1 Mechanics: Newtonian

1.1 Newton's Laws of motion

- (Law of inertia) Every object continues in its state of rest or ofuniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.
- (Law of motion) If a object of mass m is moving with accelerationa, then it is under the influence of a (resultant) force F where
$$F = ma \quad [N] = [kg \cdot m/s^2].$$
- (Law of action and reaction) When two objects interacts, the force F_1 exerted on object 1 by object 2 is equal in magnitude and opposite in directed to the force F_2 exerted on object 2 by object 1, that is
$$F_1 = -F_2.$$

Inertial coordinate systems: Postulate of special relativity. Every law of physics must be such that if it holds in any coordinate system, it holds also in any other coordinate system moving at a constant velocity with respect to the first. You need to experiment to determine which coordinate system are inertial.

By assuming that $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \mathbf{v}, \mathbf{a})$ we see that the equation of motion is a second order ordinary differential equation
$$\ddot{\mathbf{r}} = \frac{1}{m}\mathbf{F}$$
We always assume $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \mathbf{v})$.
$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \frac{1}{m}\mathbf{F}(t, \mathbf{x}, \mathbf{v}) \\ &\Updownarrow \\ \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{pmatrix} &= \begin{pmatrix} \frac{1}{m}\mathbf{F}(t, \mathbf{x}, \mathbf{v}) \end{pmatrix} = f(t, \mathbf{x}, \mathbf{v}) \\ \dot{\mathbf{z}} &= f(t, \mathbf{z}) \end{aligned}$$

1.2 Work and Energy

Assume that a force $\mathbf{F}$ acts on a particle between time t_1 and t_2. The work $W = W(t_1, t_2)$ done on the particle by the force is defined as
$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \, \mathrm{d}t \quad [Nm] = [kg \cdot m^2/s^2] = [j]$$
with $\mathbf{v}$ the velocity of the particle.

Note that
$$\begin{aligned} m\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} \cdot \mathbf{v}) &= m(\dot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v} \cdot \dot{\mathbf{v}}) = m2\mathbf{v} \cdot \mathbf{v} = 2\mathbf{F} \cdot \mathbf{v} \\ \mathbf{F} \cdot \mathbf{v} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right). \end{aligned}$$

and

$$v = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \|\mathbf{v}\|$$

for which $W = T_2 - T_1$ with $T_i = T(t_i)$ and

$$T = \frac{1}{2} m v^2$$

called the kinetic energy (of the particle).

The power $P = P(t)$ supplied by the force $\mathbf{F}$ at time t is

$$P = \mathbf{F} \cdot \mathbf{v} \quad [Nm/s] = [j/s]$$

hence

$$W = \int_{t_1}^{t_2} P \,\mathrm{d}t \quad \text{or} \quad P = \frac{\mathrm{d}W}{\mathrm{d}t_2} \quad \text{or} \quad P = \frac{\mathrm{d}T}{\mathrm{d}t}.$$

If the force only depends on position, $\mathbf{F} = \mathbf{F}(\mathbf{r})$, then

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \,\mathrm{d}t = \int_C \mathbf{F} \cdot \frac{\mathbf{v}}{v} \,\mathrm{d}s$$

where $\mathrm{d}s$ denote the (infinitesimal) arc-length element and the integral is to be taken along the curve C described by the particle between t_1 and t_2.

Note that

$$\mathbf{v} \,\mathrm{d}t = \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t = \mathrm{d}x,$$

and thus

$$\int_{t_1}^{t_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \,\mathrm{d}t.$$

The work is difficult to compute as the position over time must be known, which requires the ability to solve the differential equation.

Conservative force: If $\mathbf{F} = \mathbf{F}(\mathbf{r})$ and $\mathbf{F}$ can be expressed as the gradient of a (real valued) function $V, \mathbf{F} = \boldsymbol{\nabla} V$, then

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = V_2 - V_1 \qquad V_i = V(\mathbf{r}_i)$$

by the fundamental theorem for line integrals:

A force $\mathbf{F}$ which is a function of position alone and which can be expressed as minus the gradient of a (real valued) function U is said to be conservative

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) \quad \mathbf{F} = -\boldsymbol{\nabla} U$$

The function U is said to be a potential energy (or function). Which is equivalent to saying:

A force $\mathbf{F}$ is a function of positionalone and whose curl vanishes is said to be conservative

$$0 = \operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F}$$

for which the existence of a potential energy is guaranteed by Stroke's Theorem.

A force $\mathbf{F}$ which is a function of position alone and which can be expressed as minus the gradient of a (real valued) function U is said to be conservative

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) \quad \mathbf{F} = -\boldsymbol{\nabla} U.$$

The function U is said to be a potential energy (or function).

For a conservative force $\mathbf{F}$ we have for any two times t_1 and t_2

$$T_2 - T_1 = W = U_1 - U_2 \Leftrightarrow T_2 + U_2 = T_1 + U_1.$$

The quantity

$$E = T + U$$

is called the total energy and we conclude that:

1. Conservation of Energy I: If the (total) force acting on a particle is conservative then the total energy $E = T + U$ is conserved (is constant).
2. Conservation of Energy II: If no work is done on the particle by nonconservative foces then the total energy $E = T + U$ is conserved (is constant).

1.3 Linear Momentum

The linear momentum (impuls) $\mathbf{p} = \mathbf{p}(t)$ of a particle (of mass m) is defined as

$$\mathbf{p} = m\mathbf{v}.$$

Hence

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \dot{\mathbf{p}} = \mathbf{F} \quad \text{or} \quad \mathbf{p}_2 - \mathbf{p}_1 = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t$$

(Note that $\mathbf{F}$ is depending on position and velocity, both of which depent on the time t) and we have:

- Conservation of Linear Momentum: If the (total) force acting on a particle is zero then the linear momentum $\mathbf{p} = m\mathbf{v}$ is conserved (is constant).

A collision between two particles is called (completely) elastic if the kinetic energy T is conserved:

$$\begin{aligned} T_1 + Q &= T_2 \\ Q &> 0 \qquad \text{super elastic} \\ Q &= 0 \qquad \text{(completely) elastic} \\ Q &< 0 \qquad \text{inelastic.} \end{aligned}$$

1.4 Summary

For given constant m

$$\begin{aligned} T = T(v) &= \frac{1}{2}mv^2 \quad \text{kinetic energy} \\ \mathbf{p} = \mathbf{p}(\mathbf{v}) &= m\mathbf{v} \quad \text{linear momentum} \end{aligned}$$

For a given $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \mathbf{v})$

$$P = P(t, \mathbf{r}, \mathbf{v}) = \mathbf{F}(t, \mathbf{rv}) \cdot \mathbf{v} \quad \text{power}$$

If $\mathbf{F} = \mathbf{F}(\mathbf{r})$ and $0 = \operatorname{curl} \mathbf{F}$

$$\begin{aligned}
U = U(\mathbf{r}) &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \quad \text{potential energy (relative to } \mathbf{r}_0\text{)} \\
E = E(\mathbf{r}, \mathbf{v}) &= T(\mathbf{v}) + U(\mathbf{r}) \quad \text{energy} \\
W = W(\mathbf{r}_2, \mathbf{r}_1) &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \quad \text{work.}
\end{aligned}$$

