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COLORING A FAMILY OF CIRCULAR ARCS*

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Abstract. This paper presents a collection of results about coloring a family of circular arcs. We prove that the strong perfect graph conjecture is valid for circular-arc graphs. We give some upper bounds on the number of colors needed to color various families of arcs. Finally, we convert the problem of determining whether a family of arcs can be q-colored into a multicommodity flow problem.

1. Introduction. This paper presents a collection of results about coloring a family of circular arcs. In applications the arcs usually represent the spans of time involved in performing certain jobs and the colors distinguish different men or machines. For example, suppose that during each hour (of an 8-hour day) there are five jobs whose time spans are shown in Fig. 1. If a job can be assigned to different men during different hours, then we need only three men. However, if a job must be assigned to the same man each hour, then we will need five men. In other words, we require five colors for a *coloring of the arcs* in Fig. 1, i.e., overlapping arcs get different colors. Minimal coloring of arcs also arises in the design

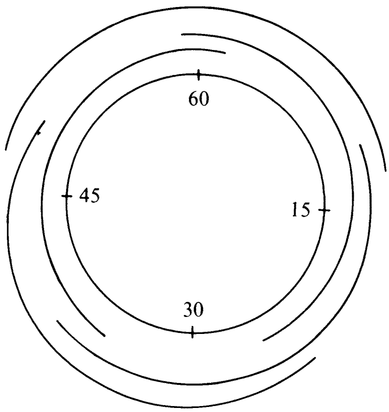

FIG. 1. *An example of time spans for 5 jobs to be performed hourly. These arcs make up the family* $F(5, 3)$.

of compilers. In a frequently used inner loop, one has certain variables which should be available in index registers during various periods in the loop. Determining the number of index registers needed during the loop is a minimal arc coloring problem.

First, we shall prove that a well-known conjecture of Gilmore about perfect graphs applies to families of circular arcs. Next we give some results about upper bounds on the number of colors needed by a family of arcs. Then we convert the

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problem of determining whether a family of arcs can be q-colored into a multicommodity flow problem. The resulting flow problem is seen to be a type of algebraic factorization problem.

A *clique* in a family of arcs is a set of pairwise overlapping arcs. An *overlap set* is the set of all arcs that contain some given point on the circle. Thus every overlap set is a clique. A *maximal* overlap set is an overlap set that is not properly contained in any other overlap set. The minimum number of colors needed to color a family F of arcs is its *chromatic number*, $\gamma(F)$. We assume that all arcs are closed (contain both endpoints), that no arc consists of the whole circle, and that the families of arcs are all finite.

With any family F of arcs we can associate a graph $G(F)$ having a vertex for each arc in F and having an edge between two vertices when the corresponding arcs overlap. A graph G is called a *circular-arc graph* if there exists a family F of arcs such that $G = G(F)$. F is called a *circular-arc model* for G. See Gavril [5] and Tucker [8] for discussions about circular-arc graphs and their applications. A coloring of a graph assigns adjacent vertices different colors. The chromatic number $\gamma(G)$ of a graph G is defined as for arcs. So $\gamma(F) = \gamma(G(F))$. A clique in a graph is any set of pairwise adjacent vertices.

If no arc in the family F contains another arc, F is called a *proper* family of arcs and $G(F)$ is called a proper circular-arc graph. See Tucker [8], [10] for characterizations of proper circular-arc graphs. We define $F(n, k)$ to be a proper family of n arcs such that when the arcs have been indexed in clockwise order (starting from an arbitrarily chosen arc A_1), A_i overlaps A_{i-k+1} through A_{i+k-1} (subscripts are mod n). Fig. 1 shows $F(5, 3)$. We define $G(n, k)$ to be $G(F(n, k))$.

