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# LHC EFFECTIVE MODEL FOR OPTICS CORRECTIONS

Measurements and corrections of high-order non-linear optics

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MAËL LE GARREC



*A Smart Quote.*

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# Abstract

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# Zusammenfassung

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# Acknowledgements

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## *Acknowledgements*

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# Glossary

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## Nomenclature

**AC-Dipole** Dipole magnet generating a variable oscillating field. Used to force beam oscillations for optics measurements .

**Aperture** Maximum physical transverse size the beam can take in the accelerator without suffering losses.

**ATS Factor** Equivalent to the ratio of the virgin  $\beta$ -function to the  $\beta$ -function used in the current ATS scheme, at the edge of the arc.

**Beta-function** Variable of the twiss-parameters:  $\beta$  as a function of the longitudinal position  $s$ . Related to the transverse beam size:  $\sigma(s) = \sqrt{\epsilon \cdot \beta(s)}$  .

**BPM** Beam Position Monitor, gives the transverse position of the beam.

**Chromaticity** Tune change with momentum offset. Usually denoted as three orders:  $Q'$ ,  $Q''$  and  $Q'''$ .

**Coupling** Correlation between the motion of particles in horizontal or vertical plane to the other. Strong coupling negatively impacts the optics and is usually avoided .

**Crosstalk** Interferences between two electronic circuits.

**Dipole** Magnets with two poles, responsible for bending the particles in the accelerator..

**Dispersion** Change of orbit with momentum offset, mainly in the horizontal plane, created by the dipoles.

**DOROS** Low noise BPM. Currently can't be used with other BPMs due to synchronization issues.

**Dynamic Aperture** Maximum stable aperture. Above that size, the particles become unstable and become lost.

**Emittance** ( $\epsilon$ ) Unit describing the beam in phase space. A low emittance indicates a beam with a small momentum offset and confined to a small distance.



## **Glossary**

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**Laundau Octupole** Octupoles that introduce a spread in the beam, making it more stable.

**LBDS** LHC Beam Dump System.

**Orbit Feedback** System responsible for acquisition and correction of the orbit.

**Rigid Waist Shift** Doing a waist shift by powering all the triplets at once. No individual trim.

**Waist** Location where the  $\beta$ -function is at its minimum in an IP.  $\beta^*$  refers to  $\beta_{waist}$ .

**Waist Shift** Changing the waist to have  $\beta^* = \beta_{IP}$ .

## **Acronyms**

**LHC** Large Hadron Collider.

## **Symbols**

**action** Action used as coordinate blabla.



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# Introduction

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## 1.1. Motivation

## 1.2. Thesis Outline

## 1.3. Particle Accelerators

Particle accelerators are a relatively recent development, driven by the particle physics field [1]. The first accelerators, at the beginning of the 20th century, were able to accelerate particles up to energies of a few MeV using electric fields.

The Cockcroft Walton generator was powerful enough to split an atom for the first time in 1932 [2], less than 100 years ago. Its design used capacitors and diodes to double the voltage at each stage, with its main limitation being the breakdown voltage of the capacitors. The Van de Graaff generator, created around the same time, was able to accelerate particles up to tens of MeV. They are still in use today [3] due to their capability of producing a wide variety of ion beams with energies ranging from hundreds of KeV to hundreds of MeV in the *Tandem* form [4].

Radiofrequency generators with alternating electric fields and drift tubes, first created by Rolf Wideröe in the late 1920s, mark the beginning of modern accelerator technology [5]. Rapidly evolving, particle accelerators progressed from accelerating particles to a few keV in linear accelerators to now TeV in circular accelerators, called synchrotrons.

While single-beam synchrotrons can be used for fixed-target experiments, only a fraction of the energy is available on impact. Dual-beam machines were more suitable for high energy physics experiments. The first hadron collider, the ISR, was built at CERN in 1971 with an energy of 62 GeV, taking its beam from the Proton Synchrotron (PS) [6].

Particle accelerators are still rapidly evolving. Several colliders have been built in the past, and



## 1. Introduction

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several are now either under construction or in the design study phase. New acceleration and focusing techniques are being developed, making the machines smaller. It is an exciting time for all fields that might benefit from energetic particles, whether in fundamental research, medical, industrial or security applications.

### 1.3.1. The CERN Complex

CERN is a large laboratory located on the border of France and Switzerland, near Geneva. Although well-known for discoveries in particle physics, studies are also conducted on medical applications, biology, radiation hardness or material science. Several accelerators are part of the accelerator complex, as illustrated in Fig. 1.1. A large number of fixed target experiments exist, whose beams are delivered by various accelerators depending on their needs. These experiments are often renewed <sup>1</sup>.

The largest part of the CERN accelerator complex is the LHC. The LINAC4, PSB, PS and SPS accelerators serve as pre-injectors to the LHC, taking priority over their other experiments.

### 1.3.2. The Large Hadron Collider

The Large Hadron Collider (LHC) is a circular particle accelerator primarily designed to collide protons for fundamental particle physics research. It can also occasionally collide ions such as oxygen or lead for specific studies. At the time of writing, in 2024, it holds several records, such as being the largest and most powerful accelerator in the world, at nearly 27 km long. The LHC is composed of two beam pipes, capable of accelerating two particle beams from an injection energy of 450 GeV to an energy of 6,800GeV, before colliding them in four detectors: ATLAS, CMS, Alice and LCHb.

Well-publicized, the LHC is often depicted via its superconducting dipole magnets, housed in blue cryostats, aimed at cooling the coils. Fig. 1.2 shows a 3D cut of such magnets. The LHC is mostly composed of these *main* dipoles, holding 1,232 of them, each about 14 meters long. Superconducting materials like niobium-titanium (NbTi) are utilized, as conventional materials such as copper would melt under the current strain. Around 12,000 amperes are supplied to generate the magnetic fields necessary for bending the trajectory of the particles. These particles travel at nearly the speed of light (99.9999905% of it), effectively going around the tunnel about 11,200 times per second.

#### Straight Sections and Arcs

The LHC is not a perfect circle. It is indeed composed of four *straight* sections, called the *Interaction Regions* (IPs) where detectors or specific instrumentation are placed. Connecting those sections, the *arcs* are where the majority of the magnets and their correctors are located along with some instrument

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<sup>1</sup>Up to date information can be found on <https://home.cern/science/experiments>.



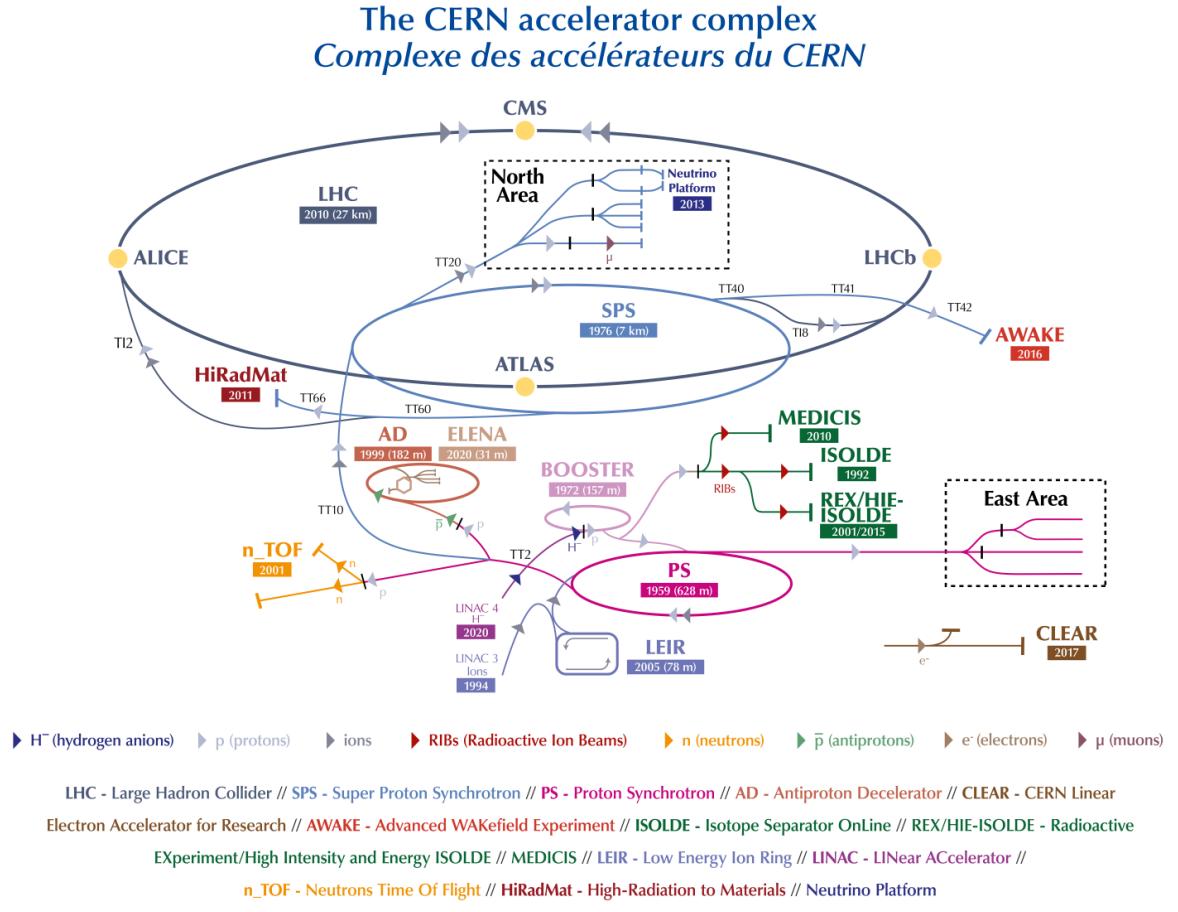


Figure 1.1.: Schematic illustration of the accelerator complex at CERN. Most accelerators are both used as injectors for the LHC or to provide beams to fixed target experiments [7].

like beam position monitors. Fig. 1.3 shows the arcs as well as the purpose of each straight section, housing either specific instrumentation or detectors.

### Arc Cells

Each arc is made up of 23 cells. Magnets are organized in a standard FoDo structure (see 2.2.2), as shows Fig. 1.4. *Dipoles* are responsible for bending the trajectory of the particles. Their associated correctors, the orbit correctors, mitigate any possible drift in path. *Quadrupoles* are used to control the beam size along the ring. Their effect is focusing in one plane and defocusing in the other. Their associated correctors control the oscillations of the beam (see tune, 2.2.2) and possible field

## 1. Introduction

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Figure 1.2.: 3D cut of a main LHC dipole [8]. Both beam pipes can be seen surrounded by the coils, strongly clamped by the yokes.

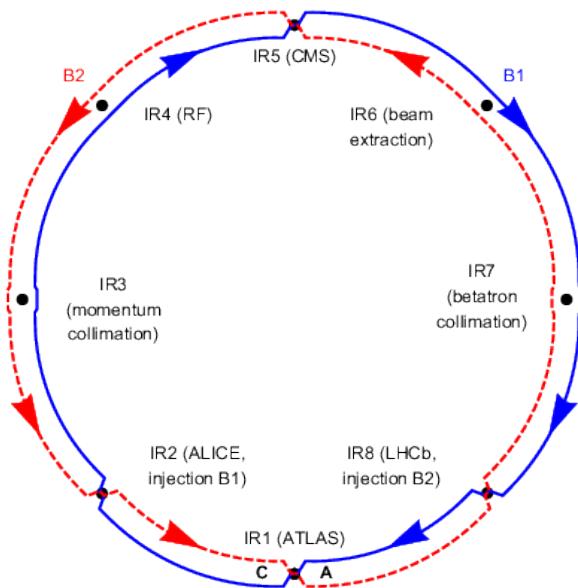


Figure 1.3.: Schematic of the LHC layout.

imperfections. *Sextupoles* correct chromaticity, being a misfocus from quadrupoles due to particles

4

having a different momentum than the reference particle. *Octupoles* are used to stabilize the beam by introducing Landau Damping [9]. The associated correctors correct higher order chromaticity effects as well as amplitude dependant tune shifts. *Decapoles* correctors aim at correcting an even higher chromaticity order.

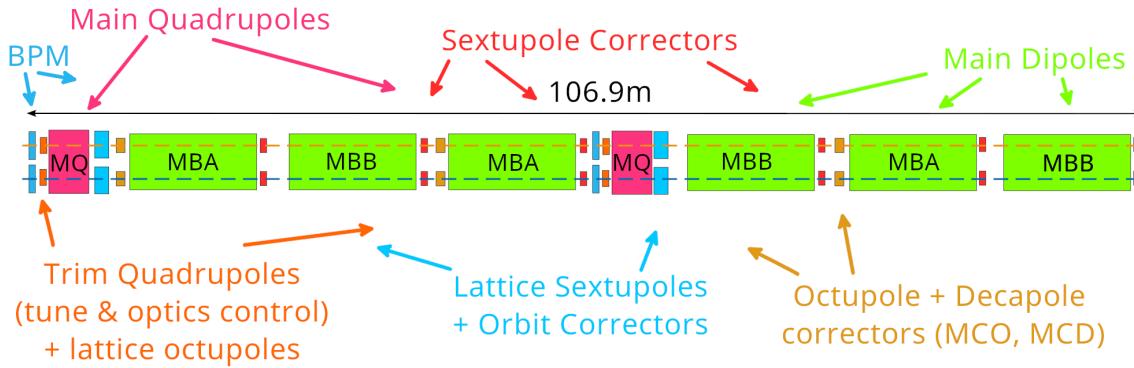


Figure 1.4.: Schematic of an LHC Arc cell [10].

## Cycles

During the operation of the LHC, the machine goes through several states. Those states of the machine [11] have been defined for specific scenarios.

A common example is the operational cycle of the LHC, shown in Fig. 1.5. The magnets are first pre-cycled [12] without any beam circulating, to get them back to a reproducible state. Their current is then increased to accept particles at the injection energy of 450GeV. In order to assess the good working condition of the machine, a probe bunch of reduced intensity is first injected. The number of bunches and their intensity is then ramped up to attain the desired scheme needed for collisions. This scheme varies throughout the year depending on the demands of the experiments. The number of bunches and intensity can also be lowered to keep the machine safe. A common scheme in 2024 is to inject about 2350 bunches with around  $10^{11}$  particles each for collisions. Optics measurements, due to their destructive nature, typically use between one and three *pilot* bunches at a lower intensity of  $10^{10}$ .

The magnets current is then ramped along with the voltage injected in the RF system, to accelerate particles to an energy of 6.8TeV. While doing so, the beam is squeezed in a first pass at the Interaction Points. A second pass is done after the ramp is over, to reach a  $\beta^* = 30\text{cm}$  at the ATLAS and CMS experiments to achieve a small beam size. Crossing-angles are then introduced to make the beams colliding.

## 1. Introduction

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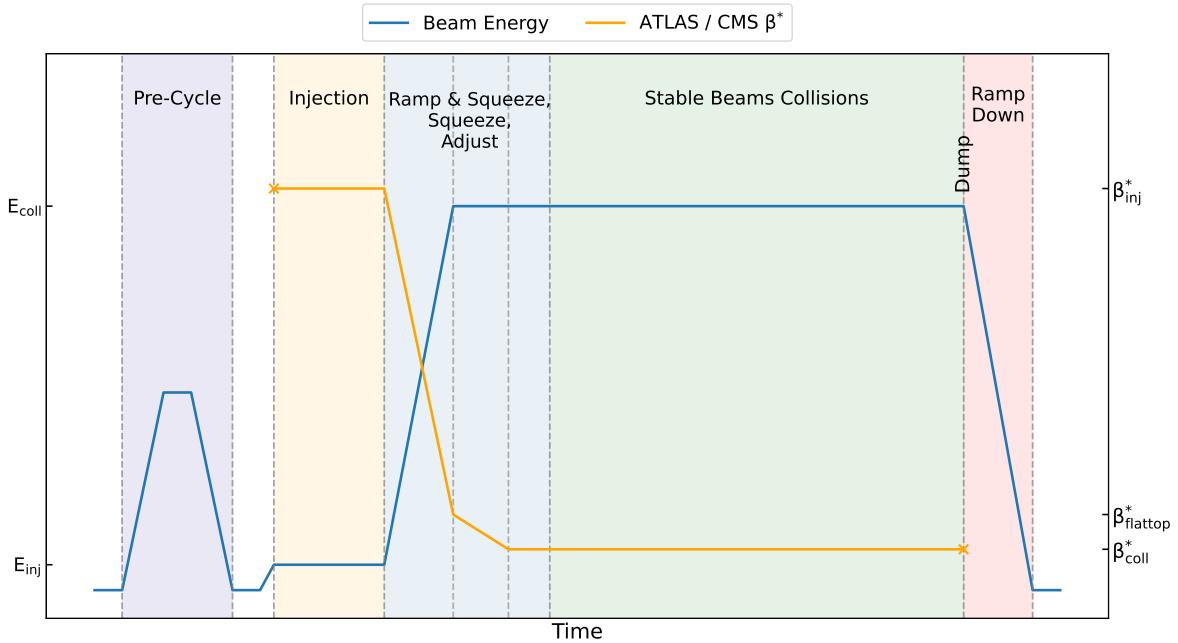


Figure 1.5.: Simplified illustration of a standard LHC cycle. Courtesy of Félix Soubelet [13].

## Magnetic Fields

The magnetic fields of the LHC are created via the coils of the magnets. Real-life magnets never have a single field as one would like. Instead, so called *allowed harmonics* exist due to the geometry of the coil. As such, the main dipoles of the LHC can exhibit fields similar to sextupoles, decapoles, decatetrapoles and so on [14]. Manufacturing imperfections also add fields errors outside of the scope of the allowed ones. Dipoles are indeed found to generate octupolar field errors.

During the design of the LHC, the main dipoles have been identified to generate significant field errors. Magnetic measurements of those various fields were thus taken and magnetic tables built based on real-life magnets nowadays installed in the machine. Those magnetic tables, computed for each LHC configuration by *WISE* [15] are used by simulation softwares. Predictions of field errors and compensating strength for the correctors is computed by the Field Description for the LHC (*FiDeL*, [16]). *FiDeL* is used in the LHC control system in operation to steer the beams.

# Concepts of Accelerator Physics

## 2.1. Magnetic Fields

### 2.1.1. Nomenclature

Several notations coexist to denote magnetic fields. In this thesis, the *European Convention* [17] is used for field indices, as shown in Tab. 2.1. MAD-X, and MAD-NG, however, use the *American Convention*.

Multipole	Index	Normalized Strength	MAD-X
Dipole	1	$K_1$	0
Quadrupole	2	$K_2$	1
Sextupole	3	$K_3$	2
Octupole	4	$K_4$	3
Decapole	5	$K_5$	4
Dodecapole	6	$K_6$	5
Decatetrapole	7	$K_7$	6
Decahexapole	8	$K_8$	7

Table 2.1.: Relation between field indices and multipoles.

As such, unless explicitly stated, quantities such as the magnetic strength  $b$  and normalized strength  $K$  will be expressed with this notation.

## 2. Concepts of Accelerator Physics

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### 2.1.2. Multipole Expansion

A 2 dimension magnetic field in the planes  $x$  and  $y$  can be described as a sum of the normal and skew field gradients  $\mathcal{B}$  and  $\mathcal{A}$  with multipoles of order  $n$ , given by [18]:

$$B_y + iB_x = \sum_{n=1}^{\infty} (\mathcal{B}_n + i\mathcal{A}_n) (x + iy)^{n-1} \quad (2.1)$$

An ideal magnet would produce either a sole normal or skew field. However, this is not applicable to real-life magnets that are imperfect, due to design and manufacturing constraints. Field errors are thus introduced, relative to the main field of the ideal 2N-pole magnet at a reference radius  $r_{ref}$  [17], as shown in Eq. (2.2). The coefficients of the normal and skew relative field errors, referred to as  $a_n$  and  $b_n$ , are dimensionless but often given in units of  $10^{-4}$ .

$$B_y + iB_x = \begin{cases} \mathcal{B}_N \cdot \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{r_{ref}}\right)^{n-1}, & \text{for normal magnets} \\ \mathcal{A}_N \cdot \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{r_{ref}}\right)^{n-1}, & \text{for skew magnets} \end{cases} \quad (2.2)$$

The normal and skew field components of order  $n$  for an imperfect 2N-pole magnet is thus given by the following equation:

$$\begin{aligned} \mathcal{B}_n &= \mathcal{B}_N \cdot \frac{b_n}{r_{ref}^{n-1}}, \\ \mathcal{A}_n &= \mathcal{A}_N \cdot \frac{a_n}{r_{ref}^{n-1}}. \end{aligned} \quad (2.3)$$

The unit of the field is relative to the multipole order  $n$ : [Tm $^{1-n}$ ].

### 2.1.3. Beam Rigidity and Normalization

#### Beam Rigidity

The beam rigidity refers to the resistance of a particle moving through the accelerator to the bending applied by the magnetic fields. It is derived from the Laurentz force [17] and relates the magnetic field  $B$ , the radius of curvature  $\rho$  to the momentum  $p$  and charge  $q$  of the particle:

$$B\rho = \frac{p}{q} \quad (2.4)$$

It is of interest when designing an accelerator to set the maximum field as well as the required radius of curvature for a specific momentum and particle. An interesting metric of an accelerator is also its *filling factor*, or percentage of dipoles in the machine. It can be calculated via the radius of curvature:  $f = \rho/r$ . A low filling factors means more space for other magnets, collimators, beam instrumentation, etc.

## Field Normalization

The Beam Rigidity is also used as a way to normalize magnetic field strengths in particle accelerators where the momentum of the particle changes (i.e. acceleration). Normalized Normal and Skew components  $K_n$  and  $J_n$  are given by [18]:

$$\begin{aligned} K_n &= \frac{q}{p}(n-1)!\mathcal{B}_n, \\ J_n &= \frac{q}{p}(n-1)!\mathcal{A}_n. \end{aligned} \quad (2.5)$$

### 2.1.4. Hamiltonian

The Hamiltonian describing the motion for the transverse planes of a given multipole or order  $n$  is given by [19–21]:

$$\begin{aligned} H &= \frac{q}{p}\Re\left[\sum_{n>1}(\mathcal{B}_n + i\mathcal{A}_n)\frac{(x+iy)^n}{n}\right] \\ &= \Re\left[\sum_{n>1}(K_n + iJ_n)\frac{(x+iy)^n}{n!}\right]. \end{aligned} \quad (2.6)$$

Quite often, when studying the effect of a magnet on the beam, only one component is required, and the sum can thus be dropped. The normal and skew fields can also be isolated in order to consider their effect only:

$$\begin{aligned} N_n &= \frac{1}{n!}K_n\Re[(x+iy)^n] \\ S_n &= -\frac{1}{n!}J_n\Im[(x+iy)^n]. \end{aligned} \quad (2.7)$$



## 2. Concepts of Accelerator Physics

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### 2.2. Coordinate Systems

In circular accelerators, particle dynamics are represented using a traveling coordinate system. A reference orbit is determined by the lattice and its magnet strengths, forming the *optics*. In the case of a synchrotron, like the LHC, where the particles return to their original location after some turns, the reference orbit is also called the closed orbit.

#### 2.2.1. Frenet-Serret System

The Frenet-Serret coordinate system moves along the ring on the reference orbit. The coordinates are then transverse:  $x$  and  $y$ , and longitudinal in the direction of travel:  $s$ . Figure 2.1 shows those coordinates.

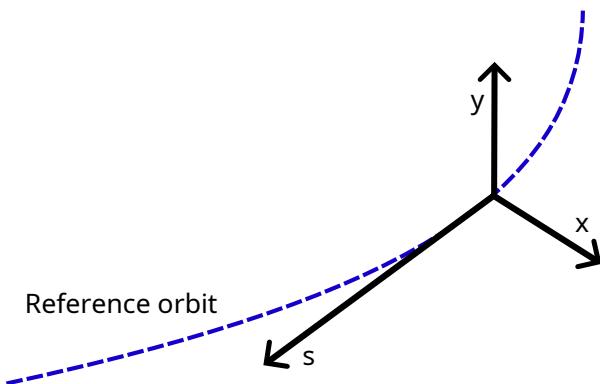


Figure 2.1.: Frenet-Serret coordinate system, commonly used in accelerator physics. The system moves along the reference orbit.

This coordinate system is widely used to simply describe either an element's or a particle's position in the accelerator. Without any explicit mention, those are coordinates used in this thesis. It is frequent to use the variable  $z$  to refer to either  $x$  or  $y$  in equations.

#### 2.2.2. Linear Lattice

##### Courant-Snyder Parameters

A circular accelerator is composed of many multipoles of different orders. A basic design only requires dipoles and quadrupoles in order to operate. Dipoles are used to bend the particles in order to form the ring, whereas quadrupoles are used to focus the beam to a focal point, similar to light optics. Those elements can be arranged in a particular order, to form a FoDo cell. Such cells present an alternating



placement of focusing and defocusing quadrupoles with dipoles in between, as shown in Fig.2.2, and are usually repeated many times along the ring.

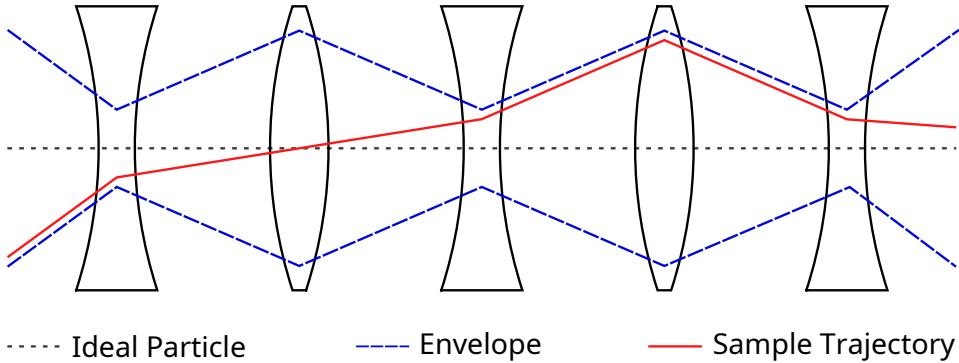


Figure 2.2.: Line composed of FoDo cells, a basic cell present in most accelerators, composed of a Focusing and a Defocusing quadrupole. The envelope is a factor of the  $\beta$ -function and the action  $J$ .

A lattice composed of only dipoles and quadrupoles, is referred to as a *linear* lattice. In a synchrotron, a circular particle accelerator, particles undergo transverse and longitudinal oscillations. As such, particles do not go back to their initial position before a certain number of turns. Taking into account those oscillations, the phase-space ellipse of a particle at a position  $s$  in the ring can be described with a new system: the Courant-Snyder parameters, also known as Twiss parameters or the *optics functions* [22], as shown in Fig. 2.3.

$J$ , the action, an invariant of motion at a given energy, is related to the other quantities by:

$$J_z = \frac{1}{2}(\gamma_z \cdot z^2 + 2\alpha_z p_z \cdot z + \beta_z p_z^2). \quad (2.8)$$

The action can be related to the area in phase space, called the emittance:  $\epsilon = 2J$ . As the  $\beta$  parameter varies along the ring, it is referred to as the  $\beta$ -function and is related to the amplitude of the oscillations. Thus, the smaller is the  $\beta$ -function, the smaller is also the envelope of the beam. The number of oscillations per turn is called the *tune*, and is closely related to the  $\beta$ -function:

$$Q_{x,y} = \frac{1}{2\pi} \oint \frac{1}{\beta_{x,y}(s)} ds. \quad (2.9)$$

It is common to express the position of a particle using *action-angle* variables, allowing to switch between the Courant-Snyder parameters and the Frenet-Serret system:

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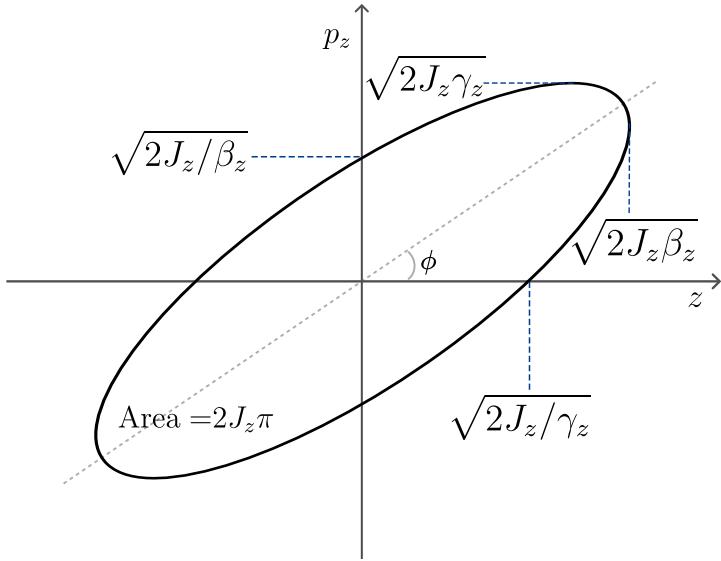


Figure 2.3.: Phase-space ellipse of a linear machine, parametrized by the Courant-Snyder parameters  $\alpha$ ,  $\beta$  and  $\gamma$ .

$$\begin{aligned} z &= \sqrt{2J_z\beta_z} \cos \phi_z \\ p_z &= -\sqrt{\frac{2J_z}{\beta_z}} (\sin \phi_z + \alpha_z \cos \phi_z). \end{aligned} \quad (2.10)$$

### Normalized Coordinates

In order to simplify the description of the linear motion in a ring, a transformation can be applied to the previously seen coordinates. Figure Fig. 2.4 shows a phase-space described in both coordinates. The new coordinates,  $\hat{z}$ , and  $\hat{p}_z$ , are then expressed as factors of the  $\alpha$  and  $\beta$  functions:

$$\begin{pmatrix} \hat{z} \\ \hat{p}_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta_z}} & 0 \\ \frac{\alpha_z}{\sqrt{\beta_z}} & \sqrt{\beta_z} \end{pmatrix} \begin{pmatrix} z \\ p_z \end{pmatrix}. \quad (2.11)$$

This allows to describe the motion as a simple rotation, the new coordinates being only dependent on the invariant  $J_z$  and the phase  $\phi_z$ :

$$\begin{aligned} \hat{z} &= \sqrt{2J_z} \cos (\phi_z), \\ \hat{p}_z &= \sqrt{2J_z} \cos (\phi_z). \end{aligned} \quad (2.12)$$



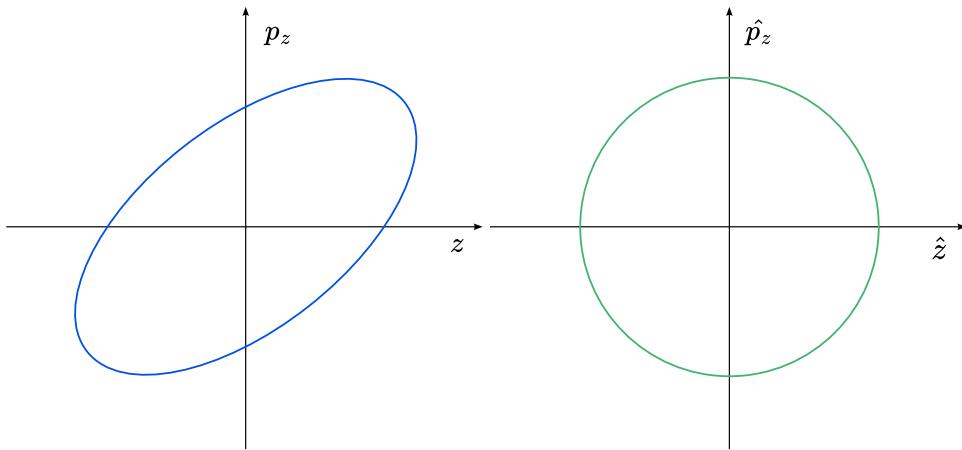


Figure 2.4.: Phase space described in both regular and normalized coordinates

### Linear Transfer Maps

It is possible to describe the final position of a particle after going through an element via *transfer maps*. In linear optics, such linear maps are matrices. Those maps are symplectic, meaning they preserve the phase-space area. For a matrix  $M$  and positions  $z$  at the initial location and  $s$ , the general formula reads [23]:

$$\begin{pmatrix} z \\ z' \end{pmatrix}_s = M \cdot \begin{pmatrix} z \\ z' \end{pmatrix}_0 \quad (2.13)$$

This formalism assumes that the magnetic field is constant along the element in the longitudinal direction. Basic elements such as drifts, dipoles, quadrupoles can then be described by a simple  $2 \times 2$  matrix:

$$M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad (2.14)$$

$$M_{dipole} = \begin{pmatrix} \cos(L/\rho) & \rho \sin(L/\rho) \\ -1/\rho \sin(L/\rho) & \cos(L/\rho) \end{pmatrix}, \quad (2.15)$$

$$M_{focusing quad.} = \begin{pmatrix} \cos(\sqrt{k_2}L) & 1/\sqrt{k_2} \sin(\sqrt{k_2}L) \\ -\sqrt{k_2} \sin(\sqrt{k_2}L) & \cos(\sqrt{k_2}L) \end{pmatrix}, \quad (2.16)$$

$$M_{defocusing\ quad.} = \begin{pmatrix} \cosh(\sqrt{|k_2|}L) & 1/\sqrt{|k_2|}\sinh(\sqrt{|k_2|}L) \\ \sqrt{|k_2|}\sinh(\sqrt{|k_2|}L) & \cosh(\sqrt{|k_2|}L) \end{pmatrix}, \quad (2.17)$$

where  $L$  is the length of the element,  $\rho$  the radius of curvature of the orbit and  $k_2$  the normalized strength of quadrupoles. In the case of quadrupoles, a focusing matrix should be used in the horizontal plane for focusing quadrupoles, where defocusing matrices should be used in the vertical plane. The opposite goes for defocusing quadrupoles.

Transfer matrices can be combined together to describe a larger group of elements, as the FoDo cell seen previously. Its transfer matrix can then be expressed as:

$$M_{FoDo} = M_{focusing\ quad.} \cdot M_{drift} \cdot M_{defocusing\ quad.} \cdot M_{drift}. \quad (2.18)$$

For a closed machine, a full revolution can be described by a so-called *one-turn map*, being the transfer matrix of the whole machine, denoted  $\mathcal{M}$ . Such a map can potentially contain thousands of elements.

**Symplecticity** An important property of any transformation is that they need to be symplectic. A symplectic transformation preserves the volume in phase space.

### 2.2.3. Non-Linear Lattice

So far, Courant-Snyder parameters were a good way to describe the distribution of positions and velocities of particles in the transverse plane. One caveat of using this formalism is that it is restrained to linear optics and does not describe non-linear beam dynamics such as resonances or the effects arising from an arrangement of several multipoles together.

#### Lie Algebra

One way to describe non-linear effects is to introduce Lie Algebra [24], a powerful algebra able to describe transformations, symmetries and their associated conserved quantities.

The Lie algebra is a vector space, denoted  $\mathfrak{g}$ , equipped with a binary operation called the *Lie bracket* and denoted  $[x, y]$  for two vectors  $x$  and  $y$ . Any vector space equipped with a Lie bracket (or commutator) satisfying the following conditions is called a Lie algebra:

- Bilinearity:

$$\begin{aligned} [ax + by, z] &= a[x, z] + b[y, z], \\ [z, ax + by] &= a[z, x] + b[z, y], \quad \forall x, y, z \in \mathfrak{g} \text{ and } a, b \text{ scalars} \end{aligned} \quad (2.19)$$

- Alternativity:

$$[x, x] = 0, \quad \forall x \in \mathfrak{g} \quad (2.20)$$

- Anticommutativity:

$$[x, y] = -[y, x], \quad \forall x, y \in \mathfrak{g} \quad (2.21)$$

- The Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g} \quad (2.22)$$

The *Lie bracket*, plays a central role in the Lie algebra. It describes how dynamical variables evolve under infinitesimal symplectic transformations.

## Poisson Brackets

To create a Lie algebra, an operation satisfying the previous conditions needs to be found. In accelerator physics, *Poisson brackets* are chosen [24, 25]. Poisson brackets are used to describe continuous symmetries, conserved quantities, and the evolution of the dynamical variables in the system.

