Implementing the Core Chase for the Description Logic ALC

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1 Introduction

An important problem in database is the conjunctive query entailment. This problem can be described in a first order logic background: Given a knowledge base $O = (R, F)$ where F is a set of conjunctive formulas (that are formulas constructed only with conjunction and existential quantification) and R is a set of rules (that are formulas of the form $\forall\vec{x}.\forall\vec{y}.(A(\vec{x},\vec{y}) \to \exists\vec{z}.B(\vec{x},\vec{z}))$ where $\vec{u}$ represents a tuple of variables), and given a query Q (that is a conjunctive formula), determine if the knowledge base O entails the query Q. We usually use reasoning algorithms to answer this problem. We can notice that in practice, a lot of formulas can be express via conjunctive formulas.

Example 1.1. For the knowledge base O composed of the set of conjunctive formulas $F = \{\mathit{Father}(\mathit{Michel})\}$ and the set of rules $R = \{\forall x.\mathit{Father}(x) \to \exists y.\mathit{IsTheSonOf}(y,x)\}$, and for the query $Q = \exists y.\mathit{IsTheSonOf}(y, Michel)$, the knowledge base O entails Q.

By definition, a knowledge base O entails a query Q if every model of O is a model of Q. It is not practical because a knowledge base can have an infinite number of model.

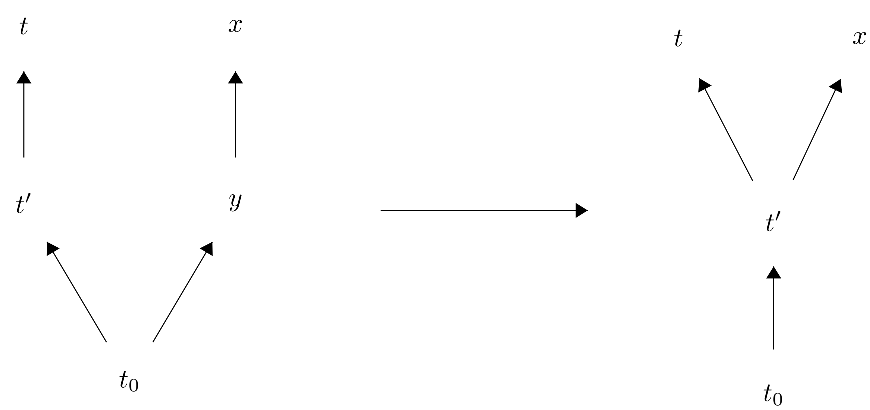
To deal with this problem, we can compute an universal model of the knowledge base O. That is a model of O that is entailed by all the models of O. If a such model U exists, we just need to show that U entails Q to conclude that O entails Q. Hence, to solve query entailment, we just have to compute a finite universal model U for a given input knowledge base and look if U entails Q.

To compute these models, we can use algorithms called the chase. We will present in this document the oblivious chase, the restricted chase and then the core chase. The last chase is the best in the sense of it terminates if and only if there exists a finite universal model.

Nevertheless, it is the slowest so it it is never used for practical applications.

Therefore, we will focus on a restricted type of knowledge base $O = (R, F)$ where the set of conjunctive formulas F is ground (that is there is no variable in the formula) and where the rule contained in R are Horn-$\mathcal{ALC}$ axioms. It is an interesting restriction because we can represent the results of the chase by a tree.

We will create a quicker chase, that we called the merge chase, that would produce the same output than the core chase. It is based on the idea that in a tree, there exists a sibling relation between the nodes (that is two nodes are siblings if they have the same father) and we will see that, in some conditions, a sibling of a node can be merged on this node like in the figure below where y is a sibling of t' and is merged on it.



We will then try to extend the merge chase on other type of knowledge bases.

Contents

First of all, we will define the usefull notions for this paper and give some well known properties. Then, we will move on our contributions.

2 Background

This section deals with a lot of first order logic notions like interpretations and formulas.

2.1 Facts

2.1.1 Syntax

We considered a set of variables **Vars** (often noted $x, y, x_1, \ldots$), a set of constants **Csts** (often noted $a, b, c, c_1, \ldots$), and a set of predicates **Preds** ($P, Q, R, P_1, \ldots$). **Csts**, **Vars**, and **Preds** are pairwise disjoint. A *term* (often noted $t, t_1, \ldots$) is a variable or a constant. We note **Terms** the set of terms.

Definition 2.1. If $t_1, \ldots, t_n$ are terms and P is a predicate of arity n, then $P(t_1, \ldots, t_n)$ is an *atom*. The atom $P(t_1, \ldots, t_n)$ is *ground* if $t_1, \ldots, t_n$ are constants.

Definition 2.2. A *factbase* F is an existentially closed conjunction of atoms, that is, a formula that does not contain occurrences of free variables and is of the form $\exists x_1, \ldots, x_n.P_1(t_1^1, \ldots, t_{k_1}^1) \wedge \ldots \wedge P_m(t_1^m, \ldots, t_{k_m}^m)$ where t_i^j are terms and P_i are predicates. A factbase is *ground* if each of its atoms is ground.

In some articles, the factbases are always considered as ground but in this document, we consider factbases that may not be ground. Consequently, a boolean conjonctive query will be a factbase, so we will only talk about factbases and not introduce the notion of query.

For convenience, we identify factbases as sets of atoms, which allows us to use set notions such as set inclusion. For example, we identify the factbase

$$\exists x, x_1, x_2, x_3.P(x) \wedge Q(x, a) \wedge R(x_1, x_2, x_3, b)$$

with the set of facts

$$\{P(x), Q(x, a), R(x_1, x_2, x_3, b)\}$$

For a formula A, let $\boldsymbol{Vars}(A)$, $\boldsymbol{Csts}(A)$, and $\boldsymbol{Terms}(A)$ be the sets of variables, constants, and terms that occur in A, respectively.

Definition 2.3. A factbase F *entails* another factbase F' (often noted $F \models F'$) if each interpretation satisfying F satisfies F'.

Definition 2.4. A factbase F is equivalent to another factbase F' if $F \models F'$ and $F' \models F$.

2.1.2 Homomorphism

Definition 2.5 (Substitution)**.** A *substitution* $\sigma : X \to$ **Terms** is a function where X is a set of variables. For example $\{x \mapsto z, y \mapsto a\}$ is a substitution from $\{x, y\}$ to **Terms**. By extension:

- if $c \in$ **Csts**, then $\sigma(c) = c$;

- if $x \in \mathbf{Vars} \setminus X$, $\sigma(x) = x$;
- if $f = P(t_1, \ldots, t_n)$ is an atom, then $\sigma(f) = P(\sigma(t_1), \ldots, \sigma(t_n))$; and
- if $F = \{f_1, \ldots, f_n\}$ is a factbase, then $\sigma(F) = \{\sigma(f_1), \ldots, \sigma(f_n)\}$.

Definition 2.6. For two factbases F and F', a *homomorphism* from F to F' is a substitution $\sigma : \mathbf{Vars}(F) \to \mathbf{Terms}$ where $\sigma(F) \subseteq F'$. Sometimes, we will say that we *map* a variable x to a term t if $\sigma(x) = t$.

