Assignment - 11

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Q-1

(a)

For poisson distribution, we have

$$E[X] = \lambda$$

Now to obtain the method of moments estimator we simply equate the first population mean to the first sample mean.

$$E[X] = \bar{x} \quad \lambda = \bar{x}$$

After solving,
we obtain the method of moments estimator

$$\hat{\lambda} = \bar{x}$$

$$\hat{\lambda} = \bar{x} = \frac{1}{4}(1+2+4+2) = 2.25$$

(b) $x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 2$

$$L(\lambda) = \prod_{i=1}^{n} f(x_i \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{\lfloor x_i}$$

$$= \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} \lfloor x_i}$$

$$\log L(\lambda) = \left(\sum_{i=1}^{n} x_i\right) \log \lambda - n\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{d}{d\lambda} \log L(\lambda) = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0$$

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\frac{d^2}{d\lambda^2} \log L(\lambda) = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2} < 0$$

we have the estimator

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{4}(1+2+4+2)$$

$$= 2.25$$

(c) we use the invariance property of the MLE.

$$\hat{P}[X=4] = \frac{\hat{\lambda}^4 e^{-\hat{\lambda}}}{4!}$$

is the max likelihood estimator for $P(X=4)$

$$\hat{P}[X=4] = \frac{\hat{\lambda}^4 e^{-\hat{\lambda}}}{4!} = \frac{2.25^4 e^{-2.25}}{4!}$$

$$= 0.1124$$

Q.2 Since we are estimating two parameters we will need two population and sample moments

$$E[X] = \theta$$

$$E[X^2] = Var[X] + (E[X])^2 = \sigma^2 + \theta^2$$

$$\overline{x^2} = \frac{1}{n} \sum_{i=1}^{n} x_i^2$$

$$E[X] = \overline{x}$$

$$E(X^2) = \overline{x^2}$$

For that example, $\theta = \overline{x}$

$$\sigma^2 + \theta^2 = \overline{x^2}$$

Solving this system of equations for θ and σ^2

$$\hat{\theta} = \overline{x}$$

$$\hat{\sigma}^2 = \overline{x^2} - (\overline{x})^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

Q3 The first moment is not useful because it is not a function of σ^2

$$E[X] = 1$$

As a result, we instead use the second moment

$$E[X^2] = Var[X] + (E[X])^2$$

$$= \sigma^2 + 1^2 = \sigma^2 + 1$$

We equate this second population moment to the second population moment

$$\overline{X^2} = \frac{1}{n} \sum_{i=1}^{n} x_i^2$$

$$E[X^2] = \overline{X^2}$$

$$\sigma^2 + 1 = \overline{X^2}$$

Now solving for σ^2, we obtain

$$\hat{\sigma}^2 = \frac{1}{n} \left(\sum_{i=1}^{n} x_i^2 \right) - 1$$

Q-7

$$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} x_i^{(1-\theta)/\theta}$$

$$= \theta^{-n} \left(\prod_{i=1}^{n} x_i \right)^{\frac{1-\theta}{\theta}}$$

$$\log L(\theta) = -n \log \theta + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \log x_i$$

$$= -n \log \theta + \frac{1}{\theta} \sum_{i=1}^{n} \log x_i$$

$$\frac{d}{d\theta} L(\theta) = \frac{-n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} \log x_i = 0$$

$$\theta = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

$\hat{\theta} > 0 \quad \log x_i < 0$

Since $0 < x_i < 1$

$$\frac{d^2}{d\theta^2} \log L(\theta) = \frac{n}{\theta^2} + \frac{2}{\theta^3}$$

$$\sum_{i=1}^{n} \log x_i$$

$$\frac{d^2}{d\theta^2} \log L(\hat{\theta}) = \frac{n}{\hat{\theta}^2} + \frac{2(-n\hat{\theta})}{\hat{\theta}^3}$$

$$= \frac{n}{\hat{\theta}^2} - \frac{2n}{\hat{\theta}^2} = -\frac{n}{\hat{\theta}^2} < 0$$

$$\hat{\theta} = \frac{-1}{n} \sum_{i=1}^{n} \log x_i$$

$$= \frac{-1}{4} \log(0.10 \times 0.22 \times 0.54 \times 0.36)$$

$$= 1.3636$$

(b) $E(X) = \int_0^1 u \frac{1}{\theta} u^{1-\theta/\theta} \, du = \frac{1}{\theta+1}$