Let's consider position and momentum coordinates  $q_1 \cdots q_n$  and  $p_1 \cdots p_n$  of a  $2n$ -dimensional phase space. Usually, those would be  $x, y, p_x$  and  $p_y$  for transverse coordinates. The Poisson brackets of two functions  $f$  and  $g$  is then defined by:

$$[f, g] = \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \quad (2.23)$$

The evolution of coordinates and momenta in time is described by Hamilton's equations of motion, which can be naturally expressed with Poisson brackets:

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} = [q_i, H] \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} = [p_i, H]. \end{aligned} \quad (2.24)$$

## Lie Operator

Given a function  $f$ , a differential operator called *Lie operator* is defined, and is closely related to the previously seen Poisson bracket:



$$:f := \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i}. \quad (2.25)$$

The action of this operator on a function  $g$  is equivalent to the Poisson brackets, as in:

$$:f:g = [f, g]. \quad (2.26)$$

A particular power series of this Lie operator can now be defined, called *Lie transformation*:

$$\begin{aligned} e^{:f:}g &= \sum_{l=0}^{\infty} \frac{1}{l!} :f :^l g \\ &= g + [f, g] + \frac{1}{2!} [f, [f, g]] + \dots . \end{aligned} \quad (2.27)$$

### Non-Linear Transfer Maps

As introduced in 2.2.2, the dynamics of a particle beam in a circular accelerator can be described by *transfer maps*. A symplectic *One Turn Map*  $\mathcal{M}$  that includes  $N$  non-linear elements is defined [24] as:

$$\mathcal{M} = e^{:h_N:} \cdot e^{:h_{N-1}:} \cdots e^{:h_1:} \cdot \mathcal{R} \quad (2.28)$$

where  $\mathcal{R}$  is a matrix describing the linear motion over one turn and the  $h_i$  terms representing the Hamiltonian of each non-linear elements of the machine. Via the Baker-Campbell-Hausdorff (BCH) theorem [26, 27], previous Lie transformations can be combined and simplified via Eq. (2.29) and Eq. (2.30). Further orders can be found and computed via [27].

$$e^{:h_1:} \cdot e^{:h_2:} = e^{:h:} \quad (2.29)$$

with

$$\begin{aligned} h &= h_1 + h_2 && \Rightarrow 1^{\text{st}} \text{ order} \\ &+ \frac{1}{2}[h_1, h_2] && \Rightarrow 2^{\text{nd}} \text{ order} \\ &+ \frac{1}{12}[h_1, [h_1, h_2]] - \frac{1}{12}[h_2, [h_1, h_2]] && \Rightarrow 3^{\text{rd}} \text{ order} \\ &+ \dots . \end{aligned} \quad (2.30)$$

The one turn map is thus expressed as a single Lie transformation:

$$\mathcal{M} = e^{\cdot h \cdot} \cdot \mathcal{R}. \quad (2.31)$$

In most cases, were the non-linear perturbations are small, the above series converges quickly and only the two first terms of Eq. (2.30) are used [28]. The resulting expression is then more elegant, being a simple sum of the Hamiltonians of the  $N$  non-linear elements:

$$\mathcal{M} = e^{\cdot h_1 + h_2 + \dots + h_N \cdot} \cdot \mathcal{R}. \quad (2.32)$$

It is though to be noted that in this thesis experimental measurements show the evidence of higher order contributions. In order to fully understand the combined effect of multipoles, the BCH expansion needs to be expended further than the first two terms.

When transporting coordinates for a machine element, its length must be taken into account:

$$\begin{aligned} e^{\cdot LH \cdot} x_0 &= x_1, \\ e^{\cdot LH \cdot} p_0 &= p_1. \end{aligned} \quad (2.33)$$

## Normal Form

As non-linearities are introduced in the machine, the phase-space becomes distorted, resulting in  $J_z$  no longer being an invariant of motion. The previously seen normalization does not work anymore and the phase-space is no longer a simple circle. A new normalization is then introduced, called the *normal form*, with complex coordinates  $\zeta$ , depending on new action and angle coordinates  $I_z$  and  $\psi_z$ :

$$\zeta_{z,\pm} = \sqrt{2I_z} e^{\mp i\psi_z}. \quad (2.34)$$

An exaggerated vision of such a phase-space in Courant-Snyder, normalized, and normal form coordinates can be seen in Fig. 2.5.

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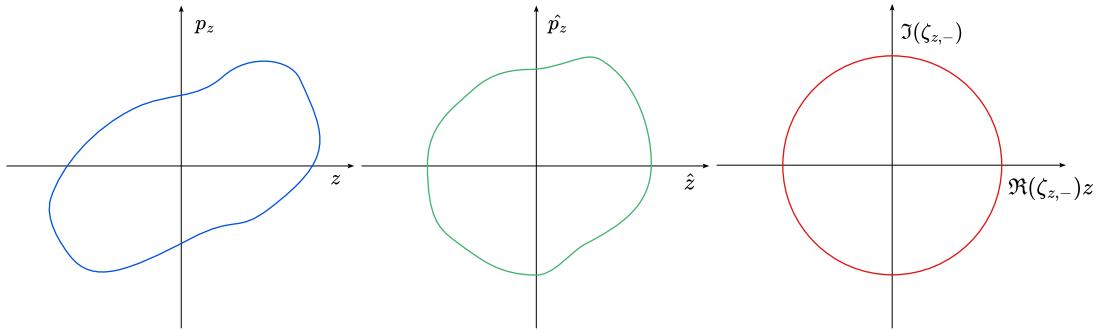


Figure 2.5.: Exaggerated phase space distorted by non-linearities described in regular, normalized and normal form coordinates.

The map defined previously in Eq. (2.31) can be rewritten in order to retrieve an invariant of motion  $I_z$  by introducing a generating function  $F$ :

$$\tilde{\mathcal{M}} = e^{-F} \mathcal{M} e^{F} \quad (2.35)$$

Such a generating function includes all the non-linearities, simplifying the calculations. Going back and forth from normalized to normal forms coordinates is then straightforward, as depicted in Fig. 2.6. The hamiltonian  $H$  is now only dependent on the action  $I_z$ .

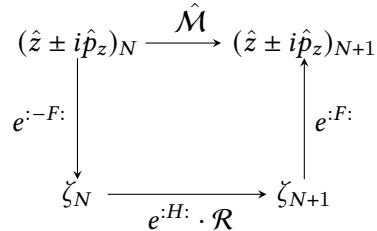


Figure 2.6.: A one turn map from turn  $N$  to  $N+1$  solved using a generating function  $F$ , transforming to normal form coordinates  $\zeta$ , applying the linear rotation  $R$  and transforming back to normalized coordinates.

The function  $F$  is defined as

$$F = \sum_{jklm} f_{jklm} \zeta_{x,+}^j \zeta_{x,-}^k \zeta_{y,+}^l \zeta_{y,-}^m, \quad (2.36)$$

where  $f_{jklm}$  are the so-called Resonance Driving Terms (RDTs). The summation  $jklm$  is done over all the combinations of  $j, k, l$  and  $m$  with  $j + k + l + m = n$  for a multipole of order  $n$ , as shown in

Eq. (2.37):

$$\sum_{jklm} = \sum_{j=0}^n \sum_{k=0}^n \sum_{l=0}^n \sum_{m=0}^n ; \quad j+k+l+m = n. \quad (2.37)$$

The expression of the resonance driving terms is given by the global hamiltonian term  $h_{jklm}$  by

$$f_{jklm} = \frac{h_{jklm}}{1 - e^{i2\pi[(j-k)Q_x + (l-m)Q_y]}}, \quad (2.38)$$

where this coefficient is a summation over the hamiltonian terms of elements  $w$  in the lattice,

$$h_{jklm} = \sum_w h_{w,jklm} e^{i[(j-k)\Delta\phi_x + (l-m)\Delta\phi_y]}. \quad (2.39)$$

The expression of  $h_{w,jklm}$  is itself derived from the general hamiltonian of Eq. (2.6) by applying a binomial expansion on the coordinates [21] as shows Eq. (2.40). Derivations and more information on resonance driving terms can be found in Appendix C.

$$h_{w,jklm} = -\Re \left[ \frac{K_{w,n} + iJ_{w,n}}{j!k!l!m!2^{j+k+l+m}} i^{l+m} \beta_{w,x}^{\frac{j+k}{2}} \beta_{w,y}^{\frac{l+m}{2}} \right] \quad (2.40)$$

Transforming from the normal form coordinates back to the original normalized coordinates can be done using the right side of Fig. 2.6. Which is written, to second order, as:

$$\begin{aligned} h_z^\pm &= e^{iF} \cdot \zeta_z^\pm \\ &\approx \zeta_z^\pm + [F, \zeta_z^\pm] + \frac{1}{2!} [F, [F, \zeta_z^\pm]]. \end{aligned} \quad (2.41)$$

Using this equation to the first order and Eq. (2.34), the normalized coordinates can be expressed after  $N$  turns in Eq. (2.42).

$$\begin{aligned} (x - ip_x)(N) &= \sqrt{2I_x} e^{i(2\pi Q_x N + \psi_{x0})} - \\ &\quad 2i \sum_{jklm} j f_{jklm} (2I_x)^{\frac{j+k-1}{2}} (2I_y)^{\frac{l+m}{2}} e^{i[(1-j+k)(2\pi Q_x N + \psi_{x0}) + (m-l)(2\pi Q_y N - \psi_{y0})]} \\ (y - ip_y)(N) &= \sqrt{2I_y} e^{i(2\pi Q_y N + \psi_{y0})} - \\ &\quad 2i \sum_{jklm} l f_{jklm} (2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m-1}{2}} e^{i[(k-j)(2\pi Q_x N + \psi_{x0}) + (1-l+m)(2\pi Q_y N - \psi_{y0})]}. \end{aligned} \quad (2.42)$$

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It is to be observed that some  $f_{jklm}$  terms will not contribute to the motion of the particle in a given plane due to the dependence on  $j$  or  $l$ .

2

### 2.3. Examples of Maps

It is important to remember that two expansions are used when creating non linear transfer maps. When referring to the order of a map, it is the order of the BCH formula, used to combine Hamiltonians, that is referred to. The Lie transformation to transport the coordinates themselves is usually only taken to the first order.

#### 2.3.1. Non-Linear Transfer of a Single Sextupole

Here, we are interested on the effect of a single sextupole on the regular frenet-serret coordinates  $x$ ,  $y$ ,  $p_x$  and  $p_y$ . Let's consider a sextupole with strength  $K_3$  and a normal field,

$$H_3 = \frac{1}{6}K_3(x^3 - 3xy^2). \quad (2.43)$$

A transfer map, from longitudinal coordinate  $s_0$  to  $s_1$ , consisting of only this element is the following:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_1} = e^{L:H_3:} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_0}, \quad (2.44)$$

where L is the length of the multipole. Using Eq. (2.27) to expand the Lie transformation to the first order, it can be rewritten as

$$\begin{aligned} e^{L:H_3:}x &= x + [L \cdot H_3, x], \\ e^{L:H_3:}p_x &= p_x + [L \cdot H_3, p_x], \\ e^{L:H_3:}y &= y + [L \cdot H_3, y], \\ e^{L:H_3:}p_y &= p_y + [L \cdot H_3, p_y]. \end{aligned} \quad (2.45)$$

Applying the poisson bracket of Eq. (2.23) on  $x$  or  $y$  yields 0, as neither the Hamiltonian nor  $x$  and  $y$  are dependent on  $p_x$  and  $p_y$ .

$$[L \cdot H_3, x] = \underbrace{\frac{\partial(L \cdot H_3)}{\partial x} \frac{\partial x}{\partial p_x}}_0 - \underbrace{\frac{\partial(L \cdot H_3)}{\partial p_x} \frac{\partial x}{\partial x}}_0 + \underbrace{\frac{\partial(L \cdot H_3)}{\partial y} \frac{\partial x}{\partial p_y}}_0 - \underbrace{\frac{\partial(L \cdot H_3)}{\partial p_y} \frac{\partial x}{\partial y}}_0 = 0. \quad (2.46)$$

The poisson bracket applied on  $p_x$  or  $p_y$  though evaluates to a non-zero value, as the momentum is present in  $p_{x,y}$  while  $x, y$  are present in the Hamiltonian:

$$\begin{aligned} [L \cdot H_3, p_x] &= \frac{\partial(L \cdot H_3)}{\partial x} \frac{\partial p_x}{\partial p_x} - \underbrace{\frac{\partial(L \cdot H_3)}{\partial p_x} \frac{\partial p_x}{\partial x}}_0 + \underbrace{\frac{\partial(L \cdot H_3)}{\partial y} \frac{\partial p_x}{\partial p_y}}_0 - \underbrace{\frac{\partial(L \cdot H_3)}{\partial p_y} \frac{\partial p_x}{\partial y}}_0 \\ &= \frac{1}{2} K_3 L (x^2 - y^2) \end{aligned} \quad (2.47)$$

The same method is used for  $p_y$ . The final form of the transfer map Eq. (2.44) is then the following:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_1} = \begin{pmatrix} 1 & 1 + \left( \frac{1}{2p_x} K_3 L (x^2 - y^2) \right) & & \\ & 1 & & \\ & & 1 & \\ & & & 1 - \left( \frac{1}{p_y} K_3 L xy \right) \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_0}. \quad (2.48)$$

It is not necessary to go higher than the first order, as the second order of the expansion of the Lie transformation is 0 ;  $p_x$  is indeed not present in the result of the first poisson bracket:

$$\begin{aligned} \frac{1}{2!} [L \cdot H_3, [L \cdot H_3, p_x]] &= \frac{1}{2!} \left[ L \frac{1}{6} K_3 (x^3 - 3xy^2), \left[ L \frac{1}{6} K_3 (x^3 - 3xy^2), p_x \right] \right] \\ &= \frac{1}{2!} \left[ L \frac{1}{6} K_3 (x^3 - 3xy^2), \frac{1}{2} K_3 L (x^2 - y^2) \right] \\ &= \frac{1}{2!} \cdot 0 \\ &= 0 \end{aligned} \quad (2.49)$$

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### 2.3.2. Non-Linear Transfer of Two Sextupoles

We saw previously that a single sextupole only acts as a sextupole when it is alone in the transfer map, which is expected. Let's now consider two sextupoles which hamiltonians are denoted  $H_1$  and  $H_2$ .

Creating a map consisting of only two sextupoles does not make much sense, as it finally results in one sextupole as their coordinates are the same. Instead, a drift is added between the two elements. The Hamiltonian of a drift of length  $L_D$  is given by [29],

$$D = -\frac{L_D}{2}(p_x^2 + p_y^2). \quad (2.50)$$

The application of the lie transformation on the canonical coordinates is then very simple, as no higher orders arise ( $[D, [D, x]] = 0$ ):

$$\begin{aligned} e^{:D:}x &= x + L_D p_x, \\ e^{:D:}p_x &= p_x. \end{aligned} \quad (2.51)$$

The transfer map of such a line is then the following,

$$\mathcal{M} = e^{:Z:} = e^{:H_2:} \cdot e^{D:H_1:}, \quad (2.52)$$

describing the evolution of coordinates from a longitudinal position  $s_0$  to  $s_1$ ,

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_1} = \mathcal{M} \cdot \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_0} \quad (2.53)$$

In order to combine those elements, the BCH formula from Eq. (2.30) is used, presented here to the third order for two elements,

$$Z = \underbrace{H_2 + H_1}_{\text{First order}} + \underbrace{\frac{[H_2, H_1]}{2}}_{\text{Second order}} + \underbrace{\frac{[H_2, [H_2, H_1]]}{12} - \frac{[H_1, [H_2, H_1]]}{12}}_{\text{Third order}} \quad (2.54)$$

**First Order** To the first order, the resulting effective hamiltonian is only the summation of two sextupoles.

**Second Order** The drift added to change the coordinates of  $H_1$  allows the poisson bracket to evaluate to a non-zero value. Octupolar-like terms indeed appear in the effective hamiltonian. From this, it can be inferred that two sextupoles will interact together and introduce effects like amplitude detuning, second order chromaticity and RDTs. Details of the derivation can be found in Appendix A.

**Third Order** To the third order, even higher orders such as decapolar-like effects appear. Such effects include the third order chromaticity, chromatic amplitude detuning and RDTs.

**Remark** It is to be noted that while sextupoles do introduce higher-order terms, these are typically small in comparison to those brought by the actual higher-order multipoles, making them thus often negligible.

## 2.4. Observables

### 2.4.1. Dispersion

Treating a beam as a single particle having the design momentum  $p_0$  leads to a machine with no apparent ill effect related to that momentum. However, when considering a particle beam where each particle follows a distribution in momentum, a few effects arise from this deviation, called the *momentum offset*,  $\delta$ . It is defined as a relative difference to the design momentum:

$$\delta = \frac{p - p_0}{p_0}. \quad (2.55)$$

Those effects are referred to as *chromatic aberrations*. The first and most important to consider is the *dispersion*. Dispersion results from a particle with a momentum offset being deflected differently by the dipoles compared to a particle at the design momentum. Figure 2.7 shows an example of deflection.

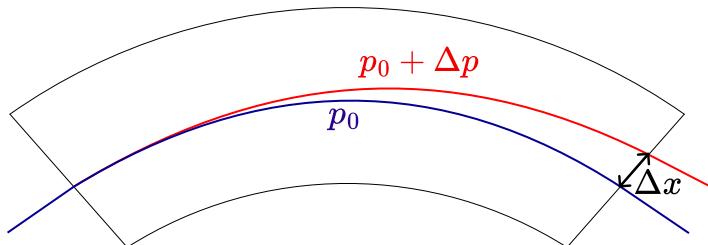


Figure 2.7.: Particles with a momentum offset will be deflected differently by dipoles. This offset in position can be described by the dispersion function.

The particle is still subject to the other properties of the lattice, but with a different orbit, described by Eq. (2.7).

$$\begin{aligned} D_x(s) &= \frac{\Delta x(s)}{\delta} \\ D_y(s) &= \frac{\Delta y(s)}{\delta} \end{aligned} \quad (2.56)$$

### 2.4.2. $\beta$ -function

As seen previously in 2.2.2, the  $\beta$ -function is related to the amplitude of oscillations of the beam. Figure 2.8 shows how the  $\beta$ -function oscillates along the ring due to quadrupoles focusing and defocusing properties. The  $\beta$ -function is an important quantity found as a factor in several other observables that will be described later in this thesis.

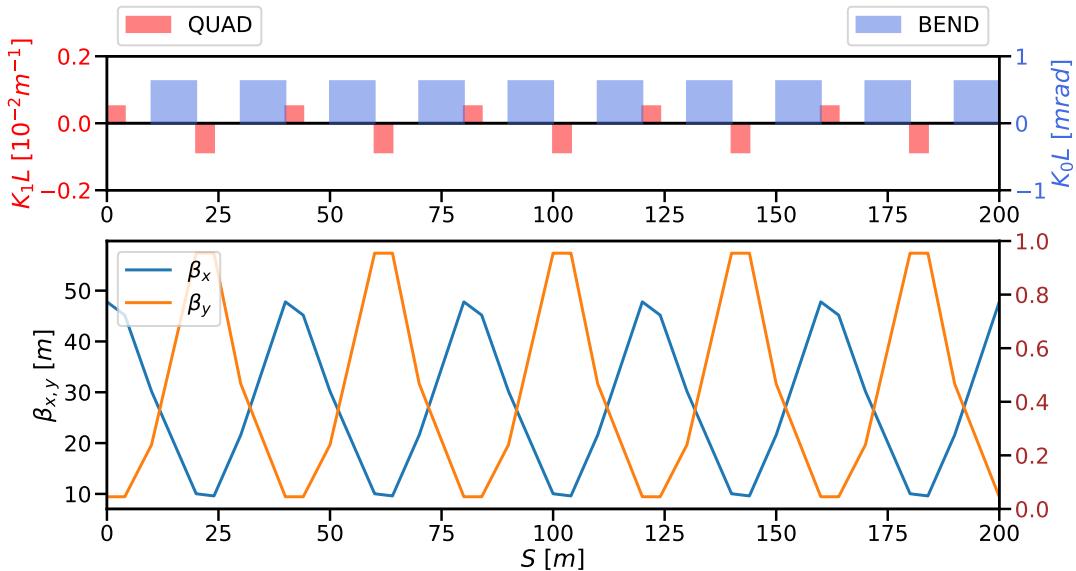


Figure 2.8.: Evolution of the  $\beta$ -function along the lattice. Horizontal and vertical beatings are usually opposite given the focusing and defocusing properties of quadrupoles in each plane.

A difference in  $\beta$ -function compared to the design leads to possibly unstable and larger beams, degrading its properties and making it harder to control. The relative difference in  $\beta$ -function is called the beta-beating, expressed in percents:

$$\beta - \text{beating} [\%] = \frac{\beta_z(s) - \beta_z(s)_{\text{model}}}{\beta_z(s)_{\text{model}}} \cdot 100\%. \quad (2.57)$$

### 2.4.3. Coupling

In a perfect scenario, the particle motion of each transverse plane is independent, or *uncoupled*. In practice, this transverse motion can be altered by some magnetic elements, giving rise to *betatron coupling* where the motion of each plane is not independent anymore. Such elements can be quadrupoles, mounted with a roll error introducing skew-quadrupolar fields, which are the main source of coupling in the LHC [13]. Field imperfections, solenoids and feed-down from higher orders can also contribute to coupling.

The resonances  $Q_x + Q_y$  and  $Q_x - Q_y$ , called the *sum* and *difference* resonances, are mainly excited by skew quadrupoles. When coupling is present in the machine, the former may lead to unstable motion while the latter introduces an periodic exchange of emittance between the planes, keeping it stable. They can be characterized by the RDTs  $f_{1010}$  and  $f_{1001}$ .

Effects of normal multipoles start showing their skew counterpart (and vice versa) as the motion of transverse planes become coupled. This is demonstrated in Chapter 6 with skew-octupolar RDTs contributed to by normal octupoles in the presence of coupling.

### 2.4.4. Momentum Compaction Factor

In synchrotrons, particles with a deviation in momentum with respect to the reference particle will experience a different path length due to the bending of the dipoles. This effect is characterized by the *momentum compaction factor* [19],

$$\alpha_c = \frac{\Delta C/C}{\delta}, \quad (2.58)$$

relating the circumference of the ring  $C$  to the momentum offset  $\delta$ . A positive momentum compaction factor indicates a longer path traveled by particles with a positive momentum offset, and vice versa.

The momentum compaction factor can also be broken down into orders as an infinite sum, where the constant term is often referred to as the first order,

$$\alpha_c = \underbrace{\alpha_{c,0}}_{\text{Constant term}} + \underbrace{\sum_{i \geq 1} \alpha_{c,i} \delta^i}_{\text{Linear and non linear terms}}. \quad (2.59)$$

In the LHC, the contribution from the non constant terms is negligible [19]. Further details can be found in Section 5.2.1.

## 2.5. Detuning Effects

### 2.5.1. Chromaticity

Chromaticity is the tune change  $\Delta Q$  relative to the momentum offset  $\delta$ . Chromaticity can be described by a Taylor expansion, given by

$$Q(\delta) = Q_0 + Q' \delta + \frac{1}{2!} Q'' \delta^2 + \frac{1}{3!} Q''' \delta^3 + O(\delta^4). \quad (2.60)$$

Or, more generally, rewritten as a sum to include all orders up to  $n$ :

$$Q(\delta) = Q_0 + \sum_{i=1}^n \frac{1}{i!} Q^{(i)} \delta^i. \quad (2.61)$$

The first order of the chromaticity expansion,  $Q'$ , is generally simply referred to as *chromaticity*, sometimes as *linear chromaticity*. The other terms are thus referred to as *non-linear chromaticity*. It is to be noted when referring to a chromaticity order, that the preceding fraction and  $\delta$  are usually *not* included in the term.

The chromaticity change induced by a single element of order  $n$  and length  $L$  can be derived from the Hamiltonian of Eq. (2.6), averaging over the phase variables and differentiating relative to the actions  $J_{x,y}$  and the momentum offset  $\delta$ :

$$\Delta Q_{x,y}^{(n)} = \frac{\partial^n Q_{x,y}}{\partial^n \delta} = \frac{1}{2\pi} \int_L \left\langle \frac{\partial^{(n+1)} H}{\partial J_{x,y} \partial^n \delta} \right\rangle ds. \quad (2.62)$$

Detailed derivations can be found in [30].

**Natural Chromaticity from Quadrupoles** In a purely linear lattice, the linear chromaticity,  $Q'$ , is a result of the momentum dependence of the quadrupoles' focusing. It is in this case called the *natural chromaticity* and can be derived from the normal hamiltonian of Eq. (2.7) and expressing the normalized magnet strength  $K$  with a dependence on  $\delta$  via  $P$  as  $P_0(1 + \delta)$ :

$$K_n = \frac{q}{P_0} \frac{1}{1 + \delta} (n - 1)! B_n \quad (2.63)$$

The normal field of a quadrupole is then given by

$$\mathcal{N}_2(x, y) = \frac{1}{2} \frac{q}{P_0} \frac{1}{1 + \delta} B_2(x^2 - y^2) \quad (2.64)$$

By operating a variable change to the angle coordinates ( $x \rightarrow \sqrt{2J_x\beta_x} \cos \phi_x$  and  $y \rightarrow \sqrt{2J_y\beta_y} \cos \phi_y$ ), the following equation linking the  $\beta$ -function and  $\delta$  to the normal field is obtained:

$$\mathcal{N}_2(x, y) = \frac{1}{2} \frac{q}{P_0} \frac{1}{1+\delta} B_2 \left[ \left( \sqrt{2J_x\beta_x} \cos \phi_x \right)^2 - \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^2 \right]. \quad (2.65)$$

Following Eq. (2.62), the natural chromaticity  $Q'$  induced by quadrupoles is given by:

$$\begin{aligned} \Delta Q'_x &= \frac{1}{2\pi} \int_L \frac{\partial^2 \langle \mathcal{N}_2 \rangle}{\partial J_x \partial \delta} ds \quad ; \quad \Delta Q'_y &= \frac{1}{2\pi} \int_L \frac{\partial^2 \langle \mathcal{N}_2 \rangle}{\partial J_y \partial \delta} ds \\ &= -\frac{1}{4\pi} \frac{q}{P_0} B_2 L \beta_x & &= \frac{1}{4\pi} \frac{q}{P_0} B_2 L \beta_y \end{aligned} \quad (2.66)$$

**Linear Chromaticity from Sextupoles** The first order chromaticity  $Q'$  is contributed to by sextupoles in the presence of off-momentum particles. The normal field of a sextupole, following Eq. (2.7) is given by

$$\mathcal{N}_3(x, y) = \frac{1}{3!} (x^3 - 3xy^2). \quad (2.67)$$

As the momentum offset  $\delta$  introduces a change in orbit via dispersion [31], a variable change can be operated on both  $x$  and  $y$ , as shown in Eq. (2.68). In this thesis, vertical dispersion will be though neglected.

$$\begin{aligned} x &\rightarrow x + \Delta x = x + D_x \delta \\ y &\rightarrow y + \Delta y = y + D_y \delta \end{aligned} \quad (2.68)$$

The positions  $x$  and  $y$  can once be replaced, using the twiss parameters, giving the full expression:

$$\begin{aligned} \mathcal{N}_3(x + \Delta x, y) &= \frac{1}{6} K_3 \left[ \left( \sqrt{2J_x\beta_x} \cos \phi_x \right)^3 \right. \\ &\quad + 3 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right)^2 D_x \delta \\ &\quad + 3 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right) D_x^2 \delta^2 \\ &\quad + D_x^3 \delta^3 \\ &\quad - 3 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right) \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^2 \\ &\quad \left. - 3D_x \delta (\sqrt{2J_y\beta_y} \cos \phi_y)^2 \right] \end{aligned} \quad (2.69)$$

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Averaging over the phase variables removes any odd cosine:

$$\begin{aligned} \langle \mathcal{N}_3(x + \Delta x, y) \rangle &= \frac{1}{6} K_3 \left( 3 J_x \beta_x D_x \delta \right. \\ &\quad + D_x^3 \delta^3 \\ &\quad \left. - 3 D_x \delta J_y \beta_y \right) \end{aligned} \quad (2.70)$$

The chromaticity can then be obtained by differentiating relative to the action  $J_{x,y}$  to retrieve the tune, and finally by the momentum offset  $\delta$ .

$$\begin{aligned} \Delta Q'_x &= \frac{1}{2\pi} \int_L \frac{\partial^2 \langle \mathcal{N}_3 \rangle}{\partial J_x \partial \delta} ds \quad ; \quad \Delta Q'_y &= \frac{1}{2\pi} \int_L \frac{\partial^2 \langle \mathcal{N}_3 \rangle}{\partial J_y \partial \delta} ds \\ &= \frac{1}{2\pi} L \frac{1}{2} K_3 \beta_x D_x && = -\frac{1}{2\pi} L \frac{1}{2} K_3 \beta_y D_x \\ &= \frac{1}{4\pi} K_3 L \beta_x D_x && = -\frac{1}{4\pi} K_3 L \beta_y D_x \end{aligned} \quad (2.71)$$

From this last equation, it is apparent that sextupoles are not a source of chromaticity of higher orders in the presence of linear dispersion. In the presence of second order dispersion [31], sextupoles can generate some amount of  $Q''$ , usually negligible.

**Non-Linear Chromaticity** Higher orders of the chromaticity function are described in [30] and follow the same logic as for the linear chromaticity from sextupoles. A general formula can be found for the chromaticity of order  $n, n > 2$ :

$$\begin{aligned} \Delta Q_x^{(n)} &= \frac{1}{4\pi} K_{n+2} L \beta_x D_x^n \\ \Delta Q_y^{(n)} &= -\frac{1}{4\pi} K_{n+2} L \beta_y D_y^n \end{aligned} \quad (2.72)$$

### 2.5.2. Amplitude Detuning

Amplitude detuning is a tune shift induced by the amplitude of oscillations of a particle. This detuning is directly related to the emittance and can be described via a Taylor expansion around the emittance of both planes,  $\epsilon_x$  and  $\epsilon_y$ . Equation (2.73) shows this expansion up to the second order. Further expansions can be found in [30].

$$Q_z(\epsilon_x, \epsilon_y) = Q_{z0} + \left( \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y \right) + \frac{1}{2!} \left( \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 \right) + \dots \quad (2.73)$$

The first order terms of amplitude detuning are generated by octupoles, and to some extent by sextupoles when considering their higher order contributions. Those higher contributions are usually measurable but small compared to the ones of normal octupoles. It is to be noted that each order does not correspond directly to a multipole order, like for chromaticity seen previously. While it is the case for the simple partial derivatives, the crossterms are instead generated by multipoles of higher orders. Further derivations can be found in Appendix B.

### 2.5.3. Chromatic Amplitude Detuning

Similar to amplitude detuning, *chromatic amplitude detuning* is a tune shift induced by the amplitude of oscillations of a particle but with an additional dependence on the momentum offset. This effect can be described by a Taylor expansion around the emittance of both planes  $\epsilon_x$ ,  $\epsilon_y$ , and the momentum offset  $\delta$ . Eq. (2.74) shows this expansion up to the second order. Both the emittance  $\epsilon$  and the action  $J$  can be seen to describe the chromatic amplitude detuning. Terms are interchangeable with  $\epsilon_{x,y} = 2J_{x,y}$ .

$$Q_z(\epsilon_x, \epsilon_y, \delta) = Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] + \dots \quad (2.74)$$

**Sextupolar contributions** To the first order, the terms of the chromatic amplitude detuning are shared with the classic amplitude detuning, which are not contributed to by sextupoles. The last term however is the linear chromaticity, seen previously in 2.5.1.

**Octupolar contributions** Similar to the sextupolar contributions, to the first order, the terms are shared with amplitude detuning. The first terms  $\frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x$  and  $\frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y$  are then contributed to by octupoles. The second order chromaticity  $Q''$  appears when expanding to the second order.

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**Decapolar contributions** So far, only terms with amplitude detuning and chromaticity have been seen. The terms highlighted in orange, in Eq. (2.75), are the terms contributed to by decapoles. Terms depending on both the emittance and the momentum offset are present, as well as the third order chromaticity  $Q'''$ .

$$\begin{aligned}
 Q_z(\epsilon_x, \epsilon_y, \delta) = & Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\
 & + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\
 & \quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] \\
 & + \frac{1}{3!} \left[ \frac{\partial^3 Q_z}{\partial \delta^3} \delta^3 + \frac{\partial^3 Q_z}{\partial \epsilon_x^3} \epsilon_x^3 + \frac{\partial^3 Q_z}{\partial \epsilon_y^3} \epsilon_y^3 + \dots \right]
 \end{aligned} \tag{2.75}$$

Further derivations can be found in Appendix B.

### 2.5.4. Feed-Down

When a particle passes off-center through a magnet, an effect called *feed-down* appears. Feed-down is a lower order contribution created by either mis-aligned magnets or an off-center orbit of the beam. A particle with an orbit offset will then experience the main field of the magnet and effects similar to those of lower order multipoles.

Opposed to feed-down is feed-up, the higher order effects introduced by combinations of multipoles. This effect is though not a consequence of the beam position in a given field. This effect is explained in great details in Appendix A.

## 2.6. Resonances

### 2.6.1. Tune Diagram

The resonances discussed in this thesis are related to the optics of the accelerator. Such resonances create unstable motion and can lead to loosing particles. Those perturbations arise from particles oscillating at frequencies excited by certain multipoles.

Fig. 2.9 shows a tune diagram where the fractional part of tunes  $Q_x$  and  $Q_y$  can be related to resonance lines excited by multipoles up to decapoles ( $n = 5$ ). It becomes apparent that the diagram fills quickly when considering further orders. Thankfully, the higher the multipole order, the weaker



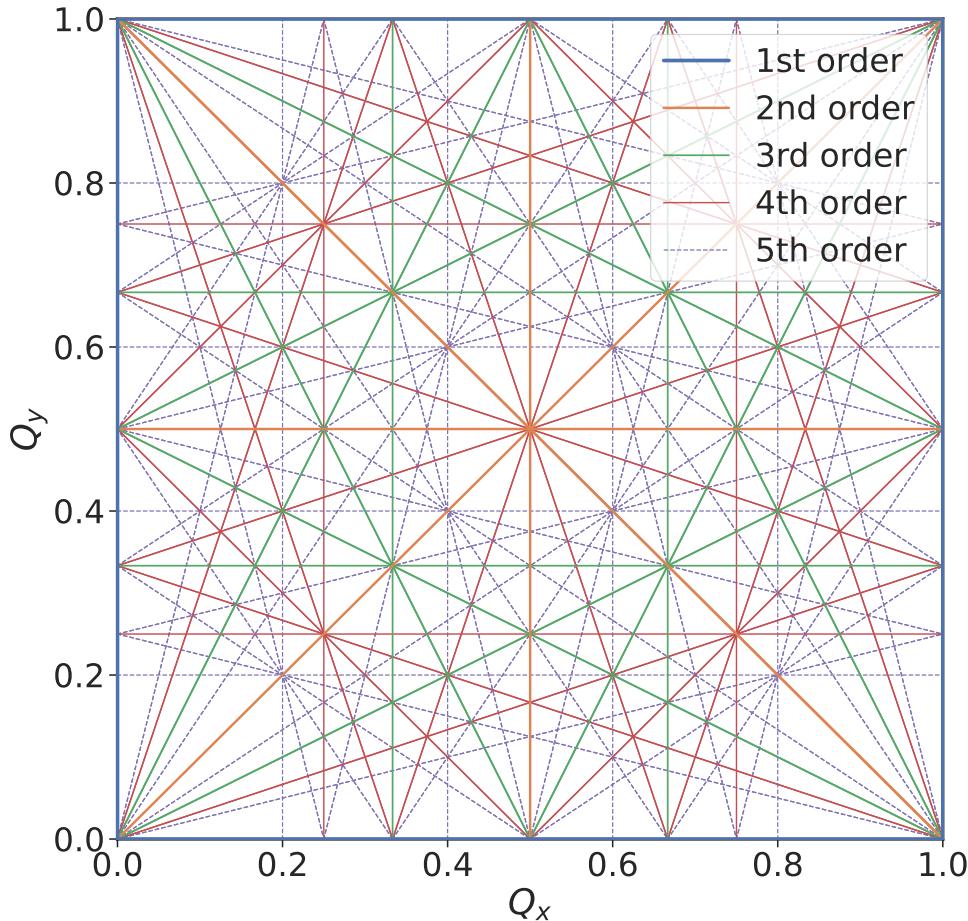


Figure 2.9.: Tune diagram with resonances lines excited by multipoles up to decapole ( $n \leq 5$ ). The working point of the machine is chosen in an area where few lines are present.

the resonances are. This makes choosing a working point possible, even if some particles are hitting resonance lines.

When considering the resonance driving terms  $f_{jklm}$  from Eq. (2.38), it can be noted that the term diverges for particular tune values. This leads to a disproportionate increase in particles position in phase-space, eventually leading to loosing them. Resonant conditions due to the tunes can thus be described by the following condition:

$$(j - k)Q_x + (l - m)Q_y = p \quad , \quad j, k, l, m, p \in \mathbb{Z}. \quad (2.76)$$

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### 2.6.2. Frequency Spectrum

As seen in Eq. (2.42), resonance driving terms have an impact on the transverse position of a particle. This means that performing a FFT on such a signal will reveal spectral lines in the frequency spectrum. Each RDT  $f_{jklm}$  can thus be observed in either one or both the frequency spectrums of the horizontal and vertical planes, at multiples of  $Q_x \pm Q_y$ . Eq. (2.77) shows where those lines would appear:

$$\begin{aligned} H_{jklm} &\quad \text{at } (1 - j + k)Q_x + (m - l)Q_y \quad ; \quad j \neq 0 \\ V_{jklm} &\quad \text{at } (k - j)Q_x + (1 - l + m)Q_y \quad ; \quad l \neq 0. \end{aligned} \quad (2.77)$$

The RDT  $f_{3000}$  coming from sextupoles can for example be seen in the horizontal spectrum at  $(1 - 3 + 0)Q_x + (0 - 0)Q_y = -2Q_x$ . For a value  $Q_x = 0.27$ , the line is seen at 0.46. in a spectrum bound in  $[0, 0.5]$ . No line can be seen in the vertical spectrum due to  $l = 0$ . Detailed tables of such lines for RDTs up to order 6 can be found in Appendix C.