Definition 2.7. For two factbases F and F', an *isomorphism* h from F to F' is a bijective homomorphism where its inverse is a homomorphism from F' to F.

For the remainder of this paper, we identify sets of facts that are unique up to isomorphism.

Theorem 2.1 (Homomorphism Theorem)**.** A factbase F *entails* another factbase Q if and only if there exists a homomorphism from Q to F.

The previous theorem has been proved in ([**?**],theorem 6.2.3).

Example 2.1. The factbase $F = \{P(b, a), A(x)\}$ entails the factbase $Q = \{P(x, a), P(y, z)\}$ due to the homomorphism $\{x \mapsto b, y \mapsto b, z \mapsto a\}$.

2.1.3 Core

For a factbase F, let $id_{|F}$ be the substitution mapping each variable in $\mathbf{Vars}(F)$ to itself. And for a subsitution σ defined on a factbase F' containing F, let $\sigma_{|F}$ be the subsitution σ defined only on $\mathbf{Vars}(F)$. In practice, it is better to work on smaller factbases. It leads us to the notions of retracts and cores:

Definition 2.8. A factbase F' is a *retract* of another factbase F if $F' \subseteq F$ and $F' \models F$. A *retractation* from F to F' is a homomorphism σ from F to F' such that $\sigma_{|F'} = id_{|F'}$. F' is a *strict retract* of F if F' is a retract of F and $F' \neq F$.

Proposition 2.1. *The factbase F' is a retract of the factbase F if and only if $F' \subseteq F$ and there exists a retractation from F to F'.*

Definition 2.9. If a factbase F does not contain a strict retract, then we say that F is a *core*. A *core* of a factbase F (noted *core*(F)) is a subset of F that is a core.

Proposition 2.2. *The cores of a finite factbase F are unique up to isomorphism.*

Hence, we speak of "the" core of a factbase.

Example 2.2. $F_1 = \{B(x, y), R(y, z)\}$ is the core of

$$F = \{B(x, y), R(y, z), B(x, w), R(w, z)\}$$

because:

- $F_1 \subseteq F$;
- $\{x \mapsto x, y \mapsto y, z \mapsto z, w \mapsto y\}$ is a homomorphism from F to F_1, so F_1 is a retract of F;
- all strict subsets of F_1 are not retracts of F_1.

Note that $F_2 = \{B(x, w), R(w, z)\}$ is also the core of F and is indeed isomorphic to F_1 due to the homomorphism $\{x \mapsto x, y \mapsto w, z \mapsto z\}$.

2.2 Existential Rules

2.2.1 Syntax

Definition 2.10. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be some tuples of variables that are pairwise disjoint. An *(existential) rule* α is a first-order formula of the form

$$\forall \vec{x}.\forall \vec{y}.(A(\vec{x}, \vec{y}) \rightarrow \exists \vec{z}.B(\vec{x}, \vec{z}))$$

where A and B are conjunctions of atoms. We define $\mathit{body}(\alpha) = A$ and $\mathit{head}(\alpha) = B$.

We omit the universal quantifiers when representing existential rules.

Definition 2.11. A *knowledge base* O is a pair (R, F) where R is a set of existential rules and F is a ground factbase.

2.2.2 Semantics

Definition 2.12 (Entailment)**.** A factbase F *entails* a rule α if each interpretation satisfying F satisfies α. We will note $F \models R$ if F entails each rule of the rule set R.

Theorem 2.2. A factbase F *entails* a rule $\alpha = A(\vec{x}, \vec{y}) \rightarrow \exists \vec{z}.B(\vec{x}, \vec{z})$ if and only if for every homomorphism σ from A to F, there exists an extension of σ that is a homomorphism from B to F.

Definition 2.13 (Model)**.** A factbase M is a *model* for a knowledge base $O = (R, F)$ if $M \models F$ and $M \models R$.

Example 2.3. We pose $O = (\{\alpha\}, F)$ where $\alpha = A(x) \rightarrow \exists z.R(x, z) \wedge A(z)$ and $F = \{A(b)\}$. An universal model of O is

$$U = \{A(b), R(b, x_0)\} \cup \{A(x_i) \mid i \in \mathbb{N}\} \cup \{R(x_i, x_{i+1}) \mid i \in \mathbb{N}\}$$

The knowledge base O does not admit finite universal models.

Definition 2.14 (Entailment)**.** A knowledge base O *entails* a factbase B (often noted $O \models B$) if for each model M of O, $M \models B$.

Definition 2.15 (Universal model)**.** A factbase U is an *universal model* for a knowledge base $O = (R, F)$ if for every model M of O, $M \models U$.

We are interested by universal models because they can be used to solve fact entailment:

Proposition 2.3. *A knowledge base O entails a factbase B if there exists a universal model U for O such that $U \models B$.*

An important problem that this document has to deal with is: Given a knowledge base $O = (R, F)$ and a factbase Q, does $O \models Q$? It is well-known that this problem is undecidable ([**?**], theorem 4).

2.3 The Chase

The process of applying rules on a factbase in order to infer more knowledge is called forward chaining. Forward chaining in existential rules is usually achieved via a family of algorithms called the chase. It can be seen as a two-steps process. It first repeatedly applies rules to the set of facts (and eventually computes sometimes the core to supress redundant facts). Then it looks for an answer to the query in this saturated set of facts. This saturated set of facts is a universal model of the knowledge base. The chase is sound and complete; so it must be non-terminating since the problem of entailment is undecidable. To determine how we apply a rule to a set of fact, we introduce the notion of trigger:

Definition 2.16 (Trigger)**.** Let T be a rule set, α be a rule, σ be a substitution, and F be a factbase. The tuple $t = (\alpha, \sigma)$ is an *oblivious trigger* for F if:

- the domain of σ is the set of all variables occurring in $Body(\alpha)$.
- σ is a homomorphism from $Body(\alpha))$ to F.

In this case, we say that t is *applicable* on F.

The tuple $t = (\alpha, \sigma)$ is a *restricted trigger* for F if t is an oblivious trigger for F and if for all $\hat{\sigma}$ that extend σ over $\mathbf{Vars}(Head(\alpha))$, $\hat{\sigma}(Head(\alpha)) \nsubseteq F$.

Notice that a restricted trigger is also an oblivious trigger. We will therefore use the term trigger to talk about the oblivous and the restricted trigger.

The chase will considere triggers to infer new knowledge from an initial factbase. We explain now how it would apply a trigger, giving rise to the notion of application.

Definition 2.17 (application)**.** For a trigger $t = (\alpha, \sigma)$ for the factbase F where α is of the form $A(\vec{x}, \vec{y}) \rightarrow \exists \vec{z}.B(\vec{x}, \vec{z})$, we pose σ^s the substitution that extends σ over $\mathbf{Vars}(Head(\alpha))$ such that for $z \in \vec{z}$, $\sigma^s(z) = f^z_\alpha(\sigma(\vec{x}))$ where $f^z_\alpha(\sigma(\vec{x}))$ is a fresh variable unique with respect to the trigger t and the variable z. The factbase $\mathbf{appl}(F, t) = F \cup \sigma^s(Head(\alpha))$ is called an *application* on the factbase F through the trigger $t = (\alpha, \sigma)$.

We say that the trigger t *has been applied* on F if $\mathbf{appl}(F, t) = F$.

