1989 30th IMO Day-1

1. Prove that the set $\{1, 2, ..., 1989\}$ can be expressed as the disjoint union of subsets $A_i (i = 1, 2, ..., 117)$ such that :
   (i) Each A_i contains 17 elements ;
   (ii) The sum of all the elements in each A_i is the same .

2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

   (i) The area of the triangle $A_0 B_0 C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$

   (ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

3. Let n and k be positive integers and let S be a set of n points in the plane such that

   (i) No three points of S are collinear, and

   (ii) For any point P of S there are at least k points of S equidistant from P.

   Prove that:

   $$k < \frac{1}{2} + \sqrt{2n}.$$

1989 30th IMO Day-2

4. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point. P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

   $$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

   .

5. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

6. A permutation $(x_1, x_2, ..., x_m)$ of the set {1,2.....,2n}. where a is a positive integer, is said to have property P if $\left|x_i - x_{i+1}\right| = n$ for at least one in{1,2,....,2n-1}. Show that, for each n, there are more permitations with property P than without.

1990 31st IMO Day-1

7. Chords AB and CD of a circle imersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through $D, E.$ and M intersects the lines BC and AC at F and $G.$ respectively, If

$$\frac{AM}{AB} = t$$

find

$$\frac{EG}{EF}$$

in terms of t .

8. Let n_3 and consider a set E of 2_{n-1} distinct points on a circle. Suppose that exactly k of these points are to he colored black. Such a coloring is "*good*" if there is at least one pair of black points such that the interior of one of the ares between them contains exactly in points from E. Find the smallest value of k so that every such coloring of k points of E is good

9. Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is integer.

1990 31st IMO Day-2

10. Let Q^+ be the set of positive rational numbers. Construct a function f : $Q^+ \to Q^+$ such that

$$f\left(xf(y)\right) = \frac{f(x)}{y}$$

for all x, y in Q^+.

11. Given an initial integer $n_0 > 1$, two players. A and B, choose integers $n_1, n_2, n_3, ...$ alternately according to the following rules:
Knowing n_{2k}, A chooses any integer n_{2k+2} such that

$$n_{2k} \le n_{2k+1} \le n_2^2 k$$

Knowing n_{2k+1} , B chooses any integer n_{2k+2} such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.
Plaver A wins the game by choosing the number 1990: player B wins by choosing the number 1. For which n_0 does:
(a)A have a winning strategy?
(b) B have a winning strategy?
(c) Neither player have a winning strategy?

12. Prove that there exists a convex 1990-gon with the following two properties
(a) All angles are equal.
(b) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, ...,1990^2$ in some order.

1991 32nd IMO Day-1

13. Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI.BI.CI.}{AA'.BB'.CC'.} \le \frac{8}{27}$$

.

14. Let $n > 6$ be an integer and $a_1, a_2, ..., a_k$ be all the natura numbers less than n and relatively prime to n If

$$a_2 - a_1 = a_3 - a_2 = ... = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

15. Let $S = \{1, 2, 3, ..., 280\}$. Find the smallest integer n such that each $n-$ element subset of S contains five numbers which are pairwise relatively prime.

1991 32nd IMO Day-2

16. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2.....k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.
[A graph consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices. u, v belongs to at most one edge. The graph G is connected if for cach pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, ..., v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.]

17. Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

18. An infinite sequence $x_0, x_1, x_2, ...$of real numbers is said to be bounded if there is a constant C such that $\left|x_i\right| \leq C$ for every $i \geq 0$.
Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, ...$ Such that

$$\left|x_i - x_j\right| \left|i - j\right|^a \geq 1$$

for every pair of distinct nonnegative integers i, j.