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Circular chromatic number: a survey ☆

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Abstract

The circular chromatic number $\chi_c(G)$ of a graph G (also known as 'the star-chromatic number'), is a natural generalization of the chromatic number of a graph. In this paper, we survey results on this topic, concentrating on the relations among the circular chromatic number, the chromatic number and some other parameters of a graph. Some of the results and/or proofs presented here are new. The last section is devoted to open problems. We pose 28 open problems, and discuss partial results and give references (if any) for each of these problems.

MSC: 05C15

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1. Definition and basic properties

Consider the problem of designing a traffic control system at a road intersection. We need to assign to each traffic flow an interval of time during which it has the right of the road, i.e., it faces a green light. A complete traffic period is a time interval during which each traffic flow gets a turn of green light. We need to design a red–green light pattern for a complete traffic period and the pattern will be repeated forever. Suppose each interval of green light is of unit length. Our objective is to minimize the total length of a complete traffic period.

Graph is an ideal model for this problem. Each traffic flow is represented by a vertex, and two vertices are adjacent if the corresponding traffic flows are not compatible, i.e., their green light intervals must not overlap.

It seems natural to solve the traffic control problem by finding the chromatic number of the corresponding graph G. We partition the graph into minimum number of inde-

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pendent sets and then assign to each independent set a unit length interval of green light such that distinct independent sets correspond to disjoint intervals. Then the total length of a complete traffic period is just the chromatic number of the graph G. However, we shall see that this method does not provide the optimal solution.

We may view a complete traffic period as a circle C, and each vertex (i.e., each traffic flow) is assigned an interval of C of unit length, which is the time interval during which the corresponding traffic flow has the green light. Thus adjacent vertices of the graph are assigned to disjoint intervals of C, and our objective is to minimize the total length of the circle C. This leads to the following definition of circular coloring and the circular chromatic number of a graph.

Definition 1.1. Let C be a circle of (euclidean) length r. An r-circular coloring of a graph G is a mapping c which assigns to each vertex x of G an open unit length arc $c(x)$ of C, such that for every edge (x,y) of G, $c(x) \cap c(y) = \emptyset$. We say a graph G is r-circular colorable if there is an r-circular coloring of G. The circular chromatic number of a graph, denote by $\chi_c(G)$, is defined as

$$\chi_c(G) = \inf\{r\colon G \text{ is } r\text{-circular colorable}\}.$$

It is easy to see that if $\chi_c(G) = r$, then for any $r' \geqslant r$, there is an r'-circular coloring of G. Another trivial observation is that if H is a subgraph of G then $\chi_c(H) \leqslant \chi_c(G)$.

We may cut the circle C at an arbitrary point to obtain an interval of length r, which we may identify with the interval $[0,r)$. For each arc $c(x)$ of C, we let $c'(x)$ be the initial point of $c(x)$ (where $c(x)$ is viewed as going around the circle C in the clockwise direction). Then c' is a mapping of $V(G)$ to $[0,r)$ such that for every edge (x,y) of G, $1 \leqslant |c'(x) - c'(y)| \leqslant r-1$. The process above of obtaining c' from c can also be reversed. Therefore, we may identify an r-circular coloring of G with a mapping $c' : V \to [0,r)$ such that $1 \leqslant |c'(x) - c'(y)| \leqslant r-1$ for every edge (x,y) of G.

We define an r-interval coloring of a graph G to be a mapping g which sends each vertex x of G to a unit length open sub-interval $g(x)$ of the interval $[0,r]$, such that adjacent vertices are sent to disjoint sub-intervals. It is well-known [30] that the chromatic number $\chi(G)$ of G is the least real number r such that there is an r-interval coloring of G. Similarly an r-interval coloring of G corresponds to a mapping f from V to $[0,r)$ such that $1 \leqslant |f(x) - f(y)| \leqslant r-1$ for every edge (x,y) of G, and moreover $f(x) \leqslant r-1$ for all $x \in V$. Therefore any r-interval coloring of G corresponds to an r-circular coloring of G. On the other hand, for an r-circular coloring $c'\colon V \to [0,r)$, let $s = \max\{c'(x) : x \in V\}$, then c' can be viewed as an $(s+1)$-interval coloring of G. As $s < r$, we obtain the following result:

Theorem 1.1. *For any finite graph G, $\chi(G) - 1 < \chi_c(G) \leqslant \chi(G)$.*

Theorem 1.1 holds for infinite graphs as well, only the proof above is not valid anymore, because the set $\{c'(x)\colon x \in V\}$ may have no maximum, and the supremum of $\{c'(x)\colon x \in V\}$ could be equal to 1. The reason that $\chi(G) - 1 < \chi_c(G)$ for infinite graphs G (with finite chromatic number) is that if $\chi_c(G) \leqslant n-1$, then $\chi(H) \leqslant n-1$ for every finite subgraph of G, which then implies that $\chi(G) \leqslant n-1$.

Theorem 1.1 tells us that $\chi_c(G)$ contains more information about the structure of the graph G than $\chi(G)$ does. If we know the circular chromatic number of a finite graph, then we obtain its chromatic number simply by taking the ceiling of the circular chromatic number. On the other hand, two finite graphs of the same chromatic number may have different circular chromatic numbers. Thus $\chi_c(G)$ is a refinement of $\chi(G)$, and that $\chi(G)$ is an approximation of $\chi_c(G)$.

We shall see that the inequality $\chi_c(G) \leqslant \chi(G)$ could be strict. Therefore, the chromatic number approach may not provide the optimal solution to the traffic control problem. The relation between the circular chromatic number and the chromatic number of a graph has been one of the main subjects of all the research [1,11,20,26,34,48,75,78,86, 88,89,96,97], and will also be a main subject of this paper.

We shall now show that for finite graphs G, the infimum in the definition of the circular chromatic number is attained, and the circular chromatic numbers $\chi_c(G)$ are always rational. A result equivalent to this one (but using a different language) was first proved by Vince [78], by using method of continuous mathematics. A pure combinatorial proof of that result was given by Bondy and Hell [6].

We need the following lemma, which is equivalent to a lemma proved by Guichard [34], and which was also implicitly used in [78,88].

Let c be an r-circular coloring of a graph G. Then we define a directed graph $D = D_c(G)$ as follows: It has vertex set $V(D) = V(G)$, and there is a directed edge from x to y if and only if $(x, y) \in E(G)$ and the left end-point of $c(y)$ is equal to the right end-point of $c(x)$. Here we view the intervals $c(v)$ as going in the clockwise direction and that the left (right) end-point of $c(v)$ is the starting (terminating) point of $c(v)$.

Lemma 1.1. *Suppose G is a finite graph, and that c is an r-circular coloring of a graph G. If $D_c(G)$ is acyclic then there is an r'-circular coloring c' of G such that $r' < r$ and $D_{c'}(G)$ contains a directed cycle.*

Proof. Suppose $D_c(G)$ is acyclic. For each vertex x, define the level $\ell(x)$ to be the length of a longest directed path in D which ends at x. (Since $D_c(G)$ is acyclic, such a path exists.) Let x_0 be a vertex with maximum level. Then the interval $c(x_0)$ can be shifted to the right (i.e., to the clockwise direction) by a small distance, without violating the condition that adjacent vertices receive disjoint intervals. After the shifting, the vertex x_0 becomes an isolated vertex in the corresponding digraph. By repeating this process, we obtain another r-circular coloring c'' such that the digraph $D_{c''}(G)$ has no arcs. Therefore, each interval $c''(x)$ can be stretched to a longer interval, say an interval of length $s > 1$, and still satisfying the condition that adjacent vertices corresponds to disjoint intervals. We now uniformly shrink the circle C into a circle C' of length r/s. Each interval of C of length s is shrank to an interval of C' of length 1. Thus we obtain an r/s-circular coloring of G.

We may repeat this process, if needed, to obtain an r'-circular coloring c' with $r' < r$ and such that $D_{c'}(G)$ contains a directed cycle. (This can be achieved in a single step, if the process is applied properly. But we would not mind taking ultra-limit if needed.) This completes the proof of Lemma 1.1. □

Suppose c is an r-circular coloring of G, and that $(x_0, x_1, \ldots, x_{p-1}, x_0)$ is a directed cycle of $D_c(G)$. It follows from the definition of $D_c(G)$ that the union of the intervals $c(x_0), c(x_1), \ldots, c(x_{p-1})$ winds around the circle C exactly q times for some integer q. Because the length of the circle C is r, we conclude that the sum of the lengths of these intervals is equal to qr. On the other hand, each interval $c(x_i)$ has length 1, hence $p = qr$ which implies that $r = p/q$. Note that in this case G contains a cycle of length p, and every vertex of G is contained in an independent set of size q (as the union of the intervals $c(x_i)$ winds around the circle C exactly q times, hence each point of the circle C is 'covered' by an independent set of size q).

By applying Lemma 1.1, we conclude that if G has an r-circular coloring, then G has an r'-circular coloring such that $r' \leqslant r$ and $r' = p/q$ for some integers p, q, where p is at most the circumference number (i.e., the longest cycle) of G and q is at most the independency number of G. Therefore, to determine the circular chromatic number $\chi_c(G)$ of G, it suffices to determine whether G is r-circular colorable or not for each of those rational numbers $r = p/q$ for which $q \leqslant \alpha(G)$ and $p \leqslant \text{circumference}(G) \leqslant |V(G)|$. Since there are only finite many such rational numbers, the infimum in the definition is attained, and that $\chi_c(G)$ is a rational p/q, where $p \leqslant \text{circumference}(G)$ and $q \leqslant \alpha(G)$, [86].

The problem of circular colorings of infinite graphs was discussed in [3,88]. The circular chromatic number of an infinite graph could be irrational. However, it was shown in [3] that the infimum in the above definition of circular chromatic number is also attained for infinite graphs. In other words, if $\chi_c(G) = r$ then G is r-circular colorable.

The converse of Lemma 1.1 is also true. Namely we have the following lemma:

Lemma 1.2. *If G is r-circular colorable, and for every r-circular coloring c of G, $D_c(G)$ contains a directed cycle, then $\chi_c(G) = r$. Therefore, a graph G has $\chi_c(G) = r$ if and only if G is r-circular colorable and for every r-circular coloring c of G, $D_c(G)$ contains a directed cycle.*

For a (p,q)-coloring ϕ of a graph G, let $D_\phi(G)$ be the digraph with vertex set $V(G)$ and xy is a directed edge if and only if $\phi(y) - \phi(x) = q \pmod p$. The digraph $D_\phi(G)$ is analog to $D_c(G)$ for a p/q-circular coloring, and they have similar properties. In particular, Lemma 1.2 is true for $D_\phi(G)$, i.e., we have the following lemma, which was proved in [34].

Lemma 1.3. *For a graph G, $\chi_c(G) = p/q$ if and only if G is (p,q)-colorable, and for any (p,q)-coloring of G, $D_\phi(G)$ contains a directed cycle.*

Lemmas 1.2 and 1.3 are very useful in determining the circular chromatic number of a graph. Theorem 1.2 below illustrates an application of Lemma 1.3.

Theorem 1.2. *Suppose G is a graph and $X \subset V(G)$ is a subset of the vertex set of G. If G is (p,q)-colorable, and for any (p,q)-coloring f of G, $f(X) = \{0, 1, \ldots, p-1\}$, and the restriction $f|X$ is unique up to a permutation of the colors, then $\chi(G) = p/q$.*

Proof. Assume G is a graph satisfying the conditions above. Since G is (p,q)-colorable, we have $\chi_c(G) \leqslant p/q$. Assume to the contrary that $\chi_c(G) < p/q$. Then by Lemma 1.3, G has a (p,q)-coloring ϕ such that $D_\phi(G)$ is acyclic. We define the level of a vertex v of $D_\phi(G)$ to be the length of a longest directed path ending at v (such a path exists, because $D_\phi(G)$ is acyclic). Let $v^* \in X$ be a vertex of X whose level is maximum. Let ϕ' be the mapping defined as follows: if there is a directed path in $D_\phi(G)$ from v^* to x, then $\phi'(x) = \phi(x) + 1 \pmod p$. Otherwise $\phi'(x) = \phi(x)$. Then it is straightforward to verify that ϕ' is also a (p,q)-coloring of G. Moreover, $\phi'|X = \phi|X$, except that $\phi'(v^*) \neq \phi(v^*)$. Therefore, $\phi'|X$ cannot be obtained from $\phi|X$ by a permutation of colors (because some vertices which are colored by different colors by ϕ are now colored by the same color by ϕ'). This is in contrary to our assumption. □

2. Equivalent formulations

The circular chromatic number $\chi_c(G)$ of a graph G was introduced by Vince in 1988 [78], as 'the star-chromatic number'. However, the definition given in the previous section is not the original definition of Vince, but an equivalent definition given by the author in [88], in a slightly different form. The original definition of Vince is as follows:

For two integers $1 \leqslant d \leqslant k$, a (k,d)-coloring of a graph G is a coloring c of the vertices of G with colors $\{0, 1, 2, \ldots, k-1\}$ such that

$$(x, y) \in E(G) \Rightarrow d \leqslant |c(x) - c(y)| \leqslant k - d.$$

The circular chromatic number is defined as

$$\chi_c(G) = \inf\{k/d\colon \text{ there is a } (k,d)\text{-coloringof } G\}.$$

For any integer k, a $(k,1)$-coloring of a graph G is just an ordinary k-coloring of G.

Suppose c is a (k,d)-coloring of G. Let $c' : V(G) \to [0, k/d)$ be the mapping defined as $c'(x) = c(x)/d$. Then for every edge (x, y) of G, we have $1 \leqslant |c'(x) - c'(y)| \leqslant k/d - 1$. Therefore a (k,d)-coloring of G corresponds to a k/d-circular coloring of G. On the other hand, it is straightforward to verify that if c' is a k/d-circular coloring of G (viewed as a mapping from $V(G)$ to $[0,r)$), then the mapping c defined as $c(x) = \lfloor c'(x)d \rfloor$ is a (k,d)-coloring of G. Therefore Vince's definition of the circular chromatic number is indeed equivalent to the one appeared in Section 1.

