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# LHC EFFECTIVE MODEL FOR OPTICS CORRECTIONS

Measurements and corrections of high-order non-linear optics

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# Abstract

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This thesis investigates the crucial role of higher-order magnetic fields and non-linear optics in the stability and performance of particle accelerators, focusing on the Large Hadron Collider (LHC) at CERN. The control of non-linear optics, which deals with the interaction of charged particle beams with complex magnetic fields such as sextupolar, octupolar, decapolar, and so on, is essential for managing beam dynamics. The LHC, as the world's most powerful accelerator, provides a unique opportunity to study these high-order effects, serving as a testbed for future accelerator designs.

These higher-order fields significantly affect the beam's dynamic aperture and lifetime, especially at injection energy, where precise correction of magnetic field errors is required. Managing these challenges is not only vital for optimizing LHC performance but also for guiding the design and operation of next-generation machines.

A key contribution of this work is the development of correction methods, based on a response matrix approach, for Resonance Driving Terms (RDTs), a critical factor in beam lifetime and dynamic aperture limitations. New corrective strategies for RDTs have led to notable improvements in beam lifetime and dynamic aperture at both injection and top energy operation. This thesis also addresses the discrepancies observed between experimental measurements and models of beam observables.

These findings highlight the importance of precise modeling and correction of non-linear magnetic fields, offering insights that will benefit both the LHC and future high-energy particle accelerators.



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# Extended Summary

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The Large Hadron Collider (LHC) at CERN, situated in a 27-kilometer tunnel beneath the Swiss-French border, is the world's largest and most powerful particle accelerator. Its primary mission is to recreate the conditions of the universe just moments after the Big Bang, enabling scientists to explore the fundamental forces and particles that constitute the cosmos. This remarkable facility accelerates protons and heavy ions to nearly the speed of light before colliding them with immense energy, providing profound insights into the building blocks of matter and the fundamental interactions that govern our universe. The LHC represents not only a monumental engineering achievement but also a crucial tool for advancing our understanding of high-energy physics.

Operating such a complex and powerful machine requires overcoming numerous technical challenges, particularly in maintaining the precise control and stability of its particle beams. One significant challenge lies in managing the effects of higher-order magnetic fields, which often arise from field errors in the magnets used to guide and focus the particle beams. These field errors can profoundly impact beam dynamics and its lifetime. Addressing these challenges is essential to ensuring that the LHC operates effectively and continues to produce valuable scientific results. Furthermore, non-linear optics control is crucial for the success of future accelerators like the HL-LHC and FCC, which will increasingly rely on high-order non-linear optics corrections to achieve their performance targets. The following extended summary synthesizes findings from three detailed studies investigating various aspects of higher-order multipole effects and their implications on the LHC's beam dynamics.

This thesis employs the Hamiltonian formalism to describe particle motion in the transverse planes under the influence of various multipole fields. In non-linear lattices, the complexity of beam dynamics increases significantly, necessitating the use of advanced mathematical tools such as Lie Algebra and

Poisson Brackets to accurately characterize non-linear effects. The study derives explicit higher-order non-linear transfer maps and provides a comprehensive summary of multipole combinations. These non-linearities in the lattice lead to complex phenomena such as high-order chromaticity, amplitude detuning, chromatic amplitude detuning, and resonances driven by Resonance Driving Terms (RDTs), all of which are thoroughly derived and supported by detailed measurement techniques.

Optics measurements are conducted using a wide range of techniques and software tools. Turn-by-turn data acquisition via Beam Position Monitors (BPMs) is emphasized as a crucial method for evaluating beam optics, where an AC-dipole excites the beam, and the resulting oscillations are analyzed using Fourier transforms to extract tunes and identify resonances. Further treatment is done via the oscillation amplitudes and the magnitude of spectral lines to retrieve linear and non-linear observables such as the phase advance, beta function, dispersion, coupling, orbit, and RDTs. Chromaticity measurements involve inducing momentum offsets by varying the RF frequency and observing the corresponding tune shifts.

To advance the understanding of high-order non-linear fields, new measurement and analysis methods have been developed. One such tool, the Non-Linear Chromaticity GUI, simplifies the process of analyzing and correcting chromaticity during operation. Additionally, a novel response matrix approach has been introduced, enabling efficient direct corrections of Resonance Driving Terms (RDTs) in the LHC. This marks a shift from empirical correction adjustments to a more quantitative and systematic method. These techniques have proven effective in correcting several key observables, as discussed throughout this thesis. The development of these methods has been crucial in reducing commissioning time, enabling a greater focus on high-order multipoles, from octupoles to decatetrapoles, some of which have never been studied before. These investigations were further facilitated by improvements in the collimator sequence, allowing the exploration of a higher action and momentum offsets. Additionally, enhancements in dynamic aperture, as presented in this thesis, provided the required amplitudes to probe these higher-order effects more effectively.

The first chapter explores the origins and consequences of skew octupolar fields within the LHC. These fields significantly influence the dynamic aperture of the accelerator, a parameter that defines the amplitudes within which the particle beam remains stable. Skew octupolar correctors are installed around key detectors, such as ATLAS and CMS, to manage these fields and mitigate their effects on beam stability. The study focuses on measuring these fields with optics designed for top energy, at 6.8 TeV per particle. Corrections were performed using a response matrix based approach, a different method than

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what was used a few years prior, marking a shift from empirical corrections to a more quantitative and systematic approach for the first time in the LHC. This method effectively addresses skew octupolar RDTs using the available corrector magnets, although its performance is limited by the absence of one corrector, which constrains the achievable correction strength. Consequently, the RDTs of interest,  $f_{1012}$  and  $f_{1210}$ , can either be effectively corrected or maintained at a constant level depending on the corrector configuration.

Additionally, the study investigates the unexpected influence of Landau octupoles on skew octupolar RDTs at injection energy, at 450 GeV per particle. Landau octupoles are powerful magnets used at injection energy to introduce damping of multi-particle coherent instabilities via a tune spread. They have been observed to generate a significant shift in skew octupolar RDTs during measurements under various powering configurations. Skew octupolar resonances have previously been identified as a source of emittance growth in the presence of electron clouds. The more intuitive explanation for the generation of skew fields by normal multipoles had been the misalignments of these octupoles, specifically roll errors. However, simulations indicate that octupole misalignments have minimal impact on skew octupolar fields. Instead, transverse coupling has been identified as a crucial factor. The combination of coupling and the strong powering of Landau octupoles at injection energy is expected to be a major contributor to skew octupolar RDTs. Therefore, precise modeling of coupling is essential for predicting the behavior of skew octupolar RDTs. Accurate modeling and correction of skew octupolar fields in the LHC are essential to suppress resonances and improve both the beam's dynamic stability and lifetime.

The second chapter focuses on decapolar fields in the LHC, particularly at injection energy. As the FCC is expected to rely on precise control of decapolar fields, their accurate study in the LHC is essential to increase confidence in its design and operational strategies. Consequently, this study addresses previously observed discrepancies between measurements and models related to third-order chromaticity, a critical parameter that describes how particles with an energy deviation experience different oscillation frequencies than the reference particle. Accurate control of chromaticity is vital for maintaining beam stability. The introduction of previously unobserved observables, has provided a clearer understanding of these discrepancies. Among these observables are the bare chromaticity, which represents the chromaticity of the machine without any correctors powered on to observe the bare influence of field errors, and chromatic amplitude detuning, a detuning function of both momentum deviations and oscillation amplitudes. Several approaches to measuring the same fields help clarify the various contributions. The research reveals that the decay of the decapolar component in the main dipoles is a significant factor contributing to

these discrepancies. When the LHC was designed, this decay was deemed too small to be significant and thus was not included in the magnetic error tables used for simulation. However, as the machine's parameters are pushed further each year and the effects of higher-order fields become better understood, it becomes clear that accurate control and modeling of these fields are necessary.

For the first time in the LHC, measurements and corrections of the decapolar Resonance Driving Term  $f_{1004}$  were carried out at injection energy. Corrections are based on a response matrix approach, effectively implementing combined corrections of third-order chromaticity, chromatic amplitude detuning, and RDT  $f_{1004}$ , leading to a 3% improvement in beam lifetime. Conversely, deliberately degrading the RDT alone resulted in a 10% decrease in beam lifetime, underscoring the importance of this resonance corrections for stable beam operation. The study also explored how sextupoles and octupoles interact to generate decapolar-like fields. It was found that sextupoles, both alone and in combination with Landau octupoles, produce a substantial  $f_{1004}$  decapolar RDT when powered to small currents. Therefore, in an operational context, decapolar resonances, largely generated by strong octupoles, would benefit from adapted corrections. These findings suggest that further advancements in correction methods could lead to even greater improvements in beam lifetime.

The third chapter investigates very-high-order fields in the LHC, specifically dodecapolar and decatetrapolar fields. Using a newly implemented collimation setup and custom post-processing techniques, this study successfully observed these higher-order fields. Studies were conducted to estimate the effect of the non-linearity of the momentum compaction factor on the chromaticity function during its computation from the RF frequency. The results indicate that while the momentum compaction factor expansion shows a second order in the LHC, its effects on the resulting chromaticity are negligible even at large momentum offsets. Several chromaticity measurements with varying configurations of octupolar and decapolar corrections then revealed the presence of fourth and fifth-order terms ( $Q^{(4)}$  and  $Q^{(5)}$ ). These measurements consistently identified these higher-order terms with similar values, demonstrating their robustness. Additionally, it is emphasized that accurately characterizing the lower-order terms requires good measurement of these higher-order terms. The study identifies, through simulations, dodecapolar and decatetrapolar fields as primary contributors to these higher-order effects, originating from field errors in the main dipoles and quadrupoles. The LHC's field error model appears to be in relative agreement with the measurements once the decay of decatetrapolar components is considered.

For the first time at injection energy, the dodecapolar Resonance Driving Term  $f_{0060}$  was measured. This measurement shows clear repeatability, even

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when performed with different configurations of octupolar and decapolar corrections. The measured values were found to be in good agreement with the model. The research concludes that further investigations are needed to address limitations in the measurement range of the chromaticity function and to refine estimates of higher-order chromaticity terms. Additionally, studying the impact of lower-order multipoles on the dodecapolar RDT and its effect on beam lifetime would be valuable for optimizing the LHC's performance.

A EAJADE secondment at SuperKEKB during its February 2024 commissioning utilized optics measurement techniques from CERN on the HER and LER rings. Linear optics measurements showed good agreement with the Closed Orbit Distortion (COD) method, demonstrating repeatability over time. For the first time, vertical plane measurements with an injection offset were conducted, yielding promising results. The study extended to non-linear optics, including chromaticity and amplitude detuning, revealing some discrepancies between measurements and model predictions, particularly concerning potential unmodeled sources. Resonance Driving Terms (RDTs) were measured successfully for the first time, although challenges remained due to factors like decoherence and damping. Overall, the findings align with alternative KEK methods, indicating that CERN's techniques are effective for enhancing understanding of SuperKEKB and future accelerators like the FCC-ee.

In summary, the research detailed in these studies underscores the critical importance of understanding and managing higher-order multipole effects in the LHC. Skew octupolar fields, decapolar fields, and other higher-order fields have a significant impact on the beam's dynamic aperture and lifetime. Developing and implementing advanced measurement techniques and correction methods is essential for enhancing the understanding of non-linear optics. The insights gained from these studies are crucial for optimizing the LHC's performance.

As particle accelerators continue to evolve, the challenges associated with higher-order multipole components will persist. Ongoing research in this field is vital for addressing these challenges and ensuring that future accelerators achieve the precision required for new scientific discoveries. The lessons learned from the LHC's experience with the complex interactions of multipole fields will inform the design and operation of next-generation accelerators, such as the HL-LHC and FCC, which will increasingly rely on precise non-linear optics control. These advancements will ensure that these accelerators remain at the forefront of exploring fundamental questions about the universe.

The work presented in these chapters represents a significant contribution to the field of accelerator physics by offering practical solutions for current operational challenges and paving the way for future advancements.



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# Zusammenfassung

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Der Large Hadron Collider (LHC) am CERN, der sich in einem 27 Kilometer langen Tunnel unter der Schweizer-französischen Grenze befindet, ist der größte und leistungsstärkste Teilchenbeschleuniger der Welt. Seine Hauptmission besteht darin, die Bedingungen des Universums kurz nach dem Urknall nachzubilden, wodurch Wissenschaftler die grundlegenden Kräfte und Teilchen untersuchen können, aus denen das Universum besteht. Diese bemerkenswerte Einrichtung beschleunigt Protonen und schwere Ionen nahezu auf Lichtgeschwindigkeit, bevor sie mit enormer Energie kollidiert werden, was tiefgreifende Einblicke in die Bausteine der Materie und die grundlegenden Wechselwirkungen, die unser Universum regieren, ermöglicht. Der LHC stellt nicht nur einen monumentalen technischen Erfolg dar, sondern ist auch ein entscheidendes Werkzeug für das Verständnis der Hochenergiephysik.

Der Betrieb einer so komplexen und leistungsstarken Maschine erfordert das Überwinden zahlreicher technischer Herausforderungen, insbesondere bei der präzisen Kontrolle und Stabilität ihrer Teilchenstrahlen. Eine bedeutende Herausforderung besteht darin, die Auswirkungen höherer magnetischer Felder zu verwalten, die häufig durch Feldfehler in den Magneten entstehen, die zur Führung und Fokussierung der Teilchenstrahlen verwendet werden. Diese Feldfehler können die Strahldynamik und dessen Lebensdauer erheblich beeinflussen. Die Bewältigung dieser Herausforderungen ist entscheidend, um sicherzustellen, dass der LHC effektiv arbeitet und weiterhin wertvolle wissenschaftliche Ergebnisse liefert. Darüber hinaus ist die Kontrolle nicht-linearer Optik für den Erfolg zukünftiger Beschleuniger wie dem HL-LHC und FCC entscheidend, die zunehmend auf Korrekturen höherer nicht-linearer Optik angewiesen sein werden, um ihre Leistungsziele zu erreichen. Die folgende erweiterte Zusammenfassung synthetisiert die Ergebnisse von drei detaillierten Studien, die verschiedene Aspekte höherer multipolarer Effekte und deren Auswirkungen auf

die Strahldynamik des LHC untersuchen.

Diese Dissertation verwendet die Hamiltonsche Formulierung, um die Bewegung von Teilchen in den transversalen Ebenen unter dem Einfluss verschiedener multipolarer Felder zu beschreiben. In nicht-linearen Lattices erhöht sich die Komplexität der Strahldynamik erheblich, sodass der Einsatz fortgeschrittener mathematischer Werkzeuge wie Lie-Algebra und Poisson-Klammern erforderlich ist, um nicht-lineare Effekte genau zu charakterisieren. Die Studie leitet explizite höhere nicht-lineare Transferkarten ab und bietet eine umfassende Zusammenfassung multipolarer Kombinationen. Diese Nichtlinearitäten im Lattice führen zu komplexen Phänomenen wie höherer chromatischer Aberration, Amplitudendämpfung, chromatischer Amplitudendämpfung und Resonanzen, die durch Resonance Driving Terms (RDTs) hervorgerufen werden, die alle gründlich abgeleitet und durch detaillierte Messtechniken unterstützt werden.

Optikmessungen werden mit einer Vielzahl von Techniken und Software-Tools durchgeführt. Die Datenerfassung „turn-by-turn“ über Strahlpositionsmonitore (BPMs) wird als entscheidende Methode zur Evaluierung der Strahloptik hervorgehoben, bei der ein AC-Dipol den Strahl anregt und die resultierenden Schwingungen mit Fourier-Transformationen analysiert werden, um die Tuningwerte zu extrahieren und Resonanzen zu identifizieren. Weitere Behandlungen erfolgen über die Schwingungsamplituden und die Größe der Spektrallinien, um lineare und nicht-lineare Observablen wie die Phasenverschiebung, die Beta-Funktion, die Dispersion, die Kopplung, die Bahn und die RDTs zu erhalten. Chromatisitätsmessungen beinhalten das Einführen von Impulsverschiebungen durch Variieren der RF-Frequenz und das Beobachten der entsprechenden Tuningverschiebungen.

Um das Verständnis höherer nicht-linearer Felder zu erweitern, wurden neue Mess- und Analysemethoden entwickelt. Ein solches Werkzeug, die Non-Linear Chromaticity GUI, vereinfacht den Prozess der Analyse und Korrektur der Chromatik während des Betriebs. Darüber hinaus wurde ein neuartiger Response-matrix ansatz eingeführt, der effiziente direkte Korrekturen von Resonance Driving Terms (RDTs) im LHC ermöglicht. Dies markiert einen Wechsel von empirischen Korrekturanpassungen zu einer quantitativeren und systematischeren Methode. Diese Techniken haben sich als effektiv erwiesen, um mehrere wichtige Observablen zu korrigieren, wie in dieser Dissertation besprochen wird. Die Entwicklung dieser Methoden war entscheidend, um die Inbetriebnahmezeit zu verkürzen und einen stärkeren Fokus auf höhere Multipole zu ermöglichen, von Oktupolen bis zu Dekatetrapolen, von denen einige nie zuvor untersucht wurden. Diese Untersuchungen wurden durch Verbesserungen in der Kollimatorsequenz weiter erleichtert, was die Erkundung höherer Aktionen und Impulsverschiebungen ermöglichte. Darüber hinaus boten die in dieser Disserta-

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tion präsentierte Verbesserungen der dynamischen Apertur die erforderlichen Amplituden, um diese höherwertigen Effekte effektiver zu untersuchen.

Das erste Kapitel untersucht die Ursprünge und Konsequenzen skew Oktupolfelder im LHC. Diese Felder beeinflussen erheblich die dynamische Apertur des Beschleunigers, ein Parameter, der die Amplituden definiert, innerhalb derer der Teilchenstrahl stabil bleibt. Schiefe Oktupolkorrekturen sind rund um wichtige Detektoren, wie ATLAS und CMS, installiert, um diese Felder zu verwalten und deren Auswirkungen auf die Strahlstabilität zu mindern. Die Studie konzentriert sich auf die Messung dieser Felder mit einer für die höchste Energie ausgelegten Optik bei 6,8 TeV pro Teilchen. Korrekturen wurden unter Verwendung eines Ansatzes basierend auf der Response-matrix durchgeführt, einer anderen Methode als die, die einige Jahre zuvor verwendet wurde, was einen Wechsel von empirischen Korrekturen zu einem quantitativeren und systematischeren Ansatz im LHC darstellt. Diese Methode adressiert effektiv schiefe Oktupol-RDTs unter Verwendung der verfügbaren Korrektormagneten, obwohl ihre Leistung durch das Fehlen eines Korrektors eingeschränkt ist, was die erreichbare Korrekturkraft einschränkt. Folglich können die von Interesse sind,  $f_{1012}$  und  $f_{1210}$ , entweder effektiv korrigiert oder auf einem konstanten Niveau gehalten werden, abhängig von der Konfiguration des Korrektors.

Darüber hinaus untersucht die Studie den unerwarteten Einfluss von Landau-Oktupolen auf schiefe Oktupol-RDTs bei Injektionsenergie von 450 GeV pro Teilchen. Landau-Oktupole, die bei Injektionsenergie verwendet werden, um kohärente Instabilitäten durch eine Tuningstreuung zu dämpfen, erzeugten während der Messungen unter verschiedenen Stromkonfigurationen eine signifikante Verschiebung in den schiefen Oktupol-RDTs. Schiefe Oktupol-Resonanzen wurden zuvor als Quelle für Emittanzwachstum in Anwesenheit von Elektronenwolken identifiziert. Die intuitivere Erklärung für die Erzeugung skew Felder durch normale Multipole war die Fehlausrichtung dieser Oktupole, insbesondere Rollfehler. Simulationen zeigen jedoch, dass die Fehlausrichtung von Oktupolen nur minimale Auswirkungen auf schiefe Oktupolfelder hat. Stattdessen wurde die transversale Kopplung als entscheidender Faktor identifiziert. Die Kombination aus Kopplung und der starken Feldstärke von Landau-Oktupolen bei Injektionsenergie wird als wesentlicher Beitrag zu schießen Oktupol-RDTs angesehen. Daher ist eine präzise Modellierung der Kopplung entscheidend, um das Verhalten der schiefen Oktupol-RDTs vorherzusagen. Die genaue Modellierung und Korrektur der schiefen Oktupolfelder im LHC sind entscheidend, um Resonanzen zu unterdrücken und sowohl die dynamische Stabilität als auch die Lebensdauer des Strahls zu verbessern.

Das zweite Kapitel konzentriert sich auf Dekapolfelder im LHC, insbesondere

bei Injektionsenergie. Da der FCC voraussichtlich auf eine präzise Kontrolle von Dekapolfeldern angewiesen ist, ist ihre genaue Untersuchung im LHC entscheidend, um das Vertrauen in sein Design und seine Betriebsstrategien zu erhöhen. Folglich befasst sich diese Studie mit zuvor beobachteten Diskrepanzen zwischen Messungen und Modellen, die sich auf die dritte chromatische Aberration beziehen, einen kritischen Parameter, der beschreibt, wie Teilchen mit einer Energiedifferenz unterschiedliche Schwingungsfrequenzen im Vergleich zum Referenzteilchen erfahren. Eine präzise Kontrolle der Chromatik ist für die Aufrechterhaltung der Strahlstabilität von entscheidender Bedeutung. Die Einführung zuvor unobservierter Observablen hat ein klareres Verständnis dieser Diskrepanzen ermöglicht. Zu diesen Observablen gehört die rohe Chromatik, die die Chromatik der Maschine darstellt, ohne dass Korrekturen eingeschaltet sind, um den rohen Einfluss von Feldfehlern zu beobachten, und die chromatische Amplitudendämpfung, eine Dämpfungsfunktion sowohl von Impulsabweichungen als auch von Schwingungsamplituden. Mehrere Ansätze zur Messung derselben Felder helfen, die verschiedenen Beiträge zu klären. Die Forschung zeigt, dass der Rückgang der dekapolaren Komponente in den Hauptdipolen ein wesentlicher Faktor ist, der zu diesen Diskrepanzen beiträgt. Als der LHC entworfen wurde, wurde dieser Rückgang als zu klein erachtet, um signifikant zu sein, und daher nicht in die für Simulationen verwendeten Tabellen der magnetischen Fehler aufgenommen. Da jedoch die Parameter der Maschine jedes Jahr weiter vorangetrieben werden und die Auswirkungen höherer Felder besser verstanden werden, wird klar, dass eine präzise Kontrolle und Modellierung dieser Felder notwendig sind.

Erstmals im LHC wurden Messungen und Korrekturen des dekapolaren Resonance Driving Terms  $f_{1004}$  bei Injektionsenergie durchgeführt. Korrekturen basieren auf einem Ansatz der Response-matrix, der effektiv kombinierte Korrekturen der dritten chromatischen Aberration, der chromatischen Amplitudendämpfung und des RDT  $f_{1004}$  implementiert, was zu einer Verbesserung der Strahllebensdauer um 3% führt. Umgekehrt führte eine absichtliche Verschlechterung des RDT allein zu einem Rückgang der Strahllebensdauer um 10%, was die Bedeutung dieser Resonanzkorrekturen für einen stabilen Strahlbetrieb unterstreicht. Die Studie untersuchte auch, wie Sextupole und Oktupole interagieren, um dekapolare ähnliche Felder zu erzeugen. Es wurde festgestellt, dass Sextupole, sowohl allein als auch in Kombination mit Landau-Oktupolen, einen erheblichen  $f_{1004}$  dekapolare RDT erzeugen, wenn sie auf kleine Ströme betrieben werden. Daher würden im Betrieb dekapolare Resonanzen, die größtenteils durch starke Oktupole erzeugt werden, von angepassten Korrekturen profitieren. Diese Ergebnisse deuten darauf hin, dass weitere Fortschritte in den Korrekturmethoden zu noch größeren Verbesserungen der Strahllebensdauer führen könnten.

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Das dritte Kapitel untersucht sehr hochgradige Felder im LHC, insbesondere dodecapolare und dekatetrapolare Felder. Mithilfe einer neu implementierten Kollimationssanordnung und maßgeschneideter Nachbearbeitungstechniken konnte diese Studie erfolgreich diese höhergradigen Felder beobachten. Es wurden Studien durchgeführt, um den Effekt der Nichtlinearität des Impulsabhängige pfadlänge auf die chromatische Funktion während ihrer Berechnung aus der RF-Frequenz zu schätzen. Die Ergebnisse zeigen, dass, obwohl die Erweiterung des Impulsabhängige pfadlänge im LHC zweiter Ordnung zeigt, seine Auswirkungen auf die resultierende Chromatik selbst bei großen Impulsverschiebungen vernachlässigbar sind. Mehrere Chromatisitätsmessungen mit variierenden Konfigurationen von oktupolarer und dekapolarer Korrektur ergaben dann das Vorhandensein von vierten und fünften Ordnungstermen ( $Q^{(4)}$  und  $Q^{(5)}$ ). Diese Messungen identifizierten diese höhergradigen Terme konsistent mit ähnlichen Werten und demonstrieren ihre Robustheit. Darüber hinaus wird betont, dass eine genaue Charakterisierung der niedergeschichteten Terme eine gute Messung dieser höhergradigen Terme erfordert. Die Studie identifizierte durch Simulationen dodecapolare und dekatetrapolare Felder als Hauptbeiträge zu diesen höhergradigen Effekten, die aus Feldfehlern in den Hauptdipolen und Quadrupolen stammen. Das Modell der Feldfehler des LHC scheint in relativer Übereinstimmung mit den Messungen zu sein, sobald der Rückgang der dekatetrapolaren Komponenten berücksichtigt wird.

Erstmals bei Injektionsenergie wurde der dodecapolare Resonance Driving Term  $f_{0060}$  gemessen. Diese Messung zeigt eine klare Wiederholbarkeit, selbst wenn sie mit unterschiedlichen Konfigurationen von oktupolarer und dekapolarer Korrektur durchgeführt wird. Die gemessenen Werte stimmten gut mit dem Modell überein. Die Studie kommt zu dem Schluss, dass weitere Untersuchungen erforderlich sind, um Einschränkungen im Messbereich der chromatischen Funktion zu beheben und Schätzungen höhergradiger chromatischer Terme zu verfeinern. Darüber hinaus wäre es wertvoll, den Einfluss niedergeschichteter Multipole auf den dodecapolaren RDT und dessen Auswirkungen auf die Strahllebensdauer zu untersuchen, um die Leistung des LHC zu optimieren.

Ein EAJADE-Transfer bei SuperKEKB während der Inbetriebnahme im Februar 2024 nutzte Optikmesstechniken vom CERN an den HER- und LER-Ringen. Lineare Optikmessungen zeigten eine gute Übereinstimmung mit der Methode der geschlossenen Bahnverzerrung (COD) und demonstrierten die Wiederholbarkeit über die Zeit. Erstmals wurden vertikale Messungen im Ebnene mit einer Injektionsabweichung durchgeführt, die vielversprechende Ergebnisse erbrachten. Die Studie erstreckte sich auf nicht-lineare Optik, einschließlich Chromatik und Amplitudendämpfung, und zeigte einige Diskrepanzen zwischen Messungen und Modellvorhersagen, insbesondere

## *Zusammenfassung*

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hinsichtlich potenzieller nicht modellierter Quellen. Resonance Driving Terms (RDTs) wurden erstmals erfolgreich gemessen, obwohl Herausforderungen aufgrund von Faktoren wie Dekohärenz und Dämpfung blieben. Insgesamt stimmen die Ergebnisse mit alternativen KEK-Methoden überein, was darauf hindeutet, dass die Techniken des CERN effektiv sind, um das Verständnis von SuperKEKB und zukünftigen Beschleunigern wie dem FCC-ee zu verbessern.

Zusammenfassend verdeutlicht die in diesen Studien detaillierte Forschung die entscheidende Bedeutung des Verständnisses und der Behandlung höhergradiger multipolarer Effekte im LHC. Schiefe Oktupolfelder, Dekapolfelder und andere höhergradige Felder haben einen erheblichen Einfluss auf die dynamische Apertur und die Lebensdauer des Strahls. Die Entwicklung und Implementierung fortschrittlicher Mess- und Korrekturtechniken sind entscheidend, um das Verständnis der nicht-linearen Optik zu verbessern. Die aus diesen Studien gewonnenen Erkenntnisse sind von entscheidender Bedeutung für die Optimierung der Leistung des LHC.

Da Teilchenbeschleuniger weiterhin weiterentwickelt werden, werden die Herausforderungen im Zusammenhang mit höheren multipolaren Komponenten bestehen bleiben. Fortlaufende Forschung in diesem Bereich ist von entscheidender Bedeutung, um diese Herausforderungen zu bewältigen und sicherzustellen, dass zukünftige Beschleuniger die erforderliche Präzision für neue wissenschaftliche Entdeckungen erreichen. Die aus den Erfahrungen des LHC mit den komplexen Wechselwirkungen der multipolaren Felder gewonnenen Erkenntnisse werden das Design und den Betrieb zukünftiger Beschleuniger der nächsten Generation wie den HL-LHC und FCC informieren, die zunehmend auf eine präzise Kontrolle der nicht-linearen Optik angewiesen sein werden. Diese Fortschritte werden sicherstellen, dass diese Beschleuniger an der Spitze der Erforschung grundlegender Fragen zum Universum bleiben.

Die in diesen Kapiteln präsentierte Arbeit stellt einen bedeutenden Beitrag zum Bereich der Beschleunigerphysik dar, indem sie praktische Lösungen für aktuelle Betriebsherausforderungen bietet und den Weg für zukünftige Fortschritte ebnet.

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# Acknowledgements

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Ok, so this is the place where I get to talk about my life and where you'll be looking for you name to see what I've got to say about you. This would need to be a blog post given how many things I want to say, so it will be *relatively* brief. Let's start first by saying that I'm incredibly happy to have to done this thesis at CERN, which I honestly found quite fun. Not only because the subject was interesting, but also because I have been surrounded by kind, caring and smart people through all these years.

To give a little perspective, I have originally studied computer science at EPITA (École Pour l'Informatique et les Techniques Avancées) in Paris and earned an engineering degree in that field. Nothing really predestined me to work in any field related to physics, let alone to do a PhD on accelerator physics, but here we are! After my degree, I applied for a trainee position at CERN, basically working on software development for optics measurements and corrections. After two years of training, I was proposed with a PhD on the topics I've been learning, which I gladly accepted. This is how I embarked on a 3-year journey about accelerator physics. For this, I would never be thankful enough to Rogelio Tomas, who recruited and trusted me for that first contract, without which my life would probably be vastly different. I am also immensely grateful to Ewen Maclean, who has been a fantastic supervisor, even probably the best I've had yet to meet. He has always been present to help with me accelerator concepts in a very intelligible manner. Being kind, following up with my studies and always proposing new ideas made me feel like a complete member of the OMC team. On the university side, at Goethe-Universität Frankfurt, I would like to thank Giuliano Franchetti for his guidance and ideas about this thesis.

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Through my now five years at CERN, I have had the privilege to meet incredible people who changed not only my career path but also who I am for the best. And even if it sounds a bit weird, I am incredibly happy you have all been there.

Upon starting my very first contract, I was paired with Max Mihailescu and Sébastien Joly in an office, all three beginning our journey at CERN. I am very thankful to Max for having been a good friend during his short stay here, and for his insights on mathematics. I consider myself remarkably lucky to have met Sébastien, who probably should be listed as a supervisor here given how much he helped me when I struggled to learn the basics of accelerator physics. I would also like to thank Michael Hofer and Félix Carlier, who were there to answer my countless questions, be it when I started or even now at the end of my PhD. Closely related to my thesis subject were Félix Soubelet and Joschua Dilly who greatly helped me in various ways. Thanks a lot as well to Jacqueline Keintzel and Frank Zimmerman, to have given me the opportunity to work for a month at KEK in Japan, and expand my knowledge of lepton accelerators.

Many thanks to all the members of the OMC team who were great colleagues, be it in the office or during barely-legal night and week-end shifts in the control room. Specifically, I would like to thank Wietse Van Goethem, Leon Van Riesen-Haupt, Elena Fol, Andreas Wegscheider, Vittorio Ferrentino and Tobias Persson. Talking about the control room, I want to thank the LHC operation team and specifically Michi Hostettler for our conversations and his deep knowledge of CERN and the LHC. CERN being a vast laboratory, I had the pleasure to meet remarkable people who I enjoyed to be around. I cannot detail how much you all mean to me as this is already quite long, but you get the idea. For this, thanks a lot to Sofia, Christophe, David, Jean-Baptiste, Lisa, Joanna, Roxana, Dora, Joséphine, Jack, Ellie, Tirsi, Kostas, Björn, Christian, Laura, Roland, Luca, Wainer, Tiziana and Pierre.

This thesis would not have been possible without the support of people from outside of CERN. Namely I would like to thank my IRC friends, bonswouar, WhatIsGoingDown, pankkake, ShameOnYou, href and bishop. Special thanks to a large friend group I have always loved to be part of, *la crème de la crème de la cataphilie*. Thanks a ton as well to Clotilde, Matthias, Alpha, Paul, Antoine, Nicolas, Youness, Ksenia, Gérard, Vivien, Baptiste, Félix, Benjamin, Fabian and Tsu for everything that happened during those years, making them memorable. I would like to thank a lot my parents and family who have always been there to support me in every possible way. Thanks Antje, Michel, and Yoann.

*Il faut savoir douter où il faut,  
se soumettre où il faut,  
croire où il faut.*

BLAISE PASCAL



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# Glossary

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## Nomenclature

**AC-Dipole** Dipole magnet capable of generating variable oscillating fields. Used to increase the transverse amplitude via forced oscillations for optics measurements.

**Amplitude detuning** Tune shift dependent on the amplitude of transverse oscillations. Corrected via octupoles.

**Aperture** Physical aperture, i.e. where the beam can pass, of an element in the accelerator.

**Beam** Short for "Particle beam". Beam 1 and Beam 2 refer to either of the two beams travelling in opposite directions in the LHC.

**Beta-beating** Relative difference of the beta function between measurement and model. Often expressed in percents:  $(\beta_{meas.} - \beta_{mdl})/\beta_{mdl} \cdot 100$ .

**Beta-function** Twiss parameter  $\beta$  as a function of  $s$ , the longitudinal position. Related to the amplitude of transverse oscillations of the beam and its size.

**Chromatic Amplitude Detuning** Tune shift dependent on both the amplitude of transverse oscillations and the momentum offset. Corrected via decapoles to the first order.

**Chromaticity** Tune shift dependent on the momentum offset. Corrected via sextupoles to the first order.

**Closed orbit** Path of the reference particle through the accelerator, closing back on itself.

**Coupling** Coupling of a particle's motion in the transverse planes. Corrected via skew quadrupoles to the first, linear, order.

**Crosstalk** Unwanted magnetic interference between adjacent magnetic circuits.

**Drift** Drift space, a field-free region.

**Dynamic Aperture** Maximum region in phase space where particle motion remains stable over time, beyond which particles may be lost.

**Feed-down** Lower-order-like effects induced by a particle passing off-center through a multipole.

**Feed-up** Higher-order-like effects induced by a combination of lower-order multipoles.

**Landau Octupole** Strong octupoles present in the LHC to introduce Landau Damping. The tune spread created via amplitude detuning helps suppress the coherent instabilities of many particles.

**MAD-X** Current version of the Methodical Accelerator Design framework developed in BE-ABP. Used for beam dynamics simulations.

**Online** Refers to actions, often corrections, performed in the control room during measurements, where time is limited. Opposed to *offline*.

## Acronyms

**BBQ** Base Band Tune – Precise tune measurement system consisting of a pick-up and filters.

**BCH** Baker-Campbell-Hausdorff theorem – Formula for the combination of exponentials in a Lie algebra,  $e^X \cdot e^Y = e^Z$ .

---

<b>BPM</b>	Beam Position Monitor – Instrumentation used to retrieve both position and intensity of the beam via its induced electric field.
<b>IP</b>	Interaction Point – Center of the straight sections of the LHC. Beams collides in four of them where they cross (IP1, 2, 5, 8).
<b>IR</b>	Insertion Region – Part of the straight sections, between the dispersion suppressors. The IP is located at its center.
<b>LHC</b>	Large Hadron Collider – Largest and most powerful particle collider in the world.
<b>LSA</b>	LHC Software Architecture – Software used to operate the particle accelerators at CERN. Based on an online database to manage high and low level parameter settings.
<b>MAD-X</b>	Methodical Accelerator Design – Current version of the framework developed in BE-ABP. Used for beam dynamics simulations.
<b>MD</b>	Machine Development – Dedicated studies aimed at improving the accelerator parameters or testing new operational configurations.
<b>PTC</b>	Polymorphic Tracking Code – Framework used by MAD-X to perform calculations in the non-linear regime.
<b>RDT</b>	Resonance Driving Term – Coefficients related to the strength of a resonance.
<b>RF</b>	Radio Frequency – Shorthand for the acceleration system of the accelerator.

## Symbols

$B\rho$	Magnetic rigidity – Quantifies the ability of a field to deviate a particle [Tm].
$J_n$	Skew magnetic field strength – Skew field component of a multipole of order $n$ , normalized to the magnetic rigidity [ $m^{-n}$ ].
$J_{x,y}$	Action – Phase space coordinate in the Courant-Snyder normalization [m].
$K_n$	Normal magnetic field strength – Normal field component of a multipole of order $n$ , normalized to the magnetic rigidity [ $m^{-n}$ ].

## Glossary

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- $Q$  Tune – Number of betatron oscillations per turn in a circular accelerator [1].
- $Q^{(n)}$  Chromaticity of order  $n$  – Orders up to three are generally denoted  $Q'$ ,  $Q''$  and  $Q'''$  [1].
- $[ , ]$  Poisson brackets operator – Commonly used in accelerator physics as commutator in the Lie algebra.
- $\alpha_c$  Momentum compaction factor – Characterizes the change of particles' path length with the momentum offset [1].
- $\beta^*$   $\beta$ -function at a given IP [m].
- $\delta$  Momentum offset – Deviation of a particle's momentum relative to the reference one [1].
- $f_{jklm}$  Resonance Driving Term – Specific term related to a multipole of order  $n = j+k+l+m$  [ $m^{-n/2+1}$ ].
- $|C^-|$  Minimum tune separation – Global quantification of the linear coupling [1].

# Introduction

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## 1.1. Motivations

Optics control in accelerators is a key factor for optimizing the performance and stability of high-energy particle colliders like the Large Hadron Collider (LHC). As accelerators push the boundaries of operational parameters, the influence of higher-order magnetic fields becomes more pronounced, requiring precise corrections to ensure optimal performance. Addressing non-linearities, which arise from magnetic field errors, is crucial not only for the LHC's current operations but also for future collider designs.

The LHC acts as a central testbed for investigating higher-order non-linearities that directly influence beam stability, dynamic aperture, and beam lifetime. Correcting these non-linearities is crucial for sustaining high performance in the LHC and for informing the design of future machines. This research is driven by the need to develop advanced beam-based techniques for measuring and correcting these higher-order effects, shifting from traditional empirical methods to more precise and quantitative approaches. Accurately quantifying the fields present in the LHC is essential for achieving a comprehensive understanding of the machine. Such confidence in managing non-linear dynamics is vital for the development and operation of future accelerators like the High Luminosity LHC (HL-LHC) and the Future Circular Collider (FCC).

A key issue highlighted during the LHC's Run 2 is the observed discrepancy at injection energy between the measured third-order chromaticity  $Q'''$  and the predictions

made by existing models. The magnetic measurements of the LHC's magnets, conducted during its construction phase, have served as the foundation for simulations, beam steering, and non-linear correction computations. However, this discrepancy suggests the presence of previously unaccounted-for field errors not captured by the initial magnetic measurements. Identifying and addressing these unknown sources of higher-order magnetic errors is crucial for improving the LHC's operational parameters, particularly during beam injection, to ensure optimal performance and stability.

This thesis aims to address these challenges by developing improved methods for characterizing higher-order magnetic fields, such as decapolar components, and their impact on beam dynamics. Through direct, quantitative approaches, the research seeks to refine correction strategies and better understand the interplay of non-linear magnetic fields. These advancements will contribute not only to enhancing the LHC's performance but also to informing the design and operation of future high-energy colliders.

## 1.2. Thesis Outline

The thesis starts by giving the motivations for this thesis work, as well as its outline, in Chapter 1. Key concepts of accelerator physics are presented in Chapter 2. The CERN accelerator complex and the LHC are then detailed. Measurement and correction techniques are presented in Chapter 3.

The first results chapter, starting at Chapter 4, examines the skew octupolar fields which have been shown to limit the dynamic aperture, especially during beam excitation with the AC-Dipole. A response matrix method was developed to correct skew octupolar Resonance Driving Terms (RDTs) at top energy. The study also explores the influence of Landau octupoles on skew octupolar RDTs at injection energy, revealing the importance of accurate coupling modeling in predicting these effects.

Chapter 5 delves into the decapolar fields at injection energy, addressing discrepancies between measurements and model predictions of third-order chromaticity. Through a series of novel measurements and simulations, including the introduction of chromatic amplitude detuning, the research identifies the decay of the decapolar component in the main dipoles as a key factor in these discrepancies. Corrective strategies were developed for decapolar RDTs, leading to measurable improvements in beam lifetime and stability.

Chapter 6 builds up on the achievements of the decapolar corrections and focuses on the measurement and analysis of dodecapolar and decatetrapolar fields, made possible by improved dynamic aperture, measurement techniques and refinements in post-processing. The study successfully measures higher-order chromaticity terms and dodecapolar RDTs, demonstrating their significant contribution to the overall field errors and allowing the first ever benchmarking of these effects in the LHC. The findings underscore the need for further investigation into these higher-order fields and their impact on beam dynamics to optimize the LHC's performance.

The final chapter, Chapter 7, which serves as a supplementary section, explores the application of optics measurement techniques employed at CERN to the SuperKEKB rings (HER and LER) at KEK in Tsukuba, Japan. This study is the outcome of a one-month secondment as part of the EAJADE collaboration.

Finally, conclusions for these studies are drawn at the end of the thesis.



# Concepts of Accelerator Physics

## 2.1. Particle Accelerators

Particle accelerators have come a long way since their invention in the 20th century, when they could only reach a few MeV of energy using electric fields [1]. Today, they achieve much higher energies, up to the TeV range, allowing for more detailed investigations into particle physics.

One of the major milestones in accelerator development was the shift to circular accelerators, known as synchrotrons. By the 1970s, machines like the ISR at CERN could accelerate particles to 62 GeV using beams from the Proton Synchrotron (PS) [2], enabling significant advances in particle physics experiments.

As accelerators have evolved, so has the need for better control of particle beams. Managing beam stability and lifetime has become crucial, especially in modern machines like the LHC.

Considerable advancements have been made in both radiofrequency acceleration technologies, collimation, superconducting magnets and beam dynamics to keep beams stables and on track, even at high energies. These improvements are not only important for physics research but are also used in medical and industrial fields.

In modern accelerators, precise beam control is necessary to ensure efficient and accurate collisions. Correcting beam optics has moved from being just a theoretical exercise to an essential part of operations. This has led to a shift away from empirical methods, where trial and error were common, towards more direct, quantitative approaches that provide a detailed and accurate correction of beam dynamics. These advancements not only improve performance but also play a key role in designing future accelerators with better stability and efficiency.

### 2.1.1. The CERN Complex

CERN is a large laboratory located on the border of France and Switzerland, near Geneva. Although well-known for discoveries in particle physics, studies are also conducted on medical applications, biology, radiation hardness or material science. Several accelerators are part of the accelerator complex, as illustrated in Fig. 2.1. A large number of fixed target experiments exist, whose beams are delivered by various accelerators depending on their needs. These experiments are often renewed<sup>1</sup>.

The largest part of the CERN accelerator complex is the LHC. The LINAC4, PSB, PS and SPS accelerators serve as pre-injectors to the LHC.

### 2.1.2. The Large Hadron Collider

The Large Hadron Collider (LHC) is a circular particle accelerator primarily designed to collide protons for fundamental particle physics research. It can also occasionally collide ions such as oxygen or lead for specific studies. At the time of writing, in 2024, it holds several records, such as being the largest and most powerful accelerator in the world, at nearly 27 km long. The LHC is composed of two beam pipes, capable of accelerating two particle beams from an injection energy of 450 GeV to an energy of 6,800 GeV, before colliding them in four detectors: ATLAS, CMS, Alice and LCHb.

Well-publicized, the LHC is often depicted via its superconducting dipole magnets, housed in blue cryostats, aimed at cooling the coils. Figure 2.2 shows a 3D cut of such magnets. The LHC is mostly composed of these *main* dipoles, holding 1,232 of

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<sup>1</sup>Up to date information can be found on <https://home.cern/science/experiments>.

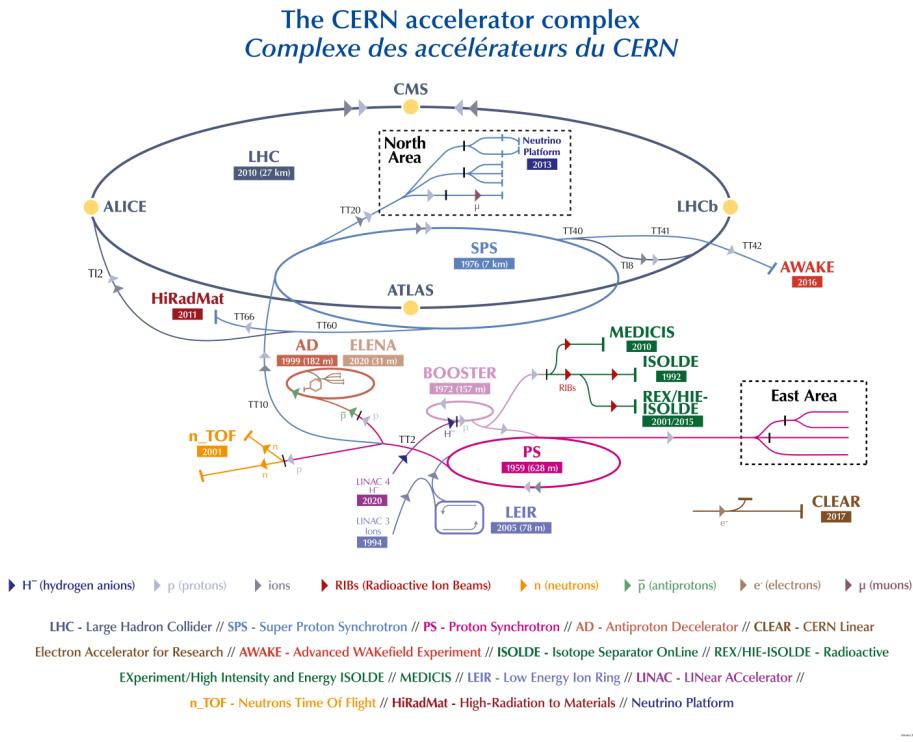


Figure 2.1.: Schematic illustration of the accelerator complex at CERN. Most accelerators are both used as injectors for the LHC or to provide beams to fixed target experiments [3].

them, each about 14 meters long. Particles in the LHC travel at nearly the speed of light (99.9999905%), completing roughly 11,200 turns around the tunnel each second. To generate the intense magnetic fields needed to deflect such high-momentum particles, currents of approximately 12,000 amperes are supplied to the magnets. However, conventional materials like copper would overheat and melt under this current load, necessitating the use of superconducting materials such as niobium-titanium (NbTi).



Figure 2.2.: 3D cut of a main LHC dipole [4]. Both beam pipes can be seen surrounded by the coils, strongly clamped by the yokes.

### Straight Sections and Arcs

The LHC is not a perfect circle. It is indeed composed of eight *straight* sections, called the *Insertion Regions* (IRs) where detectors or specific instrumentation are placed. Connecting those sections, the *arcs* are where the majority of the magnets and their correctors are located, along with some instrumentation like beam position monitors. Figure 2.3 shows the arcs as well as the purpose of each straight section.

### Arc Cells

Each arc is made up of 23 cells. Magnets are organized in a standard FODO structure (see Section 2.3.1), as shown in Fig. 2.4. *Dipoles* are responsible for bending the trajectory of the particles. Their associated correctors, the orbit correctors, mitigate any possible drift in the path. *Quadrupoles* are used to control the beam size along the ring. Their effect is focusing in one plane and defocusing in the other. Their associated correctors control the frequency of oscillations of the beam (see tune, Section 2.3.1) and possible

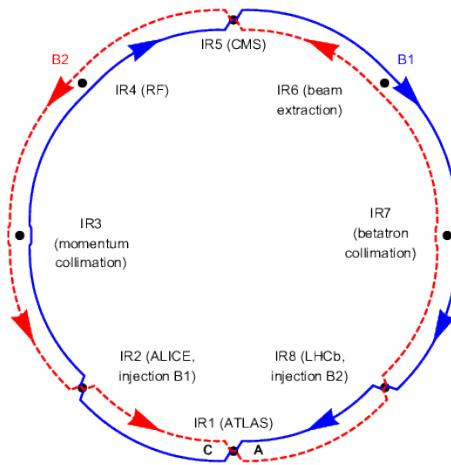


Figure 2.3.: Schematic of the LHC layout.

field imperfections. *Sextupoles* correct chromaticity, a misfocus from quadrupoles due to particles having a different momentum than the reference particle. *Octupoles* are used to stabilize the beam by introducing Landau Damping [5]. The associated correctors correct higher-order chromaticity effects as well as amplitude-dependant tune shifts. *Decapoles* correctors aim at correcting an even higher chromaticity order.

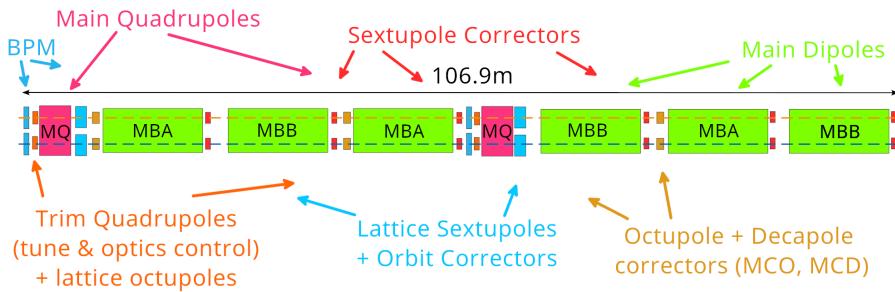


Figure 2.4.: Schematic of an LHC Arc cell [6].