Example 2.4. . If $\alpha = A(x, y) \rightarrow \exists z.B(x, z)$, $F = \{A(b, c)\}$, and $\sigma = \{x \mapsto b, y \mapsto c\}$ then (α, σ) is a restricted trigger for F. We have

$$\mathbf{appl}(F, (\alpha, \sigma)) = \{A(b, c), B(b, f^z_\alpha(b, c)\}$$

Definition 2.18 (Derivation)**.** An *oblivious derivation* (respectively a *restricted derivation*) for a knowledge base $O = (F, R)$ is a (possibly infinite) sequence $D = F_0, t_1, F_1, t_2, F_2, \ldots$ where

- $F_0, F_1, \ldots$ are factbases such that $F_i \subsetneq F_{i+1}$.
- $t_1, t_2, \ldots$ are oblivious triggers (resp. restricted triggers).
- $F_0 = F$.
- For all $i > 0$, $F_i = \mathbf{appl}(F_{i-1}, t_i)$.

Definition 2.19 (Fairness)**.** The oblivious (resp. restricted) derivation $D = F_0, t_1, F_1, t_2, F_2, \ldots$ is *fair* if for every i and every oblivious (resp. restricted) trigger t for F_i, there exists $k \geq i$ such that $\mathbf{appl}(F_k, t_k) = F_k$ (resp. t is not anymore a restricted trigger for F_k).

A fair derivation garantees that we consider every possible application. An easy way to have a fair derivation is to do a breadth-first search (BFS) on the terms, that is, always apply the oldest triggers before applying other triggers.

We will now define the oblivious and restricted chase, It is defined in [**?**].

Definition 2.20. An *oblivious chase* (resp. a restricted chase) for a knowledge base $O = (F, R)$ is a fair oblivious (resp. restricted) derivation $D = F_0, t_1, F_1, t_2, F_2, \ldots$

For every oblivious chase $D = F_0, t_1, F_1, t_2, F_2, \ldots$ for O, we have $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$. The result of an oblivious chase does not depend on the derivation D so we can pose *Obl(O)* $= \cup_{i \in \mathbb{N}} F_i$. We say that the oblivious chase *terminates* if *Obl(O)* is finite.

It is well known that:

Theorem 2.3. For a knowledge base O, *Obl*(O) is an universal model of O.

Consequently, the oblivious chase can be used to solve query entailment.

It is more difficult to define the result of the restricted chase because the result depends on the order of the application of the rules. We define

$$\textit{Res(O)} = \{\cup_{i \in \mathbb{N}} F_i \mid D = F_0, t_1, F_1, t_2, F_2, \ldots \text{is a retricted chase for } O\}$$

We say that the restricted chase *terminates* if there exists an element in *Res(O)* that is finite.

It is well known that:

Theorem 2.4. For a knowledge base O and for every $U \in \mathit{Res}(O)$, U is an universal model of O.

The oblivious chase can do a lot of applications that are useless:

Example 2.5. Suppose that we have the knowledge base $O = (\{\alpha\}, F)$ where $\alpha = A(x, y) \rightarrow \exists z.A(y, z) \wedge A(z, y)$ and $F = \{A(a, b)\}$. An oblivious chase derivation is $F_0, t_1, F_1, t_2, F_2, ...$ where

- $F_0 = F$,
- $t_1 = (\alpha, \{x \mapsto a, y \mapsto b\})$,
- $F_1 = \{A(a, b), A(b, z_1), A(z_1, b)\}$ where $z_1 = f^z_\alpha(a, b)$,
- $t_2 = (\alpha, \{x \mapsto b, y \mapsto z_1\})$,
- $F_2 = F_1 \cup \{A(z_1, z_2), A(z_2, z_1)\}$ where $z_2 = f^z_\alpha(b, z_1)$,
- $\ldots$

It will never terminate because each new atom brings new rule applications. So the oblivious chase does not terminate on O whereas the restricted chase terminates. A restricted chase derivation is F_0, t_1, F_1 where $F_0 = F$, $t_1 = (\alpha, \{x \mapsto a, y \mapsto b\})$, and $F_1 = \{A(a, b), A(b, z_{t_1}), A(z_{t_1}, b)\}$. This derivation is fair because there is not anymore any restricted trigger for F_1. We have $F_1 \in$ *Res(O)*.

Theorem 2.5. A knowledge base O entails a factbase B if and only if $\mathit{Obl}(O) \models B$ and O entails B if and only if for every $U \in \mathit{Res}(O)$, $U \models B$.

2.3.1 The core chase

It has been firstly defined in [**?**].

Definition 2.21 (Core derivation)**.** A *core derivation* for a knowledge base $O = (R, F)$ is a (possibly infinite) sequence $D = F_0, F_1, F_2, \ldots$ where $F_0 = F$, and for $i > 0$, either $F_i = \mathbf{appl}(F_{i-1}, t_i)$ is obtained by an application with t_i an oblivious trigger, or F_i is the core of F_{i-1}.

Definition 2.22 (Fairness)**.** A core derivation $D = F_0, F_1, F_2, \ldots$ is *fair* if:

- For every i, for every oblivious trigger t for F_i, there exists $k \geq i$ such that $\mathbf{appl}(F_k, t) = F_k$.

- For every i, there exists $k \geq i$ such that F_k is a core.

Definition 2.23. A *core chase* for a knowledge base $O = (R, F)$ is a fair core derivation $D = F_0, F_1, F_2, \ldots$ The core chase *terminates* on O if it is a finite core derivation.

The article [**?**] has proven that the result of the core chase on a knowledge base is unique up to isomorphism. Therefore, we can define the result of the core chase:

Definition 2.24. If the core chase terminates on O due to the finite core derivation $D = F_0, F_1, F_2, \ldots, F_i$, then we pose $C(O) = F_i$. Otherwise, if the core chase does not terminate, $C(O)$ is undefined.

The following theorem has been proven in ([**?**], theorem 7)

Theorem 2.6. The knowledge base $O = (R, F)$ admits a finite universal model if and only if the core chase algorithm terminates on O.

There exists knowledge bases where the restricted chase does not terminate whereas the core chase terminates.

Example 2.6. Suppose that we have the knowledge base $O = (\{\alpha\}, F)$ where $\alpha = A(x, y) \rightarrow \exists z.(A(x, x) \land A(y, z))$ and $F = \{A(a, b)\}$. A restricted chase derivation is $F_0, t_1, F_1, t_2, F_2, \ldots$ where

- $F_0 = F$,
- $t_1 = (\alpha, \{x \mapsto a, y \mapsto b\})$,
- $F_1 = F_0 \cup \{A(a, a), A(b, z_1)\}$ where $z_1 = f^z_\alpha(a, b)$,
- $t_2 = (\alpha, \{x \mapsto b, y \mapsto z_1\})$,
- $F_2 = F_1 \cup \{A(b, b), A(z_1, z_2)\}$ where $z_2 = f^z_\alpha(b, z_1)$,
- $t_3 = (\alpha, \{x \mapsto z_1, y \mapsto z_2\})$,
- $F_3 = F_2 \cup \{A(z_1, z_1), A(z_2, z_3)\}$ where $z_3 = f^z_\alpha(z_1, z_2)$,
- $\ldots$

It will never terminate because each new atom brings new restricted triggers.

