INF280
Graph Traversals & Paths

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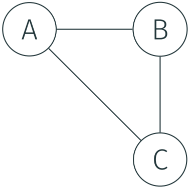
Introduction

You all know graphs:

- Set of nodes N
- Set of edges $E \subseteq N \times N$
- Edges can be undirected or directed, i.e., $(a, b) \neq (b, a)$



$$
\begin{array}{ll}
N & \{A, B, C\} \\
E & \{(A, B), (A, C), (B, C)\}
\end{array}
$$

Data Structures

Several options to represent graphs:

- Adjacency matrix:
  - `bool G[MAXN][MAXN];`
  - `G[x][y]` is **`true`** if an edge between node `x` and `y` exists
  - Replace **`bool`** by **`int`** to represent weighted edges
- Adjacency list:
  - `vector<int> Adj[MAXN];`
  - `y` is in `Adj[x]` if an edge between node `x` and `y` exists
  - Pairs to represent weights
- Edge list:
  - `vector<pair<int, int> > Edges;`
  - `Edges` contains a pair of nodes if an edge exists between them
- Nodes and edges may also be custom structs or classes

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Depth-First Search

Visit each node in the graph once:

- Recursive implementation below
- Manage stack yourself for iterative version
- First visit child nodes then siblings

```
bool Visited[MAXN] = {};
void DFS(int u) {
  if (Visited[u])
    return;
  Visited[u] = true;
  // maybe do something with u (pre-order) ...
  for (auto v : Adj[u])
    DFS(v);
  // or do something here (post-order)
}
```

Applications of DFS

- Determine a topological order of nodes
  - Only works for graphs without cycles (i.e., $x \to y \to z \to x$)
  - Add visited node at the head of an ordered list
    (at the end of `DFS`: `ordering.push_front(u)`)
- Detect if a cycle exists:
  - Check if the currently visited node is on the stack
    A) Use three states for `Visited` array:
    `UNVISITED`, `ONSTACK`, `VISITED`
    B) Explicitly search in the stack of the iterative algorithm
- Examples: `https://visualgo.net/dfsbfs`

Breadth-First Search

Visit each node in the graph once:

- Similar to DFS, but replaces **stack** by **queue**

```
queue<int> Q;
bool Visited[MAXN] = {};
void BFS(int root) {
  Q.push(root);
  while (!Q.empty()) {
    int u = Q.front();
    Q.pop();
    if (Visited[u])
      continue;
    Visited[u] = true;
    for (auto v : Adj[u])
      Q.push(v);        // usually do something with v
  }
}
```

Applications of BFS

- Shortest path search
    - Stop processing when a given node `d` was found
    - Minimizes number of hops, i.e., all edges have same weight
    - Generalization follows next
- Examples: `https://visualgo.net/dfsbfs`

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Dijkstra

BFS can only be used if edge weights are uniform

- Dijkstra's algorithm generalizes this
- Constraint: all edges need to have non-negative weights
- Use a priority queue to choose which node to examine next
  - Would require a function to update the priority of an element
    - Would need to update order in the priority queue
  - We'll use the standard `priority_queue` of STL
    - Simply insert a new element in the queue (no update)
    - Ok since priorities decrease monotonically
    - This slightly diverges from Dijkstra's algorithm
- May revisit nodes several times

Dijkstra's Algorithm

```
unsigned int Dist[MAXN];
typedef pair<unsigned int, int> WeightNode;          // weight goes first
priority_queue<WeightNode, std::vector<WeightNode>,
               std::greater<WeightNode> > Q;
void Dijkstra(int root) {
  fill_n(Dist, MAXN, MAXLEN);
  Dist[root] = 0;
  Q.push(make_pair(0, root));
  while (!Q.empty()) {
    int u = Q.top().second;                 // get node with least priority
    Q.pop();
    for (auto tmp : Adj[u]) {
      int v = tmp.second;
      unsigned int weight = tmp.first;
      if (Dist[v] > Dist[u] + weight) {             // shorter path found?
        Dist[v] = Dist[u] + weight;
        Q.push(make_pair(Dist[v], v));    // simply push, no update here
      }
    }
  }
}
```

https://visualgo.net/sssp

Bellman-Ford

- Dijkstra’s algorithm is limited to non-negative edge weights
- Bellman-Ford extends this to general edge weights
- Constraint: no cycle with negative total costs
- May again revisit nodes several times