2. Strong perfect graph theorem for circular-arc graphs. A graph G is called *perfect* if for every vertex-generated subgraph H (including $H = G$), $\gamma(H)$ equals the size of the largest clique in H. Berge [1], [2], [3], Fulkerson [4] and Lovasz [6], [7] have examined perfect graphs. Using results of Fulkerson, Lovasz has proved that a graph G is perfect if and only if its complement G' is perfect (G' has the same vertices as G, but two vertices are adjacent in G' if and only if they are not adjacent in G). A k-coloring of G' corresponds in G to a partition of G into k cliques. A clique of G' corresponds to a set of pairwise nonadjacent vertices in G, called an *independent set*. Thus, perfection of G' means that for all subgraphs H of G, the minimum size of a clique partition of H is the size of a largest independent set of H. Gilmore conjectured that G is perfect if and only if G and G' contain no odd holes (an odd hole is a chordless n-circuit, for $n \geqq 5$, odd). This has been called the strong perfect graph conjecture. Tucker [9] has shown that this conjecture is true for planar graphs.

THEOREM 1. *Let G be a circular-arc graph. Then G is perfect if and only if G and G' have no odd holes.*

Proof. Necessity of the odd-hole condition is obvious since odd holes are not perfect. The proof of sufficiency is by induction on the number of vertices. If G has one vertex, the result is trivial. Assume that G has $n > 1$ vertices and that the theorem is true for all proper (vertex generated) subgraphs of G. Suppose G and G' have no odd holes but that G and, by Lovasz's result, G' are not perfect. If k is the size of the largest independent set of G, then the nonperfection of G'

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implies that there is no partition of G into k cliques. On the other hand, for any vertex x, $G' - x$ is perfect by the induction hypothesis. So there is a partition of $G - x$ into k cliques.

Suppose that there exist vertices x_1, x_2 in G such that x_1 is adjacent to every vertex that x_2 is (including x_2). Then any clique containing x_2 can be enlarged to a clique containing x_1 also. Using this argument, a partition of $G - x_1$ into k cliques can be extended to a partition of G into k cliques. Thus G must contain no such x_1, x_2. This means that in any circular-arc model F for G, no arc contains another arc, i.e., F is a proper family of arcs. If c is the size of the largest clique in G, and hence in F, then any arc of F can overlap at most $2c - 2$ other arcs: $c - 1$ arcs at its clockwise end, and $c - 1$ arcs at its counterclockwise end. So each vertex of G is adjacent to at most $2c - 2$ other vertices.

We shall show that each vertex is adjacent to exactly $2c - 2$ other vertices. To this end, we must first show that for every x, y in G, x is in a clique of size c in $G - y$. Consider a c-coloring of $G - x$, for any vertex x. Let y be any vertex of $G - x$ and let α be the color of y. Let K be the set of vertices with color α. Then $G - x - K$ can be $(c - 1)$-colored and so has no clique of size c. However, $G - K$ cannot be $(c - 1)$-colored (or G could be c-colored contradicting the imperfection of G). Since $G - K$ is assumed to be perfect, it must contain a clique of size c which contains x. That is, x is in clique of size c in $G - y$.

Take any vertex x and look at a clique K of size c containing x. Now pick any vertex y that is not adjacent to x, and consider a c-coloring of $G - y$. Each vertex of K must have a different color. We claim that for any color α, different from the color of x, x is adjacent to at least two vertices of this color. Let z be the vertex of color α in K. A clique K' containing x in $G - z$ must contain a vertex of color α (here we use the same coloring as for $G - y$; y is uncolored, but this does not matter since y is not in K'). Then x is adjacent to at least two vertices of each other color or, in total, at least $2c - 2$ vertices.

