Assignment 1

21 September, 2022

Due date: 4 October, by 13:20 in class

Instructions

- If you write your solutions by hand, please ensure your handwriting is legible. We may subtract marks for hard-to-read solutions.
- Submit a hardcopy of your assignment in class.
- **After 13:20, an assignment will be accepted only with an MSAF.**
- Name your MATLAB files **exactly** as specified.
- Do not submit zipped files to Avenue. We will **ignore any compressed file** containing your files.
- Submit **only what is required**.

Problem 1 [2 points] (p. 32, exercise 25) For small x, the approximation $\sin x \approx x$ is often used. For what range of x is this good to a relative accuracy of $\frac{1}{2}10^{-14}$?

Problem 2 [4 points] (p. 33, exercise 41) Write the Taylor series for

a. e^{x+2h}

b. $\sin(x - 3h)$

Problem 3 [3 points] Suppose you approximate e^x by its truncated Taylor series. For given $x = 0.5$, derive how many terms of the series are needed to achieve accuracy of 10^{-10}.

Problem 4 [2 points] Consider the expression $(1 - a)(1 + a)$. In double precision, for what values of a does this expression evaluate to 1 ?

Problem 5 [2 points] Give an example in base-10 computer arithmetic when

a. $(a + b) + c \neq a + (b + c)$

b. $(a * b) * c \neq a * (b * c)$

Problem 6 [8 points] Suppose you need to generate $n + 1$ equally spaced points in the interval $[a, b]$ with spacing $h = (b - a)/n$, $n > 1$. You can use either

$$x_0 = a, \quad x_i = x_{i-1} + h, \quad i = 1, \dots, n \quad \text{or} \tag{1}$$

$$x_i = a + ih, \quad i = 0, \dots, n. \tag{2}$$

Denote by $\widetilde{x}_i$ the computed value in (1) and by $\widehat{x}_i$ the computed value in (2).

a. [2 points] Which of $|x_i - \widetilde{x}_i|$ and $|x_i - \widehat{x}_i|$ is more accurate? Explain why.

b. [2 points] Write a MATLAB program that implements both methods and illustrates the difference between them.

Submit to Avenue your MATLAB code under file name `spacing.m` and also include this file in your hardcopy.

Problem 7 [6 points] Consider the approximation, $h > 0$

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h}.$$

Assume that $f'''(x)$ in continuous on $[x, x + h]$.

a. [2 points] If we approximate $f'(x)$ by $(f(x + h) - f(x - h))/(2h)$, what is the truncation error of this approximation?

b. [2 points] When evaluated on a computer, for what value of h the error of this approximation is the smallest?

c. [2 points] For the function $f(x) = \sin x e^{\cos x}$ plot the error

$$\left| f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right|$$

versus h for appropriate values of h. Plot on a `loglog` scale. Submit your plot in the hardcopy. How does the error match your derivation in the previous part?

Problem 8 [4 points] Consider

$$f(x) = \frac{e^x - x - 1}{x^2}.$$

When evaluated with $|x| < 1$, the relative error can be large.

a. [1 point] Explain why this error can be large.

b. [3 points] Write a MATLAB function

```
function y = expm1mx(x)
% Evaluates (exp(x) − 1 − x)/x^2 accurately for |x| < 1.
```

that evaluates $f(x)$ as accurately as possible when $|x| < 1$. You must use only double precision and must not use any of the MATLAB's built-in functions. Store your function in a file with name `expm1mx.m` and submit it to Avenue and also include it in your hardcopy.

Then run the MATLAB script

```
clear all; close all;
f = @(x) (exp(x)-1-x)./x.^2;
N = [-16:1:0];
x = 10.^N;
% compute accurate values in higher precision
accurate = f(vpa(x));
% relative error in f(x)
error_f = abs((f(x)-accurate)./accurate);
loglog(x,error_f, 'o--');
y = expm1mx(x);
% relative error in expm1mx
error = abs((accurate-y)./accurate);
hold on;
loglog(x,error, 'o--');
legend("rel. error in f(x)", "rel. error in expm1mx")
print("-depsc2", "expm1mx.eps")
```

and submit the produced plot in your hard copy.

Problem 9 [8 points] The following MATLAB script

```
g = @(x) (exp(x)-1-x)./x.^2;
h = @(x) (exp(x)-x-1)./x.^2;
x = 1e-10;
fprintf('x=%.16e\ng(x)=%.16e\nh(x)=%.16e\n', x, g(x), h(x))
x = 2^(-33);
fprintf('x=%.16e\ng(x)=%.16e\nh(x)=%.16e\n', x, g(x), h(x))
```

produces (on my machine)

```
x=1.0000000000000000e-10
g(x)=8.2740370962658164e+02
h(x)=0.0000000000000000e+00
x=1.1641532182693481e-10
g(x)=0.0000000000000000e+00
h(x)=0.0000000000000000e+00
```

(3, 2, 1,1 points) Explain the values for each of the `g(x)` and `h(x)`.