The core chase terminates on O: a core chase derivation is $F_0, F_1, F_2, F_3, F_4, F_5$ where

- $F_0 = F$,
- $t_1 = (\alpha, \{x \mapsto a, y \mapsto b\})$,

- $F_1 = \mathbf{appl}(F_0, t_1) = F_0 \cup \{A(a,a), A(b, z_1)\}$ where $z_1 = f_\alpha^z(a,b)$,
- $t_2 = (\alpha, \{x \mapsto b, y \mapsto z_1\})$,
- $F_2 = \mathbf{appl}(F_1, t_2) = F_1 \cup \{A(b,b), A(z_1, z_2)\}$ where $z_2 = f_\alpha^z(b, z_1)$,
- $t_3 = \{x \mapsto a, y \mapsto a\}$, $F_3 = \mathbf{appl}(F_2, t_3)$, $t_4 = \{x \mapsto b, y \mapsto b\}$, and $F_4 = \mathbf{appl}(F_3, t_4) = F_2 \cup \{A(a, f_\alpha^z(a,a)), A(b, f_\alpha^z(b,b))\}$
- $F_5 = \mathit{Core}(F_4) = \{A(a,a), A(a,b), A(b,b)\}$.

There is not anymore any oblivious trigger for F_5 so the derivation is fair. Consequently the core chase terminates on O.

Definition 2.25. For a knowledge base O and an operation that transforms a factbase F in another factbase $f(F)$, we say that this operation *preserves the universality* if for each model M of O, $M \models F$ implies $M \models f(F)$.

Proposition 2.4. *The application of a rule and computing a core are operations that preserve the universality.*

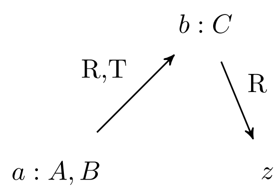
Proposition 2.5. *For a knowledge base O, the result of a chase for O is a universal model if the result is a model of O and if the operations used on the factbases preserve the universality.*

3 The Merge Chase

The core chase always terminates when there exists a finite universal model but this core chase is very expensive in time because computing the core of a factbase is hard. Therefore we are presenting a more efficient way to solve this problem in the particular case of Horn-$\mathcal{ALC}$.

Definition 3.1. For a factbase F and a term t, we note $\boldsymbol{Preds}_F^1(t)$ the set of unary predicates P such that $P(t) \in F$. For two terms t and t', we note $\boldsymbol{Preds}_F^2(t,t')$ the set of binary predicates P such that $P(t,t') \in F$.

In this section, we only consider factbases with predicates of arity one or two. Hence, we will represent a factbase F by a labelled graph $G = (V, E)$ where $V = \{t \mid t \in \mathbf{Terms}\}$ and $E = \{(t_1, t_2) \mid t_1, t_2 \in \mathbf{Terms} \wedge \mathbf{Preds}_F(t_1, t_2) \neq \emptyset\}$. We label the vertex $v \in V$ by exactly the predicates in $\mathbf{Preds}_F^1(v)$ and we label the edge between the terms t_1 and t_2 by exactly the predicates in $\mathbf{Preds}_F^2(t_1, t_2)$. For example with $F = \{A(a), B(a), R(a,b), T(a,b), C(b), R(b,z)\}$:



3.1 Horn-$\mathcal{ALCH}$

Horn-$\mathcal{ALCH}$ has been introduced in [**?**]

Definition 3.2 (Horn-$\mathcal{ALCH}$ axioms)**.** A *Horn-$\mathcal{ALCH}$ axiom* is an existential rule of the form:

$$A_1(x) \wedge A_2(x) \to B(x) \tag{1}$$
$$A(x) \wedge R(x,y) \to B(y) \tag{2}$$
$$A(x) \to \exists y.R(x,y) \wedge B(y) \tag{3}$$
$$R(x,y) \wedge B(y) \to A(x) \tag{4}$$
$$R_1(x,y) \wedge R_2(x,y) \to S(x,y) \tag{5}$$

We fix $O = (R, G)$ a knowledge base for this section where R is a Horn-$\mathcal{ALCH}$ rule set.

We can notice that all the variables are introduced by a Horn-$\mathcal{ALCH}$ axiom of the form (3).

We will now introduce some notions in order to introduce the new chase and prove that it does what we want.

Definition 3.3. Let $D = F_0, F_1, \ldots, F_k$ be a chase derivation of the knowledge base O. For a term t and a variable x appearing in F_k, we say that $t \prec x$ if there exists a rule α and a variable y such that $x = f^y_\alpha(t)$. We write $\prec^+$ to denote the transitive closure of $\prec$. We note $\boldsymbol{Succ}(t)$= $\{y \in \mathbf{Vars} \mid t = y \vee t \prec^+ y\}$

Proposition 3.1. *Let F be a factbase that occurs in a chase derivation of the knowledge base O. For a term t and a variable x, if there exists a rule α and a variable y such that $x = f^y_\alpha(t)$, then α is of the form* (3)*.*

Proposition 3.2. *Let F be a factbase that occurs in a chase derivation of the knowledge base O. For every variable x, there exists exactly one predecessor for $\prec$.*

Proposition 3.3. *Let $D = F_0, F_1, \ldots, F_k$ be a chase derivation of the knowledge base O. $\prec^+$ is a strict partial order over the set of terms of F_k.*

Proof. Suppose for a contradiction that there exists a term t such that $t \prec^+ t$. There exists then $n \in \mathbb{N} \setminus \{0\}$ and terms $t_0, \ldots, t_n$ such that $t = t_0 \prec t_1 \prec \cdots \prec t_n = t$. By Proposition **??**, there exists rules $\alpha_1, \ldots, \alpha_n$ and variables $v_1, \ldots, v_n$ such that $t_n = f_{\alpha_n}^{v_n}(f_{\alpha_{n-1}}^{v_{n-1}}(\cdots f_{\alpha_1}^{v_1}(t_0)\cdots))$. It is a contradiction since $t_0 = t$ and $t_n = t$.

Therefore $\prec^+$ is irreflexive. By construction, $\prec^+$ is transitive; So $\prec^+$ is a strict partial order

□

We have shown in the proof that the graph induced by $\prec$ does not contain any cycle. Therefore, with the last proposition and Proposition **??**:

Proposition 3.4. *Let F be a factbase that occurs in a chase derivation of the knowledge base O. The graph induced by $\prec$ is a forest of trees.*

We can now define the notion of tree of a term t that is all the facts containing only variables in $\mathbf{Succ}(t)$:

Definition 3.4 (Tree)**.** Let F be a factbase that occurs in a chase derivation of the knowledge base O. For a term t, we pose

$$\begin{aligned} \mathit{Tree}_F(t) =& \{A(t') \mid A(t') \in F \land t' \in \mathbf{Succ}(t)\} \\ & \cup \{R(t', x) \mid R(t', x) \in F \land t', x \in \mathbf{Succ}(t)\} \end{aligned}$$

We now define the notion of mergeable variable in order to consider a new operation.