The amplitude of each line will depend on the action  $I_z$  and the amplitude of the RDT [32]:

$$\begin{aligned} |H_{f_{jklm}}| &= 2j(2I_x)^{\frac{j+k-1}{2}}(2I_y)^{\frac{l+m}{2}}|f_{jklm}| \\ |V_{f_{jklm}}| &= 2l(2I_x)^{\frac{j+k}{2}}(2I_y)^{\frac{l+m-1}{2}}|f_{jklm}|. \end{aligned} \quad (2.78)$$

### 2.6.3. Resonance Driving Terms

By reworking the previous Eq. (2.78), it can be seen that RDTs are factors of the line amplitude and the actions  $I_x$  and  $I_z$ :

$$\begin{aligned} |f_{jklm}| &= \frac{|H_{f_{jklm}}|}{2j(2I_x)^{\frac{j+k-1}{2}}(2I_y)^{\frac{l+m}{2}}} \\ |f_{jklm}| &= \frac{|V_{f_{jklm}}|}{2l(2I_x)^{\frac{j+k}{2}}(2I_y)^{\frac{l+m-1}{2}}}. \end{aligned} \quad (2.79)$$

In practice, an approximation of  $J = I$  is done. The RDT is then related to the fit of the line amplitude versus the action.

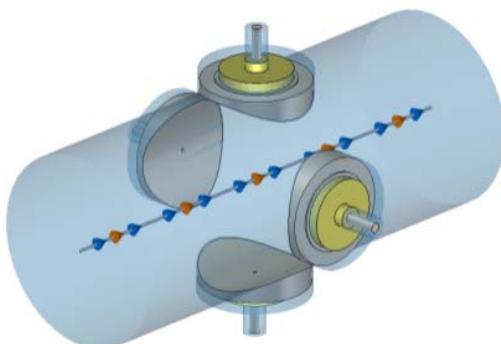


# Optics Measurements and Corrections

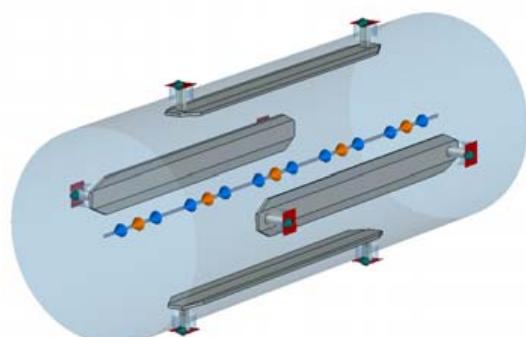
## 3.1. Beam Instrumentation

### 3.1.1. Beam Position Monitors

Beam Position Monitors (BPMs) are one of the most utilized and essential elements of beam diagnostics in particle accelerators. In the LHC, most of the BPMs are dual plane, and thus composed of four electrodes, distributed as two per plane. The BPM system consists of over than 550 BPMs per beam, distributed along the ring, in the arcs and the IPs. The most common type, the *curved-button*, shown in Fig. 3.1a, is typically placed near quadrupoles [33].



(a) Curved-button "BPM" type BPM of the LHC [33].



(b) Stripline "BPMSW" type BPM of the LHC [33].

Other pickups such as the *stripline*, shown in Fig. 3.1b, albeit more complex and expensive, offer a better signal to noise ratio and are capable of identifying the direction of the beam [33]. Such features

### 3. Optics Measurements and Corrections

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are essential for the LHC, were both beams travel through the same aperture at the IPs. The BPM response is not linear with the beam position, which requires a post-processing not systematically implemented in accelerators beam diagnostics systems. LHC's BPMs have been simulated and polynomials fitted to minimize this response error [34].

#### 3.1.2. Collimators

Collimators are a crucial part of the LHC. Their purpose is to protect the machine against beam losses and clean the outer parts of the beam [35]. The energy of the beams in the LHC is high enough to not only quench the magnets, but to also damage the elements.

At injection energy, with a low intensity pilot bunch, the consequences of a loss are though less severe. During Run 3, in 2022, a new collimator sequence was introduced, making a safe exploitation of the machine possible with more retracted collimators. This made measurements with higher kick amplitudes and larger orbit offsets, and thus momentum offsets, possible.

#### 3.1.3. Beam Loss Monitors

Beam Loss Monitors are detectors mounted on various elements of the accelerator, such as magnets or collimators, to detect abnormal losses of particles. They play a crucial role in the protection of the machine, triggering a dump when losses exceed the threshold set for their respective element. BLMs use ionization chambers, working on the same principle as simple Geiger counters: a tube filled with gas, in presence of a high voltage.

Dashboards in the control room are regularly used to monitor the losses along the ring when performing optics measurements, as those prove to often be destructive.

#### 3.1.4. AC-Dipole

The AC dipole of the LHC is a crucial component for optics studies. Its primary function is to excite the beam into large coherent oscillation, achieved by applying a sinusoidally oscillating dipole field [36]. By ramping up and down adiabatically the field, large coherent oscillations can be produced without any decoherence or emittance growth. Figure 3.2 shows an example of a measurement made with an AC-Dipole. Exciting the beam to large amplitudes make the study of linear optics, such as beta-beating easier, and that of non linear optics such as resonances possible.

The AC-Dipole is set to oscillate at a frequency  $Q_d$ , different from the natural tune of the machine  $Q$  and thus introduces systematic effects that needs to be compensated during the optics analysis. The new orbit of a particle under the influence of the AC-Dipole, at turn number  $n$  and observation point  $s$ , is given by [37]:

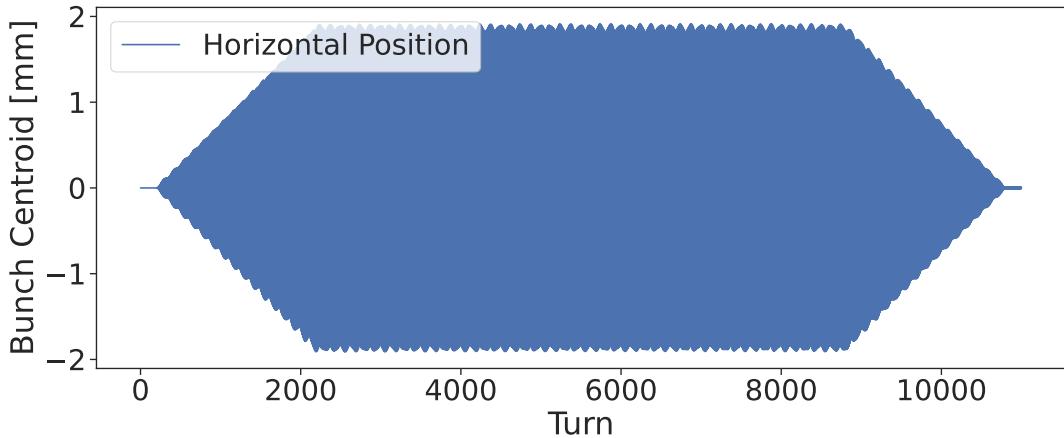


Figure 3.2.: Simulated turn by turn data with an AC-Dipole first ramping up then down.

$$z(s, n) = \frac{BL}{4\pi\rho\delta_z} \cdot \sqrt{\beta_z(s)\beta_{z,0}} \cdot \cos(2\pi Q_{d,z}n + \phi_z(s) + \phi_{z,0}), \quad (3.1)$$

where  $B$  is the amplitude of the oscillating magnetic field,  $L$  the length of the AC-Dipole,  $B\rho$  the magnetic rigidity,  $\delta$  the difference between  $Q_d$  and  $Q$ ,  $\beta$  and  $\beta_0$  the beta function at the observed point and the AC-Dipole,  $\phi$  and  $\phi_0$  the phase advance at the observed point and of the AC-Dipole.

## 3.2. Correction Principles

### 3.2.1. Response Matrix

A response matrix is a linear equation system that describes the change of an observable for a set of individual multipole strengths. By taking the pseudo-inverse of this matrix and multiplying it to the measured observables, a set of corrector strengths if obtained that can replicate the measured value. Taking the opposite sign then gives a correction. This technique is routinely used to correct, amongst others,  $\beta$ -beating as well as Resonance Driving Terms. In situations where measurements are taken at each BPM for a particular observable, the corresponding response matrix ends up containing over 500 values per corrector, for a single beam.

Individual MAD-X simulations are run with a single multipole powered at a time. The resulting parameter values (e.g.  $\beta$ -beating) are then compared to those obtained from a simulation without any powering, allowing to determine the specific impact of each multipole.

A response matrix is thus created following Eq. (3.2), for a matrix of observables  $O$ , a reference matrix of observables without any corrector  $O_R$  and a fixed multipole strength  $k$ . Given measured data

### 3. Optics Measurements and Corrections

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$M$ , the set of correctors needed to compensate the values can be obtained by taking the pseudo-inverse of the matrix in Eq. (3.3).

$$R = (O - O_R) \cdot \frac{1}{k} \quad (3.2)$$

3

$$\begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = -(R^+ \cdot M) \quad (3.3)$$

Response matrices are very versatile and can combine several observables to be corrected by the same multipoles. One example, detailed later in this thesis, is the third order chromaticity and the resonance driving term  $f_{1004}$ , both contributed to by decapoles.

#### Example

In this example, simulations are run with MAD-X PTC to correct the third chromaticity in the LHC.  $Q'''$  is taken from `ptc_normal` for each beam and axis, with MCDS, decapole correctors, powered with a fixed strength one at a time. A scaling factor is applied to get the change of chromaticity for one unit of  $K_5$ . 8 correctors are used, which strengths are denoted  $k_1$  through  $k_8$ . Transposes are only used to make the equations easier to display.

The values in Tab.3.1 are corrected via Eq. (3.5) after having built the response matrix in Eq. (3.4).

Observable	Value
$Q_x'''$	-666111
$Q_y'''$	121557

Table 3.1.: Example chromaticity values to correct via a response matrix



$$R = \left( \text{Individual simulations} \begin{pmatrix} Q_x''' & Q_y''' \\ \underbrace{\begin{bmatrix} -155899 & 122004 \\ -254584 & 138368 \\ -122715 & 106709 \\ -218597 & 110686 \\ -134140 & 106463 \\ -245791 & 118951 \\ -147035 & 116544 \\ -219537 & 112317 \end{bmatrix}}^T & \underbrace{\begin{bmatrix} 5135 \\ 8470 \end{bmatrix}}_{\text{Reference}} \end{pmatrix} - \underbrace{\begin{bmatrix} Q_x''' \\ Q_y''' \end{bmatrix}}_{\text{Measured values}} \right) \cdot \underbrace{\frac{1}{-1000}}_{\text{Corrector strength}} \quad (3.4)$$

$$\begin{aligned} k_1 & \begin{pmatrix} -1235 \\ 1032 \\ -1394 \end{pmatrix} \\ k_2 & \begin{pmatrix} 1449 \\ -1043 \\ 1864 \\ -1187 \\ 1369 \end{pmatrix} = -R^+ \cdot \begin{pmatrix} -666111 \\ 121557 \end{pmatrix} \end{aligned} \quad (3.5)$$

### 3.2.2. Chromaticity

As per the placement of the MCO and MCD spool piece correctors in the LHC layout [38],  $\beta$ -functions at their location are slightly different from arc to arc. This slight imbalance leads theoretically to the possibility of correcting the horizontal and vertical axes of the second and third order chromaticity independently, via a response matrix approach. In practice, the required strength to do so would exceed those of the design of the correctors.

Another way to correct the chromaticity is via a global uniform trim, where every available corrector is powered to the same strength. Simulations are run with `ptc_normal` via MADX-PTC to obtain the response in chromaticity for a given strength. Chromaticity being linear with multipole strength, an affine function can be determined for each axis. Figure 3.3 shows a simulation with several MCD strengths, highlighting this linear relation between  $Q'''$  and  $K_5$ , while Equation (3.6) shows an example of such functions computed at injection energy for the 2022 optics.

### 3. Optics Measurements and Corrections

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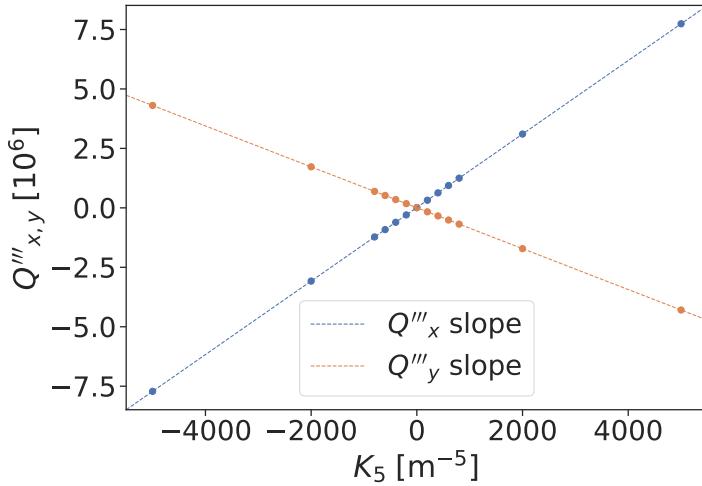


Figure 3.3.: Linear relation between the third order chromaticity and decapole corrector strengths, simulated with MADX-PTC.

$$\begin{aligned} Q'''_x &= 1533 \cdot \Delta K_5 + 6680 \\ Q'''_y &= -860 \cdot \Delta K_5 + 5647 \end{aligned} \quad (3.6)$$

Only the linear part is relevant, as the offset is generated by other multipoles and field errors. It is thus constant for a configuration where only the relevant spool pieces are used.

Corrections involve minimizing both axes, typically where  $Q'''_x$  meets  $Q'''_y$ :

$$\Delta K_5 = -\frac{(Q'''_x - Q'''_y)}{\text{slope}_{Q'''_x} - \text{slope}_{Q'''_y}} \quad (3.7)$$

## 3.3. Optics Measurements

### 3.3.1. Tools and Softwares

In order to perform the measurements, analysis and simulations presented in this thesis, various tools and softwares have been developed, used and contributed to.

Optics simulations have been done mainly in MAD-X [39] and PTC. MAD-NG [40] and Xsuite [41] have also been explored for specific tasks such as free RDT simulations and GPU tracking.

Analysis of chromaticity measurements are done via a newly developed graphical interface [42] written in Python. This tool makes cleaning of the raw signal data, its analysis and results export more



reliable and easier.

Overall, analysis of turn-by-turn measurements is supported by a large panel of libraries written by the OMC team in Python and Java. Contributions have mainly been made to extend the following packages:

- **Beta-Beat GUI** [43], Graphical interface for turn-by-turn measurements visualization and analysis.
- **OMC3** [44], Main optics analysis and corrections software.
- **Beta-Beat.src** [45], Old analysis software, now replaced by OMC3.
- **pylhc.github.io** [46], Website of the OMC team with package documentation, examples and useful resources.

### 3.3.2. Turn-by-Turn Data

Data is typically acquired turn by turn while exciting the beam with an AC-Dipole. This is usually done with a *pilot bunch*, with a reduced intensity of  $10^{10}$  protons compared to an operation bunch of  $10^{11}$ . This allow for higher amplitude oscillations while remaining safe for the machine.

A spectral analysis is then performed via a *FFT*, making apparent the driven tunes from the AC-Dipole, the transverse tunes and the possible resonance lines, as shown in Fig. 3.4.

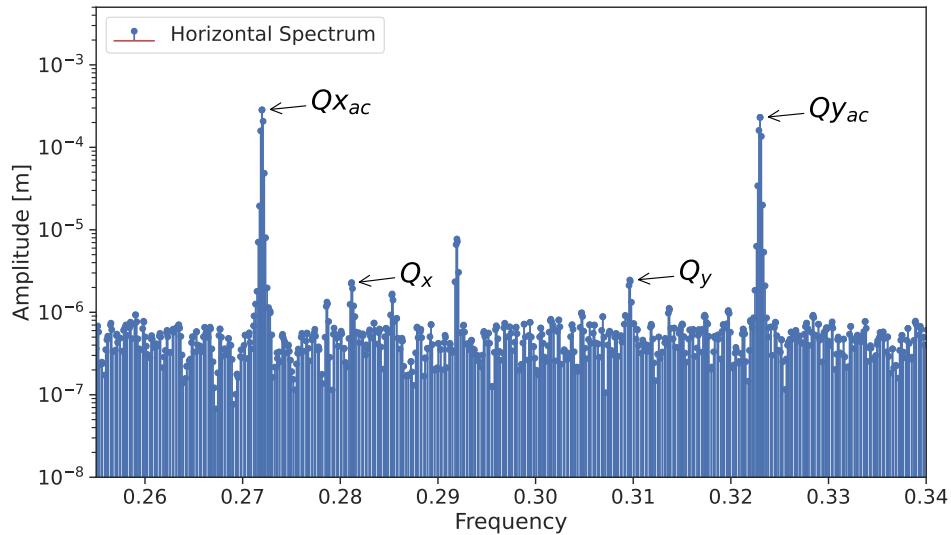


Figure 3.4.: Horizontal frequency spectrum of a turn-by-turn measurement in the LHC. The *driven* tunes of the AC-Dipoles have the highest amplitudes while the natural tune can be seen close to it. Other lines are often created by resonances.

### 3. Optics Measurements and Corrections

From this, the *optics* such as  $\beta$ -beating, dispersion, coupling and Resonance Driving Terms can be reconstructed [47].

#### 3.3.3. Chromaticity

##### Procedure

3

Chromaticity measurements are typically performed by varying the RF frequency to induce a change of momentum offset  $\delta$ , while measuring the tune. The momentum offset  $\delta$  being related to the RF frequency, the Lorentz factor  $\gamma$  and the momentum compaction factor  $\alpha_c$  [19]:

$$\delta = - \left( \frac{1}{\gamma^{-2} + \alpha_c} \right) \cdot \frac{\Delta f_{\text{RF}}}{f_{\text{RF,nominal}}} \quad (3.8)$$

In the LHC, the Lorentz factor  $\gamma$  is here negligible, as the energy is large even at injection. At 450GeV,  $\gamma^{-2} \approx 10^{-6}$ , which is two orders of magnitude smaller than  $\alpha_c$ .

Frequency steps of 20Hz every 30 secondes are usually taken to compromise between number of data points, precision of the tune estimate, and duration of the measurement. Once beam losses, registered by the BLMs are deemed too high, the frequency is reverted back to its nominal value in larger steps. The same procedure is then re-applied in the negative. Figure 3.5 shows a typical RF scan performed to measure chromaticity in the LHC.

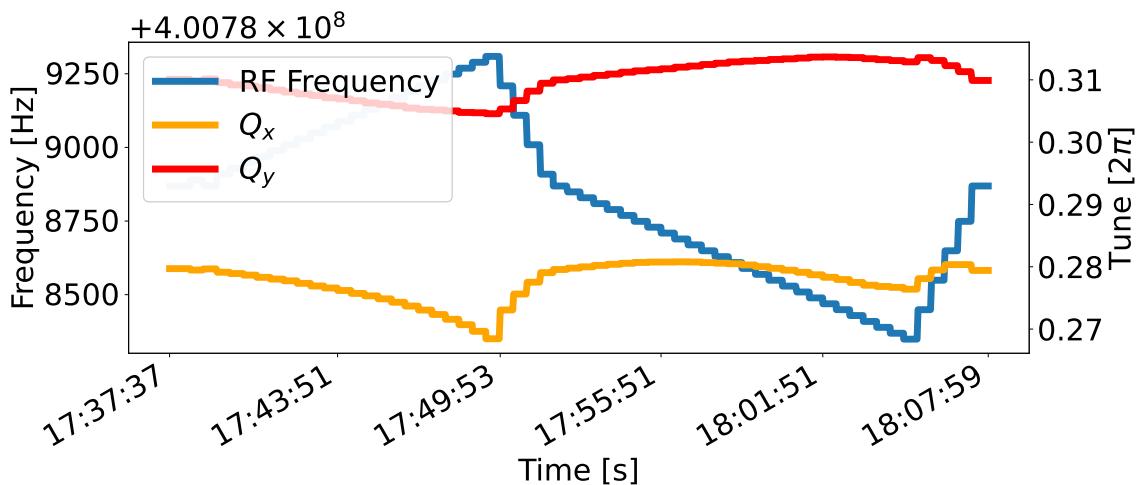


Figure 3.5.: Observation of the tune dependence on momentum offset, created by a shift of RF frequency.

## Analysis

Once the tunes have been acquired and the momentum offset computed via Eq. (3.8), the chromaticity function (see Eq. (2.60)) can be used to fit the measured data and retrieve each order.

As part of work for this thesis, a custom tool was developed, in order to ease such analysis of chromaticity measurements. This tool, the Non-Linear Chromaticity GUI [42], is composed of several parts:

- Data extraction from CERN data servers (Timber, NXCAL)
- Tune cleaning and standard deviation calculation
- Chromaticity fit up to 7th order
- Corrections of chromaticity and resonance driving terms



# Decapolar Fields

## 4.1. Introduction

### 4.1.1. Motivation

Beam-based measurements have been carried in the LHC since Run 1 to better understand the decapolar fields. Those have been carried out via chromaticity measurements [48–50]. The third order of the non-linear chromaticity,  $Q'''$ , generated for the most part by decapoles, has shown a consistent discrepancy at injection energy between its expected value from simulations and that observed. Figure 4.1 highlights this discrepancy.

The FiDeL magnetic model is used during operation to correct various multipole errors, including octupolar and decapolar. The operational corrections being based on this magnetic model and simulations, the residual  $Q'''$  value is expected to be small, which is however not the case. Chromaticity measurements have thus been repeated during LHC’s Run 3 and corrections made routine, aimed at correcting the observed discrepancy.

The study of non-linear chromaticity has proven valuable in quantifying decapolar fields, yet it does not permit alone to understand the exact origins of the observed discrepancy. In an effort to gain deeper insights, additional measurements were performed focusing on novel observables that had not been previously explored. *Bare chromaticity* involves measuring chromaticity with the octupolar and decapolar correctors deactivated ; this approach aims to isolate the machine effects from those of the correctors. *Chromatic amplitude detuning*, evaluates how the tune varies with both the beam’s action and the momentum offset ; this method has the benefit of having a different expression than that of the chromaticity.

Complementing those measurements, studies of decapolar Resonance Driving Terms have been undertaken for the first time in the LHC. Contributing to resonances close to the working, those RDTs also have benefited from corrections.

#### 4. Decapolar Fields

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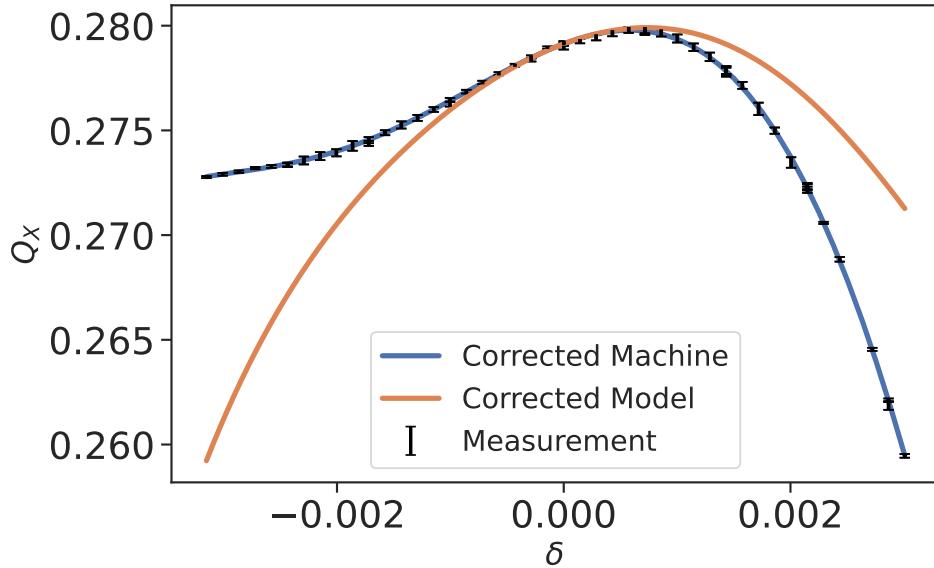


Figure 4.1.: Measured and simulated chromaticity with application of the nominal decapolar corrections from FiDeL. It can be seen that although the corrections should diminish  $Q'''$ , it is not well corrected in practice.

#### 4.1.2. Decapolar Correctors

As seen in Fig. 1.4, the LHC is equipped with decapoles. Those magnets are part of the design report, aiming at correcting the field errors of the main dipoles. Those correctors, denominated *MCD*, are specific to each beam and are placed after every second dipole, totaling 1232 in number [51]. MCDs are nested with octupolar correctors, *MCO*. The pair of those correctors often referred to as *MCDO*. It is not possible to individually power each corrector. Rather, a circuit consists of a whole arc. There are in total 16 circuits to control the correctors of both beams and 8 arcs. Fig. 4.2 shows a picture taken of decapoles on a test bench.

The important characteristics of the magnetic fields of correctors are their main field transfer function (or *response*), the field quality and possible crosstalk, as MCOs and MCDs are nested [51]. In order to better understand the decapolar fields, the decapole correctors themselves need to be studied.

**Strengths at Injection** At injection energy, the decapoles are powered to a static strength. New optics introduced throughout the years often have for effect to vary slightly the  $\beta$ -function along the ring, having thus an impact on the chromaticity, as seen in Eq. (2.72). New corrections are then computed via FiDeL to account for it. Although those corrections vary throughout the years, the shift is in practice fairly negligible. Table 4.10, a bit further in this chapter, shows the strength of the correctors and the related circuits at injection energy for the optics deployed in 2024.



Figure 4.2.: Decapoles on a test bench, being inspected after manufacturing [52].

Circuit	$K_5 [\text{m}^{-5}]$
Beam 1	
RCD.A12B1	-4582
RCD.A23B1	-5106
RCD.A34B1	-4855
RCD.A45B1	-4577
RCD.A56B1	-4125
RCD.A67B1	-5166
RCD.A78B1	-6827
RCD.A81B1	-5500
Beam 2	
RCD.A12B2	-4490
RCD.A23B2	-5155
RCD.A34B2	-4825
RCD.A45B2	-4619
RCD.A56B2	-4064
RCD.A67B2	-5066
RCD.A78B2	-6866
RCD.A81B2	-5446

Table 4.1.: Strength of decapolar correctors at injection energy for FiDeL corrections.

## 4.2. Response of correctors

**Expression** The full third term of the chromaticity function is highlighted in Eq. (4.1). Details on chromaticity are given in Section 2.5.1.

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$$Q(\delta) = Q_0 + Q'\delta + \frac{1}{2!}Q''\delta^2 + \frac{1}{3!}Q'''\delta^3 + O(\delta^4). \quad (4.1)$$

This third order, mainly contributed to by decapoles, is related to the  $\beta$ -function, the dispersion and the strength of the multipole:

$$\begin{aligned}\Delta Q_x''' &= \frac{1}{4\pi} K_5 L \beta_x D_x^3 \\ \Delta Q_y''' &= - \frac{1}{4\pi} K_5 L \beta_x D_x^3.\end{aligned} \quad (4.2)$$

**Measurements and Corrections** In order to assess the accuracy of corrections, measurements have to be done to gauge the response of the decapolar correctors, *MCDs*. During Run 3's commissioning, measurements and corrections of  $Q''$  and  $Q'''$  have been made routine. Those corrections give the opportunity to study the response of the correctors. Fig. 4.3 shows the chromaticity function measured during Run 3's commissioning in 2022 with the nominal corrections via FiDeL and beam-based corrections computed analytically based on top of FiDeL.



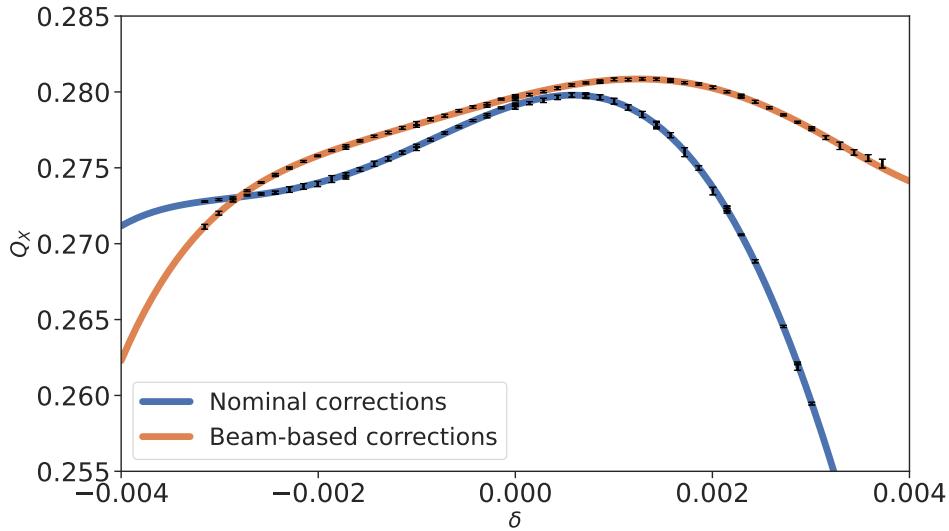


Figure 4.3.: Chromaticity of the horizontal plane of Beam 1 during Run 3's commissioning, with nominal corrections based on the magnetic model and beam-based corrections aimed at correcting  $Q''$  and  $Q'''$ .

The nominal and corrected  $Q'''$  values are shown in Table 4.2. The shift in  $Q'''$  is shown in Table 4.3 for each beam and axis, and is showing a good agreement between the measurements and simulations. The slight imbalance can be attributed to higher order effects of the octupole correctors, whose correction was implemented after that of  $Q'''$ .

Plane	$Q'''[10^6]$	
	Nominal	Beam-Based
Beam 1		
X	$-3.36 \pm 0.04$	$-1.02 \pm 0.03$
Y	$1.62 \pm 0.05$	$0.12 \pm 0.02$
Beam 2		
X	$-2.72 \pm 0.08$	$-0.64 \pm 0.03$
Y	$1.54 \pm 0.06$	$0.14 \pm 0.03$

Table 4.2.: Third order chromaticity obtained during Run 3 commissioning, with nominal and beam-based corrections aimed at correcting  $Q''$  and  $Q'''$ .

#### 4. Decapolar Fields

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		$\Delta Q'''[10^6]$	
Plane	Measurement	Simulation	
Beam 1			
X	$2.3 \pm 0.1$	2.5	
Y	$-1.5 \pm 0.1$	-1.4	
Beam 2			
X	$2.1 \pm 0.1$	2.5	
Y	$-1.4 \pm 0.1$	-1.4	

Table 4.3.: Response of the third order chromaticity upon application of corrections on top of the nominal ones used in operation for both measurements and simulations.

4

This agreement between the simulations and the measurements indicates that our decapole correctors function as intended.

### 4.3. Bare Chromaticity

Previous studies [50] have demonstrated that octupole and decapole correctors were contributing to an observed octupolar discrepancy in the machine via hysteresis and feed-down. To evaluate the possible effect of decapole correctors on the third order chromaticity  $Q'''$ , a measurement was taken with these elements turned off.



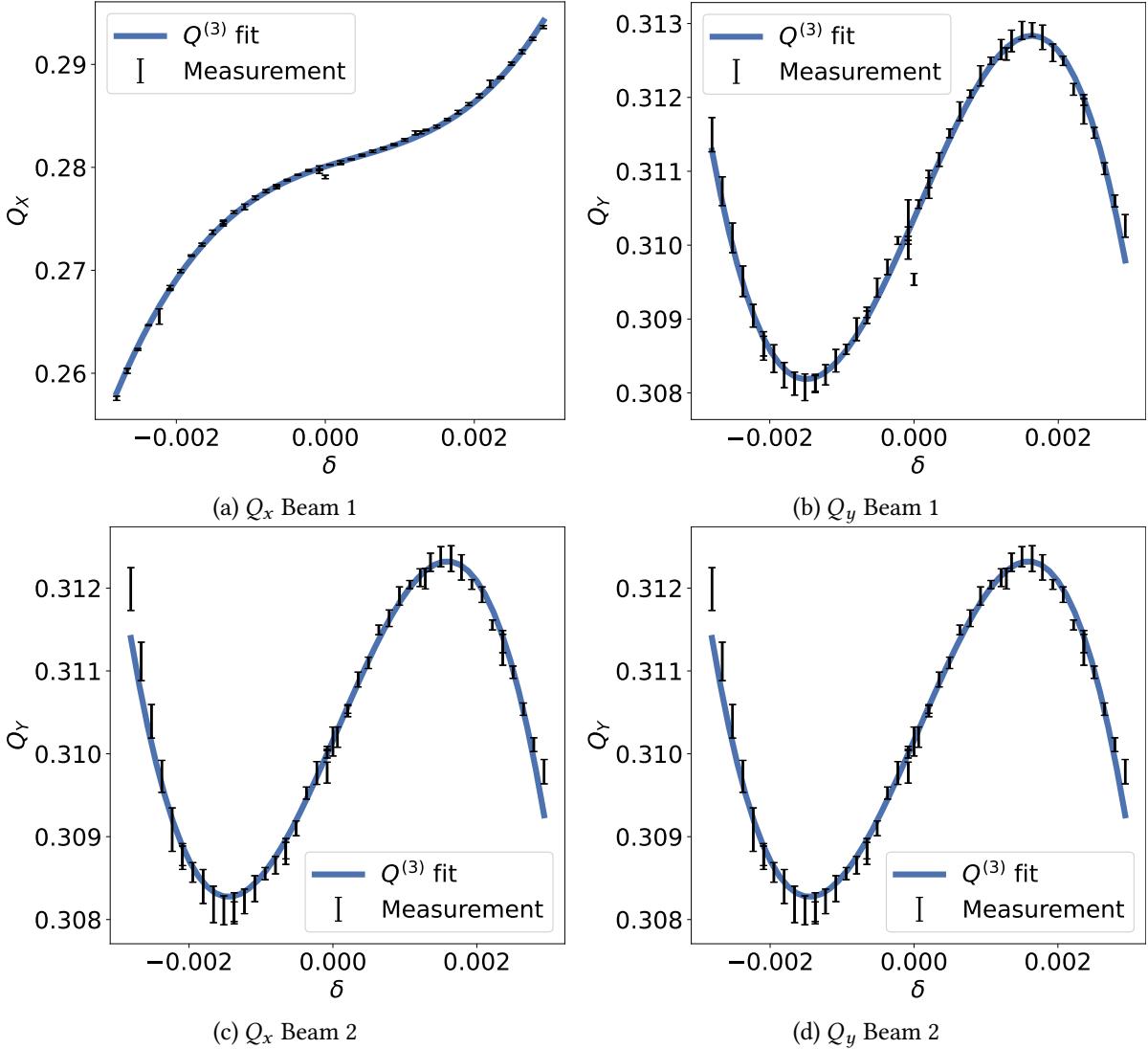


Figure 4.4.: Fit of the chromaticity function for the chromaticity measurement performed with octupole and decapole correctors powered off. The fit includes all orders up to third.

Simulations have been run with MAD-X and PTC including fields errors from normal sextupole to decahexapole. The expected  $Q''$  values are presented in Table 4.4 and compared to the measured ones along with the ratio between the two.

A consistent ratio is observed for every beam and plane between the measurement and the model. This result, supplemented by the correct response of the correctors, indicates that the decapolar correctors do no generate unwanted fields. Those correctors can thus be discarded as the potential source of discrepancy.



#### 4. Decapolar Fields

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Quantity	Measured [ $10^6$ ]	Simulated [ $10^6$ ]	Ratio
Beam 1			
$Q_x'''$	$2.95 \pm 0.04$	$6.94 \pm 0.02$	$0.43 \pm 0.01$
$Q_y'''$	$-1.82 \pm 0.04$	$-4.29 \pm 0.01$	$0.42 \pm 0.01$
Beam 2			
$Q_x'''$	$3.06 \pm 0.07$	$7.03 \pm 0.02$	$0.44 \pm 0.01$
$Q_y'''$	$-1.72 \pm 0.02$	$-4.27 \pm 0.01$	$0.42 \pm 0.01$

Table 4.4.: Measured and simulated third order chromaticity with octupole and decapole correctors turned off. The simulations include field errors from sextupoles to decahexapole (normal sextupole to decahexapole).

