Bayesian Decision Theory and Information Value Theory

Modèles et algorithmes pour la décision dans l'incertain

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Today's topics

- Decision-making under under uncertainty with maximum expected utility principle (review)
- Foundations of subjective probabilities
- Bayesian reasoning with subjective probabilities
- Information value theory

Rational Decisions in Face of Uncertainty

Sources of Uncertainty

Agents need to handle uncertainty, arising from:

- partial observability
- non determinism

The right decision depends on both the relative importance of goals and the likelihood that they will be achieved.

Example: You need to catch a fly. What plan is best for you?

- Leave 90 minutes before plane; probability of catching flight is 0.95
- Leave 180 minutes before plane; probability raises to 0.99
- Leave two days before; probability $1 - 0.1^{10}$

The best choice depends on: *How much important is to catch the flight? What's the cost (in terms of utility) associated with long wait at the airport? "Cost" of sleeping at the airport?*

Utilitarian Agents

Utility Theory + Probability Theory $\rightarrow$ Decision-theoretic agents

- Rational decisions based on what the agent believes and wants
- One-shot decision vs sequential decision-making
- $P_d(x)$: probability of outcome x conditioned on taking decision d
- The agent's preferences are captured by a utility function $u(s)$ that expresses the desiderability of a state

The Principle of Maximum Expected Utility (MEU)

According to MEU, a rational agent should choose the action that maximizes the agent's expected utility.

Expected utility of decision d:

$$\mathrm{EU}(d) = \sum_{x \in S} P_d(x) U(x). \tag{1}$$

Then choose action $d^* = \arg\max_d EU(d)$.

If utility correctly specifies the performance measures, than the agent will achieve the highest possible output averaged *over all possible environments*.

The St. Petersburg Lottery

Repeatedly toss a (fair) coin:

- For each "head" the pot is doubled (the pot starts at 2).
- The game ends right after the first "tail" and the player is awarded the current pot.

The monetary reward is 2^n where n is the number of tosses.

$$EV = \sum_{i=1}^{\infty} \frac{1}{2^i} 2^i = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + ... = 1 + 1 + 1 + \ldots = \infty$$

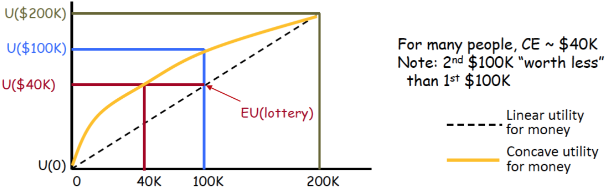
Its expected value is infinite.

To most people this game will be worth only a very small amount.
The "paradox" stems from the discrepancy between what the typical player is willing to pay to enter the game and the ∞ EV.

A solution of the paradox: utility of money is concave

Utility of Money

- What would you choose ?
  - $l_1 = 100K\$$ (with certainty)
  - $l_2 = [200K\$, 0.5; 0\$, 0.5]$
- What if l_2 offered $300K\$$, $400K\$$ or $500K\$$?
- Utility of money generally is not linear
  Usually $U(EMV(l)) > U(l)$ where EMV is the expected monetary value
- Certainty equivalent of l is $CE(l)$ such that $U(CE(l)) = U(l)$; or $CE(l) = U^{-1}(EU(l))$.



Risk Attitudes

Different Risk Attitudes

- **Risk premium**: $EMV(I) - CE(I)$
  (expected monetary value less its certainty equivalent; how much one gives up in order to remove the risk of losing)
- **Risk averse** decision maker: $EMV(I) - CE(I) > 0$ (utility of money is concave)
- **Risk neutral**: $EMV(I) - CE(I) = 0$ (utility of money is linear)
- **Risk seeking**: decision maker has negative risk premium (utility of money is convex)

Most people are risk averse: foundation of the insurance business.
In small ranges (small money gain/loss) linear is a good approximation.

St. Peterburg Game (again)

Recall that the gain is 2^n where n is the number of tosses.

$$EV = \sum_{i=1}^{\infty} \frac{1}{2^i} 2^i = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + ... = 1 + 1 + 1 + \ldots = \infty$$

If we consider that U that is concave; for example $U(x) = \log_2(x)$:

$$\mathrm{EU} = \sum_{i=1}^{\infty} \frac{1}{2^i} log_2(2^i) = \sum_{i=1}^{\infty} \frac{i}{2^i} = 2.$$

That corresponds to 4\$ as $U(4) = \log_2(4) = 2$.

Moreover, taking into consideration the current wealth w:

$\mathrm{EU} = \sum_{i=1}^{\infty} \frac{(log(w+2^i-c)-log(w))}{2^i} < \infty$

One should be willing to pay at most the maximum c such that EU is > 0: a millionaire would pay up 10.94\$, a person with 1000\$ up to 5.94\$.