Definition 3.5 (Mergeable variable)**.** Let F be a factbase that occurs in a chase derivation of the knowledge base O. For two terms t_1 and t_2 such that $t_1 \neq t_2$, t_1 is *mergeable* on t_2 in F if:

- t_1 is a variable,
- $\mathbf{Preds}^1_F(t_1) \subseteq \mathbf{Preds}^1_F(t_2)$,
- there exists a term t such that
  - $\mathbf{Preds}^2_F(t, t_1) \neq \emptyset$, and
  - $\mathbf{Preds}^2_F(t, t_1) \subseteq \mathbf{Preds}^2_F(t, t_2)$.

In this case, we say that t_1 is a *mergeable variable.*

We imposed that t_1 is mergeable on t_2 only if t_1 is a variable because, we will latter map t_1 on t_2 and it is not possible to map a constant to another term.

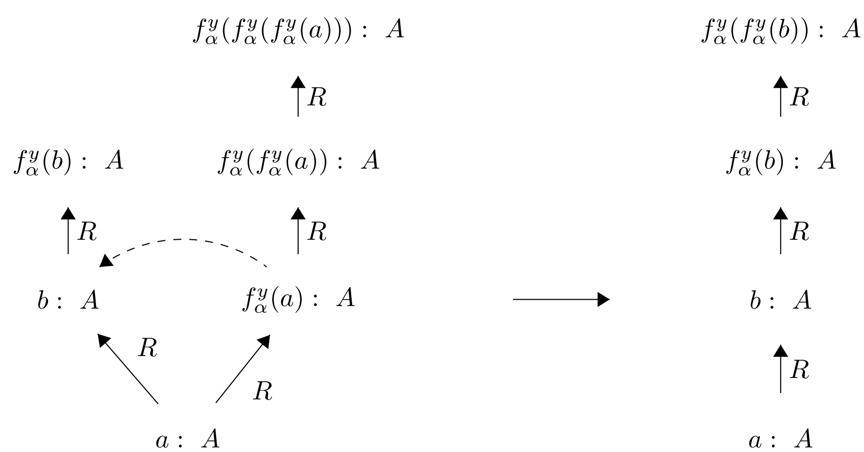
When x is mergeable on t in F, we want to say that all the triggers applied on x and on the $\prec$-successor of x can be or have already been applied on t

and on the $\prec$-successor of t. Therefore, all the facts containing x and the $\prec$-successor of x are or will be redundant. But we do not want to supress all these facts because we know that if they are not yet in the tree of t, they would be computed so we would like to "recycle" these facts. That is this intuition that leads us to define atomic merging:

Definition 3.6 (atomic merging)**.** If a variable x is mergeable on a term t in a factbase F, then we note $h_{x/t}$ the substitution defined on $\mathbf{Vars}(F)$ such that for every variable y in $\mathbf{Succ}(x)$, $h_{x/t}(y)$ is the variable y where the occurence of x is replaced by t, and for every variable y not in $\mathbf{Succ}(x)$, $h(y) = y$.

The *atomic merging* of x on t in F produces $h_{x/t}(F)$

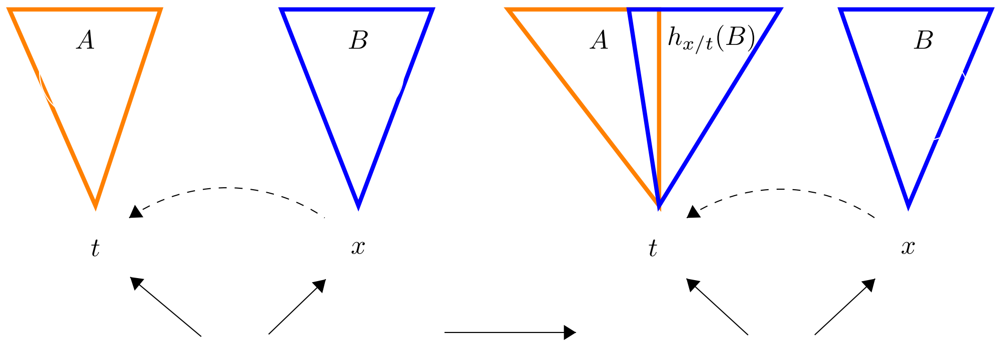
For example, in the figure below, $f^y_\alpha(a)$ is mergeable on b in the left factbase. The atomic merging of $f^y_\alpha(a)$ on b in this factbase gives the right factbase.



We want to show that the atomic merge is an operation that preserves the universality. It is the goal of the two next propositions.

Proposition 3.5. *Let $D = F_0, trig_1, F_1, \ldots, trig_m, F_m$ be an oblivious chase derivation of the knowledge base O. Let x be mergeable on t in F_m. There exists an oblivious chase derivation $D' = F_0, trig_1, F_1, \ldots, trig_k, F_k$ of O prolonging D such that $F_k = F_m \cup h_{x/t}(Tree_{F_m}(x))$. We pose $\mathbf{Fut}(F_m, t, x)= F_k$.*

Graphically, if we note $A = Tree_{F_m}(t)$ and $B = Tree_{F_m}(x)$, then we want do an oblivious derivation from the left factbase to the right factbase:



To prove the result, we will look all the triggers dealing with the variables in $\mathbf{Succ}(x)$ and apply them on the variables in $\mathbf{Succ}(t)$ if the triggers have not already been applied.

Proof. Let $tr_1 = (\alpha_1, \sigma_1), \dots, tr_n = (\alpha_n, \sigma_n)$ be all the oblivious triggers in $\{trig_1, \dots, trig_m\}$ such that

- the range of each σ_i contains only variables in $\mathbf{Succ}(x)$
- if $tr_i = trig_r$ and $tr_j = trig_l$, then $i < j$ implies $r < l$ (that is, $tr_1, \dots, tr_n$ are oredered by application time).

We note $h_{x/t}(tr_i) = (\alpha_i, h_{x/t} \circ \sigma_i)$ and

$$B'_i = h_{x/t} \circ \sigma_1^s(Head(\alpha_1)) \cup \ldots \cup h_{x/t} \circ \sigma_i^s(Head(\alpha_i))$$

We show by induction on $i \in \{0, \dots, n\}, H(i)$: There exists oblivious triggers $tr'_1, \dots, tr'_r \in \{h_{x/t} \circ \sigma_1, \dots, h_{x/t} \circ \sigma_n\}$ such that, if we pose $F_{m+1} = \mathbf{appl}(F_m, tr'_1), \dots, F_{m+r} = \mathbf{appl}(F_{m+r-1}, tr'_r)$, the sequence $D_i = D, tr'_1, F_{m+1}, \dots, tr'_r, F_{m+r}$ is an oblivious derivation and $B'_i \subseteq F_{m+r}$.

For i $= 0$, $D_0 = D$ is an oblivious derivation and $B_0 = \emptyset$, so $H(0)$ is true.

Suppose that $H(i-1)$ is true for $i \in \{1, \dots, n\}$: $D_{i-1} = D, tr'_1, F_{m+1}, \dots, tr'_r, F_{m+r}$ is an oblivious derivation and $B'_{i-1} \subseteq F_{m+r}$.