If $\mathbf{r} = \mathbf{r}(t),\ \dot{\mathbf{r}} = \mathbf{v}$ and $\mathbf{v} = |\mathbf{v}|$ then all functions above are functions of (time) t only and

$$W = W(t_2, t_1) = \int_{t_1}^{t_2} \mathbf{F}(t, \mathbf{r}, \mathbf{v}) \cdot \mathbf{v} \mathrm{d}t \quad \text{work (general definition)}$$

If $\mathbf{r} = \mathbf{r}(t)$ describes the motion of a particle with mass $m > 0$ and $\mathbf{F}(t, \mathbf{r}, \mathbf{v})$ is the resultant force on the particle then $\mathbf{F} = m\mathbf{a}$ with $\ddot{\mathbf{r}} = a$ and

$$\frac{\mathrm{d}T}{\mathrm{d}t} = P = \frac{\mathrm{d}W}{\mathrm{d}t_2}.$$

See Principle of Virtual Work (not in curriculum) which is useful for movement equations.

2 Center of Mass

Consider a system of particles. The center of mass $\mathbf{R}$ of this system is defined as

$$\begin{aligned}
M\mathbf{R} &= \sum m_i \mathbf{r}_i \\
\mathbf{R} &= \frac{1}{M} \sum m_i \mathbf{r}_i = \sum \alpha_i \mathbf{r}_i, \quad \alpha_i = \frac{m_i}{M} \Rightarrow 0 < \alpha_i \leq 1 \Rightarrow \sum \alpha_i = 1
\end{aligned}$$

where $M = \sum m_i$ and m_i and $\mathbf{r}_i$ are the mass and position vector (relative to some origin) of particle i. Note that $\mathbf{R}$ is independent of the choice of coordinate system and independent of choice of origin. Note that $\sum \alpha_i \mathbf{r}_i$ is called the convex hull:"konvekse hylster"–den mindste konvekse mængde som indeholder alle punkter)

If $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$ is the linear momentum of particle i then $P = \sum \mathbf{p}_i$ designate the total linear momentum of the system of particles and we have

$$P = \sum m_i \dot{\mathbf{r}}_i = M\dot{R}.$$

The equation of motion of particle i is

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^I + \mathbf{F}_i^E = \text{Internal Force } + \text{ External Force.}$$

We assume(implied by Newton's third law) that in any virtual translation $\delta\mathbf{r}$ of the system the internal forces would do no net work, implying that $\sum \mathbf{F}_i^I = 0$.

If $\mathbf{F}_i = m_i \ddot{\mathbf{r}}$ is the total (external) force acting on particle i then $\mathbf{F} = \sum \mathbf{F}_i$ is the total external force acting on the system and we have

$$\frac{\mathrm{d}P}{\mathrm{d}t} = M\ddot{\mathbf{R}} = \sum m_i \ddot{\mathbf{r}}_i = \mathbf{F}.$$

The center of mass of a system of particles moves like a single particle, whose mass is the total mass of the system, acted on by the force equal to the total external force acting on the system.

For a rigid body, B, having a continuous (uniform) distribution of mass, the center of mass $\mathbf{R}$ is defined as

$$\mathbf{R} = \frac{1}{M}\int_B \mathbf{r}\rho \,\mathrm{d}V$$

with uniform density $\rho = \frac{(\text{total mass of } B)}{(\text{total volume of } B)}$, $[kg/m^3]$.

2.1 Torque

From the knowledge obtained so far we can determine the movement of the center of mass of a (rigid) body by knowing the (total) force acting upon it. However this is not sufficient to determine the movement of the object. To get a complete description we also need the notion of "a(vector) torque/moment, about a point, of a force acting on a particle".

Let O denote a fixed point, $\mathbf{r}$ the vector from O to a particle and $\mathbf{F}$ a force acting on the particle. The torque $N = N_O$, or moment, about the point O of the force $\mathbf{F}$ acting on the particle is defined as

$$\mathbf{N}_O = \mathbf{r} \times \mathbf{F}$$
$$N = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{F})$$

where N is the torque about the line $\bar{AB}$ with $\mathbf{n}$ being the unit vector in the direction of $\bar{AB}$. Note that it is only the perpendicular distance to the point from the line that has any impact on the torque, the torque of the parallel distance from the point is zero and therefore neglected in the equations.

The analog of linear momentum $\mathbf{p} = m\mathbf{v}$ for a rotational motion about the fixed point O is called angular momentum $\mathbf{L} = \mathbf{L}_O$ (about O) and is defined as

$$\mathbf{L}_O = \mathbf{r} \times \mathbf{p}.$$

It follows that

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F} = \mathbf{N} \Rightarrow \mathbf{L}_2 - \mathbf{L}_1 = \int_{t_1}^{t_2} \mathbf{N}\,\mathrm{d}t$$

and we have

- Conservation of Angular Momentum: If the (total) torque acting on a particle is zero then the angular momentum $L = \mathbf{r} \times \mathbf{p}$ is conserved (is constant).

The angular momentum (impulsmoment) about the line $\bar{AB}$ with $\mathbf{n}$ being the unit vector in the direction of $\bar{AB}$ is given by

$$L = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{p}).$$

The angular momentum $\mathbf{L} = \mathbf{L}_{O^*}$ of a particle about a point O^* not necessarily fixed is given by

$$\mathbf{L}_{O^*} = m(\mathbf{r} - \mathbf{r}_{O^*}) \times (\dot{\mathbf{r}} - \dot{\mathbf{r}}_{O^*}) = (\mathbf{r} - \mathbf{r}_{O^*}) \times (\mathbf{p} - m\dot{\mathbf{r}}_O^*)$$

with $\mathbf{r}$ and $\mathbf{r}_{O^*}$ being position vectors of the particle and point O^* both with respect to some fixed point O. It follows that

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{L}} &= m(\mathbf{r}-\mathbf{r}_{O^*})\times(\dot{\mathbf{r}}-\dot{\mathbf{r}}_{O^*}) + (\mathbf{r}-\mathbf{r}_{O^*})\times m\ddot{\mathbf{r}} - (\mathbf{r}-\mathbf{r}_{O^*})\times m\ddot{\mathbf{r}}_{O^*}\\ &= (\mathbf{r}-\mathbf{r}_{O^*})\times\mathbf{F} - m(\mathbf{r}-\mathbf{r}_{O^*})\times\ddot{\mathbf{r}}_{O^*}\end{aligned}$$

with $\mathbf{N}_{O^*} = (\mathbf{r}-\mathbf{r}_{O^*})\times\mathbf{F}$ being the total torque about O^*. The equations can be written more compactly by defining $\mathbf{r}^* = \mathbf{r}-\mathbf{r}_{O^*}$ as:

$$\mathbf{r}^*\times m\ddot{\mathbf{r}}^* = \dot{\mathbf{L}} = \mathbf{N}_{O^*} - m\mathbf{r}^*\times\ddot{\mathbf{r}}_{O^*}.$$

As long as $\ddot{\mathbf{r}}_{O^*} = 0$ Newton's laws applies, but if it is nonzero, then the other inertial system is accelerated with respect to the other and as such Newton's laws will not apply:

- Conservation of Angular Momentum: If the (total) torque $\mathbf{N}_{O^*}$ acting on a particle is zero then the angular momentum $\mathbf{L}_{O^*}$ is conserved (is constant).