It follows that each vertex of G is adjacent to exactly $2c - 2$ other vertices, or each arc of F overlaps exactly $c - 1$ other arcs at each of its ends. Thus $F = F(n, c)$. Since every c successive arcs of $F(n, c)$ form a c-clique, and since $F(n, c)$ cannot be partitioned into k cliques, but $F(n, c)$ minus any one arc can be so partitioned, it follows that $n = kc + 1$. So $G = G(kc + 1, c)$. If k or $c = 1$, we get contradictions. If $c = 2$, $G(2k + 1, 2)$ is an odd hole. If $c \geqq 2$ and $k = 2$, then $G(2c + 1, c)$ corresponds to an odd hole in G'. Now let $k, c \geqq 3$ and $n = kc + 1$, and let vertex x_j of $G(n, c)$ correspond to arc A_j (cyclicly indexed). Then we claim that $G(n, c)$ contains the odd hole $H = (x_{j_1}, x_{j_2}, \cdots, x_{j_m})$ of length $m = 2k - 1$, where $j_r = \lceil rn \rceil/(2k - 1)$. (We let $\lceil s \rceil$ indicate the smallest integer $k \geqq s$ and $\lfloor s \rfloor$ the largest integer $h \leqq s$.) Thus if $k = 3$, $c = 5$, and $n = 16$, then $j_r = \lceil r(16/5) \rceil$ and $H = (x_4, x_7, x_{10}, x_{13}, x_{16})$. In general, we must show that successive pairs of vertices in H are adjacent, i.e., $j_{r+1} - j_r \leqq c - 1$, and that nonsuccessive vertices in H are not adjacent, i.e., $j_{r+2} - j_r \geqq c$. In these tests, $j_m = n$ is equivalent to a $j_0 = 0$. Now

$$j_{r+1} - j_r = \left\lceil \frac{(r+1)n}{2k-1} \right\rceil - \left\lceil \frac{rn}{2k-1} \right\rceil \leqq \left\lceil \frac{(r+1)n - rn}{2k-1} \right\rceil = \left\lceil \frac{n}{2k-1} \right\rceil.$$

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We need to show that

$$\frac{n}{2k-1} = \frac{kc+1}{2k-1} \leqq c - 1 = (c-1)\frac{(2k-1)}{2k-1} = \frac{2ck - c - 2k + 1}{2k-1},$$

or that $kc + 1 \leqq 2ck - c - 2k + 1$. Observe $k, c \geqq 3$ implies $c + 2k \leqq ck$. Then $2ck - c - 2k + 1 \geqq 2ck - ck + 1 = kc + 1$. Thus $j_{r+1} - j_r \leqq c - 1$. Next we have

$$j_{j+2} - j_r = \left\lfloor \frac{(r+2)n}{2k-1} \right\rfloor - \left\lfloor \frac{rn}{2k-1} \right\rfloor = \left\lfloor \frac{(r+2)n - rn}{2k-1} \right\rfloor$$
$$= \left\lfloor \frac{2n}{2k-1} \right\rfloor \geqq \left\lfloor \frac{2kc+2}{2k} \right\rfloor = \left\lfloor c + \frac{1}{k} \right\rfloor \geqq c.$$

One case not checked is $x_{j_{m-1}}$ and x_{j_1}—the reader can verify that those two vertices are

$$\left\lceil \frac{n}{2k-1} \right\rceil + \left\lfloor \frac{n}{2k-1} \right\rfloor \geqq \left\lfloor \frac{2n}{2k-1} \right\rfloor \geqq c$$

positions apart in $G(n, c)$. Q.E.D.

3. Upper bounds for the chromatic number of a family of arcs. In this section, we give some upper bounds on $\gamma(F)$, the chromatic number of a family F of arcs. Let $r_{\mathrm{sup}}(F)$ be the largest size of an overlap set in F and $r_{\mathrm{inf}}(F)$ be the smallest size of an overlap set in F. The following straightforward bound is often the best available.

THEOREM 2. *For a family F of arcs, $\gamma(F) \leqq r_{\mathrm{sup}}(F) + r_{\mathrm{inf}}(F) \leqq 2r_{\mathrm{sup}}(F) - 1$.*

Proof. Let p be a point whose overlap set S has size $r_{\mathrm{inf}}(F)$. Each arc of S is given a different color. To color $F - S$ we break the circle at p and treat $F - S$ as a family of intervals. It is a simple task to verify that the chromatic number of a family F' of intervals is exactly $r_{\mathrm{sup}}(F')$. The second inequality follows from the fact that $r_{\mathrm{inf}}(F) \leqq r_{\mathrm{sup}}(F) - 1$. Q.E.D.