## Cycles

During the operation of the LHC, the machine goes through several states, each defined for specific scenarios [7].

A common example is the operational cycle of the LHC, illustrated in Fig. 2.5. Initially, the magnets are pre-cycled [8] without any beam circulating, to ensure a reproducible magnetic state. Their current is then increased to accept particles at the injection energy of 450 GeV. To verify the machine's proper functioning, a probe bunch of reduced intensity is first injected, at around  $10^{10}$  particles. The number of bunches and their intensity are then gradually increased to obtain the desired scheme needed for collisions, which varies throughout the year based on experimental demands. The number of bunches and their intensity can be adjusted as needed to ensure the machine's safety. A common scheme in 2024 is to inject about 2350 bunches, with around  $1.5 \cdot 10^{11}$  particles in each bunch, for collisions. Optics measurements, due to their destructive nature, typically use between one and three *pilot* bunches at a lower intensity of  $10^{10}$  particles.

The current in the magnets is then increased along with adjustments in the phase and voltage of the RF system to accelerate the particles to an energy of 6.8 TeV. During this ramp, the beam is initially squeezed at the Interaction Points. However, further squeezing is limited by the detectors' resolution, which is insufficient to accurately reconstruct a large number of collision events simultaneously. Collisions then start, while levelling the  $\beta^*$  at the ATLAS and CMS experiments relative to the remaining beam intensity, ensuring optimal performance of the detectors.

## 2.2. Magnetic Fields

### 2.2.1. Nomenclature

Several notations coexist to denote magnetic fields. In this thesis, the *European Convention* [10] is used for field indices, as shown in Table 2.1. MAD-X and MAD-NG, simulation softwares, however, use the *American Convention*.

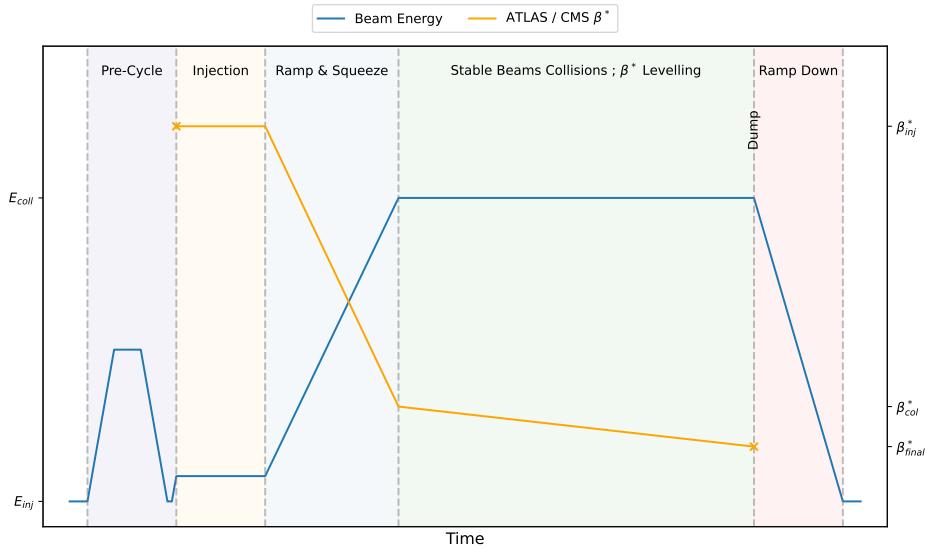


Figure 2.5.: Simplified illustration of a standard LHC cycle. Adapted from [9].

Multipole	Index	MAD-X
Dipole	1	0
Quadrupole	2	1
Sextupole	3	2
Octupole	4	3
Decapole	5	4
Dodecapole	6	5
Decatetrapole	7	6
Decahexapole	8	7

Table 2.1.: Relation between the field indices used in this thesis and multipoles.

As such, unless explicitly stated, quantities such as the magnetic strength  $b$  and normalized strength  $K$  presented latter will be expressed with the notation used in the

first column. A schematic representation of magnets up to dodecapole, order 6, is given in Fig. 2.6.

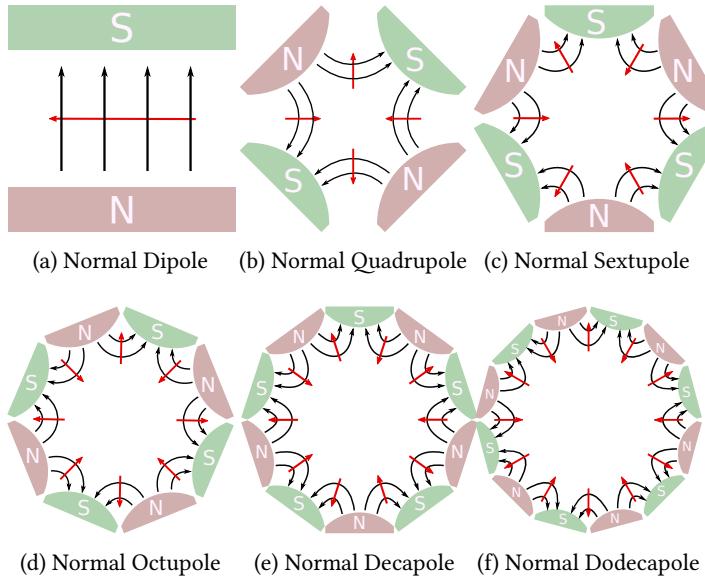


Figure 2.6.: Schematics of magnetic multipoles. In black the magnetic field lines, which extend also to the inner part of the magnets, but drawing them there has been omitted for clarity of the figure. Red arrows indicate the direction of force on a positive charge moving out of the page towards the reader. Courtesy of Joschua Dilly [10].

## 2.2.2. Multipole Expansion

In order to force the particles to form a closed orbit, they are subjected to magnetic fields that deflect their trajectories. The force exerted on a charged particle via electromagnetic fields is the Lorentz force  $\vec{F}$  [11],

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}), \quad (2.1)$$

where  $\vec{p}$  is the momentum of the particle,  $q$  its charge,  $\vec{v}$  its velocity,  $\vec{E}$  the electric field and  $\vec{B}$  the magnetic field. With particles close to the speed of light, it becomes apparent that magnetic fields are dominant in the resulting force and are thus used to act on the particles trajectories. The guiding magnetic field in the transverse planes  $x$  and  $y$  can be described using a *multipole expansion*, where the components of the magnetic field can be written as a series of terms corresponding to different orders of multipoles. The multipole expansion is then given in terms of the normal and skew field gradients  $\mathcal{B}$  and  $\mathcal{A}$  with multipoles of order  $n$  [12],

$$B_y + iB_x = \sum_{n=1}^{\infty} (\mathcal{B}_n + i\mathcal{A}_n) (x + iy)^{n-1}. \quad (2.2)$$

The normal and skew field gradients  $\mathcal{B}$  and  $\mathcal{A}$  for a multipole of order  $n$  can then be calculated from this complex field,

$$\mathcal{B}_n + i\mathcal{A}_n = \frac{1}{(n-1)!} \cdot \left. \frac{\partial^{n-1} (B_y + iB_x)}{\partial(x+iy)^{n-1}} \right|_{x=0,y=0}. \quad (2.3)$$

Expanded up to octupoles, the magnetic field acting on the horizontal plane reads,

$$B_y = \underbrace{B_{y0}}_{\text{dipole}} + \underbrace{\frac{\partial B_y}{\partial x} \cdot x}_{\text{quadrupole}} + \underbrace{\frac{1}{2!} \frac{\partial^2 B_y}{\partial x^2} \cdot x^2}_{\text{sextupole}} + \underbrace{\frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3} \cdot x^3}_{\text{octupole}} + \dots \quad (2.4)$$

An ideal magnet would often produce either a sole normal or skew field. However, this is not applicable to real-life magnets that are imperfect, due to design and manufacturing constraints. Field errors are thus introduced, relative to the main field of the ideal 2N-pole magnet at a reference radius  $r_{ref}$ , as shown in Eq. (2.5). The coefficients of the normal and skew relative field errors, referred to as  $a_n$  and  $b_n$ , are dimensionless but often given in units of  $10^{-4}$ .

$$B_y + iB_x = \begin{cases} \mathcal{B}_N \cdot \sum_{n=1}^{\infty} (b_n + ia_n) \left( \frac{x+iy}{r_{ref}} \right)^{n-1}, & \text{for normal magnets} \\ \mathcal{A}_N \cdot \sum_{n=1}^{\infty} (b_n + ia_n) \left( \frac{x+iy}{r_{ref}} \right)^{n-1}, & \text{for skew magnets} \end{cases} \quad (2.5)$$

The total normal and skew field components of order  $n$  for an imperfect 2N-pole magnet is thus given by the following equation:

$$\begin{aligned}\mathcal{B}_n &= \mathcal{B}_N \cdot \frac{b_n}{r_{ref}^{n-1}}, \\ \mathcal{A}_n &= \mathcal{A}_N \cdot \frac{a_n}{r_{ref}^{n-1}}.\end{aligned}\tag{2.6}$$

The unit of the field is relative to the multipole order  $n$ : [Tm $^{1-n}$ ].

### 2.2.3. Beam Rigidity and Normalization

#### Beam Rigidity

In order to bend particles to form a ring, the force acting on the particles given by the Lorentz force must be equal to the centrifugal force [11, 13]:

$$q(\vec{E} + \vec{v} \times \vec{B}) = \frac{\gamma mv^2}{\rho},\tag{2.7}$$

with  $q$  the charge of the particle,  $\vec{E}$  and  $\vec{B}$  respectively the electric and magnetic field strengths,  $\gamma$  the Lorentz factor,  $\vec{v}$  the velocity of the particle,  $m$  its mass and  $\rho$  the radius of the circular path. The beam rigidity is a quantification of the ability of a magnetic field to bend the trajectory of a particle. It is derived from the previous equation and relates the magnetic field  $B$ , the radius of curvature  $\rho$  to the relativistic momentum  $p$  and charge  $q$  of the particle. By neglecting the electrical force and using  $p = \gamma mv$ ,

$$\begin{aligned}qvB &= \frac{\gamma mv^2}{\rho}, \\ \rightarrow qB &= \frac{p}{\rho}, \\ \rightarrow B\rho &[T.m] = \frac{p}{q}.\end{aligned}\tag{2.8}$$

The beam rigidity can also be expressed via the momentum in GeV. The usual approximation is then given by,

$$B\rho \approx 3.33 \cdot pc. \quad (2.9)$$

This quantity is of interest when designing an accelerator to set the maximum field as well as the required radius of curvature for a specific momentum and particle. An interesting metric of an accelerator is also its *filling factor*, or percentage of dipoles in the machine. It can be calculated via the radius of curvature  $r$ :  $f = \rho/r$ . A low filling factors means more space for other magnets, collimators, beam instrumentation, etc.

### Field Normalization

The Beam Rigidity is also used as a way to normalize magnetic field strengths in particle accelerators where the momentum of the particle changes (i.e. acceleration or deceleration). The normalized Normal and Skew magnetic strengths  $K$  and  $J$  for a multipole of order  $n$  are thus given by [12] the following,

$$\begin{aligned} K_n &= \frac{q}{p}(n-1)!\mathcal{B}_n, \\ J_n &= \frac{q}{p}(n-1)!\mathcal{A}_n, \end{aligned} \quad (2.10)$$

with  $p$  the momentum of the particle,  $q$  its charge and  $\mathcal{B}$  and  $\mathcal{A}$  the field gradients.

#### 2.2.4. Hamiltonian

The Hamiltonian describing the motion of relativistic particles in static electromagnetic fields is given by [14],

$$H = c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2} + q\phi. \quad (2.11)$$

where  $\phi$  is the scalar potential,  $\mathbf{A}$  the vector potential,  $q$ ,  $\mathbf{p}$  and  $m$  respectively the charge, canonical momentum and mass of the particle and  $c$  the celerity of light.

After having changed this Hamiltonian to be dependent on curved coordinates, introduced in the next section, the motion in the transverse planes for a given multipole of order  $n$  (other than dipoles) can be described by the following [14],

$$\begin{aligned} H &= \frac{q}{p} \Re \left[ \sum_{n>1} (\mathcal{B}_n + i\mathcal{A}_n) \frac{(x+iy)^n}{n} \right] \\ &= \Re \left[ \sum_{n>1} (K_n + iJ_n) \frac{(x+iy)^n}{n!} \right]. \end{aligned} \tag{2.12}$$

Quite often, when studying the effect of a magnet on the beam, only one component is required, and the sum can thus be dropped. The normal and skew fields can also be isolated in order to consider their sole effect as shown in the following,

$$\begin{aligned} N_n &= \frac{1}{n!} K_n \Re [(x+iy)^n], \\ S_n &= -\frac{1}{n!} J_n \Im [(x+iy)^n]. \end{aligned} \tag{2.13}$$

## 2.2.5. Harmonics

The magnetic fields in the LHC are generated by the coils of its magnets. However, real-world magnets never produce a perfect, single field as desired. Instead, some field errors, known as *allowed harmonics*, naturally arise due to the geometric symmetries of the coils. As a result, the main dipoles of the LHC can generate fields resembling those of sextupoles, decapoles, decatetrapoles, and so on [15]. Additionally, manufacturing imperfections contribute to field errors beyond the allowed ones, an example being octupolar errors seen in the LHC dipoles.

During the design of the LHC, the main dipoles have been identified to generate significant field errors. Magnetic measurements of those various fields were thus taken and magnetic tables built based on real-life magnets nowadays installed in the machine. Those magnetic tables, computed for each LHC configuration by *WISE* [16] are used by simulation softwares. Predictions of field errors and compensating strength for the correctors is computed by the Field Description for the LHC (*FiDeL*, [17]). *FiDeL* is used

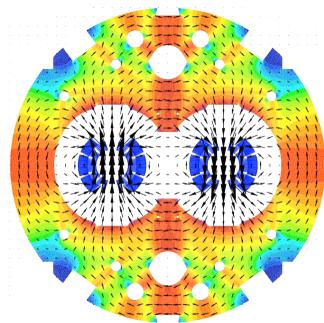


Figure 2.7.: Magnetic field in a dipole magnet [15].

## 2.3. Coordinate Systems

In circular accelerators, particle dynamics are represented using a comoving coordinate system. A reference orbit is determined by the lattice and its magnet strengths. A set of given strengths for a lattice is called *optics*. In the case of a synchrotron, like the LHC, the reference orbit is also called the closed orbit.

The Frenet-Serret coordinate system moves along the ring on the reference orbit. The coordinates are then transverse:  $x$  and  $y$ , and longitudinal in the direction of travel:  $s$ . Figure 2.8 shows these coordinates.

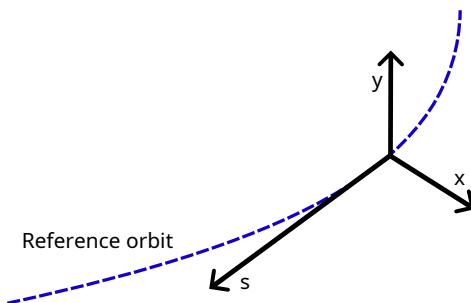


Figure 2.8.: Frenet-Serret coordinate system, commonly used in accelerator physics. The system moves along the reference orbit.

This coordinate system is widely used to simply describe either an element's or a particle's position in the accelerator. Without any explicit mention, these are the coordinates used in this thesis. It is frequent to use the variable  $z$  to refer to either  $x$  or  $y$  in equations. In order to describe the motion of particles through a lattice, different coordinate systems can be used. To better describe the motion of particles through a lattice, various coordinate systems can be employed. First, the formalism for a linear lattice will be introduced, followed by an explanation of how motion in a machine with non-linear elements can be characterized.

Linear optics refer to the regime where the forces acting on particles, such as those from dipoles and quadrupoles, are directly proportional to the particle's displacement from the reference trajectory, resulting in simple, predictable motion described by linear equations and transfer matrices. Non-linear optics, on the other hand, involves higher-order magnetic elements like sextupoles and octupoles, where the forces are non-linear with respect to displacement. This leads to more complex particle motion, which can result in phase-space distortions and chaotic trajectories.

### 2.3.1. Linear Lattice

#### Courant-Snyder Parameters

A circular accelerator is composed of many multipoles of different orders. A very basic design only requires dipoles and quadrupoles in order to operate. Dipoles are used to bend the particles to form the ring, whereas quadrupoles are used to focus the beam to a focal point, similar to photons with lenses. Those elements can be arranged in a particular order, to form a FoDo cell. Such cells present an alternating placement of focusing and defocusing quadrupoles with dipoles in between, as shown in Fig. 2.9, and are usually repeated many times along the ring.

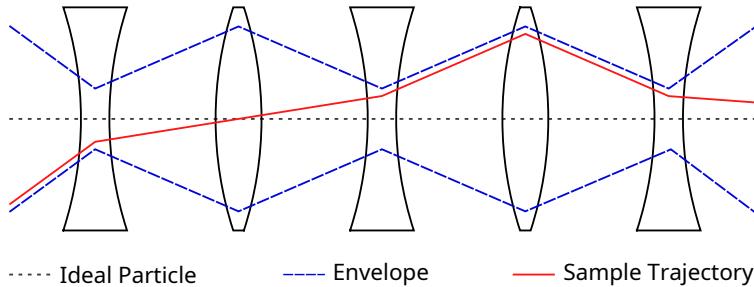


Figure 2.9.: Line composed of FoDo cells, a basic cell present in most accelerators, composed of a Focusing and a Defocusing quadrupole. The envelope of the beam is influenced by the  $\beta$ -function and the action  $J$ .

A lattice composed of only dipoles and quadrupoles, is referred to as a *linear lattice*. In a synchrotron, a circular particle accelerator, particles undergo transverse and longitudinal oscillations. These oscillations usually describe an ellipse in phase-space, a system with position  $z$  and momentum  $p_z$  as coordinates. Taking into account those oscillations, the phase-space ellipse of a particle at a position  $s$  in the ring can be described with a new system: the Courant-Snyder parameters, also known as Twiss parameters or the *optics functions* [18], as shown in Fig. 2.10.

$J$ , the action, is the invariant of linear motion at a given energy and describes the amplitude of oscillations. It is related to the other quantities by:

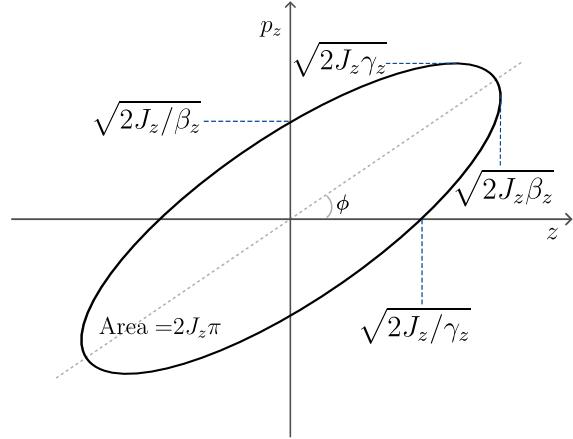


Figure 2.10.: Phase-space ellipse of a linear machine, parametrized by the Courant-Snyder parameters  $\alpha$ ,  $\beta$  and  $\gamma$ .

$$J_z = \frac{1}{2}(\gamma_z \cdot z^2 + 2\alpha_z p_z \cdot z + \beta_z p_z^2). \quad (2.14)$$

The area of the phase-space ellipse is also called the single particle emittance, and is described by the action:  $\epsilon = 2J$ . As the  $\beta$  parameter varies along the ring, it is referred to as the  $\beta$ -function and is related to the amplitude of the oscillations. Thus, the smaller is the  $\beta$ -function, the smaller is also the envelope of the beam. The number of oscillations per turn is called the *tune*, and is closely related to the  $\beta$ -function:

$$Q_{x,y} = \frac{1}{2\pi} \oint \frac{1}{\beta_{x,y}(s)} ds. \quad (2.15)$$

It is common to express the position of a particle using *action-angle* variables:

$$\begin{aligned} z &= \sqrt{2J_z\beta_z} \cos \phi_z \\ p_z &= -\sqrt{\frac{2J_z}{\beta_z}} (\sin \phi_z + \alpha_z \cos \phi_z). \end{aligned} \quad (2.16)$$

## Normalized Coordinates

In order to simplify the description of the linear motion in a ring, a transformation can be applied to the previously seen coordinates. Figure 2.11 shows a phase-space described in both coordinates. The new coordinates,  $\hat{z}$ , and  $\hat{p}_z$ , are then expressed as factors of the  $\alpha$  and  $\beta$  functions:

$$\begin{pmatrix} \hat{z} \\ \hat{p}_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta_z}} & 0 \\ \frac{\alpha_z}{\sqrt{\beta_z}} & \sqrt{\beta_z} \end{pmatrix} \begin{pmatrix} z \\ p_z \end{pmatrix}. \quad (2.17)$$

This allows to describe the motion as a simple rotation, the new coordinates being only dependent on the invariant  $J_z$  and the phase  $\phi_z$ :

$$\begin{aligned} \hat{z} &= \sqrt{2J_z} \cos(\phi_z), \\ \hat{p}_z &= \sqrt{2J_z} \sin(\phi_z). \end{aligned} \quad (2.18)$$

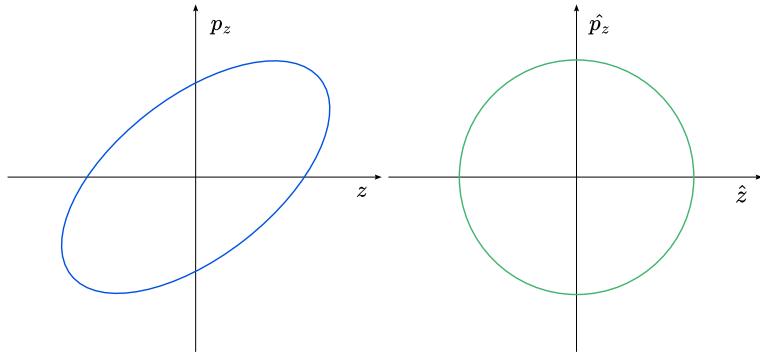


Figure 2.11.: Phase space described in both regular and normalized coordinates.

## Linear Transfer Maps

The final position of a particle after passing through an accelerator element can be described using *transfer maps*. In the case of linear optics, these maps take the form

of matrices. Importantly, these maps should be symplectic, meaning the area of the phase-space ellipse is preserved, ensuring that the particle's motion is accurately represented. Symplecticity guarantees that key properties of the beam, such as its emittance, remain constant, which is crucial for maintaining the stability and integrity of particle trajectories in tracking simulations. For a matrix  $M$  and positions  $z$  at the initial location and at  $s$ , the general formula reads [19]:

$$\begin{pmatrix} z \\ z' \end{pmatrix}_s = M \cdot \begin{pmatrix} z \\ z' \end{pmatrix}_0 \quad (2.19)$$

This formalism assumes that the magnetic field is linear or constant along the element in the longitudinal direction. Basic elements such as drifts, dipoles, quadrupoles can then be described by a simple  $2 \times 2$  matrix:

$$M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad (2.20)$$

$$M_{dipole} = \begin{pmatrix} \cos(L/\rho) & \rho \sin(L/\rho) \\ -1/\rho \sin(L/\rho) & \cos(L/\rho) \end{pmatrix}, \quad (2.21)$$

$$M_{focusing\ quad.} = \begin{pmatrix} \cos(\sqrt{k_2}L) & 1/\sqrt{k_2} \sin(\sqrt{k_2}L) \\ -\sqrt{k_2} \sin(\sqrt{k_2}L) & \cos(\sqrt{k_2}L) \end{pmatrix}, \quad (2.22)$$

$$M_{defocusing\ quad.} = \begin{pmatrix} \cosh(\sqrt{|k_2|}L) & 1/\sqrt{|k_2|} \sinh(\sqrt{|k_2|}L) \\ \sqrt{|k_2|} \sinh(\sqrt{|k_2|}L) & \cosh(\sqrt{|k_2|}L) \end{pmatrix}, \quad (2.23)$$

where  $L$  is the length of the element,  $\rho$  the radius of curvature of the orbit and  $k_2$  the normalized strength of quadrupoles. In the case of quadrupoles, a focusing matrix should be used in the horizontal plane for focusing quadrupoles, where defocusing matrices should be used in the vertical plane. The opposite goes for defocusing quadrupoles.

Transfer matrices can be combined together to describe a larger group of elements, as the FoDo cell seen previously. Its transfer matrix can then be expressed as,

$$M_{FoDo} = M_{focusing\ quad} \cdot M_{drift} \cdot M_{defocusing\ quad} \cdot M_{drift}. \quad (2.24)$$

For a closed machine, a full revolution can be described by a so-called *one-turn map*, being the transfer matrix of the whole machine, denoted  $\mathcal{M}$ . Such a map can potentially contain thousands of elements.

### 2.3.2. Non-Linear Lattice

So far, Courant-Snyder parameters were a good way to describe the distribution of positions and velocities of particles in the transverse plane. One caveat of using this formalism is that it is restrained to linear optics and does not address non-linear elements, like octupoles. These elements generate forces that are not directly proportional to a particle's displacement from the reference trajectory. Effects such as resonances or those arising from an arrangement of several multipoles together can be described by the concepts introduced in this section. An overview of the needed mathematical tools is first given, before introducing maps.

#### Lie Algebra

One way to describe non-linear effects is to introduce Lie Algebra [20], a powerful algebra able to describe transformations, symmetries and their associated conserved quantities. The Lie algebra is a vector space, denoted  $\mathfrak{g}$ , equipped with a binary operation called the *Lie bracket* and denoted  $[x, y]$  for two vectors  $x$  and  $y$ . Any vector space equipped with a Lie bracket (or commutator) satisfying the following conditions is called a Lie algebra:

- Bilinearity:

$$\begin{aligned} [ax + by, z] &= a[x, z] + b[y, z], \\ [z, ax + by] &= a[z, x] + b[z, y], \quad \forall x, y, z \in \mathfrak{g} \text{ and } a, b \text{ scalars} \end{aligned} \quad (2.25)$$

- Alternativity:

$$[x, x] = 0, \quad \forall x \in \mathfrak{g} \quad (2.26)$$

- Anticommutativity:

$$[x, y] = -[y, x], \quad \forall x, y \in \mathfrak{g} \quad (2.27)$$

- The Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g} \quad (2.28)$$

The *Lie bracket*, plays a central role in the Lie algebra. It describes how dynamical variables evolve under infinitesimal symplectic transformations.

## Poisson Brackets

In accelerator physics, following the Hamiltonian formalism and classical mechanics, the commutator is represented by the *Poisson brackets*, which satisfy the necessary conditions for describing particle motion [20, 21]. Poisson brackets are used to express continuous symmetries, conserved quantities, and the time evolution of dynamical variables within the system.

Let's consider position and momentum coordinates  $q_i \cdots q_n$  and  $p_i \cdots p_n$  of a  $2n$ -dimensional phase space. Usually, those would be  $x, y, p_x$  and  $p_y$  for transverse coordinates. The Poisson brackets of two functions  $f$  and  $g$  is then defined by:

$$[f, g] = \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \quad (2.29)$$

The evolution of coordinates and momenta in time is described by Hamilton's equations of motion, which can be naturally expressed with Poisson brackets:

$$\begin{aligned} \frac{\partial H}{\partial p_i} = \frac{dq_i}{dt} &= [q_i, H] \\ -\frac{\partial H}{\partial q_i} = \frac{dp_i}{dt} &= [p_i, H]. \end{aligned} \quad (2.30)$$

## Lie Operator

Given a function  $f$ , a differential operator called *Lie operator* is defined, and is closely related to the previously seen Poisson bracket:

$$:f := \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i}. \quad (2.31)$$

The action of this operator on a function  $g$  is equivalent to the Poisson brackets, as in:

$$:f:g = [f, g]. \quad (2.32)$$

A particular power series of this Lie operator can now be defined, called *Lie transformation*:

$$\begin{aligned} e^{:f:}g &= \sum_{l=0}^{\infty} \frac{1}{l!} :f^l g \\ &= g + [f, g] + \frac{1}{2!} [f, [f, g]] + \dots \end{aligned} \quad (2.33)$$

## Non-Linear Transfer Maps

As introduced in Section 2.3.1, the dynamics of a particle beam in a circular accelerator can be described by *transfer maps*. A symplectic *One Turn Map*  $\mathcal{M}$  that includes  $N$  non-linear elements is defined [20] as,

$$\mathcal{M} = e^{:h_N:} \cdot e^{:h_{N-1}:} \cdots e^{:h_1:} \cdot \mathcal{R} \quad (2.34)$$

where  $\mathcal{R}$  is a matrix describing the linear motion over one turn and the  $h_i$  terms representing the Hamiltonian of each non-linear elements of the machine. Via the Baker-Campbell-Hausdorff (BCH) theorem [22, 23], previous Lie transformations can be combined and simplified via Eq. (2.35) and Eq. (2.36). Further orders can be found and computed via [23].

with

$$\begin{aligned}
 h &= h_1 + h_2 && \Rightarrow 1^{\text{st}} \text{ order} \\
 &+ \frac{1}{2}[h_1, h_2] && \Rightarrow 2^{\text{nd}} \text{ order} \\
 &+ \frac{1}{12}[h_1, [h_1, h_2]] - \frac{1}{12}[h_2, [h_1, h_2]] && \Rightarrow 3^{\text{rd}} \text{ order} \\
 &+ \dots
 \end{aligned} \tag{2.36}$$

The one turn map is thus expressed as a single Lie transformation:

$$\mathcal{M} = e^{ih} \cdot \mathcal{R}. \tag{2.37}$$

In most cases, where the non-linear perturbations are small, the above series converges quickly and only the first order of Eq. (2.36) is used [24]. The resulting expression is then more elegant, being a simple sum of the Hamiltonians of the  $N$  non-linear elements:

$$\mathcal{M} = e^{ih_1+h_2+\dots+h_N} \cdot \mathcal{R}. \tag{2.38}$$

It is to be noted that, in this thesis, experimental measurements show the evidence of higher order contributions. In order to fully understand the combined effect of multipoles, the BCH expansion needs to be expended further than the first order.

When transporting coordinates from one point to another, the length of the elements must be taken into account:

$$\begin{aligned}
 e^{iLH}x_0 &= x_1, \\
 e^{iLH}p_0 &= p_1.
 \end{aligned} \tag{2.39}$$

### Normal Form

As non-linearities are introduced in the machine, the phase-space becomes distorted, resulting in  $J_z$  no longer being an invariant of motion. The previously seen normalization

does not work anymore and the phase-space ellipse is no longer a simple circle. A new normalization is then introduced, called the *normal form*, with complex coordinates  $\zeta$ , depending on new action and angle coordinates  $I_z$  and  $\psi_z$ :

$$\zeta_{z,\pm} = \sqrt{2I_z} e^{\mp i\psi_z}. \quad (2.40)$$

An exaggerated vision of such a phase-space in Courant-Snyder, normalized, and normal form coordinates can be seen in Fig. 2.12.

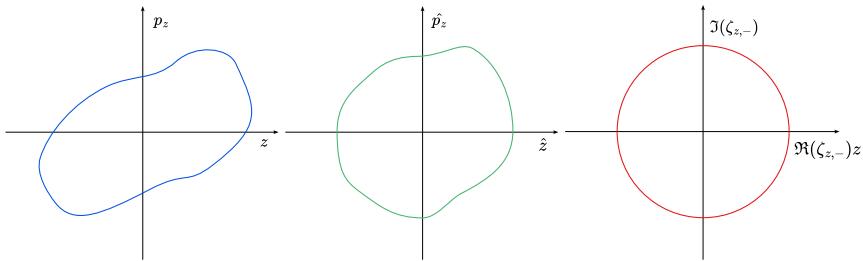


Figure 2.12.: Exaggerated phase space distorted by non-linearities described in regular (left), normalized (middle) and normal form (right) coordinates.

The map defined previously in Eq. (2.37) can be rewritten in order to retrieve an invariant of motion  $I_z$  by introducing a generating function  $F$ :

$$\tilde{\mathcal{M}} = e^{-F} \mathcal{M} e^{F} \quad (2.41)$$

Such a generating function includes all the non-linearities, simplifying the calculations. Going back and forth from normalized to normal forms coordinates is then straightforward, as depicted in Fig. 2.13. The hamiltonian  $H$  is now only dependent on the action  $I_z$ .

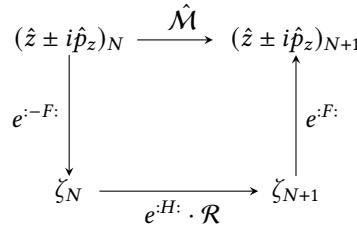


Figure 2.13.: A one turn map from turn N to N+1 solved using a generating function  $F$ , transforming to normal form coordinates  $\zeta$ , applying the linear rotation  $R$  and transforming back to normalized coordinates.

The function  $F$  is defined as

$$F = \sum_{jklm} f_{jklm} \zeta_{x,+}^j \zeta_{x,-}^k \zeta_{y,+}^l \zeta_{y,-}^m, \quad (2.42)$$

where  $f_{jklm}$  are the so-called Resonance Driving Terms (RDTs). The summation  $jklm$  is done over all the combinations of  $j, k, l$  and  $m$  with  $j + k + l + m = n$  for a multipole of order  $n$ , as shown in Eq. (2.43),

$$\sum_{jklm} = \sum_{j=0}^n \sum_{k=0}^n \sum_{l=0}^n \sum_{m=0}^n ; \quad j + k + l + m = n. \quad (2.43)$$

The expression of the Resonance Driving Terms is given by the global hamiltonian term  $h_{jklm}$  by

$$f_{jklm} = \frac{h_{jklm}}{1 - e^{i2\pi[(j-k)Q_x + (l-m)Q_y]}}, \quad (2.44)$$

where this coefficient is a summation over the hamiltonian terms of elements  $w$  in the lattice,

$$h_{jklm} = \sum_w h_{w,jklm} e^{i[(j-k)\Delta\phi_x + (l-m)\Delta\phi_y]}. \quad (2.45)$$

The expression of  $h_{w,jklm}$  is itself derived from the general hamiltonian of Eq. (2.12) by applying a multinomial expansion on the coordinates [25] as shows Eq. (2.46). Derivations and more information on Resonance Driving Terms can be found in Appendix C.

$$h_{w,jklm} = -\Re \left[ \frac{K_{w,n} + iJ_{w,n}}{j!k!l!m!2^{j+k+l+m}} i^{l+m} \beta_{w,x}^{\frac{j+k}{2}} \beta_{w,y}^{\frac{l+m}{2}} \right] \quad (2.46)$$

Transforming from the normal form coordinates back to the original normalized coordinates can be done using the right side of Fig. 2.13. Which is written, to second order, as:

$$\begin{aligned} (\hat{z} \pm i\hat{p}_z) &= e^{iF} \cdot \zeta_z^\pm \\ &\simeq \zeta_z^\pm + [F, \zeta_z^\pm] + \frac{1}{2!} [F, [F, \zeta_z^\pm]]. \end{aligned} \quad (2.47)$$

Using this equation to the first order and Eq. (2.40), the normalized coordinates can be expressed after  $N$  turns in Eq. (2.48).

$$\begin{aligned} (x - ip_x)(N) &= \sqrt{2I_x} e^{i(2\pi Q_x N + \psi_{x_0})} - \\ &2i \sum_{jklm} j f_{jklm} (2I_x)^{\frac{j+k-1}{2}} (2I_y)^{\frac{l+m}{2}} e^{i[(1-j+k)(2\pi Q_x N + \psi_{x_0}) + (m-l)(2\pi Q_y N - \psi_{y_0})]} \\ (y - ip_y)(N) &= \sqrt{2I_y} e^{i(2\pi Q_y N + \psi_{y_0})} - \\ &2i \sum_{jklm} l f_{jklm} (2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m-1}{2}} e^{i[(k-j)(2\pi Q_x N + \psi_{x_0}) + (1-l+m)(2\pi Q_y N - \psi_{y_0})]}. \end{aligned} \quad (2.48)$$

This equation highlights the contribution of the non-linear elements to the motion of the particles. Spectral lines arising from these contributions are discussed in Section 2.7.2. It is though to be observed that some  $f_{jklm}$  terms will not contribute to the motion in a given plane due to the dependence on  $j$  or  $l$ .

## 2.4. Examples of Maps

It is important to remember that two expansions are used when creating non linear transfer maps. When referring to the order of a map, it is the order of the BCH formula, used to combine hamiltonians, that is referred to. The Lie transformation to transport the coordinates themselves is usually only taken to the first order.

### 2.4.1. Non-Linear Transfer of a Single Sextupole

Here, we are interested on the effect of a single sextupole on the regular Frenet-Serret coordinates  $x, y, p_x$  and  $p_y$ . Let's consider a sextupole with strength  $K_3$  and a normal field,

$$H_3 = \frac{1}{6}K_3(x^3 - 3xy^2). \quad (2.49)$$

A transfer map, from longitudinal coordinate  $s_0$  to  $s_1$ , consisting of only this element is the following:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_1} = e^{L:H_3:} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_0}, \quad (2.50)$$

where L is the length of the multipole. Using Eq. (2.33) to expand the Lie transformation to the first order, it can be rewritten as

$$\begin{aligned} e^{L:H_3:}x &= x + [L \cdot H_3, x], \\ e^{L:H_3:}p_x &= p_x + [L \cdot H_3, p_x], \\ e^{L:H_3:}y &= y + [L \cdot H_3, y], \\ e^{L:H_3:}p_y &= p_y + [L \cdot H_3, p_y]. \end{aligned} \quad (2.51)$$

Applying the Poisson bracket of Eq. (2.29) on  $x$  or  $y$  yields 0, as neither the hamiltonian nor  $x$  and  $y$  are dependent on  $p_x$  and  $p_y$ ,

$$\begin{aligned}
 [L \cdot H_3, x] &= \overbrace{\frac{\partial(L \cdot H_3)}{\partial x} \frac{\partial x}{\partial p_x}}^0 - \overbrace{\frac{\partial(L \cdot H_3)}{\partial p_x} \frac{\partial x}{\partial x}}^0 + \underbrace{\frac{\partial(L \cdot H_3)}{\partial y} \frac{\partial x}{\partial p_y}}_0 - \underbrace{\frac{\partial(L \cdot H_3)}{\partial p_y} \frac{\partial x}{\partial y}}_0 \\
 &= 0.
 \end{aligned} \tag{2.52}$$

The Poisson bracket applied on  $p_x$  or  $p_y$  though evaluates to a non-zero value, as the momentum is present in  $p_{x,y}$  while  $x, y$  are present in the hamiltonian:

$$\begin{aligned}
 [L \cdot H_3, p_x] &= \frac{\partial(L \cdot H_3)}{\partial x} \frac{\partial p_x}{\partial p_x} - \overbrace{\frac{\partial(L \cdot H_3)}{\partial p_x} \frac{\partial p_x}{\partial x}}^0 + \underbrace{\frac{\partial(L \cdot H_3)}{\partial y} \frac{\partial p_x}{\partial p_y}}_0 - \underbrace{\frac{\partial(L \cdot H_3)}{\partial p_y} \frac{\partial p_x}{\partial y}}_0 \\
 &= \frac{1}{2} K_3 L (x^2 - y^2)
 \end{aligned} \tag{2.53}$$

The same method is used for  $p_y$ . The final form of the transfer map from Eq. (2.50) is then the following,

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_1} = \begin{pmatrix} 1 & & & \\ & 1 + \left( \frac{1}{2p_x} K_3 L (x^2 - y^2) \right) & & \\ & & 1 & \\ & & & 1 - \left( \frac{1}{p_y} K_3 L x y \right) \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_0}. \tag{2.54}$$

It is not necessary to go higher than the first order, as the second order of the expansion of the Lie transformation is 0 ;  $p_x$  is indeed not present in the result of the first Poisson bracket,

$$\begin{aligned}
\frac{1}{2!} [L \cdot H_3, [L \cdot H_3, p_x]] &= \frac{1}{2!} \left[ L \frac{1}{6} K_3(x^3 - 3xy^2), \left[ L \frac{1}{6} K_3(x^3 - 3xy^2), p_x \right] \right] \\
&= \frac{1}{2!} \left[ L \frac{1}{6} K_3(x^3 - 3xy^2), \frac{1}{2} K_3 L(x^2 - y^2) \right] \\
&= \frac{1}{2!} \cdot 0 \\
&= 0
\end{aligned} \tag{2.55}$$

### 2.4.2. Non-Linear Transfer of Two Sextupoles

It has been seen previously that a single sextupole simply acts as a sextupole when it is alone in the transfer map, which is expected. Let's now consider two sextupoles whose hamiltonians are denoted  $H_1$  and  $H_2$ .

Creating a map consisting of only two sextupoles does not make much sense, as it finally results in one sextupole as their position is the same. Instead, a drift is added between the two elements. The hamiltonian of a drift of length  $L_D$  is given by [26],

$$D = -\frac{L_D}{2}(p_x^2 + p_y^2). \tag{2.56}$$

The application of the lie transformation on the canonical coordinates is then very simple, as no higher orders arise ( $[D, [D, x]] = 0$ ):

$$\begin{aligned}
e^{D:}x &= x + L_D p_x, \\
e^{D:}p_x &= p_x.
\end{aligned} \tag{2.57}$$

The transfer map of such a line is then the following,

$$\mathcal{M} = e^{Z:} = e^{H_2:} \cdot e^{D:H_1:}, \tag{2.58}$$

describing the evolution of coordinates from a longitudinal position  $s_0$  to  $s_1$ ,

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_1} = \mathcal{M} \cdot \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_{s_0} \quad (2.59)$$

In order to combine those elements, the BCH formula from Eq. (2.36) is used, presented here to the third order for two elements,

$$Z = \underbrace{H_2 + H_1}_{\text{First order}} + \underbrace{\frac{[H_2, H_1]}{2}}_{\text{Second order}} + \underbrace{\frac{[H_2, [H_2, H_1]]}{12} - \frac{[H_1, [H_2, H_1]]}{12}}_{\text{Third order}} \quad (2.60)$$

**First Order** To the first order, the resulting effective hamiltonian is only the summation of two sextupoles.

**Second Order** The drift added to change the coordinates of  $H_1$  allows the Poisson bracket to evaluate to a non-zero value. Octupolar-like terms indeed appear in the effective hamiltonian. From this, it can be inferred that two sextupoles will interact together and introduce effects like amplitude detuning, second order chromaticity and RDTs. Details of the derivation can be found in Appendix A.

**Third Order** To the third order, even higher orders such as decapolar-like effects appear. Such effects include the third order chromaticity, chromatic amplitude detuning and RDTs.

**Remark** It is to be noted that while sextupoles do introduce higher-order terms, these are often designed to be small in comparison to those brought by the actual higher-order multipoles, making them thus often negligible. Such is the case in the LHC.

## 2.5. Observables

### 2.5.1. Dispersion

Treating a beam as a single particle having the design momentum  $p_0$  leads to a machine with no apparent ill effect related to that momentum. However, when considering a particle beam where each particle follows a distribution in momentum, a few effects arise from this deviation, called the *momentum offset*,  $\delta$ . It is defined as a relative difference to the design momentum,

$$\delta = \frac{p - p_0}{p_0}. \quad (2.61)$$

Those effects are referred to as *chromatic aberrations*. The first and most important to consider is the *dispersion*. Dispersion results from a particle with a momentum offset being deflected differently by the dipoles compared to a particle at the design momentum. Figure 2.14 shows an example of deflection. The particle is still subject to the other properties of the lattice, but with a different orbit, described by the dispersion function in Eq. (2.62),

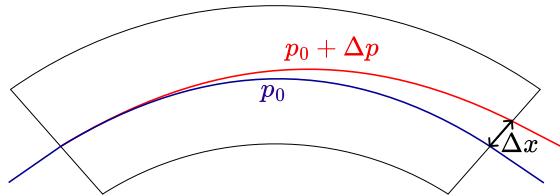


Figure 2.14.: Particles with a momentum offset will be deflected differently by dipoles. This offset in position can be described by the dispersion function.

$$\begin{aligned} D_x(s) &= \frac{\Delta x(s)}{\delta} \\ D_y(s) &= \frac{\Delta y(s)}{\delta}. \end{aligned} \quad (2.62)$$

**Momentum Compaction Factor** In synchrotrons, particles with a deviation in momentum with respect to the reference particle will experience a different path length due to the bending of the dipoles. This effect is characterized by the *momentum compaction factor* [11],

$$\alpha_c = \frac{\Delta C/C}{\delta}, \quad (2.63)$$

relating the circumference of the ring  $C$  to the momentum offset  $\delta$ . A positive momentum compaction factor indicates a longer path traveled by particles with a positive momentum offset, and vice versa.

The momentum compaction factor can also be broken down into orders as an infinite sum, where the constant term is often referred to as the first order,

$$\alpha_c = \underbrace{\alpha_{c,0}}_{\text{Constant term}} + \underbrace{\sum_{i \geq 1} \alpha_{c,i} \delta^i}_{\text{Linear and non linear terms}}. \quad (2.64)$$

In the LHC, the contribution from the non-constant terms is negligible [27]. Further details can be found in Section 6.3.

## 2.5.2. $\beta$ -function

As seen previously in Section 2.3.1, the  $\beta$ -function is related to the amplitude of oscillations of the beam. Figure 2.15 shows how the  $\beta$ -function oscillates along the ring due to quadrupoles focusing and defocusing properties. The  $\beta$ -function is an important quantity found in several other observables that will be described later in this thesis.

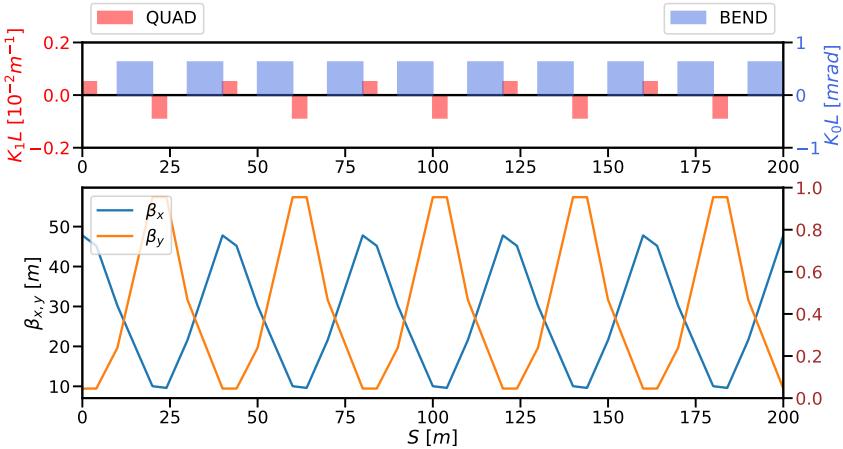


Figure 2.15.: Evolution of the  $\beta$ -function along the lattice. Horizontal and vertical beta-functions are usually opposite given the focusing and defocusing properties of quadrupoles in each plane.

A difference in  $\beta$ -function compared to the design leads to possibly unstable and larger beams, degrading its properties and making it harder to control. The relative difference in  $\beta$ -function is called the beta-beating, expressed in percents:

$$\beta\text{-beating [\%]} = \frac{\beta_z(s) - \beta_z(s)_{model}}{\beta_z(s)_{model}}. \quad (2.65)$$

### 2.5.3. Coupling

In a perfect scenario, the particle motion of each transverse plane is independent, or *uncoupled*. In practice, this transverse motion can be altered by some magnetic elements, giving rise to *betatron coupling* where the motion of each plane is not independent anymore. Such elements can be quadrupoles, mounted with a roll error introducing skew-quadrupolar fields, which are the main source of linear coupling in the LHC [9]. Field imperfections, solenoids and feed-down from higher orders can also contribute to coupling.

The resonances  $Q_x + Q_y$  and  $Q_x - Q_y$ , called the *sum* and *difference* resonances, are mainly excited by skew quadrupoles. When coupling is present in the machine, the former may lead to unstable motion while the latter introduces an periodic exchange of emittance between the planes, keeping it stable. They can be characterized by the RDTs  $f_{1010}$  and  $f_{1001}$ .

Effects of normal multipoles start showing as their skew counterpart (and vice versa) as the motion of transverse planes become coupled. This is demonstrated in Chapter 4 with octupoles.

## 2.6. Detuning Effects

### 2.6.1. Chromaticity

Chromaticity is the tune change  $\Delta Q$  relative to the momentum offset  $\delta$ . Chromaticity can be described by a Taylor expansion, given by

$$Q(\delta) = Q_0 + Q' \delta + \frac{1}{2!} Q'' \delta^2 + \frac{1}{3!} Q''' \delta^3 + O(\delta^4). \quad (2.66)$$

Or, more generally, rewritten as a sum to include all orders up to  $n$ :

$$Q(\delta) = Q_0 + \sum_{i=1}^n \frac{1}{i!} Q^{(i)} \delta^i. \quad (2.67)$$

The first order of the chromaticity expansion,  $Q'$ , is generally simply referred to as *chromaticity*, sometimes as *linear chromaticity*. The other terms are thus referred to as *non-linear chromaticity*.

The chromaticity change induced by a single element of order  $n$  and length  $L$  can be derived from the Hamiltonian of Eq. (2.12), averaging over the phase variables and differentiating relative to the actions  $J_{x,y}$  and the momentum offset  $\delta$ :

$$\Delta Q_{x,y}^{(n)} = \frac{\partial^n Q_{x,y}}{\partial^n \delta} = \frac{1}{2\pi} \int_L \left\langle \frac{\partial^{(n+1)} H}{\partial J_{x,y} \partial^n \delta} \right\rangle ds. \quad (2.68)$$

Detailed derivations are provided in [28], and an example is presented in the following section.