Another equivalent definition of the circular chromatic number was given by Goddyn et al. [29]. This definition relates the circular chromatic number to Nowhere-Zero-Flows of a graph. Given a (k,d)-coloring c of a graph G, we obtain an integer flow f of the cocyclic matroid $M^*(G)$, by giving an arbitrary orientation of G, and then by letting $f(e) = c(x) - c(y)$, where $e = (x, y)$ is an edge oriented from x to y. This flow has the property that the absolute value of the flow ranges between d and $k - d$. This process can be reversed to obtain a (k,d)-coloring of G from any integer flow f of $M^*(G)$ with absolute value ranges between d and $k - d$. Thus, a graph G has a (k,d)-coloring if and only if the cocyclic matroid $M^*(G)$ has an integer flow with absolute value ranges between d and $k - d$. It follows from a lemma of Hoffman [46] that the cocyclic matroid $M^*(G)$ admits an integer flow with absolute value ranges between d and $k - d$ if and only if there is an orientation of G such that for every cycle C of G,

$$d/(k-d) \leqslant |C^+|/|C^-| \leqslant (k-d)/d,$$

where C^+ is the set of edges of C of positive orientation, and C^- is the set of edges of C of negative orientation, along an arbitrary traversal of C.

Therefore, we obtain the following equivalent definition of the circular chromatic number of a graph: for an orientation D of a graph G, let

$$\xi(D) = \max\{|C|/|C^-|, |C|/|C^+|\text{: } C \text{ is an oriented cycle of } D\}.$$

Then

$$\chi_c(G) = \min\{\xi(D)\text{: } D \text{ is an orientation of } G\}.$$

Each of the three equivalent definitions of the circular chromatic number of a graph has its advantages. Sometimes it is just a personal taste to choose any of the above definitions. Yet there are proofs that are nicer and shorter if we use a particular definition instead of the others. We shall use all the three definitions in our proofs in this paper.

3. Graphs G for which $\chi_c(G) = \chi(G)$

The question that for which graphs G we have $\chi_c(G) = \chi(G)$ was raised by Vince [78], and investigated in [1,26,34,75,78,86,88,89,96,97]. It was shown by Guichard [34] that it is NP-hard to determine whether or not an arbitrary graph G satisfies $\chi_c(G) = \chi(G)$. Indeed, using an oracle which determines whether or not an arbitrary graph G satisfies $\chi_c(G) = \chi(G)$, we can easily determine the chromatic number of a graph G as follows: Let $G \cup H$ denote the disjoint union of graphs G and H, and let G_k^d denote the graph with vertex set $\{0, 1, \dots, k-1\}$ and in which i is adjacent to j when $d \leqslant |i - j| \leqslant k - d$. It was shown in [78] that $\chi_c(G_k^d) = k/d$. Using this fact, it is straightforward to verify that a graph G is n-colorable if and only if the following two statements are true:

(i) $\chi_c(G \cup K_n) = \chi(G \cup K_n)$;
(ii) $\chi_c(G \cup G^2_{2n+1}) \neq \chi(G \cup G^2_{2n+1})$.

The following decision problem seems more interesting:

Instance: A graph G of chromatic number n.
Question: Is it true that $\chi_c(G) = n$?

It is likely that this problem is also NP-hard for general graphs. The problem maybe more interesting if restricted some special classes of graphs, say for planar graphs, line graphs, etc.

We note that an argument similar to the proof of Guichard shows that even for planar graphs G, if the chromatic number of G is unknown, then it is NP-hard to determine whether or not $\chi_c(G) = \chi(G)$. To see this, it suffices to observe that by using an oracle which determines whether or not $\chi_c(G) = \chi(G)$ for any planar graph G, one can easily (in polynomial time) determine the chromatic number of any planar graph, while it is known that the problem of determining the chromatic number of a planar graph is NP-complete.

For the similar reason, it is also NP-hard to decide whether or not $\chi_c(G) = \chi(G)$ if G is an arbitrary line graph and $\chi(G)$ is unknown.

In light of the above results, it is unlikely that there is a simple characterization of graphs G with $\chi_c(G) = \chi(G)$. Nevertheless, some sufficient conditions are found in [1,26,34,75,78,86,88] under which a graph G satisfies $\chi_c(G) = \chi(G)$. The following result, which was proved in [75], is the strongest result along this line:

Theorem 3.1. *Suppose $\chi(G) = n$. If there is a non-trivial subset A of V (i.e., $A \neq V$ and $A \neq \emptyset$) such that for any n-coloring c of G, each color class X of c is either contained in A or disjoint from A, then $\chi_c(G) = \chi(G)$.*

Proof. Assume that A is a subset of V satisfying the condition above, and assume to the contrary of Theorem 3.1 that $\chi_c(G) = r < n$. Let c be an r-circular coloring of G. First we show that for any $x \in A, y \in V - A$, $c(x) \cap c(y) = \emptyset$. Otherwise, let $p \in c(x) \cap c(y)$ for some $x \in A$ and $y \in V - A$. Starting from the point p, we evenly put n points $p = p_1, p_2, \ldots, p_n$ on the circle C. Thus the length of the arc from p_i to p_{i+1} is $r/n < 1$. Therefore each arc $c(z)$ contains at least one of the points $p_1, p_2, \ldots, p_n$. We color a vertex z of G with color i for some $p_i \in c(z)$; and in particular, color x and y with color 1. This is an n coloring of G which has a color class, the class with color 1, that is neither contained in A nor disjoint from A, contrary to our assumption.

Let $P = \{p \in C\colon p \in c(x)$ for some $x \in A\}$. Then $c(y) \cap P = \emptyset$ for any $y \in V - A$. As A is a non-trivial subset of V, we know that P is a non-trivial subset of C. Let q be a boundary point of P, then it is easy to see that $q \notin c(z)$ for any $z \in V$ (note that each arc $c(z)$ is an open subset of C). Therefore, we may cut the circle C at q to obtain an r-interval coloring of G, contrary to our assumption that $\chi(G) = n > r$. $\square$

The Petersen graph P is an example which does not satisfy the condition of this theorem, and yet $\chi_c(P)=\chi(P)=3$. So the condition of this theorem is not a necessary condition. There are other sufficient conditions for a graph G to satisfy $\chi_c(G)=\chi(G)$, and it turns out that all the other known sufficient conditions can be easily derived from Theorem 3.1.

Suppose the complement $\bar{G}$ of a graph G is disconnected, and that $A\subset V(G)$ is a connected component of $\bar{G}$. Then for any coloring c of G, any color class is either contained in A or disjoint from A. Therefore, we have the following corollary, which was proved by Gao et al. [26], and by Abbott and Zhou [1].

Corollary 3.1. *If the complement of G is disconnected, then $\chi_c(G)=\chi(G)$.*

A special case of this corollary is that G has a universal vertex, i.e., a vertex which is adjacent to every other vertices of G. This special case was proved earlier by Zhu [88], and by Guichard [34],

Corollary 3.2. *If G has a universal vertex then $\chi_c(G)=\chi(G)$.*

If a graph G is uniquely n-colorable, then $\chi(G)=n$ and any color class can be used as the set A in Theorem 3.1. Therefore we have the following corollary [75].

Corollary 3.3. *If G is uniquely n-colorable, then $\chi_c(G)=\chi(G)$.*

Of course, Corollary 3.3 is a special case of Theorem 1.2.

It was proved by Bollobás and Sauer [5] and by Erdős [64] that for any integers $n\geqslant 1$ and $g\geqslant 3$, there is a graph G of girth at least g and that G is uniquely n-colorable. Therefore we have the following corollary [75]:

Corollary 3.4. *For any integers $n\geqslant 1$ and $g\geqslant 3$, there is a graph G of girth at least g and $\chi_c(G)=\chi(G)=n$.*

A special case of Corollary 3.4 answers the following question of Abbott and Zhou [1]: Does there exist triangle-free graphs G with $\chi_c(G)=\chi(G)=n$?

The proof of Bollobás and Sauer [5] uses probability method. A constructive proof of the existence of uniquely n-colorable graphs with large girth was given in [64,65]. By using the categorical product, Greenwell and Lovász [31] gave a simple method of constructing uniquely n-colorable graphs with arbitrarily large odd girth. A method of constructing triangle-free uniquely n-colorable graphs was also given in [66]. It follows from Corollary 3.3 that these methods provide constructions of triangle-free graphs G with $\chi_c(G)=n$. Recently, Zhou [86] gave another method of constructing triangle-free graphs G with $\chi_c(G)=n$, and this method was modified in [96] to construct uniquely n-colorable triangle-free graphs.

Here we introduce a new method of constructing graphs with given circular chromatic number. Using this method, we show a very simple way of constructing graphs G of girth at least g and with $\chi_c(G) = r$ for any rational number $r \geqslant 2$.

Suppose $r = p/q > 2$ is a rational number (where $\gcd(p,q) = 1$), Q is a graph and a, b are two distinct vertices of Q. We call the triple $(Q; a, b)$ a *strong r-circular superedge* if the following are true:

1. for any $|i-j|_p \geqslant q$, there is a (p,q)-coloring c of Q such that $c(a) = i$ and $c(b) = j$;
2. for any $\varepsilon > 0$ and for any $(r-\varepsilon)$-circular coloring f of Q we have $f(a) \cap f(b) = \emptyset$.

Given a graph G, an edge $e = xy$ of G and a triple $(Q; a, b)$ (where a, b are distinct vertices of Q), when we say *replace the edge e by* $(Q; a, b)$, we mean take the disjoint union of $G - e$ and Q then identify x with a and y with b. The resulting graph is denoted by $G(e, (Q; a, b))$.

Theorem 3.2. *If $\chi_c(G) = r$, $e = xy$ is an edge of G and $(Q; a, b)$ is a strong r-circular superedge, then by replacing the edge e by $(Q; a, b)$, the resulting graph $G(e, (Q; a, b))$ also has circular chromatic number r.*

Proof. Let f be a (p,q)-coloring of G. Then $|f(x) - f(y)|_p \geqslant q$. By the definition of a strong r-circular superedge, the coloring f can be extended to a (p,q)-coloring of $G(e, (Q; a, b))$. Therefore $\chi_c(G(e, (Q; a, b))) \leqslant p/q$.

It remains to show that $\chi_c(G(e, (Q; a, b)) \geqslant p/q$. Assume to the contrary that there is an $\varepsilon > 0$ and there is an $(r-\varepsilon)$-circular coloring c of $G(e, (Q; a, b))$. By the definition of a strong r-circular superedge, we know that $c(a) \cap c(b) = \emptyset$. Hence c is an $(r-\varepsilon)$-circular coloring of G, contrary to the assumption that $\chi_c(G) = r$. $\square$

Theorem 3.3. *For any rational number $r = p/q \geqslant 2$, and for any integer $g \geqslant 3$, there is a graph G of girth at least g and $\chi_c(G) = r$.*

Proof. If there is a strong r-circular superedge $(Q; a, b)$ such that Q has girth at least g and the distance between a and b is at least g, then we replace each edge of G_p^q by a copy of $(Q; a, b)$. Denote the resulting graph by G. Then it follows from Theorem 3.2 that $\chi_c(G) = p/q$. On the other hand, it is easy to see that G has girth at least g.

Thus, it remains to show that there is a strong r-circular superedge $(Q; a, b)$ such that Q has girth at least g and the distance between a and b is at least g.

We shall only consider the case that $r = n$ is an integer. The case r is non-integer is technically more difficult, but use a similar idea.

Let H be a graph of girth at least g with $\chi(H) = n+1$. Moreover for any edge $e = aa'$ of H, $\chi(H - e) = n$. (There are a few known methods of constructing such graphs.) Add a new vertex b and connect b to a' by an edge. Denote the resulting graph by Q. We shall show that $(Q; a, b)$ is the required strong n-circular superedge. Obviously Q has girth at least g and a, b has distance at least g. Since H is not n-colorable and

$H - e$ is n-colorable, we conclude that there is an n-coloring f of $H - e$ such that $f(a) = f(a')$. Therefore, for any two distinct colors i, j, there is an n-coloring f of Q such that $f(a) = i$ and $f(b) = j$. It remains to show that for any $\varepsilon > 0$, if f is an $(n - \varepsilon)$-circular colouring of Q, then $f(a) \cap f(b) = \emptyset$. This would follow if we can prove that $f(a) = f(a')$, because $f(b) \cap f(a') = \emptyset$ by definition. Assume to the contrary that there is an $(n - \varepsilon)$-circular colouring f of Q then $f(a) \neq f(a')$. Recall that f maps each vertex v of Q to a unit length arc of a circle C of length $n - \varepsilon$. Let p_0 be a point of C lies in the arc $f(a) - f(a')$. Starting from p_0, we put n points $p_0, p_1, \dots, p_{n-1}$ on the circle C (consecutively along the clockwise direction) such that the distance from p_i and p_{i-1} is equal to $(n - \varepsilon)/n < 1$. Since for any v, $f(v)$ is a unit length arc, so $f(v)$ contains at least one of the points p_i. Define an n-coloring c of Q as follows: $c(v) = i$ if and only if $p_i \in f(v)$ and $p_j \notin f(v)$ for any $j < i$. This is indeed an n-colouring of Q, as every vertex of Q is coloured by one of the n colours, and two adjacent vertices have distinct colours. But $c(a) = 0$ and $c(a') \neq 0$. This means that c is indeed an n-colouring of H, contrary to our assumption that H has chromatic number $n + 1$. $\square$

The case $r = p/q$ is not an integer is more complicated. By following the argument above, one can easily convince oneself that it suffices to prove the existence of a triple $(Q; a, a')$ such that Q has girth at least g, the distance between a and a' is at least $g - 1$ such that H is (p, q)-colorable, and for any (p, q)-coloring c of Q, $f(a) = f(a')$.

