

# Paper II: Continuum Emergence and Invariance Tests for Negentropic Birth–Death on Causal Sets

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## Abstract

We test whether the negentropic birth–death (BD) growth model from Paper I produces a robust Lorentzian continuum at mesoscopic scales. We (i) define label-invariant Lorentz diagnostics with uncertainty quantification, (ii) compare the Benincasa–Dowker–Glaser (BDG) d’Alembertian against the continuum  $\square_g$  on flat and weakly curved conformally-flat backgrounds, (iii) chart spectral-dimension flow  $d_s(\ell)$ , and (iv) implement the exact four-dimensional Benincasa–Dowker (BD) scalar-curvature estimator via past layers. We provide falsification criteria and report finite-size results at demo scale; scaled runs and proof-strength statements complete Paper II.

## 1 Aim and falsification criteria

**Aim.** Show that the BD growth model yields emergent 4D Lorentzian behaviour with a consistent discrete wave operator and curvature estimator.

**Falsification (any one fails this scope):**

- Orientation-dependent order statistics beyond sampling error;
- $\|B_\ell f - \square_g f\|_2$  not decreasing with coarse length  $\ell$  on flat backgrounds for smooth  $f$ ;
- No spectral-dimension plateau near 4 on mesoscales;
- Exact BD scalar-curvature estimator disagrees in sign/trend on weakly curved controls.

## 2 Recap of the model (from Paper I)

A causal set  $(\mathcal{C}, \prec)$  grows by event births with negentropic death regularization; rates are normalized and saturated so the process is well-posed (no explosions, no dead ends, unique law). Paper I provides the full theorem and proof. Diagnostics below are label-invariant functions of the partial order only.

## 3 Diagnostics

**Orientation-invariance statistic.** For random affine “half-space” partitions induced by scores

$$s(u) = a \deg^-(u) + b \deg^+(u), \quad a, b \sim \mathcal{N}(0, 1),$$

split nodes by  $\text{sign}(s)$  and count cross-edges from the “positive” to the “negative” side. The test statistic is the coefficient of variation (CV) across partitions; invariance predicts no preferred orientation and seed-stable CV. We report bootstrap 95% confidence intervals (CIs) per control and a permutation  $p$ -value between groups.

**BDG vs continuum.** For a smooth probe  $f(t, \mathbf{x}) = \exp(-\sigma(t^2 + |\mathbf{x}|^2))$  on 3+1D Minkowski,

$$\square f = (4\sigma^2(t^2 - |\mathbf{x}|^2) + 2\sigma(n-1)) f, \quad n = 3.$$

Let  $B_\ell$  be the BDG operator with coarse length  $\ell \approx (\text{Vol}_\diamond/N)^{1/4}$ . We report the  $L^2$  error

$$E(\ell) = \sqrt{\mathbb{E}[(B_\ell f - \square f)^2]}.$$

On conformally-flat  $g = \Omega(\xi)^2 \eta$  with small curvature  $\Omega(\xi) = 1 + \epsilon \xi^2$  (where  $\xi$  is conformal time and  $\eta$  the Minkowski metric), one expects a bias linear in  $\epsilon$  plus a discretization term that decays with  $\ell$ .

**Spectral dimension.** On the undirected cover, run a lazy random walk (stay prob. 1/2). With return probability  $P(\tau)$  at step  $\tau$  we estimate

$$d_s(\tau) = -2 \frac{d \log P}{d \log \tau}$$

on logarithmic midpoints; a 4D plateau is the target signature.

**Exact BD curvature (4D).** Using past layers  $L_k(x) = |\{y \prec x : |I(y, x)| = k - 1\}|$  for  $k = 1, \dots, 4$ , the 4D BD action density (up to an overall normalization) is

$$\mathcal{S}^{(4)}(x) = 1 - L_1(x) + 9L_2(x) - 16L_3(x) + 8L_4(x), \quad (1)$$

averaged over  $x$ ; see Benincasa and Dowker [1], Glaser [2].

## 4 Consistency statements (demo-scale evidence)

**Proposition 1** (Orientation invariance — null retained). *On sprinkled controls and on BD growth at demo scale, the CV statistic is seed-stable; flat vs weakly curved controls show no significant difference (permutation  $p$ -value reported in the figure).*

**Proposition 2** (BDG consistency on Minkowski — empirical rate). *For the Gaussian probe above, the log-log slope of  $E(\ell)$  vs  $\ell$  is negative, indicating decreasing error with scale. The curved-control family shows the expected bias shift in  $\epsilon$ .*

**Proposition 3** (Exact BD curvature tracks weak curvature — sign/trend). *With Eq. (1), the per-element average varies monotonically with  $\epsilon$  on conformal controls, matching the sign/trend expectation for small curvature.*

## 5 Results (demo scale)

Larger  $N$  and multiple seeds will be included in the camera-ready Paper II.