**Linear Chromaticity from Sextupoles** The first order chromaticity  $Q'$  is contributed to by sextupoles in the presence of off-momentum particles. The normal field of a sextupole, following Eq. (2.13) is given by

$$\mathcal{N}_3(x, y) = \frac{1}{3!} (x^3 - 3xy^2). \quad (2.69)$$

As the momentum offset  $\delta$  introduces a change in orbit via dispersion [29], a variable change can be operated on both  $x$  and  $y$ , as shown in Eq. (2.70). In this thesis, vertical dispersion will be though neglected.

$$\begin{aligned} x &\rightarrow x + \Delta x = x + D_x \delta \\ y &\rightarrow y + \Delta y = y + D_y \delta \end{aligned} \quad (2.70)$$

The positions  $x$  and  $y$  can once again be replaced, using the twiss parameters, giving the full expression:

$$\begin{aligned} \mathcal{N}_3(x + \Delta x, y) &= \frac{1}{6} K_3 \left[ \left( \sqrt{2J_x \beta_x} \cos \phi_x \right)^3 \right. \\ &\quad + 3 \left( \sqrt{2J_x \beta_x} \cos \phi_x \right)^2 D_x \delta \\ &\quad + 3 \left( \sqrt{2J_x \beta_x} \cos \phi_x \right) D_x^2 \delta^2 \\ &\quad + D_x^3 \delta^3 \\ &\quad - 3 \left( \sqrt{2J_x \beta_x} \cos \phi_x \right) \left( \sqrt{2J_y \beta_y} \cos \phi_y \right)^2 \\ &\quad \left. - 3D_x \delta (\sqrt{2J_y \beta_y} \cos \phi_y)^2 \right] \end{aligned} \quad (2.71)$$

Averaging over the phase variables removes any odd cosine:

$$\langle \mathcal{N}_3(x + \Delta x, y) \rangle = \frac{1}{6} K_3 \left( 3 J_x \beta_x D_x \delta + D_x^3 \delta^3 - 3 D_x \delta J_y \beta_y \right) \quad (2.72)$$

The chromaticity can then be obtained by differentiating relative to the action  $J_{x,y}$  to retrieve the tune, and finally by the momentum offset  $\delta$ , as presented in Eq. (2.73). In the presence of second order dispersion [29], sextupoles will generate some amount of  $Q''$ , usually negligible in the LHC.

$$\begin{aligned} \Delta Q'_x &= \frac{1}{2\pi} \int_L \frac{\partial^2 \langle \mathcal{N}_3 \rangle}{\partial J_x \partial \delta} ds ; \quad \Delta Q'_y &= \frac{1}{2\pi} \int_L \frac{\partial^2 \langle \mathcal{N}_3 \rangle}{\partial J_y \partial \delta} ds \\ &= \frac{1}{2\pi} L \frac{1}{2} K_3 \beta_x D_x && = -\frac{1}{2\pi} L \frac{1}{2} K_3 \beta_y D_x \\ &= \frac{1}{4\pi} K_3 L \beta_x D_x && = -\frac{1}{4\pi} K_3 L \beta_y D_x \end{aligned} \quad (2.73)$$

**Non-Linear Chromaticity** Higher orders of the chromaticity function are described in [28] and follow the same logic as for the linear chromaticity from sextupoles. A general formula can be found relating the chromaticity of order  $n, n > 2$  to a multipole of order  $n + 2$ ,

$$\begin{aligned} \Delta Q_x^{(n)} &= \frac{1}{4\pi} K_{n+2} L \beta_x D_x^n \\ \Delta Q_y^{(n)} &= -\frac{1}{4\pi} K_{n+2} L \beta_y D_x^n \end{aligned} \quad (2.74)$$

## 2.6.2. Amplitude Detuning

Amplitude detuning is a tune shift induced by the amplitude of oscillations of a particle. This detuning is directly related to the single particle emittance and can be described

via a Taylor expansion around the emittance of both planes,  $\epsilon_x$  and  $\epsilon_y$ . Equation (2.75) shows this expansion up to the second order. Further expansions can be found in [28].

$$\begin{aligned} Q_z(\epsilon_x, \epsilon_y) &= Q_{z0} + \left( \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y \right) \\ &\quad + \frac{1}{2!} \left( \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 \right) + \dots \end{aligned} \quad (2.75)$$

The first order terms of amplitude detuning are generated by octupoles, and to some extent by sextupoles when considering their higher order contributions. These higher contributions are usually measurable but small compared to the ones of normal octupoles in the LHC. Further derivations can be found in Appendix B.

### 2.6.3. Chromatic Amplitude Detuning

Similar to amplitude detuning, *chromatic amplitude detuning* is a tune shift induced by the amplitude of oscillations of a particle but with an additional dependence on the momentum offset. This effect can be described by a Taylor expansion around the emittance of both planes  $\epsilon_x$ ,  $\epsilon_y$ , and the momentum offset  $\delta$ . Equation (2.76) shows this expansion up to the second order. Both the emittance  $\epsilon$  and the action  $J$  can be used to describe the chromatic amplitude detuning as terms are interchangeable with  $\epsilon_{x,y} = 2J_{x,y}$ . This expansion being more general than the ones of chromaticity or amplitude detuning, is it to be expected to retrieve these terms here as well.

$$\begin{aligned} Q_z(\epsilon_x, \epsilon_y, \delta) &= Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\ &\quad + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\ &\quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] \\ &\quad + \dots \end{aligned} \quad (2.76)$$

**Sextupolar contributions** To the first order of the expansion, the only term coming from sextupoles is the linear chromaticity, seen previously in Section 2.6.1.

**Octupolar contributions** To the first order, the amplitude detuning terms  $\frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x$  and  $\frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y$  are contributed to by octupoles. The second order chromaticity  $Q''$  appears when expanding to the second order.

**Decapolar contributions** The terms highlighted in orange, in Eq. (2.77), are the terms contributed to the first order by decapoles. Terms depending on both the emittance and the momentum offset are present, as well as the third order chromaticity  $Q'''$ . Further derivations can be found in Appendix B.

$$\begin{aligned}
 Q_z(\epsilon_x, \epsilon_y, \delta) = & Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\
 & + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\
 & \quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] \\
 & + \frac{1}{3!} \left[ \frac{\partial^3 Q_z}{\partial \delta^3} \delta^3 + \frac{\partial^3 Q_z}{\partial \epsilon_x^3} \epsilon_x^3 + \frac{\partial^3 Q_z}{\partial \epsilon_y^3} \epsilon_y^3 + \dots \right]
 \end{aligned} \tag{2.77}$$

## 2.6.4. Feed-Down

When a particle passes off-center through a magnet, an effect called *feed-down* appears. Feed-down is a lower order effect created by either mis-aligned magnets or an off-center orbit of the beam. A particle with an orbit offset will then experience the main field of the magnet and effects similar to those of lower order multipoles. The combined contributions of multipoles up to order  $P$  on a field of order  $n$  is given by the following and greatly detailed in [10]:

$$(K_n + iJ_n)^{w/feed-down} = \sum_{p=0}^P (K_{n+p} + iJ_{n+p}) \frac{(\Delta x + i\Delta y)^p}{p!} \quad (2.78)$$

Opposed to feed-down is feed-up, the higher order effects introduced by combinations of multipoles, and is not a consequence of the beam position. This effect is explained in great details in Appendix A.

## 2.7. Resonances

### 2.7.1. Tune Diagram

The resonances discussed in this thesis are related to the optics of the accelerator. Such resonances create unstable motion and can lead to loosing particles.

Figure 2.16a shows a tune diagram where the fractional part of tunes  $Q_x$  and  $Q_y$  can be related to resonance lines excited by multipoles up to decapoles ( $n = 5$ ). It becomes apparent that the diagram fills quickly when considering further orders, as shows Fig. 2.16b. Thankfully, the higher the multipole order, the weaker the resonances usually are as their introduced perturbations are usually smaller. This makes choosing a working point possible, even if some particles are hitting resonance lines.

When considering the Resonance Driving Terms  $f_{jklm}$  from Eq. (2.44), it can be noted that the term diverges for particular tune values. This leads to a disproportionate increase in particles position in phase-space, eventually leading to loosing them. Resonant conditions due to the tunes can thus be described by the following condition:

$$(j - k)Q_x + (l - m)Q_y = p \quad , \quad j, k, l, m, p \in \mathbb{Z}. \quad (2.79)$$

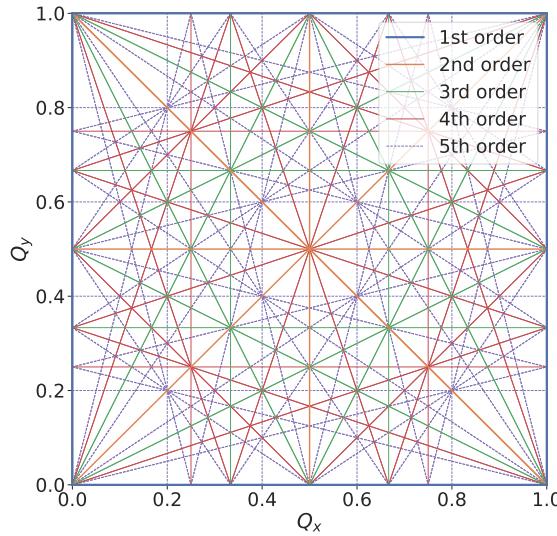
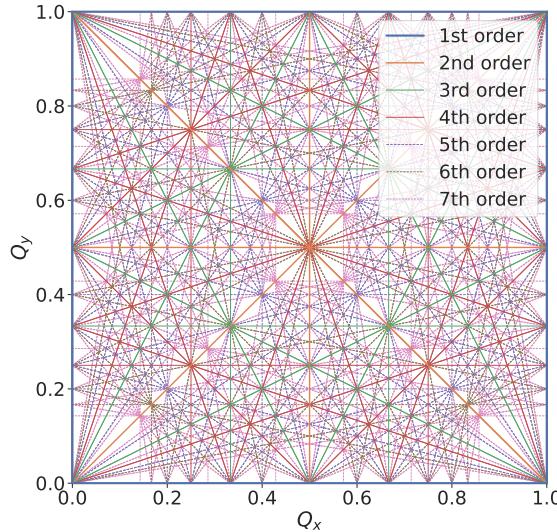
(a) Resonances lines up to decapoles ( $n \leq 5$ ).(b) Resonances lines up to decatetrapole ( $n \leq 7$ ).

Figure 2.16.: Tune diagram with resonance lines excited by different multipole orders. The working point is chosen in an area where few lines are present. When considering higher orders, it becomes apparent that the beam will inevitably hit several resonances.

## 2.7.2. Frequency Spectrum

As seen in Eq. (2.48), resonance driving terms have an impact on the transverse motion of a particle. This means that performing a FFT on the turn-by-turn signal will reveal spectral lines linked to specific resonance driving terms. Each RDT  $f_{jklm}$  can thus be observed in either one or both the frequency spectrums of the horizontal and vertical planes, at multiples of  $Q_x \pm Q_y$ . Equation (2.80) shows where these lines would appear:

$$\begin{aligned} H_{jklm} &\quad \text{at } (1 - j + k)Q_x + (m - l)Q_y \quad ; \quad j \neq 0 \\ V_{jklm} &\quad \text{at } (k - j)Q_x + (1 - l + m)Q_y \quad ; \quad l \neq 0. \end{aligned} \quad (2.80)$$

The RDT  $f_{3000}$  coming from sextupoles can for example be seen in the horizontal spectrum at  $(1 - 3 + 0)Q_x + (0 - 0)Q_y = -2Q_x$ . For a value  $Q_x = 0.27$ , the line is seen at 0.46. No line can be seen in the vertical spectrum due to  $l = 0$ . Detailed tables of such lines for RDTs up to order 6 can be found in Appendix C.

The amplitude of each line will depend on the action  $I_z$  and the amplitude of the RDT [30]:

$$\begin{aligned} |H_{f_{jklm}}| &= 2j(2I_x)^{\frac{j+k-1}{2}}(2I_y)^{\frac{l+m}{2}}|f_{jklm}|, \\ |V_{f_{jklm}}| &= 2l(2I_x)^{\frac{j+k}{2}}(2I_y)^{\frac{l+m-1}{2}}|f_{jklm}|. \end{aligned} \quad (2.81)$$

By reworking the previous Eq. (2.81), it can be seen that RDTs are factors of the line amplitude and the actions  $I_x$  and  $I_z$ :

$$\begin{aligned} |f_{jklm}| &= \frac{|H_{f_{jklm}}|}{2j(2I_x)^{\frac{j+k-1}{2}}(2I_y)^{\frac{l+m}{2}}} \\ |f_{jklm}| &= \frac{|V_{f_{jklm}}|}{2l(2I_x)^{\frac{j+k}{2}}(2I_y)^{\frac{l+m-1}{2}}}. \end{aligned} \quad (2.82)$$

In practice, an approximation of  $J = I$  is done. The RDT is then related to the fit of the line amplitude versus the action.

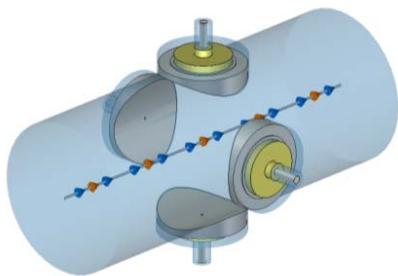
# Optics Measurements and Corrections

## 3.1. Beam Instrumentation

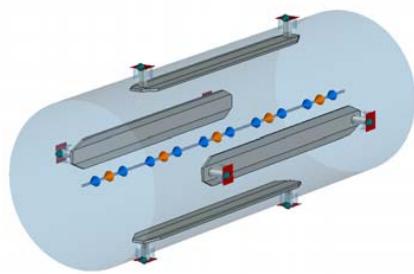
### 3.1.1. Beam Position Monitors

Beam Position Monitors (BPMs) are one of the most utilized and essential elements of beam diagnostics in particle accelerators. In the LHC, most of the BPMs are dual plane, and thus composed of four electrodes, distributed as two per plane. The BPM system consists of over than 550 BPMs per beam, positioned along the ring, in the arcs and the IPs. The most common type, the *curved-button*, shown in Fig. 3.1a, is typically placed near quadrupoles [31].

Other pickups such as the *stripline*, shown in Fig. 3.1b, albeit more complex and expensive, offer a better signal to noise ratio and are capable of identifying the direction of the beam [31]. Such features are essential for the LHC, were both beams travel through the same aperture at the IPs.



(a) Button "BPM" type BPM of the LHC [31].



(b) Stripline "BPMSW" type BPM of the LHC [31].

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### 3.1.2. Collimators

Collimators are a crucial part of the LHC. Their purpose is to protect the machine against beam losses and clean the outer parts of the beam [32]. The energy of the beams in the LHC is high enough to not only quench the magnets, but to also damage the elements. At injection energy, with a low intensity pilot bunch, the consequences of a loss are less severe.

### 3.1.3. Beam Loss Monitors

Beam Loss Monitors are detectors mounted on various elements of the accelerator, such as magnets or collimators, to detect abnormal losses of particles. They play a crucial role in the protection of the machine, triggering a dump when losses exceed the threshold set for their respective element. BLMs use ionization chambers, working on the same principle as simple Geiger counters: a tube filled with gas, in presence of a high voltage [33]. A picture of BLMs mounted on the LHC is given in Fig. 3.2.

Dashboards in the control room are regularly used to monitor the losses along the ring when performing optics measurements, as those prove to often be destructive. An example of such a dashboard is given in Fig. 3.3.

### *3.1. Beam Instrumentation*



Figure 3.2.: Beam Loss Monitors (BLM), in yellow, on the LHC [33].

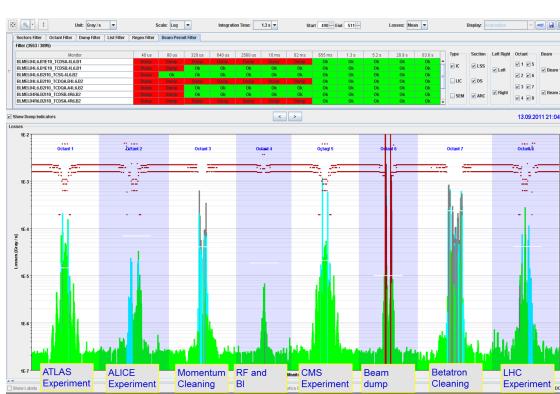


Figure 3.3.: Graphical interface used in the CERN Control Center (CCC) for instantaneous losses in the LHC [33]. Different parts of the accelerator have varying dump thresholds.

### 3.1.4. Beam Current Transformer

The Beam Current Transformer (BCT) is a device used to measure the intensity of a particle beam by detecting the current induced by the moving charge of the beam as it passes through the coil of the BCT. The beam effectively acts as a primary coil and induces a current in the secondary coil of the transformer. The BCTs are designed to

### 3. Optics Measurements and Corrections

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be able to measure intensities from pilot bunches of  $8\mu\text{A}$  to total beams of more than  $860\text{mA}$  [34]. During optics measurements, beam intensity is often closely monitored to ensure data quality, as certain observables may not be detectable at low intensities.

#### 3.1.5. BBQ System

The Base-Band Tune (BBQ) system in the LHC is designed to measure the beam's tune via its turn-by-turn signal. It operates by detecting and analyzing the signals of diode peak-detectors [35, 36]. The system further implements processing hardware and software, transmitting the acquired data to the control and logging systems. The system can operate with no explicit excitation, relying on the residual beam oscillations, or by using tune kickers or frequency sweeps [35].

#### 3.1.6. AC-Dipole

The AC dipole of the LHC is a crucial component for optics studies. Its primary function is to excite the beam into large coherent oscillation, achieved by applying a sinusoidally oscillating dipole field [37]. By ramping up and down adiabatically the amplitude, large coherent oscillations can be produced without any decoherence or emittance growth. Figure 3.4 shows an example of a simulation made with an AC-Dipole. Exciting the beam to large amplitudes make the study of linear optics, such as beta-beating easier, and that of non linear optics such as resonances possible.

The AC-Dipole is set to oscillate at a frequency  $Q_d$ , different from the natural tune of the machine  $Q$  and thus introduces systematic effects that needs to be compensated during the optics analysis. The transverse position of a particle, starting on the reference orbit, under the influence of the AC-Dipole, at turn number  $n$  and observation point  $s$ , is given by [38–40]:

$$z(s, n) = \frac{BL}{4\pi\rho\delta_z} \cdot \sqrt{\beta_z(s)\beta_{z,0}} \cdot \cos(2\pi Q_{d,z}n + \phi_z(s) + \phi_{z,0}), \quad (3.1)$$

where  $B$  is the amplitude of the oscillating magnetic field,  $L$  the length of the AC-Dipole,  $B\rho$  the magnetic rigidity,  $\delta$  the difference between  $Q_d$  and  $Q$ ,  $\beta$  and  $\beta_0$  the

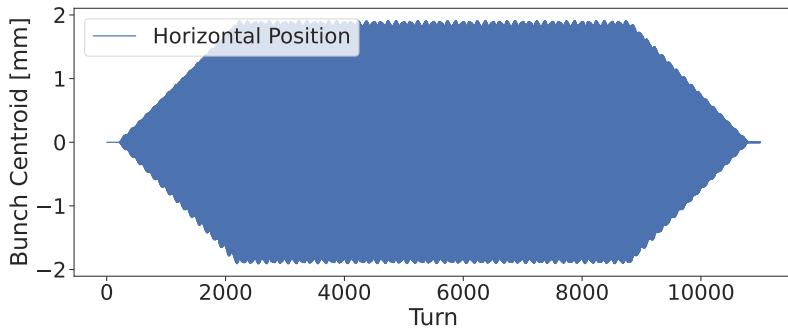


Figure 3.4.: Simulated turn-by-turn data with an AC-Dipole first ramping up then down.

beta-function at the observed point and the AC-Dipole,  $\phi$  and  $\phi_0$  the phase advance at the observed point and at the AC-Dipole.

## 3.2. Optics Measurements

To perform optics measurements, several tools and techniques are used. This section details the various implemented methods to measure specific observables.

### 3.2.1. Tools and Softwares

In order to perform the measurements, analysis and simulations presented in this thesis, various tools and softwares have been developed, used and contributed to.

Optics simulations have been done mainly in MAD-X [41] and PTC. MAD-NG [42] and Xsuite [43] have also been explored for specific tasks such as free RDT simulations and GPU tracking.

Analysis of chromaticity measurements are done via a newly developed graphical interface [44] written in Python. This tool makes cleaning of the raw signal data, its analysis and results export more reliable and easier. Overall, analysis of turn-by-

### 3. Optics Measurements and Corrections

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turn measurements is supported by a large panel of libraries written by the Optics Measurements and Corrections (OMC) team, in Python and Java [45].

#### 3.2.2. Turn-by-Turn Signal

One of the key data acquisition methods for optics measurements in the LHC is turn-by-turn acquisition, where beam position is collected on a per-turn basis. This process involves exciting the beam using an AC-Dipole to induce forced oscillations. Typically, a pilot bunch is used for this purpose, containing a reduced intensity of  $10^{10}$  protons, compared to the standard operational bunch intensity of  $10^{11}$ . The lower intensity allows for larger amplitude oscillations, enabling more precise measurements while ensuring the safety of the machine and minimizing the risk of damaging components.

A spectral analysis is then performed via a *FFT* on the signal, making apparent the driven tunes from the AC-Dipole, the transverse tunes and the possible resonance lines, as shown in Fig. 3.5.

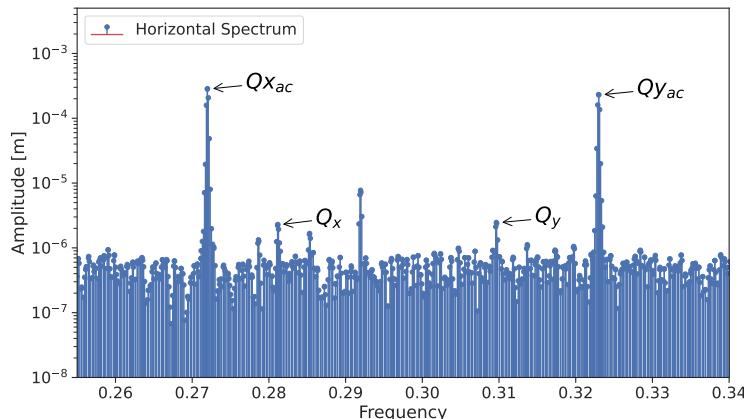


Figure 3.5.: Horizontal frequency spectrum of a turn-by-turn measurement in the LHC. The *driven* tunes of the AC-Dipoles have the highest amplitudes while the natural tune can be seen close to it. Other lines are often created by resonances.

From these oscillations and spectral lines, the optics observables, such as  $\beta$ -beating, dispersion, coupling, and resonance driving terms, can be reconstructed [46]. These key quantities provide valuable insights into the beam's dynamics and will be detailed further in the following sections.

**Betatron Phase** The betatron phase can be determined at a given BPM by performing a FFT on the turn-by-turn data. The argument of the retrieved complex number with the largest magnitude, the tune, is then taken as the phase. The phase advance between two BPMs is simply their phase difference.

**$\beta$ -Beating** The  $\beta$ -beating can be reconstructed via the phase advance between BPMs. The computation involves using a model created via a simulation software such as MAD-X and makes use of transfer matrices between elements. The  $\beta$ -function at a given BPM can be calculated from the measured phases of 3 BPMs denoted  $i, j, k$  [47],

$$\beta(s_i) = \beta^{model}(s_i) \cdot \frac{\cot(\Delta\phi_{i,j}) + \cot(\Delta\phi_{i,k})}{\cot(\Delta\phi_{i,j}^{model}) + \cot(\Delta\phi_{i,k}^{model})} \quad (3.2)$$

with  $\Delta_{i,j}$  being the phase advance between BPMs  $i$  and  $j$ . Using specific phase advances between BPMs enhances the precision of the  $\beta$ -function measurement. This approach can also be extended to use an arbitrary number  $N$  of BPMs [48, 49]. The measurement of the  $\beta$ -function via phase advance is independent from BPM calibration and relies solely on the accuracy of the phase measurement.

**Dispersion** To retrieve the linear dispersion, turn-by-turn measurements are performed at certain momentum offsets. This allows the computation of the dispersion via the shift in mean orbit  $\Delta z$  and the momentum offset  $\delta$ ,

$$D_z = \frac{\Delta z}{\delta}. \quad (3.3)$$

**Action** The action in a given plane and BPM, located at position  $s$  is calculated from the amplitude  $\mathcal{A}$  of the main peak in the frequency spectrum, which corresponds to the tune, along with the beta-function at that BPM derived from the model,

$$2J_{BPM} = \frac{\mathcal{A}_{BPM}^2}{\beta_{model,BPM}}. \quad (3.4)$$

This method of action computation is directly influenced by BPM calibration errors [50]. The overall action is then the average over  $N$  BPMs:

$$2J = \frac{1}{N} \sum_n^N \frac{\mathcal{A}_n^2}{\beta_{model,n}}. \quad (3.5)$$

**Transverse Coupling** Coupling can be calculated by comparing the amplitude of the tune in the frequency spectrum of a plane to the same tune in the other plane. The coupling RDTs  $f_{1001}$  and  $f_{1010}$  can be reconstructed with these amplitudes [51, 52],

$$\begin{aligned} |f_{1001}| &= \frac{1}{2} \sqrt{\frac{H(0, 1)V(1, 0)}{V(0, 1)H(1, 0)}}, \\ |f_{1010}| &= \frac{1}{2} \sqrt{\frac{H(0, -1)V(0, -1)}{V(0, 1)H(1, 0)}}, \end{aligned} \quad (3.6)$$

where H and V refer respectively to the amplitude of a  $Q_x, Q_y$  line in the horizontal and vertical frequency spectrum. For instance,  $H(1, 0)$  is the amplitude of  $Q_x$  in the horizontal spectrum while  $H(0, 1)$  corresponds to  $Q_y$  in the same spectrum. The phases of these RDTs is given by,

$$\begin{aligned} q_{1001} &= \phi_{V(1, 0)} - \phi_{H(1, 0)} + \frac{\pi}{2}, \\ q_{1010} &= \phi_{H(0, -1)} - \phi_{V(0, 1)} + \frac{\pi}{2}. \end{aligned} \quad (3.7)$$

The final expression of the coupling RDTs is then,

$$\begin{aligned} f_{1001} &= |f_{1001}| e^{i \cdot q_{1001}}, \\ f_{1010} &= |f_{1010}| e^{i \cdot q_{1010}}. \end{aligned} \quad (3.8)$$

**Amplitude Detuning** Amplitude detuning measurements in the LHC are usually taken with varying AC-Dipole kick amplitudes. A linear function is then fitted on the natural tunes  $Q_x$  and  $Q_y$  versus the action of either planes  $2J_x$  or  $2J_y$ .

**Resonance Driving Terms** Resonance Driving Terms are measured in the LHC with varying AC-Dipole kick amplitudes. The amplitude of the resonance line of interest in the frequency spectrum can then be fitted to the corresponding action dependence of the RDT, as detailed in Section 2.7.2 and Appendix C. As a reminder, the amplitude of the RDT will be given by,

$$\begin{aligned} |f_{jklm}| &= \frac{|H_{jklm}|}{2j(2I_x)^{\frac{j+k-1}{2}} (2I_y)^{\frac{l+m}{2}}} \\ |f_{jklm}| &= \frac{|V_{jklm}|}{2l(2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m-1}{2}}}. \end{aligned}$$

The phase of the RDT can then be reconstructed via the momentum and phase advances. In practice, *forced resonance driving terms* are actually measured, as the oscillations are driven by the AC-Dipole. Simulations are thus always made via tracking with an AC-Dipole to match. Studies linking *forced* RDTs to *free* RDTs is though ongoing to better understand the machine without excitation [53].

**Lifetime** The beam lifetime is a measure of how long the beam can remain circulating without significant losses of particles. It is typically expressed in hours and is related to the rate at which particles are lost from the beam. Particles might be lost due to resonant conditions, instabilities or vacuum quality. The beam intensity is measured via the BCTs. The lifetime  $\tau$  relates then the number  $N$  of particles at a time  $t$  to its rate of change,

$$\tau = N(t) \cdot \frac{dN^{-1}}{dt}. \quad (3.9)$$

Additionally, Beam Loss Monitors (BLMs) provide local information about beam quality by detecting and localizing particle losses. These losses can as well be integrated to give a lifetime estimate. At injection energy in the LHC, the beam lifetime with a pilot bunch is typically around 3 hours. Sudden particle losses can significantly reduce the measured lifetime, even if the remaining beam stabilizes and does not lose particles as rapidly. This introduces challenges in accurately measuring the lifetime, as the signal needs to stabilize and saturate after any trim to ensure reliable data.

### 3.2.3. Chromaticity

Chromaticity measurements are performed by varying the RF frequency to induce a change of momentum offset  $\delta$ , while measuring the tune. The momentum offset  $\delta$  being related to the RF frequency, the Lorentz factor  $\gamma$  and the momentum compaction factor  $\alpha_c$  [27]:

$$\delta = - \left( \frac{1}{\gamma^{-2} + \alpha_c} \right) \cdot \frac{\Delta f_{\text{RF}}}{f_{\text{RF,nominal}}} \quad (3.10)$$

In the LHC, the Lorentz factor  $\gamma^{-2}$  is here negligible, as the energy is large even at injection. At 450 GeV,  $\gamma^{-2} \approx 10^{-6}$ , which is two orders of magnitude smaller than  $\alpha_c$ .

During operation, where the linear chromaticity often needs to be measured, a sinusoidal function with a period of  $\approx 20$  seconds is applied on the RF frequency. This induces a change of momentum offset to measure the first order of the chromaticity function, via a harmonic fit [54]. For non-linear measurements, a large range of momentum offset is required. In order to do so, a new procedure has been developed. Dense frequency scans with steps of 20Hz every 30 seconds are usually taken to compromise between number of data points, precision of the tune estimate, and duration of the measurement. Once beam losses, registered by the BLMs are deemed too high, the frequency is reverted back to its nominal value in larger steps. Attaining the limits of the BLMs ensures a large momentum-offset range. The same procedure is then re-applied in the negative. Figure 3.6 shows a typical RF scan performed to measure non-linear chromaticity in the LHC.

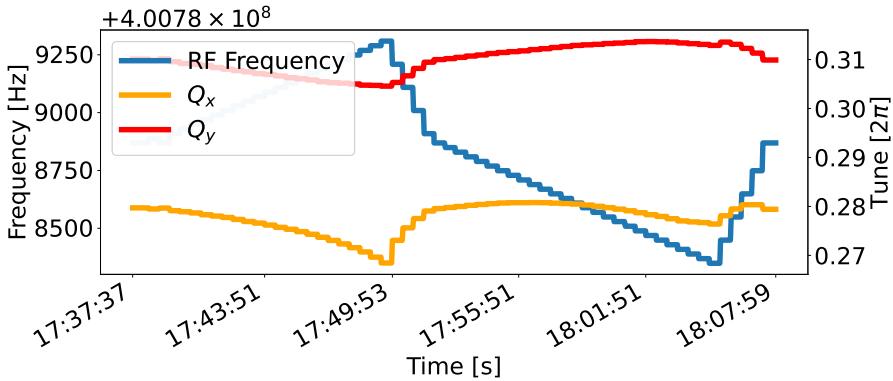


Figure 3.6.: Observation of the tune dependence on momentum offset, created by a shift of RF frequency.

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Once the tunes have been acquired and the momentum offset computed via Eq. (3.10), the chromaticity function (see Eq. (2.66)) can be used to fit the measured data and retrieve each order.

As part of the work for this thesis, a new tool, was developed. in order to ease and improve the analysis of chromaticity measurements. The Non-Linear Chromaticity GUI [44] showcases new analysis techniques using the raw signal from the BBQ system along with custom signal cleaning that are detailed later on in Chapter 6. Fits to very high chromaticity orders are also now possible along with their computed corrections and that of Resonance Driving Terms via a combined response matrix approach. Automatic data extraction from the CERN data servers (Timber, NXCALIS) is also included.

### 3.3. Correction Principles

Several ways exist to correct the following observables: phase,  $\beta$ -beating, dispersion, transverse coupling, amplitude detuning and RDTs. Two common methods to do so are here detailed.

### 3.3.1. Response Matrix

A response matrix is a linear equation system that describes the change of an observable for a set of individual multipole strengths. By taking the inverse of this matrix and multiplying it to the measured observables, a set of corrector strengths is obtained that can replicate the measured value. Taking the opposite sign then gives a correction. This technique is used to correct, amongst others,  $\beta$ -beating, chromaticity as well as Resonance Driving Terms for the first time in the LHC with this thesis. For some observables, like RDTs, it is common for the response matrix to contain over 500 values per corrector and beam, one for each BPM.

Individual MAD-X simulations are run with a single specific circuit powered at a time. The resulting parameter values (e.g.  $\beta$ -beating) are then compared to those obtained from a simulation without any powering, allowing to determine the impact of each circuit on that observable.

A response matrix  $R$  is thus created following Eq. (3.11), for a matrix of observables  $O$  with correctors powered on individually, a reference matrix of observables without any corrector  $O_R$ , and a corrector strength  $k$ , the same for each simulation. Given measured data  $M$ , the set of correctors needed to compensate the values can be obtained by taking the inverse of the response matrix in Eq. (3.12).

$$R = (O - O_R) \cdot \frac{1}{k} \quad (3.11)$$

$$\begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = -(R^+ \cdot M) \quad (3.12)$$

Response matrices are very versatile and can combine several observables to be corrected by the same multipoles. One example, detailed later in this thesis, is the correction of both the third order chromaticity  $Q'''$  and the Resonance Driving Term  $f_{1004}$ .

## Example

In this example, developed for the decapolar corrections presented in Chapter 5, simulations are run with MAD-X PTC to correct the third chromaticity in the LHC.  $Q'''$  is taken from MADX-PTC for each beam and axis, with MCDS, decapole correctors, powered with a fixed strength one at a time. A scaling factor is applied to get the change of chromaticity for one unit of  $K_5$ . 8 correctors are used, which strengths are denoted  $k_1$  through  $k_8$ . Transposes are only used to make the equations easier to display.

The values in Table 3.1 are corrected via Eq. (3.14) after having built the response matrix in Eq. (3.13).

Observable	Value
$Q_x'''$	-666111
$Q_y'''$	121557

Table 3.1.: Example of chromaticity values to be corrected via a response matrix.

$$R = \left( \begin{array}{l} \text{Individual} \\ \text{simulations} \end{array} \right) \underbrace{\begin{pmatrix} Q_x''' & Q_y''' \end{pmatrix}}_{\text{Reference}}^T - \underbrace{\begin{pmatrix} 5135 \\ 8470 \end{pmatrix}}_{\begin{pmatrix} Q_x''' \\ Q_y''' \end{pmatrix}} \cdot \underbrace{\frac{1}{-1000}}_{\text{Corrector strength}} \quad (3.13)$$

$$\begin{pmatrix} -155899 & 122004 \\ -254584 & 138368 \\ -122715 & 106709 \\ -218597 & 110686 \\ -134140 & 106463 \\ -245791 & 118951 \\ -147035 & 116544 \\ -219537 & 112317 \end{pmatrix}$$

$$\begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \end{matrix} \left( \begin{matrix} -1235 \\ 1032 \\ -1394 \\ 1449 \\ -1043 \\ 1864 \\ -1187 \\ 1369 \end{matrix} \right) = -R^+ \cdot \left( \begin{matrix} -666111 \\ 121557 \end{matrix} \right) \left\{ \begin{array}{l} \text{Measured} \\ \text{values} \end{array} \right\} \quad (3.14)$$

### 3.3.2. Global Trims for Chromaticity

As per the placement of the octupolar and decapolar correctors in the LHC layout [55],  $\beta$ -functions at their location are slightly different from arc to arc. This slight imbalance leads theoretically to the possibility of correcting the horizontal and vertical axes of the second and third order chromaticity independently, via a response matrix approach. In practice, the required strength to do so would exceed those of the design of the correctors.

Another way to correct the chromaticity is via a global uniform trim, where every available corrector is powered to the same strength. Simulations are run with MADX-PTC to obtain the response in chromaticity for a given strength. Chromaticity being linear with multipole strength, a linear function can be determined for each axis. Figure 3.7 shows a simulation with several decapolar strengths, highlighting this linear relation between the third order chromaticity  $Q'''$  and  $K_5$ , while Equation (3.15) shows an example of such functions computed at injection energy for the 2022 optics.

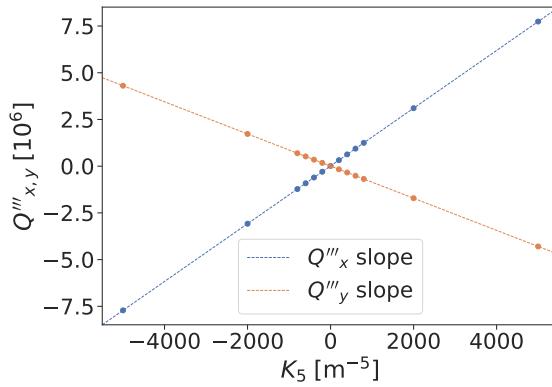


Figure 3.7.: Linear relation between the third order chromaticity and decapole corrector strengths, simulated with MADX-PTC.

$$\begin{aligned} Q'''_x &= 1533 \cdot \Delta K_5 + 6680 \\ Q'''_y &= -860 \cdot \Delta K_5 + 5647 \end{aligned} \quad (3.15)$$

Only the linear part is relevant, as the offset is generated by other multipoles and field errors. It is thus constant for a configuration where only the relevant correctors are used. Corrections involve minimizing both  $Q'''_x$  and  $Q'''_y$ , typically where both functions cross:

$$\Delta K_5 = -\frac{(Q'''_x - Q'''_y)}{\text{slope}_{Q'''_x} - \text{slope}_{Q'''_y}} \quad (3.16)$$

3

# Skew Octupolar Fields

## 4.1. Introduction

Skew-octupolar fields represent an important yet comparatively less-studied aspect of the LHC's non-linear optics. Their impact, although well-understood in simulations, presents unique challenges in real-world operation, especially as we approach closer to the High-Luminosity Large Hadron Collider (HL-LHC), where dynamic aperture is expected to be significantly affected. Unlike quadrupole and sextupole components, which have well-established methods for measurement and correction, skew-octupolar fields pose a significant challenge to measure and quantify.

In contrast to normal octupoles, which have straightforward observables such as amplitude detuning, skew octupolar fields are significantly harder to measure. Historically, these fields have been measured indirectly through feed-down effects on coupling and tune, but this approach has proven unreliable in the presence of other errors [56, 57]. Resonance Driving Terms (RDTs) have shown to be an effective method for measuring skew octupolar fields [53], although obtaining accurate measurements remains challenging due to the high kick amplitudes required.

Previous efforts to correct these resonances using RDT measurements have largely relied on empirical methods, involving extensive numerical simulations and manual

adjustments of correctors to identify a combination that reduces the RDTs. While this approach has been functional, it has proven to be time-consuming, labor-intensive, and inadequate for the fast-paced demands of LHC operations. As the machine's performance continues to reach new limits, more efficient and real-time correction techniques are becoming increasingly essential. This need has driven the development of a response matrix based approach to skew-octupolar RDT corrections, offering a quicker and more reliable method for diagnosing and addressing these field errors. These advancements will be needed to facilitate shorter commissioning times and optimize performance in future HL-LHC runs. The RDTs of interest in this chapter are  $f_{1012,y}$  and  $f_{1210,x}$ , with their associated resonances and frequency lines presented in Table 4.1. Given that the working point of the LHC is set near the resonance  $Q_x - Q_y$ , which is excited by  $f_{1012}$ , more detailed studies and corrections of these fields would be beneficial.

RDT	Resonance	H-line	V-line
$f_{1012}$	$1Q_x - 1Q_y$	$1Q_y$	$-1Q_x + 2Q_y$
$f_{1210}$	$-1Q_x + 1Q_y$	$2Q_x - 1Q_y$	$1Q_x$

Table 4.1.: Skew octupolar RDTs of interest, their associated resonances and the frequency spectrum lines they contribute to.

Skew octupolar RDTs are primarily expected to arise from skew octupolar errors and their corresponding correctors. However, significant discrepancies have been observed between the LHC's magnetic and alignment models and the measured RDTs. Normal octupoles, in the presence of coupling, have been identified as contributors to the skew octupolar RDTs. This unexpected contribution amplifies the impact of skew octupolar fields, particularly on beam lifetime and dynamic aperture.

The following sections present the measurements and corresponding corrections performed at top energy, establishing the methods for reliable measurements and corrections of skew-octupolar RDTs. Subsequently, measurements at injection energy, focusing on the unaccounted for contributions of normal octupoles to skew-octupolar RDTs are examined.

## 4.2. Response Matrix Based RDT Corrections at Top Energy

The first skew octupolar RDT corrections in the LHC were implemented in 2018 during Run 2 [53]. These corrections were calculated by simulating all possible combinations of correctors and selecting the one that best matched the measurements. In practice, this approach proved to be less effective, requiring extensive manual adjustments. Such a method is highly time-consuming and cannot be easily replicated for different optics configurations.

In this section, a different approach is taken, based on response matrices. This type of correction is explained in details in Section 3.3.1. The real and imaginary responses of the RDTs for each corrector at an arbitrary strength are simulated through tracking. These responses are collected into a matrix, allowing the determination of the required strengths to match the RDT level observed in the measurements. Inverting these values result in a correction. This approach has the benefit of being remarkably faster, as only a few simulations are needed. Online corrections in the control room are thus possible in the span of a few hours instead of a whole shift, as no manual adjustments are needed.

### 4.2.1. Measurement

Initial measurements of the machine at top energy ( $\beta^* = 30$  cm) were conducted without any skew octupolar correctors to establish a baseline for the RDTs. Measuring skew octupolar RDTs poses challenges due to the high amplitude kicks required to distinguish the RDT lines in the frequency spectrum. Consequently, the LHC's working point had to be optimized specifically for these measurements to enhance the forced dynamic aperture, as the lines of interest are not observable at the nominal working point.

Several measurements were conducted at different working points to identify a configuration that enabled high amplitude kicks while maintaining beam losses below the dump threshold. The selection of such a tune combination facilitates repeated kicks and enhances the accuracy of RDT measurements. A summary of the tested working points and the resulting AC-Dipole kick percentages is presented in Table 4.2.

#### 4. Skew Octupolar Fields

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$Q_x$	$Q_y$	$\Delta Q_{x,ac}$	$\Delta Q_{y,ac}$	Maximum Kick [%]	
				Beam1	Beam 2
0.310	0.320	-0.008	+0.010	13	16
0.295	0.305	-0.008	+0.010	30	45
0.285	0.292	-0.008	+0.010	55	50

Table 4.2.: Working points tested for skew-octupolar RDT measurements, together with the maximum diagonal kick percentages achieved without reaching the dump threshold.

A suitable configuration was identified that enables high amplitude kicks. Consequently, the measurements were conducted at natural tunes of  $Q_x = 0.285$  and  $Q_y = 0.292$ . The driven tunes were adjusted to  $Q_{x,ac} = Q_x - 0.008$  and  $Q_{y,ac} = Q_y + 0.01$ . This selection of tunes has proven to be the most effective for measuring skew octupolar RDTs while keeping losses under control and is now established as the standard for commissioning activities. Figure 4.1 shows the horizontal frequency spectrum taken with the skew octupolar correctors turned off. In this plane, the line of the RDT  $f_{1210}$  is visible at  $2Q_x - 1Q_y$ , while that of  $f_{1012}$  is visible in the vertical one at  $-1Q_x + 2Q_y$ . The measured RDTs are shown later on, in blue, in Fig. 4.4.

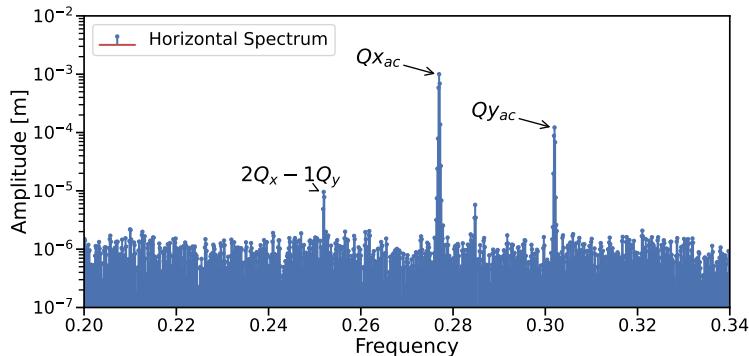


Figure 4.1.: Horizontal spectrum of Beam 1 at top energy before and after application of skew-octupolar RDT corrections. The RDT  $f_{1210}$  is seen at  $2Q_x - 1Q_y$ .

### 4.2.2. Response Matrix and Corrections

To create a response matrix, simulations were conducted with the tunes and AC-Dipole deltas set to those used for measurements. The natural tunes are  $Q_x = 0.285$  and  $Q_y = 0.292$  while the driven tunes are  $Q_{x,ac} = Q_x - 0.008$  and  $Q_{y,ac} = Q_y + 0.01$ . Each corrector is then powered individually for each tracking simulation. As measurements near the IP can be quite noisy, those are not utilized in the calculation of the correction. The polar plot of Figure 4.2 illustrates the response of each corrector at a given BPM and can be used to get an intuition of the effect of the correctors.

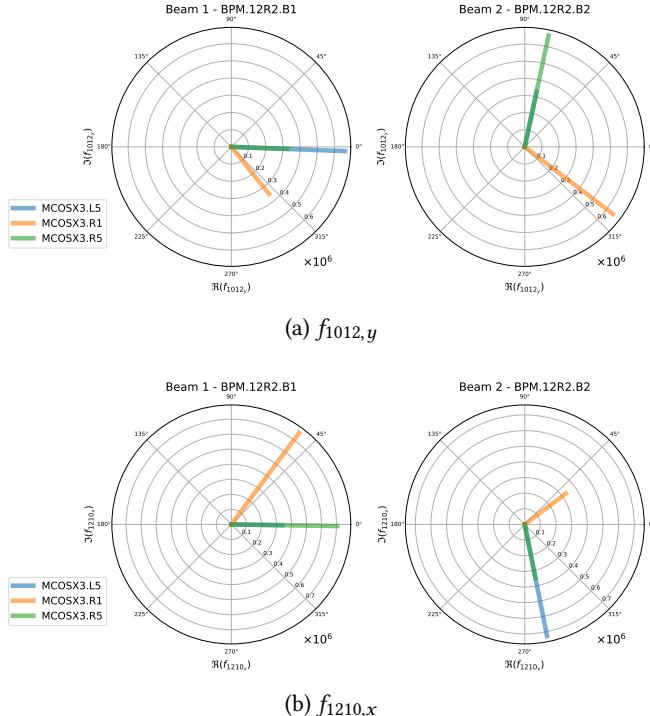


Figure 4.2.: Simulated RDTs response of the available skew octupolar correctors at top energy. Each corrector is powered at  $J_4 = 1[m^{-4}]$ . The orthogonality of R1 and L5/R5 allows to independently control the real and imaginary parts.

Correctors L5 and R5 having the same angle indicates that either can be used for corrections. Effectively having only two correctors with different angles, it makes computing corrections for four RDTs (two per beam) very challenging. In this context, a response matrix based approach is most effective in determining the optimal combination of corrector strengths. Figure 4.3 shows the real part of the RDTs from these simulations for Beam 1. Beam 2 shows a similar level of response for these correctors.

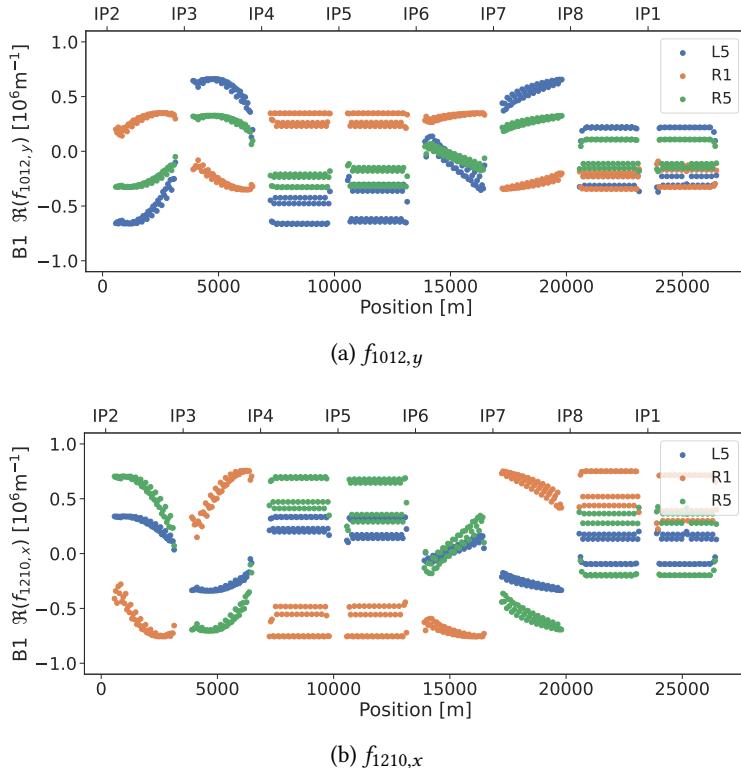


Figure 4.3.: Simulation of the RDT response of the skew octupolar correctors at top energy for Beam 1. Each corrector is powered on at  $J_4 = 1[\text{m}^{-4}]$ .

Previous simulations with each corrector powered on are compiled into a matrix, as detailed in Section 3.3.1, to capture the RDT response. By inverting this matrix, the

appropriate set of correctors needed to replicate the measurement is determined. Finally, multiplying by  $-1$  allows for the computation of the required strengths for the set of correctors. These corrector strengths, for the measured RDTs, are presented in Table 4.3.

These corrections have proven to be effective in reducing the RDT amplitudes and have been applied during 2023's commissioning and later on kept for regular operation. The RDT levels of the bare machine measured previously and with these corrections are shown in Fig. 4.4. It can be observed that both RDT amplitudes are reduced, with the exception of  $f_{1210,x}$  for Beam 2, which stays constant, while being at an already low level. The reduction of RDT amplitude is comparable to those obtained in 2018, where the same RDT for Beam 2 did neither improve or worsen.

Corrector	Strength [m <sup>-4</sup> ]
MCOSX3.L1	broken
MCOSX3.R1	-0.50
MCOSX3.L5	0.42
MCOSX3.R5	-0.01

Table 4.3.: Computed corrections for skew octupolar RDTs at top energy. The corrector L1 has been broken for several years and can not be used.