For a system of particles the angular momentum $\mathbf{L}_{iO^*}$, and its derivative, of particle i about a point O^* not necessarily fixed is

$$\begin{aligned}\mathbf{L}_{iO^*} &= m_i(\mathbf{r}_i-\mathbf{r}_{O^*})\times(\dot{\mathbf{r}}_i-\dot{\mathbf{r}}_{O^*})\\ \dot{\mathbf{L}}_{iO^*} &= (\mathbf{r}_i-\mathbf{r}_{O^*})\times\mathbf{F}_i - m_i(\mathbf{r}_i-\mathbf{r}_{O^*})\times\ddot{\mathbf{r}}_{O^*}\\ &= (\mathbf{r}_i-\mathbf{r}_{O^*})\times\mathbf{F}_i^I + (\mathbf{r}_i-\mathbf{r}_{O^*})\times\mathbf{F}_i^E - m_i(\mathbf{r}_i-\mathbf{r}_{O^*})\times\ddot{\mathbf{r}}_{O^*}\end{aligned}$$

summed over all particles we obtain

$$\begin{aligned}\dot{\mathbf{L}}_[O^*] &= \sum(\mathbf{r}_i-\mathbf{r}_{O^*})\times\mathbf{F}_i^I + \mathbf{N}_{O^*} - M(\mathbf{R}-\mathbf{r}_{O^*})\times\ddot{\mathbf{r}}_{O^*}\\ \mathbf{L}_{O^*} &= \sum\mathbf{L}_{iO^*}\\ \mathbf{N}_{O^*} &= \sum(\mathbf{r}_i-\mathbf{r}_{O^*})\times\mathbf{F}_i^E.\end{aligned}$$

The total internal torque $\sum(\mathbf{r}_i-\mathbf{r}_{O^*})\times\mathbf{F}_i^I$ is zero whenever the following applies

- Newton's third law (strong form): The forces $\mathbf{F}_{ij}^I$ (internal force on particle i due to particle j) and $\mathbf{F}_{ji}^I$ are equal in magnitude and opposite in direction and acts along the line joining them.

- Which is equivalent to saying that no net work is done by the internal force in a small virtual rotation about any axis through the point O^*.

In any case, if $\sum(\mathbf{r}_i-\mathbf{r}_{O^*})\times\mathbf{F}_i^I = 0 = -M(\mathbf{R}-\mathbf{r}_{O^*})\times\ddot{\mathbf{r}}_{O^*}$, then the momentum is conserved:

- Conservation of Angular Momentum: If the (total) external torque $\mathbf{N}_{O^*}$ acting on the system of particles is zero then the angular momentum $\mathbf{L}_{O^*}$ is conserved (is constant).

The derivative of the angular momentum of a system of particles can be rewritten with $\mathbf{r}_i^* = \mathbf{r}_i - \mathbf{r}_{O^*}$ as

$$\sum\mathbf{r}_i^*\times m_i\ddot{\mathbf{r}}_i^* = \dot{\mathbf{L}}_{O^*} = \sum\mathbf{r}_i^*\times\mathbf{F}_i^I + \mathbf{N}_{O^*} - M(\mathbf{R}-\mathbf{r}_{O^*})\times\ddot{\mathbf{r}}_{O^*}.$$

To summarize: The motion of a rigid body in space is determined by

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = M\ddot{\mathbf{R}} = \mathbf{F}$$
$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{N}$$

with

- M being the mass of the body
- $\mathbf{R}$ being the position of the center of mass relative to a point O^*
- $\mathbf{F}$ being the total force on the body
- $\mathbf{L}$ being the total angular momentum about the point O^*
- $\mathbf{N}$ being the total torque about the point O^*
- for unconstrained movement O^* is to be taken as the center of mass (implying $0 = -M(\mathbf{R} - \mathbf{r}_{O^*}) \times \ddot{\mathbf{r}}_{O^*}$)
- if the body is constrained to rotate about a fixed point, that point is to be taken as O^*.

2.2 Rotation of a Rigid Body

Consider a (rigid) body, idealized as a a system of particles, rotating about a fixed axis chosen to be the z-axis. Only one coordinate is needed to specify the orientation of the body. Let this be the angle θ between the x-axis and a fixed line $\bar{OA}$ in the body through the z-axis and lying in the xy-plan (or parallel to it). In cylindrical coordinates the angular momentum of the body aboutthe axis is then

$$L = \sum m_i r_i^2 \dot{\varphi}_i = \sum m_i r_i^2 \frac{\mathrm{d}}{\mathrm{d}t}(\theta + \beta_i) = \sum m_i r_i^2 \dot{\theta} = I_z \dot{\theta}$$

where the quantity $I_z = \sum m_i r_i^2$ is called the moment of inertia about the axis. The equation of motion for rotation of a body about the axis is thus

$$N_z = \frac{\mathrm{d}L}{\mathrm{d}t} = I_z \ddot{\theta} \quad (N_z \text{ the torque of the body about the axis})$$

similar to the (one dimensional) equation of motion for translatory motion.

The moment of inertia of a plane lamina about an axis is

$$I_y = \int_B \mathbf{r}^2 \rho \, \mathrm{d}A = \int_B \mathbf{r}^2 \frac{M}{ab} b \, \mathrm{d}\mathbf{r} = \frac{M}{a} \int_0^a \mathbf{r}^2 \, \mathrm{d}\mathbf{r} = \frac{1}{3} M a^2.$$

Now the moment of inertia about an axis going through the center of mass of the lamina is given by

$$I_G = I_y - M\mathbf{R}^2 = \frac{1}{3} M a^2 - M \left(\frac{a}{2} \right)^2 = \frac{1}{12} M a^2.$$

By using the Perpendicular Axis Theorem the moment of inertia about a third perpendicular axis can be computed if the moment of inertia is known for two axis perpendicular on each other:

$$I_z = I_y + I_z.$$

3 Mechanics: Lagranian

The number of independent quantities needed to specify uniquely the position of a system is called the degree of freedom (d.o.f.). If a system has s d.o.f. then the independent qualities, usually denoted $q = (q_1, \dots, q_s)$, needed to specify the position of the system are called generalized coordinates.

To determine a system completely, its velocities also has to be specified: the derivative $\dot{q} = (\dot{q}_1, \dots, \dot{q}_s)$ of the generalized coordinates are called the generalized velocities.

3.1 Principle of least action (Hamilton's Principle)

Is there an underlying structure from which we get Newton's laws? All equations of motion can be obtained from certain fundamental action principles (extremum of the action).

- The position of (static) equilibrium of a particle is the minimum of the potential energy.
- The state of a system at thermodynamic equilibrium is the one for which some thermodynamic potential is minimized, or for which the entropy is maximized.