A difficulty in achieving a good bound is that large cliques can occur which are not overlap sets. In the family $F' = F(2r - 1, r)$, every pair of arcs overlap but $r_{\mathrm{sup}}(F') = r$. So the worst case possible in Theorem 2 occurs with F', i.e., $\gamma(F') = 2r_{\mathrm{sup}}(F') - 1$. See Fig. 1 which shows $F(5, 3)$. Thus we need a condition that eliminates cliques that are not overlap sets. We say that arcs A_1, A_2, A_3 *cover the circle* if they are mutually overlapping and collectively they contain every point on the circle.

THEOREM 3. *If no three arcs in the family F cover the circle, then $\gamma(F) \leqq \frac{3}{2}r_{\mathrm{sup}}(F)$.*

Proof. If $r_{\mathrm{sup}}(F) = 1$, the result is trivial. Assume that the theorem is valid when $r_{\mathrm{sup}}(F) \leqq r - 1$, for some $r \geqq 2$, and we shall prove the theorem for any F with $r_{\mathrm{sup}}(F) = r$. Pick a point p such that S_p, the set of arcs containing p, has size r. Let A_1 be the arc in S_p that extends the shortest length on the counterclockwise side of p. Then any arc A^* overlapping the counterclockwise end of A_1 is in S_p, or else $A^* \cup S_p$ is an overlap set of size $r + 1$.

Starting with A_1, we move clockwise round and round the circle indexing the arcs $A_1, A_2, A_3, \cdots$ by the rule: if A_i is the arc previously indexed, then A_{i+1} is

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the first unindexed arc to begin after the (clockwise) end of A_i. Let d be the number of the index for the arc being indexed when we pass point p the second time (the first time it was A_1); if on the second pass, p is in the space between the previously indexed arc and the next arc to be indexed, let d be the index of that next arc indexed. Let e be the number of the index for the arc being indexed when we first repass the counterclockwise end of A_d; as above, if A_d begins in a space on repassing then e is the index of the next arc indexed. See Fig. 2. Let f be the index for the arc being indexed when we pass point p a third time; or if p is in a space on the third pass, let f be the index of the *last* arc indexed. If we run out of arcs before A_e, it is easy to see that $r = 2$ and F can be 2-colored.

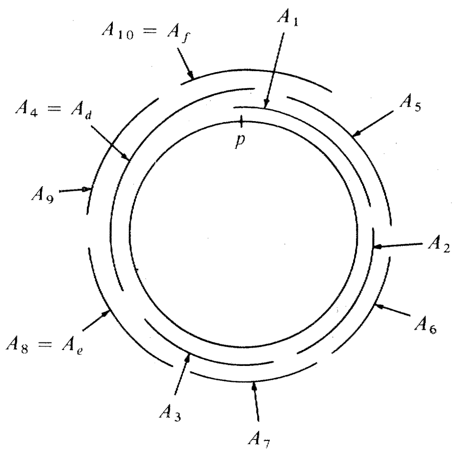

FIG. 2. *See proof of Theorem* 3.

If T_i is the overlap set of arcs with index $> i$ which contain (and extend clockwise beyond) the clockwise end of A_i, then T_i is clearly smaller than the largest overlap set in F containing A_i (in particular, $T_i \subset T_i \cup A_i$). It follows that $r(F_d) \leqq r(F) - 1$, where we let F_i denote $F - \{A_1, A_2, \cdots, A_i\}$. Similarly $r(F_f) \leqq r(F) - 2$. By assumption, F_f can be colored with $\frac{3}{2}r_{\text{sup}}(F_f)$ colors. So we are finished if we can show that $A_1, A_2, \cdots, A_f$ can be 3-colored.

