SIAM. J. ALG. DISC. METH.
Vol. 2, No. 2, June 1981

0196-5212/81/0202-0002 $01.00/0

AN $O(n^2)$ ALGORITHM FOR COLORING PROPER CIRCULAR ARC GRAPHS*

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Abstract. A graph is a circular arc graph if each vertex of the graph is associated with an arc on a circle in such a way that two vertices of the graph are adjacent if and only if the corresponding arcs overlap. A circular arc graph is proper if none of the representing arcs is contained within another. An $O(n^2)$ algorithm is given for determining whether a proper circular arc graph with n nodes may be colored with k colors.

1. Introduction. A *circular arc family* is a set $F = \{A_1, \cdots, A_n\}$ of arcs on a circle. A circular arc family is *proper* if no arc is contained within another. A graph is a (*proper*) *circular arc graph* if there is a 1 : 1 correspondence between the vertices of the graph and the arcs of a (proper) circular arc family such that two vertices of the graph are adjacent if and only if the corresponding arcs overlap. For example, the graph in Fig. 1a is a proper circular arc graph, and Fig. 1b gives a proper circular arc model of this graph. The diagram in Fig. 1c is also a circular arc model; however, it is not proper because arc A_2 is contained in arc A_1.

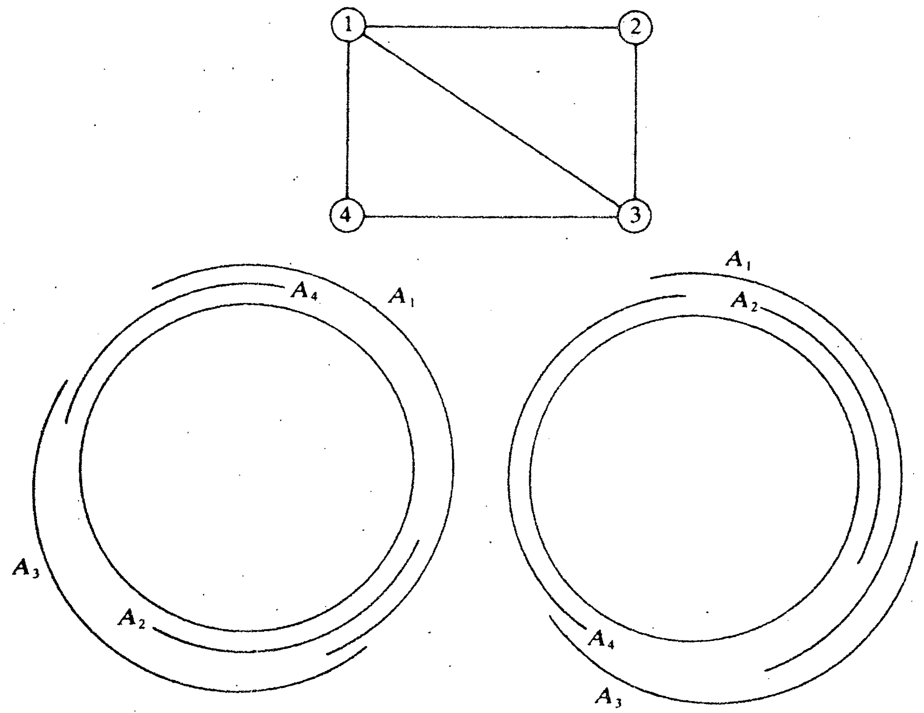

FIG. 1. a) *A proper circular arc graph.* b) *A proper circular arc representation for the graph in* a). c) *A (nonproper) circular arc representation for the graph in* a).

Circular arc graphs have been studied extensively. Tucker [12] has recently given a polynomial algorithm for recognizing these graphs. Gavril [6], [7] has given polynomial algorithms for finding a maximum independent set, a maximum clique and a minimum covering by cliques for circular arc graphs. The problem of coloring circular arc graphs

* Received by the editors April 1, 1980, and in revised form June 30, 1980.

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has been investigated by Tucker [11], and this problem was recently proved to be NP-complete by Garey, Johnson, Miller and Papadimitriou [5]. In this latter paper the problem of coloring proper circular arc graphs was mentioned as being a significant open problem. In [3] the problem of selecting the minimum number of node disjoint paths in a circular arc graph has been solved with an $O(n \log n)$ algorithm.