First, we observe that it is easy to construct a graph H of girth at least g such that H is not (p, q)-colorable, but $H - e$ is (p, q)-colorable for all edge e. (We may simply choose an H which has girth at least g and chromatic number $\lceil r \rceil + 1$, and then delete some edges of H if necessary.) Now let $Q = H - e$, where $e = aa'$ is an edge of H, we obtain a triple $(Q; a, a')$ such that Q is (p, q)-colorable, and moreover for any (p, q)-coloring f of Q, $|f(a) - f(a')|_p \leqslant q - 1$. Let $m(Q; a, b) = \max\{|f(a) - f(a')|_p : f$ is a (p, q)-coloring of $Q\}$. If $m(Q; a, b) = 0$, then we are done. If not, we may use $(Q; a, b)$ to construct another triple $(Q'; a', b')$ with $m(Q'; a', b') < m(Q; a, b)$. The construction is a little bit technical, and we refer the readers to [70] to the details, where such triples are constructed and are used to prove the following much stronger result.

Theorem 3.4. *Suppose* $r = p/q \geqslant 3$ (*where* $\gcd(p, q) = 1$), $g \geqslant 3$ *is an integer*, X *is a finite set and* $\{f_i : i = 1, 2, \dots, m\}$ *is a set of mappings from* X *to the colors* $\{0, 1, \dots, p - 1\}$. *Then there is a graph* G *with vertex set* $V \supset X$, *which has the following properties*:

1. *The girth of* G *at least* g.
2. $\chi_c(G) = p/q$.
3. *There are exactly* m (p, q)-*colorings* (*up to an isomorphism of the colors*) $c_1, c_2, \dots, c_m$ *of* G *and for each* i, *the restriction of* c_i *to* X *is equal to* f_i.

Theorem 3.4 is a generalization of a result of Muller concerning vertex coloring of graphs.

For historical note, Theorem 3.3 was first proved in [89] by probability method. The probablistic argument is parallel to that of [5,68].

If instead of large girth, we only require the graph to have large odd girth, then the construction could be simpler. By using the categorical product, Kirsch [52] gave a construction of graphs G with arbitrarily large odd girth, and with $\chi_c(G)=r$ for an arbitrary rational $r\geqslant 2$. This result was generalized in [96] where uniquely H-colorable graphs of arbitrarily large odd girth were constructed.

By applying Theorem 3.2, with one graph of circular chromatic number r and with one strong r-circular superedge, we can construct infinitely many new r-circular chromatic graphs.

The construction of strong n-circular superedges (where n is an integer) was discussed in [97]. The following results were proved:

if G is an n-chromatic graph, x, y are non-adjacent vertices of G, and $\chi(G|_{xy})=n+1$, then $(G;x,y)$ is a strong n-circular superedge;

if $(G;x,y)$ and $(G';x',y')$ are both strong n-circular superedges, then replace any edge e of G with the strong superedge $(G';x',y')$, the triple $(G(e,(G';x',y'));x,y)$ is again a strong n-circular superedge.

Many interesting examples of graphs G with $\chi_c(G)=\chi(G)$ were constructed in [97], by using the above method. Section 5 contains some examples constructed there which answer some questions about the circular chromatic numbers of planar graphs.

It was noted in [97] that the Hajos sum of two n-chromatic graphs may be regarded as replacing one edge of an n-chromatic graph by an *n-superedge* formed from the other n-chromatic graph, where an n-superedge is a triple $(G;x,y)$ such that $G|_{xy}$ is not n-colorable. Hajos theorem provides a method of constructing all graphs G with $\chi(G)\geqslant n$. The question of constructing or characterizing all graphs G with $\chi_c(G)\geqslant n$ remains an open problem. However, there is an analog of Hajos theorem obtained by Nešetřil [67] that construct, for each $r\geqslant 2$, all the graph with circular chromatic number strictly greater than r from a finite set of graphs by adding vertices and edges, identifying non-adjacent edges, and a 'generalized Hajos sum'. There are also some interesting characterizations of graphs with circular chromatic numbers at least 3. These characterizations follow from a recent result of Brandt [7] and an earlier result of Pach [71]. Both Brandt and Pach were aiming at some other problems. However, once we start to look at their results from the point of view of circular colorings, the characterizations below follow immediately. We shall not cite their original results, but interpret them in the language of circular chromatic numbers.

Theorem 3.5 (Brandt [7]). *Let H be the graph obtained from the Petersen graph by deleting one vertex. A graph G has circular chromatic number at least* 3 *if and only if every maximal triangle-free supergraph G' of G contains H as a subgraph.*

Theorem 3.6 (Pach [71]). *A graph G has circular chromatic number at least* 3 *if and only if every maximal triangle-free supergraph G' of G has an independent set whose elements do not have a common neighbour.*

These characterizations of graphs with circular chromatic number at least 3 seem mysterious to this author. It is not clear if these characterizations can be generalized to graphs with circular chromatic numbers at least n for arbitrary integer n. It is also not clear if these characterizations will be helpful in the search for methods for constructing such graphs.

4. Graphs G with $\chi_c(G)$ close to $\chi(G) - 1$

We have shown that $\chi(G)-1 < \chi_c(G) \leqslant \chi(G)$ for every graph G. Section 3 discusses the extremal case that $\chi_c(G) = \chi(G)$. The other extremal case is that $\chi_c(G)$ is close to the lower bound $\chi(G) - 1$.

As we noted before, the parameter of the circular chromatic number of a graph maybe viewed as a refinement of the chromatic number. Corollary 3.3 in the previous section is a good illustration of this idea. If a graph is uniquely n-colorable, then there is no room to 'save' any color. Therefore $\chi_c(G) = \chi(G)$. On the other hand, if a graph is n-critical, i.e., $\chi(G) = n$ and $\chi(G - v) = n - 1$ for any vertex v of G, then the nth color is 'barely' needed. It seems that in this case there should be some room for saving colors, if we consider its circular chromatic number instead of its chromatic number. This intuition is actually true, in some sense. It was shown by Guichard [34] that if a graph G is n-critical, and has girth at least $n + 1$, then $\chi_c(G) < \chi(G)$. The following result of [75] reveals a quantitative relation between the girth and the circular chromatic number of an n-critical graph.

Theorem 4.1. *Let $m \geqslant 1$ and $t \geqslant 1$ be integers. If G has a vertex x such that $G - x$ is m-colorable, and every cycle of G containing x has length at least $m(t-1)+2$, then $\chi_c(G) \leqslant m + 1/t$.*

Proof. Let c be an m-coloring of $G - x$ with colors $1, 2, \ldots, m$. We orient the edges of G as follows: Suppose (u, v) is an edge of G. Then we orient the edge from u to v if either $u = x$ or $c(u) < c(v)$. By the definition of Goddyn et al. of the circular chromatic number, to prove Theorem 4.1, it suffices to show that for any oriented cycle C, we have $|C|/|C^-| \leqslant m + 1/t$ and $|C|/|C^+| \leqslant m + 1/t$. If the cycle C does not contain x, then this is true because $G - x$ contains no directed path of length m. If C does contain x, then $|C| \geqslant m(t-1)+2$. Hence $C - x$ is a path P of length at least $m(t-1)$. Since P contains no directed path of length m, we have

$$|P^+| \leqslant (m-1)(|P^-|+1), \quad |P^-| \leqslant (m-1)(|P^+|+1),$$

where P^+, P^- are the sets of forward edges and backward edges of P, respectively. Since $|P| = |P^+| + |P^-| \geqslant m(t-1)$, we conclude that $|P^+| \geqslant t-1$ and $|P^-| \geqslant t-1$. Therefore $|C|/|C^-| = (|P|+2)/(|P^-|+1) \leqslant m + 1/t$ and $|C|/|C^+| = (|P|+2)/(|P^+| + 1) \leqslant m + 1/t$. This completes the proof. □

Corollary 4.1. *Suppose G is* $(n+1)$*-critical, and that G has girth at least* $n(t-1)+2$*, then* $\chi_c(G) \leqslant n + 1/t$.

In some sense the bound in Corollary 4.1 is the best bound for the circular chromatic number of a critical graph in terms of its girth. For example, when $n=3$, then 3-critical graphs are just odd cycles, and $\chi_c(C_{2t+1}) = 2 + 1/t$. On the other hand, the following problem remains open:

For any integer n, let $g(n)$ be the least integer such that any n-critical graph G with girth greater than $g(n)$ satisfies $\chi_c(G) < \chi(G)$. What is the value of $g(n)$?

Certainly $g(n) \geqslant 3$. On the other hand, Corollary 4.1 implies that $g(n) \leqslant n$. Indeed, if an n-critical graph G has girth greater than n, then it follows from Corollary 4.1 that $\chi_c(G) \leqslant n - 1/2$. Therefore $g(3) = 3$. Another value of $g(n)$ we know is that $g(4) = 4$ [11]. For all other n, we know nothing more than the bounds $3 \leqslant g(n) \leqslant n$. In particular, we do not know whether or not the function $g(n)$ is non-decreasing.

The result $g(4) = 4$ is obtained in [11] as a by-product of the study of Mycielski graphs. For a graph G with vertex set $V(G)=V$ and edge set $E(G)=E$, the *Mycielskian* of G is the graph $M(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x'\colon x \in V\}$, and edge set $E \cup \{xy'\colon xy \in E\} \cup \{y'u\colon y' \in V'\}$. The vertex x' is called the *twin* of the vertex x (and x is also called the twin of x').

It is well known (see [57]) that for any graph G with at least one edge, $\omega(M(G)) = \omega(G)$ and $\chi(M(G)) = \chi(G) + 1$. Moreover, it is easy to see (cf. [11]) that if G is k-critical, then $M(G)$ is $(k+1)$-critical. The following result was proved in [11]:

Theorem 4.2. *If* $\chi(G) = 3$*, then* $\chi_c(M(G)) = 4$.

Thus take an odd cycle C_{2m+1}, which is a 3-critical graph, then $M(C_{2m+1})$ is a 4-critical graph with circular chromatic number 4. This implies that $g(4) \geqslant 4$. As $g(n) \leqslant n$ for all n, we conclude that $g(4) = 4$.

The following result concerning the circular chromatic number of Mycielski graphs was proved in [49]:

Theorem 4.3. *Suppose* $\chi_c(G) \leqslant \chi(G) - 1/d$*, then for any integer* $k \geqslant 1$ $\chi_c(M^{2k}(G)) \leqslant \chi(M^{2k}(G)) - 1/d$.

Here $M^k(G)$ is recursively defined by $M^k(G) = M(M^{k-1}(G))$. Thus starting with a graph G with $\chi_c(G)$ close to $\chi(G) - 1$, by repeatedly taking the mycielskian of G, we obtain infinitely many graphs G' such that $\chi_c(G')$ is close to $\chi(G') - 1$. When G is color-critical, then those graphs G' obtained by repeatedly taking the mycielskian of G are also color-critical.

Theorem 4.1 implies that n-critical graphs G 'tends' to have circular chromatic number close to $n-1$. However, there are many n-critical graphs G with $\chi_c(G)=\chi(G)$. Suppose $n\geqslant 4$. For any integer $1\leqslant r\leqslant n-3$ one may take an r-critical graph H and an $(n-r)$-critical graph H', then $H+H'$ is an n-critical graph with $\chi_c=\chi$ (cf. Corollary 3.1). Such constructed graphs always have a vertex of large degree. Zhou [86] asked the question whether there are arbitrarily large n-critical graph G with $\chi_c(G)=\chi(G)$ which has small maximum degree. With an elegant argument, he showed that there are arbitrarily large 4-critical 4-regular graphs G with $\chi_c(G)=\chi(G)=4$. However, for $n\geqslant 5$, it remains an open problem whether there exist arbitrarily large n-critical graphs G with $\chi_c(G)=\chi(G)=n$ such that $\Delta(G)$ is bounded by a function of n.

5. Planar graphs

The circular chromatic numbers of planar graphs were investigated in [25,26,63,73,75, 78,88,91,92,96]. The questions that have been on the focus are the following:

(1) Which planar graphs have circular chromatic number 4?
(2) Which have circular chromatic number 3?
(3) For which rational number r, there is a planar graph G with $\chi_c(G)=r$?

Question 3 has now been settled. It follows from results in [63,92] that a rational number r is the circular chromatic number of a planar graph if and only if $r=1$ or $2\leqslant r\leqslant 4$.

For questions 1 and 2, it seems difficult to characterize all graphs with circular chromatic numbers 3 or 4. However, some infinite families of such graphs have been constructed. We first present some infinite families of such graphs.

Let C_{2k+1} be the odd cycle with vertices $c_0,c_1,\dots,c_{2k}$, and with edges (c_i,c_{i+1}), $i=0,1,\dots,2k$ (where $c_{2k+1}=c_0$). Suppose W_{2k+1} is obtained from C_{2k+1} by adding a vertex u and all the edges (u,c_i). The graph W_{2k+1} is called an odd *wheel*. It is easy to see that $\chi(W_{2k+1})=4$. Since W_{2k+1} contains a universal vertex, the following result follows from Corollary 3.2:

Theorem 5.1. *For every integer* $k\geqslant 1$, $\chi_c(W_{2k+1})=4$.

For planar graphs G with $\chi_c(G)=3$, we observe that if a 3-chromatic graph contains a triangle, then its circular chromatic number is equal to 3. Thus we concentrate on triangle-free planar graphs. The following result is proved in [75]:

Theorem 5.2. *Let* G_{2k+1} *be the graph obtained from* W_{2k+1} *by subdividing each of the* $2k+1$ *spokes* (u,c_i) *into two edges, say* (u,v_i) *and* (v_i,c_i). *Then* $\chi_c(G_{2k+1})=3$.

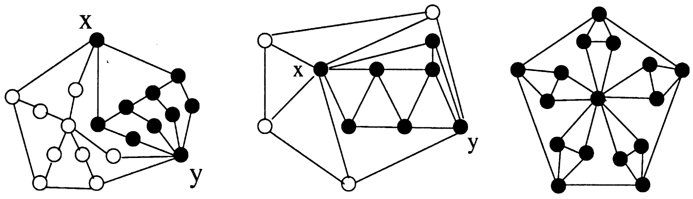

Fig. 1. Planar graphs with circular chromatic numbers 3 and 4.