A improvement in the forced dynamic aperture during AC-Dipole excitation was observed with the earlier skew-octupolar RDTs corrections implemented. Figure 4.5 shows the timeline of the various kicks amplitudes before and after corections. It can clearly seen that skew-octupolar corrections enhance the forced dynamic aperture and allow for higher-amplitude kicks with the same amount of losses.

Skew-octupolar Resonance Driving Terms have, for the first time, been directly corrected using a response matrix-based approach at top energy. This new approach allows for a more efficient use of dedicated machine time compared to the previous empirical approach. Corrections have significantly improved the forced dynamic aperture, enabling the high amplitude kicks necessary for non-linear optics measurements. Furthermore, these corrections can be computed online in the control room, reducing commissioning time and thereby increasing the integrated luminosity of the LHC.

#### 4. Skew Octupolar Fields

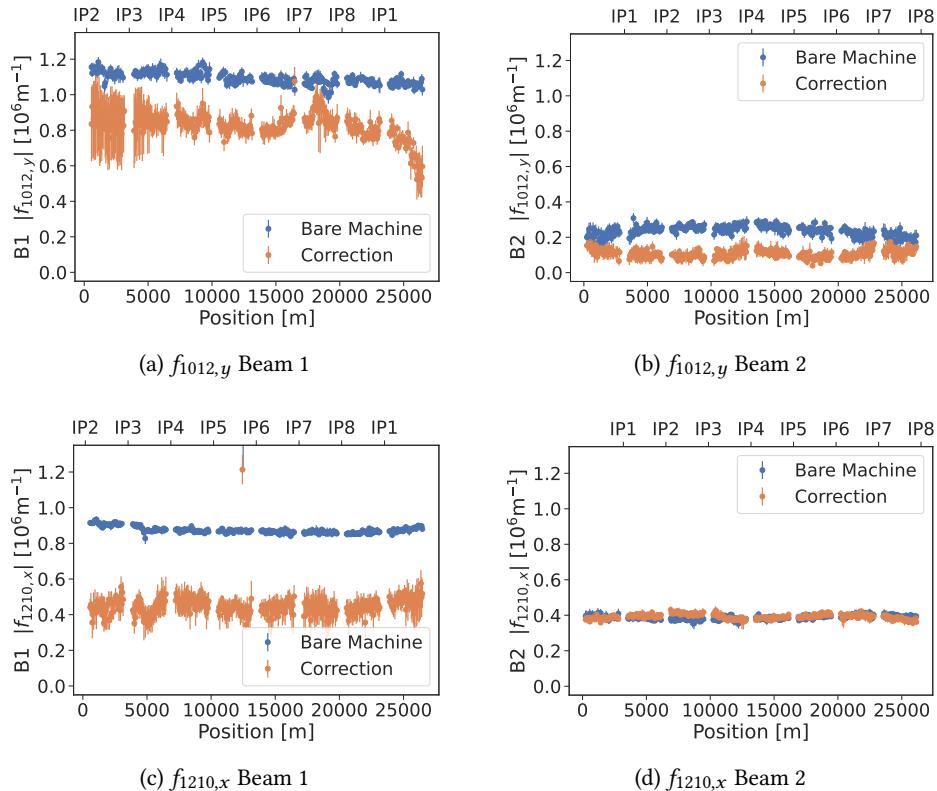


Figure 4.4.: Measured skew octupolar RDTs at top energy and  $\beta^* = 30\text{cm}$  before and after correction. A reduction if observed for all but one RDT in Beam 2.

#### *4.2. Response Matrix Based RDT Corrections at Top Energy*

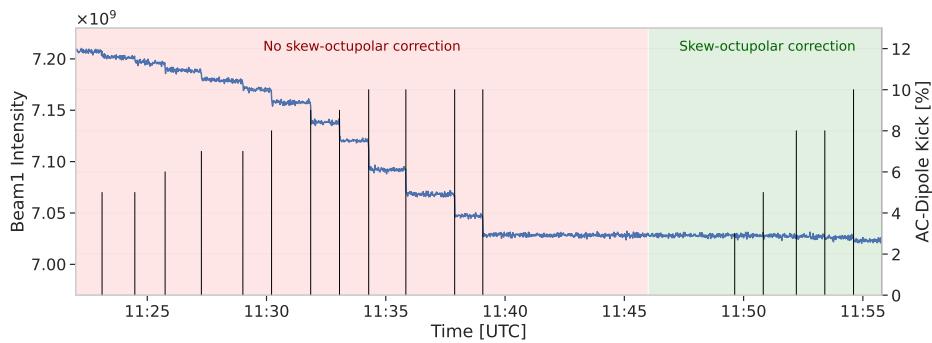


Figure 4.5.: Intensity of Beam 1 related to the kick percentage of the AC-Dipole over time, before and after having applied skew-octupolar RDT corrections.

### 4.3. Skew-Octupolar RDT driven by Landau Octupoles

The significant progress achieved in understanding skew octupolar RDTs at top energy and their corrections improving the forced dynamic aperture prompted their study at injection energy, where they had never been studied before. During these studies, large skew octupolar resonances lines were observed when Landau octupoles were powered. These magnets, being normal octupoles, are not expected to contribute to skew octupolar fields. To further understand and characterize the contribution of Landau octupoles to skew octupolar fields, subsequent measurements were conducted. Measurements of the RDTs introduced in Table 4.1, were performed with several strengths of Landau octupoles, ranging from  $-2K_4$  to  $5K_4$ .

The following Fig. 4.6 illustrates the measured amplitude for Beam 1 of the RDT  $f_{1210}$  with these varying strengths of Landau octupoles. It can be observed that normal octupoles have a large impact on this RDT at only  $K_4 = 5$ . This amplitude is expected to increase by an order of magnitude with the nominal powering of Landau octupoles at  $K_4 = 18$  and significantly supports the need for their study.

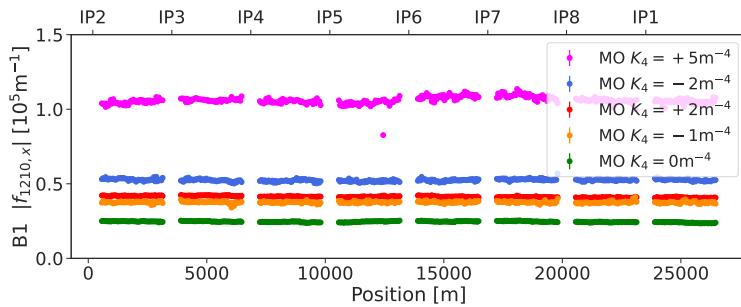


Figure 4.6.: Unexpected amplitude shift of skew-octupolar RDT  $f_{1210}$  with varying Landau octupoles strengths.

Simulations conducted with Landau powering of  $K_4 = 0$  and  $K_4 = 2$ , incorporating additional normal and skew field errors, do not replicate the measured shift. The comparison between measurements and simulations is presented in Fig. 4.7, where it is evident

that some component is absent from the magnetic model. For conciseness, all figures are shown with respect to Beam 1, although similar results are observed for Beam 2.

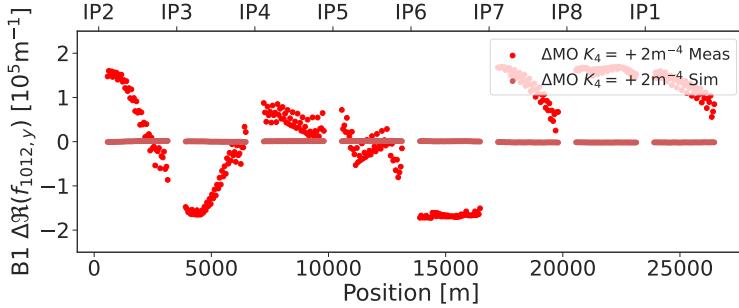


Figure 4.7.: Measured and simulated shift of skew-octupolar RDT after increase of Landau octupoles strength from zero. It is apparent that the shift in measurement is not replicated by the simulation.

### 4.3.1. Misalignments

Magnet misalignments in the LHC have been previously measured and corrected in several phases [58]. Nevertheless, residual misalignments persist and have been incorporated into the tables used for simulations. These misalignments include horizontal and vertical errors, as well as roll errors. While translation errors may induce feed-down effects, roll errors introduce the skew counterpart of normal magnetic fields, and vice versa.

To assess whether roll errors could explain the behavior observed in measurements, a tracking simulation was conducted with and without roll errors applied to the Landau octupoles, powered at  $K_4 = 2$ . The amplitude of the RDT  $f_{1012}$  is presented in Fig. 4.8, with similar results observed for  $f_{1210}$ . These results reveal an imperceptible difference between the two simulations. Roll errors are unable to account for the previously observed shifts. The average roll error of the Landau octupoles is indeed small, approximately 0.08 mrad.

#### 4. Skew Octupolar Fields

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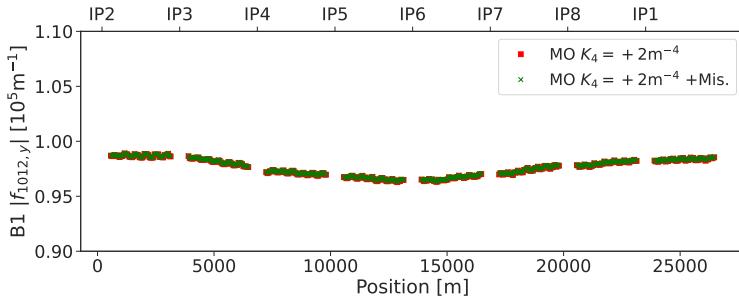


Figure 4.8.: Simulated skew octupolar RDT with normal and skew octupolar errors, and Landau octupoles powered. One simulation includes further roll errors on the Landau octupoles. No significant difference is observed between the two.

#### 4.3.2. Linear Coupling

As misalignments could not explain the discrepancy, it is necessary to consider another source of errors. According to [59], the combination of a normal octupole and a skew quadrupole can yield skew-octupolar-like fields. Linear coupling can effectively be modeled as a skew quadrupole and may be a source of the observed discrepancy. The transfer map for an octupole and a skew quadrupole is detailed in Appendix A.2.4.

To test this hypothesis, simulations were initially conducted with varying values of coupling while maintaining a fixed octupolar strength, allowing for an assessment of the impact of coupling alone. The resulting RDT  $f_{1012}$  is shown in Fig. 4.9, with a similar trend observed for  $f_{1210}$ . The presented  $C^-$  values are commonly encountered in operation and are well within the tolerances established in the LHC design [6]. These results indicate that the skew octupolar RDTs are expected to be significantly altered as coupling increases.

As demonstrated in Fig. 4.9, Landau octupoles, along with minor variations in coupling, even at low  $K_4$  strengths, can significantly influence the RDTs  $f_{1012}$  and  $f_{1210}$ , along with their associated resonances. Furthermore, at their operational powering of  $K_4 = 18$ , Landau octupoles are anticipated to generate very large skew octupolar RDTs, as simulated in Fig. 4.10, where a realistic operational coupling value is utilized. A difference of two orders of magnitude is observed when powering the octupoles and

### 4.3. Skew-Octupolar RDT driven by Landau Octupoles

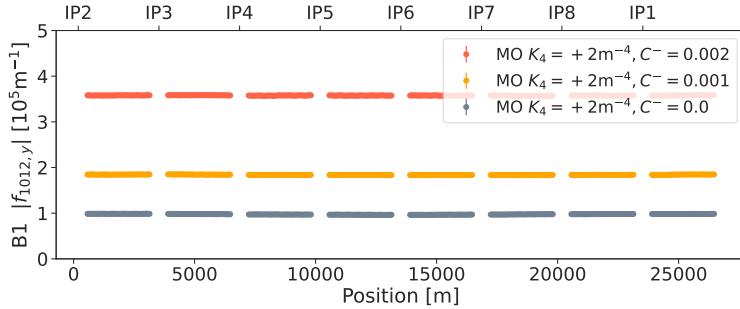


Figure 4.9.: Simulated skew octupolar RDT with fixed Landau octupole strength but varying coupling. Selected coupling values are often seen in operation.

considering coupling. Such a drastic worsening of the RDTs could prompt for further studies and corrections.

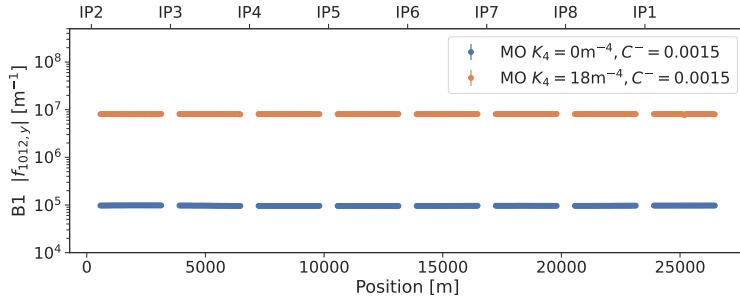


Figure 4.10.: Simulated skew octupolar RDT  $f_{1012}$  with Landau octupoles turned off and at their operational powering. Coupling is set to a value commonly seen in operation for both.

Having established that coupling can significantly contribute to the generation of skew octupolar fields from normal octupoles, further simulations were conducted to match the measured coupling for each set of observations. Initially, local coupling is assessed using coupling RDTs, followed by the definition of a set of correctors linked to a common control knob to globally correct the coupling across the machine. This global knob setting is then applied in simulations to replicate the measured coupling.

#### 4. Skew Octupolar Fields

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The measured and simulated shifts in the real part of the RDTs  $f_{1012}$  and  $f_{1210}$ , resulting from the application of various octupole powerings in the presence of coupling, are shown in Fig. 4.11. A similar trend is observed for the imaginary part.

It can be observed that simulations and measurements for positive strengths ( $K_4 = 2$  and  $K_4 = 5$ ) are now in good agreement. This suggests that the primary contribution to the skew octupolar RDTs can be attributed to the Landau octupoles and coupling. The relative deviation between measurement and simulation can be explained by the sensitivity to the coupling. A difference of  $10^{-4}$  units is enough to be noticeable, making accurate reproduction of skew octupolar RDTs not trivial. It is to be noted that measurements of  $f_{1012}$  with *negative* strength exhibit a response that is opposite to the predictions made by simulations, whereas  $f_{1210}$  agrees well with the expected results. This behavior may be attributed to magnetic hysteresis. Specifically, the measurements at positive  $K_4$  were taken after a degaussing cycle of the Landau octupoles, while the negative measurements were conducted immediately afterwards the positive ones. Additional measurements, with proper degaussing performed before each test, could help clarify this difference.

### 4.3. Skew-Octupolar RDT driven by Landau Octupoles

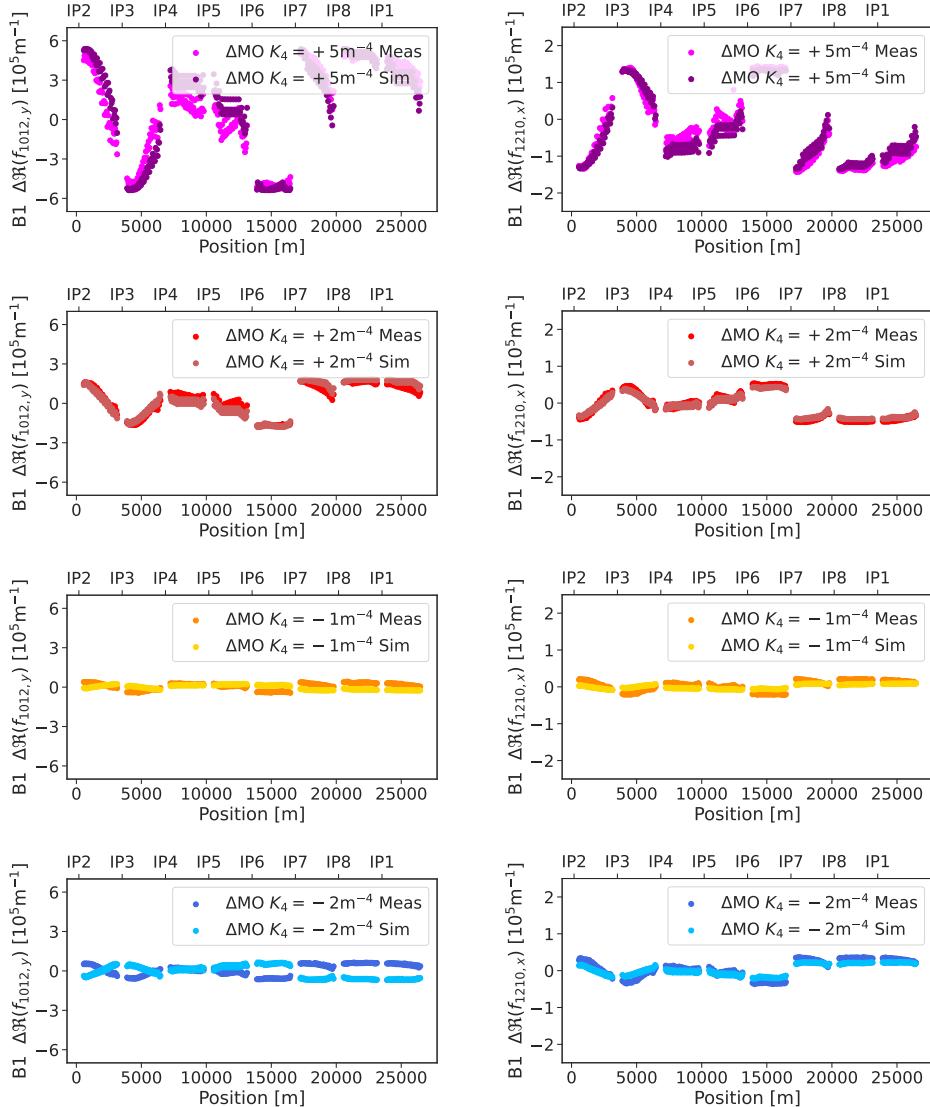


Figure 4.11.: Measured and simulated real part shift of skew octupolar RDTs induced by Landau octupoles in presence of coupling at injection energy. Left column shows  $f_{1012}$  while right shows  $f_{1210}$ .

## 4.4. Summary

This chapter investigates the origins of skew-octupolar fields in the LHC and explores methods for measuring and mitigating their effects. The studies demonstrate that these fields significantly contribute to limitations in dynamic aperture, particularly when the beam is kicked with the AC-Dipole.

For the first time, skew-octupolar Resonance Driving Terms have been directly corrected using a response matrix-based approach at top energy. This new method facilitates a more efficient use of dedicated machine time compared to previous empirical approaches. The corrections successfully correct the RDTs  $f_{1012}$  and  $f_{1210}$ .

Furthermore, the chapter addresses the unexpected influence of Landau octupoles on skew-octupolar RDTs at injection energy. Simulations reveal that misalignments of the Landau octupoles, particularly roll errors, do not have a significant impact. Instead, transverse linear coupling emerges as a crucial factor. Initial findings indeed suggest that coupling is essential for predicting the behavior of skew-octupolar RDTs in the presence of Landau octupoles. During regular operation, where the Landau octupoles are powered at  $K_4 = 18$ , significant skew-octupolar RDTs are expected to be generated and degrade the dynamic aperture.

The results presented provide valuable insights into the complex interplay of magnetic fields in the LHC and underscore the importance of accurate modeling for effective corrections. Moreover, the implemented corrections have substantially improved the forced dynamic aperture, enabling the high-amplitude kicks necessary for non-linear optics measurements. These corrections can also be computed online in the control room, thereby reducing commissioning time and increasing the integrated luminosity of the LHC.

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# Decapolar Fields

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## 5.1. Introduction

Beam-based measurements have been carried in the LHC since Run 1 to better understand the decapolar fields. Those have been carried out via chromaticity measurements [60–62]. The third order of the non-linear chromaticity,  $Q'''$ , which is expected to be generated for the most part by decapolar errors in the main dipoles, has shown a consistent discrepancy at injection energy between its expected value from simulations and that observed. Figure 5.1 illustrates this discrepancy, highlighting a factor  $-2$  between the supposedly corrected machine and the corrected model.

The FiDeL model, based on magnetic measurements, is used during operation to correct various multipole errors in the LHC, including octupolar and decapolar. The operational corrections being based on this magnetic model and simulations, the residual  $Q'''$  value is expected to be small, which is however not the case. Chromaticity measurements have thus been repeated during LHC’s Run 3 and corrections made routine, aimed at correcting the observed discrepancy.

While the study of non-linear chromaticity has proven valuable in identifying the existence of this discrepancy, it does not yet permit alone to understand its exact origins. In an effort to gain deeper insights, additional measurements were performed focusing on novel observables that had not been previously explored. *Bare chromaticity* is the measured chromaticity with the octupolar and decapolar correctors deactivated ; this

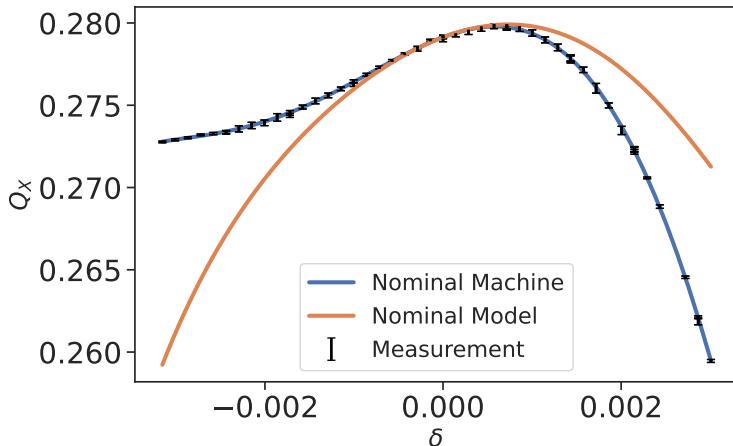


Figure 5.1.: Measured and simulated chromaticity with application of the nominal decapolar corrections from FiDeL. It can be seen that although the corrections should diminish  $Q'''$ , it is not well corrected in practice.

approach aims to isolate the machine effects from those of the correctors. *Chromatic amplitude detuning*, evaluates how the tune varies with both the beam's action and the momentum offset ; this methods has the benefit of having a different mathematical expression from that of the chromaticity, to rule out exclusively momentum dependent effects, such a higher-order dispersion.

In addition to previous measurements, direct measurements of decapolar Resonance Driving Terms (RDTs) were conducted for the first time in the LHC, at injection energy. These high-order RDTs, which influence resonances near the working point, have also benefited from corrections, marking the first instance of directly correcting such high-order terms, using a response matrix-based approach. During the study of these resonances, lower-order multipoles, such as sextupoles and octupoles, were found to be significant contributors. Their effects are thoroughly derived and examined, as they are expected to strongly impact the dynamic aperture at injection energy.

While lower-order field errors in the LHC are generally well understood and controlled, this is not the case for higher-order errors such as decapoles and beyond. Precise

control of these higher-order non-linearities is expected to be crucial for future colliders like the FCC, where decapolar field corrections will play a large role in maintaining an optimal dynamic aperture. Extending advanced beam-based measurement and correction techniques to higher-order errors is therefore of significant interest. The observed discrepancy in the correction of third-order chromaticity ( $Q'''$ ) points to an incomplete understanding of both the error sources and the magnetic model. Addressing this gap is essential for improving higher-order corrections in the LHC and preparing for the challenges of the next generation of accelerators.

## 5.2. Response of correctors

As seen in Fig. 2.4, the LHC is equipped with decapole correctors. Those magnets are part of the LHC's design, aiming at correcting the field errors of the main dipoles. These correctors, denominated *MCD*, are specific to each beam and are placed after every second dipole, totaling 1232 in number [63]. MCDs are nested with octupolar correctors, *MCO*. The pair of these correctors is often referred to as *MCDO*. It is not possible to individually power each corrector. Rather, a circuit consists of a whole arc. There are in total 16 circuits to control the correctors of both beams and 8 arcs. Figure 5.2 shows a picture taken of decapoles on a test bench.

The important characteristics of the magnetic fields of correctors are their main field transfer function (or *response*), the field quality and possible crosstalk, as MCOs and MCDs are nested [63]. To resolve the previously stated discrepancy, it is essential to investigate the decapole correctors themselves, to rule them out as the source of this difference.

The full third term of the chromaticity function is highlighted in Eq. (5.1). Details on chromaticity are given in Section 2.6.1.

$$Q(\delta) = Q_0 + Q'\delta + \frac{1}{2!}Q''\delta^2 + \boxed{\frac{1}{3!}Q''' \delta^3} + O(\delta^4). \quad (5.1)$$

This third order, generated by leading order by decapoles in a region of non-zero linear dispersion, is related to the  $\beta$ -function, the dispersion and the normalized strength

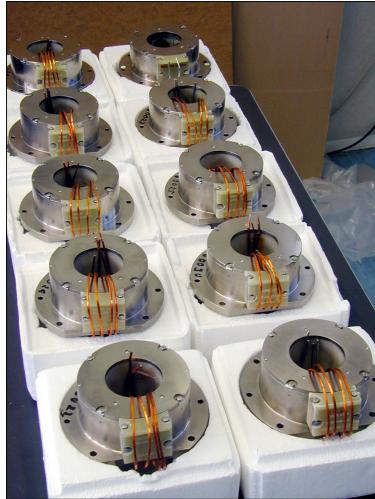


Figure 5.2.: Decapoles on a test bench, being inspected after manufacturing [64].

$K_5[m^{-5}]$  of the decapolar correctors:

$$\begin{aligned}\Delta Q''''_x &= \frac{1}{4\pi} K_5 L \beta_x D_x^3 \\ \Delta Q''''_y &= - \frac{1}{4\pi} K_5 L \beta_x D_x^3.\end{aligned}\tag{5.2}$$

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During Run 3's commissioning, the routine measurement and correction of  $Q''$  and  $Q'''$  have been established. These corrections provide a valuable opportunity to analyze the response of the decapole correctors, *MCDs*. By studying this response, it is possible to determine whether the source of the  $Q'''$  discrepancy originates from a miscalibration of the decapolar correctors or not. Figure 5.3 shows the chromaticity function measured during Run 3's commissioning in 2022 with the nominal FiDeL corrections and with further beam-based corrections computed analytically and applied on top.

The nominal and corrected  $Q'''$  values are shown in Table 5.1, with the shift in  $Q'''$  for each beam and axis. A good agreement between the measurements and simulations can be seen.

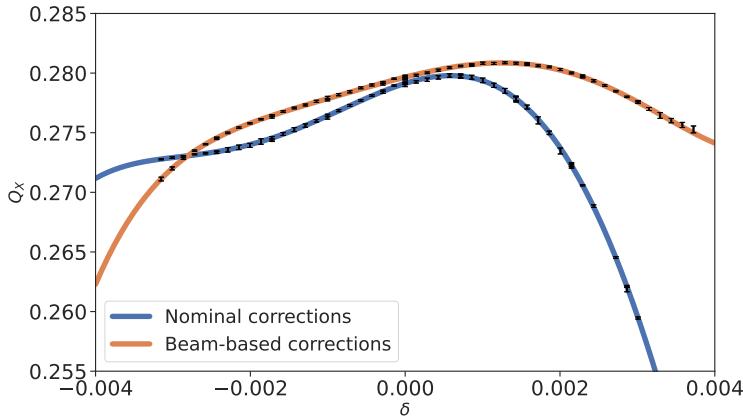


Figure 5.3.: Chromaticity of the horizontal plane of Beam 1 during Run 3's commissioning, with nominal corrections based on the magnetic model and beam-based corrections aimed at correcting  $Q''$  and  $Q'''$ .

Plane	$Q'''[10^6]$		$\Delta Q'''[10^6]$	
	Nominal	Beam-Based	Meas.	Simulation
<b>Beam 1</b>				
X	$-3.36 \pm 0.04$	$-1.02 \pm 0.03$	$2.3 \pm 0.1$	2.5
Y	$1.62 \pm 0.05$	$0.12 \pm 0.02$	$-1.5 \pm 0.1$	-1.4
<b>Beam 2</b>				
X	$-2.72 \pm 0.08$	$-0.64 \pm 0.03$	$2.1 \pm 0.1$	2.5
Y	$1.54 \pm 0.06$	$0.14 \pm 0.03$	$-1.4 \pm 0.1$	-1.4

Table 5.1.: Third order chromaticity obtained during Run 3 commissioning, with nominal and beam-based corrections aimed at correcting  $Q''$  and  $Q'''$ . Response of the correctors between the two configurations is also given for both measurement and simulation.

The agreement between simulations and measurements confirms that the decapole correctors are functioning as intended. No errors in their calibration or response function (current to strength) have been identified that could explain the observed  $Q'''$  discrepancy.

### 5.3. Bare Chromaticity

Complementary to the measurements of  $Q'''$  shifts with varying decapolar strengths, measurements of the bare machine, without any correctors powered, can provide valuable insights. These measurements help rule out potential crosstalk between the octupolar and decapolar correctors, which are physically nested, and allow for a clearer understanding of the contribution of octupolar correctors to  $Q'''$ . Additionally, they offer a direct and clean comparison to the magnetic model. The obtained results are presented in Fig. 5.4.

Simulations have been run with MAD-X and PTC including fields errors from normal sextupole to decahexapole. The expected  $Q'''$  values are presented in Table 5.2 and compared to the measured ones along with the ratio between the two.

Quantity	Measured [ $10^6$ ]	Simulated [ $10^6$ ]	Ratio
Beam 1			
$Q_x'''$	$2.95 \pm 0.04$	$6.94 \pm 0.02$	$0.43 \pm 0.01$
$Q_y'''$	$-1.82 \pm 0.04$	$-4.29 \pm 0.01$	$0.42 \pm 0.01$
Beam 2			
$Q_x'''$	$3.06 \pm 0.07$	$7.03 \pm 0.02$	$0.44 \pm 0.01$
$Q_y'''$	$-1.72 \pm 0.02$	$-4.27 \pm 0.01$	$0.42 \pm 0.01$

Table 5.2.: Measured and simulated third order chromaticity with octupole and decapole correctors turned off. The simulations include field errors from normal sextupoles to decahexapoles.

The observed discrepancy between the measurements and simulations persists both with the FiDel corrections applied and with the correctors turned off. Similarly, measuring significantly different lower-order chromaticities and working points, the discrepancy remains consistently around a factor of 2 between the measured and expected  $Q'''$ . This suggests that the source of the discrepancy lies elsewhere. One possible explanation could be an error in the systematic decapolar component of the main dipoles in the magnetic model. However, relying solely on  $Q'''$  does not allow for a definitive conclusion, as higher-order effects like non-linear dispersion or non-linear momentum compaction factor could also contribute. To resolve this, additional observables, such as chromatic

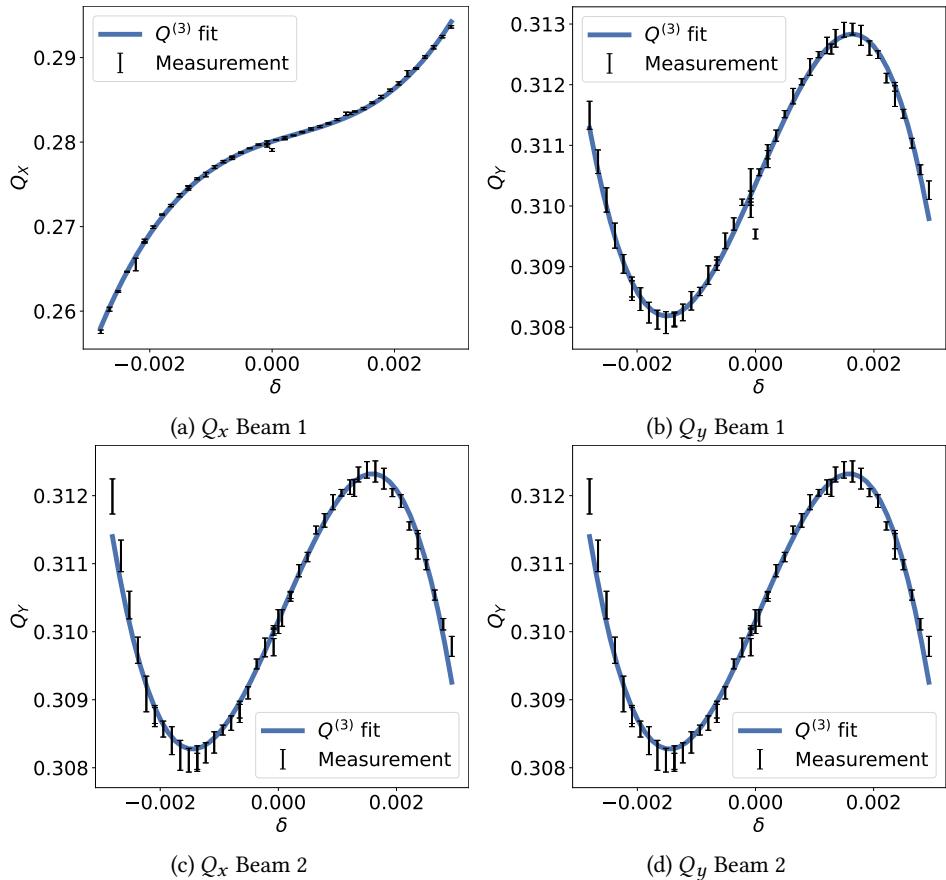


Figure 5.4.: Fit of the chromaticity function for the chromaticity measurement performed with octupole and decapole correctors powered off. The fit includes all orders up to third.

amplitude detuning, will be required for further investigation.

## 5.4. Chromatic Amplitude Detuning

The Chromatic Amplitude Detuning is the tune shift dependant on both the actions and the momentum offset, whose decapole contributed terms are described via a Taylor expansion in Eq. (5.3). More information and derivations can be found in Section 2.6.3 and Appendix B.

$$\Delta Q(J_x, J_y, \delta) = \frac{\partial^2 Q}{\partial J_x \partial \delta} \cdot J_x \delta + \frac{\partial^2 Q}{\partial J_y \partial \delta} \cdot J_y \delta + \frac{1}{3!} \frac{\partial^3 Q}{\partial \delta^3} \cdot \delta^3. \quad (5.3)$$

The last term is more commonly referred to as the third order chromaticity,  $Q'''$ . Each of these terms depend on the  $\beta$ -functions, the horizontal dispersion  $D_x$  and the normalized decapole field gradient  $K_5$  for a single source of length  $L$ ,

$$\begin{aligned} \frac{\partial^2 Q_x}{\partial J_x \partial \delta} &= \frac{1}{16\pi} K_5 L \beta_x^2 D_x, & \frac{\partial^2 Q_y}{\partial J_y \partial \delta} &= \frac{1}{16\pi} K_5 L \beta_y^2 D_x, \\ \frac{\partial^2 Q_x}{\partial J_y \partial \delta} &= -\frac{1}{8\pi} K_5 L \beta_x \beta_y D_x, & \frac{\partial^2 Q_y}{\partial J_x \partial \delta} &= -\frac{1}{8\pi} K_5 L \beta_x \beta_y D_x, \\ \frac{\partial^3 Q_x}{\partial \delta^3} &= \frac{1}{4\pi} K_5 L \beta_x D_x^3, & \frac{\partial^3 Q_y}{\partial \delta^3} &= -\frac{1}{4\pi} K_5 L \beta_y D_x^3. \end{aligned} \quad (5.4)$$

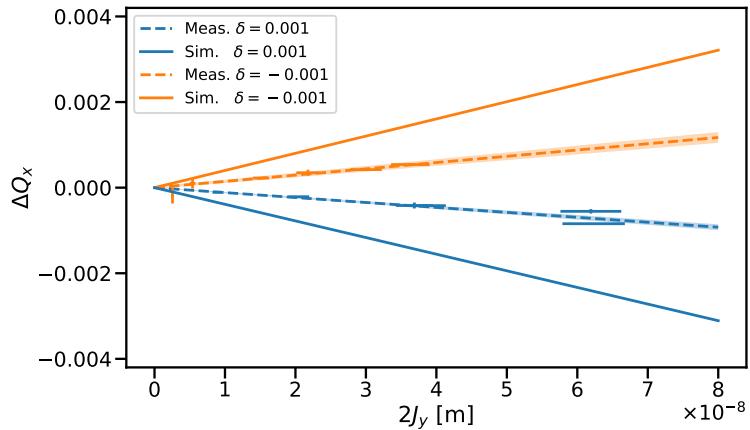
Notably, while the  $K_5$  dependence is identical to that of  $Q'''$ , the  $\beta$  and  $D_x$  dependences differ significantly. In the presence of a systematic error in the magnetic model, one would expect similar ratios for this observable as those observed for  $Q'''$ . The action dependant terms can be measured by exciting the beam with an AC-dipole with increasing strengths at different momentum-offsets. The AC-Dipole introduces effects that are corrected in the analysis [10].

Such a measurement was taken with octupole and decapole correctors turned off to measure the bare machine. The momentum-offsets measured at were  $-0.001$  and  $0.001$ , respectively roughly equal to a trim of  $+140\text{Hz}$  and  $-140\text{Hz}$  of the RF. Some data could not be collected due to machine availability issues, restricting the measurement to low amplitude kicks. Nevertheless, the terms  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$  and  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$  for beam 2 were successfully measured for the first time in the LHC.

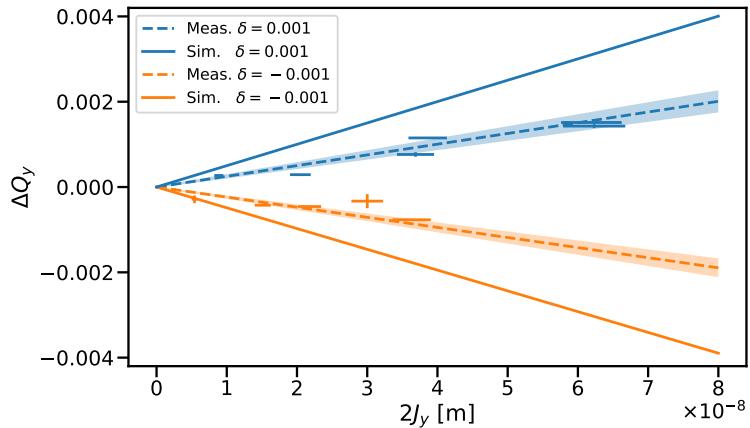
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#### *5.4. Chromatic Amplitude Detuning*

Figure 5.5a and Fig. 5.5b show a fit of those terms to measured  $Q_{x,y}$  vs  $J_y$  at two different momentum offsets. Expected shifts from MADX-PTC simulations, that include field errors ranging from sextupoles to decahexapoles ( $b_3$  to  $b_8$  and  $a_4$  to  $a_8$ ) are shown as a comparison.



(a) Horizontal tune shift depending on the vertical action:  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$ .



(b) Vertical tune shift depending on the vertical action:  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$ .

Figure 5.5.: Measured and simulated tune shift due to a change of action via an AC-Dipole at two different momentum offsets. Each fit corresponds to a chromatic amplitude detuning term evaluated at a certain  $\delta$ .

Type	$\frac{\partial^2 Q_x}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$	$\frac{\partial^2 Q_y}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$
$\delta = +0.001$		
Meas.	-1.16 ± 0.08	1.26 ± 0.15
Sim.	-3.82 ± 0.01	2.47 ± 0.01
Ratio	0.30 ± 0.02	0.51 ± 0.06
$\delta = -0.001$		
Meas.	1.47 ± 0.12	-1.18 ± 0.13
Sim.	3.92 ± 0.01	-2.41 ± 0.01
Ratio	0.38 ± 0.03	0.49 ± 0.05

Table 5.3.: Comparison of the measured and simulated terms  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$  and  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$  via PTC, at two discrete momentum offsets. Simulations include errors from normal sextupole to decahexapole and from skew octupole to decahexapole.

A similar ratio between simulation and measurement is observed across the different terms. The values and ratios of measurement to model can be found in Table 5.3. The observed ratios for the chromatic amplitude detuning are quite similar, while not exact, to those of the bare chromaticity. This similarity of the ratios between the chromatic amplitude detuning and  $Q'''$  discrepancies suggests an issue with the decapolar error model of the main dipoles, with measurements showing values about half of those predicted by the magnetic model.

## 5.5. Accounting for Decay

FiDeL, the field model used in operation, aims at correcting the field errors at various energies. Measured field errors but also decay estimates with respect to time are used when computing corrections.

Magnetic decay in superconducting magnets stems from field changes in the strands caused by current redistribution in the superconducting cables. Decay has been shown to be strongly dependent on the magnet powering history [65, 66]. This happens for example when ramping down from an energy of 6.8 TeV to 0 and then going back to

## 5. Decapolar Fields

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injection energy at 450 GeV. Some components of the magnetic field will persist and gradually change over time until they reach their final value.

While the sextupolar  $b_3$  decay is accounted for during operations, a review of the FiDeL model revealed that the decapolar component  $b_5$  has not been similarly considered. To date, all beam dynamics studies in the LHC have made use of the same set of magnetic measurement error tables, thus neglecting any decapolar decay. It has though be found that during the LHC construction, magnetic measurements were taken for a representative set of main dipoles, and decay measured. The average  $b_5$  field in the main dipoles at  $t = 0$ , upon reaching the 450 GeV energy, is approximately  $1.145 \pm 0.5$ , based on data from the magnetic reports utilized by FiDeL. For reference, decay measurements of a selection of main dipoles (each dot), over time are illustrated in Fig. 5.6. Table 5.4 shows the decay expected at injection energy after a typical cycle of machine operation. From this table, it is clear that decay accounts for a large part of the  $b_5$  at injection. About 40% of the compensated  $b_5$  at injection in the main dipoles can indeed be attributed to the decay.

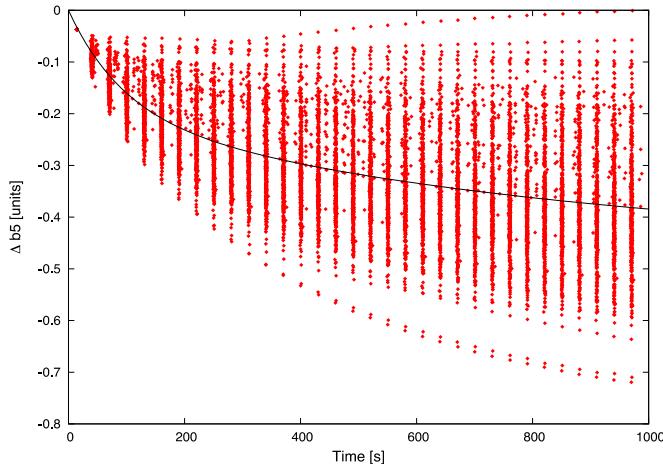


Figure 5.6.: Measured decay of the integrated decapolar field in LHC's main dipoles at injection energy. The fit is shown in black [15].

Unfortunately, individual magnet measurements conducted more than 15 years ago could not be recovered. As a result, the  $b_5$  decay in the simulations is implemented as

Time [m]	$\Delta b_5$
17	-0.38
33	-0.44
50	-0.46
67	-0.47
83	-0.47
167	-0.47

Table 5.4.: Decay of the  $b_5$  component after injection for long time periods [15].

the average decay across all the probed magnets. In the following simulations, each main dipole sees its  $b_5$  component reduced by 0.47 units.

**Chromaticity** It has been shown in the previous sections, and particularly in Table 5.2 that measurements and simulations were off regarding the third order chromaticity  $Q'''$ . New simulations have hence been conducted to evaluate the effect of this decay on chromaticity. Table 5.5 compares simulations with and without the decay of the  $b_5$  component. The simulations use the 100 available error seeds to calculate error bars.

Condition	$Q_x'''[10^6]$	$Q_y'''[10^6]$
No Decay	$6.93 \pm 0.04$	$-4.32 \pm 0.02$
$\Delta b_5 = -0.47$	$4.05 \pm 0.04$	$-2.54 \pm 0.02$

Table 5.5.: Comparison of  $Q_x'''$  and  $Q_y'''$  for Beam 1 with and without decay of the  $b_5$  component of the main dipoles at injection energy. Included fields errors range from normal and skew sextupoles ( $n = 3$ ) to icosapole ( $n = 10$ ).

Implementing the decay of the  $b_5$  component of the main dipoles in the simulations has significantly reduced the discrepancy between beam-based measurement and the magnetic model. Using the values from the bare chromaticity measurement and updating Table 5.2 with the newly obtain data provides a more accurate description of the magnetic model, as shown in Table 5.6.

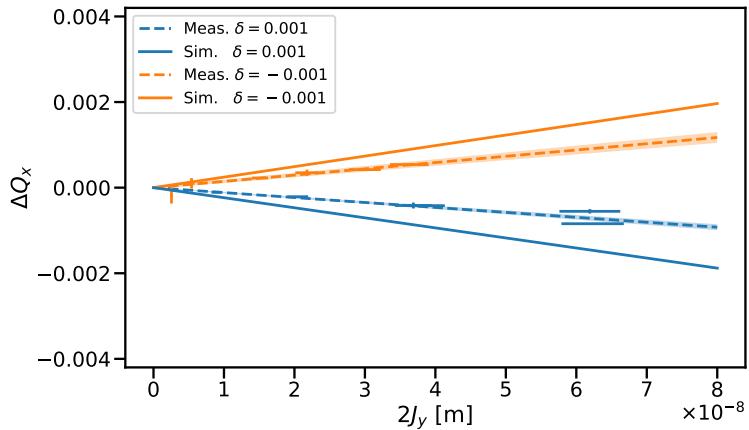
Quantity	Measured [ $10^6$ ]	Simulated [ $10^6$ ]	Ratio
Beam 1			
$Q_x'''$	$2.95 \pm 0.04$	$4.05 \pm 0.04$	$0.73 \pm 0.01$
$Q_y'''$	$-1.82 \pm 0.04$	$-2.54 \pm 0.02$	$0.72 \pm 0.02$
Beam 2			
$Q_x'''$	$3.06 \pm 0.07$	$4.27 \pm 0.03$	$0.72 \pm 0.02$
$Q_y'''$	$-1.72 \pm 0.02$	$-2.55 \pm 0.01$	$0.67 \pm 0.01$

Table 5.6.: Measured and simulated third order chromaticity with octupole and decapole correctors turned off. The simulations include field errors from normal and skew sextupoles to icosapole ( $n = 3$  to  $n = 10$ ). The  $b_5$  component of the main dipoles has been updated to include decay.

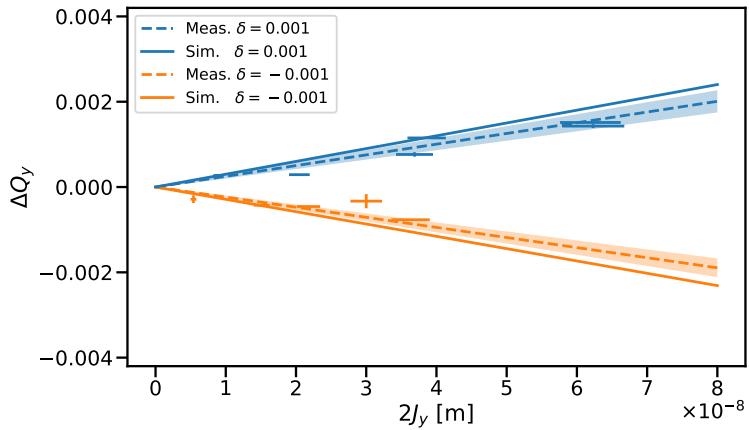
**Chromatic Amplitude Detuning** Similar to chromaticity, new simulations have been conducted for chromatic amplitude detuning. Table 5.7 gives an overview of the newly computed values and the related ratios relative to the measurements, while Fig. 5.7 gives a visual clue. A small difference in agreement can be observed between the direct term  $\partial^2 Q_y / (\partial J_y \partial \delta)$  and the crossterm. This may arise from the implementation of decay, which only considers the average  $b_5$  across all dipoles and may not fully capture the individual variations. Nevertheless, both terms now demonstrate a better agreement with the model, similar to the trend seen with chromaticity.

Type	$\frac{\partial^2 Q_x}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$	$\frac{\partial^2 Q_y}{\partial J_y \partial \delta} [10^4 \text{m}^{-1}]$
$\delta = +0.001$		
Meas.	$-1.16 \pm 0.08$	$1.26 \pm 0.15$
Sim.	$-2.35 \pm 0.01$	$1.50 \pm 0.01$
Ratio	$0.49 \pm 0.03$	$0.84 \pm 0.10$
$\delta = -0.001$		
Meas.	$1.47 \pm 0.12$	$-1.18 \pm 0.13$
Sim.	$2.46 \pm 0.01$	$-1.45 \pm 0.01$
Ratio	$0.60 \pm 0.05$	$0.82 \pm 0.09$

Table 5.7.: Comparison of the measured and simulated terms  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$  and  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$  via PTC, at two discrete momentum offsets. Simulations include errors from normal and skew sextupoles to icosapole ( $n = 3$  to  $n = 10$ ), as well as the decay of the  $b_5$  component of the main dipoles.



(a) Horizontal tune shift depending on the vertical action:  $\frac{\partial^2 Q_x}{\partial J_y \partial \delta}$ .



(b) Vertical tune shift depending on the vertical action:  $\frac{\partial^2 Q_y}{\partial J_y \partial \delta}$ .

Figure 5.7.: Measured and simulated tune shift depending on the action at two different momentum offsets. Each fit corresponds to a chromatic amplitude detuning term evaluated at a certain  $\delta$ . Estimates from simulations are lowered due to the  $b_5$  decay of the main dipoles.

## 5.6. Resonance Driving Terms

Decapoles, due to their order, contribute to many RDTs. Indeed, 25 of them can be theoretically observed in simulations and measurements. In practice, the contributions of individual multipoles become indistinguishable as many resonances or lines overlap, making it impossible to isolate certain terms. Some resonances, described in Appendix C, are unique to certain multipoles when considering not too high orders. Those resonances, provided that they are sufficiently strong and the beam close to them, can be measured via their RDTs.