Depending on the form of the rule α_i, there exists four different cases:

- If α_i is of the form $A(u) \rightarrow \exists v. R(u, v) \wedge B(v)$. We have $A(\sigma_i(u)) \in F_m$.
  - If $\sigma_i(u) = x$, then $A(x) \in F_m$. As t is a strong sibling of x in F_m, $A(t) \in F_m$. So $A(h_{x/t}(\sigma_i(u))) \in F_{m+r}$ since $h_{x/t}(\sigma_i(u)) = t$.
  - If $x \prec^+ \sigma_i(u)$, then the fact $A(\sigma_i(u)))$ has been introduced by an oblivious trigger in $\{tr_1, \dots, tr_{i-1}\}$. So, by induction hypothesis, $A(h_{x/t}(\sigma_i(u))) \in F_{m+r}$.

Therefore $h_{x/t}(tr_i)$ is an oblivious trigger for F_{m+r}.

- If α_i is of the form $A_1(u) \wedge A_2(u) \rightarrow B(u)$. We have $A_1(\sigma_i(u)), A_2(\sigma_i(u)) \in F_m$.

  - If $\sigma_i(u) = x$, then $A_1(x), A_2(x) \in F_m$. As t is a strong sibling of x in F_m, we have $A_1(t), A_2(t) \in F_m$. So $A_1(h_{x/t}(\sigma_i(u))), A_2(h_{x/t}(\sigma_i(u))) \in F_{m+r}$ since $h_{x/t}(\sigma_i(u)) = t$.
  - If $x \prec^+ \sigma_i(u)$, then the fact $A_1(\sigma_i(u))), A_2(\sigma_i(u)))$ has been introduced by an oblivious trigger in $\{tr_1, \ldots, tr_{i-1}\}$. So, by induction hypothesis, $A_1(h_{x/t}(\sigma_i(u))), A_2(h_{x/t}(\sigma_i(u))) \in F_{m+r}$.

  Therefore $h_{x/t}(tr_i)$ is an oblivious trigger for F_{m+r}.

- If α_i is of the form $A(u) \wedge R(u, v) \rightarrow B(v)$.

  - If $\sigma_i(u) = x$, then $A(x) \in F_m$. As t is a strong sibling of x in F_m, $A(t) \in F_m$. Thus $A(h_{x/t}(\sigma_i(u))) \in F_{m+r}$.
  - If $x \prec^+ \sigma_i(u)$, then the fact $A(\sigma_i(u))$ has been introduced by an oblivious trigger in $\{tr_1, \ldots, tr_{i-1}\}$. So, by induction hypothesis, $A(h_{x/t}(\sigma_i(u))) \in F_{m+r}$.

  The fact $R(\sigma_i(u), \sigma_i(v))$ has been introduced by an oblivious trigger in $\{tr_1, \ldots, tr_{i-1}\}$.So by induction hypothesis, $R(h_{x/t}(\sigma_i(u)), h_{x/t}(\sigma_i(v))) \in F_{m+r}$.

  Therefore, the facts $A(\sigma_i(x))$ and $R(\sigma_i(u), \sigma_i(v))$ are in F_{m+r}. Thus $h_{x/t}(tr_i)$ is an oblivious trigger for F_{m+r}.

- If α_i is of the form $R(u, v) \wedge B(v) \rightarrow A(u)$.

  The facts $R(\sigma_i(u), \sigma_i(v)), B(\sigma_i(v))$ has been introduced by an oblivious trigger in $\{tr_1, \ldots, tr_{i-1}\}$. So, by induction hypothesis, the facts $R(h_{x/t}(\sigma_i(u)), h_{x/t}(\sigma_i(v)))$ and $B(h_{x/t}(\sigma_i(v)))$ are in F_{m+r}. Therefore $h_{x/t}(tr_i)$ is an oblivious trigger for F_{m+r}.

If $\mathbf{appl}(F_{m+r}, h_{x/t}(tr_i)) = F_{m+r}$ then $h_{x/t} \circ \sigma_i^s(Head(\alpha_i)) \subseteq F_{m+r}$ so $B'_i = B_{i-1} \cup h_{x/t} \circ \sigma_i^s(Head(\alpha_i)) \subseteq F_{m+r}$. Therefore $D_i = D_{i-1}$ is suitable, $H(i)$ is true.

I have to add a fifth case for the rule 5

Otherwise, the oblivious trigger $h_{x/t}(tr_i)$ has not been applied on F_{m+r} we then pose $tr'_{r+1} = h_{x/t}(tr_i)$, $F_{m+r+1} = \mathbf{appl}(F_{m+r}, tr'_{r+1})$, and $D_i = D_{i-1}, tr'_{r+1}, F_{m+r+1}$. As $h_{x/t}(tr_i)$ is an oblivious trigger for F_{m+r}, D_i is an oblivious derivation. By definition of an application, we have $h_{x/t} \circ \sigma_i^s(Head(\alpha_i)) \subseteq F_{m+r+1}$ and so $B'_i = B_{i-1} \cup h_{x/t} \circ \sigma_i^s(Head(\alpha_i)) \subseteq F_{m+r}$. Therefore, $H(i)$ is true.

We have proved the heredity. So, D_n is the oblivious derivation that we was looking for. We have $F_{m+n} = F_m \cup h_{x/t}(Tree_{F_m}(x))$.

□

Proposition 3.6. *Let $D = F_0, trig_1, F_1, \ldots, trig_m, F_m$ be an oblivious chase derivation of the knowledge base O. Assume that x is mergeable on t in F_m. The atomic merging of x on t in F_m is a retract of* $\textbf{\textit{Fut}}(F_m, t, x)$ *.*

Proof. The atomic merging of x on t in F_m is $h_{x/t}(F_m)$.

By the Proposition **??**, $\textbf{Fut}(F_m, t, x) = F_m \cup h_{x/t}(\mathit{Tree}_{F_m}(x))$. So, $h_{x/t}(F_m) \subseteq \textbf{Fut}(F_m, t, x)$ since $\textbf{Fut}(F_m, t, x) = \mathit{Tree}_{F_m}(x) \cup h_{x/t}(F_m)$.

As $h_{x/t}(\textbf{Fut}(F_m, t, x)) = h_{x/t}(F_m)$, we have that $h_{x/t}(F_m) \models \textbf{Fut}(F_m, t, x)$ □

The last two propositions shows the following theorem:

Theorem 3.1. The atomic merge is an operation that preserves the universality.

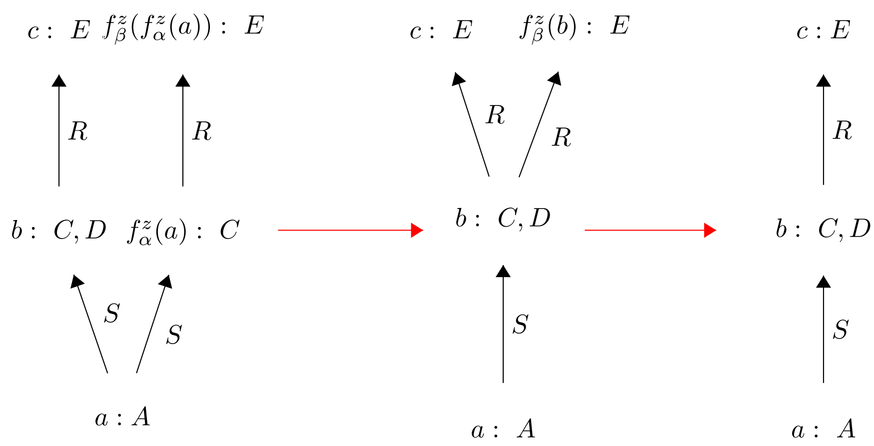
The following definition describes the operation that will replace the computation of the core in the core chase. We will show that for a chase derivation $D = F_0, ..., F_k$ of O, if we apply our operation on F_k, then it computes the core of a factbase that could have been produced by continuing the derivation D.