Proof. It is obvious that G_{2k+1} has chromatic number 3. Hence $\chi_c(G_{2k+1})\leqslant 3$. Assume to the contrary of Theorem 5.2 that $\chi_c(G_{2k+1}) < 3$. Let c be an r-circular coloring of G_{2k+1} for some $r < 3$. Consider the unit length arc $c(u)$ of the circle C. Let a, b be the two end points of $c(u)$. Then $c(c_i)\cap c(u) \neq \emptyset$ for each i, for otherwise, $c(c_i)$, $c(u)$, $c(v_i)$ would be three pairwise disjoint unit length arcs of C, contrary to the fact that C has length $r < 3$. Since $c(c_i)$ and $c(c_{i+1})$ are disjoint, we conclude that $c(c_i) \neq c(u)$ for each i. Hence $c(c_i)$ either contains a or contains b. Without loss of generality, assume that $c(c_0)$ contains a, then $c(c_1)$ contains b, for otherwise $c(c_0) \cap c(c_1) \neq \emptyset$. Similarly, we have $c(c_{2t})$ contains a and $c(c_{2t+1})$ contains b for each integer t. Then $c(c_{2k}) \cap c(c_0) \neq \emptyset$, a contradiction. □

When $k\geqslant 2$, G_{2k+1} is triangle free. For some time, the graphs G_{2k+1} were the only known triangle-free minimal planar graphs with circular chromatic number 3, and the odd wheels were the only known minimal planar graphs with circular chromatic number 4. The question that whether there are other minimal planar graphs with circular chromatic number 3 and 4 were asked in [75,88]. By using the 'n-circular superedge method' which we discussed in Section 3, one can easily construct other graphs with circular chromatic numbers 3 and 4. Indeed, it is easy to construct planar 3-circular superedges (respectively, planar 4-circular superedges) $(H; x, y)$ such that x, y lie on the outer face of H. These superedges can be used to replace any edge of a planar graph G with $\chi_c(G) = 3$ (respectively, $\chi_c(G) = 4$) to produce another planar graph G' with $\chi_c(G')=3$ (respectively $\chi_c(G')=4$). Fig. 1 below shows some examples of planar graphs constructed this way.

Next, we present the answer to Question 3. It follows from the Four Color Theorem that a necessary condition for a rational number r to be the circular chromatic number of a planar graph is that $2\leqslant r\leqslant 4$, or $r=1$. It turned out that this condition is sufficient as well. It was proved by Moser [63] that every rational number r between 2 and 3 is the circular chromatic number of a planar graph, and the method was then generalized in [92] to prove that every rational number r between 3 and 4 is also the circular chromatic number of a planar graph. We present a sketched proof of the existence of planar graphs with circular chromatic number r for those r between 2 and 3, and then briefly describe the proof for the case that r is between 3 and 4. The latter uses the same idea but is much more complicated.

Theorem 5.3. *For any rational r between* 2 *and* 3, *there is a planar graph G such that $\chi_c(G) = r$.*

Given any rational number p/q between 2 and 3, we shall construct a planar graph with circular chromatic number p/q. Assume that $(p,q) = 1$, let p', q' be the unique positive integers such that $p' < p$, $q' < q$ and $pq' - qp' = 1$. Then $p'/q' < p/q$ and that p'/q' is the largest fraction with the property that $p'/q' < p/q$ and $p' \leqslant p$. Similarly, we may find the largest fraction $p''/q'' < p'/q'$ with $p'' \leqslant p'$. Repeat this process of finding smaller and smaller fractions, we shall stop at the fraction 2/1 in a finite number of steps. Thus any rational p/q between 2 and 3 corresponds to a unique sequence of fractions

$$\frac{2}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n} = \frac{p}{q}.$$

The sequence $(p_i/q_i : i = 0, 1, \ldots, n)$ is called the *Farey sequence* of p/q [63].

Since $p_i q_{i-1} - p_{i-1} q_i = 1$, $p_{i-1} q_{i-2} - p_{i-2} q_{i-1} = 1$ and that p_{i-1}, q_{i-1} are co-prime, it follows that for $i \geqslant 2$, $\alpha_i = (p_i + p_{i-2})/p_{i-1} = (q_i + q_{i-2})/q_{i-1}$ is an integer, which is at least 2. Let $\alpha_1 = q_1$. The sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called the *alpha sequence* of p/q [63], which is obviously uniquely determined by p/q. The process of deducing the alpha sequence from the rational p/q can also be reversed. In other words, each sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \geqslant 2$ determines a rational p/q between 2 and 3. Indeed, given the alpha sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, the fractions p_i/q_i can be easily determined by solving the difference equations

$$p_i = \alpha_i p_{i-1} - p_{i-2}, \quad q_i = \alpha_i q_{i-1} - q_{i-2}, \tag{$*$}$$

with the initial condition that $(p_{-1}, q_{-1}) = (-1, 0)$ and $(p_0, q_0) = (2, 1)$.

We construct ordered graphs F_i, H_i (for $i = 1, 2, \ldots, n$) recursively as follows:

F_1 is a singleton. H_1 is a path with $2\alpha_1$ vertices, where the order of the vertices of H_1 is the order of the path. F_2 is a path with $2\alpha_1 - 2$ vertices, again with vertices ordered along the path.

For $i \geqslant 2$, the graph H_i is constructed as follows: Take α_i copies of F_{i-1} and $\alpha_i - 1$ copies of H_{i-1}. For $j = 1, 2, \ldots, \alpha_i - 1$, connect the first (resp. the last) vertex of the jth copy of F_{i-1} to the last (resp. the first) vertex of the jth copy of H_{i-1}, and connect the last (resp. the first) vertex of the jth copy of H_{i-1} to the first (resp. the last) vertex of the $(j+1)$th copy of F_{i-1}. The order of the vertices of H_i is the concatenation of the order of the copies of F_{i-1} and H_{i-1}, i.e., the vertices of the first copy of F_{i-1} in order, followed by the vertices of the first copy of H_{i-1} in order, followed by the vertices of the second copy of F_{i-1} in order, etc.

For $i \geqslant 2$, the graph F_{i+1} is constructed the same way as H_i, except that we take one less copy of F_{i-1} and H_{i-1}, i.e., we take $\alpha_i - 1$ copies of F_{i-1} and $\alpha_i - 2$ copies of H_{i-1}.

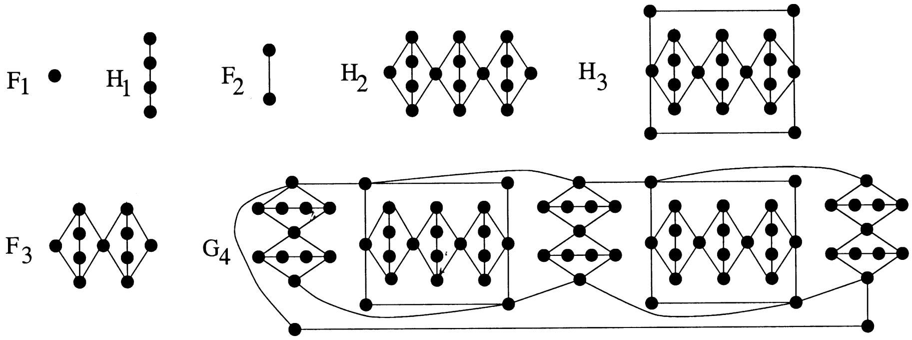

Fig. 2. The graphs F_i, H_i and G_i for the fraction 75/29.

Finally, for $i \neq 1$, let G_i be obtained from a copy of F_i and a copy of H_i by joining the first (resp. the last) vertex of F_i to the last (resp. the first) vertex of H_i. Fig. 2 below illustrate the construction of the F_i, H_i and G_i for the rational 75/29, whose alpha sequence is $(2,4,2,3)$, and whose Farey sequence is $(2/1, 5/2, 18/7, 31/12, 75/29)$.

This finishes the construction of G_i for $i = 1, 2, \ldots, n$, and the graph G_n is the one that we wanted, i.e., G_n is a planar graph with circular chromatic number $p/q = p_n/q_n$. It is easy to see that G_n is planar. To show that G_n has circular chromatic number p_n/q_n, we prove by induction that each of the graph G_i has circular chromatic number p_i/q_i.

First, it is straightforward to verify that G_i has p_i vertices (by showing that $|V(G_i)|$ and p_i satisfy the same difference equation and have the same initial values). Note that F_i and H_i are ordered graphs, where the order of the vertices of F_i (resp. H_i) gives a Hamiltonian path of F_i (resp. H_i). Thus the concatenation of the orders of F_i and H_i gives a Hamiltonian cycle of G_i. Rename the vertices of G_i so that the Hamiltonian cycle is $(x_1, x_2, \ldots, x_{p_i})$. Then it is not difficult to verify that the coloring c of G_i defined as $c(x_j) = jq_i \pmod{p_i}$ is a (p_i, q_i)-coloring of G_i. Therefore $\chi_c(G_i) \leqslant p_i/q_i$.

To show that $\chi_c(G_i)$ cannot be strictly less than p_i/q_i, we shall use the induction hypothesis that G_{i-1} has circular chromatic number p_{i-1}/q_{i-1}. Recall that H_i is constructed by using α_i copies of F_{i-1} and $\alpha_i - 1$ copies of H_{i-1}. For each $j = 1, 2, \ldots, \alpha_i - 1$, the union of the jth copy of F_{i-1} and the jth copy of H_{i-1} is a copy of G_{i-1}, and the union of the jth of copy of H_{i-1} and the $(j+1)$th copy of F_{i-1} is also a copy of G_{i-1}. Assume to the contrary that $\chi_c(G_i) < p_i/q_i$. Since $\chi_c(G_i) = k/d$ for some $k \leqslant p_i$, and p_{i-1}/q_{i-1} is the largest fraction which is less than p_i/q_i and whose numerator does not exceed p_i, we conclude that $\chi(G_i) \leqslant p_{i-1}/q_{i-1}$. As G_{i-1} is a subgraph of G_i and $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$, it follows that $\chi_c(G_i) = p_{i-1}/q_{i-1}$. Let c be a (p_{i-1}, q_{i-1})-coloring of G_i. Consider the restriction of c to H_i, which contains copies of F_{i-1} and H_{i-1}. By using the induction hypothesis that G_{i-1} has circular chromatic number p_{i-1}/q_{i-1}, and by some (non-trivial, but not that complicated) technical arguments, it can be shown that the jth copy of F_{i-1} is colored the same way as the $(j+1)$th copy of F_{i-1}. Let T_i be the graph obtained from a copy of F_{i-1} and a copy of F_i by connecting the first (resp. the last) vertex of F_{i-1} to the last (resp. the first) vertex of F_i. Then the above argument shows that c induces a (p_{i-1}, q_{i-1})-coloring of T_i. But with another induction, it can be proved that T_i has circular chromatic number greater than p_{i-1}/q_{i-1}. So we arrive at a contradiction and hence $\chi_c(G_i) = p_i/q_i$.

In the above sketched proof, the construction is the one given by Moser [63], and the proof is adopted from the simpler proof given in [91]. The proof above is generalized in [92] to prove the following result:

Theorem 5.4. *For any rational r between 3 and 4, there is a planar graph G with $\chi_c(G) = r$.*

For rationals r between 3 and 4, the construction of a planar graph with circular chromatic number r is more complicated [92]. However, the idea is similar. Given $r = p/q$ between 3 and 4, the first thing we do is also finding its Farey sequence $(3/1 = p_0/q_0 < p_1/q_1 < \cdots < p_n/q_n = p/q)$, and its alpha sequence $(\alpha_1, \ldots, \alpha_n)$. Then we recursively construct the ordered graphs F_i, H_i for $i = 1, 2, \ldots, n$. F_1 is still a singleton, H_1 is the square of a path of $3\alpha_1$ vertices, and F_2 is the square of a path of $3(\alpha_1 - 1)$ vertices. (Here the square of a path $P = (v_1, \ldots, v_t)$ is the graph with $\{v_1, \ldots, v_t\}$ as its vertices and v_iv_j is an edge if $|i-j| \leqslant 2$.) For $i \geqslant 2$ the ordered graph H_i (resp. F_{i+1}) is constructed from α_i (resp. $\alpha_i - 1$) copies of F_{i-1} and $\alpha_i - 1$ (resp. $\alpha_i - 2$) copies of H_{i-1}, by *hooking* the jth and $(j+1)$th copy of F_{i-1} to the jth copies of H_{i-1}. Finally, G_i is obtained from a copy of F_i and a copy of H_i, by hooking them together. How to hook F_{i-1} to H_{i-1} depends on a few factors, such as the parity of i, the parity of α_1 and sometimes the value of α_1. All together, six different *hooks* are used in the construction. In the case $2 < r < 3$, only one type of hook is used. The proof for the case $3 < r < 4$ is also much more complicated.

6. Relation between the circular chromatic number and other graph parameters

Besides the chromatic number of a graph, there are many other graph parameters for which the relation with the circular chromatic number have been investigated. Among these parameters are: the fractional chromatic number, the clique number, the maximum degree, the girth, the ultimate independence ratio, the connectivity, etc. We have mentioned some results and problems concerning the relation among the circular chromatic number, the girth and the maximum degree of a graph in the previous sections. In this section, we discuss the relation between the circular chromatic number and the fractional chromatic number, the connectivity, and the ultimate independence ratio.

First, we consider the fractional chromatic number. A mapping c from the collection $\mathscr{S}$ of independent sets of a graph G to the interval $[0, 1]$ is a *fractional-coloring* of G if for every vertex x of G we have $\sum_{S \in \mathscr{S},\ x \in S} c(S) = 1$. The value of a fractional-coloring c is $\sum_{S \in \mathscr{S}} c(S)$. The *fractional-chromatic number* $\chi_f(G)$ of G is the infimum of the values of fractional-colorings of G.

