *image of x under f*, and the inverse mapping $x = f^{-1}(y)$, is termed as the *original image of y*.
- A mapping can also be expressed as a relation $\underline{R}$ on the Cartesian space $X \times Y$.

Crisp Functions, and Relations

- Such a relation (crisp) can be described symbolically as

$$\underline{R} = \{(x, y) \mid y = f(x)\}$$

- with the characteristic function describing membership of specific *x*, *y* pairs to the relation $\underline{R}$ as

$$\chi_{\underline{R}}(x, y) = \begin{cases} 1, & y = f(x) \\ 0, & y \neq f(x) \end{cases}$$

Crisp Functions, and Relations

- This mapping can be extended to sets as well. Same for Power set of X and power set of Y.

$$f : P(X) \rightarrow P(Y)$$

- For a set *A* defined on universe *X*, its image, set *B* on the universe *Y*, is found from the mapping

$$B = f(A) = \{y \mid \forall x \in A, y = f(x)\}$$

- where B will be defined by its characteristic value

$$\chi_B(y) = \chi_{f(A)}(y) = \bigvee_{y = f(x)} \chi_A(x)$$

Crisp Functions, and Relations Examples

- Let A={0, 1} ⊂ X={-2, -1, 0, 1, 2}

  Find set B with the mapping y=|4x|+2

Using Zadeh's notation

$$A = \{\frac{0}{-2} + \frac{0}{-1} + \frac{1}{0} + \frac{1}{1} + \frac{0}{2}\}$$

From the mapping, it is obvious that the universe Y will be Y={2, 6, 10}. The membership values for each of the elements in Y will be:

$$\chi_{\underline{B}}(2) = \vee\{\chi_{\underline{A}}(0)\} = 1 \qquad \chi_{\underline{B}}(6) = \vee\{\chi_{\underline{A}}(-1), \chi_{\underline{A}}(1)\} = \vee\{0,1\} = 1$$

$$\chi_{\underline{B}}(10) = \vee\{\chi_{\underline{A}}(-2), \chi_{\underline{A}}(2)\} = \vee\{0,0\} = 0$$

And, the set B will be

$$B = \{\frac{1}{2} + \frac{1}{6} + \frac{0}{10}\} = \{2,6\}$$

Crisp Functions, and Relations Examples

- Let $A=\{0, 1\} \subset X=\{-2, -1, 0, 1, 2\}$

Find set $B \subset Y=\{0, 1, 2, 3, \ldots\ldots, 10)$ with the mapping $y=|4x|+2$

Relation Matrix Solution

A relation matrix solution to the problem can be found out using the composition operation on finite universe relations.

Relation Matrix Solution

The crisp relation describing the mapping is

$$R = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} \\ \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and

$$A = \{\frac{0}{-2} + \frac{0}{-1} + \frac{1}{0} + \frac{1}{1} + \frac{0}{2}\}$$

Relation Matrix Solution

Now the image B can be found by composition,

$$B = A \circ R$$

$$\chi_B(y) = \underset{x \,\square\, X}{\text{¿}} (\chi_A(x) \wedge \chi_R(x, y)) = \begin{cases} 1 & \textit{for} \quad y = 2,6 \\ 0 & \textit{otherwise} \end{cases}$$

or, in Zadeh's notation,

$$B = \{\frac{0}{0} + \frac{0}{1} + \frac{1}{2} + \frac{0}{3} + \frac{0}{4} + \frac{0}{5} + \frac{1}{6} + \frac{0}{7} + \frac{0}{8} + \frac{0}{9} + \frac{0}{10}\}$$

Extension Principle

- The extension principle, extends the concept of ***mapping*** to the fuzzy domain.
- For a fuzzy set $\underline{A}$ on universe X, the image of the set $\underline{B}$ on universe Y will also be fuzzy.
- It can be represented by $\underline{B}=f(\underline{A})$.
- The membership functions describing $\underline{A}$ and $\underline{B}$ defined on a universe of a unit interval [0,1] will be related as

$$\mu_{\underline{B}}(y) = \bigvee_{f(x)=y} \mu_{\underline{A}}(x)$$

- A ***fuzzy vector*** is a vector containing membership values of elements.
- If $\underline{A} = \{x_1, x_2, x_3, \ldots, x_n\}$ and $\underline{B} = \{y_1, y_2, y_3, \ldots, y_m\}$ the array of membership functions for $\underline{A}$ and $\underline{B}$ can be reduced to fuzzy vectors by using