Definition 3.7. For a factbase F that occurs in a chase derivation of the knowledge base O, a *merge* sequence of F is a sequence $F_0, \ldots, F_n$ of factbases such that:

- $F_0 = F$.
- For $i \in \{1, \ldots, n\}$, the factbase F_i is the atomic merging of a variable x on a term t in F_{i-1}.
- The factbase F_n does not contain a mergeable variables.

Then, $\mathit{Merge}(F) = F_n$.

Example 3.1. The figure below is an example of merging where we merge the factbase of the left. The variable $f^z_\alpha(a)$ is mergeable on b so we do an atomic merging of $f^z_\alpha(a)$ on b that gives us the factbase of the middle. Now, $f^z_\beta(b)$ is mergeable on c so we do an atomic merging of $f^z_\beta(b)$ on c that gives us the factbase of the right.



We can notice that the order of variables we choose is really important because an atomic merging can create mergeable variables. In this example, if we treat $f_\beta^z(f_\alpha^z(a))$ before $f_\alpha^z(a)$, then at the moment where we treat $f_\beta^z(f_\alpha^z(a))$, it does not have any term t yet such that $f_\beta^z(f_\alpha^z(a))$ is mergeable on t so at the end, we get the factbase of the middle and we will not have merged every possible mergeable variable.

The merge operation uses only atomic merge, so by Theorem **??**:

Theorem 3.2. The merge is an operation that preserves the universality.

We will now consider a new chase:

Definition 3.8 (derivation)**.** A *merge derivation* for a knowledge base $O = (R, F)$ is a (possibly infinite) sequence $D = F_0, F_1, F_2, \dots$ where $F_0 = F$, and for $i > 0$, either $F_i = \mathbf{appl}(F_{i-1}, t_i)$ is obtained by an application with t_i an oblivious trigger, or $F_i = \mathit{Merge}(F_{i-1})$ is obtained by the merging of F_{i-1} .

Definition 3.9 (derivation)**.** A merge derivation $D = F_0, F_1, F_2, \dots$ for a knowledge base $O = (R, F)$ is *fair* if:

- For every i, for every oblivious trigger t applicable on F_i, there exists $k > i$ such that $\mathbf{appl}(F_k, t) = F_k$.
- For every i, there exists $k \geq i$ such that F_k is a core.

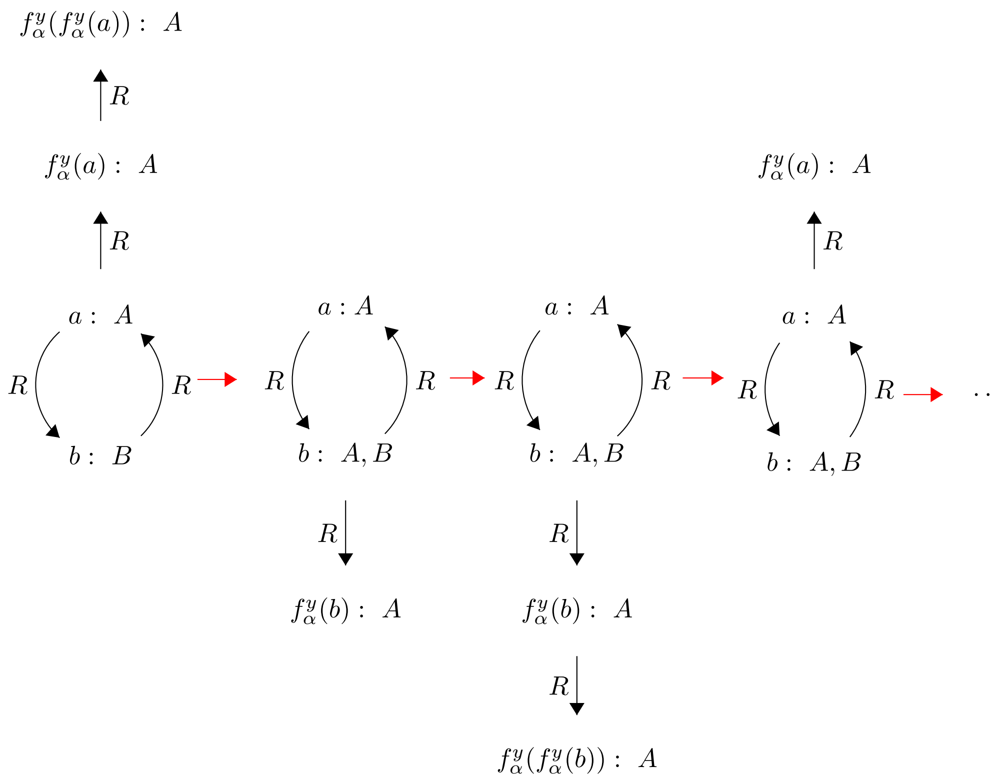
Definition 3.10. A *merge chase* for a knowledge base $O = (R, F)$ is a fair atomic merge derivation $D = F_0, F_1, F_2, \dots$

Definition 3.11. The merge chase $D = F_0, F_1, F_2, \dots$ *terminates* on O if it is a finite derivation. We pose:

It is really important that the merging does all the possible atomic merging that it can do. Otherwise, the merge chase may not terminates on some knowledge base whereas there exists a finite universal model (like in the next example) and we want that the merge chase terminates on all knowledge base that admits a finite universal model.

Example 3.2. We consider in the example that $F = \{R(a,b), R(b,a), A(a), A(b)\}$ and $R = \{\alpha, \beta\}$ where $\alpha = A(x) \rightarrow \exists y.R(x,y) \wedge A(y)$ and $\beta = B(x) \rightarrow A(x)$. We consider the atomic merge chase derivation $F_0 = F, F_1, F_2, \ldots$ We applied to F_0 the oblivious trigger $t_1 = (\alpha, \{x \mapsto a\})$, we then applied to F_1, the oblivious trigger $t_2 = (\alpha, \{x \mapsto f^y_\alpha(a)\})$ giving rise to the first factbase of the figure below. We then apply the oblivious trigger $t_3 = (\beta, \{x \mapsto b\})$ to obtain the factbase F_3. At this moment, $f^y_\alpha(a)$ is mergeable on b so we do an atomic merging of $f^y_\alpha(a)$ on b to get F_4 that is the second factbase of the figure. F_5 is obtained by the application of the oblivious trigger $t_4 = (\alpha, \{x \mapsto f^y_\alpha(b)\})$ and is the third factbase of the figure. At this moment, $f^y_\alpha(b)$ is mergeable on a so we do an atomic merging of $f^y_\alpha(b)$ on a to get F_6 that is the last factbase of the figure. We can repeat this infinitely. But O admits a finite universal model: $U = F \cup \{B(b)\}$. So the version of the chase where we consider partial merging is not what we want.

changer l'example



We want now prove that a merging computes a core:

Proposition 3.7. *For a factbase G obtained by applying some Horn-$\mathcal{ALCH}$ axioms on O, $Merge(G)$ is a core.*

Proof. Suppose for a contradiction that $Merge(G)$ is not a core. There exists $G' \subsetneq Merge(G)$ such that G' is a retract of $Merge(G)$. By Proposition [**?**], there exists then a retractation h from $Merge(G)$ to G'. We have $var(Merge(G)) \setminus var(G') \neq \emptyset$. Let x be a $\prec$-minimal variable of this set. The term x is a variable, so has been introduced by the chase due to the axiom 3. So there exists a term t such that $t \prec x$.