4

## 4.4. Chromatic Amplitude Detuning

The Chromatic Amplitude Detuning is the tune shift dependant on both the actions and the momentum offset, whose decapole contributed terms are described via a Taylor expansion in Eq. (4.3). More information and derivations can be found in Section 2.5.3 and Appendix B.

$$\Delta Q(J_x, J_y, \delta) = \frac{\partial^2 Q}{\partial J_x \partial \delta} \cdot J_x \delta + \frac{\partial^2 Q}{\partial J_y \partial \delta} \cdot J_y \delta + \frac{1}{3!} \frac{\partial^3 Q}{\partial \delta^3} \cdot \delta^3. \quad (4.3)$$

The last term is more commonly referred to as the third order chromaticity,  $Q'''$ . Each of those terms depend on the  $\beta$ -functions, the horizontal dispersion  $D$  and the normalized decapole field gradient  $K_5$  for a single source of length  $L$ ,

$$\begin{aligned} \frac{\partial^2 Q_x}{\partial J_x \partial \delta} &= \frac{1}{16\pi} K_5 L \beta_x^2 D, & \frac{\partial^2 Q_x}{\partial J_y \partial \delta} &= -\frac{1}{8\pi} K_5 L \beta_x \beta_y D, \\ \frac{\partial^3 Q_x}{\partial \delta^3} &= \frac{1}{4\pi} K_5 L \beta_x D^3, & \frac{\partial^2 Q_y}{\partial J_x \partial \delta} &= -\frac{1}{8\pi} K_5 L \beta_x \beta_y D, \\ \frac{\partial^2 Q_y}{\partial J_y \partial \delta} &= \frac{1}{16\pi} K_5 L \beta_y^2 D, & \frac{\partial^3 Q_y}{\partial \delta^3} &= -\frac{1}{4\pi} K_5 L \beta_y D^3. \end{aligned} \quad (4.4)$$

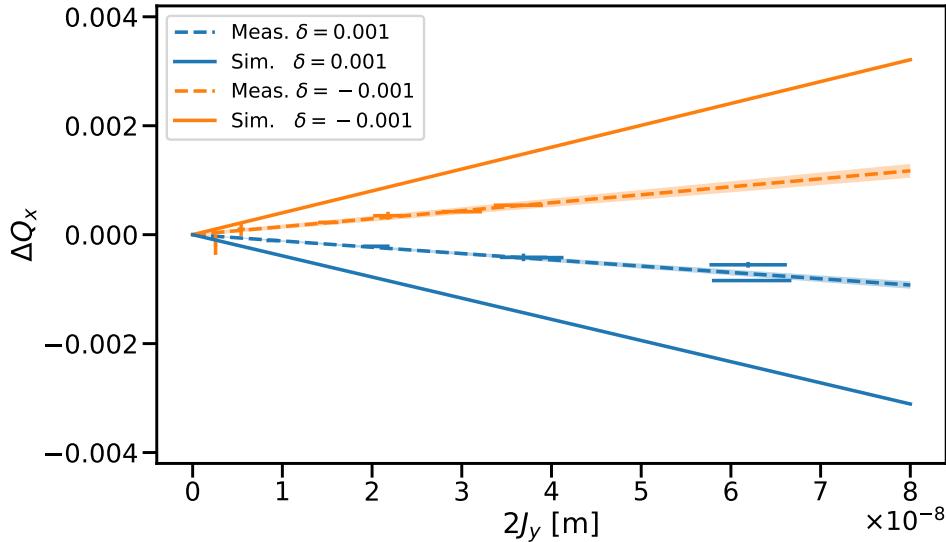
The action dependant terms can be measured by exciting the beam with an AC-dipole with increasing strengths at different momentum-offsets.

Such a measurement was taken with octupole and decapole correctors turned off to measure the bare machine. Some data could not be collected due to machine availability issues, restricting the measurement to low intensity kicks. Nevertheless, the terms  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$  and  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$  for beam 2 were measured for the first time in the LHC. The momentum-offsets measured at were  $-0.001$  and  $0.001$ , respectively roughly equal to a trim of  $+140\text{Hz}$  and  $-140\text{Hz}$  of the RF.

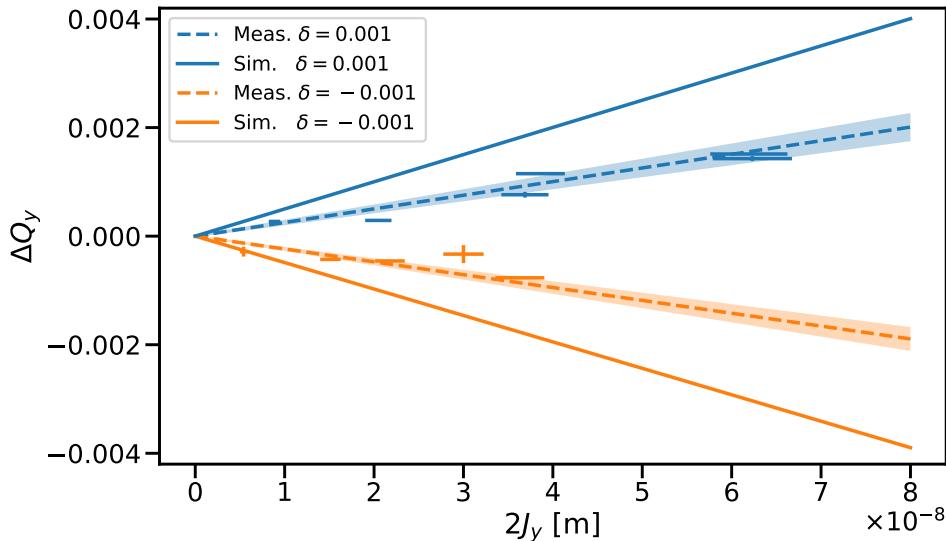


#### 4.4. Chromatic Amplitude Detuning

Fig. 4.5a and Fig. 4.5b show a fit of those terms to measured  $Q_{x,y}$  vs  $J_y$  at two different momentum offsets. Expected shifts from MADX-PTC simulations, that include field errors ranging from sextupoles to decahexapoles ( $b_3$  to  $b_8$  and  $a_4$  to  $a_8$ ) are shown as a comparison.



(a) Horizontal tune shift depending on the vertical action:  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$ .



(b) Vertical tune shift depending on the vertical action:  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$ .

Figure 4.5.: Measured and simulated tune shift due to a change of action via an AC-Dipole at two different momentum offsets. Each fit corresponds to a chromatic amplitude detuning term evaluated at a certain  $\delta$ .



#### 4. Decapolar Fields

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Type	$\frac{\partial^2 Q_x}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$	$\frac{\partial^2 Q_y}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$
$\delta = +0.001$		
Meas.	$-1.16 \pm 0.08$	$1.26 \pm 0.15$
Sim.	$-3.82 \pm 0.01$	$2.47 \pm 0.01$
Ratio	$0.30 \pm 0.02$	$0.51 \pm 0.06$
$\delta = -0.001$		
Meas.	$1.47 \pm 0.12$	$-1.18 \pm 0.13$
Sim.	$3.92 \pm 0.01$	$-2.41 \pm 0.01$
Ratio	$0.38 \pm 0.03$	$0.49 \pm 0.05$

Table 4.5.: Comparison of the measured and simulated terms  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$  and  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$  via PTC, at two discrete momentum offsets. Simulations include errors from normal sextupole to decahexapole and from skew octupole to decahexapole.

A consistent difference between simulation and measurement is observed, which values and ratios of measurement to model can be found in Table 4.5. The observed ratios of measurement to model for the chromatic amplitude detuning show slight discrepancies compared to the bare chromaticity ones. These discrepancies could be due to the low intensity kicks, which don't allow for a better fit. However, the similarity of the ratios suggests an issue with the decapolar error model of the main dipoles, with measurements showing values about half of those predicted by the magnetic model.

## 4.5. Accounting for Decay

FiDeL, the field model used in operation, aims at correcting the field errors at various energies. Measurement field errors but also decay estimates with respect to time are used when computing corrections. Decay happens after cycles in the magnets powering. This happens for example when ramping down from an energy of 6.8TeV to 0 and then going back to injection energy at 450GeV. Some components of the magnetic field will persist and gradually change over time until they reach their final value.

Such corrections thus vary with time as decay changes the field errors of the magnets. For the main dipoles, the sextupolar component  $b_3$  increased relative to time while the  $b_5$  component decreases [14].

While sextupolar decay is used by FiDeL, it has been observed that this was not the case for the decapolar component  $b_5$ . The average  $b_5$  field in the main dipoles, taken

Time [m]	$\Delta b_5$
17	-0.38
33	-0.44
50	-0.46
67	-0.47
83	-0.47
167	-0.47

Table 4.6.: Decay of the  $b_5$  component after injection for long time periods [14].

from the WISE tables and used by FiDeL, is about  $1.145 \pm 0.5$ . Decay measurements for various elements relative to time can be seen in Fig. 4.6. Table 4.6 shows the decay expected at injection energy after a typical cycle of machine operation. From this table, it is clear that decay accounts for a large part of the  $b_5$  at injection.  $\approx 40\%$  of the compensated  $b_5$  at injection in the main dipoles can indeed attributed to the decay.

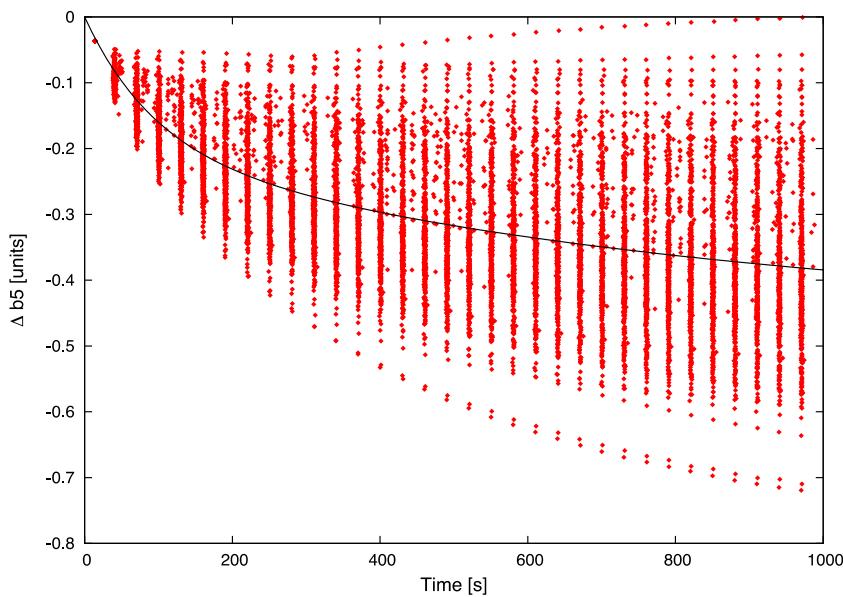


Figure 4.6.: Measured decay of the integrated decapolar field in LHC's main dipoles at injection energy. The fit is shown in black [14].

**Chromaticity** It has been shown in the previous sections, and particularly in Table 4.4 that measurements and simulations were off regarding the third order chromaticity  $Q'''$ . New simulations have hence been conducted to evaluate the effect of this decay on chromaticity. Table 4.7 compares simulations with and without the decay of the  $b_5$  component. The simulations use the 100 available error seeds to calculate error bars.

Integrating the decay of the  $b_5$  component of the main dipoles in the simulations shows a large reduction of the third order chromaticity  $Q'''$ . Using the values from the bare chromaticity measurement and updating Table 4.4 with the newly obtain data provides a more accurate comparison of the model to the measurement, as shown in Table 4.8.

#### 4. Decapolar Fields

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Condition	$Q_x'''[10^6]$	$Q_y'''[10^6]$
No Decay	$6.93 \pm 0.04$	$-4.32 \pm 0.02$
$\Delta b_5 = -0.47$	$4.05 \pm 0.04$	$-2.54 \pm 0.02$

Table 4.7.: Comparison of  $Q_x'''$  and  $Q_y'''$  for Beam 1 with and without decay of the  $b_5$  component of the main dipoles at injection energy. Fields errors range from normal and skew sextupoles ( $n = 3$ ) to icosapole ( $n = 10$ ).

Quantity	Measured [ $10^6$ ]	Simulated [ $10^6$ ]	Ratio
Beam 1			
$Q_x'''$	$2.95 \pm 0.04$	$4.05 \pm 0.04$	$0.73 \pm 0.01$
$Q_y'''$	$-1.82 \pm 0.04$	$-2.54 \pm 0.02$	$0.72 \pm 0.02$
Beam 2			
$Q_x'''$	$3.06 \pm 0.07$	$4.27 \pm 0.03$	$0.72 \pm 0.02$
$Q_y'''$	$-1.72 \pm 0.02$	$-2.55 \pm 0.01$	$0.67 \pm 0.01$

Table 4.8.: Measured and simulated third order chromaticity with octupole and decapole correctors turned off. The simulations include field errors from normal and skew sextupoles to icosapole ( $n = 3$  to  $n = 10$ ). The  $b_5$  component of the main dipoles has been updated to include decay.

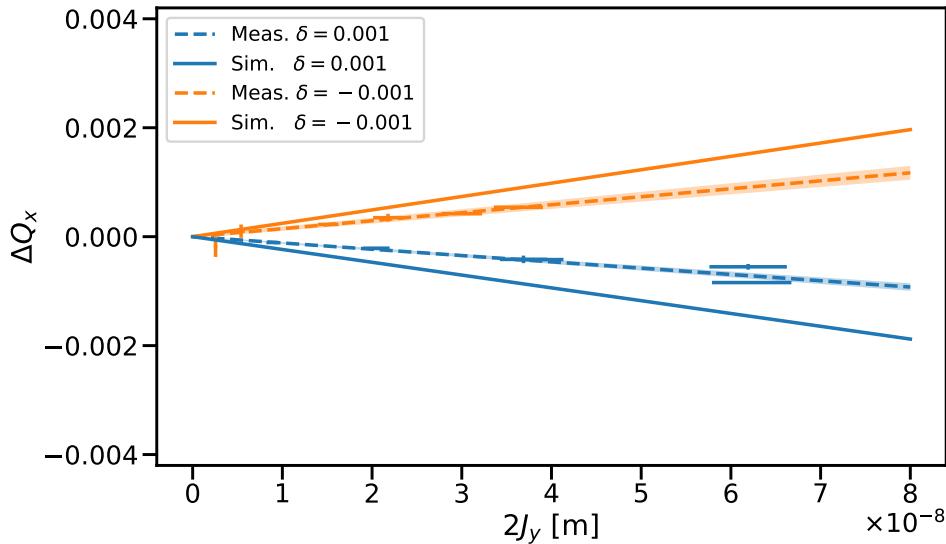
**Chromatic Amplitude Detuning** Similar to chromaticity, new simulations have been conducted for chromatic amplitude detuning. Table 4.9 gives an overview of the newly computed values and the related ratios relative to the measurements, while Fig. 4.7 gives a visual clue.

Type	$\frac{\partial^2 Q_x}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$	$\frac{\partial^2 Q_y}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$
$\delta = +0.001$		
Meas.	$-1.16 \pm 0.08$	$1.26 \pm 0.15$
Sim.	$-2.35 \pm 0.01$	$1.50 \pm 0.01$
Ratio	$0.49 \pm 0.03$	$0.84 \pm 0.10$
$\delta = -0.001$		
Meas.	$1.47 \pm 0.12$	$-1.18 \pm 0.13$
Sim.	$2.46 \pm 0.01$	$-1.45 \pm 0.01$
Ratio	$0.60 \pm 0.05$	$0.82 \pm 0.09$

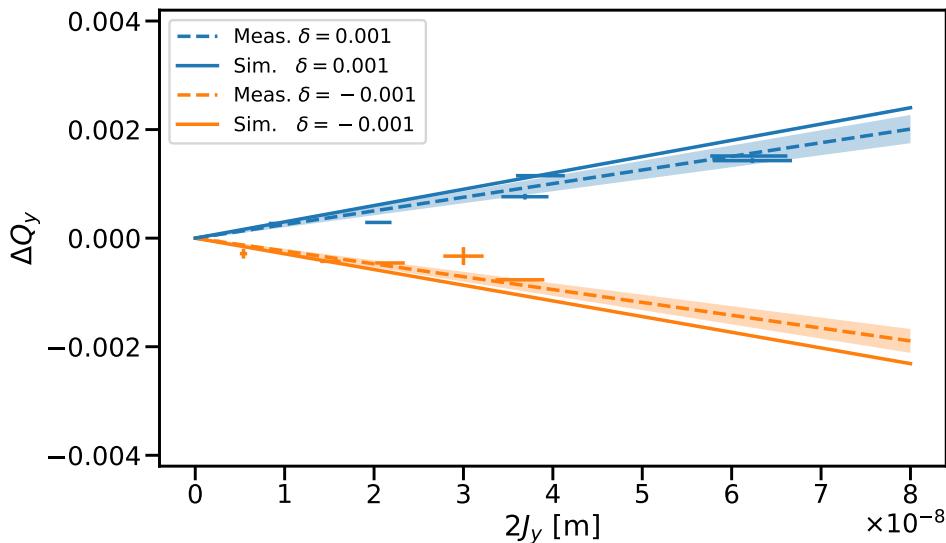
Table 4.9.: Comparison of the measured and simulated terms  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$  and  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$  via PTC, at two discrete momentum offsets. Simulations include errors from normal and skew sextupoles to icosapole ( $n = 3$  to  $n = 10$ ), as well as the decay of the  $b_5$  component of the main dipoles.



discrepancies between horizontal and vertical ?



(a) Horizontal tune shift depending on the vertical action:  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$ .



(b) Vertical tune shift depending on the vertical action:  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$ .

Figure 4.7.: Measured and simulated tune shift depending on the action at two different momentum offsets. Each fit corresponds to a chromatic amplitude detuning term evaluated at a certain  $\delta$ . Estimates from simulations are lowered due to the  $b_5$  decay of the main dipoles.

#### 4. Decapolar Fields

## 4.6. Resonance Driving Terms

Decapoles, due to their order, contribute to many RDTs. Indeed, 50 of them can be theoretically observed in simulations and measurements. In practice, the contributions of individual multipoles become indistinguishable as many resonances or lines overlap, making it impossible to isolate certain terms. Some resonances, described in Appendix C, are unique to certain multipoles when considering no too high orders. Those resonances, provided that they are sufficiently strong and the beam close to them, can be measured via their RDTs.

Of interest to the LHC Operation, is the RDT  $f_{1004}$ , driving the resonance  $1Q_x - 4Q_y$ . It can be seen in the horizontal frequency spectrum at  $-4Q_y$  with an amplitude dependence on  $J_y^2$ . Figure Fig. 4.8 shows a frequency map [53] of a simulation including decapolar field errors, where their impact on the beam is easily noticeable. The red particles evolving close to the resonance are affected by it and are subject to large tune shifts. Eventually, those particles are lost when their amplitude becomes too large.

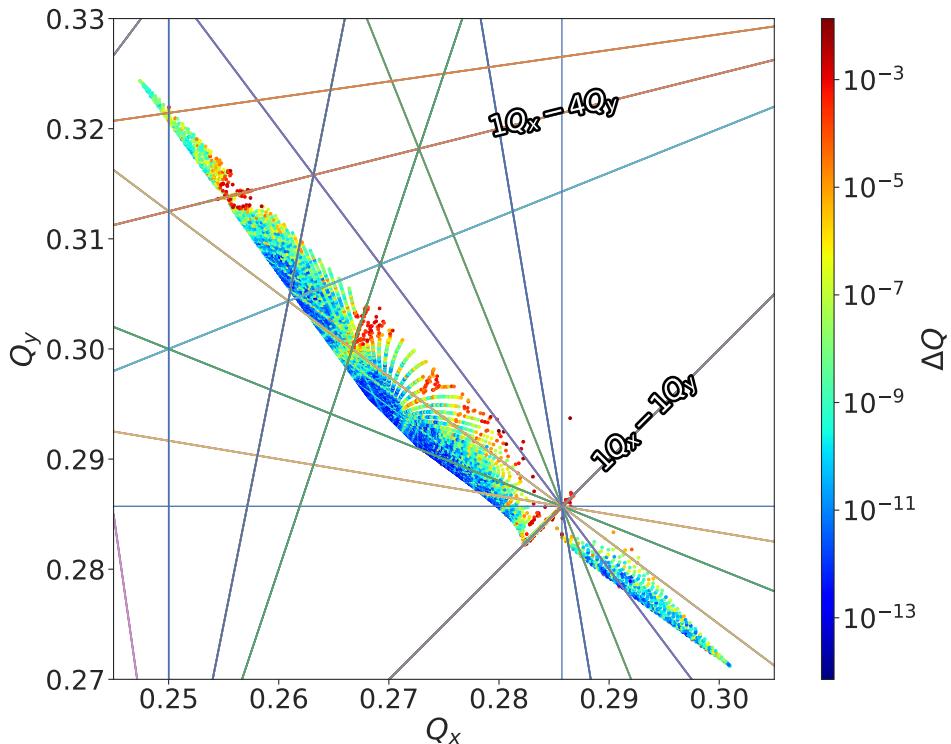


Figure 4.8.: Frequency map at injection energy, with decapolar field errors and nominal settings for landau octupoles. The highlighted resonance  $(1,-4)$ , excited by decapoles, shows a degradation over 20,000 turns. The tune shift between the start and the end of the simulation is indicated in colour. [change colormap](#)

Measuring turn-by-turn data without using any excitation is not a viable option as amplitudes are not large enough. Spectral lines are indeed usually impossible to discern from the noise floor, making RDTs not measurable. Measurements are hence taken with an AC-Dipole, introducing quadrupolar-like field errors in the linear regime [54] and more complex effects in the non linear regime. In practice, those effects are neglected. *Forced* RDTs are measured with an AC-Dipole and treated as *free* as no compensation is applied.

Such forced measurements were taken for the first time in the LHC to observe the  $f_{1004}$  RDT at injection energy. The frequency line of the resonance  $1Q_x - 4Q_y$  is seen at  $4Q_y$  in the horizontal spectrum, as shows Fig. 4.9.

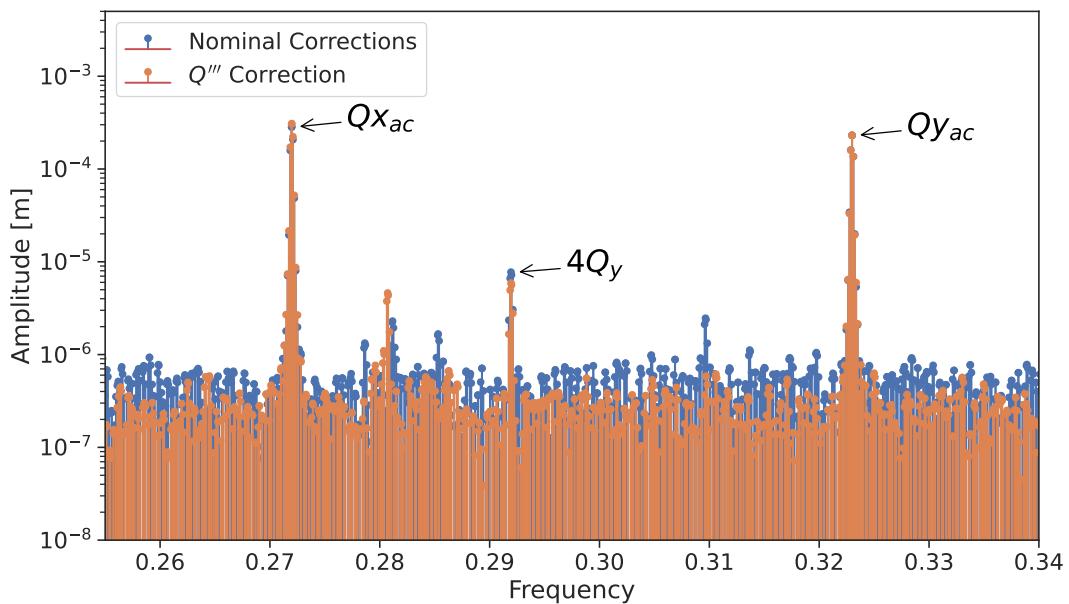


Figure 4.9.: Horizontal frequency spectrum of turn-by-turn data, with nominal and beam-based corrections for the third order chromaticity  $Q'''$ . The  $1Q_x - 4Q_y$  resonance can be seen at  $-4Q_y$  with different amplitudes for each correction scheme.

#### 4.6.1. Decapolar Contribution

Decapolar fields are the main contributors to the RDT  $f_{1004}$ . As such, powering the decapolar correctors is a good way to correct the related resonances. Being linear with the strength of the correctors, the RDT can be corrected via a response matrix.

Measurements were taken in order to attempt such a correction and get a baseline for the amplitude of the RDT without decapolar correctors and with the nominal FiDeL corrections applied. Corrections were made on the base of those nominal settings and applied on top. The strength of the decapolar

#### 4. Decapolar Fields

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correctors is shown in Table 4.10 for the FiDeL settings, the delta applied on top and the final correction values.

Circuit	$\text{FiDeL } K_5 [\text{m}^{-5}]$	$\Delta K_5 [\text{m}^{-5}]$	$K_5 [\text{m}^{-5}]$
<b>Beam 1</b>			
RCD.A12B1	-4582	6055	1473
RCD.A23B1	-5106	7	-5099
RCD.A34B1	-4855	3827	-1028
RCD.A45B1	-4577	-4746	-9323
RCD.A56B1	-4125	-4903	-9028
RCD.A67B1	-5166	2961	-2205
RCD.A78B1	-6827	3593	-3234
RCD.A81B1	-5500	2380	-3120
Total	-40738	9174	-31564
<b>Beam 2</b>			
RCD.A12B2	-4490	3639	-851
RCD.A23B2	-5155	-1147	-6302
RCD.A34B2	-4825	-1038	-5863
RCD.A45B2	-4619	3986	-633
RCD.A56B2	-4064	2944	-1120
RCD.A67B2	-5066	2357	-2709
RCD.A78B2	-6866	-2952	-9818
RCD.A81B2	-5446	1825	-3621
Total	-40531	9614	-30917

Table 4.10.: Strength of decapolar correctors with nominal FiDeL settings and after application of corrections aiming at reducing both the RDT  $f_{1004}$  and  $Q'''$ . The total value as a direct incidence on the chromaticity

This RDT correction also serves as a partial  $Q'''$  correction. To fully correct  $Q'''$  indeed approximately requires a strength of  $+13,000K_5$  distributed amongst the correctors. Therefore, this new approach reduces  $Q'''$  by about 70% compared to the previous method. The chromatic amplitude detuning terms are also expected to be decreased. Result of those measurements, as well as the inverse of the correction, are shown in Fig. 4.10

Although the FiDeL scheme was not intended to correct the RDT but rather only  $Q'''$ , it would be expected for it to lower the amplitude of the RDT. It can though be seen that it degrades the resonance compared to the machine with no decapolar correctors. On the other hand, the newly computed RDT correction does lower the amplitude of  $f_{1004}$  as expected. Its inverse has the opposite effect.

Simulations were run with decapolar correctors turned off and with the absolute value of the RDT correction. The response of the RDT between those two schemes is shown in Fig. 4.11. The difference

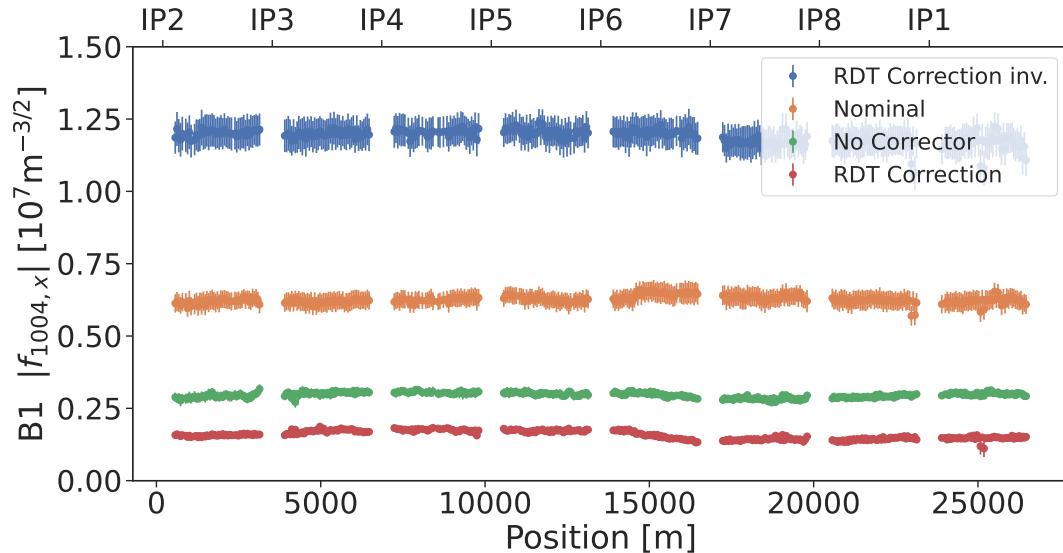
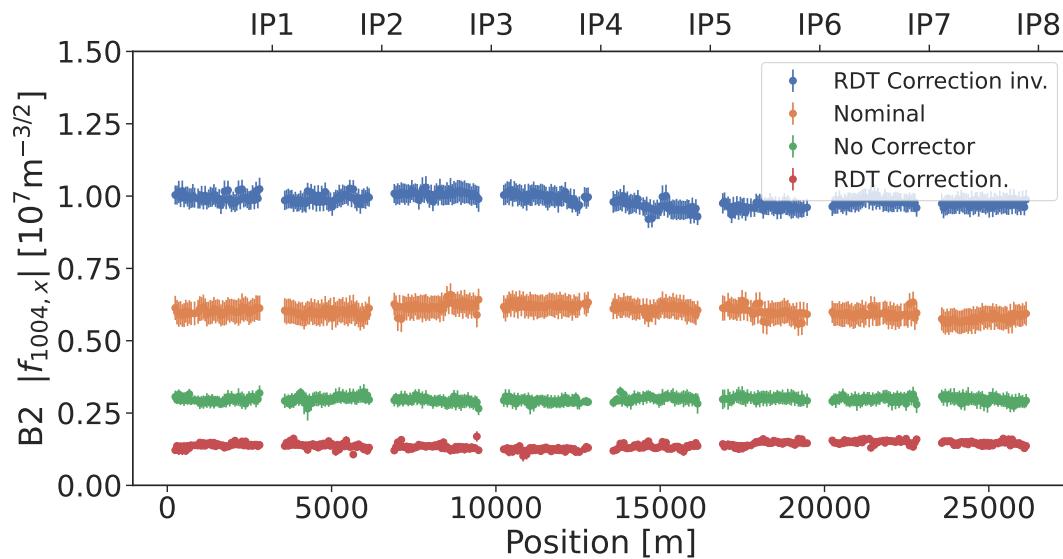
(a)  $|f_{1004}|$  for Beam 1(b)  $|f_{1004}|$  for Beam 2

Figure 4.10.: Measured  $f_{1004}$  with decapolar correctors powered off, nominal settings, and combined RDT &  $Q''''$  correction with normal and opposite signs.

between their RMS value ratio is  $\approx 6\%$ , indicating that simulations correctly model the decapolar correctors.

#### 4. Decapolar Fields

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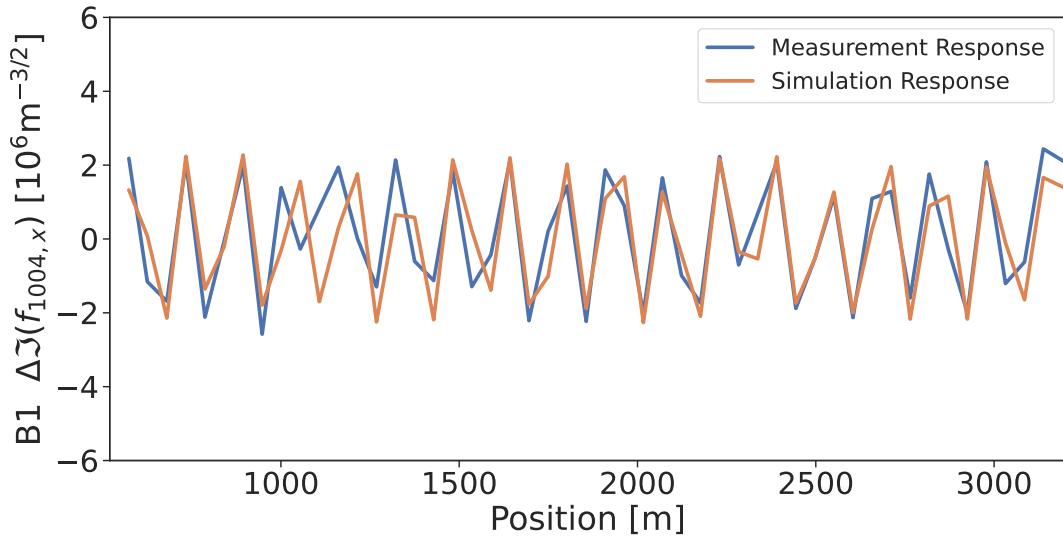


Figure 4.11.: Comparison for measurement and simulation of the response of the imaginary part of  $f_{1004}$  upon application on unpowered correctors of the RDT corrections.

#### 4.6.2. Feed-Up Contributions

##### First Observation

As described in Appendix A, multipoles can combine to create fields that are seen as higher orders when considering higher orders of the BCH expansion. For decapoles, combinations of several sextupoles and sextupoles with octupoles give rise to decapolar-like fields, as described in Table A.3. The following parts of this section will describe those combinations.

This effect was observed in 2022 during Run 3's commissioning. New corrections of the non-linear chromaticity  $Q''$  and  $Q'''$  were performed, and RDT measurements taken before and after their correction. As  $Q'''$  was corrected, the expectation was that the RDT  $f_{1004}$  would also lower with the reduction of the decapolar strengths  $K_5$ . However, an increase of the RDT was observed, as shows Fig. 4.12.



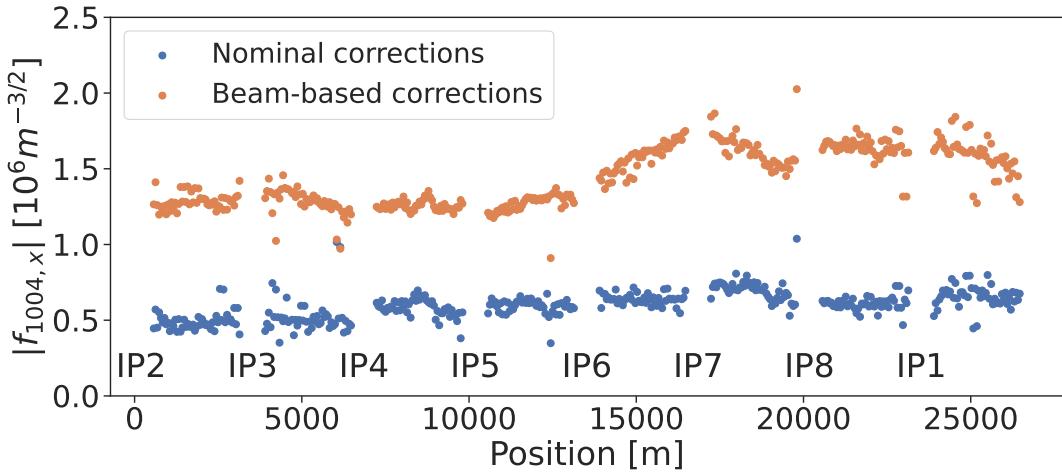


Figure 4.12.: Non intuitive increase of the RDT  $f_{1004}$  after application of both the  $Q''$  and  $Q'''$  corrections.**redo plot**

### Action Dependence and Analysis

Resonance lines in the frequency spectrum are often contributed to by several multipoles. Some lines start getting a contribution with rather high multipole orders, like the RDT  $f_{1004}$  considered here. The line  $4Q_y$  in the horizontal spectrum is indeed contributed to by decapoles and then only by decatetrapoles. When the main contributing field alone is varied, it is easy to reconstruct the RDT, as its fit is only dependant its action dependence ( $\propto J_x J_y^*$ ). Several turn-by-turn measurements at the same configuration can be taken with varying kick amplitudes, refining the RDT value with more data points for the fit.

Considering the contribution of lower order multipoles is a bit trickier, as the second order RDTs change the dependence of the frequency line [55]. In order to be able to compare the RDT from several turn by turn measurements, the same kick amplitude must then be used. Failing to do so would lead to a poor fit of the line amplitude relative to the action, resulting in an RDT with incorrect amplitude and significant noise.