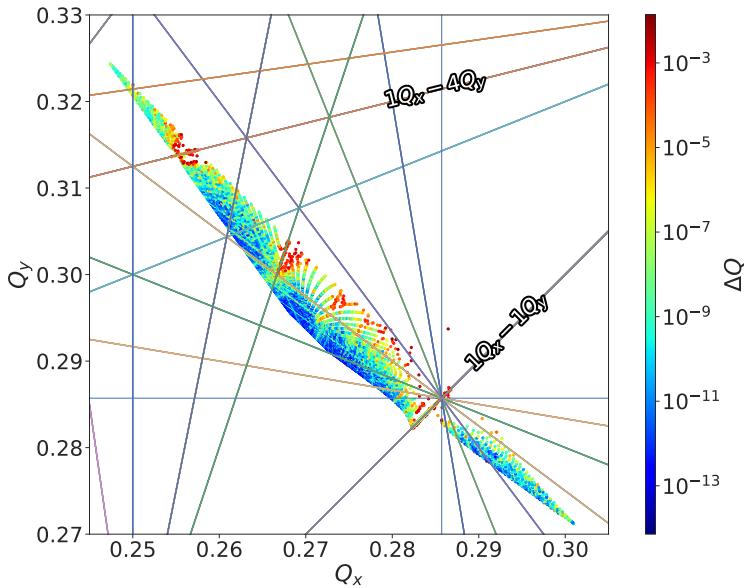


Figure 5.8.: Frequency map at injection energy, with decapolar field errors and nominal settings for Landau octupoles. The highlighted resonance  $(1,-4)$ , excited by decapoles, shows a degradation over 20,000 turns. The tune shift  $\Delta Q$  between the start and the end of the simulation is indicated in color.

Of particular interest to LHC operation is the Resonance Driving Term  $f_{1004}$ , which drives the resonance  $1Q_x - 4Q_y$ . This RDT appears in the horizontal frequency spectrum

at  $4Q_y$  with an amplitude dependence on  $J_y^2$ . Figure 5.8 shows a frequency map [67] from a simulation that includes decapolar field errors, where their impact on the beam dynamics is clearly visible. Multiple particles were tracked under varying conditions within the bunch distribution. The color map represents tune diffusion over the course of the simulation, with blue indicating small diffusion and stable particle trajectories, while red represents large diffusion, suggesting unstable or chaotic trajectories that are unlikely to survive in a real machine.

Measurements were taken for the first time in the LHC to observe the  $f_{1004}$  RDT at injection energy. The frequency line of the resonance  $1Q_x - 4Q_y$  is seen at  $4Q_y$  in the horizontal spectrum, as shown Fig. 5.9.

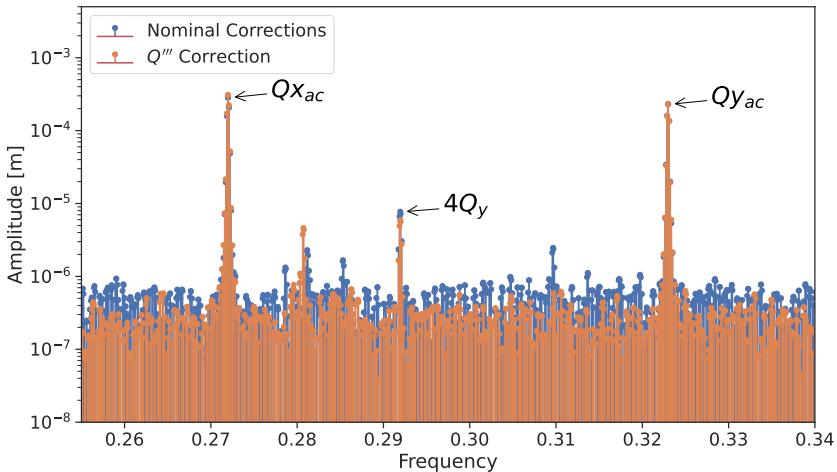


Figure 5.9.: Horizontal frequency spectrum of turn-by-turn data, with nominal and beam-based corrections for the third order chromaticity  $Q'''$ . The  $1Q_x - 4Q_y$  resonance can be seen at  $4Q_y$  with different amplitudes for each correction scheme.

### 5.6.1. Measurement and Correction

Decapolar fields are expected to be the main contributors to the RDT  $f_{1004}$ . As such, powering the decapolar correctors is a good way to correct the related resonances. Being linear with the strength of the correctors, the RDT can be corrected via a response matrix, as previously detailed in Section 3.3.1.

Measurements were conducted to establish a baseline for the amplitude of the RDT without octupolar or decapolar correctors and with the nominal FiDeL corrections. Such measurement is shown in Fig. 5.10 for Beam 1, similar results are observed for Beam 2. It can be there observed that the FiDeL corrections degrade the RDT.

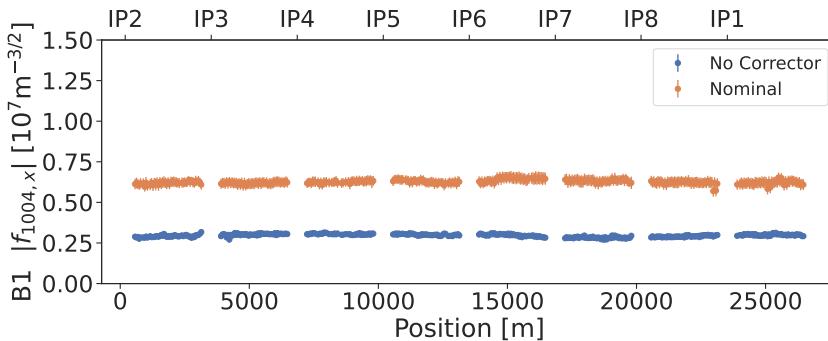


Figure 5.10.: Amplitude of the decapolar RDT  $f_{1004}$  measured with nominal FiDeL corrections for  $Q'''$  and without.

Corrections were then attempted based on the measurements obtained with the nominal FiDeL settings and were subsequently applied on top of these measurements. The strength of the decapolar correctors is shown in Table 5.8 for the FiDeL settings, the delta applied on top, and the final correctors values.

This RDT correction also serves as a partial  $Q'''$  correction. To fully correct  $Q'''$  indeed approximately requires a strength of +13,000  $K_5$  distributed amongst the correctors. Therefore, this new approach reduces  $Q'''$  by about 70% compared to the previous method. The chromatic amplitude detuning terms are also expected to be decreased. Result of these measurements, as well as the inverse of the correction, are shown in Fig. 5.11.

Circuit	$\text{FiDeL } K_5 [\text{m}^{-5}]$	$\Delta K_5 [\text{m}^{-5}]$	$K_5 [\text{m}^{-5}]$
<b>Beam 1</b>			
RCD.A12B1	-4582	6055	1473
RCD.A23B1	-5106	7	-5099
RCD.A34B1	-4855	3827	-1028
RCD.A45B1	-4577	-4746	-9323
RCD.A56B1	-4125	-4903	-9028
RCD.A67B1	-5166	2961	-2205
RCD.A78B1	-6827	3593	-3234
RCD.A81B1	-5500	2380	-3120
Total	-40738	9174	-31564
<b>Beam 2</b>			
RCD.A12B2	-4490	3639	-851
RCD.A23B2	-5155	-1147	-6302
RCD.A34B2	-4825	-1038	-5863
RCD.A45B2	-4619	3986	-633
RCD.A56B2	-4064	2944	-1120
RCD.A67B2	-5066	2357	-2709
RCD.A78B2	-6866	-2952	-9818
RCD.A81B2	-5446	1825	-3621
Total	-40531	9614	-30917

Table 5.8.: Strength of decapolar correctors with nominal FiDeL settings and after application of corrections aiming at reducing both the RDT  $f_{1004}$  and the third order chromaticity  $Q'''$ . The total value has a direct incidence on  $Q'''$ .

Although the FiDeL scheme was not intended to correct the RDT but rather only from  $Q'$  to  $Q'''$ , it would be expected for it to lower the amplitude of the RDT. It can though be seen that it degrades the resonance compared to the machine with no decapolar correctors. On the other hand, the newly computed RDT correction does lower the amplitude of  $f_{1004}$  as expected. Its inverse has the opposite effect.

Simulations were run with decapolar correctors turned off and with the RDT correction. The response of the RDT between these two schemes is shown in Fig. 5.12. The difference between their RMS value ratio is  $\approx 6\%$ , indicating that simulations correctly model the decapolar correctors.

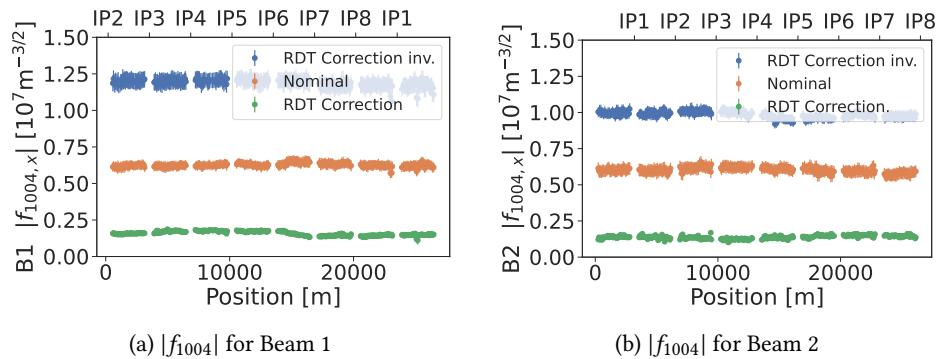


Figure 5.11.: Measured  $f_{1004}$  with nominal settings, and combined RDT &  $Q'''$  correction with normal and opposite signs.

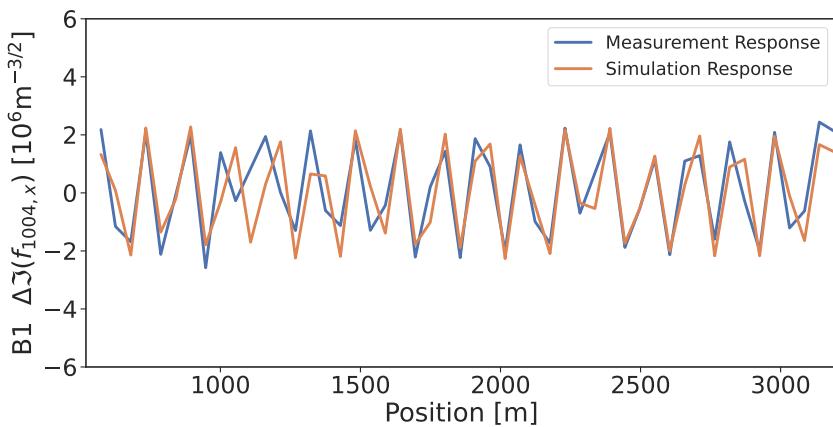


Figure 5.12.: Comparison for measurement and simulation of the response of the imaginary part of  $f_{1004}$  upon application on unpowered correctors of the RDT corrections.

In order to understand what can be gained from correcting decapolar fields, a lifetime measurement was taken with the corrections previously described in Section 5.6.1. This scheme corrects the three decapolar observables, being the RDT  $f_{1004}$  linked to the

resonance  $1Q_x - 4Q_y$ , the third order chromaticity  $Q'''$  and the chromatic amplitude detuning terms. Figure 5.13 shows the evolution of the lifetime, starting with corrections applied, removed and then trimmed to their opposite. A net change in lifetime for Beam 1 can be measured after each application. Acquired signal for Beam 2 has been deemed too noisy to be relevant, due to the shortness of the measurement.

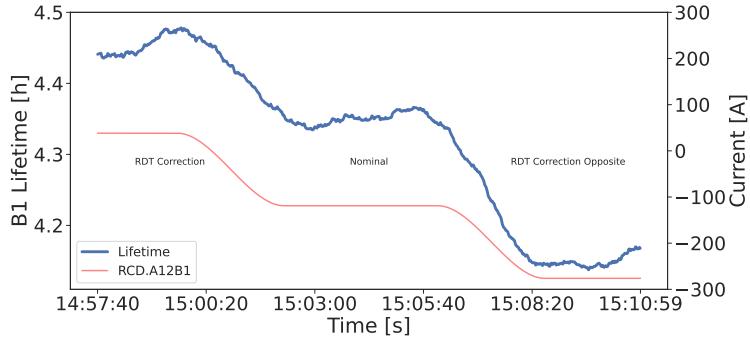


Figure 5.13.: Measured lifetime of Beam 1 with the nominal corrections for  $Q'''$ , combined correction of  $f_{1004}$  and  $Q'''$ , and its inverse.

It is apparent here that the corrections have a beneficial effect on the beam. The lifetime improvement is of  $\approx 3\%$ , while the degradation after applying the opposite is of  $\approx 5\%$ . Further developments in the correction scheme and lengthier measurements could easily improve this lifetime gain.

## 5.6.2. Feed-Up Contributions

### First Observation

As described in Appendix A, multipoles can combine to create perturbations that are seen as higher orders when considering higher orders of the BCH expansion. For decapolar RDTs, combinations of several sextupoles and sextupoles with octupoles give rise to decapolar-like fields, as described in Table A.3. Such contributions were already observed in the LHC with sextupoles contributing to amplitude detuning [68].

The effective Hamiltonian  $h$  of the transfer map of two elements  $h_1$  and  $h_2$ ,  $e^{\cdot h_1} \cdot e^{\cdot h_2} = e^{\cdot h}$ , can be expanded to an arbitrary order, which will then create perturbations akin to certain multipoles, as reminded by the following,

$$\begin{aligned} h &= h_1 + h_2 && \Rightarrow 1^{\text{st}} \text{ order} \\ &+ \frac{1}{2}[h_1, h_2] && \Rightarrow 2^{\text{nd}} \text{ order} \\ &+ \frac{1}{12}[h_1, [h_1, h_2]] - \frac{1}{12}[h_2, [h_1, h_2]] && \Rightarrow 3^{\text{rd}} \text{ order} \\ &+ \dots, \end{aligned}$$

where sextupoles would be expected to contribute to decapolar RDTs at the third order, and a combination of sextupoles and octupoles to the second.

In previous section, it was observed that the RDT response of decapolar correctors was properly modelled. However, RDT measurements taken before and after corrections of the non-linear chromaticity  $Q''$  and  $Q'''$  during the 2022's commissioning exhibited an unexpected RDT behavior. As the non-linear chromaticity corrections were applied, it was expected that the RDT  $f_{1004}$  would also lower with the reduction of the decapolar strengths  $K_5$ . However, an increase of the RDT was observed, as shows Fig. 5.14.

While decapolar errors are expected to be the main contributors to decapolar RDTs, other strong sources can indeed be identified. Figure 5.15 shows the average amplitude of the RDT  $f_{1004}$  depending on the error sources introduced in the simulations. Significant contributions to the Resonance Driving Terms (RDTs) appear to come from the sextupolar and octupolar field errors present in the main dipoles. Interestingly, the octupolar errors induce an RDT that is even larger than that generated by the decapolar errors. The following sections will provide a detailed analysis of the contributions from the sextupolar and octupolar field components to this RDT, before presenting experimental measurements.

## 5. Decapolar Fields

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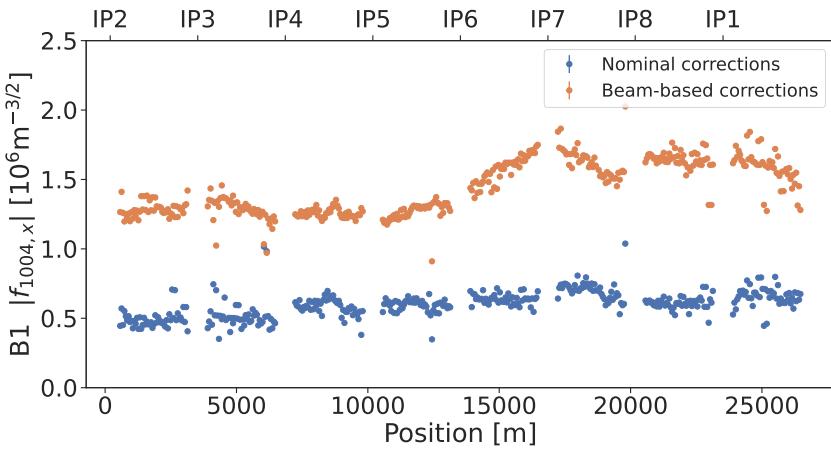


Figure 5.14.: Non-intuitive increase of the RDT  $f_{1004}$  after application of both the  $Q''$  and  $Q'''$  corrections.

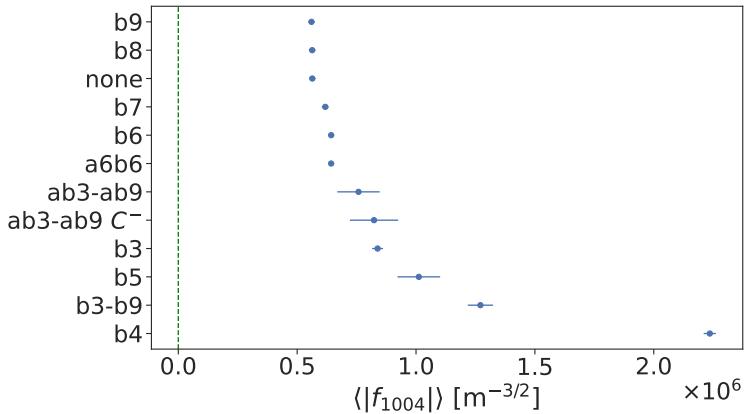


Figure 5.15.: Simulation of the amplitude of the decapolar RDT  $f_{1004}$  depending on the field errors applied on main dipoles as well as coupling ( $C^-$ ). Sextupolar ( $b_3$ ) and octupolar ( $b_4$ ) fields have a clear impact on this amplitude.

## Combination of Sextupoles

At the third order of the BCH expansion, the combination of two sextupoles yields a decapolar-like expression. This means that, during normal operation of the machine, decapolar observables will be altered when adjusting parameters such as the linear chromaticity  $Q'$ . Derivation of such a combination can be found in Appendix A.2.2. The resulting Hamiltonian indeed is similar to the terms of a decapole, dropping the  $p_{x,y}$  terms for readability:

$$(H_3)^3 \propto \frac{1}{48} (x^5 - 2x^3y^2 - 3xy^4) \\ \sim x^5 - 10x^3y^2 + 5xy^4. \quad (5.5)$$

To quantify the actual impact of such an equation on the LHC, a simulation was run with injection optics while varying this same linear chromaticity  $Q'$ . No field components higher than sextupoles, nor any additional field errors, have been introduced. The resulting effect on the RDT  $f_{1004}$  can be seen in Fig. 5.16.

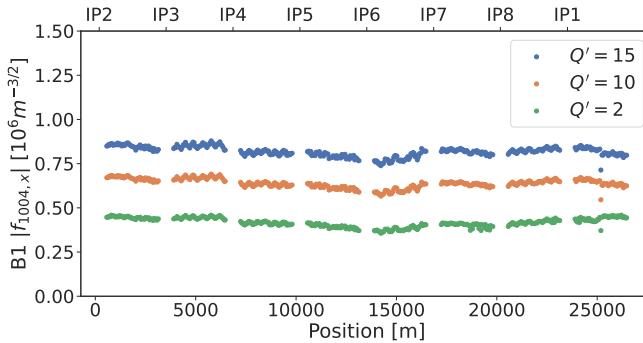


Figure 5.16.: Simulated change of the decapolar RDT  $f_{1004}$  with varying linear chromaticity  $Q'$  generated by sextupoles. The combination of sextupolar fields clearly shows an increase in decapolar RDT.

As the overall  $K_3$  strength of sextupoles changes, so does the linear chromaticity. Considering the previous Eq. (5.5), a higher chromaticity is expected to increase, in that

configuration of the LHC, the amplitude of the RDT  $f_{1004}$ , related to the last term  $xy^4$ . Figure 5.17 shows how the RDT is expected to vary, depending on the overall sextupoles strength and the linear chromaticity. It can be noted that although the relation between  $K_3$  and  $Q'$  is linear, that of  $K_3$  and the RDT varies with the cubed strength. Using the sum of the cubed strength is possible due to the chromaticity knob being a factor applied on all sextupoles at the same time.

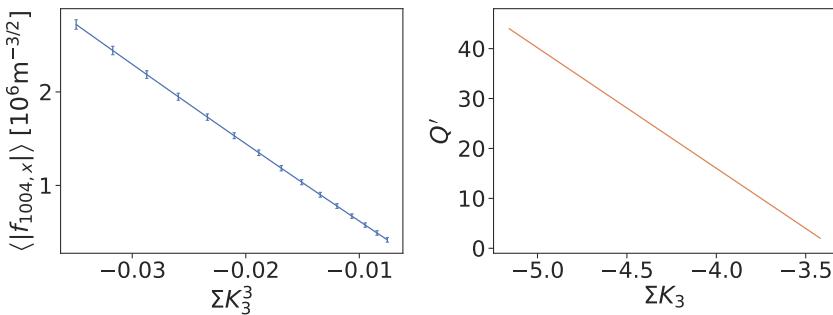


Figure 5.17.: Average amplitude of the decapolar RDT  $f_{1004}$  depending on the overall strength of the sextupoles used to control the linear chromaticity  $Q'$ . This two-plot combination is only valid for a certain LHC configuration.

## Combination Sextupoles and Octupoles

At the second order of the BCH expansion, the combination of a sextupole and an octupole yields a decapolar-like expression. Like sextupoles, octupoles are used in operation, thus contributing to decapolar RDTs. This happens, amongst other, when correcting the second order chromaticity  $Q''$  and most importantly with the Landau Octupoles, which are powered to high strengths at high strengths through the whole LHC cycle to introduce Landau damping [5]. Derivation of such a combination can be found in Appendix A.2.3. The resulting Hamiltonian indeed is similar to the terms of a decapole, dropping the  $p_{x,y}$  terms for readability:

$$H_3H_4 \propto \frac{1}{24} (x^5 + 2x^3y^2 + xy^4) \\ \sim x^5 - 10x^3y^2 + 5xy^4. \quad (5.6)$$

In order to assess the previous equation, simulations were run with several configurations. A set of two configurations was run to check the impact of octupoles alone. The first configuration is run with all sextupoles of the machine turned off, while octupoles are powered. The second configuration turns off all sextupoles and octupoles. Figure 5.18 shows the resulting RDT  $f_{1004}$  from these simulations. It is there apparent that varying octupoles without sextupoles does not have any effect on this RDT.

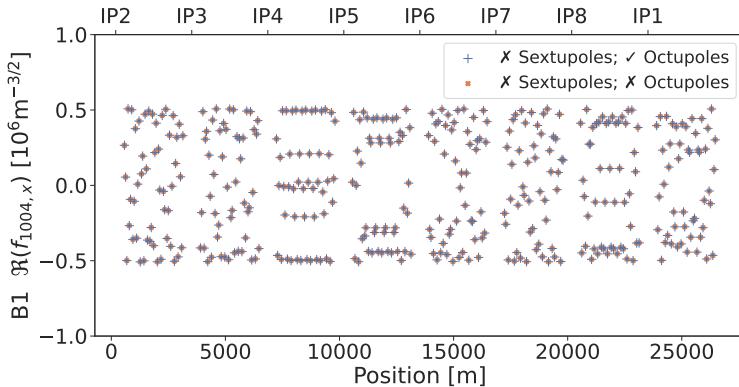


Figure 5.18.: Simulated decapolar RDT  $f_{1004}$  with two different schemes. First scheme has lattice sextupoles turned off and octupoles turned on. Second scheme has all sextupoles of the lattice turned off and octupoles turned off as well. No difference is seen, as expected from the equations.

The most powerful octupoles used in operation are the lattice octupoles, used for Landau damping. Figure 5.19 shows a simulation ran with varying strengths of those magnets. It can be noted here that the shift of the RDT is almost of an order of magnitude, making octupoles a large contributor to the decapolar fields.

Large RDT shifts due to octupoles are relevant to the operation of the LHC, as resonances can be greatly deteriorated, especially when powering landau octupoles.

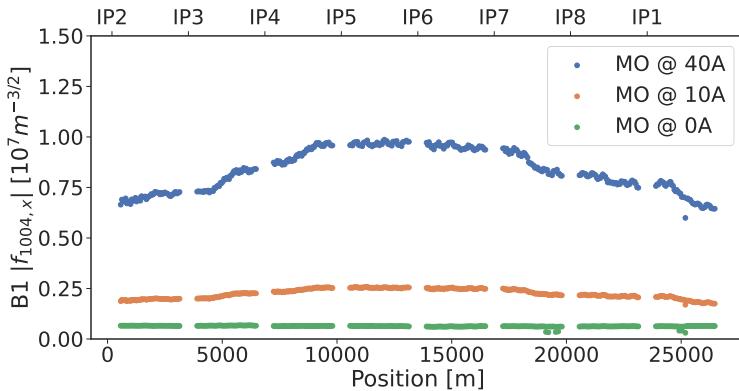


Figure 5.19.: Simulated change of the decapolar RDT  $f_{1004}$  depending on the strength of the lattice octupoles used for Landau damping.

Factor	RMS $ f_{1004} $
-10	37,308,159
-4	6,721,270
0	3,533,796
-1	2,333,384

Table 5.9.: RMS of  $|f_{1004}|$  depending on the factor of the  $Q''$  corrections.

A better understanding of the interaction between Landau octupoles and octupolar correctors could lead to improved corrections in not only octupolar but also decapolar fields in the future.

## Experimental Measurements

To confirm what is observed in simulations, measurements were performed by varying  $Q'$  and kicking the beam with the AC-Dipole. Limited by losses, up to three measurements with distinct  $Q'$  were taken, as shows Fig. 5.20.

Like in simulations, it is observed that an increase in  $Q'$  translates to an increase in

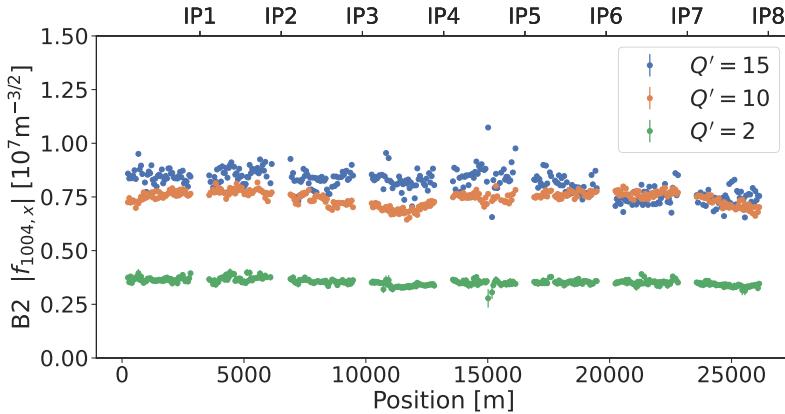


Figure 5.20.: Measured change of the decapolar RDT  $f_{1004}$  depending on the desired linear chromaticity  $Q'$  generated by sextupoles.

$|f_{1004}|$ . The observed scale of the amplitude is though one order of magnitude higher than that of simulations. An offset for all measurements could be explained by non-included field-errors. The shift between them however should be similar between machine and simulations, this could be due by the interaction of the sextupolar fields with octupoles, as detailed in the following section.

Measurements were performed to confirm and quantify the effect of octupoles coupled with sextupoles on the decapolar fields. Previous corrections, aimed at correcting the second order chromaticity  $Q''$ , via octupolar correctors MCO, were applied with varying factors. Such corrections use a uniform trim on all correctors of  $\approx +2.5K_4$ . Figure 5.21 shows a comparison of the resulting RDT with those corrections at factors  $-10, -4, -1$  and  $0$ .

Table 5.9 shows the RMS of the amplitude of this RDT for the various configurations. Similar to the shift observed when powering the Landau octupoles in simulations, the shift is of one order of magnitude between factors  $-0$  and  $-10$ . Measurements with Landau octupoles were also attempted, but losses made it impossible to obtain high enough amplitudes to correctly measure the RDT.

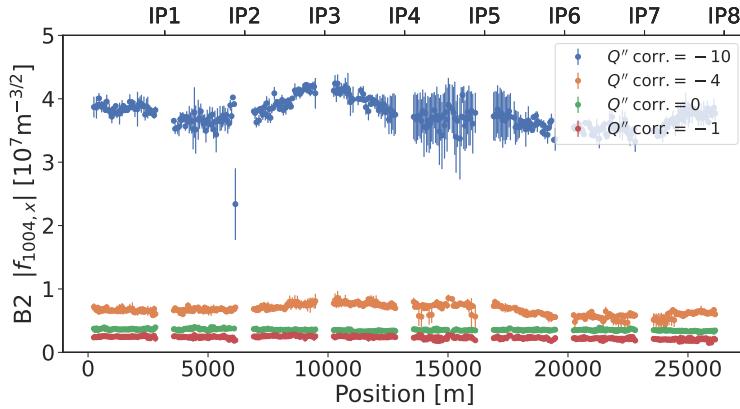


Figure 5.21.: Shift of the decapolar RDT  $f_{1004}$  depending on the factor applied on octupolar corrections for  $Q''$ .

### 5.6.3. Replicating the Landau Octupoles Contribution

$\Delta K_5$	RMS $ f_{1004} $
0	618,947
$\pm 10500$	17,566,377
$\mp 10500$	17,623,867

Table 5.10.: RMS of  $|f_{1004}|$  relative to the powering scheme of decapolar correctors.

As seen previously in Fig. 5.8, the resonance  $1Q_x - 4Q_y$  passes through the beam in tune space, deteriorating the lifetime of the nearby particles. In order to measure the impact of this resonance on the beam, a knob was created, alternating the current of all decapole correctors in the machine arc by arc. Such a powering scheme has no impact on chromaticity as the sum of the strengths  $K_5$  is zero. Rather, the RDT  $f_{1004}$  is impacted. Is it to be noted that this is not a correction, but purely a way to artificially increase the RDT in order to quantify the effect of the resonance.

Starting with nominal corrections for  $Q'''$ , a delta of  $\pm 10500$   $K_5$  is applied on each decapolar correctors. Figure 5.22 shows the response of the real part of the RDT for this

scheme and its inverse. The amplitude of the RDT is on a similar level as the shift is significantly larger than the original level of the RDT. Table 5.10 indicates the amplitude of the RDT created with each knob value.

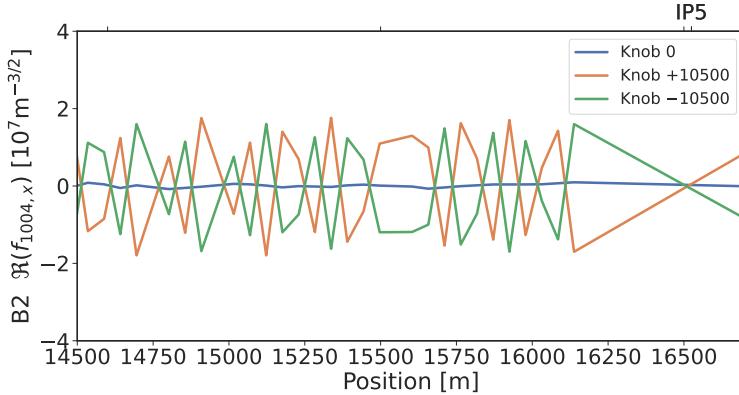


Figure 5.22.: Measured real part of the RDT  $f_{1004}$  depending on the powering scheme of the decapolar correctors.

In order to measure the lifetime, a long time window is allocated to capture a clean signal, as that returned from monitors can be jittery. The beam lifetime is expressed in hours and decreases over time due to particles being lost. It is adjusted in real-time by the beam loss monitors. Figure 5.23 shows this lifetime depending on the decapolar strength scheme applied. The current of only one circuit is shown for readability. A current of  $\approx 230$  A corresponds to a knob value of  $+10500 K_5$  while a current of  $-45$  A corresponds to 0.

It is clear from this measurement that a large RDT decreases the lifetime of the beam, more particles being lost. The first pair of trim sees the average lifetime decreasing of  $0.31 \pm 0.03$  hours, while the second one sees a decrease of  $0.36 \pm 0.03$  hours. This observed decrease of 20 minutes accounts for 10% of the beam lifetime at injection energy.

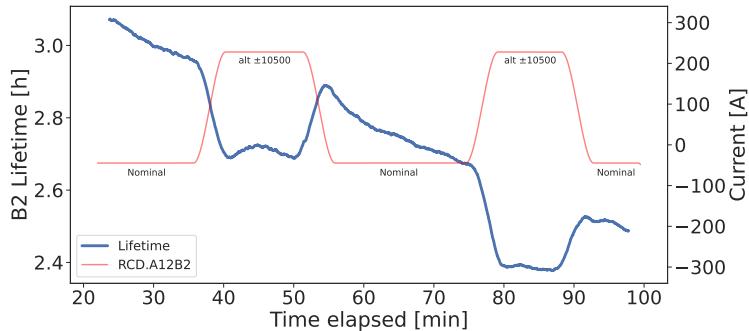


Figure 5.23.: Measured lifetime of Beam 2 upon application of two different powering schemes for decapolar correctors. One trim keeps the RDT  $f_{1004}$  at a low amplitude while the other greatly amplifies it.

## 5.7. Summary

The corrections of third order chromaticity  $Q'''$  has been observed in the past to show discrepancies to the magnetic model at injection energy, prompting an investigation of the decapolar errors using various observables to identify their sources. To better understand these discrepancies, novel observables such as bare chromaticity and chromatic amplitude detuning were introduced. Chromatic amplitude detuning, which accounts for the detuning via both horizontal and vertical actions with to momentum offset, differs in expression from standard chromaticity and provides a more comprehensive view of decapolar errors. This observable was measured for the first time at the LHC. Simulations suggest that the decay of the decapolar component in the main dipoles is the major factor contributing to these discrepancies. This component had previously been neglected in both beam dynamics studies and operational correction strategies.

For the first time, measurements and direct corrections of the decapolar Resonance Driving Term (RDT)  $f_{1004}$  were carried out. Further simulations and measurements explored how sextupoles and octupoles interact to create decapolar-like fields. The findings revealed that sextupoles, both alone and in combination with Landau octupoles, generate substantial decapolar RDTs via feed-up during machine operation that could benefit from corrections.

Applying combined corrections for third-order chromaticity, chromatic amplitude detuning, and the RDT  $f_{1004}$  led to a 3% improvement in beam lifetime. Additionally, a broader impact of decapolar RDTs on beam stability was investigated. Specifically, intentionally degrading the RDT  $f_{1004}$  resulted in a decrease in beam lifetime of about 10%. Furthermore, a significant improvement in the forced dynamic aperture during AC-Dipole excitation was observed when corrections were applied. The earlier corrections for decapolar errors were implemented alongside octupolar corrections for  $Q''$ . Figure 5.24 demonstrates the relative losses encountered when kicking at different amplitudes with the AC-Dipole. This clearly shows that octupolar and decapolar corrections enhance the forced dynamic aperture.

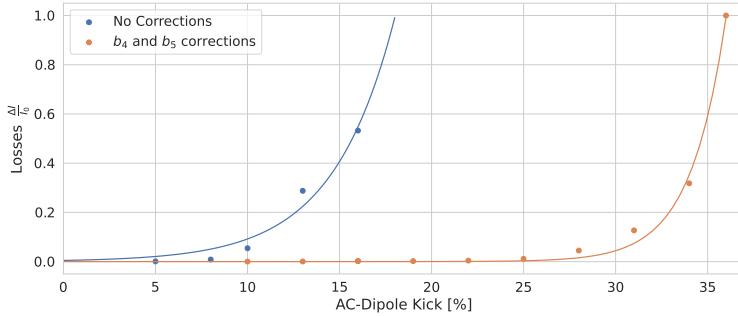


Figure 5.24.: Relative losses experienced with AC-Dipole kick amplitude with and without corrections pertaining to  $Q''$ ,  $Q'''$ , chromatic amplitude detuning, and RDT  $f_{1004}$ .

Initially, the feed-up effect was not well understood, leading to the decision to exclude any decapolar corrections from the machine. However, through studies of decapolar perturbations and the interaction of various multipoles, significant progress was made. The magnetic model for  $b_5$  errors is now more complete, enabling the implementation of these corrections in the machine, which has resulted in both increased lifetime and the possibility of higher-amplitude kicks for non-linear optics studies. Advancements in understanding the magnetic model and controlling decapolar errors are crucial not only for the HL-LHC but also for the FCC-ee, which will depend on precise control of  $b_5$  errors to optimize dynamic aperture.



# Higher-Order Fields

## 6.1. Introduction

The preceding chapters of this thesis presented significant advancements in the understanding and modeling of skew octupoles and normal decapoles, along with the consequent improvements in dynamic aperture through their correction. Non-linear optics measurements have been conducted in the LHC since its initial Run [55, 60], though only recently has attention been focused on higher-order multipoles, such as dodecapoles [69]. These indirect measurements, which utilize feed-down to amplitude detuning, are limited by their sensitivity to local errors and are applicable only in the interaction regions.

Given that field errors of this nature are anticipated to have a significant impact on beam dynamics and dynamic aperture in future accelerators such as the FCC, it becomes essential to develop methods for their study. This will not only allow benchmarking but also enhance the global magnetic model of the LHC.

Recent attempts to measure high-order multipoles have been made possible by the improvements in dynamic aperture mentioned earlier, which allowed for larger oscillation amplitudes. Additionally, the introduction of a new collimator sequence, developed by collimation experts, enabled measurements at unprecedented amplitude levels, thus allowing for higher-order multipoles to be investigated.

For the first time in the LHC, direct measurements of dodecapolar Resonance Driving Terms (RDTs) were successfully performed at injection energy, utilizing high-amplitude kicks provided by the AC-Dipole. Furthermore, fourth and fifth order chromaticities, which are related to dodecapolar and decatetrapolar errors, were also measured for the first time at injection energy. These direct measurements of novel observables related to very-high-order multipoles open up new avenues for improving the understanding of non-linear errors in the LHC, as well as for refining the techniques involved in their measurement and analysis, that could benefit future machines.

## 6.2. Dodecapolar Resonance Driving Term

During the 2024 commissioning, Resonance Driving Term measurements were conducted by applying kicks of varying strengths using the AC-Dipole, at injection energy. These measurements aimed to measure several RDTs, with a particular focus on decapoles and their associated corrections. The kicks were performed initially with nominal FiDeL settings for the octupolar and decapolar correctors, followed by additional  $Q''$  then  $Q'''$  and  $Q''''$  combined corrections.

Thanks to improvements in dynamic aperture resulting from a better understanding of lower-order multipoles, the high kick amplitudes enabled the observation of a distinct line in the vertical spectrum at  $5Q_y$ , as shown in Fig. 6.1. This line is attributed to the presence of dodecapolar fields (see Appendix C) and its amplitude scales with the vertical action.

To achieve these measurements, the kick strength of the AC-Dipole was set up to 40% of its maximum. The specific excited resonance of the observed line is the  $6Q_y$ , related to the RDT  $f_{0060}$ . Figure 6.2 highlights the real part of this RDT taken with nominal corrections at various kick percentage strengths for a given arc of the LHC. The clear observation of lines in the frequency spectrum and the agreement between the various kicks at different actions display the reliability of the measurement. The RDT reconstructed via the previous kicks is shown in Fig. 6.3, for which the RMS amplitude is of  $3.5 \cdot 10^8$  [ $\text{m}^{-2}$ ].

Similar to the decapoles discussed in Section 5.6.2, the dodecapolar RDTs can be influenced by lower-order components, as shown in Table A.3. At the second-order

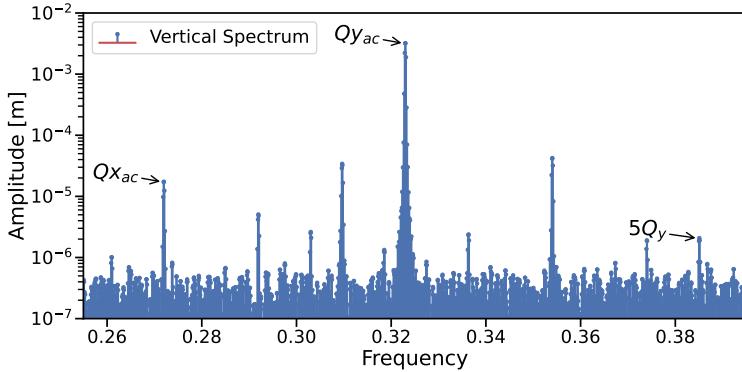


Figure 6.1.: Vertical spectrum of recorded turn-by-turn data for Beam 1 showing the tunes driven by the AC-Dipole along with a line contributed to by dodecapolar fields.

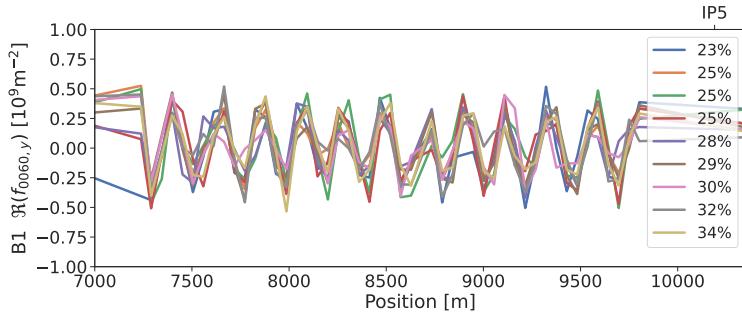


Figure 6.2.: Real part of the dodecapolar RDT  $f_{0060}$  measured with several kick strengths.

BCH, a combination of sextupoles and decapoles, as well as a combination of octupoles, generate a dodecapolar-like perturbations. At the third and fourth orders, these fields are respectively generated by a combination of sextupoles with octupoles and by sextupoles. Several tracking simulations were performed with various combinations of field errors, ranging from normal and skew sextupoles ( $a, b_3$ ) to decaoctupoles ( $a, b_9$ ), including as well coupling. The  $b_2$  errors, in the main dipoles and quadrupoles, are not added as they generate a large, unrealistic, phase error. Figure 6.4 shows the RMS amplitude of

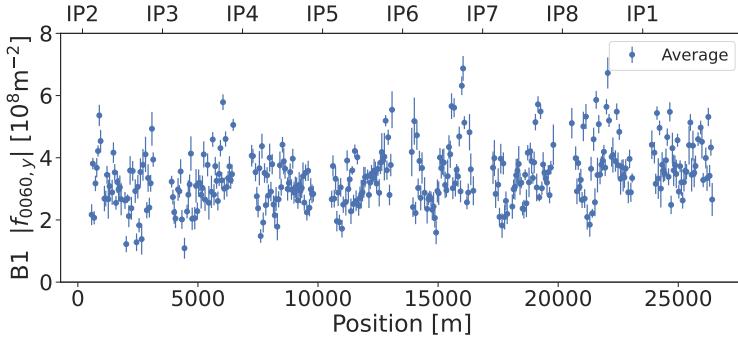


Figure 6.3.: Amplitude of the dodecapolar RDT  $f_{0060}$ . The RMS amplitude is of  $3.5 \cdot 10^8 \text{ [m}^{-2}\text{]}$ .

the RDT  $f_{0060}$  for each of these simulations. As could be expected from the analytical equations, the lower-order multipoles do contribute to this RDT and seem to cancel the  $b_6$  component. It is though apparent that Beam 1 and Beam 2 do not have the same contributions from the dodecapolar errors.

The previously referenced measurements are also compared to these simulations. The measurement for Beam 2 with the nominal corrections could not be reliably exploited and is not included. The RDTs from these measurements are similar in amplitude, despite the different configurations for the octupolar and decapolar correctors. This suggests that, at these strength variations, the high-order correctors do not significantly affect the dodecapolar RDT  $f_{0060}$ . It is however important to note that predicting the contributions from lower-order multipoles remains difficult.

### 6.3. Chromaticity

As described in Section 3.2.3, the momentum offset  $\delta$  is related to the RF frequency and the momentum compaction factor. This relation is given as a simplified form in Eq. (6.1), neglecting the Lorentz factor  $\gamma$ . The model  $\alpha_c$  for the LHC injection optics is  $3.48 \times 10^{-4}$  for beam 1 and  $3.47 \times 10^{-4}$  for beam 2. Via this relation, a change of 140Hz of the RF frequency corresponds to a momentum offset of about  $-0.001$ .

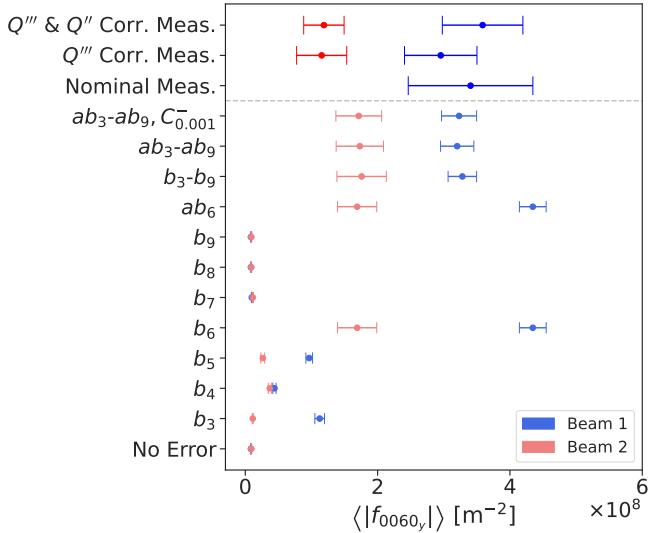


Figure 6.4.: Measured and simulated RDT  $f_{0060}$  with various normal and skew field errors. Coupling ( $C^-$ ) is set to a common value seen in operation.

$$\delta = -\frac{1}{\alpha_c} \cdot \frac{\Delta f_{RF}}{f_{RF,nominal}}. \quad (6.1)$$

The RF frequency is thus varied to induce a change in momentum-offset. The tune will then vary with this momentum-offset, as shows the chromaticity function expanded up to the fifth order,

$$Q(\delta) = Q_0 + Q'\delta + \frac{1}{2!}Q''\delta^2 + \frac{1}{3!}Q'''\delta^3 + \frac{1}{4!}Q^{(4)}\delta^4 + \frac{1}{5!}Q^{(5)}\delta^5 + O(\delta^6). \quad (6.2)$$

### 6.3.1. First Observation

Before focusing on the measurement of higher-order chromaticity such as  $Q^{(5)}$ , it is essential to consider the procedural improvements that enable such measurements.

Improvements in dynamic aperture, collimation and signal processing with noise reduction now allow for larger oscillation amplitudes and enhancements in the precision of measurements across broader amplitude ranges. This increased resolution facilitates the study of non-linear chromaticity, which has been extended to include higher-order terms, making it possible to detect contributions from dodecapolar and decatetrapolar fields. These developments are particularly interesting because they suggest the possibility of directly measuring  $Q^{(5)}$ , an observable previously inaccessible due to narrower momentum offset ranges.

A measurement was performed with the octupolar and decapolar correctors *MCO* and *MCD* set to their nominal settings. These are aimed at correcting  $Q''$  and  $Q'''$ , as previously described in Chapter 5. Results of this initial measurement are shown in Table 6.1. Lower order chromaticities such as  $Q'$  and  $Q''$  are consistent with measurements done during the previous Run [61]. The fourth and fifth order chromaticities,  $Q^{(4)}$  and  $Q^{(5)}$ , are primarily expected to originate from dodecapolar errors in the quadrupoles and decatetrapolar errors in the main dipoles, respectively, as given by the following expressions:

$$\begin{aligned}\Delta Q_x^{(4)} &= \frac{1}{4\pi} K_6 L \beta_x D_x^4, & \Delta Q_x^{(5)} &= \frac{1}{4\pi} K_7 L \beta_x D_x^5, \\ \Delta Q_y^{(4)} &= -\frac{1}{4\pi} K_6 L \beta_x D_x^4, & \Delta Q_x^{(5)} &= -\frac{1}{4\pi} K_7 L \beta_x D_x^5.\end{aligned}\tag{6.3}$$

Plane	$Q^{(2)}[10^3]$	$Q^{(3)}[10^6]$	$Q^{(4)}[10^9]$	$Q^{(5)}[10^{12}]$
<b>Beam 1</b>				
X	$-2.44 \pm 0.02$	$-3.36 \pm 0.04$	$-0.56 \pm 0.02$	$1.20 \pm 0.07$
Y	$0.97 \pm 0.02$	$1.62 \pm 0.05$	$0.15 \pm 0.03$	$-0.88 \pm 0.09$
<b>Beam 2</b>				
X	$-2.45 \pm 0.03$	$-2.72 \pm 0.08$	$-1.00 \pm 0.05$	$0.15 \pm 0.14$
Y	$0.79 \pm 0.03$	$1.54 \pm 0.06$	$0.24 \pm 0.04$	$-0.74 \pm 0.13$

Table 6.1.: Terms of the high order chromaticity obtained during Run 3's commissioning in 2022, with nominal corrections.

Due to the RF-scan method, the momentum offset crosses zero multiple times during the measurement. The absence of tune change at these points allows to conclude that the

tune drift throughout the measurement is negligible. This measurement was conducted after an extended period at injection energy, where the decay of the sextupolar fields is minimal, causing no significant change in first-order chromaticity. The fitted curve for the chromaticity function is shown in Fig. 6.5, where it is evident that a higher-order polynomial provides a better fit.