$$\underline{a} = \{a_1, a_2, a_3, \ldots, a_n\} = \{\mu_{\underline{A}}(x_1), \mu_{\underline{A}}(x_2), \mu_{\underline{A}}(x_3), \ldots, \mu_{\underline{A}}(x_n)\}$$

$$= \{\mu_{\underline{A}}(x_i)\} \quad for \quad i = 1,2,3, \ldots, n$$

$$\underline{b} = \{b_1, b_2, b_3, \ldots, b_n\} = \{\mu_B(y_1), \mu_B(y_2), \mu_B(y_3), \ldots, \mu_B(y_m)\}$$

$$= \{\mu_{\underline{B}}(y_i)\} \quad for \quad i = 1,2,3, \ldots, m$$

- Now the image of fuzzy set $\underline{A}$ as previous can be determined through the use of composition operation, i.e.,

$$\underline{B} = \underline{A} \circ \underline{R}$$

- Or when using the fuzzy vector form as

$$\underline{b} = \underline{a} \circ \underline{R}$$

  - where $\underline{R}$ is an $n \times m$ fuzzy relation matrix.

- Suppose our input universe comprises the Cartesian product of many universes. Then the mapping f is defined as

$$f : P(X_1 \times X_2 \times \ldots \times X_n) \rightarrow P(Y)$$

- Let fuzzy sets $\underline{A}_1$, $\underline{A}_2$, ..$\underline{A}_n$ defined on X_1, X_2, ..X_n.

  The mapping for these particular input sets can be defined as $\underline{B}$=F($\underline{A}_1$, $\underline{A}_2$, ..$\underline{A}_n$),

  The membership function of $\underline{B}$ by composition, [for *n* sets] is

$$\mu_{\underline{B}}(y) = \max_{y = f(x_1, x_2, \ldots, x_n)} \{\min[\mu_{\underline{A}_1}(x_1), \mu_{\underline{A}_2}(x_2), \ldots, \mu_{\underline{A}_n}(x_n)]\}$$

- This called as Zadeh's extension principle for discrete-values function if f is continuous max operator is replaced by supermum operator.

Fuzzy Transform

$$f : \underline{A} \rightarrow \underline{B}$$

- Here the input and output is fuzzy but transform is crisp
- The mapping or transformation of a crisp input into a fuzzy output is referred to as ***fuzzy transform***.

- Even if the input is a singleton crisp element, the output will be a fuzzy set of elements, i.e.,

$$\underline{f} : A \rightarrow \underline{B}; \quad \underline{B} = \underline{f}(A) \quad where \quad x \in A$$

Fuzzy Transform

- If X and Y are finite universes, the fuzzy mapping between them can be expressed as a relational matrix as

$$\underline{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ . \\ . \\ x_n \end{array} \begin{array}{c} \begin{array}{ccccc} r_{11} & y_2 & . & . & y_m \end{array} \\ \begin{bmatrix} r_{11} & r_{12} & . & . & r_{1m} \\ r_{21} & r_{22} & . & . & r_{2m} \\ . & . & . & . & . \\ . & . & . & . & . \\ r_{n1} & r_{n2} & . & . & r_{nm} \end{bmatrix} \end{array}$$

Fuzzy Transform

- For a particular single element of the input universe, x_i, its fuzzy image $\underline{B}= f(x_i)$ is given in a general symbolic form as

$$\mu_{\underline{B}_i}(y_i) = r_{ij}$$

- or, in fuzzy vector notation as

$$\underline{b}_i = \{r_{i1}, r_{i2}, r_{i3}, \ldots, r_{im}\}$$

- Thus, the fuzzy image of the element x_i is given by the elements in the i^{th} row of the fuzzy relation $\underline{R}$, defining the fuzzy mapping.