Applications. Coloring circular arc graphs has applications in both cyclic scheduling and in optimal register allocation in computer programs. Both of these applications are discussed in [11]. In cyclic scheduling we consider a number of tasks that have to be carried out periodically, and each arc represents a span of time during which the task is executed. For example, consider a limousine service at an airport that repeats its schedule every hour. Each arc represents the portion of the hour devoted to a specific (hourly repeated) round trip. The circular arc graph may be k-colored if and only if the corresponding limousine schedule can be serviced by k limousines such that each route is serviced periodically by the same limousine (see Fig. 2).

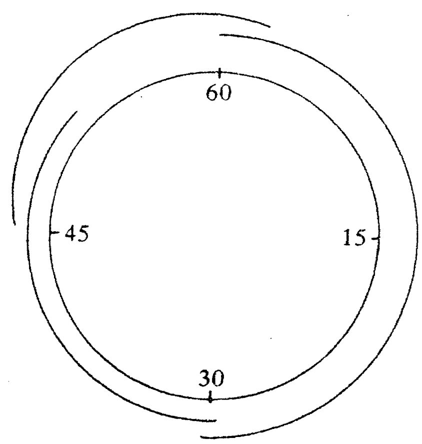

FIG. 2. *Arcs representing the timetables for three hourly repeated round trips leaving on the hour, half-past the hour, and three-quarters past the hour.*

For the computer application, consider a loop in a computer program and regard the flow of control in the loop as a circle. Each variable within the loop has a certain lifetime which may be modeled as an arc of the circle. Since it is necessary to store a variable only during its lifetime, a single register may store several variables as long as the lifetimes of any two of these variables do not intersect. There is a k-coloring of the corresponding circular arc graphs if and only if it is possible to store all the variables in k registers so that each variable is in only one register during its lifetime.

In the first application there is a restriction that each route must be traveled periodically by the same limousine. In the second application there is a restriction that each variable is stored repeatedly in the same register. If these restrictions are relaxed, the problem may be modeled as a coloring problem on "periodic interval graphs," which in turn is a special case of the periodic Dilworth's theorem, as formulated and solved in [8]. Furthermore, the problem is efficiently solvable even when one interval may contain another. Tucker [9] characterized proper circular arc graphs. A subclass which arises commonly in applications is that subclass induced by a family of arcs each with a common length.

2. Determining k-colorings for proper circular arc graphs.

Circular arc representations. Let G be a proper circular arc graph. Tucker [9] gives an efficient algorithm for creating a circular arc representation for these graphs, which runs in $O(n^2)$ time using as a subroutine the Booth and Lueker algorithm [4] for recognizing the consecutive ones property in matrices. In the following we will assume that a given proper circular arc graph G has an associated proper circular arc representation $F=\{A_1, \cdots, A_n\}$ such that $A_i=[a_i, b_i]$ and $a_i, b_i \in [0, 1)$. The interpretation is that A_i is the arc on the unit circle that stretches clockwise from point a_i to point b_i and contains both of these points. We also denote the graph as $G(F)$.

We also assume that the arcs are ordered so that $a_1 < a_2 < \cdots < a_n$, and we assume that there are at least two arcs which do not overlap (otherwise, the coloring is trivial). In the following the vertex set of G is $V=\{1, \cdots, n\}$.

Overlap cliques. An *overlap clique* of G is a maximal set of vertices of G whose corresponding arcs all intersect at a common point of the circle. It is easy to see that each overlap clique is induced by one of the points in the set $\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$. Thus there are at most $2n$ such cliques.

Set S is called *circularly consecutive* if either $S=\{i, i+1, \cdots, j\}$ or else $S=\{i, i+1, \cdots, n, 1, \cdots, j\}$ for some $i, j \in \{1, \cdots, n\}$.