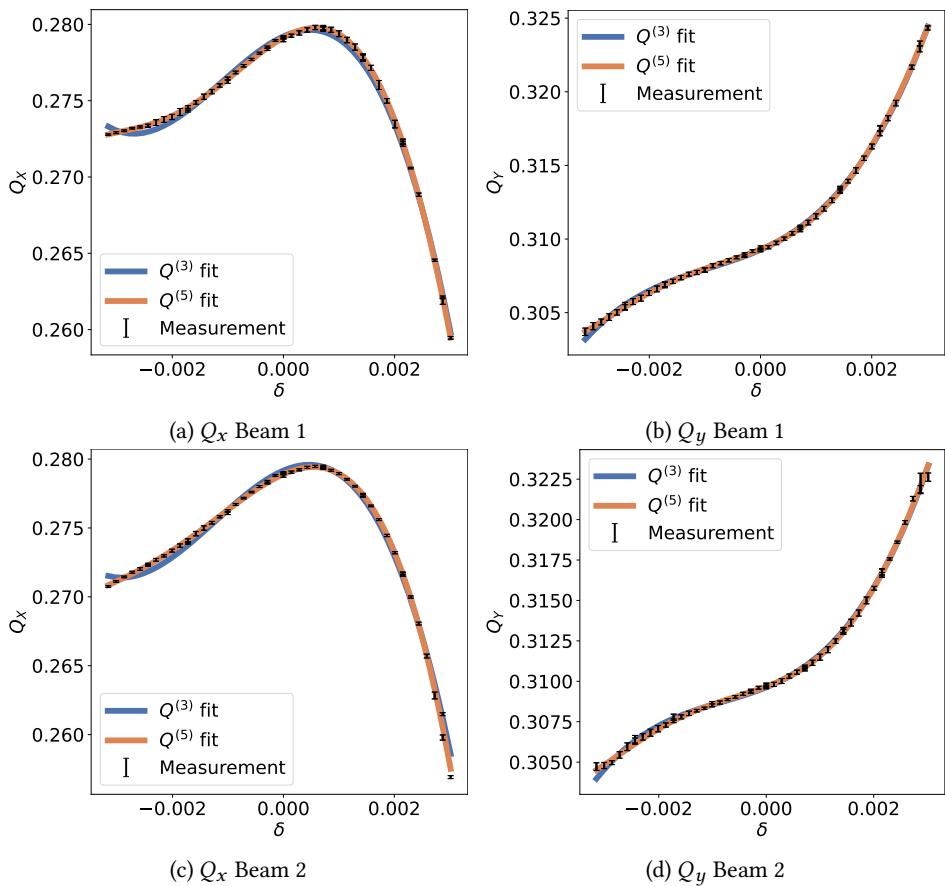


Figure 6.5.: Measurement of higher order chromaticity terms with nominal corrections used during operation. Fits are up to the third and fifth order.

The reduced chi-square for this measurement of 2022 for each fit order is detailed in Table 6.2. A value of 1 indicates a good fit. Values above indicate a poor fit while values below show an over-fit of the data. While adding terms to the chromaticity function is beneficial to the fit, it can be seen that the reduced chi-square does not substantially improve above the fifth order, indicating that such further orders are not warranted.

Plane	$Q^{(3)}$	$Q^{(4)}$	$\chi^2_\nu$	$Q^{(5)}$	$Q^{(6)}$
Beam 1					
X	17.9	12.1	1.8	1.5	
Y	3.0	2.2	0.7	0.7	
Beam 2					
X	17.3	7.1	1.8	1.8	
Y	2.9	2.8	1.0	1.0	

Table 6.2.: Reduced  $\chi^2$  values for each order of fit of the measured non-linear chromaticity, with  $Q''$  and  $Q'''$  beam-based corrections.

Given the novelty of measuring such high-order chromaticities, additional configurations were tested to ensure the consistency of the results. Potential sources contributing to high-order chromaticity, such as the non-linearities of the momentum compaction factor, dispersion, coupling, or noise, are discussed in the following sections. Comparing the fitted chromaticity across different LHC configurations allows to rule out potential issues stemming from the machine's specific state on the day of measurement.

## Momentum Compaction Factor

Rather than a constant, the momentum compaction factor can be expressed as an expansion, as detailed in Section 2.5.1. The first terms are given by the following,

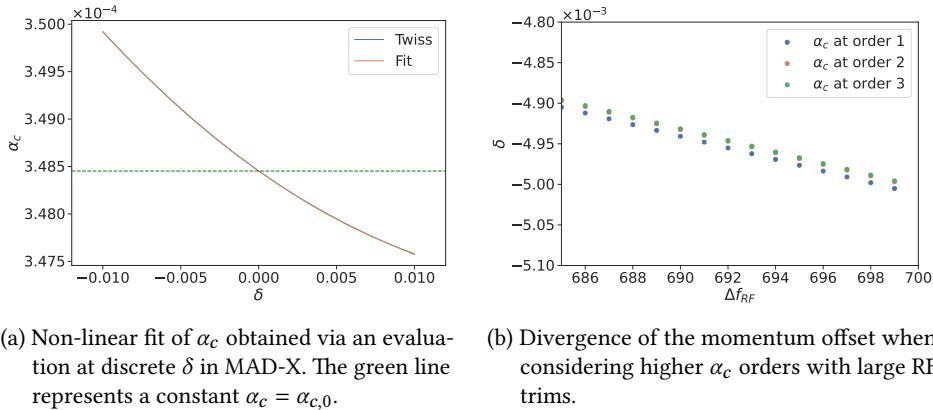
$$\alpha_c = \underbrace{\alpha_{c,0}}_{1^{\text{st}} \text{ order}} + \underbrace{\alpha_{c,1}\delta}_{2^{\text{nd}} \text{ order}} + \underbrace{\alpha_{c,2}\delta^2}_{3^{\text{rd}} \text{ order}} . \quad (6.4)$$

The expression for  $\delta$ , with  $\alpha_c$  at the first and second order in the RF formula (Eq. (6.1))

then reads,

$$\begin{aligned}\delta &= -\frac{\Delta f_{RF}}{\alpha_0 f_{RF}} && \Rightarrow \alpha_c \text{ at order 1} \\ \delta &= \frac{-\alpha_0 f_{RF} + \sqrt{f_{RF}(-4\Delta f_{RF}\alpha_1 + \alpha_0^2 f_{RF})}}{2\alpha_1 f_{RF}} && \Rightarrow \alpha_c \text{ at order 2}\end{aligned}\quad (6.5)$$

The momentum compaction factor is computed using MADX at discrete  $\delta$  values and then fitted, as shown in Fig. 6.6. Its non-linearity is clearly evident. The effect of this non-linearity on the calculated  $\delta$  via the RF frequency is also presented. Field errors have not been found to have any significant impact on the  $\alpha_c$  values. Previous studies indicated that the model  $\alpha_c$  is consistent with the one observed in the machine [70].



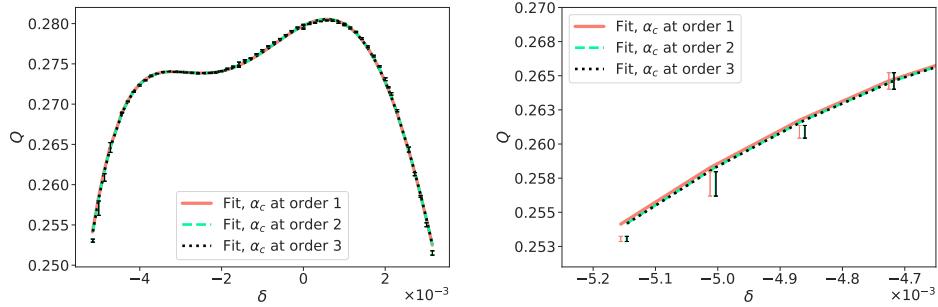
(a) Non-linear fit of  $\alpha_c$  obtained via an evaluation at discrete  $\delta$  in MAD-X. The green line represents a constant  $\alpha_c = \alpha_{c,0}$ .

(b) Divergence of the momentum offset when considering higher  $\alpha_c$  orders with large RF trims.

Figure 6.6.: Non linearity of  $\alpha_c$  and its effect on the computed  $\delta$  via RF trims. The simulations are done at injection energy of 450 GeV.

It is observed that while clearly depending on higher orders, the momentum compaction factor only has a small impact on the calculated  $\delta$ . Figure 6.7 shows a real-life measurement, comparing the fit of the chromaticity function with various  $\delta$ , computed up to the third order of  $\alpha_c$ .

The fit of the chromaticity function is barely impacted when considering the higher orders of the momentum compaction factor. The different orders of the chromaticity are



(a) Fit of the chromaticity function at the 5<sup>th</sup> order, considering the  $\alpha_c$  expansion up to the third order.

(b) Zoom on one side of the fit. The difference between the second and third order is barely noticeable.

Figure 6.7.: Fit of the chromaticity function considering several  $\alpha_c$  orders.

collected in Table 6.3. The higher order terms of  $\alpha_c$  can thus be neglected and are not a source of higher chromaticity orders.

Chromaticity	$\alpha_c$ order 1	$\alpha_c$ order 2	$\alpha_c$ order 3
$Q^{(1)}$	$2.52 \pm 0.03$	$2.53 \pm 0.03$	$2.53 \pm 0.03$
$Q^{(2)}$	$-3.04 \pm 0.05$	$-3.05 \pm 0.05$	$-3.05 \pm 0.05$
$Q^{(3)}$	$-4.75 \pm 0.03$	$-4.75 \pm 0.03$	$-4.75 \pm 0.03$
$Q^{(4)}$	$-0.33 \pm 0.07$	$-0.32 \pm 0.07$	$-0.32 \pm 0.07$
$Q^{(5)}$	$2.33 \pm 0.06$	$2.36 \pm 0.06$	$2.36 \pm 0.06$

Table 6.3.: Chromaticity values obtained for the same measurement, depending on the order of the momentum compaction factor taken into account.

## Noise and Spectral Lines

Noise lines from the electronics can be observed in the raw data obtained from the BBQ tune system. Occasionally, when these are strong, their frequency peaks can be mistaken for the actual tune and subsequently logged by the system. This leads to

significant uncertainties in the measurement, especially when these data points cannot be reliably classified as outliers. An example of a tune measurement affected by this issue is shown in Fig. 6.8. However, this incorrect tune identification was not problematic for measurements with narrower momentum offset ranges, as the induced tune shift was smaller and therefore the tune less likely to overlap with strong noise lines.

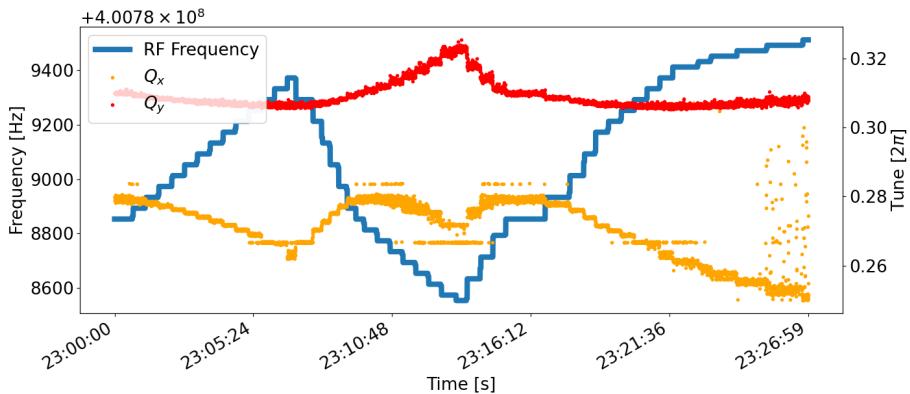


Figure 6.8.: Shift of the tune by variation of the RF. Noise lines can appear in some cases, making the tune error-bar large or downright unusable.

A solution to this issue is to use the raw data extracted from the BBQ system. From there, a spectrogram clearly shows the noise lines, as seen in Fig. 6.9. Those lines have been repeatedly identified over several measurements and confirmed to be static. The highest peak in the spectrogram can be reliably identified by removing unwanted lines, resulting in a cleaner measurement. It is also important to note that the BBQ system requires the tune window to be set, which, if overlooked, can lead to erroneous data. By analyzing the raw data, it is ensured that correct data is used, preventing the measurement from inadvertently capturing noise.

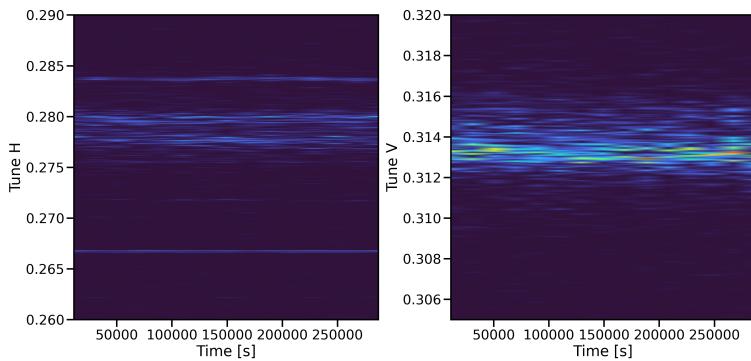


Figure 6.9.: Tune spectrogram obtained via the BBQ system. Strong noise lines can be seen above and below where the tune really is, causing the wrong frequency peak to be identified as the tune. These lines are always observed around  $Q = 0.266$  and  $Q = 0.284$

### Momentum Offset from Orbit

The momentum offset  $\delta$  can be reconstructed through several methods. One such method, based on beams's orbit, was previously stored by the LHC's logging services but is not anymore. Reconstructing the momentum offset can introduce uncertainties due to the non-linearities of both the momentum compaction factor, as discussed earlier, and the dispersion.

To evaluate this, a re-analysis of previous Run 2 measurements was conducted using the RF formula, with the goal of comparing the resulting non-linear chromaticity values. Figure 6.10 shows the chromaticity measurements for Beam 1 and Beam 2 in both planes, with chromaticity computed via both the orbit-based and RF-based methods. A numerical comparison is provided in Table 6.4.

The lack of discrepancy between the two techniques suggests that the measurements were operating within the linear regime of the dispersion and momentum compaction factor, and no detuning effects were present that could have been misinterpreted as a higher-order chromaticity.

### 6.3. Chromaticity

Plane	$\delta$ via RF		$\delta$ via orbit	
	$Q''[10^3]$	$Q'''[10^6]$	$Q''[10^3]$	$Q'''[10^6]$
Beam 1				
X	$-0.64 \pm 0.01$	$3.00 \pm 0.04$	$-0.62 \pm 0.01$	$2.91 \pm 0.04$
Y	$-0.17 \pm 0.01$	$-2.12 \pm 0.04$	$-0.14 \pm 0.01$	$-2.09 \pm 0.04$
Beam 2				
X	$-1.18 \pm 0.02$	$2.89 \pm 0.06$	$-1.23 \pm 0.03$	$3.13 \pm 0.11$
Y	$0.18 \pm 0.02$	$-1.95 \pm 0.05$	$0.20 \pm 0.02$	$-2.02 \pm 0.06$

Table 6.4.: Comparison of the chromaticity values obtained for the same measurement via two different methods to acquire  $\delta$ .

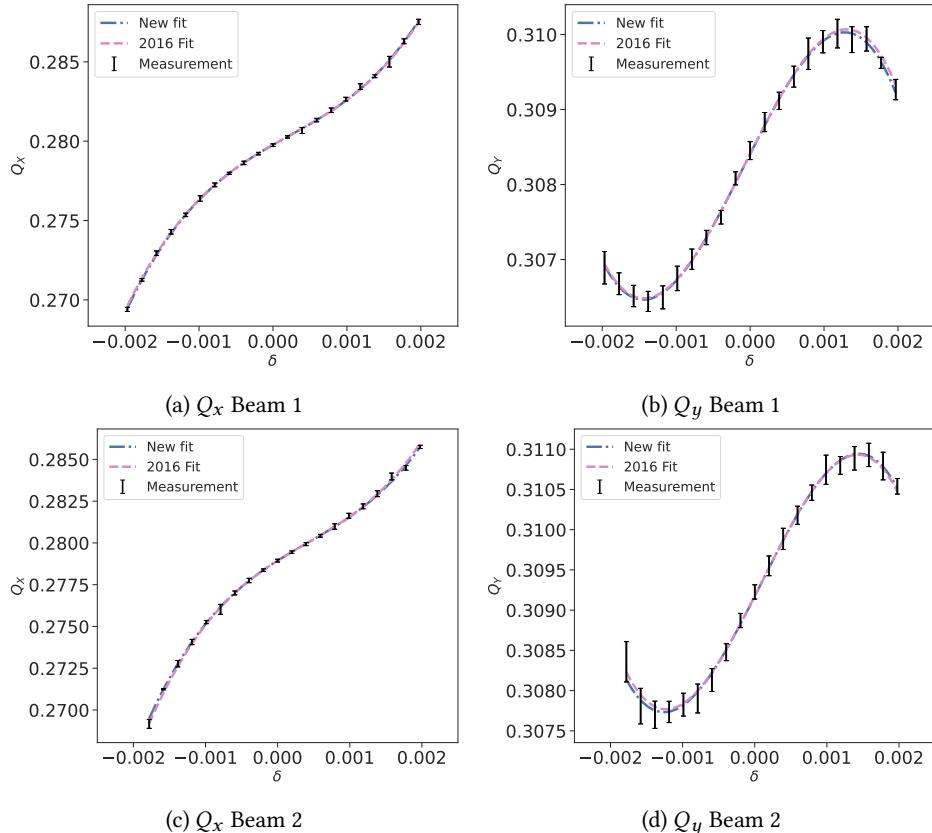


Figure 6.10.: Comparison of the non-linear chromaticity fit obtained from the computed momentum offset via the RF in 2022 and from the logged values in 2016.<sup>123</sup>

### 6.3.2. Further Measurements

In order to assess the correctness of the observation of higher chromaticity orders, measurement repeatability is needed. Two measurements were thus taken in 2022, with different configurations pertaining to the correction of the second and third order chromaticities  $Q''$  and  $Q'''$ . The first one used the nominal correction strengths for octupole and decapole corrector magnets, derived from magnetic measurements, where the second one used beam-based corrections for the same elements, computed from the previous measurement. More measurements were taken during 2024's commissioning with new optics for the same reasons, to minimize the second and third order chromaticities. Three measurements were taken: with nominal corrections, after having corrected  $Q'''$  and then  $Q''$ . The introduced new optics mainly changed the powering of the triplets at the IPs and are not expected to have a considerable impact on the chromaticity.

Table 6.5 shows a summary of those measurements with their respective achieved momentum offset ranges. While the 2024 measurements achieved greater ranges than the previous ones, those were restricted during analysis to allow suitable comparisons.

Number	Year	Corrections	$\delta$ min. [ $\times 10^{-3}$ ]	$\delta$ max. [ $\times 10^{-3}$ ]
1	2022	Nominal	-3.15	3.01
2	2022	$Q''$ & $Q'''$	-3.15	3.72
3	2024	Nominal	-5.15	3.15
4	2024	$Q'''$	-3.44	4.87
5	2024	$Q''$ & $Q'''$	-3.86	4.44

Table 6.5.: Performed chromaticity measurements with their respective momentum offset ranges.

In order to stay consistent, the horizontal and vertical tunes were respectively set to  $Q_x = 0.28$  and  $Q_y = 0.31$  for all measurements. The linear chromaticity  $Q'$  is set to a small value, around 2, to avoid large tune shifts throughout the scan. All measurements were performed during LHC's beam commissioning.

## Varying Configurations

The five previously introduced measurements were performed with very different configurations for the octupolar and decapolar correctors. Table 6.6 shows the strengths applied on every circuit for each correction scheme, in 2022. The correction is called *global* as all correctors are trimmed uniformly. The 2024 corrections are similar in order of magnitude. Figure 6.11 shows the measurements and fit of some of these measurements, to highlight their differences.

Beam	$K_4 [\text{m}^{-4}]$	$K_5 [\text{m}^{-5}]$
1	+3.2973	+1610
2	+2.1716	+1618

Table 6.6.: Corrections applied on top of the nominal octupolar and decapolar correctors strengths in 2022 for the  $Q''$  and  $Q'''$  corrections.

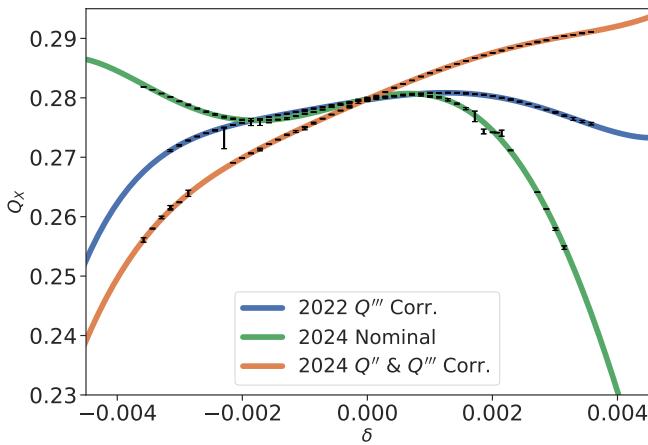


Figure 6.11.: Selection of horizontal chromaticity measurements performed with varying configurations of the octupolar and decapolar correctors for Beam 1 during the commissionings of 2022 and 2024.

A summary of the measured chromaticity orders is given in Table 6.7. The first measurement in the horizontal plane of Beam 2 suffered from low beam intensity and worse noise, resulting in a high correlation between  $Q^{(4)}$  and  $Q^{(5)}$ , and is therefore not included. The last measurement for the vertical plane experienced some tune drift, making the fit impossible and is therefore not included for both beams.

The consistent identification of fourth and fifth order chromaticity across various configurations of the LHC, with differing lower order chromaticity and coupling, enhances confidence in their identification. Previous studies have typically focused on fits limited to the third order. However, expanding the analysis to include fifth order not only increases the estimate of  $Q'''$  but also enhances the overall fit quality. Accurately measuring third order chromaticity is crucial for its correction, underscoring the importance of considering higher order terms.

Axis	Meas.	$Q''$	$Q'''$	$Q^{(4)}$	$Q^{(5)}$
<b>Horizontal</b>					
Beam 1	1	$-2.44 \pm 0.02$	$-3.37 \pm 0.04$	$-0.56 \pm 0.02$	$1.21 \pm 0.07$
	2	$-0.61 \pm 0.01$	$-1.00 \pm 0.03$	$-0.62 \pm 0.02$	$1.19 \pm 0.05$
	3	$-2.01 \pm 0.05$	$-4.49 \pm 0.10$	$-0.58 \pm 0.07$	$1.34 \pm 0.18$
	4	$-1.46 \pm 0.03$	$-0.29 \pm 0.06$	$-0.43 \pm 0.04$	$1.09 \pm 0.10$
	5	$-0.33 \pm 0.01$	$-0.31 \pm 0.03$	$-0.59 \pm 0.01$	$0.75 \pm 0.04$
	<b>Avg.</b>			$-0.56 \pm 0.07$	$1.12 \pm 0.20$
Beam 2	2	$-0.85 \pm 0.01$	$-0.66 \pm 0.03$	$-0.57 \pm 0.02$	$1.09 \pm 0.06$
	3	$-2.93 \pm 0.05$	$-4.40 \pm 0.08$	$-0.53 \pm 0.08$	$1.66 \pm 0.16$
	4	$-2.21 \pm 0.02$	$-0.00 \pm 0.03$	$-0.46 \pm 0.02$	$1.18 \pm 0.05$
	5	$-0.53 \pm 0.02$	$-0.09 \pm 0.03$	$-0.57 \pm 0.02$	$0.98 \pm 0.05$
	<b>Avg.</b>			$-0.53 \pm 0.04$	$1.23 \pm 0.26$
<b>Vertical</b>					
Beam 1	1	$0.97 \pm 0.02$	$1.62 \pm 0.05$	$0.15 \pm 0.03$	$-0.88 \pm 0.09$
	2	$-0.23 \pm 0.01$	$0.13 \pm 0.02$	$0.09 \pm 0.02$	$-0.60 \pm 0.03$
	3	$0.83 \pm 0.02$	$1.97 \pm 0.03$	$0.29 \pm 0.02$	$-0.68 \pm 0.05$
	4	$0.62 \pm 0.01$	$-0.18 \pm 0.03$	$0.00 \pm 0.02$	$-0.56 \pm 0.05$
	<b>Avg.</b>			$0.13 \pm 0.11$	$-0.68 \pm 0.12$
Beam 2	1	$0.79 \pm 0.03$	$1.54 \pm 0.06$	$0.24 \pm 0.04$	$-0.74 \pm 0.13$
	2	$-0.29 \pm 0.01$	$0.10 \pm 0.02$	$0.13 \pm 0.02$	$-0.58 \pm 0.04$
	3	$0.89 \pm 0.02$	$2.05 \pm 0.03$	$0.32 \pm 0.03$	$-0.73 \pm 0.06$
	4	$0.60 \pm 0.02$	$-0.14 \pm 0.03$	$0.04 \pm 0.02$	$-0.66 \pm 0.05$
	<b>Avg.</b>			$0.18 \pm 0.11$	$-0.68 \pm 0.06$

Table 6.7.: Summary of the chromaticity values obtained from the measurements presented in Table 6.5.

### 6.3.3. Model Estimates

Benchmarking the magnetic model of the LHC is important, in order to gauge its correctness and that of the error tables in use, and address any potential discrepancies. The model of the LHC is based on MADX and magnetic field error tables [16], containing seeds for the random errors. To compute the chromaticity, simulations are run via PTC, with a selection of these field errors.

Simulations incorporating various field errors have been conducted to evaluate the contributions of individual magnet orders to fourth and fifth order chromaticity. Field errors are applied to all magnets, with  $b_2$  errors in the main dipoles resulting in a  $\beta$ -beating of approximately 10%. Coupling is introduced using global knobs to create a  $C^-$  value of 0.001, which is commonly observed during operation. Normal and skew field errors, ranging from sextupolar ( $b_3$ ) to decahexapolar ( $b_8$ ), are added either individually or in combination to determine which has the strongest effect.

**Fourth Order Chromaticity** The results from simulations strongly imply that the do-decapolar errors are the main contributors to  $Q^{(4)}$ , as can be seen in Fig. 6.12. The most notable effect on this chromaticity order is the beta-beating, introducing a very large spread via the various error seeds. Comparing the simulation at the top with most errors added, the  $b_6$  component alone accounts for  $\approx 70\%$  of it for both axes on each beam.

**Fifth Order Chromaticity** It is seen that the the decatetrapolar errors are the main contributors to  $Q^{(5)}$ , as can be seen in Fig. 6.13. Fringe fields and skew multipoles have been found to have a negligible impact. Beta-beating and coupling are seen to increase by a small amount the chromaticity, while sextupolar errors induce a spread with the different seeds. Comparing the simulation at the top with most errors added, the  $b_7$  component alone accounts for  $\approx 70\%$  of it for both axes on each beam.

It has been noted in the previous chapter about decapoles (see Section 5.5) that the  $b_5$  component in the main dipoles was large at injection energy, and could explain most of the discrepancy between the measurements and simulations. Such a decay in the main dipoles also exists for the  $b_7$  component [71], and is shown in Fig. 6.14.

Its value is though small and settles around  $+0.0351 \pm 0.0007$ . The average  $b_7$  of the

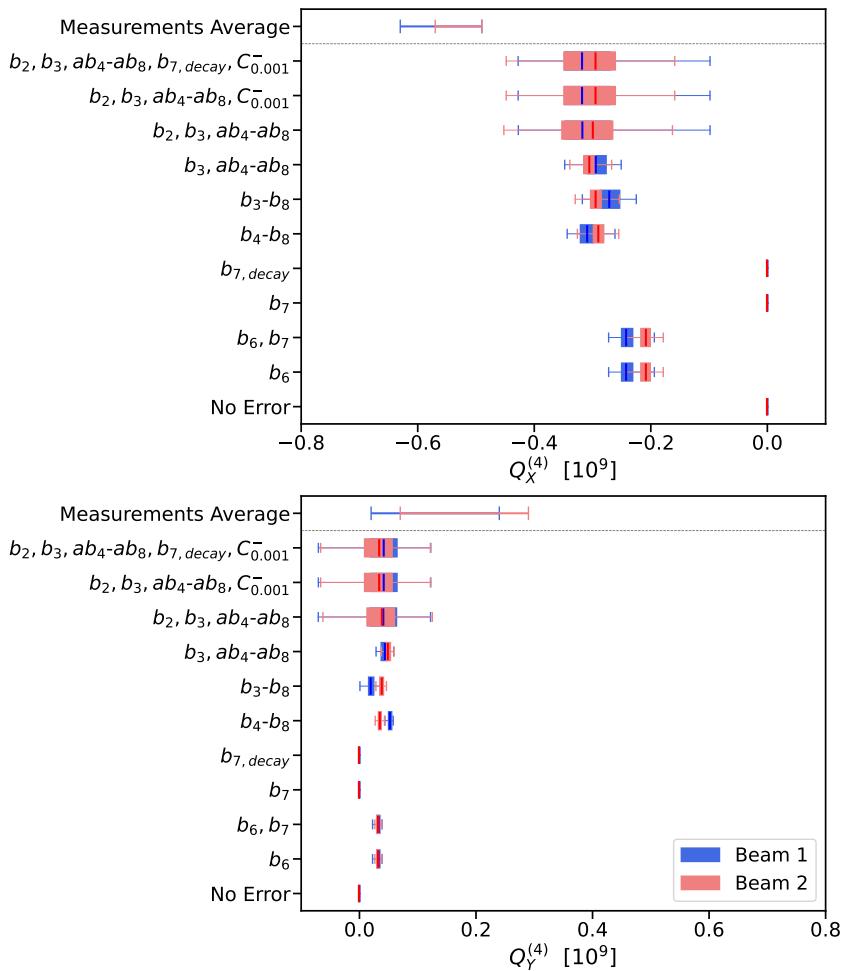


Figure 6.12.: Measured and simulated fourth order chromaticity with different multipole errors. The  $b_2$  errors, applied on dipoles and quadrupoles, generate beta-beating. Coupling is set to a value commonly seen in operation.

main dipoles is of  $0.32 \pm 0.16$ . The decay thus increase that value of only about 11%.

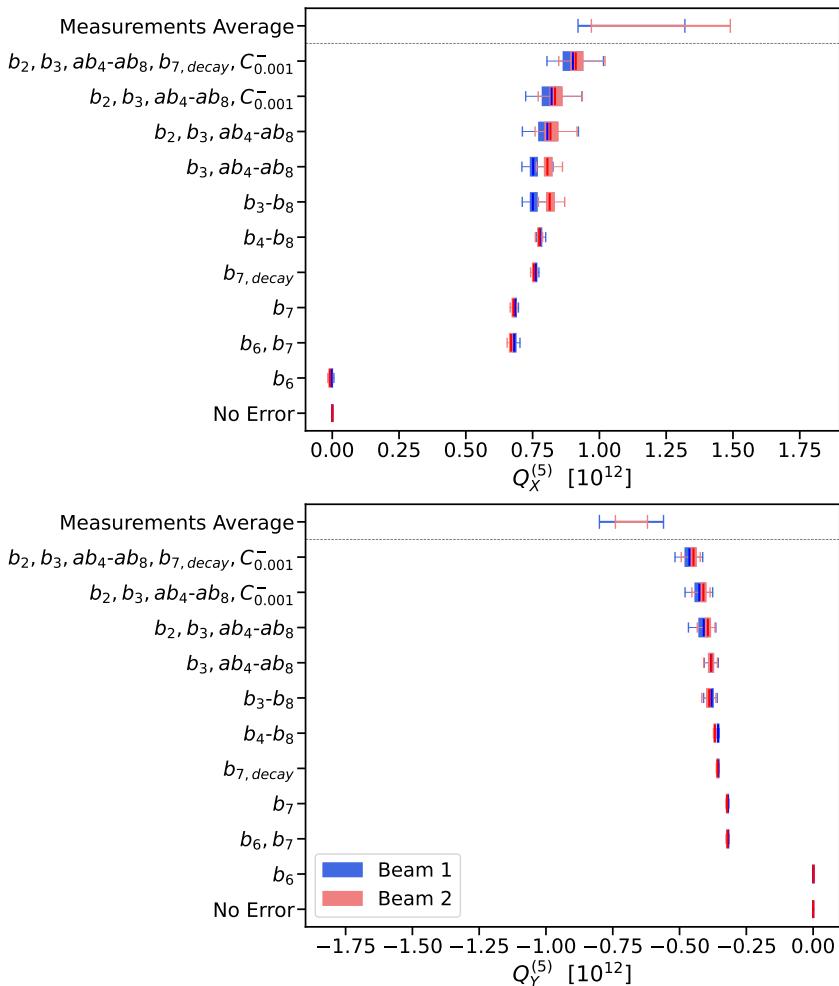


Figure 6.13.: Measured and simulated fifth order chromaticity with different multipole errors. The  $b_2$  errors, applied on dipoles and quadrupoles, generate beta-beating. Coupling is set to a value commonly seen in operation.

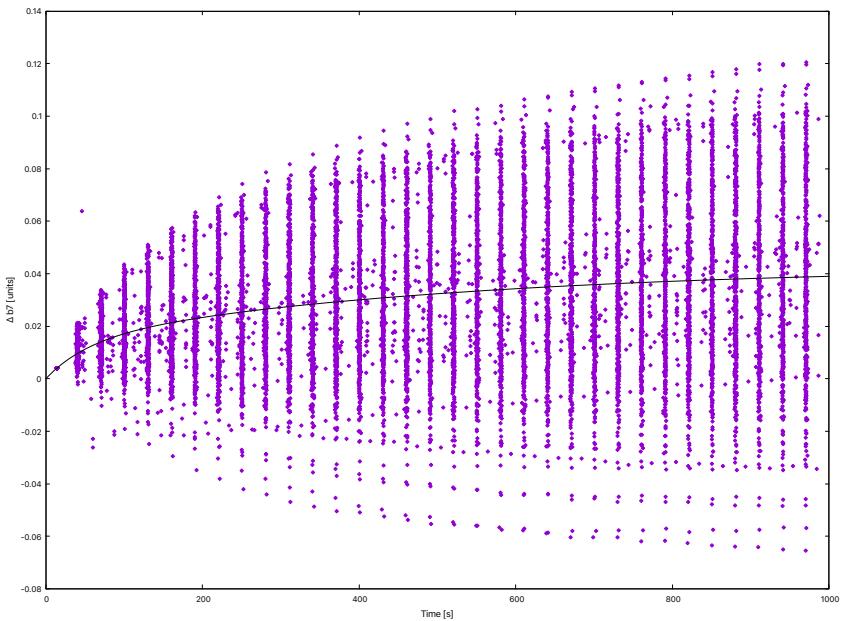


Figure 6.14.: Measured decay of the integrated decatetrapolar field in LHC's main dipoles at injection energy. The fit is shown in black [71] and settles around +0.035.

Simulations done with that decay taken into account are also present in the previous Fig. 6.13.

### Agreement with Measurements

Previous simulation results are shown in Table 6.8, taking the values from the simulation including the most effects at the top of the plots. For the fourth order, the beating is not included as the large induced  $Q^{(4)}$  is not yet explained. Table 6.9 shows the ratio between the measured average and simulated chromaticities.

The similar ratios between planes and beams for the fifth order could indicate a systematic error not modeled. The large differences observed for the fourth order are not yet explained but the difference between planes could be linked to a shift induced

Plane	$Q^{(4)} [10^9]$	$Q^{(5)} [10^{12}]$
Beam 1		
X	$-0.29 \pm 0.02$	$0.90 \pm 0.05$
Y	$0.04 \pm 0.01$	$-0.46 \pm 0.03$
Beam 2		
X	$-0.31 \pm 0.02$	$0.92 \pm 0.03$
Y	$0.05 \pm 0.00$	$-0.45 \pm 0.01$

Table 6.8.: Simulated high order chromaticity terms via PTC at injection energy, including normal and skew sextupolar to decahexapolar field errors. Are also included beta-beating, coupling and decatetrapolar decay. For the fourth order, the values do not include beta-beating as the observed spread is not yet fully understood.

Plane	$Q^{(4)}$ Ratio	$Q^{(5)}$ Ratio
Beam 1		
X	$1.91 \pm 0.28$	$1.24 \pm 0.23$
Y	$3.00 \pm 2.60$	$1.47 \pm 0.27$
Beam 2		
X	$1.74 \pm 0.16$	$1.34 \pm 0.29$
Y	$3.70 \pm 2.30$	$1.51 \pm 0.14$

Table 6.9.: Ratios of the simulated and average measured high-order chromaticity terms. The values are taken from Table 6.7 and Table 6.8.

by the decapolar corrections in the vertical plane. It indeed seems that the fourth order follows a trend with the third order.

## 6.4. Summary

Building on previous efforts at the LHC to understand non-linear multipoles like sextupoles and octupoles, this chapter extends the analysis to higher-order fields, particularly dodecapolar and decatetrapolar components at injection energy. With the

development of improved measurement techniques and analysis procedures, alongside enhancements in dynamic aperture via skew octupolar and decapolar corrections, these higher-order fields have been successfully observed for the first time. The use of a newly implemented collimation setup has played a crucial role in this, representing a significant advancement in the study of non-linear effects in the LHC.

Leveraging large kick amplitudes, the dodecapolar Resonance Driving Term  $f_{0060}$  was measured, at injection energy, for the first time at the LHC. This measurement, performed across various non-linear corrector configurations, shows strong agreement with the magnetic model. The primary source of this RDT is expected to be the dodecapolar ( $b_6$ ) field errors in the main dipoles, for which the decay has been modeled.

Chromaticity measurements over a wide range of momentum offsets, facilitated by the new collimation setup, revealed fourth and fifth order terms ( $Q^{(4)}$  and  $Q^{(5)}$ ). Repeated measurements under different LHC configurations consistently identified these higher-order terms, underscoring the robustness of their identification. The long-term effort to measure these terms, spanning several years, has been crucial in refining the understanding of high-order fields and benchmarking the magnetic model. Discrepancies observed between measurements and model predictions may stem from unmodeled systematic errors or contributions from lower-order multipoles.

The advancements in these measurement techniques not only enabled breakthroughs in RDT studies but also opened up the exploration of previously inaccessible higher-order non-linear regimes. These findings provide new pathways for understanding non-linear beam dynamics in the LHC. Improved comprehension of non-linear errors and their measurement will be crucial for ensuring the optimal performance of future LHC upgrades and next-generation accelerators, where non-linear effects are expected to have a substantial impact on dynamic aperture.



# SuperKEKB Studies

## 7.1. Introduction

A key motivation for the non-linear optics studies conducted at the LHC is to demonstrate a strong understanding and precise control of the optics through newly developed techniques, in order to benefit and inform the design of future accelerators.

KEK, or the High Energy Accelerator Research Organization (Kō Enerugi Kasokuki Kenkyū Kikō), is a Japanese institution operating the country's largest particle physics laboratory, located in Tsukuba. KEK provides particle accelerators and essential infrastructure for a wide range of research fields, including high-energy physics, material science, structural biology, and radiation science. One of its most notable facilities is SuperKEKB, the world's most advanced electron-positron collider. SuperKEKB is designed to reach an instantaneous luminosity of up to  $80 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$  and has recently completed its commissioning phase. It stores a 7 GeV electron beam in the High-Energy Ring (HER) and a 4 GeV positron beam in the Low-Energy Ring (LER), which collide at a single interaction point where the Belle II experiment is conducted. SuperKEKB currently holds the world record for instantaneous luminosity at  $4.71 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$  [72], surpassing the previous record set by the LHC. With a circumference of approximately 3 km, it is the largest lepton collider in operation. Similar to CERN, KEK is a large complex housing multiple accelerators and facilities, the injector chain for SuperKEKB being shown in Fig. 7.1.

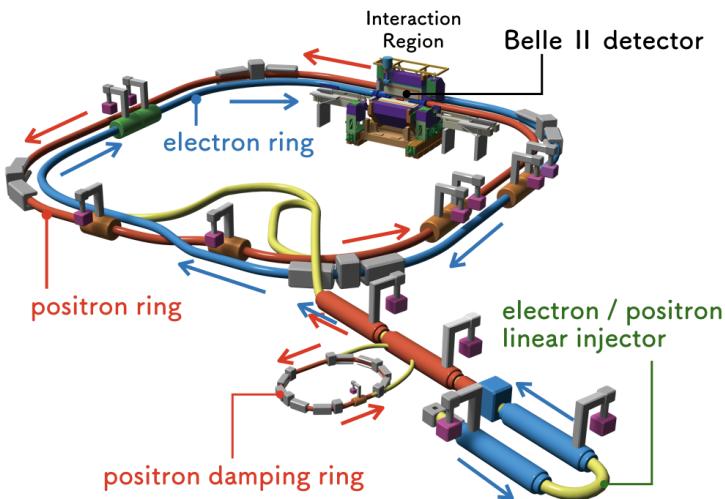


Figure 7.1.: Schematic drawing of the accelerator complex at KEK [73].

KEK maintains active collaborations with several organizations, including CERN, to advance research in various areas of interest. SuperKEKB is of particular relevance to the FCC-ee, as it serves as a prototype for testing innovative techniques in manufacturing, measurement, and analysis, whether related to mechanical prototyping, element alignment, or beam dynamics.

During a recent EAJADE [74] secondment in Japan, optics measurement techniques employed at CERN for the LHC, specifically turn-by-turn acquisitions, were applied and tested on the two rings of SuperKEKB. Previous studies on these techniques have been conducted and are extended in this chapter. Instead of providing a detailed description, the focus is on updating the methods previously used. Further details can be found in [27, 75–77].

## 7.2. Measurement Techniques for Linear Optics

In SuperKEKB, the beam optics are measured using either Closed Orbit Distortion [78] or turn-by-turn measurements. As fast optics measurements can be achieved with the Turn-By-Turn method, a measurement campaign was carried out to improve the measurement quality. This includes investigating various beam excitation techniques and settings. The necessary pre-processing steps for optics measurements have been detailed in [27], and are not covered here.

### 7.2.1. Closed Orbit Distortion

Optics measurements using the COD method [78–80] are well established and routinely performed at SuperKEKB. In COD measurements, the beam is excited using six corrector magnets, and the centroid orbit is recorded by 466 BPMs for the HER and 444 BPMs for the LER, respectively. The optics of both transverse planes are then reconstructed using analytical formulas. Since the correctors must be powered one at a time, the COD method is relatively time-consuming. Since the average particle orbit is observed, the BPM readings are dependent on precise calibration.

### 7.2.2. Turn-by-Turn

In SuperKEKB, 68 and 70 BPMs in the HER and LER rings are capable of recording turn-by-turn orbit data, typically capturing several thousand turns in both transverse planes. Turn-by-turn measurements are usually performed with a single bunch, with currents ranging from 0.2 mA to 1.5 mA. The beam can be excited using three different methods.

First, an Injection Kicker (IK) delivers a single horizontal kick, causing the beam to oscillate to large amplitudes. This amplitude then damps due to synchrotron radiation. The damping times for the positron and electron rings are 46 ms and 53 ms, corresponding to 4600 and 5300 turns, respectively [27]. However, the IK only provides horizontal kicks, limiting the precision of vertical optics measurements.

In contrast, the Phase-Locked Loop (PLL) method allows continuous excitation of the beam in both horizontal and vertical planes. The PLL tracks the natural tune of the beam and drives it at the corresponding frequency. A key advantage of the PLL system is that it can excite both planes simultaneously, enabling measurements of transverse coupling and other resonance-driving terms (RDTs). The PLL has so far been the only method used to perform vertical turn-by-turn measurements, and has not been utilized in this thesis. To induce oscillations in the vertical plane, the beam can be injected with an orbit offset. This vertical offset causes the beam to oscillate as if it had received a single kick. The oscillations then damp, and a new beam must be injected to repeat the measurement. This method has been successfully used for the first time at SuperKEKB to measure vertical optics using turn-by-turn data.

Once the turn-by-turn data is recorded, the standard analysis procedures outlined in Section 3.2 are applied. Specifically, the frequency spectrum is calculated before reconstructing the optics. The model used for this analysis is generated with the SAD (Strategic Accelerator Design) software [81]. A typical turn-by-turn signal is depicted in Fig. 7.2.

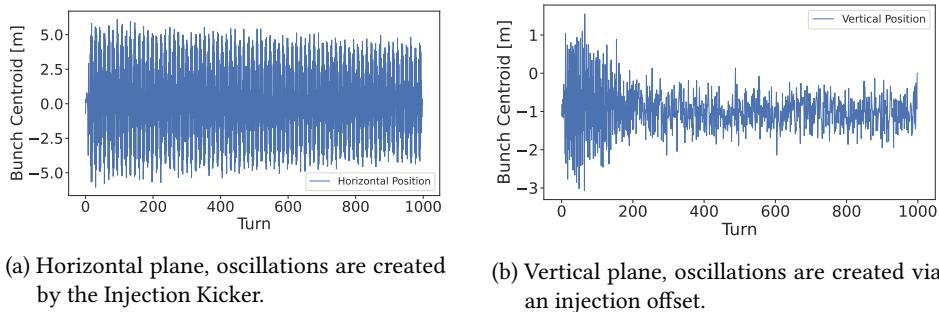


Figure 7.2.: Typical horizontal and vertical recorded turn-by-turn signals.

### 7.2.3. OMC3 Graphical User Interface

A crucial step in the analysis process is identifying faulty or noisy BPMs from the turn-by-turn data. This task is often time-consuming and susceptible to human error. In the

LHC, a Graphical User Interface (GUI) is employed to inspect the turn-by-turn signals before applying an FFT, which reveals the spectral lines associated with the tunes. This tool plays a key role in the efficient cleaning of outliers and noisy BPMs. As part of this thesis, the GUI has been updated to support both the HER and LER rings. Using the latest version of the GUI provides access to recent bug fixes and improvements, enhancing both usability and functionality. An example use case is shown in Fig. 7.3, where BPMs delivering incorrect data across multiple measurements can be easily detected.

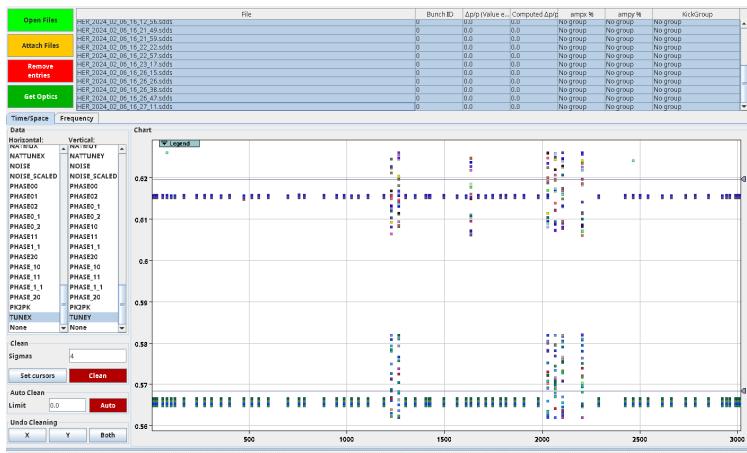


Figure 7.3.: The identification of malfunctioning BPMs across measurements is easily done with the GUI.

## 7.3. Optics Observations

All the measurements presented in this section have been performed in February 2024, during the commissioning phase of SuperKEKB and are presented in Table 7.1.

Ring	Day	$\beta_x^*$ [mm]	$\beta_y^*$ [mm]	$Q_x$	$Q_y$	Kicks
LER	06	384	<b>48.6</b>	44.556	46.635	H & V
	09	384	<b>48.6</b>	44.553	46.621	H
	20	200	<b>8</b>	44.527	46.604	H
	22	200	<b>8</b>	44.535	46.590	H
HER	06	400	<b>81</b>	45.572	43.616	H & V
	20	200	<b>8</b>	45.530	43.595	H
	22	200	<b>8</b>	45.535	43.596	V
	26	200	<b>8</b>	45.535	43.596	V

Table 7.1.: Configurations of the HER and LER rings for measurements performed via turn-by-turn acquisition. The day column informs on which date the measurement was performed in February 2024.

### 7.3.1. Beta-Beating

Several configurations of the machine were measured using turn-by-turn acquisitions, as detailed in Table 7.1. The configurations with  $\beta_y^* = 48.6$  mm in LER and  $\beta_y^* = 81$  mm in HER are referred to as *detuned*. Multiple kicks were performed for each configuration to increase measurement precision. Although additional measurements were taken, insufficient kick amplitude in some cases made it challenging to extract reliable linear optics data and are thus here not included. Specifically, vertical measurements are challenging to perform as oscillation amplitudes are often low due to the limited achievable injection offset. The low vertical action often makes measurements noisy.

Both the *detuned* and *8 mm squeezed* optics are reliably measured for both rings, with the exception of the vertical plane for the squeezed optics in LER due to the absence of injection offset. Figure 7.4 and Fig. 7.6 show the measurements taken with *detuned* optics for LER and HER. For the squeezed optics, at 8 mm, a comparison to the measurements performed via COD is done in Fig. 7.5 and Fig. 7.7, for LER and HER respectively. The agreement between the COD and the turn-by-turn techniques is a good indication of their robustness, the difference between the RMS difference of two methods being below 4% for all measurements. The agreement found during these measurements is similar to that obtained in previous studies [27].

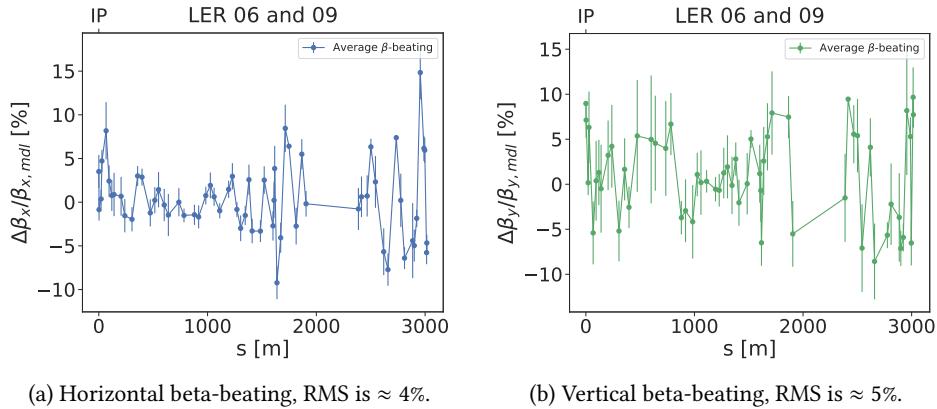


Figure 7.4.: LER horizontal and vertical beta-beating for *detuned* optics, reliably measured during two different days. The vertical plane is noticeably noisier, as oscillation amplitudes are smaller.

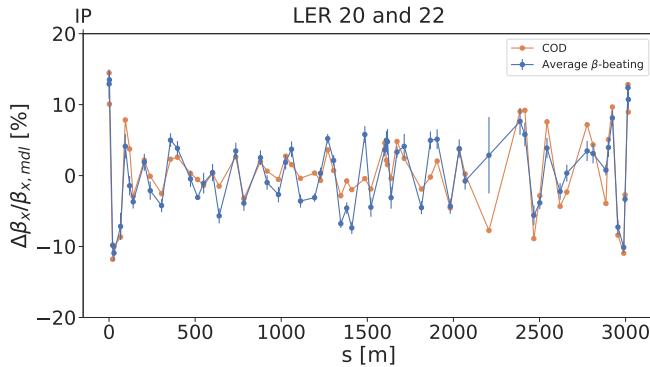


Figure 7.5.: LER horizontal beta-beating for *8 mm squeezed* optics, reliably measured during two different days. The vertical plane is absent due to amplitudes being too low to reconstruct linear optics. A comparison to the beating measured via COD is made. RMS is  $\approx 6\%$ .