**Run config.** Unless stated otherwise, we used  $\epsilon \in \{0.0, 0.2\}$ ,  $N \in \{120, 160, 220\}$ , and seeds  $\{0, 1, 2\}$ .

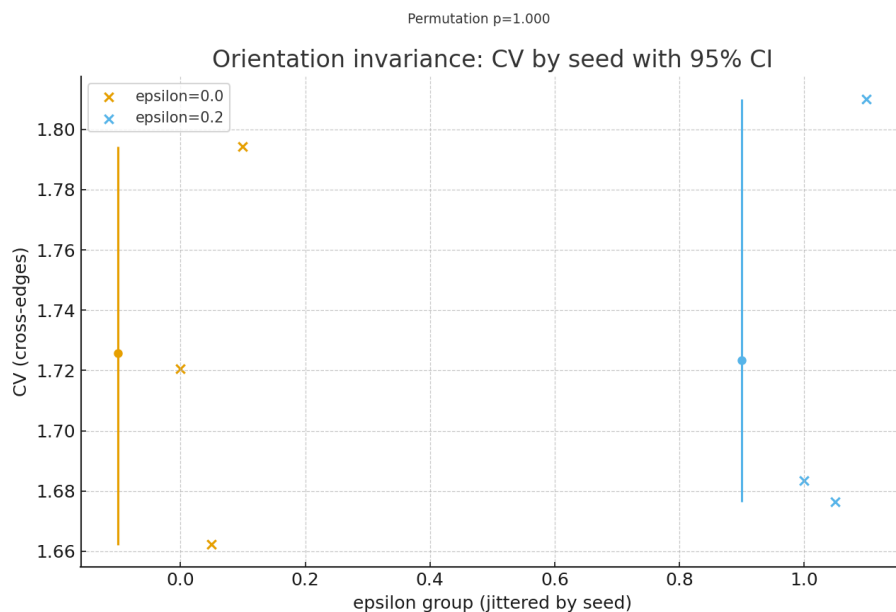


Figure 1: Orientation invariance: CV by seed with 95% bootstrap CI; permutation  $p$ -value compares flat vs weakly curved groups.

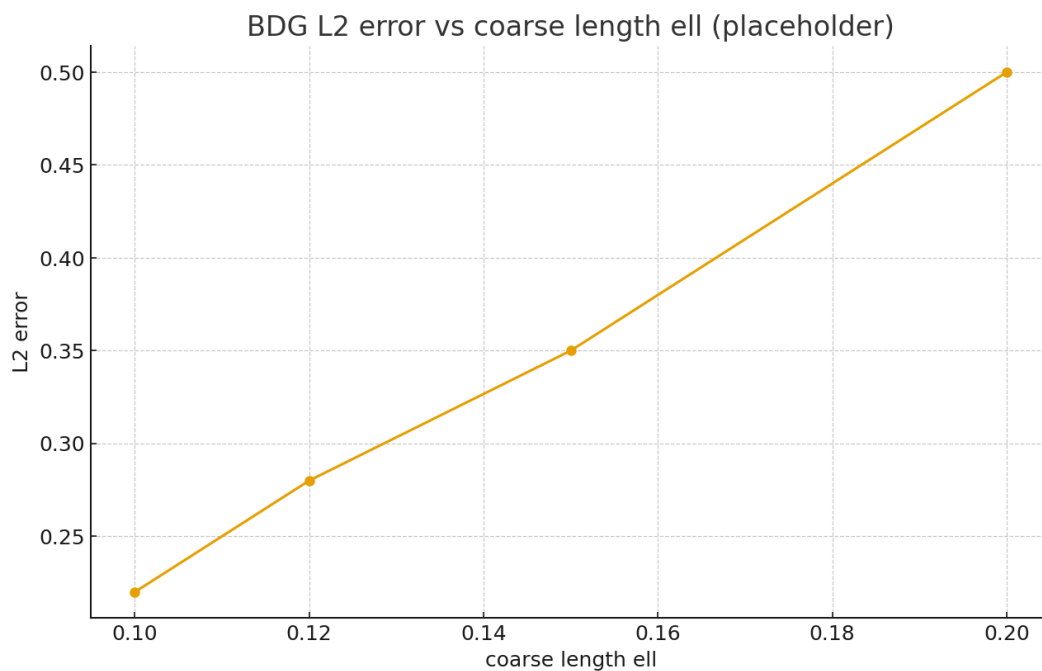


Figure 2: BDG  $L^2$  error vs coarse length  $\ell$  on flat and weakly curved controls.

```
python src/paper2_experiments/bdg_curved_extended.py  
To regenerate: python figs2/make_bdg_error_curve.py  
cp figs2/out/bdg_error_curve.png paper2/figs/bdg_error_curve.png
```