Let arcs $A_1, A_2, \cdots, A_{d-1}$ get color 1, $A_d, A_{d+1}, \cdots, A_{e-1}$ get color 2, and $A_e, A_{e+1}, \cdots, A_f$ get color 3. Since A_{d-1} does not overlap p, i.e., A_{d-1} is not in S_p, then it follows from the choice of A_1 that A_{d-1} does not overlap A_1. Thus $A_1, A_2, \cdots, A_{d-1}$ are mutually nonoverlapping. By the same argument, we see that $A_d, A_{d+1}, \cdots, A_{e-1}$ are mutually nonoverlapping. Finally, we suppose that A_e overlaps A_f. See Fig. 3. Note that if A_d does not contain p but is the first arc indexed upon repassing p, then we must have $A_e = A_f$ and there is only one arc with color 3. So we can assume that A_d contains p. But A_e does not contain p or again it follows that $A_e = A_f$. Further, A_f can only overlap A_e by reaching beyond p and thus A_f contains p. Then A_d and A_f overlap at p. Now A_d, A_e, A_f mutually overlap and cover the whole circle. Since no three such arcs are permitted A_e does not overlap A_f, and A_e, $A_{e+1}, \cdots, A_f$ are mutually nonoverlapping. Now $A_1, A_2, \cdots, A_f$ can be 3-colored. Q.E.D.

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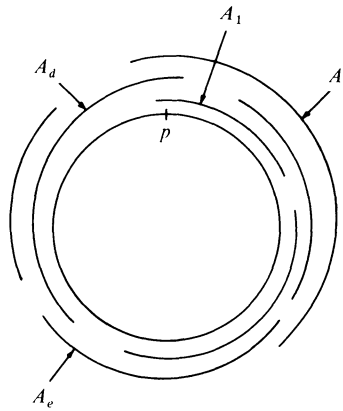

FIG. 3. *Case when A_e overlaps A_f.*

If we have a family made up of many "relatively" short arcs, it seems intuitive that $\gamma(F)$ would be very close to $r_{\text{sup}}(F)$. This result can be formalized with proper families of arcs. When F is a proper family, we can also get a bound on $\gamma(F)$ in terms of the size of the largest clique. Let $\lceil s \rceil$ and $\lfloor s \rfloor$ denote the smallest integer $m \geqq s$ and the largest integer $n \leqq s$, respectively.

THEOREM 4. *Let F be a proper family of n arcs with $r = r_{\text{sup}}(F)$ and q being the size of the largest clique. Then $\gamma(F) \leqq \frac{3}{2}q$. Also $\gamma(F) \leqq r + \lceil (r-1)/k \rceil$, where $k = \lfloor n/r \rfloor$, and for $F = F((k+1)r - 1, r)$, $k > 0$, we have $\gamma(F) = r + \lceil (r-1)/k \rceil$. Further, if $n > r^2$, then $\gamma(F) \lneqq r + 1$.*

Proof. Assume that the arcs in F are cyclically indexed $A_1, A_2, \cdots, A_n$ by the order of their counterclockwise endpoints (with A_1 chosen arbitrarily). Then the clockwise endpoint of A_i does not overlap A_{i+r} (subscript is mod n) or else $A_i, A_{i+1}, \cdots, A_{i+r}$ form an overlap set of size $r + 1$. If we write a sequence of colors for the corresponding sequence of indexed arcs, then a sufficient condition for a legal coloring is that when t is the ith color in the sequence, none of the next $r - 1$ colors in the sequence (circling to the start of the sequence if necessary) can be color t.