Table 7.2 shows a summary of the performed measurements and the RMS  $\beta$ -beating.

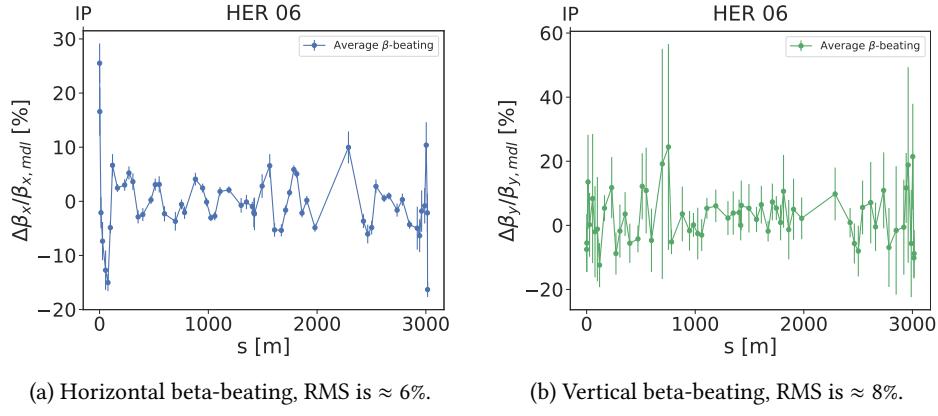


Figure 7.6.: HER horizontal and vertical beta-beating for *detuned* optics. The vertical plane is noticeably noisier, as oscillation amplitudes are smaller.

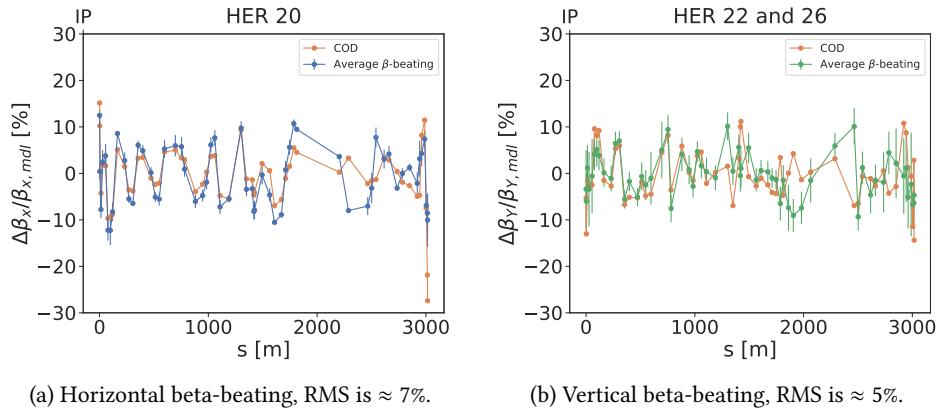


Figure 7.7.: HER horizontal and vertical beta-beating for *8 mm squeezed* optics. The vertical plane is noticeably noisier, as oscillation amplitudes are smaller. A comparison to the beating measured via COD is made.

Ring	Configuration	$\beta$ -b. rms H	$\beta$ -b. rms V
LER	Detuned	4%	5%
	Squeezed 8 mm	6%	
HER	Detuned	6%	8%
	Squeezed 8 mm	7%	5%

Table 7.2.: Summary of the measured  $\beta$ -beating with squeezed and detuned optics in the HER and LER rings.

### 7.3.2. Amplitude Detuning

#### Tune Stability

To accurately measure linear and non-linear optics, reliable measurement of the tune is essential [77]. Ensuring tune stability across consecutive kicks is needed to minimize measurement uncertainties and reflects the reproducibility of the machine. Using a suitable kicker in the horizontal plane simplifies measurements, as indicated by the tune's stability across varying kick strengths. Figure 7.8 illustrates the variation in tune from the first kick compared to subsequent kicks for both rings and planes. While reproducibility is poor in the vertical plane of the LER, all other measurements agree within few  $10^{-3}$ .

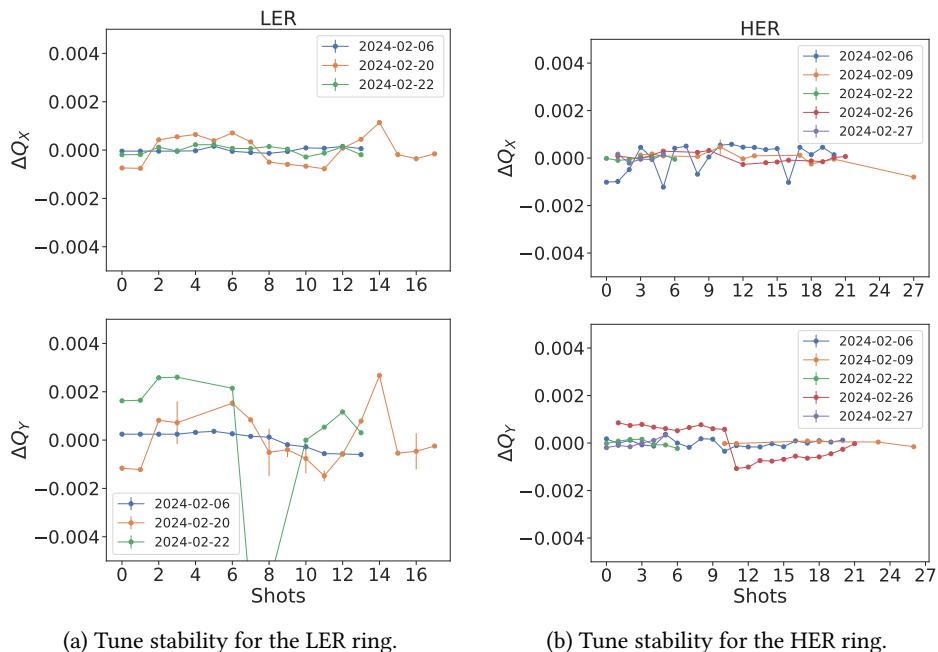


Figure 7.8.: Difference in tune between consecutive measurements, across several days.

## Amplitude Detuning in LER

Measurement with higher kick amplitudes have been performed in the LER ring with *detuned* optics. These measurements stand out compared to previous ones due to their larger action range. Figure 7.9 illustrates the various performed turn-by-turn measurements and their corresponding action. The tune is computed across each turn-by-turn measurement using a window of 200 turns, every 50 turns. A clear trend emerges with the tune increasing with the amplitude of the kicks. Over time, the oscillations gradually dampen due to synchrotron radiation, and the tune reduces accordingly.

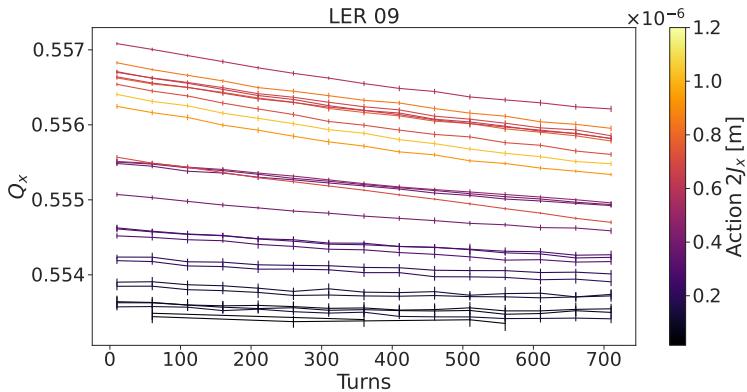


Figure 7.9.: Measured tune across the length of each turn-by-turn measurement. The kick action is shown in color. The tune is computed via a running window over 200 turns, every 50 turns.

The variation of the horizontal tune with the horizontal kick amplitude is related to the amplitude detuning term  $\frac{\partial Q_x}{\partial J_x}$ . Its value can be retrieved by fitting the tune to the action. Figure 7.10 shows the various kicks along with a fit and a comparison to the model. The measured value,  $6500 \pm 500 \text{ m}^{-1}$  is one order of magnitude higher than that simulated, of  $600 \text{ m}^{-1}$ . This observed discrepancy indicates possible unmodeled octupolar-like sources. Such a discrepancy, although not as large, already has been observed with different optics [27].

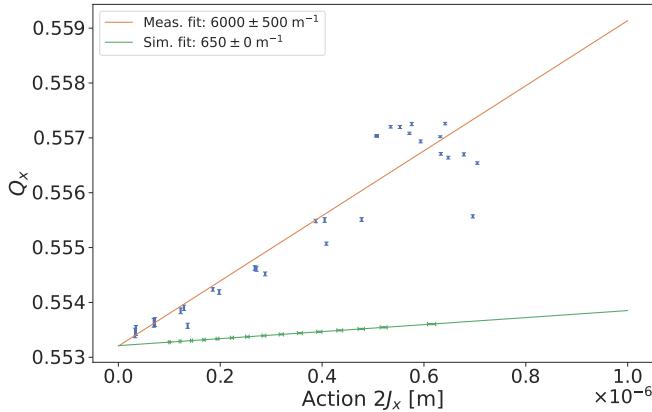


Figure 7.10.: Amplitude dependence of the tune in the LER ring. The slope of the fitted line corresponds to the amplitude detuning term  $\partial Q_x / \partial 2J_x$ . For comparison, the tune computed from the model is shown in green.

### 7.3.3. Resonance Driving Terms

Resonance Driving Terms, coefficients linked to the strength of a resonance, can be measured via their associated line amplitude in the frequency spectrum. To reliably measure RDTs, these line amplitudes must be clearly distinguishable and above the noise level for every BPM. This can be challenging to achieve and often requires high amplitude kicks. For example, lines in the vertical spectrum, attributed to octupoles, were previously observed [27] at SuperKEKB but could not lead to a successful RDT measurement due to kick amplitudes.

However, new measurements were taken with a different working point in order to be closer to the resonances with the hope of increasing their impact on particle motion. A resonance diagram, illustrating these resonances and the typical working point of LER is shown in Fig. 7.11. Although several working points were tested, no clear correlation between them and successful RDTs measurements could be established. The main determining factor remained the kick amplitude. A typical frequency spectrum from a successful measurement in HER for the vertical plane is shown in Fig. 7.12. The frequency line  $-1Q_x - 1Q_y$  is seen at each BPM. Additionally, lines near the vertical

tune are attributed to the synchrotron tune.

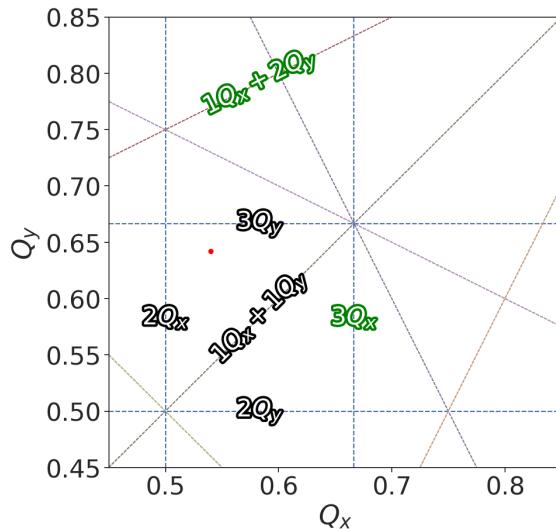


Figure 7.11.: Resonance diagram highlighting the resonances close to the HER working point, in red, and those often visible in the frequency spectra, in green.

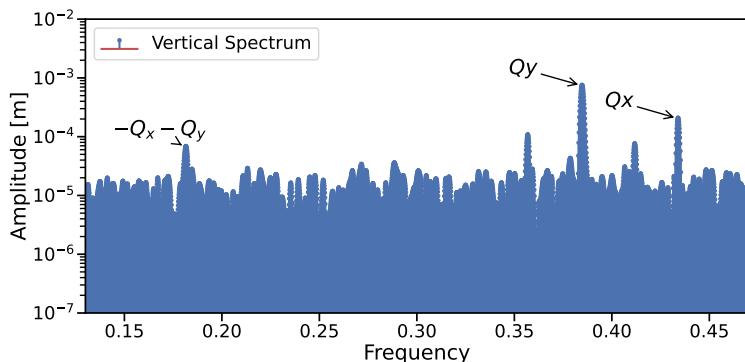


Figure 7.12.: Vertical frequency spectrum of HER, showing the natural tunes as well as a line excited by the sextupolar RDT  $f_{1020}$ .

Successful measurements of sextupolar RDTs were indeed obtained. Specifically, the RDTs  $f_{3000,x}$  in LER and  $f_{1020,y}$  in HER were measured using the *detuned* optics configuration. While additional sextupolar lines were clearly visible in both the horizontal and vertical spectra, no clear RDT measurements could be extracted as they were not seen for each BPM. The measured amplitudes, along with comparisons to the model, are presented in Fig. 7.13 and Fig. 7.14. Notably, both measurements exhibit discrepancies with the model. The average amplitude of  $f_{1020}$  in the vertical plane for HER is measured at  $21 \text{ m}^{-1/2}$ , compared to a modeled amplitude of  $7 \text{ m}^{-1/2}$ . Similarly, for  $f_{3000}$ , the average values measured in the horizontal plane for LER are  $2 \text{ m}^{-1/2}$  and  $1.3 \text{ m}^{-1/2}$ . These discrepancies remain largely unexplored and may arise from decoherence effects [82] that were not accounted for in the analysis software. Additionally, these could also arise from contributions from other multipoles through feed-down, as observed in the LHC, or unknown sextupolar sources. The measured spikes in the LER RDT are not yet explained, but could come from a badly reconstructed phase space due to non-optimal phase advances between BPMs in that region or inverted BPM plane signal [83].

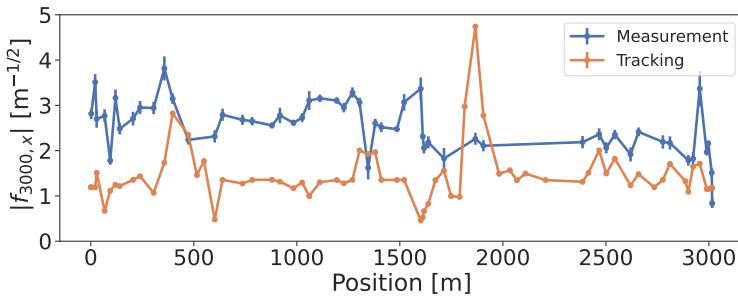


Figure 7.13.: Sextupolar RDT  $f_{3000,x}$  measured in LER with *detuned* optics and compared to the model.

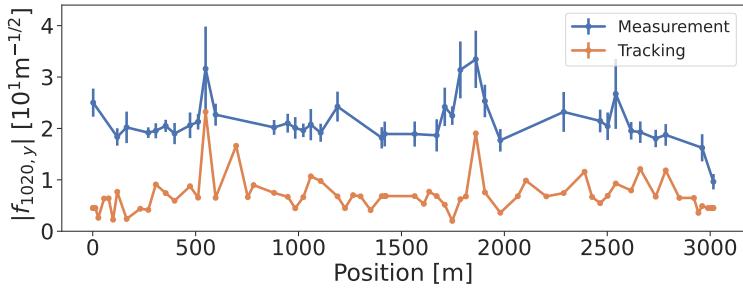


Figure 7.14.: Sextupolar RDT  $f_{1020,y}$  measured in HER with *detuned* optics and compared to the model.

### 7.3.4. Chromaticity

Chromaticity measurements were performed using the method of varying the RF frequency over large ranges while recording the tune via pickups, as described in Section 3.2.3. While this method has now been made standard at the LHC, it is not used on the SuperKEKB rings, where only a few data points are collected over a small momentum offset range. Similar to the LHC at injection energy, the Lorentz factor  $\gamma^{-2}$  is small compared to the momentum compaction factor  $\alpha_c$  and can therefore be neglected. Figure 7.15 illustrates the RF frequency method.

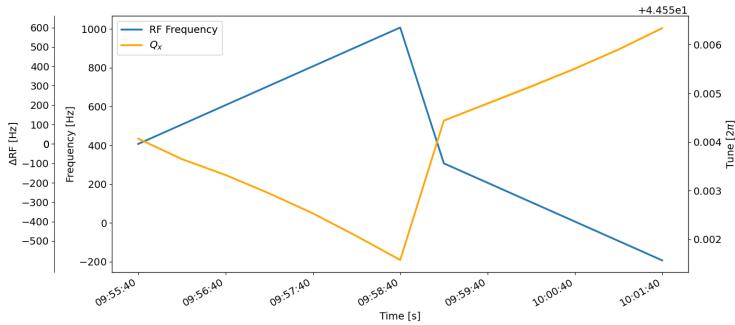


Figure 7.15.: Horizontal tune change relative to the change in RF Frequency at SuperKEKB.

Two measurements were taken with *detuned* optics for the HER and LER rings and are shown in Fig. 7.16 and Fig. 7.17. The values of the fitted chromaticity function are given in Table 7.3 and Table 7.4 for HER and LER respectively. In order to allow for a better comparison, the tune and the linear chromaticity are taken from the measurements.

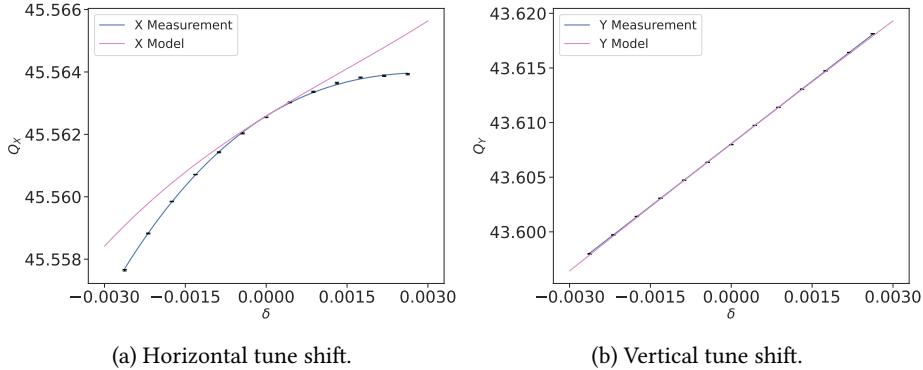


Figure 7.16.: HER chromaticity measurements with *detuned* optics.

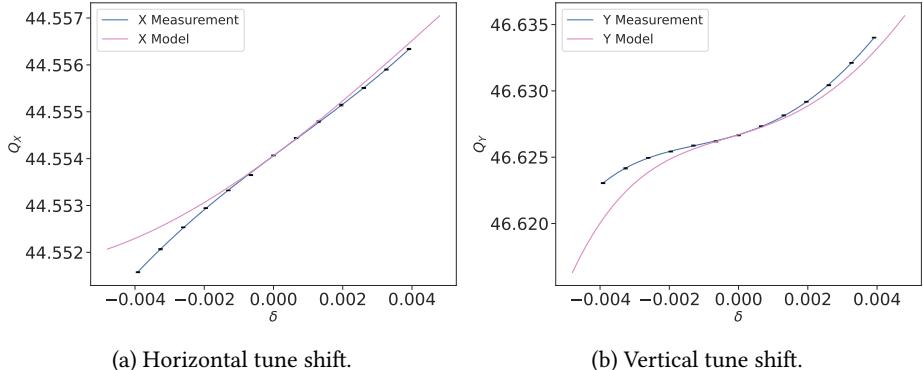


Figure 7.17.: LER chromaticity measurements with *detuned* optics.

It can be noted that most of the measured third and fourth-order chromaticities agree well with the model, whereas the second-order chromaticity appears to deviate. Discrepancies in  $Q''$  were already observed in 2022 with different optics [27]. This dis-

		$Q'' [\times 10^3]$	$Q''' [\times 10^6]$
X	Meas.	$-0.51 \pm 0.01$	$0.11 \pm 0.02$
	Model	$-0.12 \pm 0.00$	$0.09 \pm 0.00$
Y	Meas.	$0.00 \pm 0.0$	
	Model	$-0.04 \pm 0.0$	

Table 7.3.: Measured and modeled chromaticity in the HER ring with *detuned* optics in both planes.

		$Q'' [\times 10^3]$	$Q''' [\times 10^6]$	$Q^{(4)} [\times 10^9]$
X	Meas.	$-0.01 \pm 0.0$	$0.02 \pm 0.0$	
	Model	$0.04 \pm 0.0$	$-0.01 \pm 0.0$	
Y	Meas.	$0.35 \pm 0.01$	$0.25 \pm 0.01$	$-0.09 \pm 0.01$
	Model	$0.10 \pm 0.00$	$0.32 \pm 0.00$	$-0.09 \pm 0.00$

Table 7.4.: Measured and modeled chromaticity in the LER ring with *detuned* optics in both planes.

crepancy could arise from octupolar sources, higher-order contributions from sextupoles or quadrupoles and feed-down.

## 7.4. Summary

The EAJADE secondment at SuperKEKB, during its February 2024 commissioning, focused on applying the optics measurement techniques used at CERN to the HER and LER rings. SuperKEKB, an electron-positron collider, serves as a testbed for the FCC-ee, which is currently in its design phase.

Linear optics measurements, performed with turn-by-turn acquisition, showed good agreement with the conventional Closed Orbit Distortion (COD) method. These results were comparable to measurements taken several years earlier, demonstrating good repeatability across consecutive kicks and over multiple days. Additionally, vertical

plane measurements with an injection offset were performed for the first time on both rings, yielding promising results.

The scope of the optics measurements was then extended to include non-linear optics, covering chromaticity, amplitude detuning, and Resonance Driving Terms (RDTs). Chromaticity measurements for both rings were conducted over large momentum offset ranges, showing good agreement with the model for the linear term, although some discrepancies were observed in  $Q''$  for both rings.

Amplitude detuning was measured in the LER with detuned optics, and comparison with the model revealed an order of magnitude difference. When combined with the chromaticity results, this suggests potential errors due to unmodeled octupole-like sources, warranting further investigation to resolve these discrepancies.

Measuring Resonance Driving Terms (RDTs) proved challenging, as they require high-amplitude kicks and consistent repeatability across multiple kicks. Nevertheless,  $f_{1020}$  and  $f_{3000}$  were successfully measured in the HER and LER rings for the first time. While certain discrepancies with the model remain unresolved, they may be attributed to factors such as unaccounted-for decoherence, damping, or unknown sextupolar-like sources.

Overall, these measurements are consistent with those obtained through alternative methods at KEK. The successful observation of RDTs further suggests that the techniques used at CERN are effective and could enhance our understanding of not only SuperKEKB and its modeling, but also of future accelerators such as the FCC-ee.

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# Conclusions

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The primary aim of this thesis has been to extend and refine methods for the quantitative analysis of non-linear optics, an area of long-standing interest not only for the LHC but also for the field of accelerator physics in general. By pushing these studies into regimes where limited or no prior research has been conducted, this work significantly enhances the understanding of error sources in the LHC and improves its magnetic model.

The investigation into skew octupolar fields introduced, for the first time at the LHC, a response matrix approach for correcting Resonance Driving Terms (RDTs) using the available corrector magnets. This method enabled faster and more efficient commissioning of the low- $\beta$  optics, with corrections resulting in a net improvement in dynamic aperture. Further research explored the interaction between normal Landau octupoles and skew octupolar RDTs. One key insight from this work is that even small changes in coupling at injection can have a substantial impact, underscoring the need for careful consideration of this parameter in future studies and operational correction strategies. This is especially relevant for non-linear studies, which often assume a static reference value for coupling.

By leading studies on decapolar perturbations and their interaction with various multipole components, this thesis has refined the magnetic model for decapolar errors. This improved understanding has allowed for more effective correction strategies, leading to both increased machine lifetime and greater dynamic aperture, key factors for non-linear optics studies. Furthermore, previously unaccounted-for decapolar decay in the

## *Conclusions*

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main dipoles has now been identified as the main contributor to the discrepancies observed in third-order chromaticity ( $Q'''$ ) corrections.

In addition, the thesis presents the first measurement of dodecapolar ( $b_6$ ) RDTs, made possible by improvements in lifetime and dynamic aperture. The introduction of innovative measurement techniques for higher-order chromaticity terms revealed, for the first time, the contribution of both dodecapolar and decatetrapolar fields. These findings underscore the importance of further exploring higher-order fields to optimize the dynamic aperture, especially in preparation for future colliders such as the FCC-ee, which will rely heavily on the precise control of high-order fields for operational efficiency.

As a complementary study, a EAJADE secondment at SuperKEKB during its February 2024 commissioning utilized optics measurement techniques from CERN on the HER and LER rings. Linear optics measurements showed good agreement with the methods used at KEK. The study extended to non-linear optics, including chromaticity and amplitude detuning, revealing some discrepancies between measurements and model estimates, particularly with potential unmodeled octupolar-like sources. Resonance Driving Terms were measured successfully for the first time, although challenges remain due to factors like decoherence and damping. The findings align with alternative KEK methods, indicating that CERN's techniques are effective for enhancing understanding of SuperKEKB and future accelerators like the FCC-ee.

Overall, this research provides valuable insights into the complex interplay of magnetic fields within the LHC, laying crucial groundwork for future advancements in collider performance and stability.

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# Appendices



# A

# Hamiltonians and Transfer Maps

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This appendix is intended to gather and explicit the Hamiltonians of the elements used in this thesis. Non-linear transfer maps are also described for some of those elements from the first to the second order.

## A.1. Hamiltonians of Elements

The Hamiltonian of a *multipole* is the following [25, 27, 82]:

$$H = \Re \left[ \sum_{n>1} (K_n + iJ_n) \frac{(x+iy)^n}{n!} \right]. \quad (\text{A.1})$$

From this, normal and skew fields can be separated:

$$\begin{aligned} N_n &= \frac{1}{n!} K_n \Re [(x+iy)^n] \\ S_n &= -\frac{1}{n!} J_n \Im [(x+iy)^n], \end{aligned} \quad (\text{A.2})$$

where  $K$  and  $J$  are the normalized strength of the multipole and  $x, y$  the transverse coordinates. Table A.1 explicits the normal and skew Hamiltonians of multipoles up to order 8.

Name	Order	Normal and Skew Hamiltonians
Drift	-	$H = \frac{1}{2}(p_x^2 + p_y^2)$
Quadrupole	2	$N_2 = \frac{1}{2!}K_2(x^2 - y^2)$ $S_2 = -J_2xy$
Sextupole	3	$N_3 = \frac{1}{3!}K_3(x^3 - 3xy^2)$ $S_3 = -\frac{1}{3!}J_3 \cdot (3x^2y - y^3)$
Octupole	4	$N_4 = \frac{1}{4!}K_4(x^4 - 6x^2y^2 + y^4)$ $S_4 = -\frac{1}{4!}J_4 \cdot (4x^3y - 4xy^3)$
Decapole	5	$N_5 = \frac{1}{5!}K_5(x^5 - 10x^3y^2 + 5xy^4)$ $S_5 = -\frac{1}{5!}J_5 \cdot (5x^4y - 10x^2y^3 + y^5)$
Dodecapole	6	$N_6 = \frac{1}{6!}K_6(x^6 - 15x^4y^2 + 15x^2y^4 - y^6)$ $S_6 = -\frac{1}{6!}J_6 \cdot (6x^5y - 20x^3y^3 + 6xy^5)$
Decatetrapole	7	$N_7 = \frac{1}{7!}K_7(x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6)$ $S_7 = -\frac{1}{7!}J_7 \cdot (7x^6y - 35x^4y^3 + 21x^2y^5 - y^7)$
Decahexapole	8	$N_8 = \frac{1}{8!}K_8(x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8)$ $S_8 = -\frac{1}{8!}J_8 \cdot (8x^7y - 56x^5y^3 + 56x^3y^5 - 8xy^7)$

Table A.1.: Normal and skew Hamiltonians of multipoles up to order 8.

## A.2. Transfer Maps

This sections goes more in depth regarding the derivation of the examples of transfer maps introduced in Section 2.4.

As a reminder, the BCH of two elements up to order 3 is given below,

$$Z = \underbrace{H_2 + H_1}_{\text{First order}} + \underbrace{\frac{[H_2, H_1]}{2}}_{\text{Second order}} + \underbrace{\frac{[H_2, [H_2, H_1]]}{12} - \frac{[H_1, [H_2, H_1]]}{12}}_{\text{Third order}}. \quad (\text{A.3})$$

### A.2.1. Generic Effective Hamiltonian of Two Elements

In order not to have to derive every combination of multipoles, a generic approach can be taken. Two elements of orders  $n$  and  $m$  can indeed be combined together via the BCH. Their hamiltonians is thus the one from Eq. (A.1) with orders  $m$  and  $n$ . A drift  $D$  is put in between the two multipoles  $H_1$  and  $H_2$  to change the coordinates. This results in non-zero Poisson brackets as the momentum is propagated,

$$[H_2, H_1] \rightarrow [H_2, D \cdot H_1]. \quad (\text{A.4})$$

Higher orders will arise with derivations when applying the Poisson brackets. Those orders can be found by counting the number of Poisson brackets, and which operators are implicated. A Poisson bracket implicating  $H_2$  will increase the resulting order by  $m$ , and likewise for  $H_1$  and  $n$ . The resulting order is then decreased by the number of poisson brackets times 2. Equation (A.5) gives how this can be calculated when considering the written form of the Poisson brackets (eg.  $[H_2, [H_2, H_1]]$ ).

$$\text{resulting order} = \text{count}(“H_2”) \cdot m + \text{count}(“H_1”) \cdot n - 2 \cdot \text{count}(“[“]) \quad (\text{A.5})$$

Although this formula is quite helpful, it is not trivial to compute which orders will arise from each BCH order. Indeed, not all combinations of  $H_1$  and  $H_2$  appear, as explained in details in [23]. Table A.2 shows the Poisson brackets corresponding to each order of the BCH, up to order 6, along with the resulting multipole order. To keep the table readable, Poisson Brackets are not shown above that order, as duplicates of resulting orders become quite frequent. Similarly, Table A.3 shows how fields can be generated depending on the orders of multipoles and that of the BCH.

### A. Hamiltonians and Transfer Maps

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BCH Order	Poisson Brackets	Resulting Order
1	$A$ $B$	$m$ $n$
2	$\frac{[A,B]}{2}$	$m+n-2$
3	$-\frac{[B,[A,B]]}{12}$ $\frac{[A,[A,B]]}{12}$	$m+2n-4$ $2m+n-4$
4	$-\frac{[B,[A,[A,B]]]}{24}$	$2m+2n-6$
5	$-\frac{[A,[B,[A,[A,B]]]]}{120}$ $-\frac{[B,[A,[A,[A,B]]]]}{180}$ $-\frac{[A,[B,[B,[A,B]]]]}{360}$ $-\frac{[A,[A,[A,[A,B]]]]}{720}$ $-\frac{[B,[B,[A,[A,B]]]]}{180}$ $-\frac{[B,[B,[B,[A,B]]]]}{720}$	$3m+2n-8$ $3m+2n-8$ $2m+3n-8$ $4m+n-8$ $2m+3n-8$ $m+4n-8$
6	$-\frac{[A,[A,[B,[B,[A,B]]]]]}{240}$ $-\frac{[A,[B,[B,[A,[A,B]]]]]}{240}$ $-\frac{[B,[B,[A,[A,[A,B]]]]]}{360}$ $-\frac{[A,[B,[B,[B,[A,B]]]]]}{720}$ $-\frac{[B,[A,[A,[A,[A,B]]]]]}{1440}$ $-\frac{[B,[B,[B,[A,[A,B]]]]]}{1440}$	$3m+3n-10$ $3m+3n-10$ $3m+3n-10$ $2m+4n-10$ $4m+2n-10$ $2m+4n-10$
7	...	$2m+5n-12$ $m+6n-12$ $3m+4n-12$ $5m+2n-12$ $6m+n-12$ $4m+3n-12$

Table A.2.: Resulting multipole orders arising from the Poisson brackets of a given BCH order for two multipoles  $A$  and  $B$  of order  $m$  and  $n$ . At order 7, the Poisson brackets are not given as duplicates grow the list significantly.

Field	BCH Order					
	1	2	3	4	5	6
3	$H_3$					
4	$H_4$	$(H_3)^2$				
5	$H_5$	$H_3 H_4$	$(H_3)^3$			
6	$H_6$	$H_3 H_5,$ $(H_4)^2$	$(H_3)^2 H_4$	$(H_3)^4$		
7	$H_7$	$H_3 H_6,$ $H_4 H_5$	$(H_3)^2 H_5,$ $H_3 (H_4)^2$		$(H_3)^5$	
8	$H_8$	$(H_5)^2,$ $H_3 H_7,$ $H_4 H_6$	$(H_3)^2 H_6,$ $(H_4)^3$	$(H_3)^2 (H_4)^2$	$(H_3)^4 H_4$	$(H_3)^6$
9	$H_9$	$H_4 H_7,$ $H_3 H_8,$ $H_5 H_6$	$(H_3)^2 H_7,$ $(H_4)^2 H_5,$ $H_3 (H_5)^2$		$(H_3)^3 (H_4)^2,$ $(H_3)^4 H_5$	
10	$H_{10}$	$H_4 H_8,$ $H_5 H_7,$ $H_3 H_9,$ $(H_6)^2$	$(H_3)^2 H_8,$ $H_4 (H_5)^2,$ $(H_4)^2 H_6$	$(H_4)^4,$ $(H_3)^2 (H_5)^2$	$(H_3)^2 (H_4)^3,$ $(H_3)^4 H_6$	$(H_3)^4 (H_4)^2$
11	$H_{11}$	$H_4 H_9,$ $H_3 H_{10},$ $H_6 H_7,$ $H_5 H_8$	$(H_4)^2 H_7,$ $H_3 (H_6)^2,$ $(H_5)^3,$ $(H_3)^2 H_9$		$(H_3)^4 H_7,$ $H_3 (H_4)^4,$ $(H_3)^3 (H_5)^2$	
12	$H_{12}$	$H_6 H_8,$ $(H_7)^2,$ $H_5 H_9,$ $H_3 H_{11},$ $H_4 H_{10}$	$H_4 (H_6)^2,$ $(H_4)^2 H_8,$ $(H_5)^2 H_6,$ $(H_3)^2 H_{10}$	$(H_3)^2 (H_6)^2,$ $(H_4)^2 (H_5)^2$	$(H_3)^4 H_8,$ $(H_4)^5$	$(H_3)^4 (H_5)^2,$ $(H_3)^2 (H_4)^4$

Table A.3.: Correspondence of a combination of multipoles from a BCH order to multipole-like fields. The exponents indicate the order of the BCH for individual components.

## A.2.2. Transfer Map of Two Sextupoles

The transfer map of two sextupoles  $H_1$  and  $H_2$  of strength  $K_1$  and  $K_2$ , separated by a drift to introduce a change of coordinates in  $H_1$  is the following,

$$\mathcal{M} = e^{:Z:} = e^{:H_2:} \cdot e^{D:H_1:}, \quad (\text{A.6})$$

After such application of the drift on  $H_1$ , the two hamiltonians read,

$$\begin{aligned} H_1 &= \frac{1}{3!} K_1 L_1 \left( (L_D p_x + x)^3 - 3 (L_D p_x + x) (L_D p_y + y)^2 \right) \\ H_2 &= \frac{1}{3!} K_2 L_2 (x^3 - 3xy^2). \end{aligned} \quad (\text{A.7})$$

Below are detailed each term of the BCH. Each term should be added together in order to obtain the whole effective Hamiltonian  $Z$ .

### First Order

$$K_1 L_1 L_D \left\{ \begin{aligned} &\left( \frac{L_D^2 p_x^3}{6} - \frac{L_D^2 p_x p_y^2}{2} + \frac{L_D p_x^2 x}{2} - L_D p_x p_y y \right) \\ &\left( -\frac{L_D p_y^2 x}{2} + \frac{p_x x^2}{2} - \frac{p_x y^2}{2} - p_y x y \right) \\ &+ K_1 L_1 \left( \frac{x^3}{6} - \frac{xy^2}{2} \right) + K_2 L_2 \left( \frac{x^3}{6} - \frac{xy^2}{2} \right) \end{aligned} \right\} \text{sextupolar} \quad (\text{A.8})$$

### Second Order

$$K_1 K_2 L_1 L_2 L_D \left\{ \begin{aligned} &\left( \frac{L_D^2 p_x^2 x^2}{8} - \frac{L_D^2 p_x^2 y^2}{8} + \frac{L_D^2 p_x p_y x y}{2} - \frac{L_D^2 p_y^2 x^2}{8} \right) \\ &\left( + \frac{L_D^2 p_y^2 y^2}{8} + \frac{L_D p_x x^3}{4} + \frac{L_D p_x x y^2}{4} + \frac{L_D p_y x^2 y}{4} \right) \\ &\left( + \frac{L_D p_y y^3}{4} + \frac{x^4}{8} + \frac{x^2 y^2}{4} + \frac{y^4}{8} \right) \end{aligned} \right\} \text{octupolar-like} \quad (\text{A.9})$$

**Third Order**

$$\begin{aligned}
& K_1^2 K_2 L_1^2 L_2 L_D \left( \begin{array}{l} \frac{L_D^5 p_x^4 x}{48} + \frac{L_D^5 p_x^3 p_y y}{12} - \frac{L_D^5 p_x^2 p_y^2 x}{8} - \frac{L_D^5 p_x p_y^3 y}{12} \\ + \frac{L_D^5 p_y^4 x}{48} + \frac{L_D^4 p_x^3 x^2}{12} + \frac{L_D^4 p_x^3 y^2}{12} - \frac{L_D^4 p_x p_y^2 x^2}{4} \\ - \frac{L_D^4 p_x p_y^2 y^2}{4} + \frac{L_D^3 p_x^2 x^3}{8} + \frac{L_D^3 p_x^2 x y^2}{8} - \frac{L_D^3 p_x p_y x^2 y}{4} \\ - \frac{L_D^3 p_x p_y y^3}{4} - \frac{L_D^3 p_y^2 x^3}{8} - \frac{L_D^3 p_y^2 x y^2}{8} + \frac{L_D^2 p_x x^4}{12} \\ - \frac{L_D^2 p_x y^4}{12} - \frac{L_D^2 p_y x^3 y}{6} - \frac{L_D^2 p_y x y^3}{6} + \frac{L_D x^5}{48} \\ - \frac{L_D x^3 y^2}{24} - \frac{L_D x y^4}{16} \end{array} \right) \\
& + K_1 K_2^2 L_1 L_2^2 L_D \left( \begin{array}{l} \frac{L_D^2 p_x x^4}{48} - \frac{L_D^2 p_x x^2 y^2}{8} + \frac{L_D^2 p_x y^4}{48} + \frac{L_D^2 p_y x^3 y}{12} \\ - \frac{L_D^2 p_y x y^3}{12} + \frac{L_D x^5}{48} - \frac{L_D x^3 y^2}{24} - \frac{L_D x y^4}{16} \end{array} \right)
\end{aligned} \quad \text{decapolar-like} \quad (\text{A.10})$$

**A.2.3. Transfer Map of a Sextupole and Octupole**

The transfer map of a sextupole  $H_1$  and octupole  $H_2$  of strength  $K_1$  and  $K_2$ , separated by a drift like in the previous example is given by

$$\mathcal{M} = e^{[Z]} = e^{[H_2]} \cdot e^{D[H_1]} \quad (\text{A.11})$$

with  $H_1$  and  $H_2$  having as final expressions,

$$\begin{aligned}
H_1 &= \frac{1}{3!} K_{3,h1} L_1 \left( (L_D p_x + x)^3 - 3 (L_D p_x + x) (L_D p_y + y)^2 \right) \\
H_2 &= \frac{1}{4!} K_2 L_2 (x^4 - 6x^2 y^2 + y^4).
\end{aligned} \quad (\text{A.12})$$

The first two orders of the BCH of those two elements is given below.

### First Order

$$K_3 \left\{ \begin{aligned} & \left( \frac{L_D^3 p_x^3}{6} - \frac{L_D^3 p_x p_y^2}{2} + \frac{L_D^2 p_x^2 x}{2} - L_D^2 p_x p_y y - \frac{L_D^2 p_y^2 x}{2} \right) \\ & + \left( \frac{L_D p_x x^2}{2} - \frac{L_D p_x y^2}{2} - L_D p_y x y + \frac{x^3}{6} - \frac{x y^2}{2} \right) \end{aligned} \right\} \text{ sextupolar} \\ + K_4 \left( \frac{x^4}{24} - \frac{x^2 y^2}{4} + \frac{y^4}{24} \right) \left\{ \begin{aligned} & \text{octupolar} \end{aligned} \right\} \quad (\text{A.13})$$

### Second Order

$$K_3 K_4 L_D \left\{ \begin{aligned} & \left( \frac{L_D^2 p_x^2 x^3}{24} - \frac{L_D^2 p_x^2 x y^2}{8} + \frac{L_D^2 p_x p_y x^2 y}{4} - \frac{L_D^2 p_x p_y y^3}{12} \right) \\ & - \left( \frac{L_D^2 p_y^2 x^3}{24} + \frac{L_D^2 p_y^2 x y^2}{8} + \frac{L_D p_x x^4}{12} - \frac{L_D p_x y^4}{12} \right) \\ & + \left( \frac{L_D p_y x^3 y}{6} + \frac{L_D p_y x y^3}{6} + \frac{x^5}{24} + \frac{x^3 y^2}{12} + \frac{x y^4}{24} \right) \end{aligned} \right\} \text{ decapolar-like} \quad (\text{A.14})$$

### A.2.4. Transfer Map of a Skew Quadrupole and Octupole

The transfer map of a skew quadrupole  $H_1$  and octupole  $H_2$  of strength  $K_1$  and  $K_2$ , separated by a drift like in the previous examples is given by

$$\mathcal{M} = e^{:Z:} = e^{:H_2:} \cdot e^{D:H_1:} \quad (\text{A.15})$$

with  $H_1$  and  $H_2$  having as final expressions,

$$\begin{aligned} H_1 &= -J_1 L_1 (L_D p_x + x) (L_D p_y + y) \\ H_2 &= \frac{1}{4!} K_2 L_2 (x^4 - 6x^2 y^2 + y^4). \end{aligned} \quad (\text{A.16})$$

The first two orders of the BCH of those two elements is given below.

**First Order**

$$\left. \begin{aligned} & J_1 L_1 (-L_D^2 p_x p_y - L_D p_x y - L_D p_y x - xy) \\ & + K_2 L_2 \left( \frac{x^4}{24} - \frac{x^2 y^2}{4} + \frac{y^4}{24} \right) \end{aligned} \right\} \begin{array}{l} \text{skew quadrupolar} \\ \text{octupolar} \end{array} \quad (\text{A.17})$$

**Second Order**

$$J_1 K_2 L_1 L_2 L_D \left( \begin{aligned} & \left( \frac{L_D p_x x^2 y}{4} - \frac{L_D p_x y^3}{12} - \frac{L_D p_y x^3}{12} \right) \\ & + \frac{L_D p_y x y^2}{4} + \frac{x^3 y}{6} + \frac{x y^3}{6} \end{aligned} \right) \left\} \text{skew octupolar-like} \quad (\text{A.18}) \right.$$

A

# Chromatic Amplitude Detuning

This appendix details the derivations of chromatic amplitude detuning from sextupoles up to dodecapoles. *Chromatic Amplitude Detuning*, being detuning coming from both the action and the momentum offset, only starts appearing with decapoles. Below that order, only amplitude detuning and chromaticity can be observed. As those are part of the derivations, they will also be detailed here.

## B.1. Derivations

In this section, the terms of the chromatic amplitude detuning are given up to dodecapoles. Derivations are given only for sextupoles and octupoles, as the process remains fairly similar for higher orders. Vertical offsets will be neglected, as horizontal dispersion is dominant. The reasoning remains the same.

Up to the third order, the expression of the Taylor expansion of the Chromatic Amplitude Detuning around  $\epsilon_x$ ,  $\epsilon_y$  and  $\delta$ , for a tune  $Q_z$ ,  $z \in \{x, y\}$  reads:

$$\begin{aligned}
 Q_z(\epsilon_x, \epsilon_y, \delta) = & Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\
 & + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\
 & \quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] \\
 & + \frac{1}{3!} \left[ \frac{\partial^3 Q_z}{\partial \delta^3} \delta^3 + \frac{\partial^3 Q_z}{\partial \epsilon_x^3} \epsilon_x^3 + \frac{\partial^3 Q_z}{\partial \epsilon_y^3} \epsilon_y^3 \right. \\
 & \quad \left. + 3 \frac{\partial^3 Q_z}{\partial \epsilon_x \partial \delta^2} \delta^2 \epsilon_x + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \delta^2} \delta^2 \epsilon_y + 3 \frac{\partial^3 Q_z}{\partial \epsilon_x^2 \partial \delta} \delta \epsilon_x^2 \right. \\
 & \quad \left. + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y^2 \partial \delta} \delta \epsilon_y^2 + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \epsilon_x^2} \epsilon_x^2 \epsilon_y + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y^2 \partial \epsilon_x} \epsilon_x \epsilon_y^2 \right. \\
 & \quad \left. + 6 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \epsilon_x \partial \delta} \delta \epsilon_x \epsilon_y \right] + \dots
 \end{aligned} \tag{B.1}$$

### B.1.1. Principle

From [28], the detuning caused by a magnet of length L can be described by its hamiltonian with

$$\Delta Q_z = \frac{1}{2\pi} \int_L \frac{\partial \langle H \rangle}{\partial J_z} ds. \tag{B.2}$$

The usual variables  $x$  and  $y$  of Eq. (2.13) can be replaced by *action-angle* variables to introduce the action:

$$\begin{aligned}
 x &\rightarrow \sqrt{2J_x \beta_x} \cos \phi_x \\
 y &\rightarrow \sqrt{2J_y \beta_y} \cos \phi_y
 \end{aligned} \tag{B.3}$$

A momentum dependence can be introduced for a particle with a different orbit

$(\Delta z)$  [84] via dispersion. Combined with Eq. (B.3), a dependence on all required components is achieved:

$$\begin{aligned} x + \Delta x &\rightarrow \sqrt{2J_x\beta_x} \cos \phi_x + D_x \delta \\ y + \Delta y &\rightarrow \sqrt{2J_y\beta_y} \cos \phi_y + D_y \delta \end{aligned} \quad (\text{B.4})$$

After averaging over the phase variable, all that is left is to compute the partial derivatives.