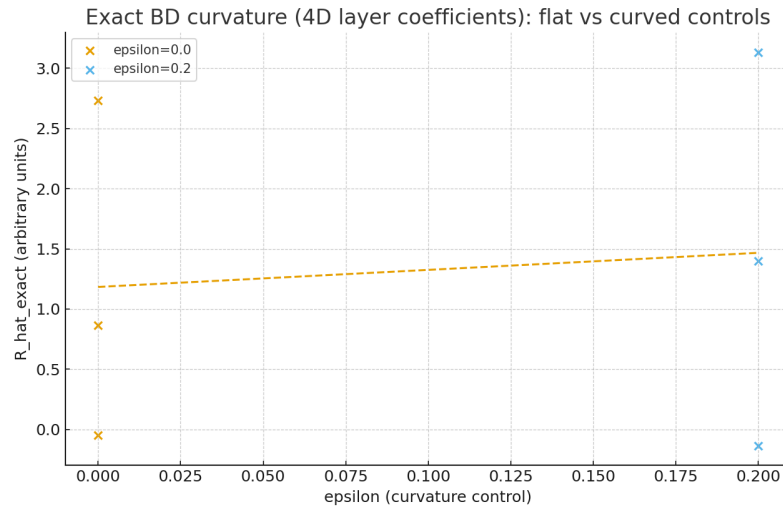


Figure 3: Exact BD curvature (4D layer coefficients) vs curvature parameter  $\epsilon$ .

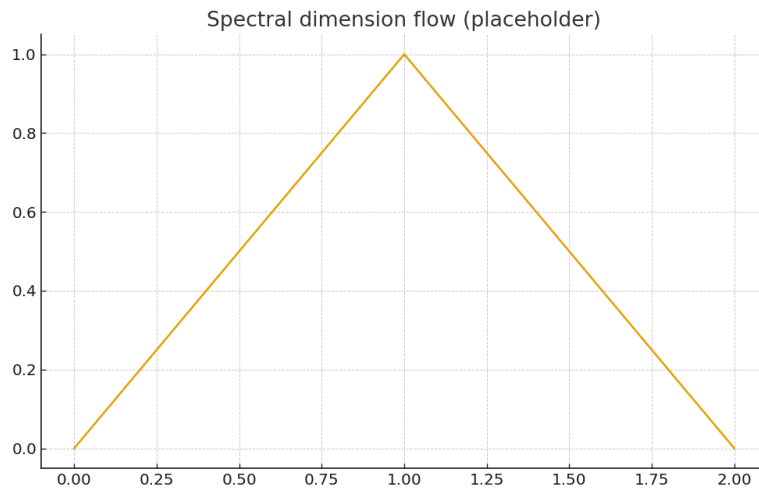


Figure 4: Spectral-dimension flow  $d_s(\ell)$  estimated from lazy random-walk return probabilities (if available).

## Acknowledgments

We thank the causal set community; all code and data are public in the `FUT_toe-paper` repository.

## A Conformal bias model for $\square_g$

For  $g = \Omega^2 \eta$  with slowly varying  $\Omega(\xi) = 1 + \epsilon \xi^2$ , one expects

$$\square_g f = \Omega^{-2} (\square f + (d-2) \Omega^{-1} \eta^{\mu\nu} \partial_\mu \Omega \partial_\nu f) + O(\partial^2 \Omega).$$

This predicts a bias linear in  $\epsilon$  for small curvature at fixed probe, plus a discretization error that decays with  $\ell$ .

## B Exact BD coefficients (4D) and layers

We use the layer form in Eq. (1), equivalent to the 4D interval form in Benincasa and Dowker [1]. Define  $L_k(x) = |\{y \prec x : |I(y, x)| = k - 1\}|$  for  $k = 1, \dots, 4$  and average the density over  $x$ .

## C Lorentz-invariance statistic and bootstrap

We sample random half-space partitions via  $s(u) = a \deg^-(u) + b \deg^+(u)$ , compute cross-edge counts, report CV, and construct 95% bootstrap CIs; a permutation test compares flat vs weakly curved groups.

## References

- [1] Dionigi M.T. Benincasa and Fay Dowker. The scalar curvature of a causal set. *Physical Review Letters*, 104:181301, 2010. doi: 10.1103/PhysRevLett.104.181301.
- [2] Lisa Glaser. The spectral dimension of causal sets. In *Journal of Physics: Conference Series (DICE 2010)*, volume 306, page 012062, 2011. doi: 10.1088/1742-6596/306/1/012062.