It is easy to see that $\chi_f(G) \leqslant \chi_c(G)$ for any graph G. Indeed, if $\chi_c(G) = k/d$ and c is a (k,d)-coloring of G, then each of the sets $S_i = \cup_{j=i}^{i+d-1} c^{-1}(j)$ is an independent set of G (where additions are modulo k). The mapping c' that assigns to each S_i the weight $1/d$ is a fractional coloring of G which has value k/d. On the other hand, it is well known that the difference between the chromatic number and the fractional chromatic number of a graph could be arbitrarily large, hence the difference between the circular chromatic number and the fractional chromatic number of a graph could be arbitrarily large.

The class of graphs G for which $\chi_c(G)$ equals $\chi_f(G)$ turns out to have some interesting properties. Such graphs are called *star-extremal graphs*. The following theorem was proved in [27]:

Theorem 6.1. *If $\chi_c(G)$ equals $\chi_f(G)$ and H is an arbitrary graph, then $\chi_c(G[H]) = \chi_c(G)\chi(H)$.*

Here $G[H]$ is the lexicographic product of G and H, whose vertex set is $V(G) \times V(H)$, and in which (g,h) is adjacent to (g',h') if either $(g,g') \in E(G)$ or $g = g'$ and $(h,h') \in E(H)$.

Theorem 6.1 raises the question that for which graphs G we have $\chi_c(G) = \chi_f(G)$. The problem was studied in [27,55], where the authors concentrated on circulant graphs.

Let p be a positive integer and let S be a subset of $\{1,2,\ldots,p-1\}$ such that $i \in S$ implies $p-i \in S$. For brevity, we write $-i$ for $p-i$. The circulant graph $G(p,S)$ has vertices $0,1,\ldots,p-1$ and $i \sim j$ if and only if $i-j \in S$, where the subtraction is carried out modulo p.

Let $\alpha(G)$ be the *independence number* of the graph G, i.e., $\alpha(G)$ is the size of a maximum independent set of G. It is known, and easy to prove, that if a graph G is vertex transitive, then $\chi_f(G) = |V(G)|/\alpha(G)$. Therefore for any circulant graph $G = (p,S)$, we have $\chi_f(G) = p/\alpha(G)$. Thus $\chi_c(G) = \chi_f(G)$ if and only if $\chi_c(G) = p/\alpha(G)$. To prove a circulant graph G satisfies $\chi_c(G) = p/\alpha(G)$, it suffices to show that G is $(p,\alpha(G))$-colorable, as we know that $\chi_c(G) \geqslant \chi_f(G) = p/\alpha(G)$.

A very useful method for proving the existence of such a coloring is the *multiplier method* [27,58,84]. Given a circulant graph $G = G(p,S)$ and an integer t, we let

$$\lambda_t(G) = \min\{|ti|_p \colon i \in S\}$$

and let

$$\lambda(G) = \max\{\lambda_t(G) \colon t = 1,2,\ldots\},$$

where the multiplications ti are carried out modulo p, and $|x|_p = \min\{|x|, p-|x|\}$. It is routine to verify that for any integer t, the mapping $c(i) \equiv ti \pmod p$ is a $(p,\lambda_t(G))$-coloring of the circulant graph G. Therefore we have the following result, which was proved in [27]:

Theorem 6.2. *Suppose G is a circulant graph. Then $\lambda(G) \leqslant \alpha(G)$. Moreover if $\lambda(G) = \alpha(G)$ then $\chi_c(G) = \chi_f(G)$.*

Note that for all circulant graphs $G = G(p, S)$, the parameter $\lambda(G)$ can be determined in polynomial time. If $\lambda(G) = \lambda_t(G) = r$, then the pre-images of the set $\{0, 1, \ldots, r-1\}$ under the mapping $c(i) = ti$ (where multiplications are carried out modular p) is an independent set of G of size r. Therefore if $\chi_c(G) = |V(G)|/\alpha(G) = |V(G)|/\lambda(G)$ for a circulant graph G, then there is a polynomial time algorithm which finds a maximum independent set and an optimal circular coloring of G simultaneously. For general circulant graphs, the complexity of determining these parameters are NP-complete, [9]. In the following, we list some circulant graphs G that were proved in [27,58] to satisfy the equality $\lambda(G) = \alpha(G)$, and hence $\chi_c(G) = \chi_f(G)$.

Theorem 6.3. *If $S = \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$ and $G = G(p, S)$, then $\lambda(G) = \alpha(G)$.*

Theorem 6.4. *If $|S| \leqslant 3$ and $G = G(p, S)$, then $\lambda(G) = \alpha(G)$.*

Theorem 6.5. *Suppose $G = G(p, S)$ is a circulant graph and $|S| = 4$.*

(1) *If $S = \{\pm 1, \pm k\}$, k is odd and $p > (k(k-3)+2)r/2$, where r is the unique number $0 \leqslant r < k$ satisfying $r \equiv p (\operatorname{mod} k)$, then $\lambda(G) = \alpha(G)$.*
(2) *If $S = \{\pm 1, \pm k\}$, k is even and $p > k(k-1)$, then $\lambda(G) = \alpha(G)$.*

We do not know whether or not for every S with $|S| = 4$, the circulant graph $G(p, S)$ satisfies the equality $\lambda(G) = \alpha(G)$.

Theorem 6.6. *Suppose that $S = \{\pm k, \pm(k+1), \ldots, \pm k'\}$. If $(5/4)k \leqslant k' \leqslant p/2$, then $\lambda(G) = \alpha(G)$.*

There are also many circulant graphs G for which $\chi_c(G) \neq \chi_f(G)$. For example, take any circulant graph H with $\chi_f(H) \neq \chi(H)$, then the complement $\bar{H}$ of H is a circulant graph, and for any integer $k \geqslant 2$, the disjoint union $\bar{G}$ of k copies of $\bar{H}$ is also a circulant graph. It was shown [27] that for such graphs $\chi_f(G) < \chi_c(G)$. It follows from the construction that the order of such circulant graphs are always composite numbers. An example of circulant graph G of prime order for which $\chi_f(G) < \chi_c(G)$ was also found in [27] through a computer search.

Star-extremal graph was also studied in [55]. It was shown in [55] for each n, the circulant graph $G = G(3n-1, \{1, 4, \ldots, 3n-2\})$ is star-extremal. The graphs G_n was then used to show that for each integer n there exists a graph G_n such that $\chi(G_n[K_n]) < n\chi(G_n)$ and $\chi(G_n[K_{n-1}]) = (n-1)\chi(G_n)$.

The relation between the circular chromatic number and the fractional chromatic number of a weighted graph seems to be a very interesting problem, and was discussed in [20]. We use the traffic control problem to motivate the definition of the circular chromatic number. In that model, each traffic flow has a unit length of time interval in which it faces the green light. However, in practical situation, different traffic flows may have different weights. Thus we may define the circular coloring and the circular chromatic number of weighted graphs as follows:

Let $G=(V,E)$ be a graph with a nonnegative weight function $w: V\to[0,\infty)$. Let C be a circle of length r in the plane. An *r-circular coloring* of (G,w) is a mapping c which assigns to each vertex of G an open arc of C such that

1. if $(x,y)\in E$ then $c(x)$ and $c(y)$ are disjoint; and
2. for all vertices $x\in V$ the length of the arc $c(x)$ is equal to $w(x)$.

The *circular-chromatic number* $\chi_c(G,w)$ of (G,w) is defined as

$$\chi_c(G,w)=\inf\{r\colon \text{ there is a } r\text{-circular coloring of } (G,w)\}.$$

Most of the basic properties of the circular chromatic number of graphs are still true for the circular chromatic number of weighted graphs. For example, it was shown in [20] that the infimum in the definition of $\chi_c(G,w)$ can be replaced by minimum; if all the weights are rationals, then the circular chromatic number is also a rational, etc.

The fractional chromatic number can also be defined for weighted graphs. A fractional coloring of a weighted graph (G,w) is an assignment c of non-negative weights to independent sets X of G so that for each vertex $x\in V(G)$, we have $\Sigma_{x\in X}c(X)\geqslant w(x)$. The *fractional chromatic number* $\chi_f(G,w)$ of (G,w) is the minimum total weight (i.e., the sum of the weights of all independent sets) of a fractional coloring of (G,w). The interval chromatic number of weighted graphs, or simply called the chromatic number $\chi(G,w)$ of weighted graphs, the maximum clique weight $\omega(G,w)$ are defined similarly (cf. [30]). It is straightforward to verify that

$$\omega(G,w)\leqslant\chi_f(G,w)\leqslant\chi_c(G,w)\leqslant\chi(G,w)$$

for any weighted graph (G,w).

A graph G is called *superperfect* if $\omega(G,w)=\chi(G,w)$ for any weight assignment w of G. Thus every superperfect graph is perfect. However, there are perfect graphs which are not superperfect. We call a graph G *circular superperfect* if $\chi_f(G,w)=\chi_c(G,w)$ for any weight assignment w of G (such graphs are called star-superperfect in [20]). It follows from the inequalities

$$\omega(G,w)\leqslant\chi_f(G,w)\leqslant\chi_c(G,w)\leqslant\chi(G,w)$$

that every superperfect graph is circular superperfect. The converse is not true. The following result was proved in [20]:

Theorem 6.7. *Odd cycles are circular superperfect, and the complement of odd cycles are also circular superperfect.*

It remains an open problem to characterize all graphs which are circular superperfect. Comparing to the situation for superperfect graphs and perfect graphs, this problem is probably not easy.

Next, we consider the relation between the circular chromatic number and the connectivity of graphs. This problem was first studied by Abbott and Zhou, and the following result was proved in [1]:

Theorem 6.8. *Let G be a k-critical graph of connectivity* 2. *Then* $\chi_c(G) \leqslant k - 1/2$.

Proof. Suppose $V(G) - \{u, v\} = V_1 \cup V_2$, where no vertex of V_1 is joined to a vertex of V_2. Since G is k-critical, each of the subgraphs induced by $V_1 \cup \{u, v\}$ and $V_2 \cup \{u, v\}$ is $(k-1)$-colorable. Moreover, the two vertices u, v must be colored the same color in one of the colorings, and colored by distinct colors in the other. Let $A_1, A_2, \ldots, A_{k-1}$ be the color classes of $V_1 \cup \{u, v\}$, and let $B_1, B_2, \ldots, B_{k-1}$ be the color classes of $V_2 \cup \{u, v\}$. Without loss of generality, we may assume that $u, v \in A_1$, $u \in B_1$, $v \in B_{k-1}$. Define a coloring $c : V \mapsto \{0, 1, \ldots, 2k-2\}$ as follows: $c(u) = 1$, $c(v) = 0$, $c(x) = 2i - 1$ if $x \in A_i - \{u, v\}$ and $c(x) = 2i$ if $x \in B_i - \{u, v\}$. Then it is straightforward to verify that c is a $(2k-1, 2)$-coloring of G. $\square$

The same authors also proved in [1] that for any $k \geqslant 4$ and for any $\varepsilon > 0$, there exist arbitrarily large k-critical 3-connected graphs G for which $\chi_c(G) < k - 1 + \varepsilon$. They asked the question whether or not there are critical graphs G of higher connectivity for which $\chi_c(G)$ are arbitrary close to $\chi(G) - 1$. This question was answered by Steffen and Zhu in affirmative. The following result was proved in [75]:

Theorem 6.9. *For any integer* $m \geqslant 4$ *and for any* $\varepsilon > 0$ *there is an m-connected,* $(m+1)$-*critical graph G for which* $\chi_c(G) \leqslant m + \varepsilon$.

Finally, we briefly mention the relation between the ultimate independence ratio and the circular chromatic number. The ratio $\alpha(G)/|V(G)|$, denoted by $i(G)$, is called the *independence ratio* of a graph G. Let G^k be the Cartesian product $G \square G \square \cdots \square G$ of k copies of G, in which a vertex $(g_1, g_2, \ldots, g_k)$ is adjacent to $(g'_1, g'_2, \ldots, g'_k)$ if and only if there is an index j such that g_j is adjacent to g'_j in G and $g_t = g'_t$ for $t \neq j$. The *ultimate independence ratio*, denoted by $I(G)$, is defined as the limit of $i(G^k)$ as k goes to infinity. The ultimate independence ratio was first studied by Hell et al. [43], where it was proved, among other things, that the limit $\lim_{k\to\infty} i(G^k)$ exists, and is between $i(G)$ and $1/\chi(G)$. It was proved in [87] that

$$I(G) \lim_{k\to\infty} \chi_f(G^k) = 1$$

for every graph G. Since $\chi_f(G) \leqslant \chi_c(G)$ and $\chi_c(G^k) = \chi_c(G)$, we conclude that $I(G) \geqslant 1/\chi_c(G)$. There are graphs G for which $I(G) = 1/\chi_c(G)$, and there are also graphs G for which $I(G) > 1/\chi_c(G)$. It seems a difficult problem to characterize all graphs G for which the equality $I(G) = 1/\chi_c(G)$ holds.

7. Circular chromatic number of special graphs

In order to understand the circular chromatic number of general graphs, it is necessary to investigate the circular chromatic number of special classes of graphs. In the previous sections, we have already discussed the circular chromatic number of many special graphs: Mycielski graphs, planar graphs, circulant graphs, etc. There are many other special graphs whose circular chromatic number have been investigated. In this section, we survey some more results concerning the circular chromatic number of Mycielski graphs, and some results concerning the circular chromatic number of Kneser graphs and distance graphs.

The chromatic number and clique number of the mycielskian $M(G)$ of G is determined by the chromatic number and the clique number of G by the simple formulae $\omega(M(G)) = \omega(G)$ and $\chi(M(G)) = \chi(G) + 1$ (provided that G has at least one edge). Recently, Larsen et al. [57] showed that $\chi_f(M(G)) = \chi_f(G) + 1/\chi_f(G)$ for any graph G, where $\chi_f(G)$ is the fractional chromatic number of a graph (cf. Section 6 for the definition). Then it is quite natural to ask how is the circular chromatic number of $M(G)$ related to the circular chromatic number of G. It turns out that there is no simple formula that relates $\chi_c(M(G))$ to $\chi_c(G)$. However, there does exist some connection. We have mentioned some relations in Section 4, and many open problems remains.