### Sextupoles

At the third order of the BCH expansion, the combination of two sextupoles yields a decapolar-like expression. This means that, during normal operation of the machine, decapolar observables will be altered when adjusting parameters such as the linear chromaticity  $Q'$ . Derivation of such a combination can be found in Appendix A.2.2. The resulting Hamiltonian indeed is similar to the terms of a decapole, dropping the  $p_{x,y}$  terms for readability:

#### 4. Decapolar Fields

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$$(H_3)^3 \propto \frac{1}{48} \left( x^5 - 2x^3y^2 - 3xy^4 \right) \\ \sim x^5 - 10x^3y^2 + 5xy^4. \quad (4.5)$$

To quantify the actual impact of such an equation on the LHC, a simulation was run with injection optics while varying this same linear chromaticity  $Q'$ . No higher fields than sextupoles are have been included, including field errors. The resulting effect on the RDT  $f_{1004}$  can be seen in Fig. 4.13.

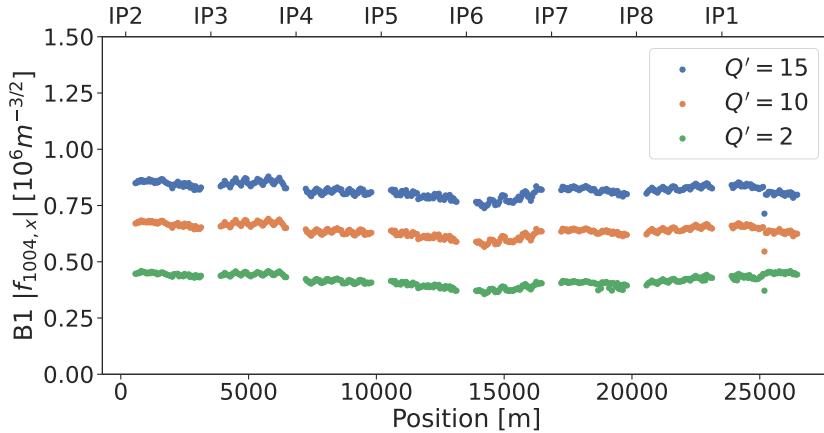


Figure 4.13.: Simulated change of the decapolar RDT  $f_{1004}$  with varying linear chromaticity  $Q'$  generated by sextupoles. The combination of sextupolar fields clearly shows a increase in decapolar RDT.

As the linear chromaticity increases, the overall  $K_3$  strength of sextupoles actually becomes more negative. Considering the previous Eq. (4.5), a higher chromaticity is expected to increase the amplitude of the RDT  $f_{1004}$ , related to the last term  $xy^4$ . Fig. 4.14 shows how the RDT is expected to vary, depending on the overall sextupoles strength and the linear chromaticity. It can be noted that although the relation between  $K_3$  and  $Q'$  is linear, that of  $K_3$  and the RDT varies with the cubed strength. Using the sum of the cubed strength is possible due to the chromaticity knob being a factor applied on all sextupoles at the same time.

To confirm what is observed in simulations, measurements were performed by varying  $Q'$  and kicking the beam with the AC-Dipole. Limited by losses, up to three measurements with distinct  $Q'$  were taken, as shows Fig. 4.15.

Like in simulations, it is observed that an increase in  $Q'$  translates to an increase in  $|f_{1004}|$ . The scale of the amplitude is though one order of magnitude higher than that of simulations. An offset for all measurements could be explained by non-included field-errors. The shift between them however should be similar between machine and simulations, this could be due by the interaction of the sextupolar

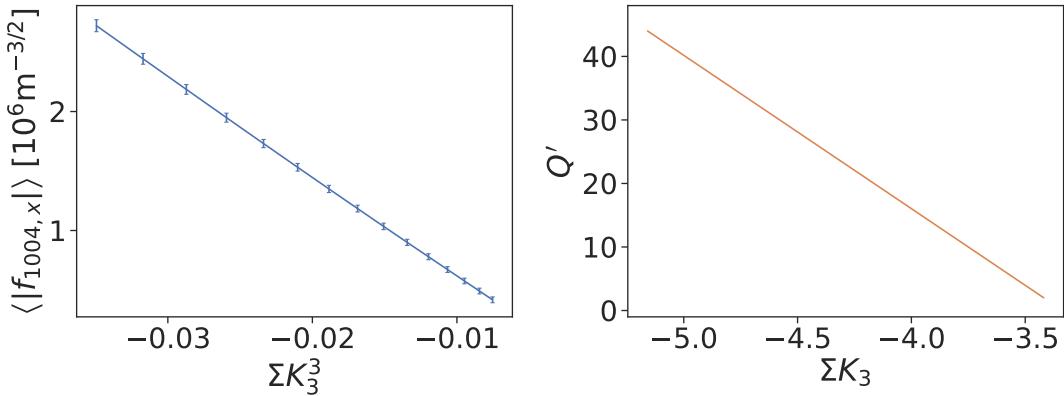


Figure 4.14.: Average amplitude of the decapolar RDT  $f_{1004}$  depending on the overall strength of the sextupoles used to control the linear chromaticity  $Q'$ . The right plot can be used to relate the RDT amplitude to a specific  $Q'$  value.

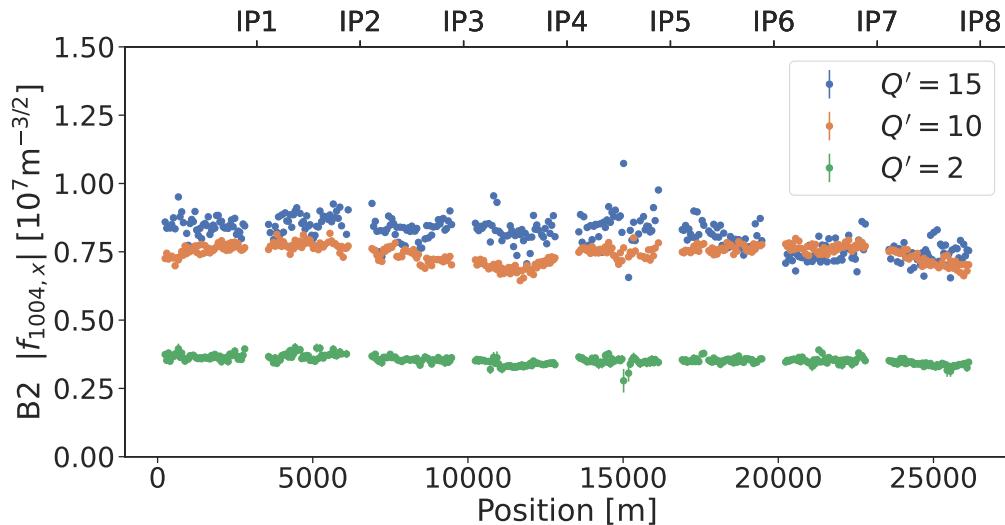


Figure 4.15.: Measured change of the decapolar RDT  $f_{1004}$  depending of the desired linear chromaticity  $Q'$  generated by sextupoles. It is to be noted that the vertical axis is one order of magnitude higher than the previous simulations' plot.

fields with octupoles, as detailed in the following section. More data points with varying  $Q'$  at similar kick amplitudes would be required to further investigate.

#### 4. Decapolar Fields

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### Sextupoles and Octupoles

At the second order of the BCH expansion, the combination of a sextupole and an octupole yields a decapolar-like expression. Like sextupoles, octupoles are used in operation, thus contributing to decapolar fields. This happens amongst other when correcting the second order chromaticity  $Q''$  and most importantly with the Landau Octupoles, which are powered to high strengths at injection energy to introduce Landau damping [9]. Derivation of such a combination can be found in Appendix A.2.3. The resulting Hamiltonian indeed is similar to the terms of a decapole, dropping the  $p_{x,y}$  terms for readability:

4

$$H_3H_4 \propto \frac{1}{24} (x^5 + 2x^3y^2 + xy^4) \\ \sim x^5 - 10x^3y^2 + 5xy^4. \quad (4.6)$$

In order to assess the previous equation, simulations were run with several configurations. As seen previously, a combination of two sextupoles creates a decapolar-like field, varying their field is thus not needed. Rather, a set of two configurations was run to check the impact of octupoles alone. The first configuration is ran with all sextupoles of the machine turned off, while octupoles are powered. The second configuration turns off all sextupoles and octupoles. Fig. 4.16 shows the resulting RDT  $f_{1004}$  from these simulations. It is there apparent that varying octupoles without sextupoles does not have any effect on this RDT.

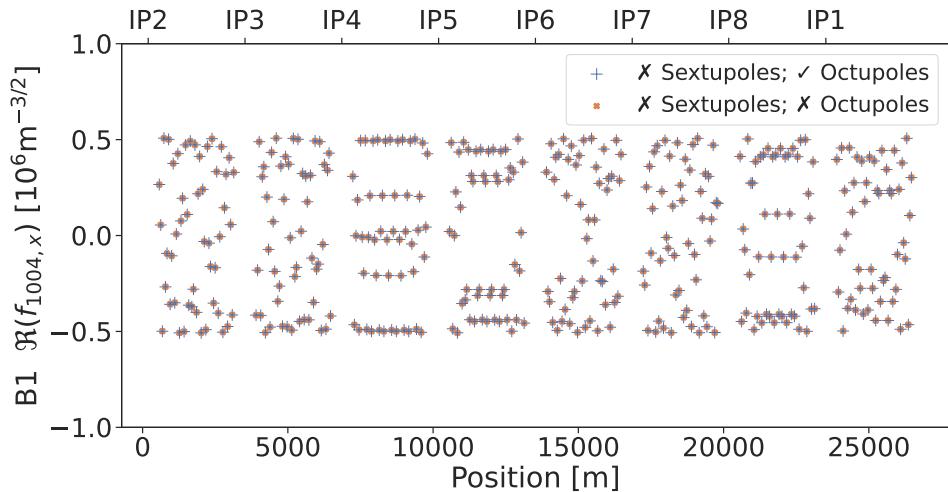


Figure 4.16.: Simulated decapolar RDT  $f_{1004}$  with two different schemes. First scheme has lattice sextupoles turned off and octupoles turned on. Second scheme has all sextupoles of the lattice turned off and octupoles turned off as well. No difference is seen, as expected from the equations.

The most powerful octupoles used in operation are the lattice octupoles, used for Landau damping. Fig. 4.17 shows a simulation ran with varying strengths of those magnets. It can be noted here that the shift of the RDT is almost of an order of magnitude, making octupoles a large contributor to the decapolar fields.

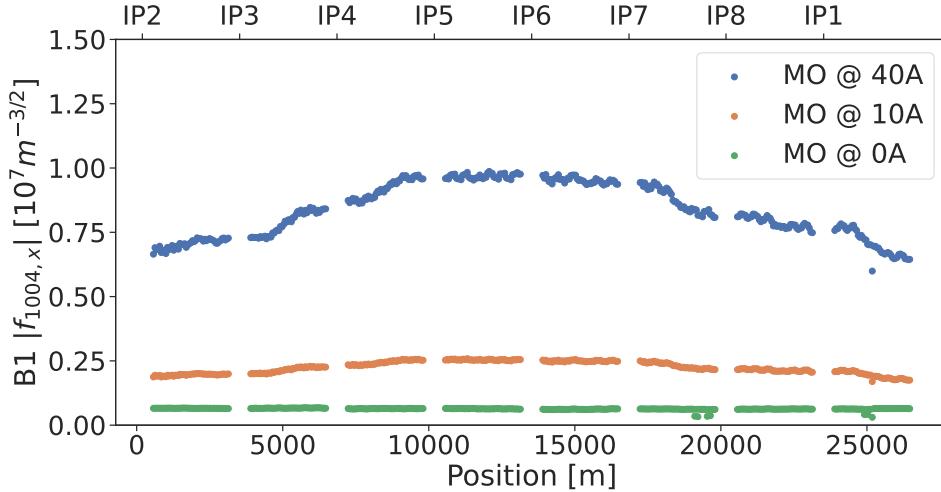


Figure 4.17.: Simulated change of the decapolar RDT  $f_{1004}$  depending of the strength of the lattice octupoles used for Landau damping.

While decapoles are expected to be the main contributors to decapolar fields, other strong sources can indeed be identified. Fig. 4.18 shows the average amplitude of the RDT  $f_{1004}$  depending on the error sources introduced in the simulations. A large contribution comes from the octupolar errors in the main dipoles, being actually larger than the decapolar errors in those magnets.

Measurements were performed to confirm and quantify the effect of octupoles coupled with sextupoles on the decapolar fields. Previous corrections, aimed at correcting the second order chromaticity  $Q''$ , via octupolar correctors MCO, were applied with varying factors. Such corrections apply use a uniform trim on all correctors of  $\approx +2.5K_4$ . Fig. 4.19 shows a comparison of the resulting RDT with those corrections at factors  $-10, -4, -1$  and  $0$ .

Table 4.11 shows the RMS of the amplitude of this RDT for the various configurations. Similar to the shift observed when powering the landau octupoles in simulations, the shift is of one order of magnitude between factors  $-0$  and  $-10$ . Measurements with landau octupoles were also attempted but losses made it impossible to obtain high enough amplitudes to correctly measure the RDT.

Factor	RMS $ f_{1004} $
-10	37, 308, 159
-4	6, 721, 270
0	3, 533, 796
-1	2, 333, 384

Table 4.11.: RMS of  $|f_{1004}|$  depending on the factor of the  $Q''$  corrections.

#### 4. Decapolar Fields

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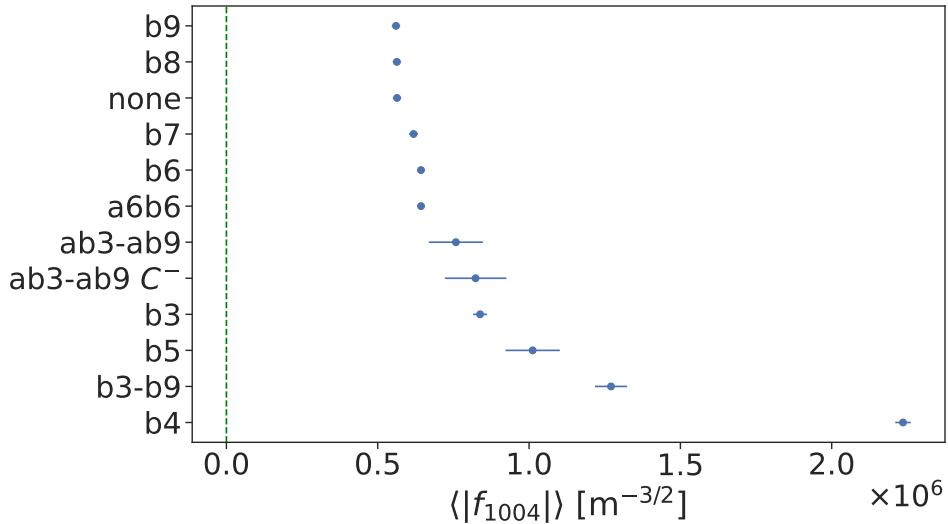


Figure 4.18.: Simulation of the amplitude of the decapolar RDT  $f_{1004}$  depending on the field errors applied on main dipoles as well as coupling ( $C^-$ ). While it is apparent that some multipolar errors drive the resonance higher, some combinations actually seem to cancel each other.

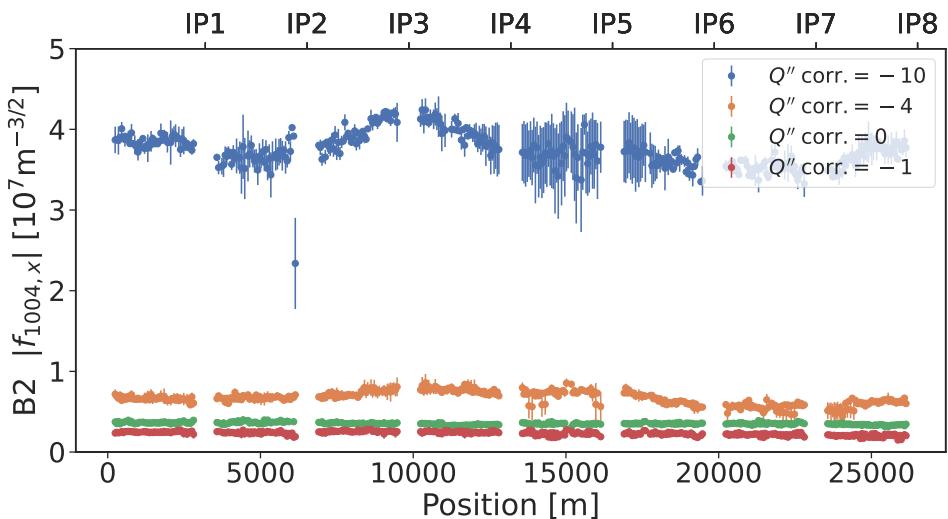


Figure 4.19.: Shift of the decapolar RDT  $f_{1004}$  depending on the factor applied on octupolar corrections for  $Q''$ .

Simulated and observed large shifts due to octupoles are relevant to the operation of the LHC, as resonances can be greatly deteriorated, especially when powering Landau octupoles. A better understanding of the interaction between Landau octupoles and octupolar correctors could lead to improved corrections in not only octupolar but also decapolar fields in the future.



### 4.6.3. Feed-Down Contributions

#### Simulation at injection with/without crossing angles.

To produce collisions at top energy, *crossing angles* are introduced via the orbit correctors located in the triplets, before the separation dipoles and the matching section of the interaction regions (MCBX, MCBY and MCBC) [56]. Those collisions happen with a small  $\beta^*$ , currently 30cm, requiring strong quadrupolar fields from the triplets.

At such  $\beta$ , those triplets also generate strong dodecapolar field errors. Because of the crossing-angles, feed-down appears and lower-order fields can be observed. Such feed-down to decapolar fields was observed during the first commissioning of Run 3, in 2022 [57]. Fig. 4.20 shows how the RDT  $f_{1004}$ , normally affected by decapoles, varies with the application of crossing angles.

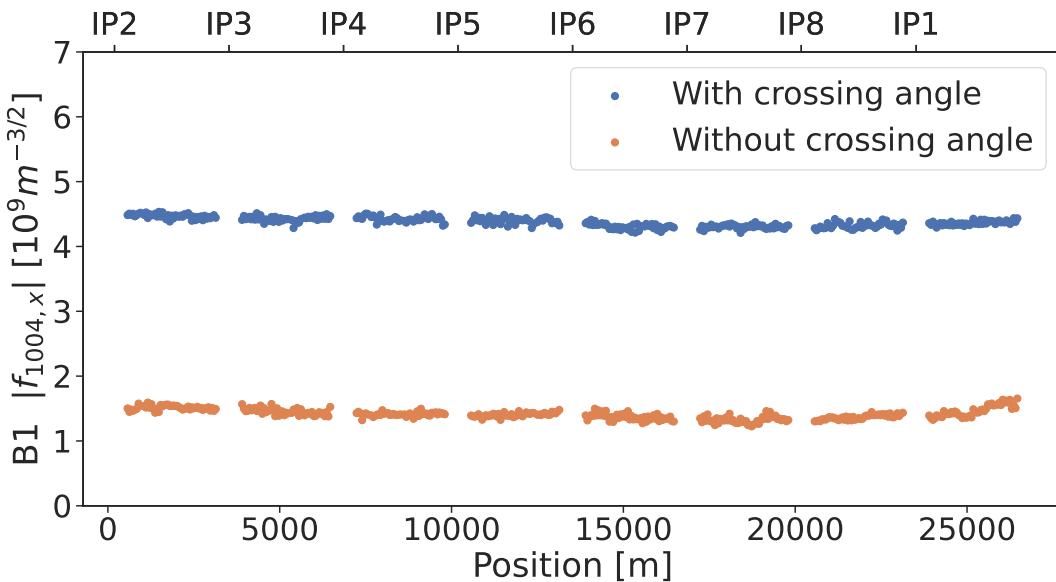


Figure 4.20.: Varying amplitude of the decapolar RDT  $f_{1004}$  at top energy depending on the activation or not of the crossing angle at the IP. Offsets in orbit create feed-down from higher orders.

Such a contribution is though not expected at injection energy, as the triplets aren't powered as much as at top energy,  $\beta^*$  being set at around 10m.

## 4.7. Impact of Decapolar Fields

Decapolar fields can influence the beam lifetime in several ways. Chromatic amplitude detuning and chromaticity will induce a tune shift, relative to either or both the action and the momentum offset.

#### 4. Decapolar Fields

After such detuning, particles may move closer to certain resonances in tune space, causing their oscillations to grow, eventually leading to their loss.

##### 4.7.1. Large RDT

As seen previously in Fig. 4.8, the resonance  $1Q_x - 4Q_y$  passes through the beam in tune space, deteriorating the lifetime of the nearby particles. In order to measure the impact of this resonance on the beam, a knob was created, alternating the current of all decapole correctors in the machine arc by arc. Such a powering scheme has no impact on chromaticity as the sum of the strengths  $K_5$  is zero. Rather, the RDT  $f_{1004}$  is impacted. Is it to be noted that this is not a correction, but purely a way to artificially increase the RDT in order to quantify the effect of the resonance.

Starting with nominal corrections for  $Q'''$  corrections, a delta of  $\pm 10500K_5$  is applied on each decapolar correctors. Fig. 4.21 shows the response of the real part of the RDT for this scheme and its inverse. The amplitude of the RDT is on a similar level as the shift is significantly larger than the original level of the RDT. Table 4.12 indicates the amplitude of the RDT created with each knob value.

$\Delta K_5$	RMS $ f_{1004} $
0	618,947
$\pm 10500$	17,566,377
$\mp 10500$	17,623,867

Table 4.12.: RMS of  $|f_{1004}|$  relative to the powering scheme of decapolar correctors.

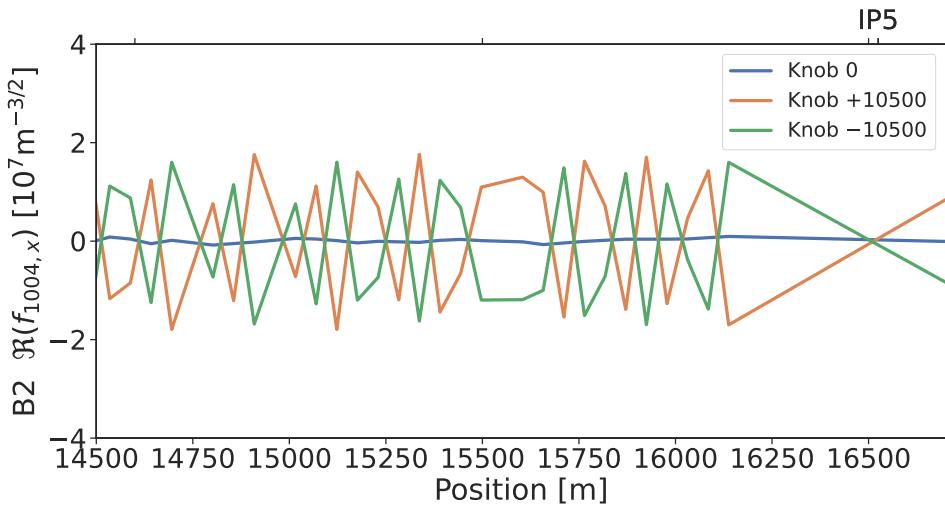


Figure 4.21.: Measured real part of the RDT  $f_{1004}$  depending on the powering scheme of the decapolar correctors.

In order to measure the lifetime, a long period must be allocated as the signal returned from monitors can be jittery. Fig. 4.22 shows this lifetime depending on the decapolar strength scheme applied. The

current of only one circuit is shown for readability. A current of  $\approx 230$  corresponds to a knob value of  $+10500$  while a current of  $-45$  corresponds to 0.

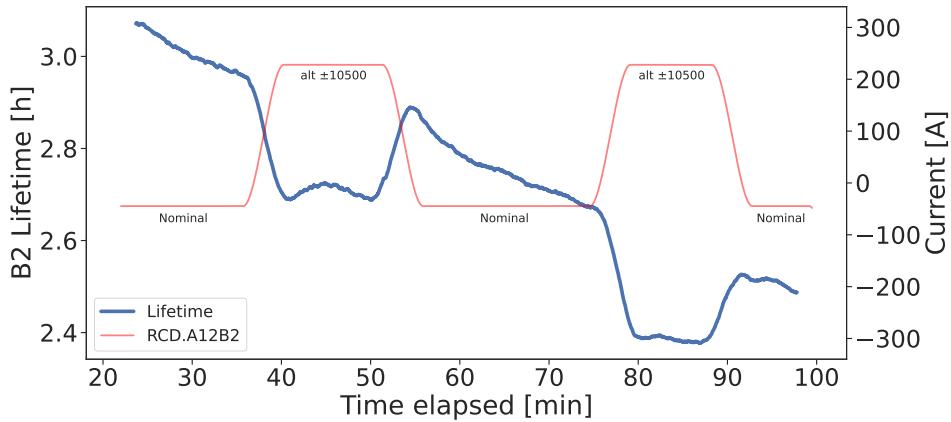


Figure 4.22.: Measured lifetime of Beam 2 upon application of two different powering schemes for decapolar correctors. One trim keeps the RDT at a low amplitude while the other greatly amplifies it.

It is clear from this measurement that a large RDT decreases the lifetime of the beam. The first pair of trim sees the average lifetime decreasing of  $0.31 \pm 0.03$  hours, while the second one sees a decrease of  $0.36 \pm 0.03$  hours. This observed decrease of 20 minutes accounts for 10% of the beam lifetime at injection energy.

### 4.7.2. Corrections

In order to understand what can be gained from correcting decapolar fields, a lifetime measurement was taken with the corrections described in Section 4.6.1. This scheme corrects the three decapolar observables, being the RDT  $f_{1004}$  linked to the resonance  $1Q_x - 4Q_y$ , the third order chromaticity  $Q'''$  and the chromatic amplitude detuning terms. Fig. 4.23 shows the evolution of the lifetime, starting with corrections applied, removed and then trimmed to their opposite. A net change in lifetime for Beam 1 can be measured after each application. Acquired signal for Beam 2 has been deemed too noisy to be relevant, due to the shortness of the measurement.

It is apparent here that the corrections have a beneficial effect on the beam. The lifetime improvement is of  $\approx 3\%$ , while the degradation after applying the opposite is of  $\approx 5\%$ . Further developments in the correction scheme and lengthier measurements could easily improve this lifetime gain.

#### 4. Decapolar Fields

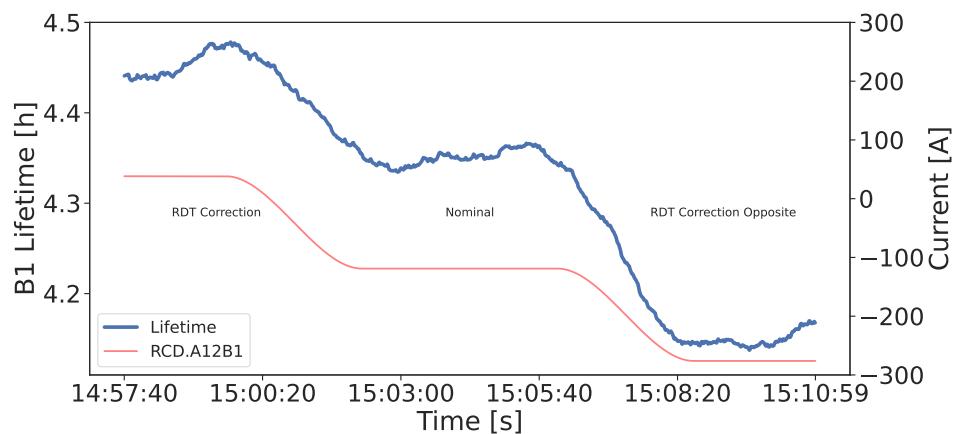


Figure 4.23.: Measured lifetime of Beam 1 with the nominal corrections for  $Q'''$ , combined correction of  $f_{1004}$  and  $Q'''$ , and its inverse.

## 4.8. Summary

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# Higher Order Fields

## 5.1. Introduction

Beam-based high order field measurements have been carried out in the LHC since its first Run [38, 48], via chromaticity studies. Those measurements, made by varying the RF frequency while observing the resulting tune change, have been performed with a momentum offset of up to  $\delta = \pm 2.2 \times 10^{-3}$ , which led to the observation of the third order term of the non-linear chromaticity.

During the commissioning of Run 3 in 2022, a new collimator sequence has been introduced, allowing wider momentum offset measurements, within  $\delta \in [-3.2 \times 10^{-3}, 3.7 \times 10^{-3}]$ . This improved setup led to the observation of the fourth and fifth order terms at injection energy. Those terms, denoted  $Q^{(4)}$  and  $Q^{(5)}$  respectively in Eq. (5.1), are produced to first order by dodecapoles and decatetrapoles. Dodecapoles being powered off at injection and decatetrapoles being absent from the lattice, those fields originate from the field errors of the various magnets installed in the LHC.

$$Q(\delta) = Q_0 + Q'\delta + \frac{1}{2!}Q''\delta^2 + \frac{1}{3!}Q'''\delta^3 + \frac{1}{4!}Q^{(4)}\delta^4 + \frac{1}{5!}Q^{(5)}\delta^5 + \mathcal{O}(\delta^6). \quad (5.1)$$

In addition to completing the measurements of high-order fields through chromaticity scans, turn-by-turn measurements were also conducted. High amplitude kicks indeed made it possible to observe dodecapolar RDTs in the LHC for the first time.

## 5.2. Chromaticity

### 5.2.1. Procedure

As described in Section 3.3.3, the momentum offset  $\delta$  is related to the RF frequency and the momentum compaction factor. This relation is given as a simplified form in Eq. (5.2). The model  $\alpha_c$  for the LHC

## 5. Higher Order Fields

injection optics is  $3.48 \times 10^{-4}$  for beam 1 and  $3.47 \times 10^{-4}$  for beam 2. Via this relation, a change of 140Hz of the RF frequency corresponds to a momentum offset of about  $-0.001$ .

$$\delta = -\frac{1}{\alpha_c} \cdot \frac{\Delta f_{RF}}{f_{RF,nominal}}. \quad (5.2)$$

To properly characterize higher orders of the chromaticity function and ensure quality measurements, several steps are required. The tune measured during chromaticity scans can exhibit jitter and resonance lines may appear, requiring thorough data cleaning to either reject problematic data points or reduce error bars. The simplified Eq. (5.2), describing  $\delta$ , has been sufficient for reliably measuring up to the third order chromaticity. However, this relation also needs verification.

### Noise and Spectral Lines

Noise lines, due to electronics, can be seen in the raw data obtained from the BBQ tune system. Occasionally, when those resonances are strong, their frequency peak can be mistaken as the tune and logged as such by the system. This yield large uncertainties in the measurement when data points can't properly be classified as outliers. A tune measurement presenting this issue is showed in Fig. 5.1.

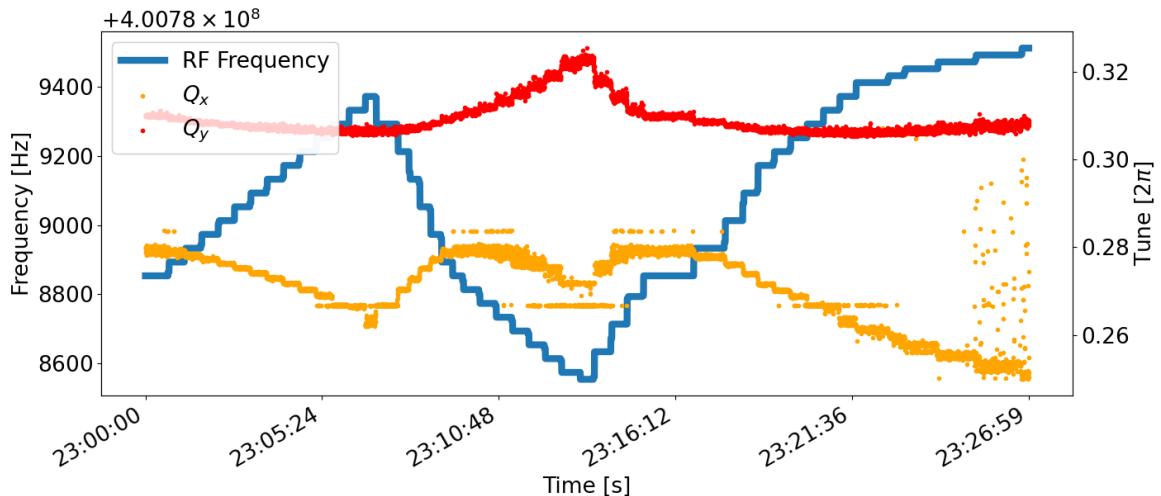


Figure 5.1.: Shift of the tune by variation of the RF. Noise lines can appear in some cases, making the tune error bar large or downright unusable.

A solution to this issue is to use the raw data extracted from the BBQ system. From there, a spectrogram clearly shows the noise lines, as seen in Fig. 5.2. Those lines have been repeatedly identified over several measurements and confirmed to be fixed. The highest peak of the spectrogram



can thus be safely identified by removing those resonances, yielding a cleaner measurement. It is also to be added that the BBQ requires to set a tune window, which can be forgotten. By analyzing the raw data, it is ensured that the measurement has usable data and does not try to measure noise.

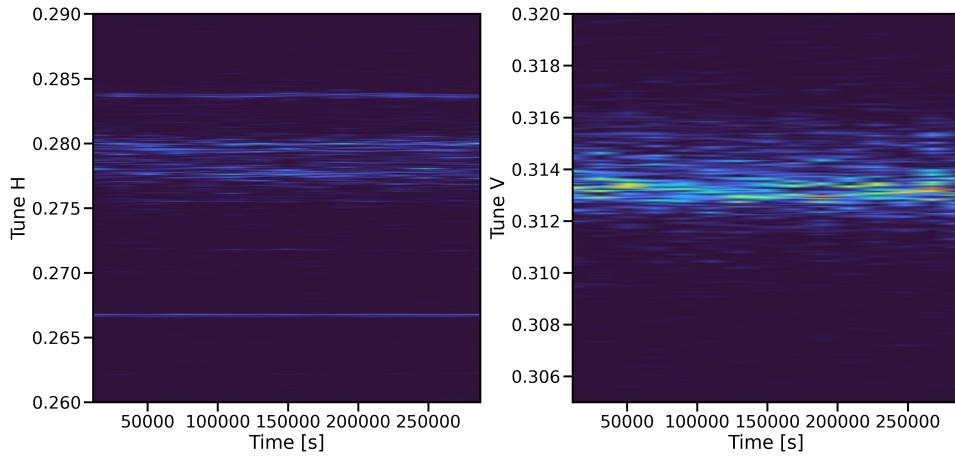


Figure 5.2.: Tune spectrogram obtained via BBQ system. Strong resonance lines can be seen above and below where the tune really is, causing the wrong frequency peak to be identified as the tune.

### Momentum Compaction Factor

Rather than a constant, the momentum compaction factor can be expressed as an expansion, as detailed in Section 2.4.4. The first terms are given by the following,

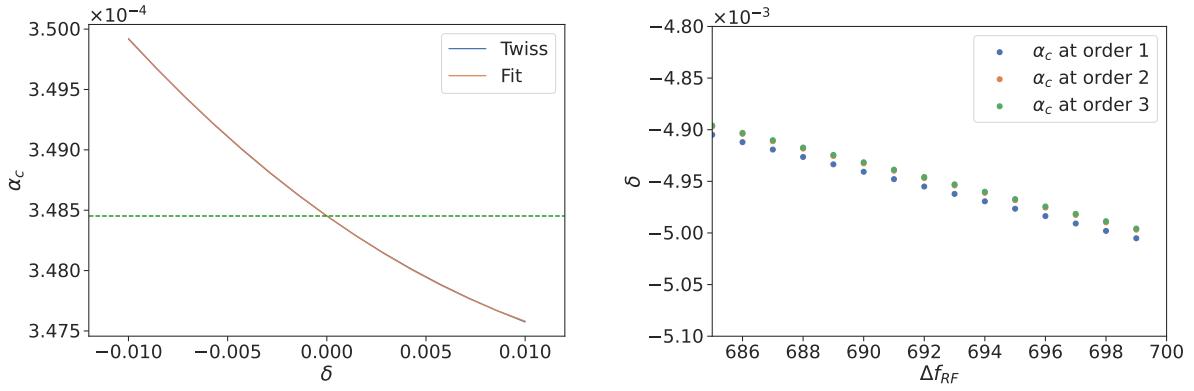
$$\alpha_c = \underbrace{\alpha_{c,0}}_{1^{\text{st}} \text{ order}} + \underbrace{\alpha_{c,1}\delta}_{2^{\text{nd}} \text{ order}} + \underbrace{\alpha_{c,2}\delta^2}_{3^{\text{rd}} \text{ order}}. \quad (5.3)$$

The expression for  $\delta$  at first and second order then reads,

$$\begin{aligned} \delta &= -\frac{\Delta f_{RF}}{\alpha_0 f_{RF}} && \Rightarrow \text{Order 1} \\ \delta &= \frac{-\alpha_0 f_{RF} + \sqrt{f_{RF}(-4\Delta f_{RF}\alpha_1 + \alpha_0^2 f_{RF})}}{2\alpha_1 f_{RF}} && \Rightarrow \text{Order 2} \end{aligned} \quad (5.4)$$

It is assumed that only the first term is relevant as the induced difference in chromaticity is negligible as will be demonstrated later on. Fig. 5.3 shows the non linearity of the momentum compaction factor and its effect on the calculated  $\delta$  via the previous formulas.