To every mechanical system there exists a function $L = L(q, \dot{q}, t)$ such that for any two instances $t_1 < t_2$ the system moves between $q(t_1)$ and $q(t_2)$ in such a way as to minimize the action $S = S(q)$ given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,\mathrm{d}t.$$

The function L is called the Lagrangian of the system.

In principle we want to minimize S by solving $\dot{S} = 0$, this proves to be problematic as q are functions. Hence we use another method.

3.2 Example of a pendulum

Find the Lagrangian for the system described by the figure

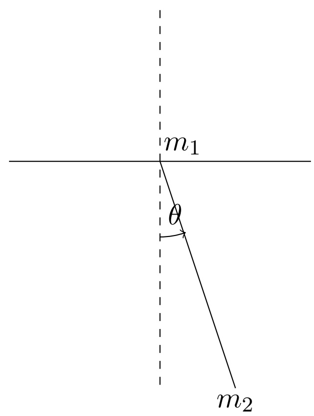


Figure 1: A pendulum consisting of a mass-less rigid rod of length l and a mass m_2 suspended from a bearing mass m_1 which can move on a horizontal in the plane in which m_2 moves.

Kinetic energy and potential energy:

$$\begin{aligned}
L &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - (l - l\cos(\theta))m_2 g \\
x_2 &= l\sin(\theta), \quad y_2 = -l\cos(\theta) \\
L &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 \\
q_1 &= x_1, \quad q_2 = \theta \\
\frac{\partial L}{\partial x_1} &= 0, \quad \frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 \Rightarrow m_1\ddot{x}_1 = 0 \\
\frac{\partial L}{\partial \theta} &= -l\sin(\theta)m_2 g, \quad \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \Rightarrow I\ddot{\theta} = -m_2 g l\sin(\theta) \\
\ddot{\theta} &= -\frac{g}{l}\sin(\theta).
\end{aligned}$$

But this is all incorrect as m_1 will start to move as m_2 moves.

The correct Lagrangian equations for movement are

$$\begin{aligned}
L &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - (l - l\cos(\theta))m_2 g \\
x_2 &= x_1 + l\sin(\theta),\ \dot{x}_2 = \dot{x}_1 + l\cos(\theta)\dot{\theta}, \quad y_2 = -l\cos(\theta),\ \dot{y}_2 = l\sin(\theta)\dot{\theta} \\
L &= \frac{1}{2}m_1\dot{x}_1 + \frac{1}{2}m_2\Big(\dot{x}_1^2 + l^2\cos^2(\theta)\dot{\theta}^2 + 2\dot{x}_1 l\cos(\theta)\dot{\theta} + l^2\sin^2(\theta)\dot{\theta}^2\Big) - (l - l\cos(\theta))m_2 g \\
&= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}I\dot{\theta}^2 + \dot{x}_1 m_2 l\cos(\theta)\dot{\theta} - (l - l\cos(\theta))m_2 g \\
\frac{\partial L}{\partial x_1} &= 0, \quad \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 + m_2 l\cos(\theta)\dot{\theta}
\end{aligned}$$

4 Opgaver (Ja på dansk)

4.1 Opgave 4: Pulley

Opstiller fritlegeme diagrammer for m_1 og m_2. Bevægelsesligningerne for m_1 med Newton er givet ved

$$m_1 a_1 = F_{g1} + F_{s1} = m_1 g + F_{s1}$$

og for m_2:

$$m_2 a_2 = F_{g2} + F_{s2} = m_2 g + F_{s2}.$$

Disse kan kombineres da $F_{s1} = F_{s2}$, og bliver da

$$m_1 a_1 = m_1 g + m_2 a_2 - m_2 g,$$

og da de to accelerationer er modsatrettet kan vi ændre det til:

$$\begin{aligned} a &= a_1 = -a_2 \\ m_1 a + m_2 a &= (m_1 - m_2) g \\ a &= \frac{(m_1 - m_2) g}{m_1 + m_2}. \end{aligned}$$

Denne opgave kan også løses ved at sætte en positiv retning langs snoren mod uret. Dermed gælder argumenterne ikke helt på samme måde, da snorkræfterne da er modsatrettet, hvilket følger aktion og reaktion. Dette medfører også til at de har samme acceleration. Vi har da

$$\begin{aligned} m_1 a_1 &= m_1 g_1 + m_2 g_2 - m_2 a_2 \\ m_1 a_1 + m_2 a_2 &= m_1 g_1 + m_2 g_2, \end{aligned}$$

men nu er $a_1 = a_2$ og $g_1 = -g_2$:

$$m_1 a_1 + m_2 a_2 = (m_1 - m_2) g \Rightarrow a = \frac{(m_1 - m_2) g}{m_1 + m_2}.$$

Alternativt kan der anvendes kraftmoment: $N = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 = 0$, og er derved givet ved

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = 0,$$

hvilket gælder hvis $\mathbf{r} = 0$, $\mathbf{F}_1 + \mathbf{F}_2 = 0$ eller hvis $\mathbf{r}$ er parallel med $\mathbf{F}_1 + \mathbf{F}_2$.

4.2 Opgave 5: Spring-damper

Opstiller fritlegeme diagram for de to masser m_1, m_2 og opskriver bevægelsesligningerne. Disse er givet ved

$$\begin{aligned} m_1 a_1 &= F_{f1} + F_d + F_u + F_{f2} = -k_1 x_1 - \alpha \dot{x}_1 + F_u - k_2 (x_1 - x_2) \\ m_2 a_2 &= F_\mu + F_{f2} = \begin{cases} -\alpha \dot{x}_2 - k_2 (x_2 - x_1) \\ -(\text{fortegn})(\dot{x}_2) \mu m g - k_2 (x_1 - x_1) \end{cases} \end{aligned}$$

4.3 Opgave 15: Vinkelhastighed

Positionsvektoren af en partikel er givet ved

$$\mathbf{r}(t) = \Big(\cos(\theta(t)), \sin(\theta(t)), z(t)\Big) = \cos(\theta(t))\mathbf{x} + \sin(\theta(t))\mathbf{y} + z(t)\mathbf{z}.$$

Bestem nu vinkelhastigheden ω. Vi vælger nu et andet koordinatsystem med koordinater $\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*$ som til $t = 0$ er identisk med ikke stjerne koordinatsystemet, men derefter roterer systemet med partiklen, men det stiger ikke. Derved har vi

$$\begin{aligned}
\mathbf{x}^*(t) &= \cos(\theta(t))\mathbf{x} + \sin(\theta(t))\mathbf{y} + 0\mathbf{z} \\
\mathbf{y}^*(t) &= -\sin(\theta(t))\mathbf{x} + \cos(\theta(t))\mathbf{y} + 0\mathbf{z} \\
\mathbf{z}^*(t) &= \mathbf{z}.
\end{aligned}$$