If there were an integer k such that $n = kr$, then a legal color sequence would be k repetitions of the subsequence $1, 2, 3, \cdots, r$. For example, if $n = 12$, $k = 3$, $r = 4$, we get the sequence 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4. In general, if $k = \lfloor n/r \rfloor$ so that $kr \leqq n < (k+1)r$, then we could have a legal color sequence made up of e repetitions of the subsequence $1, 2, \cdots, s$, where $s = \lceil n/k \rceil$ and $e = n - (s-1)k$, followed by $k - e$ repetitions of the subsequence $1, 2, \cdots, s - 1$. Note that $s - 1 \geqq r$ unless $k = n/r$, in which case $k - e = 0$. Since $n \leqq (k+1)r - 1$, we have $\gamma(F) \leqq s \leqq \lceil ((k+1)r - 1)/k \rceil = r + \lceil (r-1)/k \rceil$. The chromatic number of $F((k+1)r - 1, r)$ is, by the above, at most $r + \lceil (r-1)/k \rceil$, but no color can be used more than k times in this F and thus we need $\lceil ((k+1)r - 1)/k \rceil = r + \lceil (r-1)/k \rceil$ colors. For $n \geqq r^2$, $k = \lceil n/r \rceil \geqq r$ and so $\gamma(F) \leqq r + \lceil (r-1)/k \rceil = r + 1$.

For $n \geqq 2r$, the bound $\gamma(F) \leqq r + \lceil (r-1)/k \rceil$ implies $\gamma(F) \leqq \frac{3}{2}q$. Suppose F has fewer than $2r$ arcs. Find a pair of nonoverlapping arcs in F. Give them color 1 and remove them from F. Repeat this process until we have removed $2m$ arcs and the remaining arcs are all mutually overlapping, i.e., they form a clique.

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This way we will use $m + (n - 2m)$ colors. Observe $(n - 2m) \leqq q$ and $r \leqq q$. If $2m < r$, then $m + (n - 2m) < \frac{1}{2}r + q \leqq \frac{3}{2}q$. If $2m \geqq r$, then $m + (n - 2m) = n - m \leqq 2r - m \leqq 2r - \frac{1}{2}r \leqq \frac{3}{2}q$. Q.E.D.

Let $\theta(F)$ denote the size of the largest clique in the family F of arcs. In Theorem 3, we showed that if no 3 arcs cover the circle, then $\gamma(F) \leqq \frac{3}{2}\theta(F)$. Similarly in Theorem 4, we showed that for a proper family F of arcs, $\gamma(F) \leqq \frac{3}{2}\theta(F)$. These results suggest the following conjecture.

Conjecture. For any family F of arcs, $\gamma(F) \leqq \frac{3}{2}\theta(F)$.

4. Reformulation of arc coloring as a multicommodity flow. In this section we present a method for converting the problem of producing a q-coloring of a family F of arcs into a multicommodity flow problem. Clearly, if we cannot produce a q-coloring of F but can produce a $(q + 1)$-coloring, then the $(q + 1)$-coloring is a desired minimal coloring of F. Let $r = r_{\text{sup}}(F)$. Then we need only try values of q with $q \geqq r$. While in theory $\gamma(F)$ can be as large as $2r - 1$, in practice it is typically close to r, and thus $q = r$ or $q = r + 1$ are good values to use in the first attempt at producing a q-coloring. Then depending on whether the attempt succeeds or fails, we reduce or increase q, respectively.

We assume that each arc in the family F has a different counterclockwise endpoint (if not, slightly alter certain arcs). With each maximal overlap set C_i we associate an overlap point p_i which is contained in all arcs in C_i. Note that two arcs overlap if and only if they are in a common C_i.

Step 1. *Index the arcs and maximal overlap sets.* Let C_0 be any maximal overlap set. Index the arcs in F according to the clockwise order of their counterclockwise endpoints starting with A_1, the arc in C_0 that extends farthest on the counterclockwise side of overlap point p_0. Starting from p_0, index the maximal overlap sets from C_0 to C_{n-1} according to the clockwise order of their overlap points. See the example in Fig. 4.