It follows from Theorem 4.3 that if $\chi(G) - \chi_c(G) \geqslant (d-1)/d$ for some integer d, then $\chi(M^2(G)) - \chi_c(M^2(G)) \geqslant (d-1)/d$. It seems that the sequence $\chi(M^{2k}(G)) - \chi_c(M^{2k}(G))$ $(k = 0, 1, \ldots)$ is non-decreasing. However, it remains unknown whether this sequence would eventually increase, and approach to 1. The following two results proved in [11] reveal some information about the sequence $\chi(M^k(G)) - \chi_c(M^k(G))$ $(k = 0, 1, \ldots)$:

Theorem 7.1. *If G is a graph of chromatic number n, then $\chi_c(M^{n-1}(G)) \leqslant \chi(M^{n-1}(G)) - 1/2$.*

The following corollary follows easily from Theorems 4.3 and 7.1:

Corollary 7.1. *If G is an n-chromatic graph and k a non-negative integer, then $\chi_c(M^{n-1+2k}(G)) \leqslant \chi(M^{n-1+2k}(G)) - 1/2$.*

We do not know if Theorem 7.1 is sharp, in the sense that for any $m < n-1$ there would be an n-chromatic graph G such that $\chi_c(M^m(G)) = \chi(M^m(G))$. An affirmative answer is equivalent to the following conjecture:

Conjecture 7.1. If $m \leqslant n-2$, then $\chi_c(M^m(K_n)) = \chi(M^m(K_n))$.

The following theorem [11] confirms the above conjecture for the case that $n = 3$ and $n = 4$.

Theorem 7.2. *If $n \geqslant 3$, then $\chi_c(M(K_n)) = n+1$; if $n \geqslant 4$, then $\chi_c(M^2(K_n)) = n+2$.*

Next, we consider the circular chromatic number of Kneser graphs. For a positive integer m, let $[m]$ denote the set of non-negative integers less than m, and let I_n^m be the family of subsets of $[m]$ of cardinality n. The Kneser graph $\mathrm{KG}(m, n)$ is the graph whose vertex set is I_n^m, in which two vertices being adjacent if and only if they are disjoint. It was conjectured by Kneser [53], proved by Lovász [62], that $\chi(\mathrm{KG}(m,n))=m-2n+2$. The circular chromatic number of Kneser graphs was studied by Johnson et al. [48]. The following results were proved in [48]:

Theorem 7.3. *For all* $n\geqslant 1$, $\chi_c(\mathrm{KG}(2n+1,n)) = 3 = \chi(\mathrm{KG}(2n+1,n))$ *and* $\chi_c(\mathrm{KG}(2n+2,n)) = 4 = \chi(\mathrm{KG}(2n+2,n))$.

Theorem 7.4. *For any integer* $m\geqslant 4$, $\chi_c(\mathrm{KG}(m,2)) = m-2 = \chi(\mathrm{KG}(m,2))$.

Then Johnson et al. proposed the following conjecture:

Conjecture 7.1. For any Kneser graph $\mathrm{KG}(m,n)$, $\chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n)) = m-2n+2$.

The proof of Theorem 7.3 in [48] introduces an interesting technique. We formulate the main idea in the proof into the following lemma, where an undirected graph is regarded as a symmetric directed graph, with each edge replaced by two opposite directed edges.

Lemma 7.1. *Suppose G is a symmetric directed graph with* $\chi(G)=3$ (*resp.* $\chi(G)=4$). *If there exists a sequence of directed cycles* $C_0,C_1,\ldots,C_t$ *such that* $|C_i|=2m+1$ *for all i*, C_i *and* C_{i+1} *differs in at most* 3 (*resp.* 2) *edges, and that* C_t *is the reverse of* C_0, *then* $\chi_c(G)=3$ (*resp.* $\chi_c(G)=4$).

Proof. Suppose to the contrary that $\chi_c(G) = k/d < 3$ (resp. $k/d < 4$). Let c be a (k,d)-coloring of G. For any integer x let $\xi(x)$ be the unique non-negative integer less than $k-1$ such that $x \equiv \xi(x) \pmod k$. For any directed edge $e=(u,v)$ of G, let $\Delta(e)=\xi(c(v)-c(u))/k$. Then for any edge e, $1/3 < d/k\leqslant\Delta(e)\leqslant(k-d)/k < 2/3$. (resp. $1/4 < d/k\leqslant\Delta(e)\leqslant(k-d)/k < 3/4$.) Moreover, if e' is the reverse of e, i.e., $e'=(v,u)$, then $\Delta(e)+\Delta(e')=1$. For any directed cycle C, let $w(C)=\sum_{e\in C}\Delta(e)$. It is easy to see that $w(C)$ is a positive integer. If C' is the reverse of the directed cycle C, then $w(C)+w(C')=|C|$.

If two cycles A,B differs in s edges, where $s\leqslant 3$ (resp. $s\leqslant 2$), then

$$w(A)-w(B)=\sum_{i=1}^{s}(\Delta(e_i)-\Delta(e_i')),$$

where $\{e_1,\ldots,e_s\}=A-B$ and $\{e_1',\ldots,e_s'\}=B-A$. Since $|\Delta(e)-\Delta(e')| < 1/3$ (resp. $< 1/2$) for any two edges e and e', we conclude that $|w(A)-w(B)| < 1$. Because both $w(A)$ and $w(B)$ are integers, we have $w(A)=w(B)$.

Now consider the sequence of cycles $C_0,C_1,\ldots,C_t$. Since any two consecutive cycles differs in at most 3 (resp. 2) edges, we conclude that $w(C_i)=w(C_{i+1})$, and hence $w(C_0)=w(C_t)$. As $w(C_0)+w(C_t)=|C_0|=2m+1$, it follows that $w(C_0)=m+1/2$, contrary to the fact that $w(C_0)$ is an integer. □

To prove Theorem 7.3, it suffices to find sequences of odd cycles that satisfy the stated conditions. For the first part of Theorem 7.3 (i.e., for $\mathrm{KG}(2n+1,n)$), the sequence of cycles are found as follows:

Let $\mathbf{p} = (p_0, p_1, \ldots, p_{2n})$ be any ordering of the elements of $[2n+1]$. Let $C(\mathbf{p}) = (v_0, v_1, \ldots, v_{2n})$ be the cycle defined as follows:

$$v_0 = \{p_0, p_1, \ldots, p_{n-1}\},$$

$$v_1 = \{p_n, p_{n+1}, \ldots, p_{2n-1}\},$$

$$v_2 = \{p_{2n}, p_0, \ldots, p_{n-2}\}, \ldots,$$

$$v_{2n} = \{p_{n+1}, p_{n+2}, \ldots, p_2 n\}.$$

It is straightforward to verify that if $\mathbf{q}$ is obtained from $\mathbf{p}$ by a transposition, that is for some s, $q_s = p_{s+1}, q_{s+1} = p_s$ and $q_i = p_i$ for other i, then $C(\mathbf{q})$ and $C(\mathbf{p})$ differs in exactly three edges. Note that if $\mathbf{p}'$ is the reverse of $\mathbf{p}$, then $C(\mathbf{p}')$ is the reverse of $C(\mathbf{p})$. As any ordering $\mathbf{p}$ can be changed to any other ordering $\mathbf{q}$ by a sequence of transpositions, the required sequence of cycles exists.

Finally, we survey some recent result on the circular chromatic number of distance graphs. Let D be a set of positive integers. The graph $G(Z,D)$ has vertex set Z, and two vertices i,j are adjacent if $|i-j| \in D$. Distance graphs may be viewed as the limit of circulant graphs, where Z is viewed to form an infinite 'circle'. Distance graphs have been studied in many context.

Eggleton et al. [22] studied it as a variation of the well-known plane coloring problem: what is the least number of colors needed to color the Euclidean plane so that points of unit distance are colored with distinct colors? Cantor et al. [8,39,77] studied the supremum density of a D-set, which is a set of integers such that the difference of any two integers is not in D. The supremum density of a D-set turned out to be the fractional chromatic number of the corresponding distance graphs $G(Z,D)$ [13]. Griggs et al. [32,59,60,74], when studying the T-coloring problem or channel assignment problem, considered the parameter $R(T) = \lim_{n\to\infty} sp_T(K_n)/n$, which also turned out to be the fractional chromatic number of the distance graphs $G(Z,D)$ where $D = T - \{0\}$ [13].

The chromatic number of distance graphs have been extensively investigated. The circular chromatic number of distance graphs was studied in [12,13,50,99]. A slight modification of the 'multiplier method' used for the circulant graphs (cf. Section 6) is a very useful tool for determining the circular chromatic number of distance graphs. This is not a surprise, as distance graphs are just infinite circulant graphs. The following lemma was proved in [12]:

Lemma 7.2. *Suppose D is a set of positive integers, and that p and r are positive integers. Let*

$$d_D(p,r) = \min\{|ri(\text{mod } p)|_p \colon i \in D\}$$

and let

$$f_D = \inf\{p/d_D(p,r)\colon d_D(p,r) \geqslant 1\}.$$

Then $\chi_c(G(Z,D)) \leqslant \chi_f(G(Z,D)) = f_D$.

Very recently, the author finds that the value f_D is studied in number theory as 'the Diophantine approximation' problem [79–81]; and also studied in geometry as 'view-obstruction' problems [16]. It is also related to the investigation of flows in graphs and matroids [4]. These studies use a different notation.

For a real number x, denote by $||x||$ the distance from x to the nearest integer. Suppose $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$ is a k-tuple of nonzero reals. Let

$$\kappa(\boldsymbol{x}) = \sup_{t \in R} \min_{i=1}^{k} ||tx_i||.$$

The function $\kappa(\boldsymbol{x})$, which has been studied in number theory [2,14,80–83], in geometry [16–19], in graph theory and matroids [4,20,93,100], is actually equal to $1/f_D$, where D is the set of coordinates of the vector $\boldsymbol{x}$. Indeed, $|ti|_p = p||ti/p||$, and it follows that, if D is the set of coordinates of $\boldsymbol{x}$, then

$$1/f_D = \kappa(\boldsymbol{x}).$$

The function $\lambda(G)$ defined in Section 6 is also very close to the function $\kappa(\boldsymbol{x})$. An alternate definition of λ is as follows:

$$\lambda(G) = \sup_{t \in Z} \min_{i \in S} p||ti/p|| \leqslant p\kappa(\boldsymbol{x}),$$

where $\boldsymbol{x}$ is a vector whose coordinates form the set S.

Although the function $\kappa(\boldsymbol{x})$ has attracted attention from many fields of mathematics, not much is known about this function. There is a conjecture, proposed by Wills [81] more than 30 years ago, asserts that $\kappa(\boldsymbol{x}) \geqslant 1/(k+1)$ for any k-tuple $\boldsymbol{x}$ of nonzero numbers. This conjecture has been verified for $k \leqslant 4$ and remains open for $k \geqslant 5$. From this point of view, the previous knowledge (known by this author) of the function $\kappa(\boldsymbol{x})$ is not very helpful in the study of circular chromatic number. On the converse, the study of the circular chromatic number and chromatic number of distance graphs and circulant graphs stimulated the study of the function $\kappa(\boldsymbol{x})$, and reveals some new results concerning this function, [12,27,58,100]. Theorem 7.5 below, which was proved in [12], determines the fractional chromatic number and circular chromatic number of the distance graphs $G(Z,D)$ with $|D| = 2$. As a consequence, it also determines the value of $\kappa(\boldsymbol{x})$ for $\boldsymbol{x} = (a,b)$.

Theorem 7.5. *If* $D = \{a,b\}$, *where* $a,b \in Z^+$ *and* $\gcd(a,b) = 1$, *then*

$$\chi_c(G(Z,D)) = \chi_f(G(Z,D)) = f_D = (a+b)/\lfloor (a+b)/2 \rfloor.$$

Corollary 7.2. *If* $\boldsymbol{x} = (a,b)$, *where* $a,b \in Z^+$ *and* $\gcd(a,b) = 1$, *then* $\kappa(\boldsymbol{x}) = \lfloor (a+b)/2 \rfloor/(a+b)$.

Corollary 7.2 is also proved in [4].

For $|D|=3$, the chromatic number of the corresponding distance graphs is completely determined [93], where the argument is based on an estimation of the function $\kappa(\boldsymbol{x})$ for $\boldsymbol{x}=(a,b,c)$. However, for $|D|=3$, the circular chromatic numbers are only determined for some special cases:

Theorem 7.6. *If* $D=\{a,a+1,(2a+1)k+r\}$ *and* $r=\pm(a-1),\pm(a-2),\ldots,0$ *then* $\chi_c(G(Z,D))\leqslant((2a+1)k+r+a)/(ak)$. (*We guess the equality hold.*) If $r=a,a+1$, *then* $\chi_c(G(Z,D))=f_D=(2a+1)/a$.

Theorem 7.7. *If* $D=\{2,3,b\}$ *and* $b+2=5k+r$ *for some* $k\geqslant 1$ *and* $0\leqslant r\leqslant 4$, *then*

$$\chi_c(G(Z,D))=\chi_f(G(Z,D))=f_D=\begin{cases}(b+2)/2k & \text{if } r=1,2,\\ (b+3)/(2k+1), & \text{if } r=3,\\ 5/2 & \text{if } r=4,0.\end{cases}$$

Theorem 7.8. *Suppose* $D=\{a,b,a+b\}$, *where* $\gcd(a,b)=1$. *If* $a\equiv b \pmod 3$, *then* $\chi_c(G(Z,D))=\chi(G(Z,D))=3$. If $a\not\equiv b \pmod 3$, then $\chi(G(Z,D))=4$, *and hence* $\chi_c(G(Z,D))>3$.