Nu kan vi tage den tidsafledede og udtrykke dem i baserne:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t} &= -\sin(\theta(t))\dot{\theta}(t)\mathbf{x} + \cos(\theta(t))\dot{\theta}(t)\mathbf{y} \\
\frac{\mathrm{d}\mathbf{y}^*}{\mathrm{d}t} &= -\cos(\theta(t))\dot{\theta}(t)\mathbf{x} - \sin(\theta(t))\dot{\theta}(t)\mathbf{y} \\
\frac{\mathrm{d}\mathbf{z}^*}{\mathrm{d}t} &= 0
\end{aligned}$$

hvor vi husker på

$$\begin{aligned}
\frac{\mathrm{d}x^*}{\mathrm{d}t} &= a_{11}x^* + a_{12}y^* + a_{13}z^* \\
\frac{\mathrm{d}y^*}{\mathrm{d}t} &= a_{21}x^* + a_{22}y^* + a_{23}x^* \\
&\dots \\
\omega &= a_{23}x^* + a_{31}y^* + a_{13}z^*.
\end{aligned}$$

Vi anvender $\mathbf{x}^* \cdot \frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t} = 0$, som kun gælder når man har valdt en ortogonal base for * systemet.

$$\begin{aligned}
\mathbf{y}^* \cdot \frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t} &= \dot{\theta}(t) = a_{12} \\
\mathbf{z}^* \cdot \frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t} &= 0 = a_{13} = -a_{31} \\
\mathbf{z}^* \cdot \frac{\mathrm{d}\mathbf{y}^*}{\mathrm{d}t} &= 0 = a_{23} \\
&\Downarrow
\end{aligned}$$

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}(t) \end{pmatrix} = 0\mathbf{x}^* + 0\mathbf{y}^* + \dot{\theta}(t)\mathbf{z}^*,$$

hvilket også kunne være bestemt ved $\dot{\mathbf{r}} = \omega \times \mathbf{r}$, hvor $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ og finde de ubekendte.

Centripetal accelerationen er givet ved $\omega \times \omega \times \mathbf{r}$, men først skal $\mathbf{r}$ udtrykkes i * baserne:

$$\begin{aligned}
\mathbf{x} &= \cos(-\theta(t))\mathbf{x}^* + \sin(-\theta(t))\mathbf{y}^* + 0\mathbf{z}^* \\
&= \cos(\theta(t))\mathbf{x}^* - \sin(\theta(t))\mathbf{y}^* \\
\mathbf{y} &= \sin(\theta(t))\mathbf{x}^* + \cos(\theta(t))\mathbf{y}^* \\
\mathbf{r}^* &= 1\mathbf{x}^* + 0\mathbf{y}^* + z\mathbf{z}^*.
\end{aligned}$$

Nu kan centripetal accelerationen bestemmes

$$\omega \times (\omega \times \mathbf{r}^*) = \omega \times \begin{vmatrix} 0 & 0 & \dot{\theta}(t) \\ 1 & 0 & z \end{vmatrix} = \begin{vmatrix} 0 & 0 & \dot{\theta}(t) \\ 0 & \dot{\theta}(t) & 0 \end{vmatrix} = \begin{pmatrix} -\dot{\theta}^2(t) \\ 0 \\ 0 \end{pmatrix},$$

i * koordinatsystemet. Coriolis accelerationen er da givet ved $\omega \times \frac{\mathrm{d}^*\mathbf{r}}{\mathrm{d}t} = 0$ da de er parallelle.

4.4 Opgave 11: Pendul

Opstiller fritlegeme diagram for massen m som bliver holdt af et masseløst rør/snor. Bevægelsesligningerne er da

$$\begin{aligned}
ma &= \mathbf{F}_g + \mathbf{F}_s = \begin{pmatrix} mg \\ 0 \end{pmatrix} + \begin{pmatrix} F_{sx} \\ F_{sy} \end{pmatrix} \\
m\ddot{x} &= mg + F_{sx} = mg - \|\mathbf{F}_s\| \cos(\theta) \\
m\ddot{y} &= 0 + F_{sy} = -\|\mathbf{F}_s\| \sin(\theta) \\
\mathbf{r} &= l \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\
\dot{\mathbf{r}} &= l \begin{pmatrix} -\sin(\theta)\dot{\theta} \\ \cos(\theta)\dot{\theta} \end{pmatrix} \\
\ddot{\mathbf{r}} &= l \begin{pmatrix} -\cos(\theta)\ddot{\theta}^2 - \sin(\theta)\ddot{\theta} \\ -\sin(\theta)\ddot{\theta} + \cos(\theta)\ddot{\theta} \end{pmatrix} \\
\mathbf{F}_s &= \|\mathbf{F}_s\| \begin{pmatrix} -\cos(\theta) \\ -\sin(\theta) \end{pmatrix} \\
m\ddot{x}\sin(\theta) - m\ddot{y}\cos(\theta) &= mg\sin(\theta) \\
ml\Big(-\sin^2(\theta)\ddot{\theta} - \cos^2(\theta)\ddot{\theta}\Big) &= mg\sin(\theta) \\
ml\ddot{\theta} &= -mg\sin(\theta).
\end{aligned}$$

Det kan også løses ved

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= N \\
N = r \times F = r \times F_g + r \times F_s = r \times F_g &= \begin{vmatrix} 0 & \sin(\theta) & 0 \\ mg & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -lmg\sin(\theta) \end{pmatrix} \\
L = r \times p = m &\begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta)\dot{\theta} & \cos(\theta)\dot{\theta} & 0 \end{vmatrix} = ml^2 \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \\
\frac{\mathrm{d}L}{\mathrm{d}t} &= N \\
\begin{pmatrix} 0 \\ 0 \\ ml^2\ddot{\theta} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -lmg\sin(\theta) \end{pmatrix}.
\end{aligned}$$

4.5 Opgave 8: Inertimoment af firkant

Ligningerne er givet ved

$$I = \sum m_i r_i^2 = \sum m_i 2\left(\frac{l}{2}\right)^2 = \frac{1}{2}Ml^2$$

hvor origo er placeret i midten af firkanten. Ved angivelse af inertimoment i forhold til massemidpunktet skal der lidt mere til. Vi siger:

$$\begin{aligned}
r_i &= R + \bar{r}_i \\
I_G = \sum m_i \bar{r}_i^2 = \sum m_i (r_i - R)^2 & \\
&= \sum m_i (r_i^2 + R^2 - 2r_i \cdot R) \\
&= \frac{1}{2}Ml^2 + MR^2 - 2MR^2 = \frac{1}{2}Ml^2 - MR^2.
\end{aligned}$$

Hvilket siger massemidpunkter det punkt hvorom der gælder, at inertimomentet er mindst.

4.6 Opgave 9: Inertimoment af disk

Cylinderen er tynd: det har altså en højde man kan se bort fra. Inertimomentet er givet ved

$$\begin{aligned}
I &= \int_B r^2 \rho \, \mathrm{d}A \\
\mathrm{d}A &= \pi(r + \mathrm{d}r)^2 - \pi r^2 = \pi \mathrm{d}r^2 + 2\pi r \mathrm{d}r \\
&\Downarrow \\
I &= \int_0^R r^2 \frac{M}{\pi R^2} 2\pi r \, \mathrm{d}r \\
&= 2\frac{M}{R^2}\int_0^R r^3 \, \mathrm{d}r = \frac{1}{2}MR^2.
\end{aligned}$$

5 Electrical Systems

5.1 Resistive Circuitry – Kirchhoff's Laws

- Schematic: Physical devices and their inter-connection.
- Circuit Diagram: A model of the circuit consisting of ideal components.