Utility of Wealth

The current status of wealth may affect the attractability of a lottery

- $U(w)$: utility of current wealth
- $U(w + \text{gain})$: utility of wealth after a gain (similarly with losses)
- $\Delta U = U(w + \text{gain}) - U(w)$

St. Peterburg Game (yet again!)

Taking into consideration the current wealth w:

$$\text{EU} = \sum_{i=1}^{\infty} \frac{log(w + 2^i - c) - log(w)}{2^i} < \infty$$

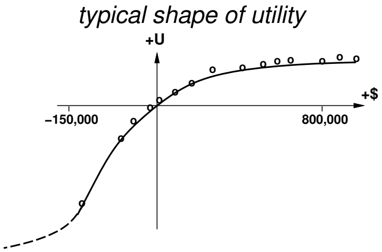
One should be willing to pay at most the maximum c such that EU is > 0:

- A millionaire would pay up 10.94$
- A person with 1000$ up to 5.94$

S-shaped Utility Functions

People usually are:

- Risk-averse wrt gains
- Risk-neutral around zero
- Risk-seeking wrt losses



For small amounts of money, linearity is a very good approximation.

What are Subjective Probabilities?

The probability of rain tomorrow is 30%

- *Frequentist* dogmatic interpretation: result of comparing current weather conditions to the past
- *Belief* dogmatic interpretation: probability expresses a subjective belief

Fundamental difference between the two approaches with practical consequences!

What are Subjective Probabilities?

What do you prefer?

- A1) Get 10 dollars if coin lands heads
- A2) Get 10 dollars if there is snow in Paris the 25 of December

If you prefer A1 to A2, it means that *your* probability of snow is less than 0.5.

Bets can be used to "objectify" personal beliefs.

What do you prefer?

- B1) Get 10 dollars if *two* coins land heads
- B2) Get 10 dollars if there is snow in Paris the 25 of December

If you prefer B1 to B2, it means that you believe that $p_{\text{snow}} < 0.25$.

A Thought Experiment

What do you prefer?

Let A be an uncertain event. You are given the choice between two possibilities:

- 1) Win $(1-q) \cdot 10\$$ if A occurs
- 2) Win $q \cdot 10\$$ if $\neg A$ occurs

You choose q but another person chooses between the two options. You get what the other person did not chose.

You should assign q so that options 1) and 2) are equally preferred: an expected utility maximizer derive

$$p(1-q) = (1-p)q$$

and therefore $p = q$.

If you consider a value q such that 1) is preferred to 2), then you should increase q.
(in theory one could *calibrate* probability values in this way)

Payoff of a bet

The stake S of a bet is the total dollar amount that is "on the table"

- I bet $b = pS$ on A
- You bet $S - b = S(1 - p)$ against me

My betting rate is $p = \frac{b}{S}$

	payoff for bet on A	payoff for bet on $\neg A$
A	$(1-p)S$	$-(1-p)S$
$\neg A$	$-pS$	pS

Payoff matrix

Example: if I bet 5$ on Snow on December 25th, and you bet 10$ on ¬Snow, then my $p = 0.33$, $S = 15$ and $b = 5$.

Fair Bets

Suppose A is an uncertain event. Consider:

- a bet on A at rate p: you win $(1-p)10\$$ if A
- a bet against A at rate $1-p$: you win $p10\$$ if $\neg A$

If betting rate p is fair, there is no advantage of betting one way or another.

The betting rate of $\neg A$ is the reverse of that of A.

Odds

Bookies advertise odds, not betting rates, but it is equivalent.

- Odds against A: $o_{\neg A} = \frac{1-p}{p}$ often written as $y : x$
- If the odds against your bet are o then your betting rate is $\frac{1}{1+o}$ (or equivalently $\frac{x}{x+y}$).

If you bet b on A at the odds $o_{\neg A}$:

- you gain $o_{\neg A} \cdot b$
- you lose the amount b if $\neg A$

Should personal probabilities be coherent ? Can I pick any number?

Dutch Book Example

Assume the following betting agency *(assuming exactly one and only one among A and B happen)* allows you to place the following bets:

event	odds against	betting ratios
A	3:1	0.25
B	1:2	0.66

If I bet 1$ on *A*, I gain 3 dollar if *A* happens and lose 1$ otherwise.
If I bet 1$ on *B*, I gain 0.5 dollar if *B* happens and lose 1$ otherwise.

I bet 3 dollars on A and 8 dollars on B

- in case of A: I win 9 and lose 8; gain of 1 dollar
- in case of B: I win 4 and lose 3; gain of 1 dollar

NOT MATTER what happen I am sure to win 1 dollar!

That's called a *Dutch book:* or *sure loss contract* (against the betting agency).

Personal Probabilities and Betting Rates

A group of beliefs can be represented by a set of betting rates

Game

Imagining advertising a set of betting rates; for each proposition A you offer the betting rate p_A.

