INF280
Graph Algorithms

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Contents

Union-Find

A data structure to track equivalence relations between elements:

- Elements are partitioned into non-overlapping sets
    - Initially only pairwise relations are known
      (i.e., X and Y are in the same set)
    - From pairwise relations, deduce the global partitioning step-wise

- Basic idea:
    - Represent partitions as trees
    - Merge trees when a new pairwise relation is discovered

Union-Find (2)

Two operations to update/query the union-find data structure:

- `Union(x,y)`:
  Add a new pairwise relation between `x` and `y` and
  update the Union-Find structure to put them in the same set
- `Find(x)`:
  Get the (current) representative of the set for element `x`

https://visualgo.net/ufds

Union-Find using Ranks and Path Compression

```
map<int, pair<int, unsigned int> > Sets;  // map to parent & rank
void MakeSet(int x) {
  Sets.insert(make_pair(x, make_pair(x, 0)));
}
int Find(int x) {
  if (Sets[x].first == x) return x;                 // Parent == x ?
  else return Sets[x].first = Find(Sets[x].first);  // Get Parent
}
void Union(int x, int y) {
  int parentX = Find(x), parentY = Find(y);
  int rankX = Sets[parentX].second, rankY = Sets[parentY].second;
  if (parentX == parentY) return;
  else if (rankX < rankY)
    Sets[parentX].first = parentY;
  else
    Sets[parentY].first = parentX;
  if (rankX == rankY)
    Sets[parentX].second++;
}
```

Contents

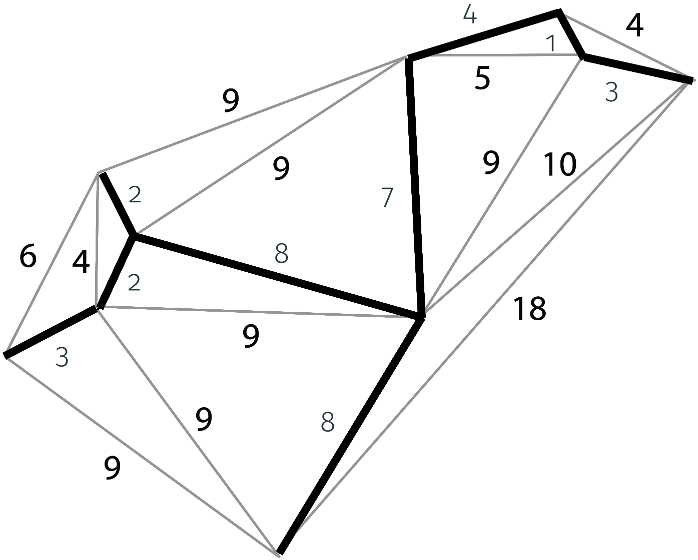
Spanning Trees

- Subgraph of undirected connected graph that forms a tree
- The subgraph contains each node of the original graph

- Computing spanning trees
  - Many possible spanning trees exist
  - Any depth-first traversal gives a spanning tree

- In weighted graphs one might need a **minimum spanning tree**
  - A spanning tree as defined above
  - Require that the total sum of edge weights is **minimal**
    (i.e., no other spanning tree with a lower total sum exists)

Example: Minimum Spanning Trees



https://commons.wikimedia.org/wiki/File:Minimum_spanning_tree.svg, Dcoetzee, public domain

Kruskal's Algorithm

```
vector<pair<int, pair<int,int> > > Edges;
set<pair<int,int> > A;               // Final minimum spanning tree

void Kruskal() {
  for(int u=0; u < N; u++)
    MakeSet(u);                                  // Initialize Union-Find

  sort(Edges.begin(), Edges.end());        // Sort edges by weight

  for(auto tmp : Edges) {
    auto edge = tmp.second;
    if (Find(edge.first) != Find(edge.second)) {
      Union(edge.first, edge.second);        // update Union-Find
      A.insert(edge);                          // include edge in MST
    }
  }
}
```

https://visualgo.net/mst

Contents

Flow Networks

- Flow networks are weighted directed graphs
- Edge weights denote the capacity of edges
  - Current in an electric circuit
  - Water in pipes
  - Trains on a railroad
- Question: What is the maximum flow between nodes s and t?
  - Assign flow to each edge respecting the edge's capacity
  - For each node, the combined in/out flows must be equal