LEMMA 1. *If $G=G(F)$ is a proper circular arc graph, then each overlap clique is circularly consecutive.*

Proof. If the point inducing the overlap clique is $p=0$, then $S=\{i: a_i > b_i \text{ or } a_i=0\}$, and this set is circularly consecutive. Let $x(\text{mod } 1)$ denote the fractional part of x. For $p \neq 0$, the overlap clique induced by p is $s=\{i: (a_i-p)(\text{mod } 1)=(b_i-p)(\text{mod } 1)$ or $a_i(\text{mod } 1)=p\}$, which is circularly consecutive. □

For a given circularly consecutive set S, the *last element* of S is the unique element $i \in S$ such that $i+1 \notin S$ (the last element of S is n if $n \in S$ and $1 \notin S$). Henceforth, we will write an overlap clique as $S=\langle i, j\rangle$ where j is the last element of S and i is the last element of $\{i, \cdots, n\}-S$. For example, if $n=5$, then $\{3, 4, 5\}=\langle 2, 5\rangle$, and $\{4, 5, 1\}=\langle 3, 1\rangle$. This choice of our representation will be made more clear in the context of the algorithm.

LEMMA 2. *Let $G=G(F)$ be a circular arc graph with n vertices, and let k be a divisor of n. Then G is k-colorable if and only if G has no overlap clique of $k+1$ vertices.*

Proof. The "only if" part is trivial, since no graph with a clique of $k+1$ vertices is k-colorable. For the "if" direction, consider the coloring of G with colors $0, 1, \cdots, k-1$ such that vertex i is assigned color $i(\text{mod } k)$. If $i<j$ and nodes i and j are assigned the same color, then either $a_i \in [a_i, b_i]$ or else $b_j \in [a_i, b_i]$. In the former case vertices $i, i+1, \cdots, j$ are in the same overlap clique; in the latter case vertices $j, \cdots, n, 1, \cdots, i$ are in the same overlap clique. In both cases the overlap clique has at least $k+1$ vertices. □

In the following, $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the least integer function and the greatest integer function.

LEMMA 3. *Let G be a proper circular arc graph that is k-colorable. Then G may be k-colored in such a way that each color class has either $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ colors.*

Proof. We first show the result for $k=2$. If n is even and $k=2$, the result is a consequence of the coloring given in Lemma 2. Suppose there is a 2-coloring of G, and assume n is odd. One of the color classes has at least $(n+2)/2$ vertices; otherwise, there is nothing to prove. This color class contains vertices i and $i+1$ for some i. Consider now the subset

$$C=\{i, i-2, i-4, \cdots\} \cup \{i+1, i+3, i+5, \cdots\}.$$

Neither C nor $V-C$ has two adjacent vertices; else it would imply a clique in G of size 3. Thus there is a 2-coloring with color classes C and $V-C$ with $(n+1)/2$ and $(n-1)/2$ vertices, thus proving the lemma for the case that $k=2$.

Consider now a k-coloring for $k>2$, and let C and D be color classes with the greatest and least number of vertices. By the above, we may partition the set $C \cup D$ into two color classes whose cardinality differs by at most one. Iteratively applying this recoloring procedure we obtain a coloring in which each color class has $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ vertices, proving the lemma. □

THEOREM 1. *Let $G = G(F)$ be a proper circular arc graph with n vertices. Let k be an integer less than n, and let $r = n \pmod{k}$ with $0 \leqq r \leqq k-1$. Graph G may be k-colored if and only if there exists a subset V' of $r \cdot \lceil n/k \rceil$ vertices such that* (1) *the subgraph of G induced by V' has no overlap clique of size $r+1$, and* (2) *the subgraph of G induced by $\{1, \cdots, n\} - V'$ has no overlap clique of size $k-r+1$.*

Proof. If k is a divisor of n, the theorem follows from Lemma 2. Suppose there is a k-coloring of G, and suppose that $r = n \pmod{k} \neq 0$. By Lemma 3, there is a k-coloring such that each color class has $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ vertices. There are r color classes of size $\lceil n/k \rceil$ since the number of vertices of G is n. Let V' be the union of these color classes. The subgraph induced by V' is r-colorable and thus has no overlap clique of size $r+1$. The subgraph induced by $\{1, \cdots, n\} - V'$ is $(k-r)$-colorable and hence has no overlap clique of size $k-r+1$.

Conversely, suppose there is a subset V' of vertices satisfying the conditions of this theorem. The subgraph of G induced by V' has $r \cdot \lceil n/k \rceil$ vertices and may be r-colored by Lemma 2. The subgraph of G induced by $\{1, \cdots, n\} - V'$ has $(k-r) \cdot \lfloor n/k \rfloor$ vertices and may be $(k-r)$-colored by Lemma 2. Hence G may be k-colored. □

There is no a priori reason why the above partition problem should appear easier to solve than the coloring problem; however, the partition problem is really a special case of the shortest-path problem, a consequence of the following integer programming formulation of the partition problem. A similar partition problem for proper circular arc graphs was solved in a similar way by Bartholdi, Orlin and Ratliff in [1].