Conducters connecting components are not physical wires, but idealised connections with no resistance. When multiple conducters (branches) meet at a connection, the connections are specified. If conducters cross each other without a specified connection, it is understood that they are not connected. Un-connected connectors are used for specifying when there is no connection to anything else at that point (terminals).

Conductance is the reciprocal value of the resistance: $K = \frac{1}{R}$.

Definition 5.1 (Kirchhoff's Current Law)
The algebraic sum of the currents leaving any enclosed volume (node) in a circuit equals zero:
$$\sum_n i_n = 0.$$

Node analysis algorithm:

1. Identify and label a reference node for other voltages in the circuit (ground).
2. Identify and label each other node for which the voltage at that node, relative to the reference node, is unknown.
3. Write Kirchhoff's current law at each such node, using Ohm's law to express the currents in terms of the node voltage and the resistance of the resistors attached to the node.
4. Solve the resulting equations for the node voltages.
5. Use the node voltages and Ohm's law to find any currents of interest.

Each independent voltage source that appears in a circuit reduces the number of unknown node voltages by one.

Definition 5.2 (Kirchhoff's Voltage Law)
The algebraic sum of the voltage drops around any closed path equals zero.

A **loop** is a closed path in or around a circuit. A **mesh** is a loop that does not enclose (encircle) any other loops.

Mesh analysis:

1. Identify each mesh and associate a mesh current with each.

2. Write Kirchhoff's voltage law around each mesh, using Ohm's law to express the voltages across resitors in the mesh in terms of the mesh currents and the resistances of the resistors.

3. Solve the resulting equations for the mesh currents.

4. Use the mesh currents and Ohm's law to find any voltages of interest.

Each independent current source that appears in a cicruit reduces the number of unknown mesh currents by one.

Voltage divider, current divider.

Superposition: $f(x_1,\ x_2) = f(x_1) + f(x_2)$. Hence the circuit is analysed for each with all other sources set to zero. The individual contributions are added at the end.

6 Equivalent Circuits

The terminal characteristic for a circuit element or a circuit is the relation between source and current at the terminals of the circuit. Some conventions:

- Seen as a source: think of current as flowing *out* of the positive terminal.
- Seen as a load: think of current as flowing *into* the positive terminal.

To determine the characteristics:

- Assign polarity.
- Assign positive direction of current.
- Determine the relation between v and i using standard analysis techniques on the circuit.

We assume for now that the circuit is resistive, i.e., it consists of resistors and sources (voltage and current).

Two circuits are equivalent at a given terminal pair if they have identical characteristics at the terminals. For two circuits where the characteristics are given by $f_1(v)$ and $f_2(v)$ respectively, then the circuits are equivalent if $f_1 = f_2$, despite any differences in topology, number and type of components/sources.

Commutable components:

- Resistors in series (derived by Ohm's law).
- Voltage sources in series (derived by Kirchhoff's voltage law).
- Resistors in parallel (derived by Ohm's law).
- Current Sources in parallel (derived by Kirchhoff's current law).

It can be convenient to approximate elements in a circuit to reduce the effective number of elements, this is typically done when $R_2 \gg R_1$ for series and $\frac{1}{R_2} \gg \frac{1}{R_1} \Leftrightarrow R_2 \ll R_1$ for parallel. In reality manufacturers do not produce resistors with any arbitrary resistance, thus approximations are often necessary in practice.

6.1 Source Transformation

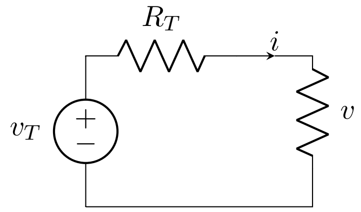
A voltage source in series with a resistor is equivalent to a current source in parallel with a resistor (of same resistance). We may replace any such part in a circuit by its equivalent. This can enable series or parallel reductions. The transformation can be achieved by evaluating the circuit as an open circuit and determining the characteristics.

In a resistive circuit we can choose any terminal pair a–b that satisfies Kirchhoff's current law, i.e., output current is equal to input current. The terminal characteristic at such a terminal pair is:

Thévenin's theorem (v_T and R_T for Thévenin resistor and voltage):

$$\frac{i}{i_{sc}} = 1 - \frac{v}{v_{oc}}$$

$$v = v_{oc} - \frac{v_{oc}}{i_{sc}} i = v_T - R_T i.$$



Thévenin equivalent circuit: from Thévenin's theorem we can multiply both sides by i_{sc} to obtain:

$$i = -\frac{i_{sc}}{v_{oc}}v + i_{sc} = -\frac{v}{R_N} + i_N,$$

where R_N and i_N are the Norton resistance and current, given by the circuit of a current source in parallel with a resistor.

Time varying voltage and current sources, for these we use the following notation:

- Constant voltages or currents: V or I.
- Time-varying voltages or currents: v or i.
- When a quantity may be constant or time-varying we use lower-case: v or i.
- Constant notation may be used for specifying a constant parameter in an expression for a time-varying quantity,:

$$v_1(t) = V_1 \cos(\omega t).$$

- We often omit the time-dependency (t) for convenience in notation.

We may also consider sources which are dependent on currents or voltages in other parts of the circuit or in another circuit. More generally, dependent sources may depend on other physical quantities (transducers) such as a microphone converting a sensed air pressure variation to a varying voltage.

For these we have a voltage-controlled voltage source(VCVS), voltage controlled current source(VCCS), current controlled voltage source(CCVS) and finally a current controlled current source(CCCS). These have the following terminal characteristics:

- VCVS: v_c, $\quad v = \mu v_c$, where v_c is controllable.
- VCCS: v_c, $\quad i = g v_c$, where v_c is controllable.
- CCVS: i_c, $\quad v = r i_c$, where i_c is controllable.
- CCCS: i_c, $\quad i = \beta i_c$, where i_c is controllable.

7 Capacitors and Inductors

7.1 Capacitor

Capacitors are elements capable of storing energy in the form of an electric field. The electric field is maintained between two conductors. Closely spaced but isolated. A capacitor also has a small inner resistance and inductance but these can for most applications be neglected. The electric field is maintained by a difference in charge between the two conductors. The two conductors are in principle “plates” facing each other. The capacitance is given by

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d},$$

where A is the area of the conductors and d is the distance between the conductors. ε is the permitivity of the medium between the plates.

The charge (on either plate) is given by

$$q(t) = Cv(t).$$

Charge cannot move directly between the plates but will have to take a “detour” through connected circuitry. Therefore, the charge and thereby voltage across the capacitor cannot change instantaneously.

7.1.1 Terminal Characteristics

The terminal characteristics of a capacitor are given by

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}.$$

Rearranging we can get $v(t)$:

$$\begin{aligned}
\mathrm{d}\, v(t) &= \frac{1}{C} i(t)\,\mathrm{d}t \\
\int_{t_0}^{t} \mathrm{d}\, v(t)\,\mathrm{d}t = v(t) - v(t_0) &= \frac{1}{C}\int_{t_0}^{t} i(t')\,\mathrm{d}t' \\
v(t) &= v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(t')\,\mathrm{d}t'.
\end{aligned}$$

The intragration constant $v(t_0)$ is the voltage on the capacitor at time t_0.