Maximum Flow / Minimum Cut

Maximum flow is limited by a cut separating s and t

- Basic idea behind cuts:
    - Partition the network's nodes into two sets S and T
    - S contains s while T contains t
    - Edges (u, v) with $u \in S$ and $v \in T$ are in the cut between S and T
    - The edges in the cut completely separate the two sets
      $\Longrightarrow$ Removing those edges gives a maximum flow of zero
- Link between cuts and flows:
    - The combined edge weights of a cut bound the flow from s to t
    - The maximum flow is thus limited by a minimum cut

Edmonds-Karp Algorithm

```
// find path from s to t in G, return true if such a path exists
bool BFS(int G[MAXN][MAXN], int s, int t, int Predecessor[MAXN]);

int EdmondsKarp(int G[MAXN][MAXN], int s, int t) {
  int GRes[MAXN][MAXN];                                  // residual graph
  copy_n((int*)G, MAXN*MAXN, (int*)GRes);           // copy original graph

  int Predecessor[MAXN];
  int Maxflow = 0;
  while (BFS(GRes, s, t, Predecessor)) {             // find residual path
    int Bottleneck = MAXFLOW;         // get minimal flow of residual path
    for (int v = t; v != s; v = Predecessor[v])
      Bottleneck = min(Bottleneck, GRes[Predecessor[v]][v]);
    for (int v = t; v != s; v = Predecessor[v]) {
      // decrease capacity along residual path
      GRes[Predecessor[v]][v] -= Bottleneck;
      GRes[v][Predecessor[v]] += Bottleneck;
    }
    Maxflow += Bottleneck;
  }
  return Maxflow;
}
```

https://visualgo.net/maxflow

Contents

Assignment Problems and Matchings

- Represented as bipartite graphs:
  - Nodes are partitioned into two disjoint sets X and Y
  - Edges always connect nodes from both sets ($G = (X \cup Y, E)$, where $E \subseteq X \times Y$)
- Basic idea:
  - Search the best assignment of elements in X to elements in Y
  - Each element may appear only in one assignment
- Problem variants
  - Maximize matching cardinality (Hopcroft-Karp – on next slide)
  - Maximize matching cost in weighted graphs

Hopcroft-Karp (data structures)

```
// Artificial node (unused otherwise) -- end of augmenting path
#define NIL 0

// "Infinity", i.e., value larger than min(|X|, |Y|)
#define INF numeric_limits<unsigned int>::max()

// Partitions X and Y
vector<int> X, Y;
// Neighbors in Y of nodes in X
vector<int> Adj[MAXY];

// Matching X-Y and Y-X
int PairX[MAXX];
int PairY[MAXY];

// Augmenting path lengths
unsigned int Dist[MAXX];
```

Hopcroft-Karp (main)

```
int HopcroftKarp() {
  fill_n(PairX, MAXX, NIL);     // initialize: empty matching
  fill_n(PairY, MAXY, NIL);

  int Matching = 0;            // count number of edges in matching

  while (BFS()) {             // find all shortest augmenting paths
    for(auto x : X)                 // update matching cardinality
      if (PairX[x] == NIL &&         // node not yet in matching?
          DFS(x))            // does an augmenting path start at x?
        Matching++;
  }
  return Matching;
}
```

Hopcroft-Karp (BFS)

```
bool BFS() {
  queue<int> Q;
  Dist[NIL] = INF;
  for(auto x : X) { // start from nodes that are not yet matched
    Dist[x] = (PairX[x] == NIL) ? 0 : INF;
    if (PairX[x] == NIL)
      Q.push(x);
  }
  while (!Q.empty()) {          // find all shortest paths to NIL
    int x = Q.front();    Q.pop();
    if (Dist[x] < Dist[NIL])  // can this become a shorter path?
      for (auto y : Adj[x])
        if (Dist[PairY[y]] == INF) {
          Dist[PairY[y]] = Dist[x] + 1;     // update path length
          Q.push(PairY[y]);
        }
  }
  return Dist[NIL] != INF;     // any shortest path to NIL found?
}
```

Hopcroft-Karp (DFS)

```
bool DFS(int x) {
  if (x == NIL)
    return true;                                     // reached NIL

  for (auto y : Adj[x])
    if (Dist[PairY[y]] == Dist[x] + 1 &&
        DFS(PairY[y])) {                       // follow trace of BFS
      PairX[x] = y;            // add edge from x to y to matching
      PairY[y] = x;
      return true;
    }
  Dist[x] = INF;
  return false;                           // no augmenting path found
}
```