In this imaginary game, you are prepared to bet:

- on A at betting rate p_A
- against A at betting rate $1 - p_A$

Real booking agencies only let you bet with their odds in "one direction". Moreover, they make money by adding an edge; for example if real odds against A are $3:1$ then:

- their betting rate on $A = \frac{1}{1+3+x}$ where x is the *edge*; and
- their betting rate on $\neg A = \frac{3}{3+1+x}$.

Dutch Book Theorem

1. Personal degrees of belief can be represented by betting rates (or equivalently by personal odds)
2. Personal betting rates should be coherent
3. A set of betting rates is coherent iff it satisfies the basic rules of probability

Therefore, personal degrees of belief should satisfies the basic rules of probability

Bruno de Finetti

" You should be consistent, otherwise you are going to give me all your money !"

Additivity

Let A and B be mutually exclusive events.

Betting rate on $A \vee B$ should be the sum of the betting rates of A and B

If this is not true, we can construct a Dutch book.

p, q, r are betting rates that do not satisfy additivity

- On A: p
- On B: q
- On $A \vee B$: $r < p + q$

Betting strategy:

- Bet p on A
- Bet q on B
- Bet $1 - r$ against $A \vee B$

In any case, the payoff is $r - p - q < 0$ (Similar observations can be made about conditional probabilities)

Manipulating Subjective Probabilities

The Importance of Bayes' Rule

For a frequentist, Bayes rule is just a rule.

But for subjective probabilities, it is the primary way to manipulate beliefs. That's why belief-type probability approaches are often called Bayesian !

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)} = \alpha\, P(X|Y)P(Y)$$
$$= \alpha\, \text{likelihood} \times \text{prior}$$

Useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

Rare Diaseases

Problem

- You have been tested positive for a dreadful disease; the disease is rare; prior $P(\text{sick}) = 0.00001$.
- The test has 99% of accuracy.
- Should you worry?

The test has 99% of accuracy $\implies$
$P(\text{positive}|\text{sick}) = P(\neg\text{positive}|\neg\text{sick}) = 0.99$.

$$P(\text{sick}|\text{positive}) = \frac{P(\text{positive}|\text{sick})P(\text{sick})}{P(\text{positive})} = \frac{0.99 \times 0.00001}{0.01} \approx 0.001$$

where

$$\begin{aligned} P(\mathit{positive}) &= P(\text{positive}, \text{sick}) + P(\text{positive}, \neg\text{sick}) = \\ &= P(\text{positive}|\text{sick})P(\text{sick}) + P(\text{positive}|\neg\text{sick})P(\neg\text{sick}) = \\ &= 0.99 \times 0.00001 + 0.01 \times (1 - 0.00001) \approx 0.01 \end{aligned}$$

Bayes' Rule and conditional independence

$$\begin{aligned}
&\mathbf{P}(Cavity \mid toothache \wedge catch) \\
&\quad = \alpha\, \mathbf{P}(toothache \wedge catch \mid Cavity)\mathbf{P}(Cavity) \\
&\quad = \alpha\, \mathbf{P}(toothache \mid Cavity)\mathbf{P}(catch \mid Cavity)\mathbf{P}(Cavity)
\end{aligned}$$

This is an example of a naive Bayes model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i \mid Cause)$$

Total number of parameters is **linear** in n

Where do Probabilities Come from?

The Bayesian approach:

1. assign self-consistent prior probabilities and
2. use Bayesian updating whenever new evidence arrives.

The Problem of Unknown Priors:

- Prior comes from prior knowledge, experts' opinion, ...
- Incorrect priors can be "washed out" given sufficient information: Bayes theorem accounts for both the prior probability and the observations
- It's possible to have distribution over probability values!

Laplace's principle of indifference

Outcomes that are "symmetric" should be accorded equal probability

Laplace's rule of succession

- Bernoulli trials (random experiment with outcomes “success” or “failure”) with unknown success probability p
- Repated n times independently, and get s successes
- What is the probability that the next repetition will succeed?

Laplace's solution: $\hat{p} = \frac{s+1}{n+2}$

What is the probability that the sun will rise tomorrow, given that it has risen every day for the past 5000 years ?

Laplace calculated the odds of 1826251:1 in favour of the sun rising tomorrow.

Bayesian interpretation: uniform prior for p between 0 and 1
Likelihood of observing s successes in n trials: $p^s(1-p)^{n-s}$
Posterior is a Beta distribution

Beta distribution

A Beta-distributed random variable x has pdf:

$$\begin{aligned} f(x;\alpha,\beta) &= \text{constant} \cdot x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \end{aligned}$$

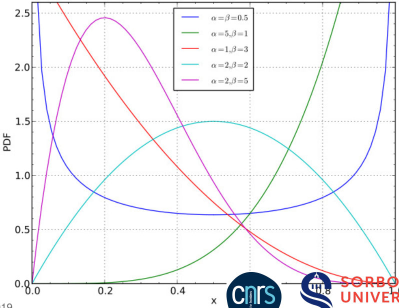
The Beta function $B(\alpha,\beta)$ can be seen as a normalization constant.