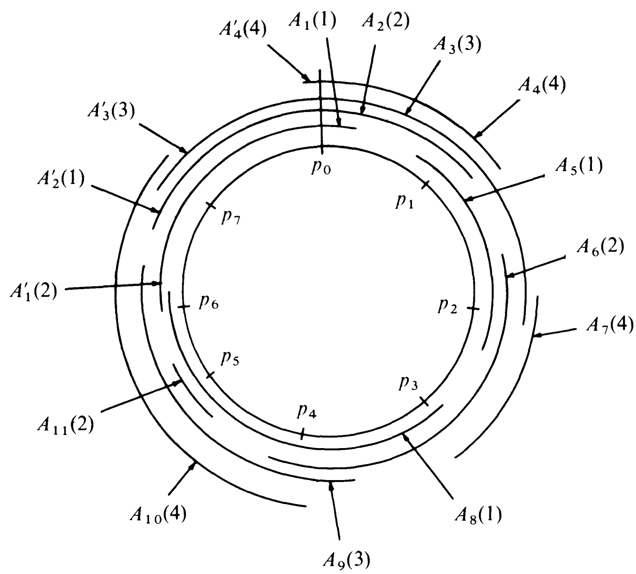

FIG. 4. *Coloring of family of arcs cut at p_0. Number in parenthesis is color number assigned to arc.*

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Step 2. *Break the circle at p_0 and q-color arcs*. Temporarily, we form a new family F' from F by cutting the circle at p_0 so that each arc in C_0 becomes two arcs; let A_k remain the name of the clockwise part of the old A_k (for A_k in C_0) and let A'_k be the name of the counterclockwise part. A q-coloring of F' is easily obtained: for A_k in C_0, assign A_k color k, and for the rest of the arcs, assign each successive arc A_k a color that does not conflict with the colors of the preceding arcs (A_k overlaps at most $r - 1$ preceding arcs or else an overlap set of size $r + 1$ or more results). One possible method for making color assignments is to give A_k the smallest conflict-free color number. See Fig. 4. For each A_k in C_0, we want A'_k to get the same color as A_k. Of course, this is not likely to happen. Below, we shall show how to change the color assignments just made in order to satisfy (when possible) these *boundary* conditions.

Step 3. *Recoloring and the reassignment matrix M*. Let D_i be the set of colors not assigned to any arc in $C_i \cap C_{i-1}$. Let D_0 be all colors. In other words, D_i is the set of colors available for coloring new arcs of C_i. We define D_n, the last D_i, to consist of the colors that are free when coming back to C_0 after going around the circle. Observe that we can color all new arcs in C_i simultaneously at p_i; the order of these arcs' counterclockwise endpoints is irrelevant since they all overlap the same set of old (previously colored) arcs, i.e., the arcs in $C_i \cap C_{i-1}$. If the coloring procedure in Step 2 produces coloring conflicts in the boundary conditions, then various color assignments must be changed. Thus the assignments of the colors in D_i to the new arcs in C_i must be altered for certain i's. To aid in this reassignment, we define a *reassignment matrix M* with entry $m_{ij} = 1$ if color j is in D_i, and $=0$ otherwise. See Fig. 5.

	Colors 1	2	3	4	5
D_0	1	1	1	1	1
D_1	1	0	0	0	1
D_2	0	1	0	1	1
D_3	1	0	1	0	1
D_4	0	0	1	1	1
D_5	0	1	0	0	1
D_6	0	1	0	0	1
D_7	1	0	1	0	1
D_8	0	0	0	1	1

FIG. 5. *The reassign-matrix M*

Note that we can reconstruct from M most of the overlap information for F': an arc corresponds to a column subsequence of a 1 followed by a string of zeros; such subsequences in column j correspond to arcs with color j. However, an arc like A_{11} which is in only one C_i does not have such a subsequence. Later, we shall see that such arcs can be eliminated from our problem.