## 5. Higher Order Fields

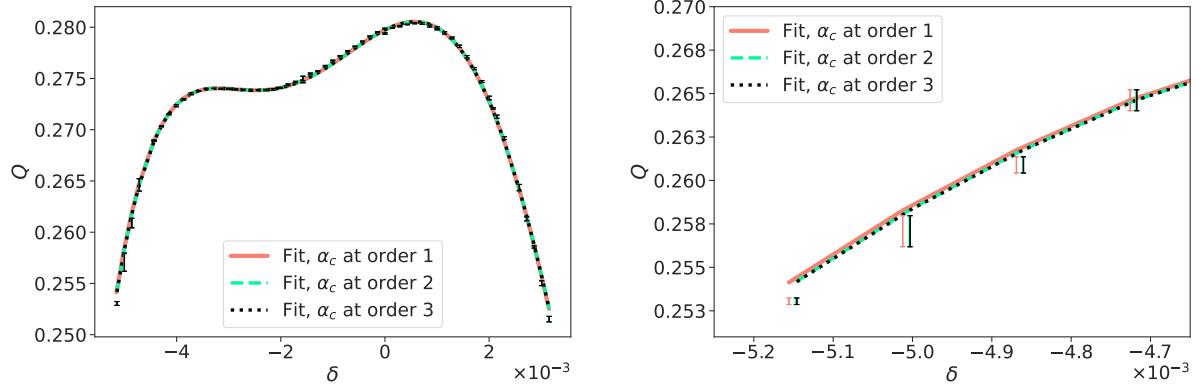


(a) Non-linear fit of  $\alpha_c$  obtained via an evaluation at discrete  $\delta$  in MAD-X. The green line represents a constant  $\alpha_c = \alpha_{c,0}$ .

(b) Divergence of the momentum offset when considering higher  $\alpha_c$  orders with large RF trims.

Figure 5.3.: Non linearity of  $\alpha_c$  and its effect on the computed  $\delta$  via RF trims. The simulations are done at injection energy of 450GeV.

It is observed that while clearly depending on higher orders, the momentum compaction factor only has a small impact on the calculated  $\delta$ . Fig. 5.4 shows a real-life measurement, comparing the fit of the chromaticity function with various  $\delta$ , computed up to the third order of  $\alpha_c$ .



(a) Fit of the chromaticity function at the 5<sup>th</sup> order, considering the  $\alpha_c$  expansion up to the third order.

(b) Zoom on one side of the fit. The difference between the second and third order is barely noticeable.

Figure 5.4.: Fit of the chromaticity function considering several  $\alpha_c$  orders.

The fit of the chromaticity function is barely impacted when considering the higher orders of the momentum compaction factor. The different orders of the chromaticity are collected in Table 5.1. The higher order terms of  $\alpha_c$  can thus be neglected and are not a source of higher chromaticity orders.



Chromaticity	$\alpha_c$ order		
	1	2	3
$Q^{(1)}$	$2.52 \pm 0.03$	$2.53 \pm 0.03$	$2.53 \pm 0.03$
$Q^{(2)}$	$-3.04 \pm 0.05$	$-3.05 \pm 0.05$	$-3.05 \pm 0.05$
$Q^{(3)}$	$-4.75 \pm 0.03$	$-4.75 \pm 0.03$	$-4.75 \pm 0.03$
$Q^{(4)}$	$-0.33 \pm 0.07$	$-0.32 \pm 0.07$	$-0.32 \pm 0.07$
$Q^{(5)}$	$2.33 \pm 0.06$	$2.36 \pm 0.06$	$2.36 \pm 0.06$

Table 5.1.: Chromaticity values obtained for the same measurement, depending on the order of the momentum compaction factor taken into account.

### Momentum Offset from Orbit

During machine operation, the momentum offset, derived from the orbit, used to be logged on the servers. It was then possible to directly compute the chromaticity that way without having to use the RF and the momentum compaction factor. In 2016, measurements of the non-linear chromaticity were performed using the former method. Fig. 5.5 shows a comparison of the obtained non-linear chromaticity from both methods, while Table 5.2 shows a numerical comparison. Results being similar, it is deemed that both methods are reliable to measure the non-linear chromaticity in the LHC.

Plane	$\delta$ via RF		$\delta$ via orbit	
	$Q''[10^3]$	$Q'''[10^6]$	$Q''[10^3]$	$Q'''[10^6]$
Beam 1				
X	$-0.64 \pm 0.01$	$3.00 \pm 0.04$	$-0.62 \pm 0.01$	$2.91 \pm 0.04$
Y	$-0.17 \pm 0.01$	$-2.12 \pm 0.04$	$-0.14 \pm 0.01$	$-2.09 \pm 0.04$
Beam 2				
X	$-1.18 \pm 0.02$	$2.89 \pm 0.06$	$-1.23 \pm 0.03$	$3.13 \pm 0.11$
Y	$0.18 \pm 0.02$	$-1.95 \pm 0.05$	$0.20 \pm 0.02$	$-2.02 \pm 0.06$

Table 5.2.: Comparison of the chromaticity values obtained for the same measurement via two different methods to acquire  $\delta$ .

## 5. Higher Order Fields

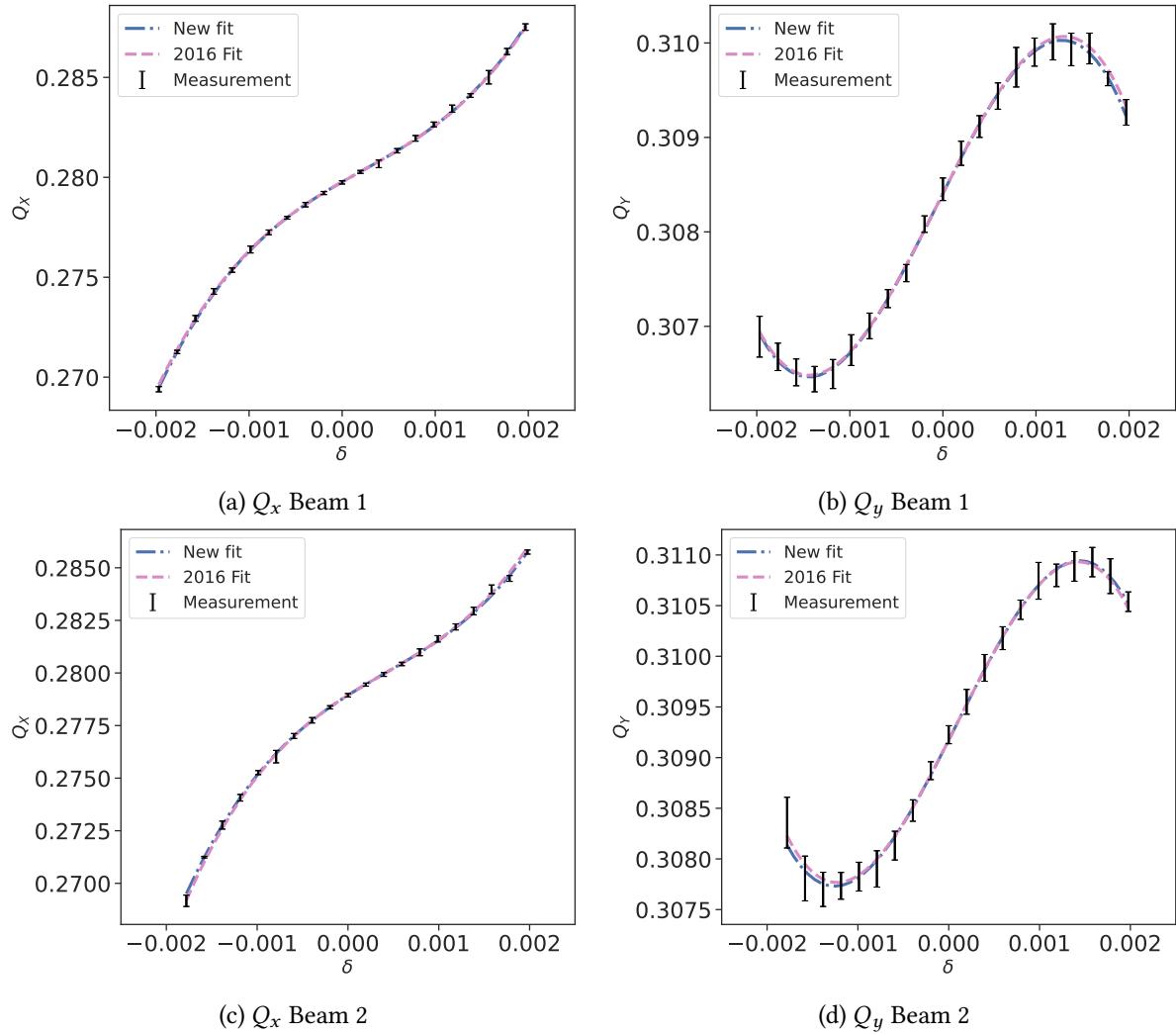


Figure 5.5.: Comparison of the non-linear chromaticity fit obtained from the computed momentum offset via the RF in 2022 and from the logged values in 2016.

### 5.2.2. Measurements

In order to assess the correctness of the observation of higher chromaticity orders, measurement repeatability is needed. Two measurements were thus taken in 2022, with different configurations pertaining to the correction of the second and third order chromaticities  $Q''$  and  $Q'''$ . The first one used the nominal correction strengths for octupole and decapole corrector magnets, derived from magnetic measurements, where the second one used beam-based corrections for the same elements, computed from the previous measurement. More measurements were taken during 2024's commissioning



with new optics for the same reasons, to minimize the second and third order chromaticities. Three measurements were taken: with nominal corrections, after having corrected  $Q'''$  and then  $Q''$ . The introduced new optics mainly changed the powering of the triplets at the IPs and are not expected to have a considerable impact on the chromaticity.

Table 5.3 shows a summary of those measurements with their respective achieved momentum offset ranges. While the 2024 measurements achieved greater ranges than the previous ones, those were restricted during analysis to allow suitable comparisons.

Number	Year	Corrections	$\delta$ min. [ $\times 10^{-3}$ ]	$\delta$ max. [ $\times 10^{-3}$ ]
1	2022	Nominal	-3.15	3.01
2	2022	$Q''$ & $Q'''$	-3.15	3.72
3	2024	Nominal	-5.15	3.15
4	2024	$Q'''$	-3.44	4.87
5	2024	$Q''$ & $Q'''$	-3.86	4.44

Table 5.3.: Performed chromaticity measurements with their respective momentum offset ranges.

In order to stay consistent, the horizontal and vertical tunes were respectively set to  $Q_x = 0.28$  and  $Q_y = 0.31$  for both measurements. The linear chromaticity  $Q'$  is set to a small value, around 2, to avoid large tune shifts throughout the scan. All measurements were performed during LHC's beam commissioning, as part of the measurements and corrections performed after technical or long shutdowns.

## First Observation

The first observation of higher order chromaticity was done with the octupolar and decapolar correctors *MCO* and *MCD* set to their nominal settings. Those are aimed at correcting  $Q''$  and  $Q'''$ , as previously described in Chapter 4. Results of this initial measurement are shown in Table 5.4. Lower order chromaticities such as  $Q'$  and  $Q''$  are consistent with measurements done during the previous Run [49].

Due to the RF-scan method, the momentum offset crosses zero several times during the measurement. Negligible change in tune at this point makes it possible to determine that the tune drift through the measurement is of no consequence. This measurement was performed after an extended period at injection energy, where the decay of the sextupolar fields is small and not causing any change in the first order chromaticity. The fitted curve for the chromaticity function is shown in Fig. 5.6. It is clear that a higher order polynomial is beneficial to the fit, as discussed further in Section 5.2.2.

## 5. Higher Order Fields

Plane	$Q^{(2)} [10^3]$	$Q^{(3)} [10^6]$	$Q^{(4)} [10^9]$	$Q^{(5)} [10^{12}]$
Beam 1				
X	$-2.44 \pm 0.02$	$-3.36 \pm 0.04$	$-0.56 \pm 0.02$	$1.20 \pm 0.07$
Y	$0.97 \pm 0.02$	$1.62 \pm 0.05$	$0.15 \pm 0.03$	$-0.88 \pm 0.09$
Beam 2				
X	$-2.45 \pm 0.03$	$-2.72 \pm 0.08$	$-1.00 \pm 0.05$	$0.15 \pm 0.14$
Y	$0.79 \pm 0.03$	$1.54 \pm 0.06$	$0.24 \pm 0.04$	$-0.74 \pm 0.13$

Table 5.4.: Terms of the high order chromaticity obtained during Run 3's commissioning in 2022, with nominal corrections.

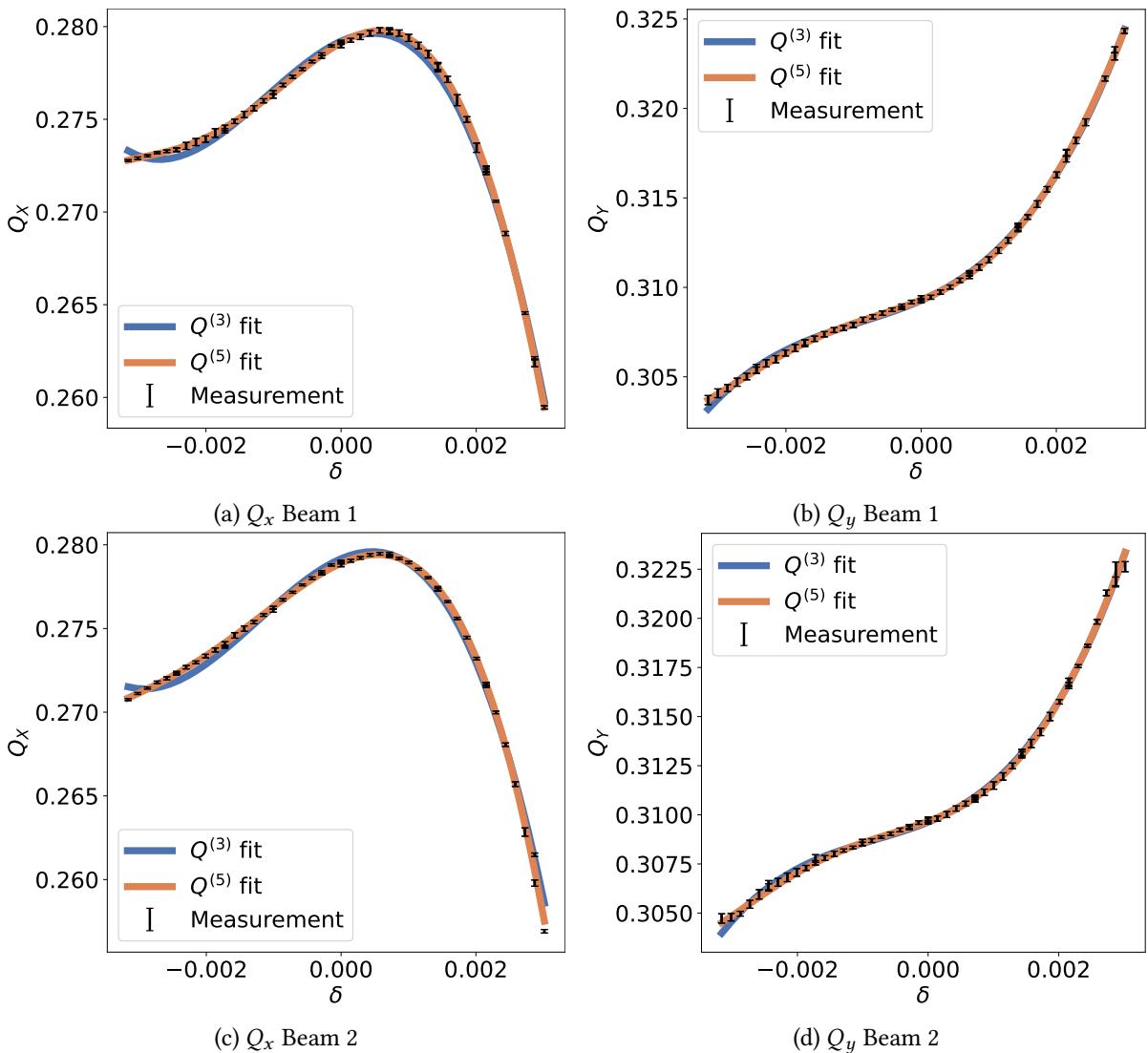


Figure 5.6.: Measurement of higher order chromaticity terms with nominal corrections used during operation. Fits are up to the third and fifth order.



Previous studies of chromaticity in the LHC only considered fits up to the third order. Expanding the fits to the fifth order increases the  $Q'''$  estimate and improves the fit quality. Accurately measuring the third order chromaticity is essential for its correction, making it important to consider the higher orders.

### Varying Configurations

The five previously introduced measurements were performed with very different configurations for the octupolar and decapolar correctors. Table 5.5 shows the strengths applied on every circuit for each correction scheme, in 2022. The correction is called *global* as all correctors are trimmed uniformly. The 2024 corrections are similar in order of magnitude. Fig. 5.7 shows the measurements and fit of some of these measurements, to highlight their differences.

Beam	$K_4 [\text{m}^{-4}]$	$K_5 [\text{m}^{-5}]$
1	+3.2973	+1610
2	+2.1716	+1618

Table 5.5.: Corrections applied on top of the nominal octupolar and decapolar correctors strengths in 2022 for the  $Q''$  and  $Q'''$  corrections.

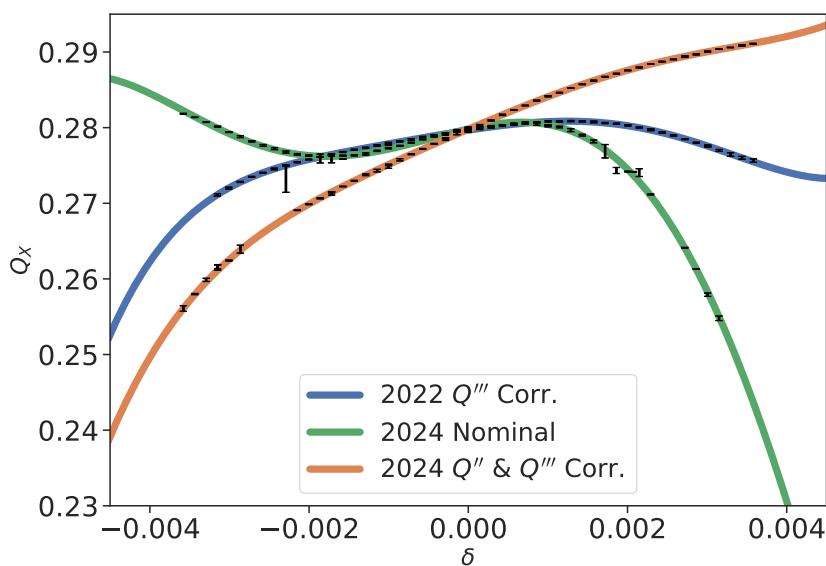


Figure 5.7.: Selection of horizontal chromaticity measurements performed with varying configurations of the octupolar and decapolar correctors for Beam 1 during the commissionings of 2022 and 2024.

## 5. Higher Order Fields

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Axis	Meas.	$Q''$	$Q'''$	$Q^{(4)}$	$Q^{(5)}$
<b>Horizontal</b>					
Beam 1	1	$-2.44 \pm 0.02$	$-3.37 \pm 0.04$	$-0.56 \pm 0.02$	$1.21 \pm 0.07$
	2	$-0.61 \pm 0.01$	$-1.00 \pm 0.03$	$-0.62 \pm 0.02$	$1.19 \pm 0.05$
	3	$-2.01 \pm 0.05$	$-4.49 \pm 0.10$	$-0.58 \pm 0.07$	$1.34 \pm 0.18$
	4	$-1.46 \pm 0.03$	$-0.29 \pm 0.06$	$-0.43 \pm 0.04$	$1.09 \pm 0.10$
	5	$-0.33 \pm 0.01$	$-0.31 \pm 0.03$	$-0.59 \pm 0.01$	$0.75 \pm 0.04$
	<b>Avg.</b>			$-0.58 \pm 0.02$	$1.00 \pm 0.06$
Beam 2	2	$-0.85 \pm 0.01$	$-0.66 \pm 0.03$	$-0.57 \pm 0.02$	$1.09 \pm 0.06$
	3	$-2.93 \pm 0.05$	$-4.40 \pm 0.08$	$-0.53 \pm 0.08$	$1.66 \pm 0.16$
	4	$-2.21 \pm 0.02$	$-0.00 \pm 0.03$	$-0.46 \pm 0.02$	$1.18 \pm 0.05$
	5	$-0.53 \pm 0.02$	$-0.09 \pm 0.03$	$-0.57 \pm 0.02$	$0.98 \pm 0.05$
	<b>Avg.</b>			$-0.53 \pm 0.02$	$1.10 \pm 0.06$
<b>Vertical</b>					
Beam 1	1	$0.97 \pm 0.02$	$1.62 \pm 0.05$	$0.15 \pm 0.03$	$-0.88 \pm 0.09$
	2	$-0.23 \pm 0.01$	$0.13 \pm 0.02$	$0.09 \pm 0.02$	$-0.60 \pm 0.03$
	3	$0.83 \pm 0.02$	$1.97 \pm 0.03$	$0.29 \pm 0.02$	$-0.68 \pm 0.05$
	4	$0.62 \pm 0.01$	$-0.18 \pm 0.03$	$0.00 \pm 0.02$	$-0.56 \pm 0.05$
	<b>Avg.</b>			$0.13 \pm 0.02$	$-0.62 \pm 0.04$
Beam 2	1	$0.79 \pm 0.03$	$1.54 \pm 0.06$	$0.24 \pm 0.04$	$-0.74 \pm 0.13$
	2	$-0.29 \pm 0.01$	$0.10 \pm 0.02$	$0.13 \pm 0.02$	$-0.58 \pm 0.04$
	3	$0.89 \pm 0.02$	$2.05 \pm 0.03$	$0.32 \pm 0.03$	$-0.73 \pm 0.06$
	4	$0.60 \pm 0.02$	$-0.14 \pm 0.03$	$0.04 \pm 0.02$	$-0.66 \pm 0.05$
	<b>Avg.</b>			$0.14 \pm 0.02$	$-0.64 \pm 0.05$

Table 5.6.: Summary of the chromaticity values obtained from the measurements presented in Table 5.3.

A summary of the measured chromaticity orders is given in Table 5.6. The first measurement in the horizontal plane of Beam 2 showed a high correlation between  $Q^{(4)}$  and  $Q^{(5)}$ , making the fit less reliable and is not included. The last measurement for the vertical plane experienced some tune drift, making the fit impossible and is therefore not included for both beams.

### Fit Quality

The values measured for  $Q^{(4)}$  and  $Q^{(5)}$  are similar across the measurements, with nominal and beam-based corrections performed with very different lower order chromaticity, and well separated in time. This reproducibility with varying configurations gives confidence that the measured values are robust. It is to be noted that one exception exists for the first measurement, the horizontal plane of beam 2 showed a high correlation between the fourth and fifth order terms, making the fit less reliable.

The reduced chi-square for the last measurement of 2022 for each fit order is detailed in Table 5.7. While adding terms to the chromaticity function is beneficial to the fit, it can be seen that the reduced

chi-square does not substantially improve above the fifth order, indicating that such further orders are not warranted.

Plane	$Q^{(3)}$	$Q^{(4)}$	$\chi^2_\nu$	$Q^{(5)}$	$Q^{(6)}$
Beam 1					
X	17.9	12.1	1.8	1.5	
Y	3.0	2.2	0.7	0.7	
Beam 2					
X	17.3	7.1	1.8	1.8	
Y	2.9	2.8	1.0	1.0	

Table 5.7.: Reduced  $\chi^2$  values for each order of fit, taken from the second measurement of 2022, with  $Q''$  and  $Q'''$  beam-based corrections.

## 5.3. Model Estimates

The model of the LHC is based on MAD-X and WISE field errors [15], containing a hundred seeds for the random errors. To compute the chromaticity, simulations are run via PTC, with various field errors.

### 5.3.1. Decatetrapolar Decay

It has been noted in the previous chapter about decapoles (see Section 4.5) that the  $b_5$  component in the main dipoles was large at injection energy, and could explain most of the discrepancy between the measurements and simulations. Such a decay in the main dipoles also exists for the  $b_7$  component [58], and is shown in Fig. 5.8.

Its value is though small and settles around  $+0.0351 \pm 0.0007$ . The average  $b_7$  of the main dipoles is of  $0.32 \pm 0.16$ . The decay thus increase that value of only about 11%. Simulations done with that decay taken into account are detailed next.

### 5.3.2. Major contributions

Simulations with various field errors have been run to assess the contribution of individual magnet order and combinations for the fourth and firth order chromaticity. Normal and skew fields errors ranging from sextupolar ( $b_3$ ) to decahexapolar ( $b_8$ ) are added alone or in combination to observe which

## 5. Higher Order Fields

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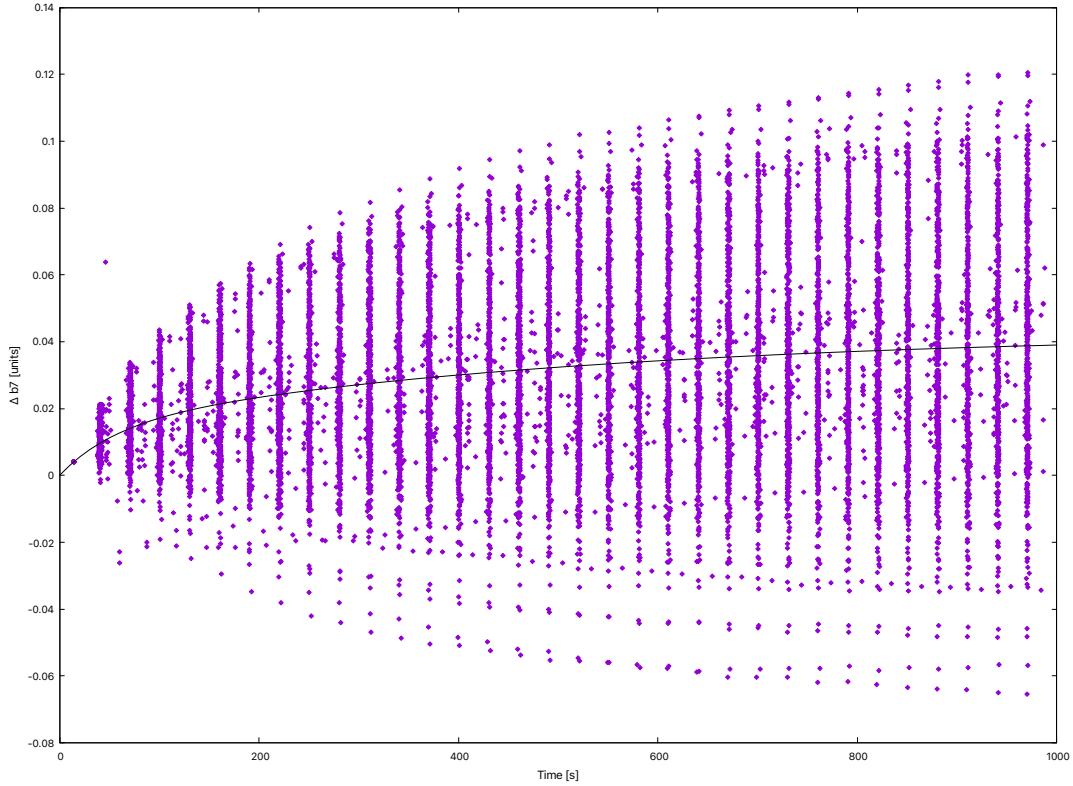


Figure 5.8.: Measured decay of the integrated decatetrapolar field in LHC's main dipoles at injection energy. The fit is shown in black [58] and settles around +0.035.

is the strongest. Quadrupolar field errors ( $b_2$ ) introduce beta-beating. Coupling is introduced via skew quadrupolar correctors.

### Fourth Order Chromaticity

The results from simulations strongly imply that the dodecapolar errors are the main contributors to  $Q^{(4)}$ , as can be seen in Fig. 5.9. Fringe fields have a negligible impact, as do skew multipoles. The most notable effect on this chromaticity order is the beta-beating, introducing a very large spread via the various error seeds. Comparing the simulation at the top with most errors added, the  $b_6$  component alone accounts for  $\approx 70\%$  of it for both axes on each beam.

### Fifth Order Chromaticity

It is seen that the the decatetrapolar errors are the main contributors to  $Q^{(5)}$ , as can be seen in Fig. 5.10. Fringe fields and skew multipoles have been found to have a negligible impact. Beta-beating and



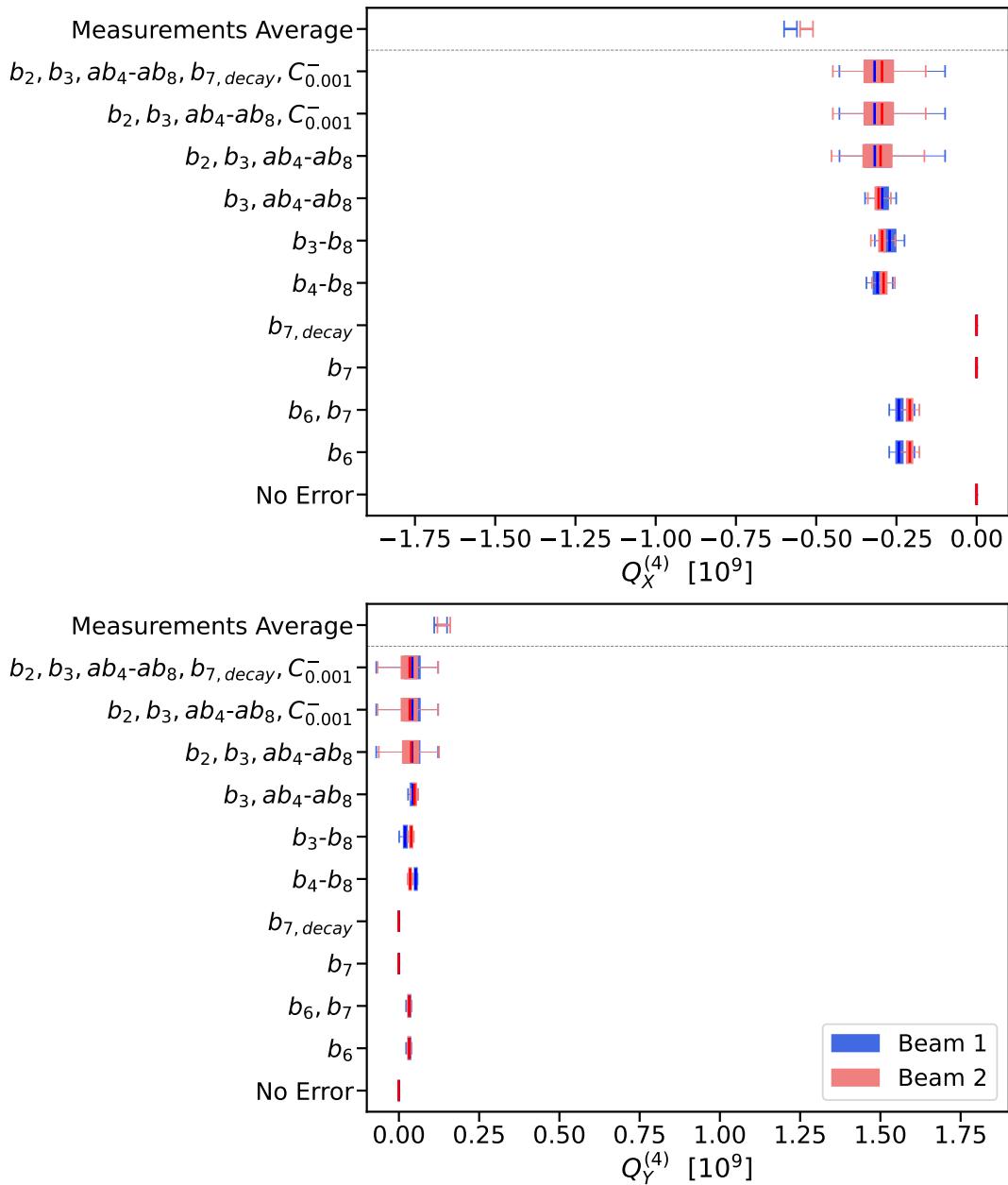


Figure 5.9.: Measured and simulated fourth order chromaticity with different multipole errors. The  $b_2$  errors, applied on dipoles and quadrupoles, generate beta-beating. Coupling is set to a value commonly seen in operation.

coupling are seen to increase by a small amount the chromaticity, while sextupolar errors induce a spread with the different seeds. As for the fourth order, comparing the simulation at the top with most

## 5. Higher Order Fields

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errors added, the  $b_7$  component alone accounts for  $\approx 70\%$  of it for both axes on each beam of the fifth order chromaticity.

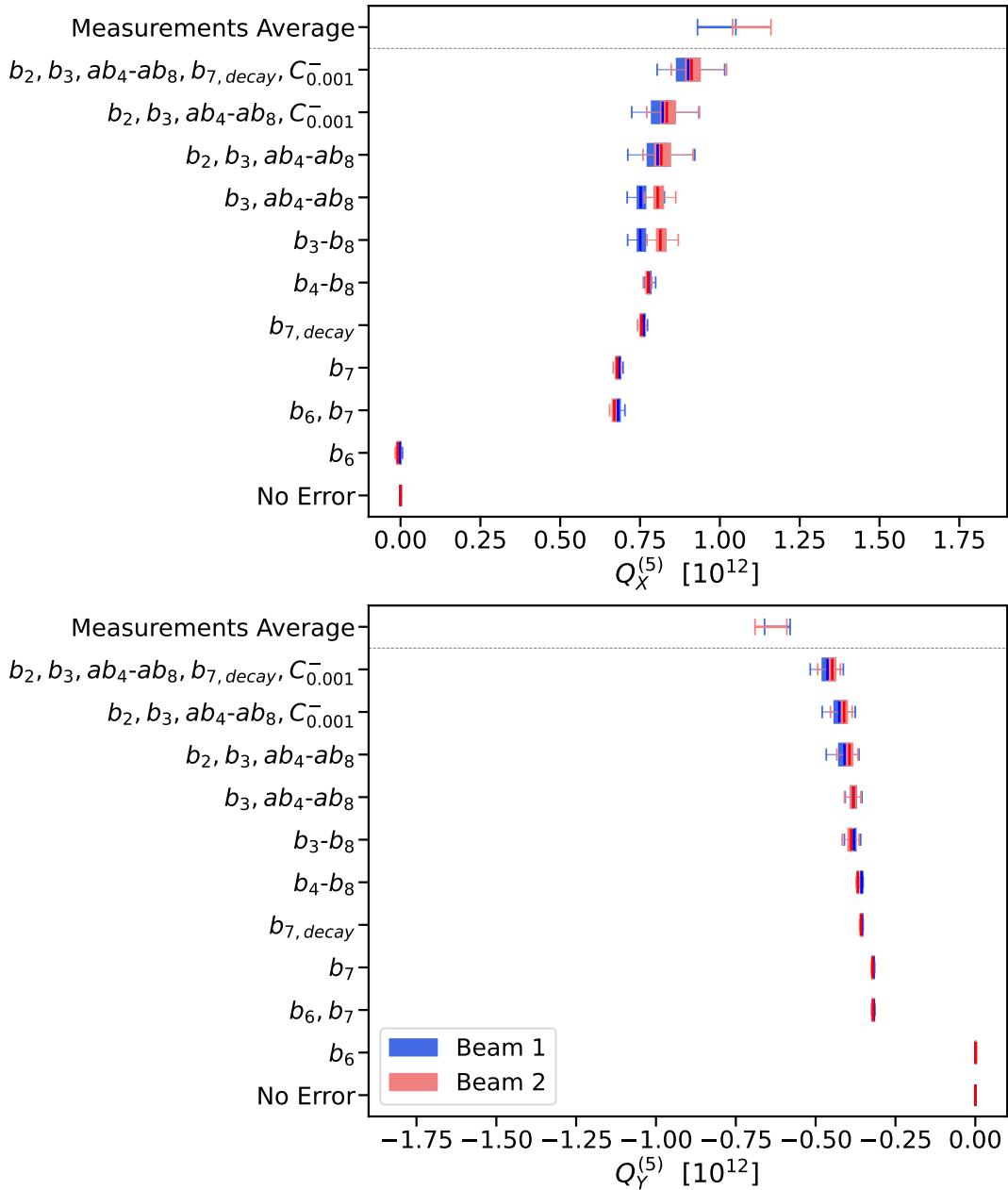


Figure 5.10.: Measured and simulated fifth order chromaticity with different multipole errors. The  $b_2$  errors, applied on dipoles and quadrupoles, generate beta-beating. Coupling is set to a value commonly seen in operation.