Bellman-Ford Algorithm

```
unsigned int Dist[MAXN];
void BellmanFord(int root) {
  fill_n(Dist, MAXN, MAXLEN);
  Dist[root] = 0;
  for(int u=0; u < N - 1; u++) {  // just iterate N - 1 times
    for (auto tmp : Edges) {
      unsigned int weight = get<0>(tmp);
      int u = get<1>(tmp);  // similar to Dijkstra before
      int v = get<2>(tmp);
      Dist[v] = min(Dist[v], Dist[u] + weight);
    }
  }
}
```

https://visualgo.net/sssp

Floyd-Warshall

- Dijkstra and Bellman-Ford compute shortest paths
  - From a single source (`root`)
  - To all other (reachable) nodes
  - This is known as: single-source shortest path problem
- Floyd-Warshall extends this to compute the shortest paths between **all pairs** of nodes
- This is known as: all-pairs shortest path problem

Floyd-Warshall Algorithm

```
int Dist[MAXN][MAXN];
void FloydWarshall() {
  fill_n((int*)Dist, MAXN*MAXN, MAXLEN);
  for(int u=0; u < N; u++) {
    Dist[u][u] = 0;
    for (auto tmp : Adj[u])
      Dist[u][tmp.second] = tmp.first;
  }
  for(int k=0; k < N; k++)        // check sub-path combinations
    for(int i=0; i < N; i++)
      for(int j=0; j < N; j++)                // concatenate paths
        Dist[i][j] = min(Dist[i][j], Dist[i][k] + Dist[k][j]);
}
```

Keeping track of the path

We only considered the length of the path so far:

- All of the above algorithms can track the actual shortest path
- This allows to *print* each edge/node along the path
- Basic idea:
  - Introduce an array `int Predecessor[MAXN]`
    (Use two-dimensional array for Floyd-Warshall)
  - Updated whenever `Dist[v]` changes
  - Simply set to the new predecessor `u`

Heuristics – A* Search

Heuristics may speed-up the path search

- Bellman-Ford and Floyd-Warshall equally explore all possibilities
- Dijkstra *prefers* nodes with shorter distance
- Basic idea behind A* Search:
  - Extension to Dijkstra’s algorithm
  - Introduce an estimation of the remaining distance
  - Prefer nodes with minimal estimated *remaining* distance
- Advantages
  - May converge faster than Dijkstra
  - Can be used to compute approximate solutions
    (trading speed for precision)

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Eulerian Circuits

We study undirected graphs and assume they are connected:

- Eulerian path:
  Use every edge of a graph exactly once. Start and end may differ
- Eulerian circuit:
  Use every edge exactly once. Start and end at the same node

- Conditions to find Eulerian path:
  - All nodes have even degree or
  - Precisely two nodes have odd degree
- For Eulerian circuit, all nodes must have even degree

Hierholzer's Algorithm for Eulerian Paths (assuming they exist)

```
set<int> Adj[MAXN]; vector<int> Circuit;
void Hierholzer() {
  int v = 0;          // find node with odd degree, else start with node 0
  for (int u=0; u < N && v == 0; u++)
    if (Adj[u].size() & 1)
      v = u;                                              // node with odd degree
  stack<int> Stack;
  Stack.push(v);
  while (!Stack.empty()) {
    if (!Adj[v].empty()) {                            // follow edges until stuck
      Stack.push(v);
      int tmp = *Adj[v].begin();
      Adj[v].erase(tmp);                        // remove edge, modifying graph
      Adj[tmp].erase(v);
      v = tmp;
    } else {                             // got stuck: stack contains a circuit
      Circuit.push_back(v);                // append node at the end of circuit
      v = Stack.top();         // backtrack using stack, find larger circuit
      Stack.pop();
    }
  }
}
```

https://www-m9.ma.tum.de/graph-algorithms/hierholzer/index_en.html