In the following we wish to determine a subset V' of vertices of G satisfying the conditions of Theorem 1. We let $x_i = |\{1, \cdots, i\} \cap V'|$, which is the number of vertices with index at most i in the set V'. Thus $i \in V'$ if and only if $x_i - x_{i-1} = 1$, where $x_0 = 0$. As before, we let $r = n \pmod{k}$. We now wish to determine a feasible solution to the system of constraints (1):

$$0 \leqq x_i - x_{i-1} \leqq 1 \quad \text{for } i = 1, \cdots, n, \tag{1a}$$

$$x_0 = 0, \tag{1b}$$

$$x_n = r \cdot \lceil n/k \rceil. \tag{1c}$$

For each overlap clique $S = \langle i, j \rangle$ with $i < j$,

$$x_j - x_i \leqq r, \tag{1d}$$

$$|S| - (x_j - x_i) \leqq k - r. \tag{1e}$$

Finally, for each overlap clique $S = \langle i, j \rangle$ with $i > j$,

$$x_j - x_i + x_n \leqq r, \tag{1f}$$

$$|S| - (x_j - x_i + x_n) \leqq k - r \tag{1g}$$

and

(1h) x_i is integer valued for $i = 1, \cdots, n$.

THEOREM 2. *Circular arc graph G may be k-colored if and only if there is a feasible solution to system* (1). *Such a solution may be determined in $O(n^2)$ steps.*

Proof. We note first that there is a 1:1 correspondence between subsets V' of vertices of G and integer vectors $x = (x_i)$ satisfying (1a) and (1b). The correspondence is given by the relation $x_i = |V' \cap \{1, \cdots, i\}|$. With this correspondence, each overlap clique $S = \langle i, j\rangle$ with $i < j$ is such that $|S \cap V'| = x_j - x_i$. If $S = \langle i, j\rangle$ with $i > j$, then $|S \cap V'| = x_j - x_i + x_n$.

We interpret the constraints of (1) as follows: (1c) requires that V' has $r \cdot \lceil n/k \rceil$ vertices; (1d) and (1f) require that each overlap clique in the subgraph induced by V' has at most r vertices; (1e) and (1g) require that each overlap clique in the subgraph induced by $\{1, \cdots, n\} - V'$ has at most $k - r$ vertices. Thus (1) is equivalent to requiring that G satisfy the conditions of Theorem 1.

Since x_n is fixed in value in the constraint (1c), we may eliminate x_n from the constraints (1f) and (1g). Each of the resulting constraints may be written as $x_j - x_i \leqq d_{ij}$ for an appropriate value d_{ij} (where i or j may be 0). Consider now a directed graph G' with vertex set $\{0, \cdots, n-1\}$, and for each constraint "$x_j - x_i \leqq d_{ij}$" of (1), there is an associated edge (i, j) of G' with distance d_{ij}. Then a feasible solution for (1) is $x'_0, \cdots, x'_{n-1}, x'_n$, where x'_j is the minimum distance in G' from node 0 to node j for $j = 0, \cdots, n-1$, and $x'_n = r \cdot \lceil n/k \rceil$. Let m denote the number of edges of G'. Then these distances may be computed in $O(n \cdot m)$ steps by the Bellman–Ford method [2], which in turn is $O(n^2)$ steps because each edge is associated with an inequality of (1), and there are at most $2n$ overlap cliques.

Once a feasible solution for (1) is determined, the coloring may be carried out in $O(n)$ steps via Lemma 2. □

COROLLARY 5. *A minimum coloring for a circular arc graph may be determined in $O(n^2 \log n)$ steps.*

Proof. It suffices to use binary search to determine the minimum value of k for which the given graph is k-colorable. This takes $O(\log n)$ iterations, each with $O(n^2)$ steps. □

The network G' of the proof of Theorem 1 is highly structured. An open question is whether the shortest-path distances may be computed faster than $O(n^2)$ using a specialized algorithm.

Acknowledgment. We gratefully acknowledge the helpful comments of Alan Tucker.

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