$$E[X] = \bar{x}$$

$$\frac{1}{\theta+1} = \bar{x}$$

$$\tilde{\theta} = \frac{1-\bar{x}}{\bar{x}}$$

$$\bar{x} = \frac{1}{4}(0.10 + 0.22 + 0.59 + 0.36)$$

$$= 0.305$$

$$\tilde{\theta} = \frac{1-\bar{x}}{\bar{x}} = 2.707$$

Q-5 $L(\theta) = \prod_{i=1} \frac{\theta}{x_i^2}$ $0 < \theta \le x_i$ for x_i

$= \frac{\theta^n}{\prod_{i=1}^n x_i^2}$ $0 < \theta \le \min\{x_i\}$

$\log L(\theta) = n \log \theta - 2 \sum_{i=1} \log x_i$

$\frac{d}{d\theta} \log L(\theta) = \frac{n}{\theta} > 0$

$\hat{\theta} = \min \{x_i\}$

Q-6

(a) $L(\alpha) = \prod_{i=1} f(x_i; \alpha)$

$= \prod_{i=1}^n \alpha^{-2} x_i e^{-x_i/\alpha}$

$= \alpha^{-2n} \left(\prod_{i=1}^n x_i \right) \exp\left(\frac{\sum_{i=1} x_i}{\alpha} \right)$

Instead of directly maximizing the likelihood, we instead maximize the log-likelihood

$\log L(\alpha) = -2n \log \alpha + \sum_{i=1} \log x_i - \frac{\sum_{i=1} x_i}{\alpha}$

$\frac{d}{d\alpha} \log L(\alpha) = -\frac{2n}{\alpha} + \frac{\sum_{i=1} x_i}{\alpha^2}$

We set this derivatives equal to zero,

then solve for α

$$-\frac{2n}{\alpha} + \frac{\sum_{i=1}^{n} u_i}{\alpha^2} = 0$$

Solving gives our estimator, which we

denote with a hat

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} u_i}{2n} = \frac{\bar{x}}{2}$$

$$\hat{\alpha} = \frac{0.25 + 0.75 + 1.5 + 2.5 + 2.0}{2.5}$$

$= 0.7$

(b) $u_1 = 0.25$, $u_2 = 0.75$, $u_3 = 1.5$,

$u_4 = 2.5$, $u_5 = 2.0$

$$f(x|\theta) = \frac{1}{\theta} e^{-u/\theta}, \quad u > 0, \theta > 0$$

$$E[x^k] = \int_0^{\infty} \frac{u^k}{\theta} e^{-u/\theta}$$

$= \lfloor k \, \theta^k$

$$E(x) = \int_0^{\infty} u \cdot \alpha^{-2} u e^{-u/\alpha} du$$

$$= \frac{1}{\alpha} \int_0^{\infty} \frac{u^2}{\alpha} e^{-u/\alpha} du = \frac{1}{\alpha}(2\alpha^2)$$

$= 2\alpha$

$$2\alpha = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\bar{\alpha} = \frac{\sum_{i=1}^{n} x_i}{2n} = \frac{\bar{x}}{2}$$

$$\bar{\alpha} = \frac{0.25 + 0.75 + 1.5 + 2.5 + 2.0}{2.5}$$

$$= 0.7$$

Now that, in that case, the MLE and MoM estimates are the same.

Q-7

$$L(\beta) = \prod_{i=1}^{n} f(x_i; \beta) = \prod_{i=1}^{n} \frac{1}{2\beta^3} x_i^2 e^{-x_i/\beta}$$

$$= 2^{-n} \beta^{-3n} \left(\prod_{i=1}^{n} x_i^2 \right) \exp\left(- \frac{\sum_{i=1}^{n} x_i}{\beta} \right)$$

$$\log L(\beta) = -n \log 2 - 3n \log \beta + \sum_{i=1}^{n} \log x_i^2 - \frac{\sum_{i=1}^{n} x_i}{\beta}$$

$$\frac{d}{d\beta} \log L(\beta) = -\frac{3n}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2}$$

$$\frac{-3n}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2} = 0$$

$$\beta = \frac{\sum_{i=1}^{n} x_i}{3n} = \frac{\bar{x}}{3}$$

$$\beta = \frac{2+4+7.9+3}{3.4} = 1.378$$

(b) $E(X) = \int_0^\infty x \frac{1}{2\beta^3} x^2 e^{-x/\beta} dx$

$$= \frac{1}{2\beta} \int_0^\infty \frac{x^3}{\beta} e^{-x/\beta} dx$$

$$= \frac{1}{2\beta^2} \cdot 6(\beta) = 3\beta$$

$$E(X) = \bar{x}$$

$$3\beta = \sum_{i=1}^{n} \frac{x_i}{n}$$

Solving for β, $\beta = \frac{\sum_{i=1}^{n} x_i}{3n} = \frac{\bar{x}}{3}$

$$\beta = \frac{2+4+7.5+3}{3.4} = 1.375$$

Q.9 $L(\alpha) = \frac{2^n}{\alpha^n} \left\{ \prod_{i=1}^{n} y_i \right\} \exp\left\{ -\frac{1}{\alpha} \sum_{i=1}^{n} y_i^2 \right\}$

$$\log L(\alpha) = n \log 2 - n \log \alpha + \sum_{i=1}^{n} \log y_i - \frac{1}{\alpha} \sum_{i=1}^{n} y_i^2$$

$$\frac{d}{d\alpha} \log L(\alpha) = -\frac{n}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^{n} y_i^2$$