We note that all these cases are based on the determination of the corresponding values of $\kappa(\boldsymbol{x})$.

For larger distance set D, only some special cases have been studied. Among these special cases is the class of distance graphs $G(Z,D)$ with D of the form $D=\{1,2,\ldots,m\}-\{k,2k,3k,\ldots,sk\}$. We shall denote by $D_{m,k,s}$ the set $\{1,2,\ldots,m\}-\{k,2k,3k,\ldots,sk\}$. Recently, the circular chromatic number of distance graphs $G(Z,D_{m,k,s})$ has been successfully determined for all values of m,k,s [99]. The determination of the circular chromatic number of these distance graphs demonstrates that the study of the circular chromatic of a graph is very natural, and that, in some cases, it may help in determining the chromatic number of the graphs under consideration. For explaining this, we briefly review the history of the problem of determining the chromatic number of the graphs $G(Z,D_{m,k,s})$. The chromatic number of graphs $G(Z,D_{m,k,s})$ was first studied by Eggleton et al. [22]. They only considered the case that $s=1$. The exact values of $\chi(G(Z,D_{m,k,s}))$ was determined for $k=s=1$. For $s=1$ and $k=2$ some partial solution was found [22]. For $s=1$ and $k\geqslant 3$, only rough upper and lower bounds for the chromatic number were found. The problem was also studied by Kemnitz and Kolberg [51], and the case $k=s=1$ was solved by another method. Then, by introducing a new coloring method-*the pre-coloring method*, Chang et al. [13] determined the value of $\chi(G(Z,D_{m,k,1}))$ for all k. Immediately after that, the circular chromatic number of these graphs was also determined [12]. For $s\geqslant 2$, the chromatic number of $G(Z, D_{m,k,s})$ was first studied in [61], where the chromatic number of $G(Z, D_{m,k,2})$ was completely solved. In [21], a very tight upper and lower bound (with gap 1) for $\chi(G(Z, D_{m,k,s}))$ was given. Then by studying the circular chromatic number of the graphs $G(Z, D_{m,k,s})$, Huang and Chang determined the chromatic number of these graphs. Huang and Chang's work can be easily translated into a discussion about the chromatic number. However, by studying the circular chromatic number instead of the chromatic number, it is easier to find the coloring patterns. The general coloring rules is nicer for the circular coloring than that for the ordinary coloring. This is due to the fact that the graphs $G(Z, D_{m,k,s})$ are 'very symmetric'. It is the opinion of this author, that in [21], we failed to find a general coloring rule for the graphs $G(Z, D_{m,k,s})$, because we were too concentrated on the ordinary coloring, instead of the circular coloring.

The following theorem is obtained in [99]. It determines the circular chromatic number of all the graphs $G(Z, D_{m,k,s})$, and hence the chromatic number of these graphs as well.

Theorem 7.9. *Suppose $D = D_{m,k,s}$, where $1 \leqslant (s+1)k \leqslant m$. Let $m' = m + sk + 1$ and let $d = \gcd(m', k)$.* Then

$$\chi_c(G(Z, D)) = \begin{cases} m'/(s+1) & \text{if } d = 1 \text{ or } d(s+1) \mid m', \\ (m'+1)/(s+1) & \text{otherwise.} \end{cases}$$

The fractional chromatic number of the graphs $G(Z, D_{m,k,s})$ was determined in [61]:

Theorem 7.10. *Suppose $D = D_{m,k,s}$, where $1 \leqslant (s+1)k \leqslant m$. Let $m' = m + sk + 1$ and let $d = \gcd(m', k)$. Then*

$$\chi_f(G(Z, D)) = m'/(s+1).$$

We remark that Theorems 7.9 and 7.10 also show that not every distance graph is star-extremal.

The distance set $D_{m,k,s}$ is the set obtained from the initial segment $\{1, 2, \ldots, m\}$ by deleting some numbers. In particular, the family of sets $D_{m,k,1}$ can be defined as the family of sets 'obtained from an arbitrary initial segment of integers by deleting an arbitrary number'. The next theorem concerns those sets which are obtained from an arbitrary initial segment of integers by adding an arbitrary number.

Theorem 7.11 (Chang et al. [12]). *If $D = \{1, 2, \ldots, m, n\}$, where $1 \leqslant m < n$, then*

$$\chi_c(G(Z, D)) = \chi_f(G(Z, D)) = f_D = \begin{cases} m+1 & \text{if } n \not\equiv 0 \pmod{m+1}, \\ m + 1 + 1/k & \text{if } n = k(m+1). \end{cases}$$

If we replace the 'initial segment of integer' by 'an arbitrary interval of integers', the circular chromatic number (as well as the chromatic number) of the corresponding distance graphs remains unknown. To be precise, for D of the form $D = \{q, q+1, q+2, \ldots, p\} \cup \{n\}$, $\chi_c(G(Z,D))$ is unknown in general. The following theorem settles the case that D is just an interval of integers.

Theorem 7.12 (Chang [12]). *If $D = \{q, q+1, \ldots, p\}$ is a set of consecutive integers, then $\chi_c(G(Z,D)) = \chi_f(G(Z,D)) = f_D = 1 + p/q$.*

8. Open problems

We shall list some open problems in this section, some of these have already been mentioned in the corresponding sections. A few of these problems could be very difficult. However, most of them seems doable.

Question 8.1. *Characterize those planar graphs G with $\chi(G) = 4$ and those planar graphs with $\chi_c(G) = 3$.*

Question 8.2. *Prove that $\chi_c(G) < 5$ for every planar graph G, without using the Four Color Theorem.*

It was proved by Hilton et al. [45] that every planar graph G has fractional chromatic number strictly less than 5, without using the Four Color Theorem. However, as $\chi_c(G) \geqslant \chi_f(G)$ for any graph G, it would be a stronger result if it were proved that every planar graph has circular chromatic number strictly less than 5.

Question 8.3. *Prove that if G is a 2-edge connected cubic planar graph, and $H = L(G)$ is the line graph of G, then $\chi_c(H) < 4$, without using the Four Color Theorem.*

Indeed, we do not know any 2-edge connected cubic graph whose line graph has circular chromatic number 4. One might expect that the line graph $L(P)$ of the Petersen graph P has circular chromatic number 4 (why not?). However, this is not the case. The graph $L(P)$ has circular chromatic number 11/3.

Question 8.4. *Are there any 2-edge connected cubic graph G whose line graph has circular chromatic number 4?*

We note that there are graphs G with maximum degree 3 such that $\chi_c(L(G)) = 4$. For example, if G is obtained from K_4 by subdividing an edge, then $\chi_c(L(G)) = 4$.

Question 8.5. *For any integer $n \geqslant 1$, what are the possible values of the circular chromatic numbers of graphs embeddable on the surface of (orientable) genus n?*

In particular, we have the following question:

Question 8.6. *Does there exists an* $\varepsilon > 0$ *and an integer* n *such that for every rational* $4 \leqslant r \leqslant 4+\varepsilon$, *there exists a graph* G *embeddable on the surface of* (*orientable*) *genus* n, *and* $\chi_c(G) = r$?

It seems that the answer to this question is more likely to be negative.

An alternate way of asking the same question as Question 8.5 is this:

Given a rational number r, *what is the minimum* n *such that there exists a graph* G *embeddable on the surface of* (*orientable*) *genus* n *and that* $\chi_c(G) = r$?

If $r > 4$, it is possible that the number n is somehow related to the length of the alpha sequence of r.

Colorings of graphs and flows of graphs are dual concepts. One of the alternate definitions of the circular chromatic number discussed in Section 2 relates it to the flows of the cocyclic matroid of G. In case the matroid is a graphic matroid, then a flow of the matroid is a flow of the graph. To be precise, we define the circular flow number of a graph G as follows:

Suppose k and d are integers such that $k \geqslant 2d$. A (k,d)-*flow* [29,85] of an oriented (2-edge connected) graph G is an assignment f of integers to the edges of G such that

- for any vertex v of G, we have
$$\sum_{e \in N^+(v)} f(e) = \sum_{e \in N^-(v)} f(e).$$
Here $N^+(v)$ denotes the set of edges with v as their head and $N^-(v)$ denotes the set of edges with v as their tail.
- for every edge e of G we have
$$d \leqslant |f(e)| \leqslant k - d.$$

The *circular flow number* $F_c(G)$ of a 2-edge connected graph G is the infimum of the ratios k/d such that an (arbitrary) orientation of G has a (k,d)-flow.

Just like the circular chromatic number of a graph, the circular flow number of a graph is a refinement of the 'flow number' of a graph, namely, a graph G has a no-where-zero n-flow if and only if $F_c(G) \leqslant n$. Also similar to the circular chromatic number, the circular flow number of a finite graph is always a rational.

It follows from Seymore's 6-flow Theorem that for any graph G we have $F_c(G) \leqslant 6$. If Tutte's 5-flow conjecture is true, then for any graph G we have $F_c(G) \leqslant 5$.

A natural question for the circular flow numbers of graphs is the following:

Question 8.7. *What are the possible values of the circular flow numbers of graphs*? *Is it true that for any rational number* $2 \leqslant r \leqslant 5$, *there is a graph* G *with circular flow number* r?

Since for a planar graph G, the circular chromatic number is equal to the circular flow number of its dual, it follows from Theorems 5.3 and 5.4 that any rational number r between 2 and 4 is indeed the circular flow number of a planar graph. It remains open whether or not every rational between 4 and 5 is the circular flow number of a graph.

It is known [28] that the Petersen graph has circular flow number 5. Recently, Steffen [76] proved that there are graphs whose circular flow numbers are greater than 4 but arbitrary close to 4. It is unknown whether or not there are graphs whose circular flow numbers are less than 5 but arbitrary close to 5.

Question 8.8. *For which rational number r, there is a graph G whose line graph $L(G)$ has circular chromatic number r? In particular, is it true that for any rational number $r > 3$, there is a graph G with $\chi_c(L(G)) = r$?*

The problem concerning the circular chromatic number of line graphs has hardly been touched. If a graph G is of type 1, then of course the circular chromatic number of the line graph $L(G)$ is equal to the maximum degree of G. Thus the only interesting case is that G is a connected graph of type 2. We note that if a connected type 2 graph G has maximum degree 2, then it is an odd cycle. Hence the line graph is also an odd cycle, which implies that $\chi_c(L(G)) = 2 + 1/k$ for some integer k. Therefore we have the following result:

Theorem 8.1. *Suppose $2 < r < 3$ is a rational number. There is a graph G with $\chi_c(L(G)) = r$ if and only if $r = 2 + 1/k$ for some integer k.*

The circular chromatic numbers of the line graphs of complete graphs K_n is easy to determine [26]. If n is even, then K_n is of Type 1, and hence $\chi_c(L(K_n)) = n - 1$. If n is odd then the line graph of K_n has independence number $(n - 1)/2$, and hence has circular chromatic number $n = 2|V(L(K_n))|/(n - 1)$, i.e., $\chi_c(L(K_n)) = n$.

A graph H is called a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting some edges. We say G is *H-minor free* if H is not a minor of G. The well-known Hadwiger conjecture asserts that any graph of chromatic number at least n contains K_n as a minor. In other words, it was conjectured that the circular chromatic number of a K_n-minor free graph G is at most $n - 1$.

Question 8.9. *Is it true that for any $n \geqslant 5$, for any rational $2 \leqslant r \leqslant n - 1$ there exists a K_n-minor free graph whose circular chromatic number is equal to r?*

The results on planar graphs shows that the answer is ‘yes’ if $n = 5$, because planar graphs are K_5-minor free. However, a recent result of Hell and Zhu [44] shows that when $n = 4$, the answer is ‘no’. For $n \geqslant 6$, it was proved in [98] that for any rational $2 \leqslant r \leqslant n-2$, there exists a K_n-minor free graph G with $\chi_c(G) = r$. The problem remains open for $n \geqslant 6$ and for $n - 2 < r < n - 1$.

The class of K_4-minor free graph is the class of series–parallel graphs. The following result was proved in [44]:

Theorem 8.2. *If a series–parallel graph G has girth as least* $2\lfloor(3k-1)/2\rfloor$, *then* $\chi_c(G)\leqslant 4k/(2k-1)$.

This result raises several questions. Since a series–parallel graph containing a triangle has circular chromatic number 3, and without a triangle has girth at least 4 and hence has circular chromatic number at most 8/3, it follows that any rational $8/3<r<3$ is not the circular chromatic number of any series–parallel graphs. In other words, the interval $(8/3,3)$ is a 'gap' among the circular chromatic numbers of series–parallel graphs.

Question 8.10. *Are there other gaps among the circular chromatic numbers of series–parallel graphs? What rationals are the circular chromatic numbers of series–parallel graphs?*

Theorem 8.2 exhibits a relationship between the girth and the circular chromatic number of series–parallel graphs. Recently, Chien and Zhu [15] proved that the girth requirement in Theorem 8.2 is sharp in the following sense:

Theorem 8.3. *For any positive integer* $k\geqslant 2$, *there is a series–parallel graph G which has girth* $2\lfloor(3k-1)/2\rfloor-1$ *and* $\chi_c(G)>4k/(2k-1)$.

We may consider other classes of graphs instead of series–parallel graphs, and ask whether or not such a relation exists. Klostermeyer and Zhang [56] recently proved the following result:

Theorem 8.4. *For any* $\varepsilon>0$, *there is an integer* $f(\varepsilon)$ *such that any planar graph of odd girth at least* $f(\varepsilon)$ *has circular chromatic number at most* $2+\varepsilon$.

If the 'odd girth' requirement is replaced by a 'girth' requirement, then the same conclusion holds for graphs embeddable on higher surfaces, as well as for H-minor free graphs for any fixed H (of course, with different functions f), [41].

The following result is proved in [69].

Theorem 8.5. *Suppose k is a positive integer. For any* $\varepsilon>0$, *there is an integer* $f(\varepsilon)$ *such that for any graph G of girth at least* $f(\varepsilon)$ *and any graph* G' *of treewidth at most k, if* G' *is homomorphic to G then* G' *has circular chromatic number at most* $2+\varepsilon$.