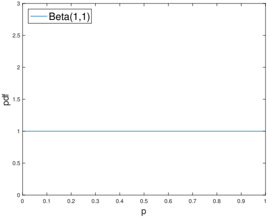
Notation: $X \sim \text{Beta}(\alpha,\beta)$

Expected value $\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$.

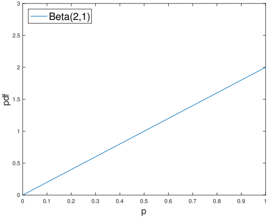
Parameters α, β are called *pseudocounts*

Beta(1,1) is the uniform distribution

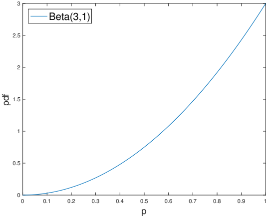




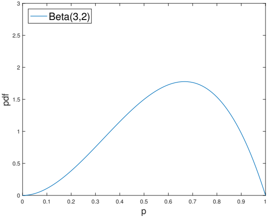
Start with Beta(1,1) (uniform)



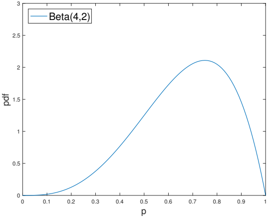
Observe sucess! $p \sim \text{Beta}(2, 1)$



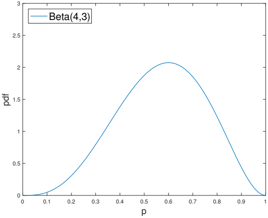
Success again! $p \sim \text{Beta}(3, 1)$



Failure $p \sim \text{Beta}(3, 2)$



Success $p \sim \text{Beta}(4, 2)$

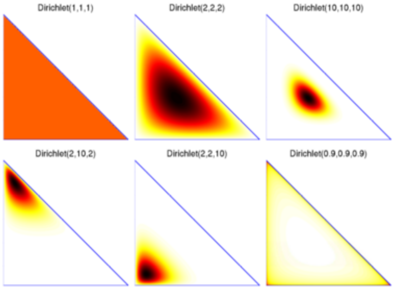


Failure Beta(4, 3)

Conjugate priors

If the posterior distribution is in the same familiy as the prior distribution

- Beta distribution is a conjugate prior for the Bernoulli distribution
- Dirichlet distribution is a conjugate prior for the categorical distribution



Dirichlet distribution of order k has density defined on the open $(k-1)$-dimensional simplex

(Used in Bayesian model-based reinforcement learning)

Eliciting Probabilities from Experts

- We ask an expert about the (unknown) probability p of an event A. The expert reports q.
- Should we trust the expert? How to give them the right incentive?
- Pay expert according to *scoring rule* $S(\cdot, q)$
  - Expert's payoff $S(A, q)$ if A happen
  - Expert's payoff $S(\neg A, q)$ if $\neg A$ happen
- Then the expected payoff for the forecaster is

$$\mathbb{E}_p[S(\cdot, q)] = pS(A, q) + (1 - p)S(\neg A, q)$$

  and expertt (if rational) reports $\arg\max_q \mathbb{E}_p[S(\cdot, q)$

- Problem: naive approach, as setting $S(A, q) = q$ and $S(\neg A, q) = 1 - q$ does not incentivate correct reporting

$$\mathbb{E}_p[S] = pq + (1 - p)(1 - q)$$

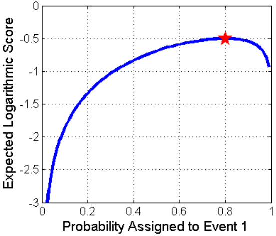
$\Longrightarrow$ Expert reports $q = 1$ if $p > 0.5$

Proper Scoring Rules

- A scoring rule is *proper* if the expert can maximize his expected payoff by reporting his true (subjective) probability.

$$p = \arg\max \mathbb{E}_{x\sim p}[S(x, q)]$$

- Many different forms, for example logarithmic



(Event happening with 0.8 probability)

Should we go to ski ?

Two choices: going to the Mountains or staying at Home.
Uncertain state of the slopes, either Snow or ¬Snow.

Outcome (Mountain, Snow) most preferred, (Mountain, ¬Snow) least preferred.