Setting this equal to 0

$$-\frac{n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2 = 0$$

$$\frac{n}{\alpha} = \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2$$

$$\alpha = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

So, our candidate for the MLE is

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

Taking the second derivative,

$$\frac{d^2}{d\alpha^2}\log L(\alpha) = \frac{n}{\alpha^2} - \frac{2}{\alpha^3}\sum_{i=1}^{n} y_i^2$$

$$= \frac{n}{\alpha^2} - \frac{2n\hat{\alpha}}{\alpha^3}$$

So that:

$$\frac{d^2}{d\alpha^2}\log L(\hat{\alpha}) = \frac{n}{\hat{\alpha}^2} - \frac{2n}{\hat{\alpha}^3}\hat{\alpha} = \frac{-n}{\hat{\alpha}^2} < 0$$

$$\hat{\alpha}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

(b) If $Z_i = Y_i^2$, then $Y_i = \sqrt{Z_i}$

and $\frac{dy_i}{dz_i} = \frac{1}{2\sqrt{z_i}}$, so that:

$$f_z(z) = \frac{2}{\alpha}\sqrt{z} \cdot \exp\left\{\frac{-z}{\alpha}\right\} \times \frac{1}{2} \times \frac{1}{\sqrt{z}}$$

$$\frac{1}{\alpha} \exp\left\{\frac{-z}{\alpha}\right\}$$

which is half of an exponential distribution with parameter

$$E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right] = E(\bar{Z}] = E[Z_i] = \alpha$$

$$F_Z(z) = P(Z \leq z) = P(Y^2 \leq z)$$

$$= P(Y \leq \sqrt{z})$$

$$= F_Y(\sqrt{z})$$

$$f_Z(z) = \frac{d}{dz}P(Z \leq z) = \frac{d}{dz} F_Y(\sqrt{z}) = f(\sqrt{z})\frac{d\sqrt{z}}{dz}$$

$$\frac{d}{dz}$$

Q-5

(a) $L(p) = \binom{n}{y} p^y (1-p)^{n-y}$

$0 < p < 1 \quad y = 0, 1, \ldots n$

The log likelihood is

$\log L(p) = \log\left\{\binom{n}{y}\right\} + y \log(p) + (n-y)\log(1-p)$

The derivative of the log likelihood is

$$\frac{d}{dp} \log L(p) = \frac{y}{p} - \frac{n-y}{1-p}$$

$$\frac{y}{p} - \frac{n-y}{1-p} = 0$$

$$y - py = n - py \Rightarrow \therefore p = \frac{y}{n}$$

Thus, $\hat{p} = \frac{y}{n}$ is our candidate

we take the second derivative

$$\frac{d^2}{dp^2} \log L(p) = \frac{-y}{p^2} - \frac{n-y}{(1-p)^2}$$

Which is always less than 0; thus is the maximum likelihood estimator for p.

$$\hat{p} = \frac{x}{n}$$

(b) we can let x be the no. of

6s in 40 rolls of the die

$X \sim Binom(40, p)$

$= \frac{8}{40} = 0.2$

is the max. likelihood estimate for p.

(c) $P(Y=0) = \binom{5}{0} p^0 (1-p)^{5-0} = (1-p)^5$

$\theta = (1-p)^5$

find the MLE $\hat{\theta} = (1-\hat{p})^5$

$(1-0.2)^5 = 0.33$

$\underline{\theta-1}$

If $\theta = 1$, x follows a poisson distribution with parameter $\lambda = 2$.

$P(\lambda = 3) = \frac{e^{-2} \cdot 2^3}{\lfloor 3} = 0.180447$

$\theta = 2$, x follows a geometric distribution parameter $p = 1/4$. Thus if $\theta = 2$

$P(x=3) = \frac{1}{4}(1-\frac{1}{4})^{3-1} = 0.140625$

Thus observing $X=3$, is more likely when $\theta=1$ $(0, 1/8)$ than when $\theta=2$ $(0, 1/8)$

So $\boxed{1}$ is the maximum likelihood estimate of θ.

Θ "

$$L(\theta) = \prod_{i=1}^{n} \frac{2\theta^2}{y_i^3} = \frac{2^n \theta^{2n}}{\prod_{i=1}^{n} y_i^3}$$

$0 < \theta \leq y_i < \infty$, for every i

~~Note that~~

$$\log L(\theta) = n\log 2 + 2n\log\theta - \log\left(\prod_{i=1}^{n} y_i^3\right)$$

$$\frac{d}{d\theta}\log L(\theta) = \frac{2n}{\theta} > 0 \text{ on } \theta \in (0, \min\{y_i\})$$

Thus the MLE is the largest possible value of θ

$$\hat{\theta} = \min\{x_i\}$$