Fuzzy Transform

- If the input is also a fuzzy set, then

$$\underline{B} = \underline{f}(\underline{A})$$

- The extension principle can also be used to find its fuzzy image, $\underline{B}$, by the expression

$$\mu_{\underline{B}}(y) = \bigvee_{x \in X} (\mu_{\underline{A}}(x) \wedge \mu_{\underline{R}}(x, y))$$

- In vector form,

$$\underline{b}_j = \max_i (\min(a_i, r_{ij}))$$

Fuzzy Transform Example

- Let a fuzzy relation between the length and mass of an article to be sent in space is given by

$$\underset{\sim}{R} = \begin{array}{c} \begin{array}{ccccc} 1.4 & 1.5 & 1.6 & 1.7 & 1.8 \end{array} \; m \\ \begin{bmatrix} 1.0 & 0.8 & 0.2 & 0.1 & 0 \\ 0.8 & 1.0 & 0.8 & 0.2 & 0.1 \\ 0.2 & 0.8 & 1.0 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.8 & 1.0 & 0.8 \\ 0 & 0.1 & 0.2 & 0.8 & 1.0 \end{bmatrix} \begin{array}{c} 40 \\ 50 \\ 60 \\ 70 \\ 80 \end{array} kg \end{array}$$

Find the fuzziness in the length of the article.

Let the fuzzy quantity mass is given by

$$\underset{\sim}{A} = \{\frac{0.8}{40} + \frac{1}{50} + \frac{0.6}{60} + \frac{0.2}{70} + \frac{0}{80}\} kg$$

Fuzzy Transform Example

In vector form, the mass of the article can be expressed as

$$\underline{a} = \{0.8, 1.0, 0.6, 0.2, 0\}$$

The fuzzy image vector $\underline{b}$ describing the fuzziness in the length of the article can be determined using the composition operation, i.e.,

$$\underline{b} = \underline{a} \circ \underline{R}$$

The output fuzzy vector $\underline{b}$ is thus,

$$\underline{b} = \{0.8, 1.0, 0.8, 0.6, 0.2\}$$

One-to-one mapping

- If there is a one-to-one mapping between the elements u, of one universe U, onto elements v, of another universe V, through a function f,

$$f : u \rightarrow v$$

- Let $\underset{\sim}{A}$ is a fuzzy set on U, i.e $\underset{\sim}{A} \subset U$,

$$\underset{\sim}{A} = \{\frac{\mu_1}{u_1} + \frac{\mu_2}{u_2} + ... + \frac{\mu_n}{u_u}\}$$

- Then the extension principle asserts a function (producing a fuzzy image) that performs a one to one mapping is given by,

$$f(\underset{\sim}{A}) = f(\frac{\mu_1}{u_1} + \frac{\mu_2}{u_2} + ... + \frac{\mu_n}{u_n}) \quad = [\frac{\mu_1}{f(u_1)} + \frac{\mu_2}{f(u_2)} + ... + \frac{\mu_n}{f(u_n)}]$$

One-to-one mapping Example

- Let $\underset{\sim}{A} = \{\frac{0.6}{1} + \frac{1.0}{2} + \frac{0.8}{3}\}$ be defined on the universe *U*={1, 2, 3}. Map the elements of this fuzzy set to another universe, *V*, under the function $v = f(u) = 2u - 1$

Clearly, *V*={1,3,5}. The fuzzy membership function for *v*=*f*(*u*)=2*u*-1 would be

$$f(\underset{\sim}{A}) = \left\{\frac{0.6}{1} + \frac{1}{3} + \frac{0.8}{5}\right\}$$

Mapping of Cartesian Products

- If there is mapping of the cartesian product of elements from two universes, U_1 and U_2 to another universe V, then

$$f(\underline{A}) = f(U_1 \times U_2) = \left\{ \sum \frac{\min[\mu_1(i), \mu_2(j)]}{f(i, j)} \mid i \in U_1, j \in U_2 \right\}$$

where, $\mu_1(i)$, $\mu_2(j)$ are noninteractive and separable fuzzy membership functions.

Mapping of Cartesian Products Example

- Let $U_1 = U_2 = \{1,2,3,....,10\}$ and

$$\underline{A} = \underline{2} = "approx \quad 2" = \{\frac{0.6}{1} + \frac{1.0}{2} + \frac{0.8}{3}\}$$

$$\underline{B} = \underline{6} = "approx \quad 6" = \{\frac{0.8}{5} + \frac{1.0}{6} + \frac{0.7}{7}\}$$