$u(\cdot)$	Snow	¬Snow
Mountain	10	-5
Home	1	1

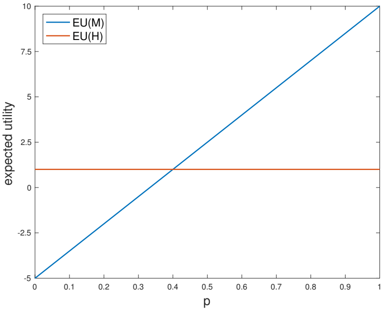
Assume $P(\text{Snow}) = 0.2$ (belief θ_0):

- $EU(\text{Mountain}) = 0.2 \cdot 10 + 0.8 \cdot (-5) = -2$
- $EU(\text{Home}) = 0.2 \cdot 1 + 0.8 \cdot 1 = 1$

EU depends on belief, we write $EU(\cdot;\theta)$ to explicit the dependence.

Let $p := P(snow)$; our belief θ contains the value of p

- $EU(\mathrm{M}; p) = 10p - 5(1 - p) = 15p - 5$
- $EU(\mathrm{H}; p) = 1$



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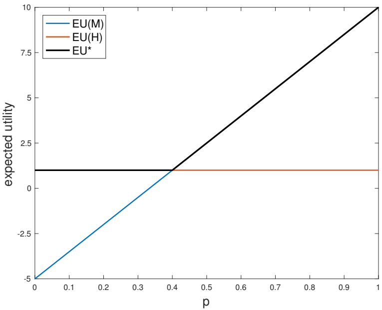
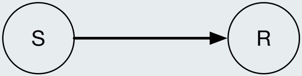


Decision rule: we should go to ski if $EU(M) > EU(H) \iff p > 0.4$

Snow bulletin



CPT of yes/no report given snow

- $P(yes|snow) = 0.6$, $P(no|snow) = 0.4$
- $P(yes|\neg snow) = 0.2$, $P(no|\neg snow) = 0.8$

From an initial prior $p = 0.2$, update the value of p given a Report of snow:

$$P(snow|yes) = \frac{P(yes|snow)P(snow)}{\sum_x P(yes|x)P(x)} = \frac{0.12}{0.12 + 0.16} = 0.43$$

The "new" value of p in the updated belief is 0.43

Given this evidence it is now recommended to go to the mountains (as p is higher than the threshold value).

Value of Information

What is the benefit of acquiring additional information ?

- With information, one's course of action can be changed to suit the *actual* situation.
- Without information, one has to do what's best on average over the possible situation
- The value of a given piece of information is the defined to be *the difference in expected value between best actions before and after* the information is obtained.

Expected Value of Perfect Information (EVPI) is the expected value of acquiring perfect information about the value of that variable

Simplest case: uncertainty *on a single variable* (scenario)

- Decisions belong to decision set D
- Uncertainty on random variable Y given by belief $P^{\theta}(Y)$
- Utility depends on the chosen d and on the realization of Y: $U : D \times Y \rightarrow \mathbb{R}$

Expected utility $\mathrm{EU}(d) = \sum_y P^{\theta}(Y{=}y)U(d,y)$
EU of best choice: $\mathrm{EU}(d^*) = \max_d \sum_y P^{\theta}(Y{=}y)U(d,y)$

EVPI Formulation

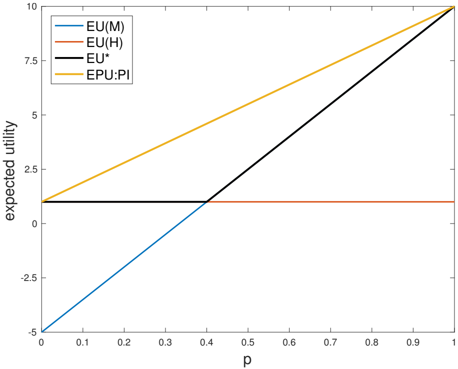
EVPI(Y), expected value of perfect information on variable Y

$$\begin{aligned} EVPI(Y) &= \sum_y P^{\theta}(Y{=}y) \max_d U(d,y) - EU(d^*) \\ &= \mathrm{EPU}(Y) - EU(d^*) \end{aligned}$$

EPU stands for Expected Posterior Utility given perfect information on Y.

- What's the value of a "perfect" snow bulletin report ?
- With perfect information, we would always do the true best thing

$\mathrm{EPU}_{\theta|PI(\mathrm{snow})}$: expected posterior utility given perfect information about variable Snow (also called *expected value of clairvoyance*)



$EVPI = \mathrm{EPU}_{\theta|PI} - EU^*$
$\mathrm{EPU} = 2.8$ for $p = 0.2 \rightarrow EVPI(\mathrm{snow}) = 1.8$ *Information value high for "middle" values of p*

EVOI and EVPI

In general, there may be several variables of uncertainty and you may observe only one (or few) of them

- Perfect information (EVPI) assumes that we can observe the true value of the random variable of interest
- In general you may access some "noisy" observation, a random variable O
- Information (even noisy) can improve the quality of our action
- You need to update your belief $P(\cdot|O)$ accordingy, with respect to the assumed "response model"

Computation of "noisy" EVOI can be reframed as computing the EVPI of some new artificial variable modeling the obsevation

Note: no source of information can be worth more than the value of perfect information.