7.1.2 RC Circuits

Cirtcuits containing only resistors and capacitors. It is fairly simple to analyse what happens in RC circuits when the current or voltage changes.

Discharging is obtained by using KCL:

$$\begin{aligned} t &= 0 \\ v(t) &= V_0\Big|_{t=0} \\ \frac{v(t)}{R} + C\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= -\frac{v(t)}{RC} \\ v(t) &= e^{-\frac{t}{RC}}K \\ t &= 0^+ : v(0^+) = K \Rightarrow V_0 = v(0^-). \end{aligned}$$

The notation $t = 0^+$ means the positive time shortly after $t = 0$. Since the voltage across a capacitor cannot change instantaneously the voltage $v(0^-) = V_0 = v(0^+)$. Finally we define $\tau = RC$ $[\Omega F]$ and rewrite:

$$v(t) = e^{-\frac{t}{\tau}}V_0,\ t > 0.$$

The time constant τ $[s]$ determines the decay rate of the voltage across the capacitor.

Charging is obtained by using KVL:

$$\begin{aligned} t &= 0 \\ V_0 &= Ri(t) + v(0^+) + \frac{1}{C}\int_{0^+}^{t} i(t')\,\mathrm{d}t' \\ v(0^+) &= 0 \\ CV_0 &= CRi(t) + \int_{0^+}^{t} i(t')\,\mathrm{d}t' \\ 0 &= CR\frac{\mathrm{d}i(t)}{\mathrm{d}t} + [i(t')]_{0^+}^{t} \\ \frac{\mathrm{d}i(t)}{\mathrm{d}t} &= -\frac{1}{\tau}i(t) \\ i(t) &= e^{-\frac{t}{\tau}}i(0^+), \quad t > 0 \\ i(0^+) &= \frac{V_0}{R}. \end{aligned}$$

Just after the switch closes, C is seen as a short circuit because the voltage cannot change instantaneously, i.e., no voltage drop across C; $i(0^+) = V_0/R$.

Capacitors in Series :

$$\begin{aligned} v(t) &= \frac{1}{C}\int_{-\infty}^{\infty} i(t')\,\mathrm{d}t' + \frac{1}{C}\int_{-\infty}^{\infty} i(t')\,\mathrm{d}t' + \cdots + \frac{1}{C_n}\int_{-\infty}^{\infty} i(t')\,\mathrm{d}t \\ v(t) &= \sum_{i=1}^{n}\frac{1}{C_i}\int_{-\infty}^{\infty} i(t')\,\mathrm{d}t' \end{aligned}$$

Capacitors in Parallel :

$$i(t) = C_1 \frac{\mathrm{d}v(t)}{\mathrm{d}t} + C_2 \frac{\mathrm{d}v(t)}{\mathrm{d}t} + \cdots + C_n \frac{\mathrm{d}v(t)}{\mathrm{d}t}$$
$$i(t) = \sum_{i=1}^{n} C_i \frac{\mathrm{d}v(t)}{\mathrm{d}t}.$$

7.2 Inductors

Inductance is in some sense a counterpart to capacitance. Inductors are elements capable of storing energy in the form of a magnetic field.

- A current flowing in a conductor induces a magnetic field.
- The magnetic field, called the flux ϕ, is proportional to the current.
- Flux Linkage is also proportional to the current $\lambda = Li(t)$.
- L is a constant of proportionality called the inductance.
- Flux Linkage depends on the magnetic field surrounding the conductor and the geometry of the conductor.

The magnetic field produced by a current in a conductor opposes a change in that current by inducing a voltage that tends to keep the current from changing. The induced voltate v is given by Faraday's law of induction

$$v(t) = \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}, \quad \lambda = Li(t).$$

If an inductor is wound from N turns, the flux linkage is $\lambda = N\phi$. Thus we have

$$\lambda = N(N\phi_1) = kN^2 i(t) = Li(t).$$

A larger inductance is obtained by

- Increasing section area.
- Increasing number of wounds.
- Inserting a core of high magnetic permeability.

7.2.1 Terminal Characteristics

The terminal characteristics of an inductor are given by:

$$v(t) = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = L \frac{\mathrm{d}i(t)}{\mathrm{d}t}$$

and for current:

$$\int_{t_0}^{t} \mathrm{d}i(t') \, \mathrm{d}t = i(t) - i(t_0) = \frac{1}{L} \int_{t_0}^{t} v(t') \, \mathrm{d}t'$$
$$i(t) = \frac{1}{L} \int_{t_0}^{t} v(t') \, \mathrm{d}t' + i(t_0).$$

We can see how the inductor complements the capacitor - their terminal characteristics are "opposite".

7.2.2 RL Circuits

Circuits containing only resistors and inductors (analogous to RC circuits). We use a switch and consider the scenario where the switch is on:

$$\begin{aligned}
i(t) &= i(0^-) + \frac{1}{L}\int_{0^-}^{t} v(t')\,\mathrm{d}t' \\
i(0^-) &= 0 \\
i(t) &= \frac{1}{L}\int_{0^-}^{t} v(t')\,\mathrm{d}t'.
\end{aligned}$$

By KCL the sum of the current must be zero and hence

$$\begin{aligned}
\frac{v(t)-V_0}{R} + \frac{1}{L}\int_{0^-}^{t} v(t')\,\mathrm{d}t' \\
\frac{L}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= -v(t) \\
v(t) &= e^{-\frac{R}{L}t}K \\
v(0^+) &= V_0 \\
K &= V_0
\end{aligned}$$

Just after the switch closes, L is seen as an open circuit because the current cannot change instantaneously, i.e., no current through L; $v(0^+) = V_0$. Again we can define the time constant $\tau = L/R$ and we have

$$v(t) = e^{-\frac{t}{\tau}}V_0.$$

7.2.3 Inductors in Series

$$\begin{aligned}
v(t) &= L_1\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \cdots + L_n\frac{\mathrm{d}i(t)}{\mathrm{d}t} \\
v(t) &= \sum_{i=1}^{n} L_i\frac{\mathrm{d}i(t)}{\mathrm{d}t}.
\end{aligned}$$

7.2.4 Inductors in Parallel

$$\begin{aligned}
i(t) &= \frac{1}{L_1}\int_{-\infty}^{\infty} v(t')\,\mathrm{d}t' + \cdots + \frac{1}{L_n}\int_{-\infty}^{\infty} v(t')\,\mathrm{d}t' \\
i(t) &= \sum_{i=1}^{n}\frac{1}{L_i}\int_{-\infty}^{\infty} v(t')\,\mathrm{d}t'.
\end{aligned}$$

8 Elektriske Systemer

I elektriske systemer har vi kigget på systemer bestående af

- resistorer
- kapacitorer
- induktorer.