Find the fuzzy number "*approx* 12" defined on the universe

$$V = \{5,6,....,18,21\}$$

Mapping of Cartesian Products Example

$$\underline{2} \times \underline{6} = \left(\frac{0.6}{1} + \frac{1.0}{2} + \frac{0.8}{3}\right) \times \left(\frac{0.8}{5} + \frac{1.0}{6} + \frac{0.7}{7}\right)$$

$$= \left\{ \frac{\min(0.6,0.8)}{5} + \frac{\min(0.6,1.0)}{6} + ... + \frac{\min(0.8,1.0)}{18} + \frac{\min(0.8,0.7)}{21} \right\}$$

$$= \left\{ \frac{0.6}{5} + \frac{0.6}{6} + \frac{0.6}{7} + \frac{0.8}{10} + \frac{1.0}{12} + \frac{0.7}{14} + \frac{0.8}{15} + \frac{0.8}{18} + \frac{0.7}{21} \right\}$$

Mapping of Cartesian Products

- If more than one combinations of the elements of universes, U_1 and U_2, mapped to another universe V, then the mapping does not remain an one-to-one mapping.
- Here, the maximum membership grades of the combinations mapping to the same output variable, are taken according to

$$\mu_{\underset{\sim}{A}}(u_1, u_2) = \max_{v = f(u_1, u_2)} [\min\{\mu_1(u_1), \mu_2(u_2)\}]$$

Mapping of Cartesian Products Example

- Let

$$\underset{\sim}{A} = \{\frac{0.2}{1} + \frac{1.0}{2} + \frac{0.7}{4}\}$$ and $$\underset{\sim}{B} = \{\frac{0.5}{1} + \frac{1.0}{2}\}$$

Find the fuzzy membership values for the algebraic product mapping

$$f(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{A} \times \underset{\sim}{B}$$

Mapping of Cartesian Products Example

$$f(\underline{A}\times\underline{B}) = \left(\frac{0.2}{1}+\frac{1.0}{2}+\frac{0.7}{4}\right)\times\left(\frac{0.5}{1}+\frac{1.0}{2}\right)$$

$$= \left\{\frac{\min(0.2,0.5)}{1}+\frac{\max[\min(0.2,1.0),\min(0.5,1.0)]}{2}\right\}$$

$$+\left\{\frac{\max[\min(0.7,0.5),\min(1.0,1.0)]}{4}+\frac{\min(0.7,1.0)}{8}\right\}$$

$$= \left\{\frac{0.2}{1}+\frac{0.5}{2}+\frac{1.0}{4}+\frac{0.7}{8}\right\}$$

Fuzzy Numbers

- A fuzzy number ($\underline{I}$), is described by a normal, convex membership function on the real line.

- An arithmetic operation on two fuzzy numbers, $\underline{I}$ and $\underline{J}$, defined on the real line in the universes X and Y respectively, will yield another fuzzy number.

- Arithmetic operations are denoted by '*', where '*' may be either of {+, -, ×, ÷}.

- Any arithmetic operation between two fuzzy numbers is a sort of mapping.

- This mapping between two fuzzy numbers, $\underline{I}$ and $\underline{J}$, denoted by $\underline{I} * \underline{J}$, will be defined on universe Z and can be obtained using the extension principle as

$$\mu_{\underline{I}*\underline{J}}(z) = \bigvee_{x*y=z} (\mu_{\underline{I}}(x) \wedge \mu_{\underline{J}}(y))$$

Example - Addition

- Perform the addition between two fuzzy ones '$\underline{1}$' defined as

$$\underline{1} = \{\frac{0.2}{0} + \frac{1.0}{1} + \frac{0.2}{2}\}$$

- Perform the addition between two fuzzy ones '$\underline{1}$' defined as

$$\underline{1} = \{ \frac{0.2}{0} + \frac{1.0}{1} + \frac{0.2}{2} \}$$

$$\underline{1} + \underline{1} = \underline{2} = (\frac{0.2}{0} + \frac{1.0}{1} + \frac{0.2}{2}) + (\frac{0.2}{0} + \frac{1.0}{1} + \frac{0.2}{2})$$

$$= \left\{ \frac{\min(0.2,0.2)}{0} + \frac{\max[\min(0.2,1.0), \min(1.0,0.2)]}{1} \right\}$$

$$+ \left\{ \frac{\max[\min(0.2,0.2), \min(1.0,1.0), \min(0.2,0.2)]}{2} \right\}$$

$$+ \left\{ \frac{\max[\min(1.0,0.2), \min(0.2,1.0)]}{3} + \frac{\min(0.2,0.2)}{4} \right\}$$

$$= \left\{ \frac{0.2}{0} + \frac{0.2}{1} + \frac{1.0}{2} + \frac{0.2}{3} + \frac{0.2}{4} \right\}$$