### 5.3.3. Ratios

Previous simulation results are shown in Table 5.8, taking the values from the simulation including the most effects at the top of the plots. For the fourth order, the beating is not included as the large beating is not yet explained. Table 5.9 shows the ratio between the measured average and simulated chromaticities.

Plane	$Q^{(4)} [10^9]$	$Q^{(5)} [10^{12}]$
Beam 1		
X	$-0.29 \pm 0.02$	$0.90 \pm 0.05$
Y	$0.04 \pm 0.01$	$-0.46 \pm 0.03$
Beam 2		
X	$-0.31 \pm 0.02$	$0.92 \pm 0.03$
Y	$0.05 \pm 0.00$	$-0.45 \pm 0.01$

Table 5.8.: Simulated high order chromaticity terms via PTC at injection energy, including normal and skew sextupolar to decahexapolar field errors. Are also included beta-beating, coupling and decatetrapolar decay. For the fourth order, the values do not include beta-beating as the observed spread is not yet fully understood.

Plane	$Q^{(4)}$ Ratio	$Q^{(5)}$ Ratio
Beam 1		
X	$1.98 \pm 0.16$	$1.10 \pm 0.09$
Y	$3.00 \pm 0.70$	$1.34 \pm 0.11$
Beam 2		
X	$1.74 \pm 0.11$	$1.20 \pm 0.08$
Y	$2.90 \pm 0.50$	$1.42 \pm 0.12$

Table 5.9.: Ratios of the simulated and average measured high-order chromaticity terms. The values are taken from Table 5.6 and Table 5.8.

The similar ratios between planes and beams for the fifth order could indicate a systematic error not modeled. The plane differences for the fourth order are not yet explained but could be linked to a lack of clear reproducibility induced by lower orders in the vertical plane. It indeed seems that the fourth order follows a trend with the third order.

## 5.4. Dodecapolar RDTs

b6 meas from 2024 commissioning

## 5.5. Going Further

2024 chromaticity measurements without range cutting.

## 5.6. Summary

A wider momentum offset range, combined with new analysis techniques permitted the observation of the fourth and fifth order of the chromaticity function for the first time in the LHC. Reproducible values were measured with different machine configurations.

Simulations show that the observed values do not match well with the LHC non-linear model. A factor  $\approx 1.5$  is observed between measurements and simulations for  $Q^{(5)}$ , which may point to a systematic error in the b7 error model.

Correction of the measured higher order chromaticity terms is not possible, due to the lack of adequate correctors in the LHC. It is nevertheless interesting to characterize the higher order errors for an effective model and understand the effect a higher order fit has on lower order terms. Precise measurement of those lower chromaticity terms is required in order to effectively correct them. Higher order terms have thus to be taken into account.

The current range of momentum offset is deemed sufficient to measure higher order chromaticity. Attempts will, however, be taken to increase that range and assess if such a wider range can refine the estimate of  $Q^{(4)}$  and  $Q^{(5)}$ .



# Skew Octupolar Fields

## 6.1. Correction of skew octupole Fields at Top Energy

1. RDT Measurements
2. Orthogonality of correctors
3. Response Matrix
4. Correction
5. Comparison to 2018

## 6. Skew Octupolar Fields

### 6.2. Correction of Skew Octupole Fields at Injection Energy

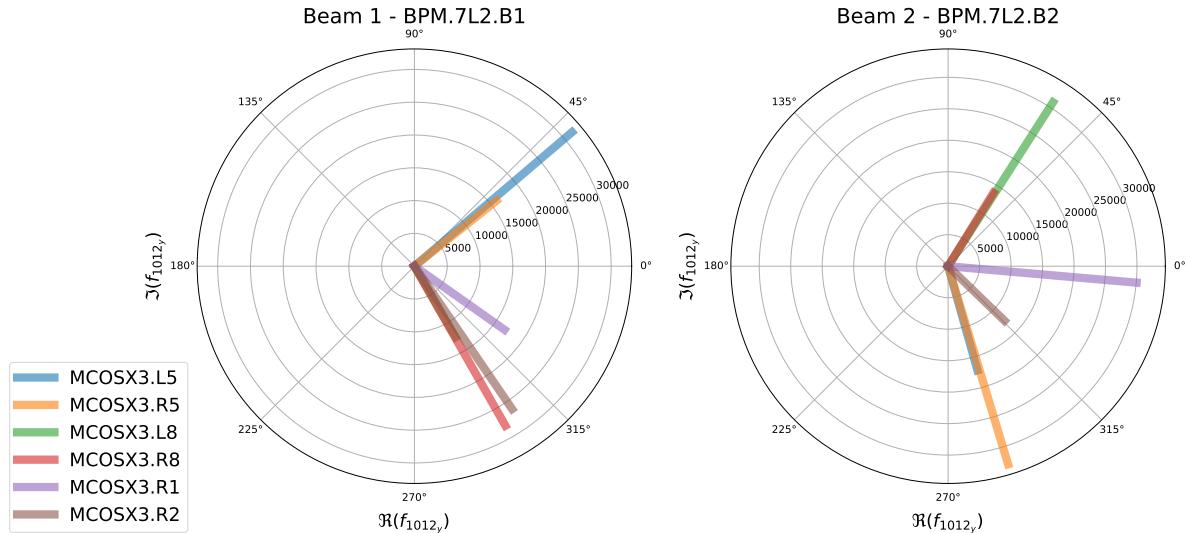


Figure 6.1.

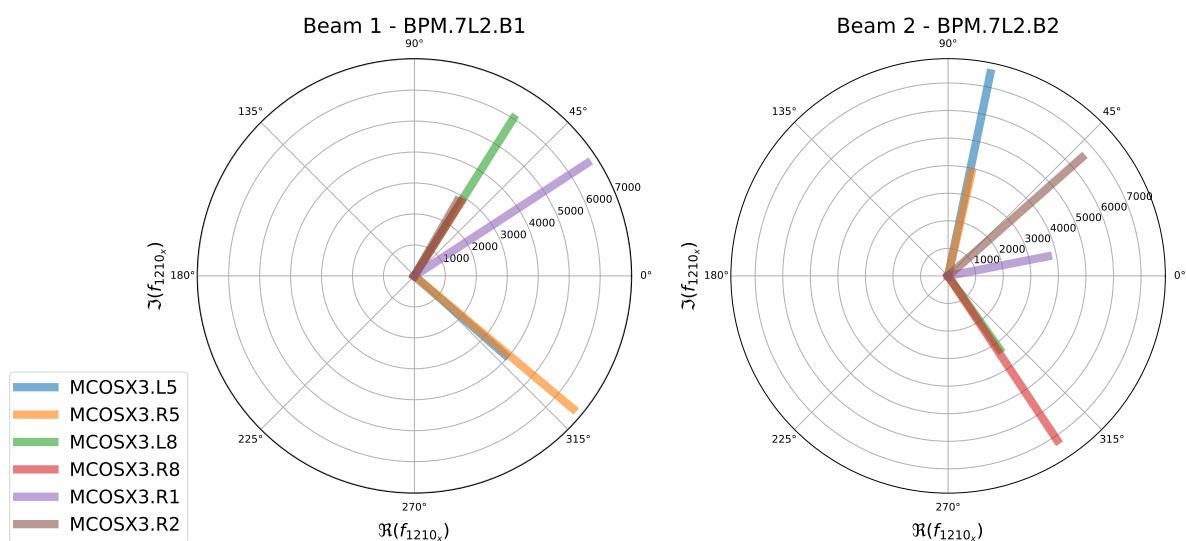


Figure 6.2.

### 6.3. Skew Octupolar Fields from Landau Octupoles

# Hamiltonians and Transfer Maps

This appendix is intended to gather and explicit the Hamiltonians of the elements used in this thesis. Non-linear transfer maps are also described for some of those elements from the first to the second order.

## A.1. Hamiltonians of Elements

The Hamiltonian of a *multipole* is the following [19–21]:

$$H = \Re \left[ \sum_{n>1} (K_n + iJ_n) \frac{(x+iy)^n}{n!} \right]. \quad (\text{A.1})$$

From this, normal and skew fields can be separated:

$$\begin{aligned} N_n &= \frac{1}{n!} K_n \Re [(x+iy)^n] \\ S_n &= -\frac{1}{n!} J_n \Im [(x+iy)^n], \end{aligned} \quad (\text{A.2})$$

where  $K$  and  $J$  are the normalized strength of the multipole and  $x, y$  the transverse coordinates. Table A.1 explicits the normal and skew Hamiltonians of multipoles up to order 8.

## A. Hamiltonians and Transfer Maps

---

Name	Order	Normal and Skew Hamiltonians
Drift	-	$H = \frac{1}{2}(p_x^2 + p_y^2)$
Quadrupole	2	$N_2 = \frac{1}{2!}K_2(x^2 - y^2)$ $S_2 = -J_2xy$
Sextupole	3	$N_3 = \frac{1}{3!}K_3(x^3 - 3xy^2)$ $S_3 = -\frac{1}{3!}J_3 \cdot (3x^2y - y^3)$
Octupole	4	$N_4 = \frac{1}{4!}K_4(x^4 - 6x^2y^2 + y^4)$ $S_4 = -\frac{1}{4!}J_4 \cdot (4x^3y - 4xy^3)$
Decapole	5	$N_5 = \frac{1}{5!}K_5(x^5 - 10x^3y^2 + 5xy^4)$ $S_5 = -\frac{1}{5!}J_5 \cdot (5x^4y - 10x^2y^3 + y^5)$
Dodecapole	6	$N_6 = \frac{1}{6!}K_6(x^6 - 15x^4y^2 + 15x^2y^4 - y^6)$ $S_6 = -\frac{1}{6!}J_6 \cdot (6x^5y - 20x^3y^3 + 6xy^5)$
Decatetrapole	7	$N_7 = \frac{1}{7!}K_7(x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6)$ $S_7 = -\frac{1}{7!}J_7 \cdot (7x^6y - 35x^4y^3 + 21x^2y^5 - y^7)$
Decahexapole	8	$N_8 = \frac{1}{8!}K_8(x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8)$ $S_8 = -\frac{1}{8!}J_8 \cdot (8x^7y - 56x^5y^3 + 56x^3y^5 - 8xy^7)$

Table A.1.: Normal and skew Hamiltonians of multipoles up to order 8.

A

## A.2. Transfer Maps

This section goes more in depth regarding the derivation of the examples of transfer maps introduced in Section 2.3.

As a reminder, the BCH of two elements up to order 3 is given below,

$$Z = \underbrace{H_2 + H_1}_{\text{First order}} + \underbrace{\frac{[H_2, H_1]}{2}}_{\text{Second order}} + \underbrace{\frac{[H_2, [H_2, H_1]]}{12} - \frac{[H_1, [H_2, H_1]]}{12}}_{\text{Third order}}. \quad (\text{A.3})$$

### A.2.1. Generic Effective Hamiltonian of Two Elements

In order not to have to derive every combination of multipoles, a generic approach can be taken. Two elements of orders  $n$  and  $m$  can indeed be combined together via the BCH. Their hamiltonians is thus the one from Eq. (A.1) with orders  $m$  and  $n$ . A drift is put in between the two multipoles to change the coordinates. This results in non-zero Poisson brackets as the momentum is propagated,

$$[H_2, H_1] \rightarrow [H_2, D \cdot H_1]. \quad (\text{A.4})$$

Higher orders will arise with derivations when applying the Poisson brackets. Those orders can be found by counting the number of Poisson brackets, and which operators are implicated. A Poisson bracket implicating  $H_2$  will increase the resulting order by  $m$ , and likewise for  $H_1$  and  $n$ . The resulting order is then decreased by the number of poisson brackets times 2. Eq. (A.5) gives how this can be calculated when considering the written form of the Poisson brackets (eg.  $[H_2, [H_2, H_1]]$ ).

$$\text{resulting order} = \text{count}("H_2") \cdot m + \text{count}("H_1") \cdot n - 2 \cdot \text{count}("[") \quad (\text{A.5})$$

A

Although this formula is quite helpful, it is not trivial to compute which orders will arise from each BCH order. Indeed, not all combinations of  $H_1$  and  $H_2$  appear, as explained in details in [27]. Table A.2 shows the Poisson brackets corresponding to each order of the BCH, up to order 6, along with the resulting multipole order. To keep the table readable, Poisson Brackets are not shown above that order, as duplicates of resulting orders become quite frequent. Similarly, Table A.3 shows how fields can be generated depending on the orders of multipoles and that of the BCH.

### A. Hamiltonians and Transfer Maps

---

BCH Order	Poisson Brackets	Resulting Order
1	$A$	$m$
	$B$	$n$
2	$\frac{[A,B]}{2}$	$m+n-2$
3	$-\frac{[B,[A,B]]}{12}$ $\frac{[A,[A,B]]}{12}$	$m+2n-4$ $2m+n-4$
4	$-\frac{[B,[A,[A,B]]]}{24}$	$2m+2n-6$
5	$-\frac{[A,[B,[A,[A,B]]]]}{120}$ $-\frac{[B,[A,[A,[A,B]]]]}{180}$ $-\frac{[A,[B,[B,[A,B]]]]}{360}$ $-\frac{[A,[A,[A,[A,B]]]]}{720}$ $-\frac{[B,[B,[A,[A,B]]]]}{180}$ $-\frac{[B,[B,[B,[A,B]]]]}{720}$	$3m+2n-8$ $3m+2n-8$ $2m+3n-8$ $4m+n-8$ $2m+3n-8$ $m+4n-8$
6	$-\frac{[A,[A,[B,[A,[A,B]]]]]}{240}$ $-\frac{[A,[B,[B,[A,[A,B]]]]]}{240}$ $-\frac{[B,[B,[A,[A,[A,B]]]]]}{360}$ $-\frac{[A,[B,[B,[B,[A,B]]]]]}{720}$ $-\frac{[B,[A,[A,[A,[A,B]]]]]}{1440}$ $-\frac{[B,[B,[B,[A,[A,B]]]]]}{1440}$	$3m+3n-10$ $3m+3n-10$ $3m+3n-10$ $2m+4n-10$ $4m+2n-10$ $2m+4n-10$
7	$\dots$	$2m+5n-12$ $m+6n-12$ $3m+4n-12$ $5m+2n-12$ $6m+n-12$ $4m+3n-12$

Table A.2.: Resulting multipole orders arising from the Poisson brackets of a given BCH order for two multipoles  $A$  and  $B$  of order  $m$  and  $n$ . At order 7, the Poisson brackets are not given as duplicates grow the list significantly.

BCH Order						
Field	1	2	3	4	5	6
3	$H_3$					
4	$H_4$	$(H_3)^2$				
5	$H_5$	$H_3 H_4$	$(H_3)^3$			
6	$H_6$	$H_3 H_5,$ $(H_4)^2$	$(H_3)^2 H_4$	$(H_3)^4$		
7	$H_7$	$H_3 H_6,$ $H_4 H_5$	$(H_3)^2 H_5,$ $H_3 (H_4)^2$		$(H_3)^5$	
8	$H_8$	$H_3 H_7,$ $H_4 H_6$	$(H_3)^2 H_6,$ $(H_4)^3$	$(H_3)^2 (H_4)^2$	$(H_3)^4 H_4$	$(H_3)^6$
9	$H_9$	$H_4 H_7,$ $H_3 H_8,$ $H_5 H_6$	$(H_3)^2 H_7,$ $(H_4)^2 H_5,$ $H_3 (H_5)^2$		$(H_3)^3 (H_4)^2,$ $(H_3)^4 H_5$	
10	$H_{10}$	$H_4 H_8,$ $H_5 H_7,$ $H_3 H_9,$ $(H_6)^2$	$(H_3)^2 H_8,$ $H_4 (H_5)^2,$ $(H_4)^2 H_6$	$(H_4)^4,$ $(H_3)^2 (H_5)^2$	$(H_3)^2 (H_4)^3,$ $(H_3)^4 H_6$	$(H_3)^4 (H_4)^2$
11	$H_{11}$	$H_4 H_9,$ $H_3 H_{10},$ $H_6 H_7,$ $H_5 H_8$	$(H_4)^2 H_7,$ $H_3 (H_6)^2,$ $(H_5)^3,$ $(H_3)^2 H_9$		$(H_3)^4 H_7,$ $H_3 (H_4)^4,$ $(H_3)^3 (H_5)^2$	$(H_3)^3 (H_4)^3$
12	$H_{12}$	$H_6 H_8,$ $(H_7)^2,$ $H_5 H_9,$ $H_3 H_{11},$ $H_4 H_{10}$	$H_4 (H_6)^2,$ $(H_4)^2 H_8,$ $(H_5)^2 H_6,$ $(H_4)^2 (H_5)^2$	$(H_3)^2 (H_6)^2,$ $(H_4)^2 (H_5)^2$	$(H_3)^4 H_8,$ $(H_4)^5$	$(H_3)^4 (H_5)^2,$ $(H_3)^2 (H_4)^4$

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Table A.3.: Correspondence of a combination of multipoles from a BCH order to multipole-like fields. The exponents indicate the order of the BCH for individual components.

### A.2.2. Transfer Map of Two Sextupoles

The transfer map of two sextupoles  $H_1$  and  $H_2$  of strength  $K_1$  and  $K_2$ , separated by a drift to introduce a change of coordinates in  $H_1$  is the following,

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$$\mathcal{M} = e^{\cdot Z \cdot} = e^{\cdot H_2 \cdot} \cdot e^{D \cdot H_1 \cdot}, \quad (\text{A.6})$$

After such application of the drift on  $H_1$ , the two hamiltonians read,

$$\begin{aligned} H_1 &= \frac{1}{3!} K_1 L_1 \left( (L_D p_x + x)^3 - 3 (L_D p_x + x) (L_D p_y + y)^2 \right) \\ H_2 &= \frac{1}{3!} K_2 L_2 \left( x^3 - 3xy^2 \right). \end{aligned} \quad (\text{A.7})$$

Below are detailed each term of the BCH. Each term should be added together in order to obtain the whole effective Hamiltonian  $Z$ .

#### First Order

$$K_1 L_1 L_D \left( \begin{array}{l} \left( \frac{L_D^2 p_x^3}{6} - \frac{L_D^2 p_x p_y^2}{2} + \frac{L_D p_x^2 x}{2} - L_D p_x p_y y \right) \\ - \frac{L_D p_y^2 x}{2} + \frac{p_x x^2}{2} - \frac{p_x y^2}{2} - p_y x y \\ + K_1 L_1 \left( \frac{x^3}{6} - \frac{x y^2}{2} \right) + K_2 L_2 \left( \frac{x^3}{6} - \frac{x y^2}{2} \right) \end{array} \right) \left. \right\} \text{sextupolar} \quad (\text{A.8})$$

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#### Second Order

$$K_1 K_2 L_1 L_2 L_D \left( \begin{array}{l} \left( \frac{L_D^2 p_x^2 x^2}{8} - \frac{L_D^2 p_x^2 y^2}{8} + \frac{L_D^2 p_x p_y x y}{2} - \frac{L_D^2 p_y^2 x^2}{8} \right) \\ + \frac{L_D^2 p_y^2 y^2}{8} + \frac{L_D p_x x^3}{4} + \frac{L_D p_x x y^2}{4} + \frac{L_D p_y x^2 y}{4} \\ + \frac{L_D p_y y^3}{4} + \frac{x^4}{8} + \frac{x^2 y^2}{4} + \frac{y^4}{8} \end{array} \right) \left. \right\} \text{octupolar-like} \quad (\text{A.9})$$

### Third Order

$$\begin{aligned}
& K_1^2 K_2 L_1^2 L_2 L_D \left( \begin{array}{l} \frac{L_D^5 p_x^4 x}{48} + \frac{L_D^5 p_x^3 p_y y}{12} - \frac{L_D^5 p_x^2 p_y^2 x}{8} - \frac{L_D^5 p_x p_y^3 y}{12} \\ + \frac{L_D^5 p_y^4 x}{48} + \frac{L_D^4 p_x^3 x^2}{12} + \frac{L_D^4 p_x^3 y^2}{12} - \frac{L_D^4 p_x p_y^2 x^2}{4} \\ - \frac{L_D^4 p_x p_y^2 y^2}{4} + \frac{L_D^3 p_x^2 x^3}{8} + \frac{L_D^3 p_x^2 x y^2}{8} - \frac{L_D^3 p_x p_y x^2 y}{4} \\ - \frac{L_D^3 p_x p_y y^3}{4} - \frac{L_D^3 p_y^2 x^3}{8} - \frac{L_D^3 p_y^2 x y^2}{8} + \frac{L_D^2 p_x x^4}{12} \\ - \frac{L_D^2 p_x y^4}{12} - \frac{L_D^2 p_y x^3 y}{6} - \frac{L_D^2 p_y x y^3}{6} + \frac{L_D x^5}{48} \\ - \frac{L_D x^3 y^2}{24} - \frac{L_D x y^4}{16} \end{array} \right) \\
& + K_1 K_2^2 L_1 L_2^2 L_D \left( \begin{array}{l} \frac{L_D^2 p_x x^4}{48} - \frac{L_D^2 p_x x^2 y^2}{8} + \frac{L_D^2 p_x y^4}{48} + \frac{L_D^2 p_y x^3 y}{12} \\ - \frac{L_D^2 p_y x y^3}{12} + \frac{L_D x^5}{48} - \frac{L_D x^3 y^2}{24} - \frac{L_D x y^4}{16} \end{array} \right)
\end{aligned} \quad \left. \right\} \text{decapolar-like} \quad (\text{A.10})$$

### A.2.3. Transfer Map of a Sextupole and Octupole

The transfer map of a sextupole  $H_1$  and octupole  $H_2$  of strength  $K_1$  and  $K_2$ , separated by a drift like in the previous example is given by

$$\mathcal{M} = e^{:Z:} = e^{:H_2:} \cdot e^{D:H_1:} \quad (\text{A.11})$$

with  $H_1$  and  $H_2$  having as final expressions,

$$\begin{aligned}
H_1 &= \frac{1}{3!} K_{3,h1} L_1 \left( (L_D p_x + x)^3 - 3 (L_D p_x + x) (L_D p_y + y)^2 \right) \\
H_2 &= \frac{1}{4!} K_2 L_2 \left( x^4 - 6x^2 y^2 + y^4 \right).
\end{aligned} \quad (\text{A.12})$$

The first two orders of the BCH of those two elements is given below.

### First Order

$$\begin{aligned}
& K_3 \left( \begin{array}{l} \frac{L_D^3 p_x^3}{6} - \frac{L_D^3 p_x p_y^2}{2} + \frac{L_D^2 p_x^2 x}{2} - L_D^2 p_x p_y y - \frac{L_D^2 p_y^2 x}{2} \\ + \frac{L_D p_x x^2}{2} - \frac{L_D p_x y^2}{2} - L_D p_y x y + \frac{x^3}{6} - \frac{x y^2}{2} \end{array} \right) \left. \right\} \text{sextupolar} \\
& + K_4 \left( \frac{x^4}{24} - \frac{x^2 y^2}{4} + \frac{y^4}{24} \right) \left. \right\} \text{octupolar}
\end{aligned} \quad (\text{A.13})$$

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### Second Order

$$K_3 K_4 L_D \left( \begin{array}{l} \left( \frac{L_D^2 p_x^2 x^3}{24} - \frac{L_D^2 p_x^2 x y^2}{8} + \frac{L_D^2 p_x p_y x^2 y}{4} - \frac{L_D^2 p_x p_y y^3}{12} \right) \\ \left( -\frac{L_D^2 p_y^2 x^3}{24} + \frac{L_D^2 p_y^2 x y^2}{8} + \frac{L_D p_x x^4}{12} - \frac{L_D p_x y^4}{12} \right) \\ \left( + \frac{L_D p_y x^3 y}{6} + \frac{L_D p_y x y^3}{6} + \frac{x^5}{24} + \frac{x^3 y^2}{12} + \frac{x y^4}{24} \right) \end{array} \right) \text{ decapolar-like} \quad (\text{A.14})$$

### A.2.4. Transfer Map of a Skew Quadrupole and Octupole

The transfer map of a skew quadrupole  $H_1$  and octupole  $H_2$  of strength  $K_1$  and  $K_2$ , separated by a drift like in the previous examples is given by

$$\mathcal{M} = e^{:Z:} = e^{:H_2:} \cdot e^{D:H_1:} \quad (\text{A.15})$$

with  $H_1$  and  $H_2$  having as final expressions,

$$\begin{aligned} H_1 &= -J_1 L_1 (L_D p_x + x) (L_D p_y + y) \\ H_2 &= \frac{1}{4!} K_2 L_2 (x^4 - 6x^2 y^2 + y^4). \end{aligned} \quad (\text{A.16})$$

The first two orders of the BCH of those two elements is given below.

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### First Order

$$\begin{aligned} J_1 L_1 \left( -L_D^2 p_x p_y - L_D p_x y - L_D p_y x - x y \right) &\} \text{ skew quadrupolar} \\ + K_2 L_2 \left( \frac{x^4}{24} - \frac{x^2 y^2}{4} + \frac{y^4}{24} \right) &\} \text{ octupolar} \end{aligned} \quad (\text{A.17})$$

### Second Order

$$J_1 K_2 L_1 L_2 L_D \left( \begin{array}{l} \left( \frac{L_D p_x x^2 y}{4} - \frac{L_D p_x y^3}{12} - \frac{L_D p_y x^3}{12} \right) \\ \left( + \frac{L_D p_y x y^2}{4} + \frac{x^3 y}{6} + \frac{x y^3}{6} \right) \end{array} \right) \} \text{ skew octupolar-like} \quad (\text{A.18})$$

# Chromatic Amplitude Detuning

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This appendix details the derivations of chromatic amplitude detuning from sextupoles up to dodecapoles. *Chromatic Amplitude Detuning*, being detuning coming from both the action and the momentum offset, only starts appearing with decapoles. Below that order, only amplitude detuning and chromaticity can be observed. As those are part of the derivations, they will also be detailed here.

## B.1. Derivations

In this section, the terms of the chromatic amplitude detuning are given up to dodecapoles. Derivations are given only for sextupoles and octupoles, as the process remains fairly similar for higher orders. Vertical offsets will be neglected, as horizontal dispersion is dominant. The reasoning remains the same.

Up to the third order, the expression of the Taylor expansion of the Chromatic Amplitude Detuning around  $\epsilon_x$ ,  $\epsilon_y$  and  $\delta$ , for a tune  $Q_z$ ,  $z \in \{x, y\}$  reads:

## B. Chromatic Amplitude Detuning

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$$\begin{aligned}
Q_z(\epsilon_x, \epsilon_y, \delta) = Q_{z0} + & \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\
& + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\
& \quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] \\
& + \frac{1}{3!} \left[ \frac{\partial^3 Q_z}{\partial \delta^3} \delta^3 + \frac{\partial^3 Q_z}{\partial \epsilon_x^3} \epsilon_x^3 + \frac{\partial^3 Q_z}{\partial \epsilon_y^3} \epsilon_y^3 \right. \\
& \quad \left. + 3 \frac{\partial^3 Q_z}{\partial \epsilon_x \partial \delta^2} \delta^2 \epsilon_x + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \delta^2} \delta^2 \epsilon_y + 3 \frac{\partial^3 Q_z}{\partial \epsilon_x^2 \partial \delta} \delta \epsilon_x^2 \right. \\
& \quad \left. + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y^2 \partial \delta} \delta \epsilon_y^2 + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \epsilon_x^2} \epsilon_x^2 \epsilon_y + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y^2 \partial \epsilon_x} \epsilon_x \epsilon_y^2 \right. \\
& \quad \left. + 6 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \epsilon_x \partial \delta} \delta \epsilon_x \epsilon_y \right] + \dots
\end{aligned} \tag{B.1}$$

### B.1.1. Principle

From [30], the detuning caused by a magnet of length L can be described by its hamiltonian with

$$\Delta Q_z = \frac{1}{2\pi} \int_L \frac{\partial \langle H \rangle}{\partial J_z} ds. \tag{B.2}$$

The usual variables  $x$  and  $y$  of Eq. (2.7) can be replaced by *action-angle* variables to introduce the action:

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$$\begin{aligned}
x &\rightarrow \sqrt{2J_x \beta_x} \cos \phi_x \\
y &\rightarrow \sqrt{2J_y \beta_y} \cos \phi_y
\end{aligned} \tag{B.3}$$

A momentum dependence can be introduced for a particle with a different orbit ( $\Delta z$ ) [59] via dispersion. Combined with Eq. (B.3), a dependence on all required components is achieved:

$$\begin{aligned}
x + \Delta x &\rightarrow \sqrt{2J_x \beta_x} \cos \phi_x + D_x \delta \\
y + \Delta y &\rightarrow \sqrt{2J_y \beta_y} \cos \phi_y + D_y \delta
\end{aligned} \tag{B.4}$$

After averaging over the phase variable, all that is left is to compute the partial derivatives.

### B.1.2. Sextupole

Taken from Table A.1, the normal field of a sextupole is  $N_3 = \frac{1}{3!} K_3 (x^3 - 3xy^2)$ . Introducing  $\delta$  via an orbit offset and changing the variables to action-angle, as given in Eq. (B.4), leads to the following expression:

$$N_3 = \frac{1}{6} K_3 \left[ -3 \left( \delta D_x + \sqrt{2J_x\beta_x} \cos \phi_x \right) \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^2 + \left( \delta D_x + \sqrt{2J_x\beta_x} \cos \phi_x \right)^3 \right] \quad (\text{B.5})$$

Averaging over the cosines means integrating over  $[0, \pi]$  and dividing by  $\pi$ :

$$\begin{aligned} \langle N_3 \rangle &= \frac{1}{\pi} \int_0^\pi N_3 \\ &= \frac{1}{6\pi^2} K_3 \cdot \left( 3\pi^2 J_x \beta_x \delta D_x - 3\pi^2 J_y \beta_y \delta D_x + \pi^2 \delta^3 D_x^3 \right) \end{aligned} \quad (\text{B.6})$$

Differentiating by either  $J_x$  or  $J_y$  gives the tune shift induced by a single sextupole:

$$\begin{aligned} \Delta Q_x &= \frac{1}{4\pi} K_3 \beta_x D_x \delta L \\ \Delta Q_y &= -\frac{1}{4\pi} K_3 \beta_y D_x \delta L \end{aligned} \quad (\text{B.7})$$

From now on, differentiating by the action would give amplitude detuning, and by  $\delta$  the chromaticity. Cross-terms exist but evidently depend on the expression of the multipole. For a sextupole, differentiating by the action does not have an effect, as no action is present in the tune shift equation. Rather, sextupoles are known to contribute to the first order chromaticity  $Q'$ . The following gives a recap of those operations:

$$\begin{aligned} \frac{\partial Q_x}{\partial J_x} &= 0 & ; & \frac{\partial Q_x}{\partial J_y} = 0 & ; & \frac{\partial Q_x}{\partial \delta} = \frac{1}{4\pi} K_3 \beta_x D_x L \\ \frac{\partial Q_y}{\partial J_x} &= 0 & ; & \frac{\partial Q_y}{\partial J_y} = 0 & ; & \frac{\partial Q_y}{\partial \delta} = -\frac{1}{4\pi} K_3 \beta_y D_x L \end{aligned} \quad (\text{B.8})$$

The overall contribution of sextupoles to the Chromatic Amplitude Detuning is then highlighted in the following:

$$Q_z(\epsilon_x, \epsilon_y, \delta) = Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \quad (\text{B.9})$$

## B. Chromatic Amplitude Detuning

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### B.1.3. Octupole

With the normal hamiltonian of octupoles being  $N_4 = \frac{1}{4!}K_4(x^4 - 6x^2y^2 + y^4)$ , applying an orbit offset and changing for action-angle variables gives the following:

$$\begin{aligned}
N_4 = & \frac{1}{24}K_4 \left[ \left( \sqrt{2J_x\beta_x} \cos\phi_x \right)^4 \right. \\
& + 4 \left( \sqrt{2J_x\beta_x} \cos\phi_x \right)^3 D_x\delta \\
& + 6 \left( \sqrt{2J_x\beta_x} \cos\phi_x \right)^2 D_x^2\delta^2 \\
& + 4 \left( \sqrt{2J_x\beta_x} \cos\phi_x \right) D_x^2\delta \\
& + D_x^4\delta^4 \\
& - 6 \left( \sqrt{2J_x\beta_x} \cos\phi_x \right)^2 \left( \sqrt{2J_y\beta_y} \cos\phi_y \right)^2 \\
& - 6 \left( \sqrt{2J_y\beta_y} \cos\phi_y \right)^2 \cdot 2 \left( \sqrt{2J_x\beta_x} \cos\phi_x \right) D_x\delta \\
& - 6 \left( \sqrt{2J_y\beta_y} \cos\phi_y \right)^2 D_x^2\delta^2 \\
& \left. + \left( \sqrt{2J_y\beta_y} \cos\phi_y \right)^4 \right]
\end{aligned} \tag{B.10}$$

Average over the phase variables gives the following:

$$\begin{aligned}
\langle N_4 \rangle &= \frac{1}{\pi} \int_0^\pi N_4 \\
&= \frac{1}{48} K_4 \left[ 3J_x^2\beta_x^2 + 12J_x\beta_x D_x^2\delta^2 + 2D_x^4\delta^4 \right. \\
&\quad \left. - 12J_y\beta_y(J_x\beta_x + D_x^2\delta^2) + 3J_y^2\beta_y^2 \right]
\end{aligned} \tag{B.11}$$

Differentiating by either  $J_x$  or  $J_y$  gives the tune shift induced by a single octupole. Contrary to sextupoles, it can be noted that some action terms  $J_{x,y}$  remain in the expression.

$$\begin{aligned}
\Delta Q_x &= \frac{1}{48\pi} K_4 L \left[ 3J_x\beta_x^2 + 6\beta_x D_x^2\delta^2 - 6\beta_x J_y \beta_y \right] \\
\Delta Q_y &= \frac{1}{48\pi} K_4 L \left[ -6J_x\beta_x \beta_y - 6\beta_y D_x^2\delta^2 + 3J_y \beta_y^2 \right]
\end{aligned} \tag{B.12}$$

Differentiating by the action  $J_{x,y}$  yields amplitude detuning, while differentiating by the momentum offset yields the second order chromaticity  $Q''$ :

$$\begin{aligned}\frac{\partial Q_x}{\partial J_x} &= \frac{1}{16\pi} K_4 \beta_x^2 L \quad ; \quad \frac{\partial Q_x}{\partial J_y} = -\frac{1}{8\pi} K_4 \beta_x \beta_y L \quad ; \quad \frac{\partial^2 Q_x}{\partial \delta^2} = \frac{1}{4\pi} K_4 \beta_x D_x^2 L \\ \frac{\partial Q_y}{\partial J_x} &= -\frac{1}{8\pi} K_4 \beta_x \beta_y L \quad ; \quad \frac{\partial Q_y}{\partial J_y} = \frac{1}{16\pi} K_4 \beta_y^2 L \quad ; \quad \frac{\partial^2 Q_y}{\partial \delta^2} = -\frac{1}{4\pi} K_4 \beta_y D_x^2 L\end{aligned}\tag{B.13}$$

The overall contribution of octupoles to the Chromatic Amplitude Detuning is then highlighted in the following:

$$\begin{aligned}Q_z(\epsilon_x, \epsilon_y, \delta) &= Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\ &\quad + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\ &\quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right]\end{aligned}\tag{B.14}$$

#### B.1.4. Decapole

Skipping the first steps detailed previously, the tune shift induced by a normal decapole is given by the following:

$$\begin{aligned}\Delta Q_x &= \frac{1}{240\pi} K_5 L \left[ 10D_x^3 \delta^3 \beta_x + 15D_x \delta J_x \beta_x^2 - 30J_y \beta_y \beta_x D_x \delta \right] \\ \Delta Q_y &= \frac{1}{240\pi} K_5 L \left[ -10D_x^3 \delta^3 \beta_y + 15D_x \delta J_y \beta_y^2 - 30J_x \beta_y \beta_x D_x \delta \right]\end{aligned}\tag{B.15}$$

The terms of the chromatic amplitude detuning can now be computed. Unlike sextupoles and octupoles, cross-terms between  $\delta$  and the action now appear, giving rise to *chromatic amplitude detuning* and the third order chromaticity  $Q'''$ :

$$\begin{aligned}\frac{\partial^2 Q_x}{\partial J_x \partial \delta} &= \frac{1}{16\pi} K_5 \beta_x^2 D_x L \quad ; \quad \frac{\partial^2 Q_x}{\partial J_y \partial \delta} = \frac{1}{8\pi} K_5 \beta_x \beta_y D_x L \quad ; \quad \frac{\partial^3 Q_x}{\partial \delta^3} = \frac{1}{4\pi} K_5 \beta_x D_x^3 L \\ \frac{\partial^2 Q_y}{\partial J_x \partial \delta} &= -\frac{1}{8\pi} K_5 \beta_x \beta_y D_x L \quad ; \quad \frac{\partial^2 Q_y}{\partial J_y \partial \delta} = \frac{1}{16\pi} K_5 \beta_y^2 D_x L \quad ; \quad \frac{\partial^3 Q_y}{\partial \delta^3} = -\frac{1}{4\pi} K_5 \beta_y D_x^3 L\end{aligned}\tag{B.16}$$

The contribution of decapoles to Chromatic Amplitude Detuning is then the following:

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$$\begin{aligned}
Q_z(\epsilon_x, \epsilon_y, \delta) = Q_{z0} + & \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\
& + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\
& \quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] \\
& + \frac{1}{3!} \left[ \frac{\partial^3 Q_z}{\partial \delta^3} \delta^3 + \dots \right]
\end{aligned} \tag{B.17}$$

### B.1.5. Dodecapole

The tune shift induced by a normal dodecapole is given by the following:

$$\begin{aligned}
Q_x = & \frac{1}{720\pi} K_6 L \left[ 15J_x^2 \beta_x^3 + 30\beta_x D_x^4 \delta^4 + 90J_x \beta_x^2 D_x^2 \delta^2 \right. \\
& \quad \left. - 180J_y \beta_y \beta_x D_x^2 \delta^2 - 90J_y \beta_y J_x \beta_x^2 + 90J_y^2 \beta_y^2 \beta_x \right] \\
Q_y = & \frac{1}{720\pi} K_6 L \left[ -30\beta_y D_x^4 \delta^4 - 180\beta_y J_x \beta_x D_x^2 \delta^2 - 90\beta_y J_x^2 \beta_x^2 \right. \\
& \quad \left. + 90J_y \beta_y^2 J_x \beta_x + 45J_y \beta_y^2 D_x^2 \delta^2 - 15J_y^2 \beta_y^3 \right]
\end{aligned} \tag{B.18}$$

The terms of the chromatic amplitude detuning can now be computed. Like for decapoles, chromatic amplitude detuning terms appear, as well as the third order chromaticity  $Q^{(4)}$ :

$$\begin{aligned}
\frac{\partial^2 Q_x}{\partial J_x^2} = & \frac{1}{96\pi} K_6 \beta_x^3 L \quad ; \quad \frac{\partial^2 Q_y}{\partial J_y^2} = -\frac{1}{96\pi} K_6 \beta_y^3 L \\
\frac{\partial^3 Q_x}{\partial J_x \partial \delta^2} = & \frac{1}{16\pi} K_6 \beta_x^2 D_x^2 L \quad ; \quad \frac{\partial^3 Q_y}{\partial J_y \partial \delta^2} = \frac{1}{16\pi} K_6 \beta_y^2 D_x^2 L \\
\frac{\partial^2 Q_x}{\partial J_y^2} = & \frac{1}{32\pi} K_6 \beta_y^2 \beta_x L \quad ; \quad \frac{\partial^2 Q_y}{\partial J_x^2} = -\frac{1}{32\pi} K_6 \beta_y \beta_x^2 L \\
\frac{\partial^3 Q_x}{\partial J_y \partial \delta^2} = & -\frac{1}{8\pi} K_6 \beta_y \beta_x D_x^2 L \quad ; \quad \frac{\partial^3 Q_y}{\partial J_x \partial \delta^2} = -\frac{1}{8\pi} K_6 \beta_y \beta_x D_x^2 L \\
\frac{\partial^2 Q_x}{\partial J_x \partial J_y} = & -\frac{1}{32\pi} K_6 \beta_y \beta_x^2 L \quad ; \quad \frac{\partial^2 Q_y}{\partial J_y \partial J_x} = \frac{1}{32\pi} K_6 \beta_y^2 \beta_x L \\
\frac{\partial^4 Q_x}{\partial \delta^4} = & \frac{1}{4\pi} K_6 \beta_x D_x^4 L \quad ; \quad \frac{\partial^4 Q_y}{\partial \delta^4} = -\frac{1}{4\pi} K_6 \beta_y D_x^4 L
\end{aligned} \tag{B.19}$$

The contribution of dodecapoles to Chromatic Amplitude Detuning is then the following:

$$\begin{aligned}
 Q_z(\epsilon_x, \epsilon_y, \delta) = & Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\
 & + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + \dots \right] \\
 & + \frac{1}{3!} \left[ 3 \frac{\partial^3 Q_z}{\partial \epsilon_x \partial \delta^2} \delta^2 \epsilon_x + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \delta^2} \delta^2 \epsilon_y + \dots \right] \\
 & + \frac{1}{4!} \left[ \frac{\partial^4 Q_z}{\partial \delta^4} \delta^4 \right]
 \end{aligned} \tag{B.20}$$

## B.2. PTC Validation

A simulation has been done with PTC to assess that those equations are correct. A dodecapole has been added to the lattice with a strength  $KL = 1e^6$ . Here are the results, confirming PTC works as intended.