- We have $\mathbf{Preds}^2_{Merge(G)}(t, x) \neq \emptyset$. By $\prec$-minimality of x, $t \in \mathbf{Vars}(G')$. So, as h is a retractation: $h(t) = t$, so for $R \in \mathbf{Preds}^2_{Merge(G)}(t, x)$, $h(R(t, x)) = R(t, h(x)) \in Merge(G)$ and so $R \in \mathbf{Preds}^2_{Merge(G)}(t, h(x))$. Thus $\mathbf{Preds}^2_{Merge(G)}(t, x) \subseteq \mathbf{Preds}^2_{Merge(G)}(t, h(x))$ and $t \prec h(x)$.
- $x \notin G'$ and $h(x) \in G'$ so $h(x) \neq x$.

- Let $A \in \mathbf{Preds}^1_{Merge(G)}(x)$. $h(A(x)) \in Merge(G)$ so $A(h(x)) \in Merge(G)$ so $\mathbf{Preds}^1_{Merge(G)}(x) \subseteq \mathbf{Preds}^1_{Merge(G)}(h(x))$.

Consequently, x is mergeable on $h(x)$ in $Merge(G)$: contradiction. So $Merge(G)$ is a core. □

$Merge(G)$ is a core but not necessarily the core of G.

Proposition 3.8. *The merge chase computes a finite universal model of O when it terminates.*

Proof. Let $D = F_0, \ldots, F_n$ be a merge chase for $O = (R, G)$.

- The merge chase never take off ground facts and $G = F_0$ so $G \subseteq F_n$. We have then $F_n \models G$. Assume for a contradiction that $F_n \nvDash R$. There exists then a rule α in R not satisfied by F_n: $F_n \vDash Body(\alpha)$ and $F_n \nvDash Head(\alpha)$. It means that there exists a substitution σ from $Body(\alpha)$ to F_n. Therefore, $t = (\alpha, \sigma)$ is an oblivious trigger for F_n. As the derivation D is fair, $\mathbf{appl}(F_n, t) = F_n$. Thus $\sigma^s(Head(\alpha)) \subseteq F_n$ and so $F_n \models \alpha$: contradiction. We have then $F_n \models R$ so F_n is a model of O.

- According to Proposition **??** and **??**, each operation of the merge chase conserves the universality so we can show by induction that F_n is a universal model of O.

- We applied a finite number of operations and each operation either add a finite number of facts or remove facts so F_n is finite.

□

Proposition 3.9. *If there exists a finite universal model for O, the merge chase terminates.*

Proof. □

Theorem 3.3. The merge chase computes an universal model if and only if there exists an universal model.

We will now describe a deterministic algorithm to merge a factbase:

Definition 3.12 (Merging)**.** Let F be a factbase that occurs in a chase derivation of the knowledge base O.

Algorithm 1: Merge(F):

1 Let **Vars**$(F) = \{x_1, \ldots, x_n\}$ be such that $(x_i \prec^+ x_j) \Rightarrow i < j$;
2 **for** $i = 1$ *to* n **do**
3   **if** *x_i is still a variable in F* **then**
4   **for** *all term t such that x_i is mergeable on t* **do**
5             $F \leftarrow$ the atomic merging of x_i on t in F.
6   **end**
7   **end**
8 **end**
9 return F

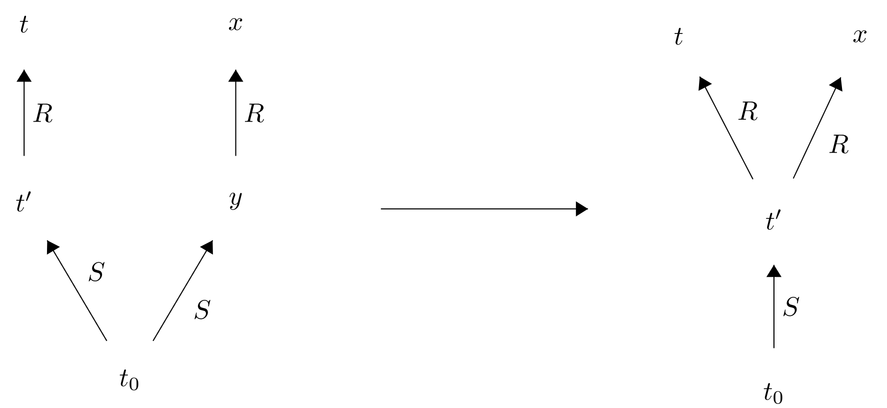
At line 1, we can sort terms like that because, by proposition **??**, $\prec^+$ is a strict partial order over the set of variables of F.

The Example **??** shows the importance of doing a total merging. We have to prove that our merging algorithm does a total merging:

Proposition 3.10. *Let G be a factbase that occurs in a chase derivation of the knowledge base O. There does not exists a term t and a variable x such that x is mergeable on t in Merge(G).*

Proof. Suppose for a contradiction that there exists a term t and a variable x such that x is mergeable on t in *Merge*(G), that is, there exists a term t' and a binary predicate R such that $R(t', x), R(t', t) \in$ *Merge*(G). This case can happen only if x became mergeable after that x has been traited by the merging algorithm. Thus, during the merging, there has been an atomic merging on t'. Let y be the variable merged on t' such that $y \prec x$. We note G^1 the factbase just before the atomic merging of y on t' and we note G^2 the factbase just after the atomic merging. There exists a term t_0 and a binary predicate S such that $S(t_0, t'), S(t_0, y) \in G^1$. G^1 is the left figure and G^2 is the right figure (we do not represent all the graphs):

I have to modify some stuff for rule 5



We have $t_0 \prec t' \prec t$ and $t_0 \prec y \prec x$ so x should have been treated by the algorithm after the merging of t' and y so the algorithm will merge x on t: contradiction. □

3.2 Horn-$\mathcal{ALCHI}$

Definition 3.13 (Horn-$\mathcal{ALCHI}$ axioms)**.** A *Horn-$\mathcal{ALCHI}$ axiom* is either a Horn-$\mathcal{ALCH}$ axiom or an existential rule of the form:

$$R_1(x, y) \wedge R_2(x, y) \rightarrow S(y, x) \tag{6}$$

We fix $O = (R, F)$ a knowledge base for this section where R is a Horn-$\mathcal{ALCHI}$ rule set and F is a ground factbase with predicates of arity one or two. We have to modify the merge chase because it doesn't work anymore:

We keep the same relation $\prec$. It is still a strict partial order over the set of variables.