- Possible outcomes S, distributed acc. to $P_d(x)$ inducing $P(U_d)$;
- θ is the current "belief" representing the set of current probabilistic estimates $P_d(\cdot)$ *(so that we don't need to make explicit conditioning on all previous observations).*

$$\mathrm{EU}_\theta(d) = \sum_{x \in S} U(x)\ P_d(x)$$

- Perform the best action $d^*_\theta = \arg\max_d EU(d)$.
- Random variable representing value (of yet to be made) observation O. Suppose we knew $O = o$, then we would choose d^*_o s.t.

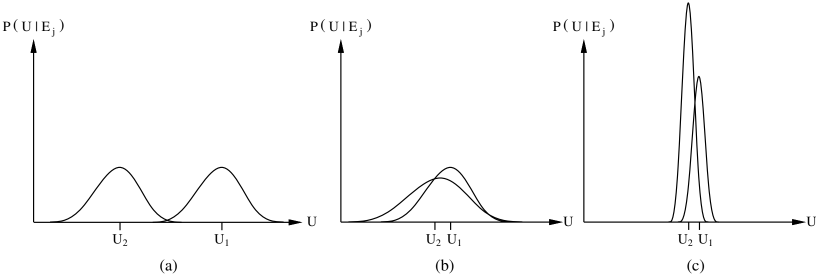
$$\mathrm{EU}_\theta(d^*_o | O = o) = \max_d \sum_x U(x)\ P_d(x | O = o)$$

EVOI General Formulation

O is a random variable whose value is *currently* unknown
$\Longrightarrow$ must compute expected gain over all possible values:

$$EVOI_\theta(O) = \left(\sum_o P(O = o)\, \mathrm{EU}(d^*_o | O = o) \right) - EU_\theta(d)$$

Qualitative behaviors



1. *(left)* Choice is obvious, information worth little
2. *(middle)* Choice is nonobvious, information worth a lot
3. *(right)* Choice is nonobvious, information worth little

Slides from the website of AIMA book of S. Russell and P. N

Exercise

Buying oil drilling rights.

- k blocks $B_1, \ldots, B_n$, exactly one has oil, worth C.
- Prior probabilities $\frac{1}{n}$ each, mutually exclusive.
- Current price of each block is C/n.

"Consultant" offers accurate survey of B_1. What is the fair price?

Compute *expected value of information*, that is:

expected value of best action given the information $-$ expected value of best action without information.

Taken from AIMA book of S. Russell and P. Norvig (Chapter 16)

Solution

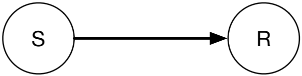
- Survey say “oil in B_1” with **probability** $\frac{1}{n}$
  - Best action is to buy B_1
  - Value is $C - \frac{C}{n}$ (gain minus cost) $= \frac{(n-1)C}{n}$
- Survey says “no oil in B_1” with **probablity** $\frac{n-1}{n}$
  - Best action: buy one randomly in $B_2, \ldots, B_n$
  - Expected value: $\frac{C}{n-1} - \frac{C}{n} = \frac{C}{n(n-1)}$
- Expected profit given survey information

$$\frac{1}{n} \cdot \frac{(n-1)C}{n} + \frac{n-1}{n} \frac{C}{n(n-1)} = \frac{C}{n}$$

- Expected value of information: *expected value of best action given the information* − *expected value of best action without information*

$$\frac{C}{n} - 0 = \frac{C}{n}$$

Expected Value of “noisy” information



Snow bulletin

CPT of yes/no report given snow

- $P(\text{yes}|\text{snow}) = 0.6$; $P(\text{no}|\text{snow}) = 0.4$
- $P(\text{yes}|\neg\text{snow}) = 0.2$; $P(\text{no}|\neg\text{snow}) = 0.8$

Current belief θ: $P(\text{snow}) = p = 0.2$.
Best action in belief θ: Home with $\text{EU}(H) = 1$

$$\text{EPU} = P^{\theta}(\text{Report}=\text{yes}) \max_d EU^*(d|\text{yes}) + P^{\theta}(\text{Report}=\text{no}) \max_d EU^*(d|\text{no})$$

Case 1) the report says there will be snow:

$$P^{\theta}(yes) = P(\text{yes}|\text{snow})P(\text{snow}) + P(\text{yes}|\neg\text{snow})P(\neg\text{snow}) = 0.28$$

$$P(\text{snow}|\text{yes}) = \frac{P(yes|snow)P(snow)}{\sum_x P(yes|x)P(x)} = 0.43$$

$$\text{EU}(M|\text{yes}) = 15 \cdot P(\text{snow}|\text{yes}) - 5 = 1.45; \quad \text{EU}(H|\text{yes}) = 1$$