If we were to make a color reassignment for A_k which first appears in C_i, say giving A_k color j' instead of j, then to avoid conflicts with later arcs, we would have to change the colors of some (perhaps, all) arcs of color j to color j' and some

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arcs of color j' to color j. (This would be like interchanging colors in a Kempe chain in graph coloring.) For simplicity, let us interchange colors j and j' in all arcs following A_k. In terms of the matrix M, columns j and j' should be permuted from row i down. More generally, reassigning colors at C_i corresponds to some permutation (from row i down) of the columns of M with a 1 in row i. Any such collection of partial-column permutations corresponds to a legal recoloring of F'. Conversely, every possible q-coloring of F' (with the given coloring of C_0) can be generated by the appropriate set of partial-column permutations, since any q-coloring can be obtained by first recoloring arcs in C_1, then C_2, etc. If the recoloring eliminates conflicts in the boundary conditions, we have a legal coloring of F.

Step 4. *The multicommodity flow model.* We can model a set of partial-column permutations with a multicommodity flow network which has a node $x_{i,j}$ of unit capacity corresponding to each entry m_{ij} in M. The node $s_j = x_{0,j}$ is the source of commodity j. For each $i < n$ and every j, there is an edge from $x_{i,j}$ to $x_{i+1,j}$. For each m_{ij} equal to 1, there are edges from $x_{i-1,j}$ to $x_{i,k}$ for all k in D_i. If A'_k has color j_k (for A_k in C_0), then the node $t_k = x_{n,k_k}$ is the sink of commodity k. See Fig. 6. The movement of unit commodity flows among the columns of the network

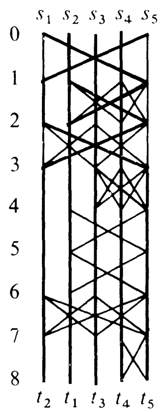

FIG. 6. *Multicommodity flow model*

corresponds to the permutations of coloring assignment mentioned in Step 3. If commodity k flows through $x_{i,j}$, this means that at C_i color k has replaced the original role of color j. So, if commodity k flows through the subsequence of $x_{i,j}$'s in column j corresponding to the subsequence in column j of M consisting of a 1 followed by zeros which in turn corresponds to an arc A_m, then A_m is being recolored with color k.

A unit flow for each commodity from its source to its sink in this network corresponds to a recoloring of F' which satisfies the boundary conditions. In Fig. 6, such a flow is shown for recoloring the arcs in Fig. 4. Then the set of arcs in Fig. 4 can be 5-colored. If a fifth color had not been allowed (and the fifth column of the reassignment matrix were removed), then it is easy to see that no feasible flow would exist in the associated multicommodity network. That is, the arcs in Fig. 4 cannot be 4-colored, and so 5 colors are minimal. On the other hand, if this family could not be 5-colored, we would add another column of ones

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to M and try 6-coloring; if this failed, then another column of ones and 7-coloring; etc.

We note that it is convenient to remove arcs in only one C_i, such as A_{11} in Fig. 4. After coloring the other arcs, an arc such as A_{11} is reinserted and simply assigned a color different from the colors of the other arcs in its C_i.

Let us give an algebraic formulation to this problem. Let j_k be the color originally assigned to A'_k (for A_k in C_0), and let S_N be the symmetric group on the set N of integers. Then we wish to express the permutation $\sigma^* \in S_{\{1,2,\cdots,q\}}$ which maps k to j_k and leaves other integers fixed as a product of the form

$$\sigma^* = \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \cdots \sigma_n,$$

where $\sigma_i \in S_{D_i}$.

This multicommodity flow model seems to be a very good approach, for it gets to the heart of the arc coloring problem. Examples of a size comparable to that in Fig. 4 can readily be solved by inspection, using the associated flow models. However, it must be pointed out that the practical uses of this model are currently quite limited. Integer multicommodity flows constitute an unexplored area of integer programming. All existing multicommodity flow algorithms work with fractional flows. Efficient implementation of large integer multicommodity flows will require development of specialized branch-and-bound techniques or the like.

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