### B.1.2. Sextupole

Taken from Table A.1, the normal field of a sextupole is  $N_3 = \frac{1}{3!}K_3(x^3 - 3xy^2)$ . Introducing  $\delta$  via an orbit offset and changing the variables to action-angle, as given in Eq. (B.4), leads to the following expression:

$$N_3 = \frac{1}{6}K_3 \left[ -3 \left( \delta D_x + \sqrt{2J_x\beta_x} \cos \phi_x \right) \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^2 + \left( \delta D_x + \sqrt{2J_x\beta_x} \cos \phi_x \right)^3 \right] \quad (\text{B.5})$$

Averaging over the cosines means integrating over  $[0, \pi]$  and dividing by  $\pi$ :

$$\begin{aligned} \langle N_3 \rangle &= \frac{1}{\pi} \int_0^\pi N_3 \\ &= \frac{1}{6\pi^2} K_3 \cdot (3\pi^2 J_x \beta_x \delta D_x - 3\pi^2 J_y \beta_y \delta D_x + \pi^2 \delta^3 D_x^3) \end{aligned} \quad (\text{B.6})$$

Differentiating by either  $J_x$  or  $J_y$  gives the tune shift induced by a single sextupole:

$$\begin{aligned} \Delta Q_x &= \frac{1}{4\pi} K_3 \beta_x D_x \delta L \\ \Delta Q_y &= - \frac{1}{4\pi} K_3 \beta_y D_x \delta L \end{aligned} \quad (\text{B.7})$$

From now on, differentiating by the action would give amplitude detuning, and by  $\delta$  the chromaticity. Cross-terms exist but evidently depend on the expression of the multipole. For a sextupole, differentiating by the action does not have an effect, as no action is present in the tune shift equation. Rather, sextupoles are known to contribute to the first order chromaticity  $Q'$ . The following gives a recap of those operations:

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$$\begin{aligned} \frac{\partial Q_x}{\partial J_x} &= 0 & ; \quad \frac{\partial Q_x}{\partial J_y} &= 0 & ; \quad \frac{\partial Q_x}{\partial \delta} &= \frac{1}{4\pi} K_3 \beta_x D_x L \\ \frac{\partial Q_y}{\partial J_x} &= 0 & ; \quad \frac{\partial Q_y}{\partial J_y} &= 0 & ; \quad \frac{\partial Q_y}{\partial \delta} &= -\frac{1}{4\pi} K_3 \beta_y D_x L \end{aligned} \quad (\text{B.8})$$

The overall contribution of sextupoles to the Chromatic Amplitude Detuning is then highlighted in the following:

$$Q_z(\epsilon_x, \epsilon_y, \delta) = Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \quad (\text{B.9})$$

### B.1.3. Octupole

With the normal hamiltonian of octupoles being  $N_4 = \frac{1}{4!} K_4 (x^4 - 6x^2y^2 + y^4)$ , applying an orbit offset and changing for action-angle variables gives the following:

$$\begin{aligned}
N_4 = & \frac{1}{24} K_4 \left[ \left( \sqrt{2J_x\beta_x} \cos \phi_x \right)^4 \right. \\
& + 4 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right)^3 D_x \delta \\
& + 6 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right)^2 D_x^2 \delta^2 \\
& + 4 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right) D_x^2 \delta \\
& + D_x^4 \delta^4 \\
& - 6 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right)^2 \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^2 \\
& - 6 \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^2 \cdot 2 \left( \sqrt{2J_x\beta_x} \cos \phi_x \right) D_x \delta \\
& - 6 \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^2 D_x^2 \delta^2 \\
& \left. + \left( \sqrt{2J_y\beta_y} \cos \phi_y \right)^4 \right] \tag{B.10}
\end{aligned}$$

Average over the phase variables gives the following:

$$\begin{aligned}
\langle N_4 \rangle = & \frac{1}{\pi} \int_0^\pi N_4 \\
= & \frac{1}{48} K_4 \left[ 3J_x^2 \beta_x^2 + 12J_x \beta_x D_x^2 \delta^2 + 2D_x^4 \delta^4 \right. \\
& \left. - 12J_y \beta_y (J_x \beta_x + D_x^2 \delta^2) + 3J_y^2 \beta_y^2 \right] \tag{B.11}
\end{aligned}$$

Differentiating by either  $J_x$  or  $J_y$  gives the tune shift induced by a single octupole. Contrary to sextupoles, it can be noted that some action terms  $J_{x,y}$  remain in the

expression.

$$\begin{aligned}\Delta Q_x &= \frac{1}{48\pi} K_4 L \left[ 3J_x \beta_x^2 + 6\beta_x D_x^2 \delta^2 - 6\beta_x J_y \beta_y \right] \\ \Delta Q_y &= \frac{1}{48\pi} K_4 L \left[ -6J_x \beta_x \beta_y - 6\beta_y D_x^2 \delta^2 + 3J_y \beta_y^2 \right]\end{aligned}\quad (\text{B.12})$$

Differentiating by the action  $J_{x,y}$  yields amplitude detuning, while differentiating by the momentum offset yields the second order chromaticity  $Q''$ :

$$\begin{aligned}\frac{\partial Q_x}{\partial J_x} &= \frac{1}{16\pi} K_4 \beta_x^2 L \quad ; \quad \frac{\partial Q_x}{\partial J_y} = -\frac{1}{8\pi} K_4 \beta_x \beta_y L \quad ; \quad \frac{\partial^2 Q_x}{\partial \delta^2} = \frac{1}{4\pi} K_4 \beta_x D_x^2 L \\ \frac{\partial Q_y}{\partial J_x} &= -\frac{1}{8\pi} K_4 \beta_x \beta_y L \quad ; \quad \frac{\partial Q_y}{\partial J_y} = \frac{1}{16\pi} K_4 \beta_y^2 L \quad ; \quad \frac{\partial^2 Q_y}{\partial \delta^2} = -\frac{1}{4\pi} K_4 \beta_y D_x^2 L\end{aligned}\quad (\text{B.13})$$

The overall contribution of octupoles to the Chromatic Amplitude Detuning is then highlighted in the following:

$$\begin{aligned}Q_z(\epsilon_x, \epsilon_y, \delta) &= Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\ &\quad + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\ &\quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right]\end{aligned}\quad (\text{B.14})$$

#### B.1.4. Decapole

Skipping the first steps detailed previously, the tune shift induced by a normal decapole is given by the following:

$$\begin{aligned}\Delta Q_x &= \frac{1}{240\pi} K_5 L \left[ 10D_x^3 \delta^3 \beta_x + 15D_x \delta J_x \beta_x^2 - 30J_y \beta_y \beta_x D_x \delta \right] \\ \Delta Q_y &= \frac{1}{240\pi} K_5 L \left[ -10D_x^3 \delta^3 \beta_y + 15D_x \delta J_y \beta_y^2 - 30J_x \beta_y \beta_x D_x \delta \right]\end{aligned}\quad (\text{B.15})$$

The terms of the chromatic amplitude detuning can now be computed. Unlike sextupoles and octupoles, cross-terms between  $\delta$  and the action now appear, giving rise to *chromatic amplitude detuning* and the third order chromaticity  $Q'''$ :

$$\begin{aligned} \frac{\partial^2 Q_x}{\partial J_x \partial \delta} &= -\frac{1}{16\pi} K_5 \beta_x^2 D_x L \quad ; \quad \frac{\partial^2 Q_x}{\partial J_y \partial \delta} = -\frac{1}{8\pi} K_5 \beta_x \beta_y D_x L \quad ; \quad \frac{\partial^3 Q_x}{\partial \delta^3} = \frac{1}{4\pi} K_5 \beta_x D_x^3 L \\ \frac{\partial^2 Q_y}{\partial J_x \partial \delta} &= -\frac{1}{8\pi} K_5 \beta_x \beta_y D_x L \quad ; \quad \frac{\partial^2 Q_y}{\partial J_y \partial \delta} = -\frac{1}{16\pi} K_5 \beta_y^2 D_x L \quad ; \quad \frac{\partial^3 Q_y}{\partial \delta^3} = -\frac{1}{4\pi} K_5 \beta_y D_x^3 L \end{aligned} \quad (\text{B.16})$$

The contribution of decapoles to Chromatic Amplitude Detuning is then the following:

$$\begin{aligned} Q_z(\epsilon_x, \epsilon_y, \delta) &= Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\ &\quad + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + \frac{\partial^2 Q_z}{\partial \delta^2} \delta^2 \right. \\ &\quad \left. + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \delta} \epsilon_x \delta + 2 \frac{\partial^2 Q_z}{\partial \delta \partial \epsilon_y} \delta \epsilon_y \right] \\ &\quad + \frac{1}{3!} \left[ \frac{\partial^3 Q_z}{\partial \delta^3} \delta^3 + \dots \right] \end{aligned} \quad (\text{B.17})$$

### B.1.5. Dodecapole

The tune shift induced by a normal dodecapole is given by the following:

$$\begin{aligned} Q_x &= \frac{1}{720\pi} K_6 L \left[ 15J_x^2 \beta_x^3 + 30\beta_x D_x^4 \delta^4 + 90J_x \beta_x^2 D_x^2 \delta^2 \right. \\ &\quad \left. - 180J_y \beta_y \beta_x D_x^2 \delta^2 - 90J_y \beta_y J_x \beta_x^2 + 90J_y^2 \beta_y^2 \beta_x \right] \\ Q_y &= \frac{1}{720\pi} K_6 L \left[ -30\beta_y D_x^4 \delta^4 - 180\beta_y J_x \beta_x D_x^2 \delta^2 - 90\beta_y J_x^2 \beta_x^2 \right. \\ &\quad \left. + 90J_y \beta_y^2 J_x \beta_x + 45J_y \beta_y^2 D_x^2 \delta^2 - 15J_y^2 \beta_y^3 \right] \end{aligned} \quad (\text{B.18})$$

## B. Chromatic Amplitude Detuning

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The terms of the chromatic amplitude detuning can now be computed. Like for decapoles, chromatic amplitude detuning terms appear, as well as the third order chromaticity  $Q^{(4)}$ :

$$\begin{aligned}
 \frac{\partial^2 Q_x}{\partial J_x^2} &= \frac{1}{96\pi} K_6 \beta_x^3 L & ; \quad \frac{\partial^2 Q_y}{\partial J_y^2} &= -\frac{1}{96\pi} K_6 \beta_y^3 L \\
 \frac{\partial^3 Q_x}{\partial J_x \partial \delta^2} &= \frac{1}{16\pi} K_6 \beta_x^2 D_x^2 L & ; \quad \frac{\partial^3 Q_y}{\partial J_y \partial \delta^2} &= \frac{1}{16\pi} K_6 \beta_y^2 D_x^2 L \\
 \frac{\partial^2 Q_x}{\partial J_y^2} &= \frac{1}{32\pi} K_6 \beta_y^2 \beta_x L & ; \quad \frac{\partial^2 Q_y}{\partial J_x^2} &= -\frac{1}{32\pi} K_6 \beta_y \beta_x^2 L \\
 \frac{\partial^3 Q_x}{\partial J_y \partial \delta^2} &= -\frac{1}{8\pi} K_6 \beta_y \beta_x D_x^2 L & ; \quad \frac{\partial^3 Q_y}{\partial J_x \partial \delta^2} &= -\frac{1}{8\pi} K_6 \beta_y \beta_x D_x^2 L \\
 \frac{\partial^2 Q_x}{\partial J_x \partial J_y} &= -\frac{1}{32\pi} K_6 \beta_y \beta_x^2 L & ; \quad \frac{\partial^2 Q_y}{\partial J_y \partial J_x} &= \frac{1}{32\pi} K_6 \beta_y^2 \beta_x L \\
 \frac{\partial^4 Q_x}{\partial \delta^4} &= \frac{1}{4\pi} K_6 \beta_x D_x^4 L & ; \quad \frac{\partial^4 Q_y}{\partial \delta^4} &= -\frac{1}{4\pi} K_6 \beta_y D_x^4 L
 \end{aligned} \tag{B.19}$$

The contribution of dodecapoles to Chromatic Amplitude Detuning is then the following:

$$\begin{aligned}
 Q_z(\epsilon_x, \epsilon_y, \delta) &= Q_{z0} + \left[ \frac{\partial Q_z}{\partial \epsilon_x} \epsilon_x + \frac{\partial Q_z}{\partial \epsilon_y} \epsilon_y + \frac{\partial Q_z}{\partial \delta} \delta \right] \\
 &\quad + \frac{1}{2!} \left[ \frac{\partial^2 Q_z}{\partial \epsilon_x^2} \epsilon_x^2 + \frac{\partial^2 Q_z}{\partial \epsilon_y^2} \epsilon_y^2 + 2 \frac{\partial^2 Q_z}{\partial \epsilon_x \partial \epsilon_y} \epsilon_x \epsilon_y + \dots \right] \\
 &\quad + \frac{1}{3!} \left[ 3 \frac{\partial^3 Q_z}{\partial \epsilon_x \partial \delta^2} \delta^2 \epsilon_x + 3 \frac{\partial^3 Q_z}{\partial \epsilon_y \partial \delta^2} \delta^2 \epsilon_y + \dots \right] \\
 &\quad + \frac{1}{4!} \left[ \frac{\partial^4 Q_z}{\partial \delta^4} \delta^4 \right]
 \end{aligned} \tag{B.20}$$

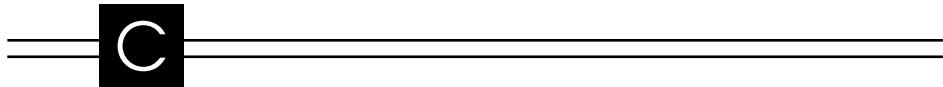
## B.2. PTC Validation

A simulation has been done with PTC to assess that those equations are correct. A dodecapole has been added to the lattice with a strength  $KL = 1e^6$ . Here are the results, confirming PTC works as intended.

The ANH numbers refer to the partial derivative relative to  $J_x, J_y$  and  $\delta$ . So ANHX 021 would for example be  $\frac{\partial^3 Q_x}{\partial J_y^2 \partial \delta}$ .

Term	Analytical	Simulation	Rel. Diff [%]
ANH X 200	4782639.96971	4782639.97	0.0
ANH X 102	86945.930342	86945.93	-0.0
ANH X 020	593469879.552116	593469880.01	0.0
ANH X 012	-1118366.433407	-1118366.433	-0.0
ANH X 110	-92277073.535598	-92277073.6	0.0
ANH X 004	1053.754809	1053.7548	-0.000001
ANH Y 200	-92277073.535598	-92277073.6	0.0
ANH Y 102	-1118366.433407	-1118366.433	-0.0
ANH Y 020	-1272278817.264865	-1272278818.913	0.0
ANH Y 012	3596325.539479	3596325.543	0.0
ANH Y 110	593469879.552116	593469880.01	0.0
ANH Y 004	-6777.108503	-6777.1085	-0.0

B



# Resonance Driving Terms

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C

This appendix intends to clarify where Resonance Driving Terms can be seen in the frequency spectrum, what resonance they contribute to and what their action dependence is.

## C.1. Expressions

The number of valid RDTs indeed grows rapidly with the magnet order  $n$ , as shows Table C.1, and is given by the following combinations:

$$C(n+3, 3) - C(n+1, 1) - [(n+1) \bmod 2] \cdot C\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 1\right). \quad (\text{C.1})$$

Multipole	Order	Number of poles	Number of RDTs
Quadrupole	2	4	5
Sextupole	3	6	16
Octupole	4	8	27
Decapole	5	10	50
Dodecapole	6	12	73
Decatetrapole	7	14	112
Decahexapole	8	16	151
Hectopole	50	100	23349
Kilopole	500	1000	$2.1 \times 10^7$

Table C.1.: Number of valid RDTs for a given multipole order

Several different RDTs can contribute to the same line, which can be observed in the horizontal or vertical spectrum. The tables below describe which RDTs contribute to a specific combination of line and plane. All tables have been computed up to the order 6, for decapoles. The line columns represents  $(Q_x, Q_y)$ . For example  $(-1, 2)$  is  $-1Q_x + 2Q_y$ .

As a reminder, for a given RDT  $f_{jklm}$ , we will observe:

$$\begin{aligned} (j - k)Q_x + (l - m)Q_y = p &\in \mathbb{N} && \text{excited resonance} \\ H(1 - j + k, m - l) & && \text{horizontal line, if } j \neq 0 \\ V(k - j, 1 - l + m) & && \text{vertical line, if } l \neq 0. \end{aligned} \quad (\text{C.2})$$

The amplitude of each line is given by:

$$\begin{aligned} |H_{f_{jklm}}| &= 2j(2I_x)^{\frac{j+k-1}{2}}(2I_y)^{\frac{l+m}{2}}|f_{jklm}| \\ |V_{f_{jklm}}| &= 2l(2I_x)^{\frac{j+k}{2}}(2I_y)^{\frac{l+m-1}{2}}|f_{jklm}|. \end{aligned} \quad (\text{C.3})$$

According to Eq. (C.2) and Eq. (C.3), it can be seen that many RDTs will not contribute to any line and thus can not be observed.

## C.2. Frequency Spectrum Lines

The following Table C.2 shows which RDTs can be seen at a factor  $Q_x \pm Q_y$  in the vertical or horizontal spectrums. This table is mostly useful when trying to identify which multipole or RDT contributes to a specific line that can appear during measurements.

$Q_x$	$Q_y$	H-jklm	V-jklm
-5	0	6000	5010
-4	-1	5010	4020
-4	0	5000	4010
-4	1	5001	4011
-3	-2	4020	3030
-3	-1	4010	3020
-3	0	4000, 4011, 5100	3010, 3021, 4110
-3	1	4001	3011
-3	2	4002	3012
-2	-3	3030	2040
-2	-2	3020	2030
-2	-1	3010, 3021, 4110	2020, 2031, 3120
-2	0	3000, 3011, 4100	2010, 2021, 3110
-2	1	3001, 3012, 4101	2011, 2022, 3111
-2	2	3002	2012
-2	3	3003	2013
-1	-4	2040	1050
-1	-3	2030	1040
-1	-2	2020, 2031, 3120	1030, 1041, 2130
-1	-1	2010, 2021, 3110	1020, 1031, 2120
-1	0	2000, 2011, 3100, 2022, 3111, 4200	1010, 1021, 2110, 1032, 2121, 3210
-1	1	2001, 2012, 3101	1011, 1022, 2111
-1	2	2002, 2013, 3102	1012, 1023, 2112
-1	3	2003	1013
-1	4	2004	1014
0	-5	1050	0060
0	-4	1040	0050
0	-3	1030, 1041, 2130	0040, 0051, 1140
0	-2	1020, 1031, 2120	0030, 0041, 1130

### C. Resonance Driving Terms

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$Q_x$	$Q_y$	H-jklm	V-jklm
0	-1	1010, 1021, 2110, 1032, 2121, 3210	0020, 0031, 1120, 0042, 1131, 2220
0	0	1011, 2100, 1022, 2111, 3200	0021, 1110, 0032, 1121, 2210
0	1	1001, 1012, 2101, 1023, 2112, 3201	
0	2	1002, 1013, 2102	0012, 0023, 1112
0	3	1003, 1014, 2103	0013, 0024, 1113
0	4	1004	0014
0	5	1005	0015
1	-4	1140	0150
1	-3	1130	0140
1	-2	1120, 1131, 2220	0130, 0141, 1230
1	0		0110, 0121, 1210, 0132, 1221, 2310
1	-1	1110, 1121, 2210	0120, 0131, 1220
1	1	1101, 1112, 2201	0111, 0122, 1211
1	2	1102, 1113, 2202	0112, 0123, 1212
1	3	1103	0113
1	4	1104	0114
2	-3	1230	0240
2	-2	1220	0230
2	-1	1210, 1221, 2310	0220, 0231, 1320
2	0	1200, 1211, 2300	0210, 0221, 1310
2	1	1201, 1212, 2301	0211, 0222, 1311
2	2	1202	0212
2	3	1203	0213
3	-2	1320	0330
3	-1	1310	0320
3	0	1300, 1311, 2400	0310, 0321, 1410
3	1	1301	0311
3	2	1302	0312
4	-1	1410	0420
4	0	1400	0410
4	1	1401	0411
5	0	1500	0510

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Table C.2.: Correspondence of Resonance Driving Terms to the lines seen in the horizontal or vertical frequency spectrum, up to dodecapoles ( $n = 6$ ).

## C.3. Amplitude, Resonances and Lines

This part focuses on individual Resonance Drivings Terms, detailing what magnet they originate from, what resonance they excite, how they can be observed and what kicks are needed in order to measure them. The amplitude columns implicitly omits the term  $|f_{jklm}|$ , which depends on  $K$  and  $J$ . The color coding helps quickly identifying dependences:

- $I_x$  : depends only on horizontal amplitude
- $I_y$  : depends only on vertical amplitude
- $I_x I_y$  : depends on both horizontal and vertical amplitudes

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
2	0020	normal	(0, 2)		(0, -1)		$4(2I_y)^{1/2}$
2	2000	normal	(2, 0)	(-1, 0)		$4(2I_x)^{1/2}$	
2	0110	skew	(-1, 1)		(1, 0)		$2(2I_x)^{1/2}$
2	1001	skew	(1, -1)	(0, 1)		$2(2I_y)^{1/2}$	
2	1010	skew	(1, 1)	(0, -1)	(-1, 0)	$2(2I_y)^{1/2}$	$2(2I_x)^{1/2}$
3	0111	normal	(-1, 0)		(1, 1)		$2(2I_x)^{1/2}(2I_y)^{1/2}$
3	0120	normal	(-1, 2)		(1, -1)		$4(2I_x)^{1/2}(2I_y)^{1/2}$
3	1002	normal	(1, -2)	(0, 2)		$2(2I_y)$	
3	1011	normal	(1, 0)	(0, 0)	(-1, 1)	$2(2I_y)$	$2(2I_x)^{1/2}(2I_y)^{1/2}$
3	1020	normal	(1, 2)	(0, -2)	(-1, -1)	$2(2I_y)$	$4(2I_x)^{1/2}(2I_y)^{1/2}$
3	1200	normal	(-1, 0)	(2, 0)		$2(2I_x)$	
3	2100	normal	(1, 0)	(0, 0)		$4(2I_x)$	
3	3000	normal	(3, 0)	(-2, 0)		$6(2I_x)$	
3	0012	skew	(0, -1)		(0, 2)		$2(2I_y)$
3	0021	skew	(0, 1)		(0, 0)		$4(2I_y)$
3	0030	skew	(0, 3)		(0, -2)		$6(2I_y)$
3	0210	skew	(-2, 1)		(2, 0)		$2(2I_x)$
3	1101	skew	(0, -1)	(1, 1)		$2(2I_x)^{1/2}(2I_y)^{1/2}$	
3	1110	skew	(0, 1)	(1, -1)	(0, 0)	$2(2I_x)^{1/2}(2I_y)^{1/2}$	$2(2I_x)$

### C. Resonance Driving Terms

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n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
3	2001	skew	(2, -1)	(-1, 1)		$4(2I_x)^{1/2}(2I_y)^{1/2}$	
3	2010	skew	(2, 1)	(-1, -1)	(-2, 0)	$4(2I_x)^{1/2}(2I_y)^{1/2}$	$2(2I_x)$
4	0013	normal	(0, -2)		(0, 3)		$2(2I_y)^{3/2}$
4	0031	normal	(0, 2)		(0, -1)		$6(2I_y)^{3/2}$
4	0040	normal	(0, 4)		(0, -3)		$8(2I_y)^{3/2}$
4	0211	normal	(-2, 0)		(2, 1)		$2(2I_x)(2I_y)^{1/2}$
4	0220	normal	(-2, 2)		(2, -1)		$4(2I_x)(2I_y)^{1/2}$
4	1102	normal	(0, -2)	(1, 2)		$2(2I_x)^{1/2}(2I_y)$	
4	1120	normal	(0, 2)	(1, -2)	(0, -1)	$2(2I_x)^{1/2}(2I_y)$	$4(2I_x)(2I_y)^{1/2}$
4	1300	normal	(-2, 0)	(3, 0)		$2(2I_x)^{3/2}$	
4	2002	normal	(2, -2)	(-1, 2)		$4(2I_x)^{1/2}(2I_y)$	
4	2011	normal	(2, 0)	(-1, 0)	(-2, 1)	$4(2I_x)^{1/2}(2I_y)$	$2(2I_x)(2I_y)^{1/2}$
4	2020	normal	(2, 2)	(-1, -2)	(-2, -1)	$4(2I_x)^{1/2}(2I_y)$	$4(2I_x)(2I_y)^{1/2}$
4	3100	normal	(2, 0)	(-1, 0)		$6(2I_x)^{3/2}$	
4	4000	normal	(4, 0)	(-3, 0)		$8(2I_x)^{3/2}$	
4	0112	skew	(-1, -1)		(1, 2)		$2(2I_x)^{1/2}(2I_y)$
4	0121	skew	(-1, 1)		(1, 0)		$4(2I_x)^{1/2}(2I_y)$
4	0130	skew	(-1, 3)		(1, -2)		$6(2I_x)^{1/2}(2I_y)$
4	0310	skew	(-3, 1)		(3, 0)		$2(2I_x)^{3/2}$
4	1003	skew	(1, -3)	(0, 3)		$2(2I_y)^{3/2}$	
4	1012	skew	(1, -1)	(0, 1)	(-1, 2)	$2(2I_y)^{3/2}$	$2(2I_x)^{1/2}(2I_y)$
4	1021	skew	(1, 1)	(0, -1)	(-1, 0)	$2(2I_y)^{3/2}$	$4(2I_x)^{1/2}(2I_y)$
4	1030	skew	(1, 3)	(0, -3)	(-1, -2)	$2(2I_y)^{3/2}$	$6(2I_x)^{1/2}(2I_y)$
4	1201	skew	(-1, -1)	(2, 1)		$2(2I_x)(2I_y)^{1/2}$	
4	1210	skew	(-1, 1)	(2, -1)	(1, 0)	$2(2I_x)(2I_y)^{1/2}$	$2(2I_x)^{3/2}$
4	2101	skew	(1, -1)	(0, 1)		$4(2I_x)(2I_y)^{1/2}$	
4	2110	skew	(1, 1)	(0, -1)	(-1, 0)	$4(2I_x)(2I_y)^{1/2}$	$2(2I_x)^{3/2}$
4	3001	skew	(3, -1)	(-2, 1)		$6(2I_x)(2I_y)^{1/2}$	
4	3010	skew	(3, 1)	(-2, -1)	(-3, 0)	$6(2I_x)(2I_y)^{1/2}$	$2(2I_x)^{3/2}$

### C.3. Amplitude, Resonances and Lines

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
5	0113	normal	(-1, -2)		(1, 3)		$2(2I_x)^{1/2}(2I_y)^{3/2}$
5	0122	normal	(-1, 0)		(1, 1)		$4(2I_x)^{1/2}(2I_y)^{3/2}$
5	0131	normal	(-1, 2)		(1, -1)		$6(2I_x)^{1/2}(2I_y)^{3/2}$
5	0140	normal	(-1, 4)		(1, -3)		$8(2I_x)^{1/2}(2I_y)^{3/2}$
5	0311	normal	(-3, 0)		(3, 1)		$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	0320	normal	(-3, 2)		(3, -1)		$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	1004	normal	(1, -4)	(0, 4)		$2(2I_y)^2$	
5	1013	normal	(1, -2)	(0, 2)	(-1, 3)	$2(2I_y)^2$	$2(2I_x)^{1/2}(2I_y)^{3/2}$
5	1022	normal	(1, 0)	(0, 0)	(-1, 1)	$2(2I_y)^2$	$4(2I_x)^{1/2}(2I_y)^{3/2}$
5	1031	normal	(1, 2)	(0, -2)	(-1, -1)	$2(2I_y)^2$	$6(2I_x)^{1/2}(2I_y)^{3/2}$
5	1040	normal	(1, 4)	(0, -4)	(-1, -3)	$2(2I_y)^2$	$8(2I_x)^{1/2}(2I_y)^{3/2}$
5	1202	normal	(-1, -2)	(2, 2)		$2(2I_x)(2I_y)$	
5	1211	normal	(-1, 0)	(2, 0)	(1, 1)	$2(2I_x)(2I_y)$	$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	1220	normal	(-1, 2)	(2, -2)	(1, -1)	$2(2I_x)(2I_y)$	$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	1400	normal	(-3, 0)	(4, 0)		$2(2I_x)^2$	
5	2102	normal	(1, -2)	(0, 2)		$4(2I_x)(2I_y)$	
5	2111	normal	(1, 0)	(0, 0)	(-1, 1)	$4(2I_x)(2I_y)$	$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	2120	normal	(1, 2)	(0, -2)	(-1, -1)	$4(2I_x)(2I_y)$	$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	2300	normal	(-1, 0)	(2, 0)		$4(2I_x)^2$	
5	3002	normal	(3, -2)	(-2, 2)		$6(2I_x)(2I_y)$	
5	3011	normal	(3, 0)	(-2, 0)	(-3, 1)	$6(2I_x)(2I_y)$	$2(2I_x)^{3/2}(2I_y)^{1/2}$
5	3020	normal	(3, 2)	(-2, -2)	(-3, -1)	$6(2I_x)(2I_y)$	$4(2I_x)^{3/2}(2I_y)^{1/2}$
5	3200	normal	(1, 0)	(0, 0)		$6(2I_x)^2$	
5	4100	normal	(3, 0)	(-2, 0)		$8(2I_x)^2$	
5	5000	normal	(5, 0)	(-4, 0)		$10(2I_x)^2$	
5	0014	skew	(0, -3)		(0, 4)		$2(2I_y)^2$
5	0023	skew	(0, -1)		(0, 2)		$4(2I_y)^2$
5	0032	skew	(0, 1)		(0, 0)		$6(2I_y)^2$
5	0041	skew	(0, 3)		(0, -2)		$8(2I_y)^2$
5	0050	skew	(0, 5)		(0, -4)		$10(2I_y)^2$

### C. Resonance Driving Terms

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n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
5	0212	skew	(-2, -1)		(2, 2)		$2(2I_x)(2I_y)$
5	0221	skew	(-2, 1)		(2, 0)		$4(2I_x)(2I_y)$
5	0230	skew	(-2, 3)		(2, -2)		$6(2I_x)(2I_y)$
5	0410	skew	(-4, 1)		(4, 0)		$2(2I_x)^2$
5	1103	skew	(0, -3)	(1, 3)		$2(2I_x)^{1/2}(2I_y)^{3/2}$	
5	1112	skew	(0, -1)	(1, 1)	(0, 2)	$2(2I_x)^{1/2}(2I_y)^{3/2}$	$2(2I_x)(2I_y)$
5	1121	skew	(0, 1)	(1, -1)	(0, 0)	$2(2I_x)^{1/2}(2I_y)^{3/2}$	$4(2I_x)(2I_y)$
5	1130	skew	(0, 3)	(1, -3)	(0, -2)	$2(2I_x)^{1/2}(2I_y)^{3/2}$	$6(2I_x)(2I_y)$
5	1301	skew	(-2, -1)	(3, 1)		$2(2I_x)^{3/2}(2I_y)^{1/2}$	
5	1310	skew	(-2, 1)	(3, -1)	(2, 0)	$2(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$
5	2003	skew	(2, -3)	(-1, 3)		$4(2I_x)^{1/2}(2I_y)^{3/2}$	
5	2012	skew	(2, -1)	(-1, 1)	(-2, 2)	$4(2I_x)^{1/2}(2I_y)^{3/2}$	$2(2I_x)(2I_y)$
5	2021	skew	(2, 1)	(-1, -1)	(-2, 0)	$4(2I_x)^{1/2}(2I_y)^{3/2}$	$4(2I_x)(2I_y)$
5	2030	skew	(2, 3)	(-1, -3)	(-2, -2)	$4(2I_x)^{1/2}(2I_y)^{3/2}$	$6(2I_x)(2I_y)$
5	2201	skew	(0, -1)	(1, 1)		$4(2I_x)^{3/2}(2I_y)^{1/2}$	
5	2210	skew	(0, 1)	(1, -1)	(0, 0)	$4(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$
5	3101	skew	(2, -1)	(-1, 1)		$6(2I_x)^{3/2}(2I_y)^{1/2}$	
5	3110	skew	(2, 1)	(-1, -1)	(-2, 0)	$6(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$
5	4001	skew	(4, -1)	(-3, 1)		$8(2I_x)^{3/2}(2I_y)^{1/2}$	
5	4010	skew	(4, 1)	(-3, -1)	(-4, 0)	$8(2I_x)^{3/2}(2I_y)^{1/2}$	$2(2I_x)^2$
6	0015	normal	(0, -4)		(0, 5)		$2(2I_y)^{5/2}$
6	0024	normal	(0, -2)		(0, 3)		$4(2I_y)^{5/2}$
6	0042	normal	(0, 2)		(0, -1)		$8(2I_y)^{5/2}$
6	0051	normal	(0, 4)		(0, -3)		$10(2I_y)^{5/2}$
6	0060	normal	(0, 6)		(0, -5)		$12(2I_y)^{5/2}$
6	0213	normal	(-2, -2)		(2, 3)		$2(2I_x)(2I_y)^{3/2}$
6	0222	normal	(-2, 0)		(2, 1)		$4(2I_x)(2I_y)^{3/2}$
6	0231	normal	(-2, 2)		(2, -1)		$6(2I_x)(2I_y)^{3/2}$
6	0240	normal	(-2, 4)		(2, -3)		$8(2I_x)(2I_y)^{3/2}$

### C.3. Amplitude, Resonances and Lines

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
6	0411	normal	(-4, 0)		(4, 1)		$2(2I_x)^2(2I_y)^{1/2}$
6	0420	normal	(-4, 2)		(4, -1)		$4(2I_x)^2(2I_y)^{1/2}$
6	1104	normal	(0, -4)	(1, 4)		$2(2I_x)^{1/2}(2I_y)^2$	
6	1113	normal	(0, -2)	(1, 2)	(0, 3)	$2(2I_x)^{1/2}(2I_y)^2$	$2(2I_x)(2I_y)^{3/2}$
6	1131	normal	(0, 2)	(1, -2)	(0, -1)	$2(2I_x)^{1/2}(2I_y)^2$	$6(2I_x)(2I_y)^{3/2}$
6	1140	normal	(0, 4)	(1, -4)	(0, -3)	$2(2I_x)^{1/2}(2I_y)^2$	$8(2I_x)(2I_y)^{3/2}$
6	1302	normal	(-2, -2)	(3, 2)		$2(2I_x)^{3/2}(2I_y)$	
6	1311	normal	(-2, 0)	(3, 0)	(2, 1)	$2(2I_x)^{3/2}(2I_y)$	$2(2I_x)^2(2I_y)^{1/2}$
6	1320	normal	(-2, 2)	(3, -2)	(2, -1)	$2(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	1500	normal	(-4, 0)	(5, 0)		$2(2I_x)^{5/2}$	
6	2004	normal	(2, -4)	(-1, 4)		$4(2I_x)^{1/2}(2I_y)^2$	
6	2013	normal	(2, -2)	(-1, 2)	(-2, 3)	$4(2I_x)^{1/2}(2I_y)^2$	$2(2I_x)(2I_y)^{3/2}$
6	2022	normal	(2, 0)	(-1, 0)	(-2, 1)	$4(2I_x)^{1/2}(2I_y)^2$	$4(2I_x)(2I_y)^{3/2}$
6	2031	normal	(2, 2)	(-1, -2)	(-2, -1)	$4(2I_x)^{1/2}(2I_y)^2$	$6(2I_x)(2I_y)^{3/2}$
6	2040	normal	(2, 4)	(-1, -4)	(-2, -3)	$4(2I_x)^{1/2}(2I_y)^2$	$8(2I_x)(2I_y)^{3/2}$
6	2202	normal	(0, -2)	(1, 2)		$4(2I_x)^{3/2}(2I_y)$	
6	2220	normal	(0, 2)	(1, -2)	(0, -1)	$4(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	2400	normal	(-2, 0)	(3, 0)		$4(2I_x)^{5/2}$	
6	3102	normal	(2, -2)	(-1, 2)		$6(2I_x)^{3/2}(2I_y)$	
6	3111	normal	(2, 0)	(-1, 0)	(-2, 1)	$6(2I_x)^{3/2}(2I_y)$	$2(2I_x)^2(2I_y)^{1/2}$
6	3120	normal	(2, 2)	(-1, -2)	(-2, -1)	$6(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	4002	normal	(4, -2)	(-3, 2)		$8(2I_x)^{3/2}(2I_y)$	
6	4011	normal	(4, 0)	(-3, 0)	(-4, 1)	$8(2I_x)^{3/2}(2I_y)$	$2(2I_x)^2(2I_y)^{1/2}$
6	4020	normal	(4, 2)	(-3, -2)	(-4, -1)	$8(2I_x)^{3/2}(2I_y)$	$4(2I_x)^2(2I_y)^{1/2}$
6	4200	normal	(2, 0)	(-1, 0)		$8(2I_x)^{5/2}$	
6	5100	normal	(4, 0)	(-3, 0)		$10(2I_x)^{5/2}$	
6	6000	normal	(6, 0)	(-5, 0)		$12(2I_x)^{5/2}$	
6	0114	skew	(-1, -3)		(1, 4)		$2(2I_x)^{1/2}(2I_y)^2$
6	0123	skew	(-1, -1)		(1, 2)		$4(2I_x)^{1/2}(2I_y)^2$

### C. Resonance Driving Terms

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n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
6	0132	skew	(-1, 1)		(1, 0)		$6(2I_x)^{1/2}(2I_y)^2$
6	0141	skew	(-1, 3)		(1, -2)		$8(2I_x)^{1/2}(2I_y)^2$
6	0150	skew	(-1, 5)		(1, -4)		$10(2I_x)^{1/2}(2I_y)^2$
6	0312	skew	(-3, -1)		(3, 2)		$2(2I_x)^{3/2}(2I_y)$
6	0321	skew	(-3, 1)		(3, 0)		$4(2I_x)^{3/2}(2I_y)$
6	0330	skew	(-3, 3)		(3, -2)		$6(2I_x)^{3/2}(2I_y)$
6	0510	skew	(-5, 1)		(5, 0)		$2(2I_x)^{5/2}$
6	1005	skew	(1, -5)	(0, 5)		$2(2I_y)^{5/2}$	
6	1014	skew	(1, -3)	(0, 3)	(-1, 4)	$2(2I_y)^{5/2}$	$2(2I_x)^{1/2}(2I_y)^2$
6	1023	skew	(1, -1)	(0, 1)	(-1, 2)	$2(2I_y)^{5/2}$	$4(2I_x)^{1/2}(2I_y)^2$
6	1032	skew	(1, 1)	(0, -1)	(-1, 0)	$2(2I_y)^{5/2}$	$6(2I_x)^{1/2}(2I_y)^2$
6	1041	skew	(1, 3)	(0, -3)	(-1, -2)	$2(2I_y)^{5/2}$	$8(2I_x)^{1/2}(2I_y)^2$
6	1050	skew	(1, 5)	(0, -5)	(-1, -4)	$2(2I_y)^{5/2}$	$10(2I_x)^{1/2}(2I_y)^2$
6	1203	skew	(-1, -3)	(2, 3)		$2(2I_x)(2I_y)^{3/2}$	
6	1212	skew	(-1, -1)	(2, 1)	(1, 2)	$2(2I_x)(2I_y)^{3/2}$	$2(2I_x)^{3/2}(2I_y)$
6	1221	skew	(-1, 1)	(2, -1)	(1, 0)	$2(2I_x)(2I_y)^{3/2}$	$4(2I_x)^{3/2}(2I_y)$
6	1230	skew	(-1, 3)	(2, -3)	(1, -2)	$2(2I_x)(2I_y)^{3/2}$	$6(2I_x)^{3/2}(2I_y)$
6	1401	skew	(-3, -1)	(4, 1)		$2(2I_x)^2(2I_y)^{1/2}$	
6	1410	skew	(-3, 1)	(4, -1)	(3, 0)	$2(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	2103	skew	(1, -3)	(0, 3)		$4(2I_x)(2I_y)^{3/2}$	
6	2112	skew	(1, -1)	(0, 1)	(-1, 2)	$4(2I_x)(2I_y)^{3/2}$	$2(2I_x)^{3/2}(2I_y)$
6	2121	skew	(1, 1)	(0, -1)	(-1, 0)	$4(2I_x)(2I_y)^{3/2}$	$4(2I_x)^{3/2}(2I_y)$
6	2130	skew	(1, 3)	(0, -3)	(-1, -2)	$4(2I_x)(2I_y)^{3/2}$	$6(2I_x)^{3/2}(2I_y)$
6	2301	skew	(-1, -1)	(2, 1)		$4(2I_x)^2(2I_y)^{1/2}$	
6	2310	skew	(-1, 1)	(2, -1)	(1, 0)	$4(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	3003	skew	(3, -3)	(-2, 3)		$6(2I_x)(2I_y)^{3/2}$	
6	3012	skew	(3, -1)	(-2, 1)	(-3, 2)	$6(2I_x)(2I_y)^{3/2}$	$2(2I_x)^{3/2}(2I_y)$
6	3021	skew	(3, 1)	(-2, -1)	(-3, 0)	$6(2I_x)(2I_y)^{3/2}$	$4(2I_x)^{3/2}(2I_y)$
6	3030	skew	(3, 3)	(-2, -3)	(-3, -2)	$6(2I_x)(2I_y)^{3/2}$	$6(2I_x)^{3/2}(2I_y)$

### C.3. Amplitude, Resonances and Lines

n	jklm	Type	Resonance	H-line	V-line	Amplitude H	Amplitude V
6	3201	skew	(1, -1)	(0, 1)		$6(2I_x)^2(2I_y)^{1/2}$	
6	3210	skew	(1, 1)	(0, -1)	(-1, 0)	$6(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	4101	skew	(3, -1)	(-2, 1)		$8(2I_x)^2(2I_y)^{1/2}$	
6	4110	skew	(3, 1)	(-2, -1)	(-3, 0)	$8(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$
6	5001	skew	(5, -1)	(-4, 1)		$10(2I_x)^2(2I_y)^{1/2}$	
6	5010	skew	(5, 1)	(-4, -1)	(-5, 0)	$10(2I_x)^2(2I_y)^{1/2}$	$2(2I_x)^{5/2}$

Table C.3.: Amplitude dependence, associated resonance and spectrum correspondence of Resonance Driving Terms up to dodecapoles ( $n = 6$ ).

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- [106] Maël Le Garrec. “Les champs décapolaires du LHC”. Roscoff, France, 2023. URL: <https://indico.ijclab.in2p3.fr/event/9312/contributions/30982/>.



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# List of Publications

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## Journal Publications (co-author)

- (1) Ewen H. Maclean et al. “Prospects for beam-based study of dodecapole nonlinearities in the CERN High-Luminosity Large Hadron Collider”. In: *The European Physical Journal Plus* 137.11 (2022). doi: [10.1140/epjp/s13360-022-03367-2](https://doi.org/10.1140/epjp/s13360-022-03367-2). URL: <https://link.springer.com/10.1140/epjp/s13360-022-03367-2>
- (2) Joschua Dilly et al. “First operational dodecapole correction in the LHC”. in: *Physical Review Accelerators and Beams* 26.12 (2023). doi: [10.1103/PhysRevAccelBeams.26.121001](https://doi.org/10.1103/PhysRevAccelBeams.26.121001). URL: <https://link.aps.org/doi/10.1103/PhysRevAccelBeams.26.121001>

## Conference Proceedings

- (1) Mael Le Garrec et al. “Measurement and modelling of decapole errors in the LHC from beam-based studies”. In: *IPAC’23*. In collab. with Assmann, Ralph et al. Venice, Italy: [object Object], 2023. doi: [10.18429/JACOW-IPAC2023-MOPL024](https://doi.org/10.18429/JACOW-IPAC2023-MOPL024). URL: <https://jacow.org/ipac2023/doi/jacow-ipac2023-mopl024>

- (2) Mael Le Garrec et al. "First measurement of fourth and fifth order chromaticity in the LHC". in: *IPAC'23*. In collab. with Assmann,Ralph et al. Venice, Italy: [object Object], 2023. doi: [10.18429/JACOW-IPAC2023-MOPL027](https://doi.org/10.18429/JACOW-IPAC2023-MOPL027). URL: <https://jacow.org/ipac2023/doi/jacow-ipac2023-mopl027>

## Conference Proceedings (co-author)

- (1) Tobias Persson et al. "LHC Optics Commissioning in 2023 and 2024". In: *IPAC'24*. Nashville, Tennessee, USA, 2024
- (2) Vittorio Ferrentino et al. "LHC 2023 Ion Optics Commissioning". In: *IPAC'24*. Nashville, Tennessee, USA, 2024
- (3) Sasha J. Horney et al. "Sextupole RDTs in The LHC at Injection and in the Ramp". In: *IPAC'24*. Nashville, Tennessee, USA, 2024
- (4) Tuuli Nissinen et al. "LHC Optics Measurements from Transverse Damper for the High Intensity Frontier". In: HB2023. In collab. with Yunhai Cai et al. [object Object], 2023. doi: [10.18429/JACOW-HB2023-THBP14](https://doi.org/10.18429/JACOW-HB2023-THBP14). URL: <https://jacow.org/hb2023/doi/JACoW-HB2023-THBP14.html>
- (5) Rogelio Tomás García et al. "Optics for Landau Damping with Minimized Octupolar Resonances in the LHC". in: HB2023. In collab. with Yunhai Cai et al. [object Object], 2023. doi: [10.18429/JACOW-HB2023-THBP20](https://doi.org/10.18429/JACOW-HB2023-THBP20). URL: <https://jacow.org/hb2023/doi/JACoW-HB2023-THBP20.html>
- (6) Jacqueline Keintzel et al. "60° phase advance optics measurements in the Large Hadron Collider at CERN". in: *IPAC'23*. In collab. with Assmann,Ralph et al. Venice, Italy: [object Object], 2023. doi: [10.18429/JACOW-IPAC2023-MOPL026](https://doi.org/10.18429/JACOW-IPAC2023-MOPL026). URL: <https://jacow.org/ipac2023/doi/jacow-ipac2023-mopl026>
- (7) Felix Carlier et al. "Challenges of K-modulation measurements in the LHC Run 3". In: *IPAC'23*. In collab. with Assmann,Ralph et al. Venice, Italy: [object Object], 2023. doi: [10.18429/JACOW-IPAC2023-MOPL014](https://doi.org/10.18429/JACOW-IPAC2023-MOPL014). URL: <https://jacow.org/ipac2023/doi/jacow-ipac2023-mopl014>

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- (8) Felix Carlier et al. “LHC Run 3 optics corrections”. In: *IPAC’23*. In collab. with Assmann,Ralph et al. Venice, Italy: [object Object], 2023. doi: [10.18429/JACOW-IPAC2023-MOPL015](https://doi.org/10.18429/JACOW-IPAC2023-MOPL015). URL: <https://jacow.org/ipac2023/doi/jacow-ipac2023-mopl015>
  - (9) Elena Fol et al. “Experimental Demonstration of Machine Learning Application in LHC Optics Commissioning”. In: *Proc. 13th Int. Part. Accel. Conf. IPAC22*. Vol. IPAC2022. Bangkok, Thailand: JACoW, 2022. doi: [10.18429/JACOW-IPAC2022-MOPOPT047](https://doi.org/10.18429/JACOW-IPAC2022-MOPOPT047)
  - (10) Tobias Persson et al. “Optics Correction Strategy for Run 3 of the LHC”. in: *Proc. 13th Int. Part. Accel. Conf. IPAC22*. Vol. IPAC2022. Bangkok, Thailand: JACoW, 2022. doi: [10.18429/JACOW-IPAC2022-WEPOST008](https://doi.org/10.18429/JACOW-IPAC2022-WEPOST008)
  - (11) Tobias Persson et al. “Optics Measurements and Correction Plans for the HL-LHC”. in: *IPAC’21*. Campinas, Brazil: JACoW Publishing, Geneva, Switzerland, 2021. doi: [10.18429/JACOW-IPAC2021-WEPAB026](https://doi.org/10.18429/JACOW-IPAC2021-WEPAB026). URL: <https://accelconf.web.cern.ch/ipac2021/doi/JACoW-IPAC2021-WEPAB026.html>
  - (12) Tobias Persson et al. “Optics Correction Strategy for Run 3 of the LHC”. in: *IPAC’21*. Campinas, Brazil: JACoW Publishing, Geneva, Switzerland, 2021. doi: [10.18429/JACOW-IPAC2021-WEPAB027](https://doi.org/10.18429/JACOW-IPAC2021-WEPAB027). URL: <https://accelconf.web.cern.ch/ipac2021/doi/JACoW-IPAC2021-WEPAB027.html>
  - (13) Ewen Maclean et al. “Optics Measurement by Excitation of Betatron Oscillations in the CERN PSB”. in: *IPAC’21*. Campinas, Brazil: JACoW Publishing, Geneva, Switzerland, 2021. doi: [10.18429/JACOW-IPAC2021-THPAB168](https://doi.org/10.18429/JACOW-IPAC2021-THPAB168)

## Notes and Reports

- (1) Maël Le Garrec et al. *MD6864 — Decapole studies at injection*. Accelerators & Technology Sector Note CERN-ACC-NOTE-2023-0018. CERN, 2023. URL: <https://cds.cern.ch/record/2879070>
- (2) Joschua Dilly and Mael Le Garrec. *On the derivation of Amplitude Detuning*

*and Chromaticity Formulas for Particle Accelerators.* 2023. arXiv: [2301.09132\[physics\]](#). URL: <http://arxiv.org/abs/2301.09132>

- (3) Xavier Buffat et al. *Optics Measurement and Correction Strategies for HL-LHC*. Accelerators & Technology Sector Note CERN-ACC-2022-0004. CERN, 2022. URL: <https://cds.cern.ch/record/2808650>
- (4) O. Aberle et al. *High-Luminosity Large Hadron Collider (HL-LHC): Technical Design Report*. CERN Yellow Reports: Monographs CERN-2020-010. Geneva: CERN, 2020. doi: [10.23731/CYRM-2020-0010](https://doi.org/10.23731/CYRM-2020-0010). URL: <https://cds.cern.ch/record/2749422>

## Seminars and Conferences

- (1) Maël Le Garrec. “Correction of Decapolar Resonances in the LHC”. Goethe Universität, Frankfurt, 2023
- (2) Maël Le Garrec. “Les champs décapolaires du LHC”. Roscoff, France, 2023. URL: <https://indico.ijclab.in2p3.fr/event/9312/contributions/30982/>



# Maël Le Garrec

CS ENGINEER · ACCELERATOR PHYSICS PHD STUDENT

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## Experience



### KEK

VISITING SCIENTIST

- Measurement of linear optics of SuperKEKB
- First observations of higher chromaticity non-linear orders
- First observations of sextupolar Resonance Driving Terms

Tsukuba, JP

2024



### CERN

PHD STUDENT - ACCELERATORS AND BEAM PHYSICS

- Measurements and corrections during LHC's commissioning
  - Linear optics:  $\beta$ -beating, dispersion, energy, coupling
  - Non-linear optics via chromaticity:  $Q''$  and  $Q'''$
- Measurement, modelling and correction of decapole fields via Chromatic Amplitude Detuning,  $Q'''$  and RDTs.
- Observation and modelling of higher order fields (dodecapole and decatetrapole).
- Measurement and correction of skew octupole fields via RDTs.
- Analytical calculations from Hamiltonians.

Geneva, CH

2021 – 2024

TRAINEE - ACCELERATORS AND BEAM PHYSICS

- Software development for Optics Measurements and Corrections for LHC Run III commissioning.
- Particle tracking simulations ; dev in Python and Java.

Geneva, CH

2019 – 2021



### Scaleway

DEVOPS

- Maintenance of Python APIs, server images and system software.
- Administration of thousand of hypervisors and servers.

Paris, FR

2018 – 2019

## Education

### Gothe Universität Frankfurt

Frankfurt am Main, DE

2021 - 2024

DOCTORAL CANDIDATE

- Thesis: LHC Effective Model for Optics Corrections
- Under the supervision of Apl. Prof. Dr. Giuliano Franchetti and Dr. Ewen H. Maclean

### CERN Accelerator School

Chavanne de Bogis, CH / Sévrier, FR

2021-2022

ACCELERATOR PHYSICS COURSES - [HTTPS://CAS.WEB.CERN.CH](https://cas.web.cern.ch)

- Introduction to Accelerator Physics, 2 weeks focused on basic accelerator physics subjects.
- Advanced Accelerator Physics, continuation of previous school, focused on advanced subjects.

### EPITA

Paris, FR

2013 - 2019

COMPUTER SCIENCE ENGINEERING

- Focus on systems and network security
- Low-level development and system administration

## Skills and Interests

<b>Coding</b>	Python, C/C++, Java, bash, LaTeX, Rust
<b>Web</b>	Flask, HTML, CSS, Apache
<b>Others</b>	Archlinux, Debian, Git, pgSQL, Scipy, Sympy, Numpy, Pandas, Matplotlib
<b>Languages</b>	French, English (C1), German (A2)
<b>Interests</b>	Electronics, CAD Design & 3D Printing, Automotive Mechanics, Climbing, Paragliding

# Professional Knowledge

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## Particle Tracking and Analysis

MAD-X, MAD-NG, OMC3, XSUITE

2019-

- LHC single particle tracking with field errors.
- Frequency and Optics analysis for beta-beating, dispersion, RDTs, etc.
- Correction of Optics based on response matrices (Chromaticity, RDTs)
- Modelling of machine errors.
- Dynamic Aperture simulations with XSuite.

## Contests

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FRANCE

2017 37<sup>th</sup>/700, *Meilleur Développeur de France* (Best French Developer), algorithms contest

*Paris*