Case 2) the report says there won't be snow:

$$P^{\theta}(no) = P(\text{no}|\text{snow})P(\text{snow}) + P(\text{no}|\neg\text{snow})P(\neg\text{snow}) = 0.72$$

$$P(\text{snow}|\text{no}) = \frac{P(no|snow)P(snow)}{\sum_x P(no|x)P(x)} = 0.13$$

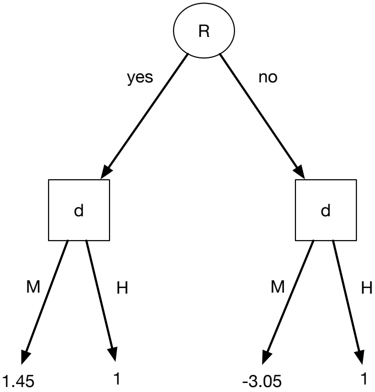
$$\text{EU}(M|\text{no}) = 15 \cdot P(\text{snow}|\text{no}) - 5 = -3.05; \quad \text{EU}(H|\text{no}) = 1$$

Expected posterior utility:

$$\begin{aligned} \text{EPU} =& P^{\theta}(\text{yes}) \cdot \max[\text{EU}(M|\text{yes}), \text{EU}(H|\text{yes})] + \\ & P^{\theta}(\text{no}) \cdot \max[\text{EU}(M|\text{no}), \text{EU}(H|\text{no})] \\ =& 0.28 \cdot \max[1.45, 1] + 0.72 \cdot \max[-3.05, 1] = 1.126 \end{aligned}$$

EVOI is EPU minus EU^*_θ (value of initial best decision): $1.126 - 1 = 0.126$
(cfr. with EVPI(Snow)=1.8)

Decision Tree View



Gambling with Biased Coins

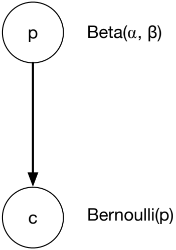
- You are offered to gamble with a biased coin; win 12$ if head, loose 12$
- The probability p of head is unknown
- Should you accept to gamble?
- Assuming risk-neutrality (utility approximately linear in monetary outcoumes)
  expected profit of gambling $\text{EU}(gamble; p) = 24p - 12$
- With uniform prior, $\hat{p} = 0.5$ and $\text{EU}(g) = 0$
- Other priors?

Now suppose your friend offers you another deal:

- He'll flip the coin once and let you see the result before you whether you accept to gamble, but *you can't win anything on this first flip.*

Question: How much should you pay to see one flip?

Credit: `http://lesswrong.com/lw/85x/value_of_information_`[illegible]`_examples/`



Coin with unknown probability $p \sim Beta(\alpha, \beta)$
We assume $\alpha = 1$, $\beta = 1$: uniform prior on p; density $P(p) = 1$.

Analysis a-posteriori:

- After seeing head:
  $p \sim Beta(2, 1)$ and density $P(p) = 2p$
  expected profit of gambling

  $$\mathbb{E}_p[\text{EU}(\text{gamble}; p)] = \int_0^1 (24p - 12)(2p)dp = 4$$

  best action is to accept to gamble
- After seeing tail:
  $p \sim Beta(1, 2)$ and $P(p) = 2 - 2p$
  expected loss is 4\$
  best action is NOT to gamble
- $\text{EVOI} = 0.5 \cdot 4 + 0.5 \cdot 0 - 0 = 2\$$

Properties of EVOI

$$EVOI_\theta(O) = \left(\sum_o P(O=o)\,\mathrm{EU}(d_o^* | O=o) \right) - EU_\theta(d)$$

Nonnegative—in **expectation**, not **post hoc**

$$\forall\, O, \theta \quad \mathrm{EVOI}_\theta(O) \geq 0$$

Nonadditive—consider, e.g., obtaining E_j twice

$$\mathrm{EVOI}_\theta(O_j, O_k) \neq EVOI_\theta(O_j) + EVOI_\theta(O_k)$$

Order-independent

$$\begin{aligned} \mathrm{EVOI}_\theta(O_j, O_k) &= EVOI_\theta(O_j) + EVOI_{\theta, O_j}(O_k) \\ &= EVOI_\theta(O_k) + EVOI_{\theta, O_k}(O_j) \end{aligned}$$

Note: when more than one piece of evidence can be gathered, maximizing EVOI for each to select one is not always optimal (it is *myopically*)
$\Longrightarrow$ evidence-gathering becomes a **sequential** decision problem