The ANH numbers refer to the partial derivative relative to  $J_x, J_y$  and  $\delta$ . So ANHX 021 would for example be  $\frac{\partial^3 Q_x}{\partial J_y^2 \partial \delta}$ .

Term	Analytical	Simulation	Rel. Diff [%]
ANH X 200	4782639.96971	4782639.97	0.0
ANH X 102	86945.930342	86945.93	-0.0
ANH X 020	593469879.552116	593469880.01	0.0
ANH X 012	-1118366.433407	-1118366.433	-0.0
ANH X 110	-92277073.535598	-92277073.6	0.0
ANH X 004	1053.754809	1053.7548	-0.000001
ANH Y 200	-92277073.535598	-92277073.6	0.0
ANH Y 102	-1118366.433407	-1118366.433	-0.0
ANH Y 020	-1272278817.264865	-1272278818.913	0.0
ANH Y 012	3596325.539479	3596325.543	0.0
ANH Y 110	593469879.552116	593469880.01	0.0
ANH Y 004	-6777.108503	-6777.1085	-0.0

B

B

# Resonance Driving Terms

This appendix intends to clarify where Resonance Driving Terms can be seen in the frequency spectrum, what resonance they contribute to and what their action dependance is.

## C.1. Expressions

The number of valid RDTs indeed grows rapidly with the magnet order  $n$ , as shows Table C.1, and is given by the following combinations:

$$C(n+3, 3) - C(n+1, 1) - [(n+1) \bmod 2] \cdot C\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 1\right). \quad (\text{C.1})$$

Multipole	Order	Number of poles	Number of RDTs
Quadrupole	2	4	5
Sextupole	3	6	16
Octupole	4	8	27
Decapole	5	10	50
Dodecapole	6	12	73
Decatetrapole	7	14	112
Decahexapole	8	16	151
Hectopole	50	100	23349
Kilopole	500	1000	$2.1 \times 10^7$

Table C.1.: Number of valid RDTs for a given multipole order

Several different RDTs can contribute to the same line, which can be observed in the horizontal or vertical spectrum. The tables below describe which RDTs contribute to a specific combination of line

### C. Resonance Driving Terms

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and plane. All tables have been computed up to the order 6, for octopoles. The line columns represents  $(Q_x, Q_y)$ . For example  $(-1, 2)$  is  $-1Q_x + 2Qy$ .

As a reminder, for a given RDT  $f_{jklm}$ , we will observe:

$$\begin{array}{ll} (j-k)Q_x + (l-m)Q_y = p \in \mathbb{N} & \text{excited resonance} \\ H(1-j+k, m-l) & \text{horizontal line, if } j \neq 0 \\ V(k-j, 1-l+m) & \text{vertical line, if } l \neq 0. \end{array} \quad (\text{C.2})$$

The amplitude of each line is given by:

$$\begin{aligned} |H_{f_{jklm}}| &= 2j(2I_x)^{\frac{j+k-1}{2}}(2I_y)^{\frac{l+m}{2}}|f_{jklm}| \\ |V_{f_{jklm}}| &= 2l(2I_x)^{\frac{j+k}{2}}(2I_y)^{\frac{l+m-1}{2}}|f_{jklm}|. \end{aligned} \quad (\text{C.3})$$

According to equations C.2 and C.3, it can be seen that many RDTs will not generate any line and thus can not be observed.

## C.2. Frequency Spectrum Lines

The following Table C.2 shows which RDTs can be seen at a factor  $Q_x \pm Q_y$  in the vertical or horizontal spectrums. This table is mostly useful when trying to identify which multipole or RDT contributes to a specific line that can appear during measurements.

$Q_x$	$Q_y$	H-jklm	V-jklm
-5	0	6000	5010
-4	-1	5010	4020
-4	0	5000	4010
-4	1	5001	4011
-3	-2	4020	3030
-3	-1	4010	3020
-3	0	4000, 4011, 5100	3010, 3021, 4110
-3	1	4001	3011
-3	2	4002	3012
-2	-3	3030	2040
-2	-2	3020	2030
-2	-1	3010, 3021, 4110	2020, 2031, 3120
-2	0	3000, 3011, 4100	2010, 2021, 3110
-2	1	3001, 3012, 4101	2011, 2022, 3111
-2	2	3002	2012

$Q_x$	$Q_y$	H-jklm	V-jklm
-2	3	3003	2013
-1	-4	2040	1050
-1	-3	2030	1040
-1	-2	2020, 2031, 3120	1030, 1041, 2130
-1	-1	2010, 2021, 3110	1020, 1031, 2120
-1	0	2000, 2011, 3100, 2022, 3111, 4200	1010, 1021, 2110, 1032, 2121, 3210
-1	1	2001, 2012, 3101	1011, 1022, 2111
-1	2	2002, 2013, 3102	1012, 1023, 2112
-1	3	2003	1013
-1	4	2004	1014
0	-5	1050	0060
0	-4	1040	0050
0	-3	1030, 1041, 2130	0040, 0051, 1140
0	-2	1020, 1031, 2120	0030, 0041, 1130
0	-1	1010, 1021, 2110, 1032, 2121, 3210	0020, 0031, 1120, 0042, 1131, 2220
0	0	1011, 2100, 1022, 2111, 3200	0021, 1110, 0032, 1121, 2210
0	1	1001, 1012, 2101, 1023, 2112, 3201	
0	2	1002, 1013, 2102	0012, 0023, 1112
0	3	1003, 1014, 2103	0013, 0024, 1113
0	4	1004	0014
0	5	1005	0015
1	-4	1140	0150
1	-3	1130	0140
1	-2	1120, 1131, 2220	0130, 0141, 1230
1	0		0110, 0121, 1210, 0132, 1221, 2310
1	-1	1110, 1121, 2210	0120, 0131, 1220
1	1	1101, 1112, 2201	0111, 0122, 1211
1	2	1102, 1113, 2202	0112, 0123, 1212
1	3	1103	0113
1	4	1104	0114
2	-3	1230	0240
2	-2	1220	0230
2	-1	1210, 1221, 2310	0220, 0231, 1320
2	0	1200, 1211, 2300	0210, 0221, 1310
2	1	1201, 1212, 2301	0211, 0222, 1311
2	2	1202	0212
2	3	1203	0213
3	-2	1320	0330
3	-1	1310	0320
3	0	1300, 1311, 2400	0310, 0321, 1410
3	1	1301	0311
3	2	1302	0312

### C. Resonance Driving Terms

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$Q_x$	$Q_y$	H-jklm	V-jklm
4	-1	1410	0420
4	0	1400	0410
4	1	1401	0411
5	0	1500	0510

Table C.2.: Correspondence of Resonance Driving Terms to the lines seen in the horizontal or vertical frequency spectrum.

## C.3. Amplitude, Resonances and Lines

This part focuses on individual Resonance Drivings Terms, expliciting what magnet they originate from, what resonance they excite, how they can be observed and what kicks are needed in order to measure them. The amplitude columns implicitly omits the term  $|f_{jklm}|$ , which depends on  $K$  and  $J$ . The color coding helps quickly identifying dependences:

- $I_x$  : depends only on horizontal amplitude
- $I_y$  : depends only on vertical amplitude
- $I_x I_y$  : depends on both horizontal and vertical amplitudes

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
2	0020	normal	(0, 2)		(0, -1)		$4(2I_y)^{1/2}$
2	2000	normal	(2, 0)	(-1, 0)		$4(2I_x)^{1/2}$	
2	0110	skew	(-1, 1)		(1, 0)		$2(2I_x)^{1/2}$
2	1001	skew	(1, -1)	(0, 1)		$2(2I_y)^{1/2}$	
2	1010	skew	(1, 1)	(0, -1)	(-1, 0)	$2(2I_y)^{1/2}$	$2(2I_x)^{1/2}$
3	0111	normal	(-1, 0)		(1, 1)		$2(2I_x)^{1/2}(2I_y)^{1/2}$
3	0120	normal	(-1, 2)		(1, -1)		$4(2I_x)^{1/2}(2I_y)^{1/2}$
3	1002	normal	(1, -2)	(0, 2)		$2(2I_y)$	
3	1011	normal	(1, 0)	(0, 0)	(-1, 1)	$2(2I_y)$	$2(2I_x)^{1/2}(2I_y)^{1/2}$
3	1020	normal	(1, 2)	(0, -2)	(-1, -1)	$2(2I_y)$	$4(2I_x)^{1/2}(2I_y)^{1/2}$
3	1200	normal	(-1, 0)	(2, 0)		$2(2I_x)$	
3	2100	normal	(1, 0)	(0, 0)		$4(2I_x)$	
3	3000	normal	(3, 0)	(-2, 0)		$6(2I_x)$	
3	0012	skew	(0, -1)		(0, 2)		$2(2I_y)$
3	0021	skew	(0, 1)		(0, 0)		$4(2I_y)$
3	0030	skew	(0, 3)		(0, -2)		$6(2I_y)$
3	0210	skew	(-2, 1)		(2, 0)		$2(2I_x)$

### C.3. Amplitude, Resonances and Lines

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
3	1101	skew	(0, -1)	(1, 1)		$2(2I_x)^{1/2}(2I_y)^{1/2}$	
3	1110	skew	(0, 1)	(1, -1)	(0, 0)	$2(2I_x)^{1/2}(2I_y)^{1/2}$	$2(2I_x)$
3	2001	skew	(2, -1)	(-1, 1)		$4(2I_x)^{1/2}(2I_y)^{1/2}$	
3	2010	skew	(2, 1)	(-1, -1)	(-2, 0)	$4(2I_x)^{1/2}(2I_y)^{1/2}$	$2(2I_x)$
4	0013	normal	(0, -2)		(0, 3)		$2(2I_y)^{3/2}$
4	0031	normal	(0, 2)		(0, -1)		$6(2I_y)^{3/2}$
4	0040	normal	(0, 4)		(0, -3)		$8(2I_y)^{3/2}$
4	0211	normal	(-2, 0)		(2, 1)		$2(2I_x)(2I_y)^{1/2}$
4	0220	normal	(-2, 2)		(2, -1)		$4(2I_x)(2I_y)^{1/2}$
4	1102	normal	(0, -2)	(1, 2)		$2(2I_x)^{1/2}(2I_y)$	
4	1120	normal	(0, 2)	(1, -2)	(0, -1)	$2(2I_x)^{1/2}(2I_y)$	$4(2I_x)(2I_y)^{1/2}$
4	1300	normal	(-2, 0)	(3, 0)		$2(2I_x)^{3/2}$	
4	2002	normal	(2, -2)	(-1, 2)		$4(2I_x)^{1/2}(2I_y)$	
4	2011	normal	(2, 0)	(-1, 0)	(-2, 1)	$4(2I_x)^{1/2}(2I_y)$	$2(2I_x)(2I_y)^{1/2}$
4	2020	normal	(2, 2)	(-1, -2)	(-2, -1)	$4(2I_x)^{1/2}(2I_y)$	$4(2I_x)(2I_y)^{1/2}$
4	3100	normal	(2, 0)	(-1, 0)		$6(2I_x)^{3/2}$	
4	4000	normal	(4, 0)	(-3, 0)		$8(2I_x)^{3/2}$	
4	0112	skew	(-1, -1)		(1, 2)		$2(2I_x)^{1/2}(2I_y)$
4	0121	skew	(-1, 1)		(1, 0)		$4(2I_x)^{1/2}(2I_y)$
4	0130	skew	(-1, 3)		(1, -2)		$6(2I_x)^{1/2}(2I_y)$
4	0310	skew	(-3, 1)		(3, 0)		$2(2I_x)^{3/2}$
4	1003	skew	(1, -3)	(0, 3)		$2(2I_y)^{3/2}$	
4	1012	skew	(1, -1)	(0, 1)	(-1, 2)	$2(2I_y)^{3/2}$	$2(2I_x)^{1/2}(2I_y)$
4	1021	skew	(1, 1)	(0, -1)	(-1, 0)	$2(2I_y)^{3/2}$	$4(2I_x)^{1/2}(2I_y)$
4	1030	skew	(1, 3)	(0, -3)	(-1, -2)	$2(2I_y)^{3/2}$	$6(2I_x)^{1/2}(2I_y)$
4	1201	skew	(-1, -1)	(2, 1)		$2(2I_x)(2I_y)^{1/2}$	
4	1210	skew	(-1, 1)	(2, -1)	(1, 0)	$2(2I_x)(2I_y)^{1/2}$	$2(2I_x)^{3/2}$
4	2101	skew	(1, -1)	(0, 1)		$4(2I_x)(2I_y)^{1/2}$	
4	2110	skew	(1, 1)	(0, -1)	(-1, 0)	$4(2I_x)(2I_y)^{1/2}$	$2(2I_x)^{3/2}$
4	3001	skew	(3, -1)	(-2, 1)		$6(2I_x)(2I_y)^{1/2}$	
4	3010	skew	(3, 1)	(-2, -1)	(-3, 0)	$6(2I_x)(2I_y)^{1/2}$	$2(2I_x)^{3/2}$
5	0113	normal	(-1, -2)		(1, 3)		$2(2I_x)^{1/2}(2I_y)^{3/2}$
5	0122	normal	(-1, 0)		(1, 1)		$4(2I_x)^{1/2}(2I_y)^{3/2}$
5	0131	normal	(-1, 2)		(1, -1)		$6(2I_x)^{1/2}(2I_y)^{3/2}$
5	0140	normal	(-1, 4)		(1, -3)		$8(2I_x)^{1/2}(2I_y)^{3/2}$

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### C. Resonance Driving Terms

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
5	0311	normal	(-3, 0)		(3, 1)		$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	0320	normal	(-3, 2)		(3, -1)		$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	1004	normal	(1, -4)	(0, 4)		$2(2I_y)^2$	
5	1013	normal	(1, -2)	(0, 2)	(-1, 3)	$2(2I_y)^2$	$2(2I_x)^{1/2}(2I_y)^{3/2}$
5	1022	normal	(1, 0)	(0, 0)	(-1, 1)	$2(2I_y)^2$	$4(2I_x)^{1/2}(2I_y)^{3/2}$
5	1031	normal	(1, 2)	(0, -2)	(-1, -1)	$2(2I_y)^2$	$6(2I_x)^{1/2}(2I_y)^{3/2}$
5	1040	normal	(1, 4)	(0, -4)	(-1, -3)	$2(2I_y)^2$	$8(2I_x)^{1/2}(2I_y)^{3/2}$
5	1202	normal	(-1, -2)	(2, 2)		$2(2I_x)(2I_y)$	
5	1211	normal	(-1, 0)	(2, 0)	(1, 1)	$2(2I_x)(2I_y)$	$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	1220	normal	(-1, 2)	(2, -2)	(1, -1)	$2(2I_x)(2I_y)$	$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	1400	normal	(-3, 0)	(4, 0)		$2(2I_x)^2$	
5	2102	normal	(1, -2)	(0, 2)		$4(2I_x)(2I_y)$	
5	2111	normal	(1, 0)	(0, 0)	(-1, 1)	$4(2I_x)(2I_y)$	$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	2120	normal	(1, 2)	(0, -2)	(-1, -1)	$4(2I_x)(2I_y)$	$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	2300	normal	(-1, 0)	(2, 0)		$4(2I_x)^2$	
5	3002	normal	(3, -2)	(-2, 2)		$6(2I_x)(2I_y)$	
5	3011	normal	(3, 0)	(-2, 0)	(-3, 1)	$6(2I_x)(2I_y)$	$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	3020	normal	(3, 2)	(-2, -2)	(-3, -1)	$6(2I_x)(2I_y)$	$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	3200	normal	(1, 0)	(0, 0)		$6(2I_x)^2$	
5	4100	normal	(3, 0)	(-2, 0)		$8(2I_x)^2$	
5	5000	normal	(5, 0)	(-4, 0)		$10(2I_x)^2$	
5	0014	skew	(0, -3)		(0, 4)		$2(2I_y)^2$
5	0023	skew	(0, -1)		(0, 2)		$4(2I_y)^2$
5	0032	skew	(0, 1)		(0, 0)		$6(2I_y)^2$
5	0041	skew	(0, 3)		(0, -2)		$8(2I_y)^2$
5	0050	skew	(0, 5)		(0, -4)		$10(2I_y)^2$
5	0212	skew	(-2, -1)		(2, 2)		$2(2I_x)(2I_y)$
5	0221	skew	(-2, 1)		(2, 0)		$4(2I_x)(2I_y)$
5	0230	skew	(-2, 3)		(2, -2)		$6(2I_x)(2I_y)$
5	0410	skew	(-4, 1)		(4, 0)		$2(2I_x)^2$
5	1103	skew	(0, -3)	(1, 3)		$2(2I_x)^{1/2}(2I_y)^{3/2}$	
5	1112	skew	(0, -1)	(1, 1)	(0, 2)	$2(2I_x)^{1/2}(2I_y)^{3/2}$	$2(2I_x)(2I_y)$
5	1121	skew	(0, 1)	(1, -1)	(0, 0)	$2(2I_x)^{1/2}(2I_y)^{3/2}$	$4(2I_x)(2I_y)$
5	1130	skew	(0, 3)	(1, -3)	(0, -2)	$2(2I_x)^{1/2}(2I_y)^{3/2}$	$6(2I_x)(2I_y)$
5	1301	skew	(-2, -1)	(3, 1)		$2(2I_x)^{3/2}(2I_y)^{1/2}$	
5	1310	skew	(-2, 1)	(3, -1)	(2, 0)	$2(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$

### C.3. Amplitude, Resonances and Lines

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
5	2003	skew	(2, -3)	(-1, 3)		$4(2I_x)^{1/2}(2I_y)^{3/2}$	
5	2012	skew	(2, -1)	(-1, 1)	(-2, 2)	$4(2I_x)^{1/2}(2I_y)^{3/2}$	$2(2I_x)(2I_y)$
5	2021	skew	(2, 1)	(-1, -1)	(-2, 0)	$4(2I_x)^{1/2}(2I_y)^{3/2}$	$4(2I_x)(2I_y)$
5	2030	skew	(2, 3)	(-1, -3)	(-2, -2)	$4(2I_x)^{1/2}(2I_y)^{3/2}$	$6(2I_x)(2I_y)$
5	2201	skew	(0, -1)	(1, 1)		$4(2I_x)^{3/2}(2I_y)^{1/2}$	
5	2210	skew	(0, 1)	(1, -1)	(0, 0)	$4(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$
5	3101	skew	(2, -1)	(-1, 1)		$6(2I_x)^{3/2}(2I_y)^{1/2}$	
5	3110	skew	(2, 1)	(-1, -1)	(-2, 0)	$6(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$
5	4001	skew	(4, -1)	(-3, 1)		$8(2I_x)^{3/2}(2I_y)^{1/2}$	
5	4010	skew	(4, 1)	(-3, -1)	(-4, 0)	$8(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$
6	0015	normal	(0, -4)		(0, 5)		$2(2I_y)^{5/2}$
6	0024	normal	(0, -2)		(0, 3)		$4(2I_y)^{5/2}$
6	0042	normal	(0, 2)		(0, -1)		$8(2I_y)^{5/2}$
6	0051	normal	(0, 4)		(0, -3)		$10(2I_y)^{5/2}$
6	0060	normal	(0, 6)		(0, -5)		$12(2I_y)^{5/2}$
6	0213	normal	(-2, -2)		(2, 3)		$2(2I_x)(2I_y)^{3/2}$
6	0222	normal	(-2, 0)		(2, 1)		$4(2I_x)(2I_y)^{3/2}$
6	0231	normal	(-2, 2)		(2, -1)		$6(2I_x)(2I_y)^{3/2}$
6	0240	normal	(-2, 4)		(2, -3)		$8(2I_x)(2I_y)^{3/2}$
6	0411	normal	(-4, 0)		(4, 1)		$2(2I_x)^2(2I_y)^{1/2}$
6	0420	normal	(-4, 2)		(4, -1)		$4(2I_x)^2(2I_y)^{1/2}$
6	1104	normal	(0, -4)	(1, 4)		$2(2I_x)^{1/2}(2I_y)^2$	
6	1113	normal	(0, -2)	(1, 2)	(0, 3)	$2(2I_x)^{1/2}(2I_y)^2$	$2(2I_x)(2I_y)^{3/2}$
6	1131	normal	(0, 2)	(1, -2)	(0, -1)	$2(2I_x)^{1/2}(2I_y)^2$	$6(2I_x)(2I_y)^{3/2}$
6	1140	normal	(0, 4)	(1, -4)	(0, -3)	$2(2I_x)^{1/2}(2I_y)^2$	$8(2I_x)(2I_y)^{3/2}$
6	1302	normal	(-2, -2)	(3, 2)		$2(2I_x)^{3/2}(2I_y)$	
6	1311	normal	(-2, 0)	(3, 0)	(2, 1)	$2(2I_x)^{3/2}(2I_y)$	$2(2I_x)^2(2I_y)^{1/2}$
6	1320	normal	(-2, 2)	(3, -2)	(2, -1)	$2(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	1500	normal	(-4, 0)	(5, 0)		$2(2I_x)^{5/2}$	
6	2004	normal	(2, -4)	(-1, 4)		$4(2I_x)^{1/2}(2I_y)^2$	
6	2013	normal	(2, -2)	(-1, 2)	(-2, 3)	$4(2I_x)^{1/2}(2I_y)^2$	$2(2I_x)(2I_y)^{3/2}$
6	2022	normal	(2, 0)	(-1, 0)	(-2, 1)	$4(2I_x)^{1/2}(2I_y)^2$	$4(2I_x)(2I_y)^{3/2}$
6	2031	normal	(2, 2)	(-1, -2)	(-2, -1)	$4(2I_x)^{1/2}(2I_y)^2$	$6(2I_x)(2I_y)^{3/2}$
6	2040	normal	(2, 4)	(-1, -4)	(-2, -3)	$4(2I_x)^{1/2}(2I_y)^2$	$8(2I_x)(2I_y)^{3/2}$
6	2202	normal	(0, -2)	(1, 2)		$4(2I_x)^{3/2}(2I_y)$	

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### C. Resonance Driving Terms

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
6	2220	normal	(0, 2)	(1, -2)	(0, -1)	$4(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	2400	normal	(-2, 0)	(3, 0)		$4(2I_x)^{5/2}$	
6	3102	normal	(2, -2)	(-1, 2)		$6(2I_x)^{3/2}(2I_y)$	
6	3111	normal	(2, 0)	(-1, 0)	(-2, 1)	$6(2I_x)^{3/2}(2I_y)$	$2(2I_x)^2(2I_y)^{1/2}$
6	3120	normal	(2, 2)	(-1, -2)	(-2, -1)	$6(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	4002	normal	(4, -2)	(-3, 2)		$8(2I_x)^{3/2}(2I_y)$	
6	4011	normal	(4, 0)	(-3, 0)	(-4, 1)	$8(2I_x)^{3/2}(2I_y)$	$2(2I_x)^2(2I_y)^{1/2}$
6	4020	normal	(4, 2)	(-3, -2)	(-4, -1)	$8(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	4200	normal	(2, 0)	(-1, 0)		$8(2I_x)^{5/2}$	
6	5100	normal	(4, 0)	(-3, 0)		$10(2I_x)^{5/2}$	
6	6000	normal	(6, 0)	(-5, 0)		$12(2I_x)^{5/2}$	
6	0114	skew	(-1, -3)		(1, 4)		$2(2I_x)^{1/2}(2I_y)^2$
6	0123	skew	(-1, -1)		(1, 2)		$4(2I_x)^{1/2}(2I_y)^2$
6	0132	skew	(-1, 1)		(1, 0)		$6(2I_x)^{1/2}(2I_y)^2$
6	0141	skew	(-1, 3)		(1, -2)		$8(2I_x)^{1/2}(2I_y)^2$
6	0150	skew	(-1, 5)		(1, -4)		$10(2I_x)^{1/2}(2I_y)^2$
6	0312	skew	(-3, -1)		(3, 2)		$2(2I_x)^{3/2}(2I_y)$
6	0321	skew	(-3, 1)		(3, 0)		$4(2I_x)^{3/2}(2I_y)$
6	0330	skew	(-3, 3)		(3, -2)		$6(2I_x)^{3/2}(2I_y)$
6	0510	skew	(-5, 1)		(5, 0)		$2(2I_x)^{5/2}$
6	1005	skew	(1, -5)	(0, 5)		$2(2I_y)^{5/2}$	
6	1014	skew	(1, -3)	(0, 3)	(-1, 4)	$2(2I_y)^{5/2}$	$2(2I_x)^{1/2}(2I_y)^2$
6	1023	skew	(1, -1)	(0, 1)	(-1, 2)	$2(2I_y)^{5/2}$	$4(2I_x)^{1/2}(2I_y)^2$
6	1032	skew	(1, 1)	(0, -1)	(-1, 0)	$2(2I_y)^{5/2}$	$6(2I_x)^{1/2}(2I_y)^2$
6	1041	skew	(1, 3)	(0, -3)	(-1, -2)	$2(2I_y)^{5/2}$	$8(2I_x)^{1/2}(2I_y)^2$
6	1050	skew	(1, 5)	(0, -5)	(-1, -4)	$2(2I_y)^{5/2}$	$10(2I_x)^{1/2}(2I_y)^2$
6	1203	skew	(-1, -3)	(2, 3)		$2(2I_x)(2I_y)^{3/2}$	
6	1212	skew	(-1, -1)	(2, 1)	(1, 2)	$2(2I_x)(2I_y)^{3/2}$	$2(2I_x)^{3/2}(2I_y)$
6	1221	skew	(-1, 1)	(2, -1)	(1, 0)	$2(2I_x)(2I_y)^{3/2}$	$4(2I_x)^{3/2}(2I_y)$
6	1230	skew	(-1, 3)	(2, -3)	(1, -2)	$2(2I_x)(2I_y)^{3/2}$	$6(2I_x)^{3/2}(2I_y)$
6	1401	skew	(-3, -1)	(4, 1)		$2(2I_x)^2(2I_y)^{1/2}$	
6	1410	skew	(-3, 1)	(4, -1)	(3, 0)	$2(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	2103	skew	(1, -3)	(0, 3)		$4(2I_x)(2I_y)^{3/2}$	
6	2112	skew	(1, -1)	(0, 1)	(-1, 2)	$4(2I_x)(2I_y)^{3/2}$	$2(2I_x)^{3/2}(2I_y)$
6	2121	skew	(1, 1)	(0, -1)	(-1, 0)	$4(2I_x)(2I_y)^{3/2}$	$4(2I_x)^{3/2}(2I_y)$

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### C.3. Amplitude, Resonances and Lines

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
6	2130	skew	(1, 3)	(0, -3)	(-1, -2)	$4(2I_x)(2I_y)^{3/2}$	$6(2I_x)^{3/2}(2I_y)$
6	2301	skew	(-1, -1)	(2, 1)		$4(2I_x)^2(2I_y)^{1/2}$	
6	2310	skew	(-1, 1)	(2, -1)	(1, 0)	$4(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	3003	skew	(3, -3)	(-2, 3)		$6(2I_x)(2I_y)^{3/2}$	
6	3012	skew	(3, -1)	(-2, 1)	(-3, 2)	$6(2I_x)(2I_y)^{3/2}$	$2(2I_x)^{3/2}(2I_y)$
6	3021	skew	(3, 1)	(-2, -1)	(-3, 0)	$6(2I_x)(2I_y)^{3/2}$	$4(2I_x)^{3/2}(2I_y)$
6	3030	skew	(3, 3)	(-2, -3)	(-3, -2)	$6(2I_x)(2I_y)^{3/2}$	$6(2I_x)^{3/2}(2I_y)$
6	3201	skew	(1, -1)	(0, 1)		$6(2I_x)^2(2I_y)^{1/2}$	
6	3210	skew	(1, 1)	(0, -1)	(-1, 0)	$6(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	4101	skew	(3, -1)	(-2, 1)		$8(2I_x)^2(2I_y)^{1/2}$	
6	4110	skew	(3, 1)	(-2, -1)	(-3, 0)	$8(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	5001	skew	(5, -1)	(-4, 1)		$10(2I_x)^2(2I_y)^{1/2}$	
6	5010	skew	(5, 1)	(-4, -1)	(-5, 0)	$10(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$

Table C.3.: Amplitude dependence, associated resonance and spectrum appearance of Resonance Driving Terms up to dodecapoles ( $n = 6$ ).



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# List of Publications

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## Journal Publications

None yet.

## Journal Publications (co-author)

- (1) Ewen H. Maclean et al. “Prospects for beam-based study of dodecapole nonlinearities in the CERN High-Luminosity Large Hadron Collider”. In: *The European Physical Journal Plus* 137.11 (2022). doi: [10.1140/epjp/s13360-022-03367-2](https://doi.org/10.1140/epjp/s13360-022-03367-2). URL: <https://link.springer.com/10.1140/epjp/s13360-022-03367-2>
- (2) Joschua Dilly et al. “First operational dodecapole correction in the LHC”. in: *Physical Review Accelerators and Beams* 26.12 (2023). doi: [10.1103/PhysRevAccelBeams.26.121001](https://doi.org/10.1103/PhysRevAccelBeams.26.121001). URL: <https://link.aps.org/doi/10.1103/PhysRevAccelBeams.26.121001>

## Conference Proceedings

- (1) Mael Le Garrec et al. “Measurement and modelling of decapole errors in the LHC from beam-based studies”. In: *IPAC’23*. Venice, Italy, 2023. doi: [10.18429/JACOW-IPAC2023-MOPL024](https://doi.org/10.18429/JACOW-IPAC2023-MOPL024). URL: <https://jacow.org/ipac2023/doi/jacow-ipac2023-mopl024>
- (2) Mael Le Garrec et al. “First measurement of fourth and fifth order chromaticity in the LHC”. in: *IPAC’23*. Venice, Italy, 2023. doi: [10.18429/JACOW-IPAC2023-MOPL027](https://doi.org/10.18429/JACOW-IPAC2023-MOPL027). URL: <https://jacow.org/ipac2023/doi/jacow-ipac2023-mopl027>



## Conference Proceedings (co-author)

- (1) Tobias Persson et al. "LHC Optics Commissioning in 2023 and 2024". In: *IPAC'24*. Nashville, Tennessee, USA, 2024
- (2) Vittorio Ferrentino et al. "LHC 2023 Ion Optics Commissioning". In: *IPAC'24*. Nashville, Tennessee, USA, 2024
- (3) Sasha J. Horney et al. "Sextupole RDTs in The LHC at Injection and in the Ramp". In: *IPAC'24*. Nashville, Tennessee, USA, 2024
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- (13) Ewen Maclean et al. “Optics Measurement by Excitation of Betatron Oscillations in the CERN PSB”. in: *IPAC’21*. Campinas, Brazil: JACOW Publishing, Geneva, Switzerland, 2021. doi: [10.18429/JACoW-IPAC2021-THPAB168](https://doi.org/10.18429/JACoW-IPAC2021-THPAB168)

## Notes and Reports

- (1) Maël Le Garrec et al. *MD6864 – Decapole studies at injection*. Accelerators & Technology Sector Note CERN-ACC-NOTE-2023-0018. CERN, 2023. URL: <https://cds.cern.ch/record/2879070>
- (2) Joschua Dilly and Mael Le Garrec. *On the derivation of Amplitude Detuning and Chromaticity Formulas for Particle Accelerators*. 2023. URL: <http://arxiv.org/abs/2301.09132>
- (3) Xavier Buffat et al. *Optics Measurement and Correction Strategies for HL-LHC*. Accelerators & Technology Sector Note CERN-ACC-2022-0004. CERN, 2022. URL: <https://cds.cern.ch/record/2808650>
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## Seminars and Conferences

- (1) Maël Le Garrec. *Correction of Decapolar Resonances in the LHC*. Goethe Universität, Frankfurt, 2023
- (2) Maël Le Garrec. *Les champs décapolaires du LHC*. Roscoff, France, 2023. URL: <https://indico.ijclab.in2p3.fr/event/9312/contributions/30982/>