Systemet bliver påvirket af et signal og udsender også et signal. Man skelner mellem input kaldet “Effort signal” og output kaldet “Flow signal”. Ved at påvirke systemet med en kraft “effort”(spænding), så får man noget “flow” (strøm/**bevægelse** af elektroner).

Effekten er givet ved produktet af spændingen og strømmen ved $P = VI$ angivet i Watt og energien er $E = \int P \, \mathrm{d}t$.

Et batteri har en elektromotorisk **kraft** og dens effort til et system er derved i overensstemmelse med en kraft.

9 Mekaniske Systemer

I mekanik har vi talt om systemer som også bliver påvirket af et “effort signal” og som giver et “flow signal”. Komponenterne i det mekaniske system er

- masse
- fjeder
- dæmper.

Ved at påvirke systemet med en kraft/moment opstår der en hastighed/vinkelhastighed. Produktet af effort signal og flow signal giver effekten: $Fv = P$ for translatorisk bevægelse, ved rotation er effekten $\tau\theta = P$, hvor τ er kraftmomentet.

10 Analyse af System

Givet er en masse som påvirkes af en kraft og får derved en translatorisk hastighed. Newtons 2. lov; $F = ma = m\dot{v}$. Altså er givet en relation mellem effort variablen og den tidsafledede af flow-variablen.

- Masse: Hvile $\Rightarrow$ hvile, bevægelse $\Rightarrow$ bevægelse medmindre massen påvirkes af en kraft.
- Konstant hastighed: Massen vil modvirke ændringer i denne hastighed.
- Massen har hukommelse: Husker fortiden ved at lagre kinetisk energi $KE = \frac{1}{2}mv^2$.

I det roterende domæne vil inertimomentet I modvirke ændringer af rotationen og forsøger dermed at bevare konstant vinkelhastighed ω. Kraftmomentet τ er givet ved $\tau = I\dot{\omega}$, som er analog til Newtons 2. lov før: der er igen en relation mellem effort og den tidsafledede af flow variablen.

- Inertimoment: Husker fortiden ved at lagre roterende kinetisk energi: $RKE : \frac{1}{2}I\omega^2$.

I det elektriske domæne modellerere vi et system bestående af en spole med selvinduktion. Spolen påtrykkes af en strøm og spændingen er da

$$\underbrace{V}_{F} = \underbrace{L}_{m}\underbrace{\dot{i}}_{\dot{v}},$$

hvilket er analog til relationerne mellem effort og flow i det mekaniske domæne.

- Spole: Producerer et magnetisk felt pga. strømmen i i gennem spolen.
- Spole: Forsøger at bibeholde det magnetiske felt og dermed strømmen gennem den (fastholde konstant flow).
- Spole: Husker fortiden ved at lagre elektrisk energi $EE = \frac{1}{2}Li^2$.

Altså svarer en masse/inerti i det mekaniske domæne til en spole i det elektriske.

For en fjeder gælder Hookes lov $F = kx = k\int v \, \mathrm{d}t$. For en fjeder er der en relation mellem effort-variablen (kraft/moment) og tids-integralet af flow-variablen (udstrækning).

- Fjeder: I steady state har en fjeder en bestemt længde/vinkel.
- Fjeder: Husker fortiden ved at lagre elastisk potentiel energi $EPE = \frac{1}{2}kx^2$.

I det elektriske domæne er fjederen analog til en kapacitor, hvor effort-variablen er en spænding, relationen til flow-variablen er da

$$V = \frac{1}{C}\int i(t) \, \mathrm{d}t = \underbrace{\frac{1}{C}}_{k}\underbrace{Q}_{x}.$$

Konstanten stivhed er i mekanik k og $\frac{1}{C}$ i elektriske systemer. Føjeligheden (compliance) svarer til kapacitansen C.

- Kapacitor: Foretrækker en upåvirket ladning.
- Kapacitor: Husker fortiden ved at lagre elektrisk potentiel energi $EPE = \frac{1}{2}\frac{1}{C}Q^2$.

I det mekaniske domæne er det sidste komponent dæmperen. Dæmperen påtrykker en hastighed med en modsatrettet kraft. En dæmper bliver bl.a. anvendt i biler, både translatorisk og rotatorisk. Relationen mellem effort-variablen (kraft) og flow-variablen (hastighed/vinkelhastighed) er givet ved

$$F = cv, \quad \tau = c\omega.$$

- Dæmper: Ingen hukommelse, den kan ikke lagre energi, men forbruger energi.

I det elektriske domæne svarer dæmperen til en resistor, med spænding som effort-variabel givet ved Ohms lov $V = RI$.

11 Summary

Mekanik	$\rightarrow$	Elektronik
Masse		Spole
Fjeder		Kapacitor
Dæmper		Resistor
Gear		Transformator

Effort-variabler(uafhængige)	Flow-variabler(afhængige)
Spænding	Strøm
Kraft	Hastighed
Moment	Vinkelhastighed
Lydtryk	Luftstrøm

Example 11.1
1) En masse påvirket af en kraft, en fjeder og en dæmper (friktion). Givet er kraften F, massen m, fjederkoefficienten c og dæmpningen r.

$$\begin{aligned} F &= ma + rv + \frac{1}{C}x \\ &= m\dot{v} + rv + \frac{1}{C}\int v \, \mathrm{d}t \end{aligned}$$

Nu anvendes Laplace transformationen:

$$\begin{aligned} F(s) &= msV(s) + rV(s) + \frac{1}{C}V(s)\frac{1}{s} \\ \frac{F(s)}{V(s)} &= \frac{\text{kraft}}{\text{hastighed}} = \frac{\text{spænding}}{\text{strøm}} = \text{impedans} \\ &= Z(s) \\ &= ms + r + \frac{1}{sC} \end{aligned}$$

som svarer til en spole, resistor og kapacitor. Det kredsløb påtrykkes af en spænding $F(s)$(kraft) og en strøm $V(s)$(strøm).

2) En generator hænger fra et loft og trækker en masseløs og stiv plade op, hvorpå en masse er koblet til pladen ved hjælp af en fjeder og en dæmper. I det system er der to hastigheder (en for massen og en for pladen) og en kraft fra generatoren. Bevægelsesligningerne er da

$$\begin{aligned} F &= m\dot{v}_2 \\ &= r(v_1 - v_2) + \frac{1}{C}\int (v_1 - v_2)\, \mathrm{d}t \\ F(s) = msV_2(s) &= r(V_1(s) - V_2(s)) + \frac{1}{C}\frac{1}{s}(V_1(s) - V_2(s)) \end{aligned}$$

hvor $msV_2(s)$ svarer til en spole med strøm $V_2(s)$, $r(V_1(s) - V_2(s))$ svarer til en modstand med strøm $V_1(s) - V_2(s)$ og til sidst $\frac{1}{sC}(V_1(s) - V_2(s))$ svarer til en kapacitor med strøm $V_1(s) - V_2(s)$. I diagrammet over systemet sidder fjeder og dæmper i parallel, som begge er i serie med massen. Det ækvivalente elektriske kredsløb vil bestå af en spole(massen) i parallel med en serieforbindelse af en resistor(dæmperen) og en kapacitor(fjeder). Denne analogi svarer kaldes for en impedans analogi.