- **utility of decision** d (random var) U_d; distributed according to $Pr(U_d)$
- **expected utility**

$$\mathrm{EU}(d) = \mathbb{E}_{Pr(U_d)}\left[U_d\right] = \int_u u\, Pr(U_d = u)\, du$$

- **current best action/decision**

$$d^* = \arg\max_{d \in \mathcal{D}} \mathrm{EU}(d);\ \mathrm{EU}^* = \max_{d \in \mathcal{D}} \mathrm{EU}(d)$$

- **conditional expected utility**

$$\mathrm{EU}(d|O = o) = \mathbb{E}_{Pr(U_d|o)}\left[U_d\right] = \int_u u\, Pr(U_d = u | O = o)\, du$$

- **expected posterior utility**

$$\mathrm{EPU}(O) = \sum_{o \in Dom(O)} Pr(O = o) \max_d \mathrm{EU}(d|o)$$

- **expected value of information** $\mathrm{EVOI}(O) = \mathrm{EPU}(O) - \mathrm{EU}^*$.

Exercise: Prove that the expected value of information is always non negative.

Assume current best action is $d^* = \arg\max_{d \in \mathcal{D}} \mathrm{EU}(d)$ where EU is calculated wrt utility distribution $Pr(U_d)$ of utility of decision d (implicitly conditioned to all previous evidence).

Note tat you can always stick to the choice made before making the observation. We show that the expected utility obtained by not switching, is the same as the expected utility $\mathrm{EU}(d^*)$ before making the observation.

$$\sum_{o \in O} P(O{=}o)\, \mathrm{EU}(d^*|o) = \mathrm{EU}(d^*).$$

Then, as the formula of EVOI allow to make a change *after* an observation is made, EVOI should necessarily be at least as much this value.

$$\mathrm{EPU}(O) = \sum_{o \in Dom(O)} P(O{=}o) \max_d \mathrm{EU}(d|o) \geq \sum_o P(O{=}o)\, \mathrm{EU}(d^*|o) = \mathrm{EU}(d^*).$$

Therefore $\mathrm{EVOI}(O) = \mathrm{EPU}(O) - \mathrm{EU}^* = \mathrm{EPU}(O) - \mathrm{EU}(d^*) \geq 0$

If we pick the same action, EU is the same

$$
\begin{aligned}
\sum_{o \in O} P(O = o)\,\mathrm{EU}(d|o) &= \sum_{o \in O} Pr(O = o) \int_u Pr(U_d = u | O = o) u\, du \\
&= \int_u \sum_o P(O = o) P(u | O = o) u\, du \\
&= \int_u \sum_o P(O = o, U_d = u) u\, du \\
&= \int_u P(U_d = u) u\, du \\
&= \mathrm{EU}(d)
\end{aligned}
$$

Note: the conditional expected utility of a decision given a *particular* observation $\mathrm{EU}(d|O = o)$ can be different from $\mathrm{EU}(d)$!

Exercise: Prove that the EVOI is strictly positive (>0) if and only if the probability of "switching" decision is >0.

Recall the definition of EVOI.

$$\begin{aligned}
\mathrm{EVOI}(O) &= \sum_{o \in Dom(O)} P(O=o) \max_{d \in \mathcal{D}} \mathrm{EU}(a|o) - \mathrm{EU}^* \\
&= \sum_{o \in Dom(O)} P(O=o) \Big(\max_{d \in \mathcal{D}} \mathrm{EU}(d|o) - \mathrm{EU}(d^*|o) \Big) \\
&= \sum_{o \in Dom(O)} P(O=o) \max_{d \in \mathcal{D}} \Big[\mathrm{EU}(d|o) - \mathrm{EU}(d^*|o) \Big]
\end{aligned}$$

- Let d^*_o be the optimal choice wrt $Pr(U_d|O=o)$.
- We may switch decision if $\exists o \in Dom(O)$ such that $Pr(O=o) > 0$ and $\mathrm{EU}(d^*_o|o) = \max_d \mathrm{EU}(a|o) > \mathrm{EU}(d^*|o)$.
- In this case, it follows $\mathrm{EVOI} > 0$.
- Otherwise, if $d^*_o = d \; \forall o \in Dom(O)$ with non null $P(O=o)$, it follows $\mathrm{EVOI} = 0$.

Conclusions

- Decision-making with expected utility and subjective probabilities
- Fundations of subjective probabilities (Dutch book), elicitation of probabilities
- How to be a "Perfect" Bayesian:
  choose a prior, update with Bayes formula, decide when and what to ask with value of information

Some References

- Ian Hacking. An Introduction to Probability and Inductive Logic. Cambridge University Press, 2001.
- Martin Peterson. An Introduction to Decision Theory. Cambridge University Press, Second Edition, 2017.
- Stuart Russell and Peter Norvig. Artificial Intelligence: A Modern Approach. Prentice Hall, 2010.