It is not known if the bounded treewidth condition can be replaced by an H-minor free condition; or the condition of the existence of homomorphism from G' to a graph G of large girth can be replaced by the condition that G' has large odd girth. To be precise, we have the following question:

Question 8.11. *Suppose H is a fixed finite graph, and $\varepsilon > 0$. Does there exist an integer $f(H,\varepsilon)$ such that the following is true: if G is a graph of girth at least $f(H,\varepsilon)$ and G' is H-minor free graph admitting a homomorphism to G then $\chi_c(G') \leqslant 2+\varepsilon$?*

Question 8.12. *Is it true that for any $\varepsilon > 0$ and for any integer k, there is an integer $f(\varepsilon,k)$ such that for any graph G of treewidth at most k and of odd girth at least $f(\varepsilon,k)$ has circular chromatic number at most $2+\varepsilon$?*

Question 8.13. *Is it true that $\chi_c(G\times H) = \min\{\chi_c(G),\ \chi_c(H)\}$?*

Here the product is the Categorical product, in which two vertices (g,h) and (g',h') are adjacent if and only if $g \sim g'$ in G and $h \sim h'$ in H. As a generalization of Hedetniemi's conjecture [40,47], it was conjectured in [88] that $\chi_c(G\times H) = \min\{\chi_c(G),\chi_c(H)\}$ for all graphs G and H. It is easy to see that for any graphs G,H, $\chi_c(G\times H) \leqslant \min\{\chi_c(G),\chi_c(H)\}$. Thus the above conjecture is equivalent to the following conjecture:

Conjecture 8.1. For any number r, if G and H are not r-circular colorable, then $G\times H$ is also not r-circular colorable.

When r is an arbitrary integer, this becomes Hedetniemi's conjecture. Conjecture 8.1 is proved to be true for $r = 2+1/k$, where k is any positive integer [24,36,88], and for $r=1$ or 2. It remains open for all other numbers. The problem seems to be difficult, and a settlement of this conjecture for any other number r would be very welcome.

Besides the categorical product, there are many other types of graph products, including the strong product, the lexicographic product, the graph bundles, the Cartesian product, etc. Among these products, we know the formula for expressing the circular chromatic number of the Cartesian product $G\square H$ in terms of $\chi_c(G)$ and $\chi_c(H)$, namely $\chi_c(G\square H) = \max\{\chi_c(G),\chi_c(H)\}$. It would be interesting to find such expressions, or lower and upper bounds for the other products. The circular chromatic number of lexicographic product of graphs was discussed in [88]. We remark that there is a rich list of literatures dealing with the chromatic number of graph products. We refer the readers to [54] for such references.

Question 8.14. *What is the least integer $g(n)$ such that any n-critical graph G with girth at least $g(n)$ has $\chi_c(G) < \chi(G)$?*

We know that $g(3)=3$, $g(4)=4$. But no other values are known. We do not know whether or not $g(n) > 3$ for some $n \geqslant 5$.

Another problem concerning critical graphs G with $\chi_c(G) = \chi(G)$ was implicitly asked by Zhou in [86]:

Question 8.15. *For an integer $n \geqslant 4$, does there exist an integer m such that there are arbitrarily large n-critical graph G which has maximum degree at most m and for which $\chi_c(G) = \chi(G)$? If there exists such an integer, let $\Delta(n)$ be the least one of such integers. What is $\Delta(n)$ for $n \geqslant 4$?*

First we note that for $n=3$, the only 3-critical graph G with $\chi_c(G)=\chi(G)$ is K_3. For $n\geqslant 4$, there do exist arbitrarily large n-critical graphs G with $\chi_c(G)=\chi(G)$. We simply take a large $(n-k)$-critical graph H and a k-critical graph H' and let $G=H+H'$ be the graph by joining every vertex of H to every vertex of H' (where $1\leqslant k\leqslant n-1$). Then $\chi_c(G)=\chi(G)$ (by Corollary 3.1), and that G is critical. However such constructed graph has a vertex of large degree. It was shown by Zhou [86] that $\Delta(4)=4$. For $n\geqslant 5$, we do not know if $\Delta(n)$ exists.

Question 8.16. *Is it true that every graph G has a vertex v such that $\chi_c(G-v)\geqslant \chi_c(G)-1$?*

The question that how much could be the decrease of the circular chromatic number when one vertex is deleted from the graph was discussed in [88]. It was shown [88] that the decrease must be less than 2, and could be arbitrary close to 2. It was conjectured in [88] that any graph G contains a vertex whose deletion decrease the circular chromatic number by at most 1, and the conjecture remains open.

Question 8.17. *Let r,r' be any rational numbers such that $2<r<r'$ and $\lceil r\rceil \geqslant \lceil r'\rceil -1$. Does there exist a graph G and a vertex v of G such that $\chi_c(G)=r'$ and $\chi_c(G-v)=r$?*

Question 8.18. *For which graph G, $\chi_c(G-v)=\chi_c(G)-1$ for each vertex v of G?*

A construction method was given in [90] that produces infinitely many such graphs. However there is no characterization of such graphs.

We know that any graph of chromatic number $n\geqslant 2$ contains a subgraph of chromatic number $n-1$. No such results are known for circular chromatic number.

Question 8.19. *Are there two rational numbers $2<r<r'$ such that any graph of circular r' contains a subgraph of circular chromatic number r?*

One particular instance of this question is this: Given a rational $r'=p/q>2$. Suppose $0<p'<p$ and $0<q'<q$ are integers such that $q'p-p'q=1$. Is it true that any graph of circular chromatic number p/q contains a subgraph of circular chromatic number p'/q'?

A positive answer to this question would imply that any graph of circular chromatic greater than n contains a subgraph of circular chromatic number n.

Question 8.20. *For which graph G, we have $\chi_c(H)=\chi(H)$ for any induced subgraph H of G?*

An odd hole of a graph G is an induced odd cycle of G, and an odd antihole of G is an induced subgraph of G which is the complement of an odd cycle. If H is an odd hole or an odd antihole, then $\chi(H) > \chi_c(H)$. Therefore a graph satisfying the property that $\chi_c(H) = \chi(H)$ for any induced subgraph of G does not contain odd holes and odd antiholes. On the other hand, perfect graphs G are easily seen to have the property that $\chi(H) = \chi_c(H) = \omega(H)$ for every induced subgraph H of G. Thus the following conjecture is implied by the Strong Perfect Graph Conjecture:

Conjecture 8.2. A graph G has the property that $\chi_c(H) = \chi(H)$ for any induced subgraph H if and only if G contains no odd hole and no odd antihole.

From this point of view, it is natural that we expect a result correspond to the Perfect Graph Theorem. However, the question is still open:

Question 8.21. *Is it true that $\chi_c(H) = \chi(H)$ for every induced subgraph of G implies that $\chi_c(H) = \chi(H)$ for every induced subgraph of the complement $\bar{G}$ of G?*

Question 8.22. *For which graph G, we have $\chi_c(G) = \chi_f(G)$? And for which graph G, we have $\chi_c(H) = \chi_f(H)$ for any induced subgraph of G? Especially for which circulant graphs, this equality hold? How about distance graphs?*

Graphs G satisfying the property $\chi_c(G) = \chi_f(G)$ are called star-extremal graphs [27]. It was shown in [27] that if a non-trivial graph G satisfy the equality $\chi_c(G) = \chi_f(G)$, then for any graph H, $\chi_c(G[H]) = \chi_c(G)\chi(H)$ for any other graph H. Many circulant graphs are proved to satisfy the equality $\chi_c(G) = \chi_f(G)$ (cf. Section 6). Also many circulant graphs do not satisfy this equality.

Question 8.23. *What is the complexity of determining whether or not $\chi_c(G) = \chi(G)$, if the chromatic number $\chi(G)$ is known?*

If the chromatic number of G is unknown then it is NP-hard to determine whether or not $\chi_c(G) = \chi(G)$, [34]. It is likely that the problem remains NP-hard even if the chromatic number is known. Probably the problem is more interesting when restricted to some special classes of graphs, such as planar graphs, line graphs, etc. Maybe the questions should be discussed in a more general setting. Instead of graph colorings, we may consider graph homomorphisms [95], which are just edge-preserving vertex mapping from one graph to another. It was proved by Hell and Nešetřil [42] that for any non-bipartite graph H, it is *NP*-complete to decide whether or not any given graph G is homomorphic to H. However, the following question remains open:

Let H_1, H_2, H_3 be arbitrary graphs such that H_1 is homomorphic to H_2 and H_2 is homomorphic to H_3. However, H_3 is not homomorphic to H_2 and H_2 is not homomorphic to H_1. Consider the following decision problem:

The instance of the problem is a graph G which is homomorphic to H_3 and not homomorphic to H_1, and the question is whether or not G is homomorphic to H_2?

Suppose H_2 is non-bipartite, is the decision problem above NP-complete?

A positive answer to this question would imply that it is NP-hard to determine whether or not $\chi_c(G) = \chi(G)$, even if the chromatic number $\chi(G)$ is known.

Question 8.24. *Let $M(G)$ be the Mycielskian of G, and let $M^k(G) = M(M^{k-1}(G))$. Is it true that $\chi(G) - \chi_c(G) \leqslant \chi(M^2(G)) - \chi_c(M^2(G))$?*

It was shown in [11] that if $\chi(G) - \chi_c(G) \geqslant 1/d$ for $d = 2,3$, then $\chi(M^2(G)) - \chi_c(M^2(G)) \geqslant 1/d$ [11]. After the preliminary version of this manuscript, the result was generalized in [49] to every integer d, i.e., for any integer $d \geqslant 1$, if $\chi(G) - \chi_c(G) \geqslant 1/d$, then $\chi(M^2(G)) - \chi_c(M^2(G)) \geqslant 1/d$. However, Question 8.24 is still not completely answered.

Question 8.25. *What determines whether $\chi_c(M(G)) = \chi(M(G))$? In particular, for coprime integers k, d with $2d < k$, let G_k^d be the graph with vertex set $\{0, 1, \ldots, k-1\}$ in which ij is an edge if and only if $d \leqslant |i-j| \leqslant k-d$. Is it true that $\chi_c(M(G_k^d)) = \chi(M(G_k^d))$?*

We have many examples G for which $\chi_c(M(G)) = \chi(M(G))$, and also many examples G for which $\chi_c(M(G)) < \chi(M(G))$. However, it seems difficult to characterize those graphs G for which $\chi_c(M(G)) = \chi(M(G))$.

One class of graphs that are of special interest are the graphs G_k^d, defined in Section 2. It is easy to see (also [6]) that a graph G is (k,d)-colorable if and only if there exists a homomorphism from G to G_k^d. Therefore, in the study of circular chromatic numbers, graphs G_k^d play the role of complete graphs, as in the study of chromatic numbers. As $\chi_c(M(K_n)) = \chi(M(K_n))$ for any integer n, it is attempting to conjecture that $\chi_c(M(G_k^d)) = \chi(M(G_k^d))$. This conjecture has been proved to be true in [49], after the circulation of the first draft of this manuscript. Namely, the second half of this question has been solved, it was proved in [49] that $\chi_c(M(G_k^d)) = \chi(M(G_k^d))$ for all $k \geqslant 2d \geqslant 1$.

Concerning the circular chromatic number of distance graphs, the case $|D| = 3$ is still not completely settled. Since the chromatic number of such distance graph has just been determined [93,94], it is natural to look forward to a solution for the circular chromatic number of such graphs.

Question 8.26. *Suppose $D = \{a,b,c\}$ with $\gcd(a,b,c) = 1$. What is the circular chromatic number of $G(Z,D)$?*

We note that the proof in [93] is very long and complicated. Also the fractional chromatic number of these distance graphs remains unknown, in spite of the fact that this problem has been approached by people from various different point of views (cf. Section 7).

Question 8.27. *Is it true that for any Kneser graph $\mathrm{KG}(m,n), \chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n)) = m - 2n + 2$?*

A positive answer to this question was conjectured in [48] and some special cases are confirmed (cf. Section 7).

Question 8.28. *Is it possible to construct all the graphs G with $\chi_c(G)\geqslant k/d$ from copies of G_k^d by following some simple rules* (*such as those in Hajos theorem*)?

This question was discussed in [96] (see also Section 3).

Remark 1. When Vince [78] introduced the concept of circular chromatic number, the name *star-chromatic number* and the notation $\chi^*(G)$ were used instead of circular chromatic number and $\chi_c(G)$. That have been the name and notation used in the literatures. However, the sign χ^* is very much abused, and still many papers use χ^* to denote the fractional chromatic number, or other graph parameters. Also the name 'the star-chromatic number' is misleading, as it seems to suggest that it has something to do with the star graph. There have been a few rounds of discussions about changing to a better name on the net, provoked by Hahn and others. As more and more papers on this topic appear in the journals, and more and more variations of graph colorings are discussed (such as the acyclic chromatic number, the oriented chromatic number, the list chromatic number, etc.), it seems more urgent that we coin a better name for it, to avoid confusion. Thus we use the name 'circular chromatic number' in this paper, which has been used in a few papers, and it seems to be more acceptable.

Remark 2. After the submission of the paper, two of the questions listed in Section 8 have been answered: Questions 8.10 and 8.28. It was proved in [72,73] that for every rational number r in the interval $[2, 8/3]$, there is a series–parallel graph G with $\chi_c(G)=r$. In [101,102], and analogue of Hajós' theorem for the circular chromatic number is proved for $k/d\geqslant 3$. To be precise, for $k/d\geqslant 3$, a set of simple graph operations was given in [102] so that starting from G_d^k, by repeatedly applying these operations, one can construct all graphs of circular chromatic number at least k/d.

Added in proof: Question 8.7 has been answered in affirmative by Pan and Zhu. Question 8.9 has been answered in affirmative by Liaw, Pan and Zhu.

9. Uncited references

[10,23,33,35,37,38]

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