Homework I

- The deadline for on time submission is November 17, 2024. The maximum grade in case of late submission is 3 points, and the deadline for late submission is 5 days before the oral exam. The solutions have to be uploaded on the course website under the name Homework1.
- The exercises may be solved by using numerical codes or by hand. You may implement your codes in any language, but the teacher assistant can only guarantee you support with MATLAB and Python. Your code should be written in a quite general manner, i.e., if a question is slightly modified, it should only require slight modifications in your code as well. Upload a PDF lab-report together with your code.
- The PDF should read like a standard lab-report, including a description of what you are doing and proper presentation of results (including readable figures with axis labels, if any). Writing the report in Latex is strongly encouraged. Clarity of the presentation (especially if the report is hand-written) and ability to synthesize are part of the evaluation of the homework.
- Comment your code well. Clarity is more important than efficiency.
- Collaboration such as exchange of ideas among students is encouraged. You can work in group of up to 5 students and write a unique PDF report. However, every student has to submit her/his copy of the final report and code, and specify whom she/he has collaborated with and on what particular part of the work.

Exercise 1. Consider the network in Figure 1 with link capacities

$$c_1 = c_3 = c_5 = 3, \qquad c_6 = c_7 = 1 \qquad c_2 = c_4 = 2.$$

(a) Compute the capacity of all the cuts and find the minimum capacity to be removed for no feasible flow from o to d to exist.

(b) You are given $x > 0$ extra units of capacity ($x \in \mathbf{Z}$). How should you distribute them in order to maximize the throughput that can be sent from o to d? Plot the maximum throughput from o to d as a function of $x \geq 0$.

(c) You are given the possibility of adding to the network a directed link e_8 with capacity $c_8 = 1$ and $x > 0$ extra units of capacity ($x \in \mathbf{Z}$). Where should you add the link and how should you distribute the additional capacity in order to maximize the throughput that can be sent from o to d? Plot the maximum throughput from o to d as a function of $x \geq 0$.

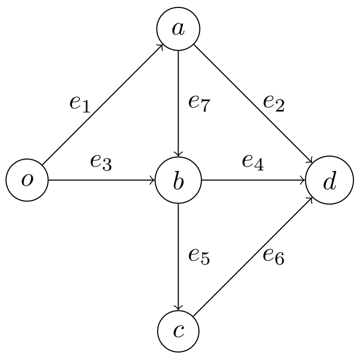

Figure 1

Exercise 2. There are a set of people $\{a_1, a_2, a_3, a_4\}$ and a set of foods $\{b_1, b_2, b_3, b_4\}$. Each person is interested in a subset of foods, specifically

$$a_1 \to \{b_1, b_2\}, \quad a_2 \to \{b_2, b_3\}, \quad a_3 \to \{b_1, b_4\}, \quad a_4 \to \{b_1, b_2, b_4\}.$$

(a) Exploit max-flow problems to find a perfect matching (if any).

(b) Now, assume that there are multiple portions of every food, and the distribution of the portions is $(2, 3, 2, 2)$. Each person can take an arbitrary number of *different* foods. Exploit the analogy with max-flow problems to establish how many portions of food can be assigned in total.

(c) Now, assume that a_1 wants 3 portions of food, a_i (for every $i \neq 1$) want 2 portions of food, every person can take multiple portions of the same food, and the distribution of the portions is $(2, 3, 2, 2)$. Exploit the analogy with max-flow problems to establish how many portions of food can be assigned in total.

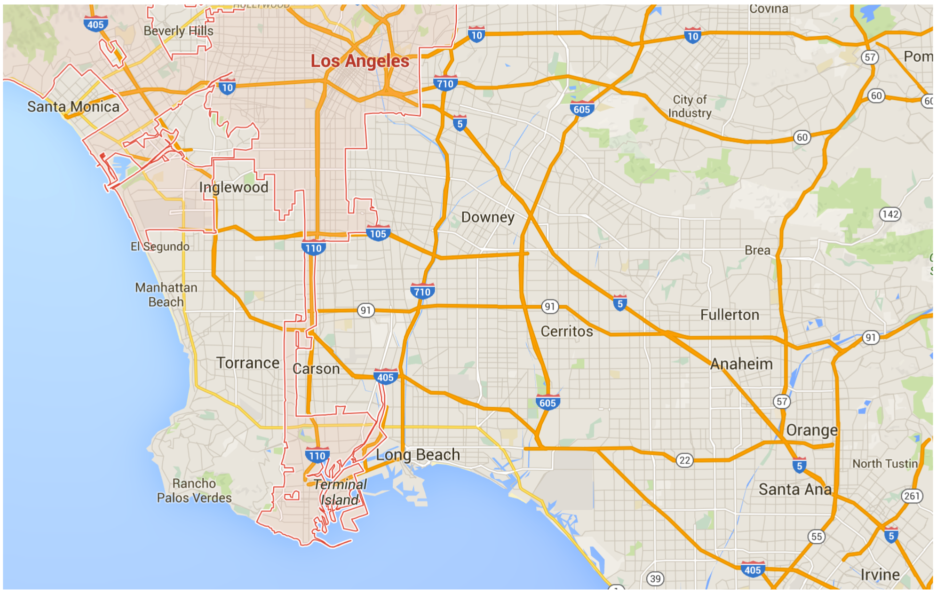

Figure 2: The highway network in Los Angeles.

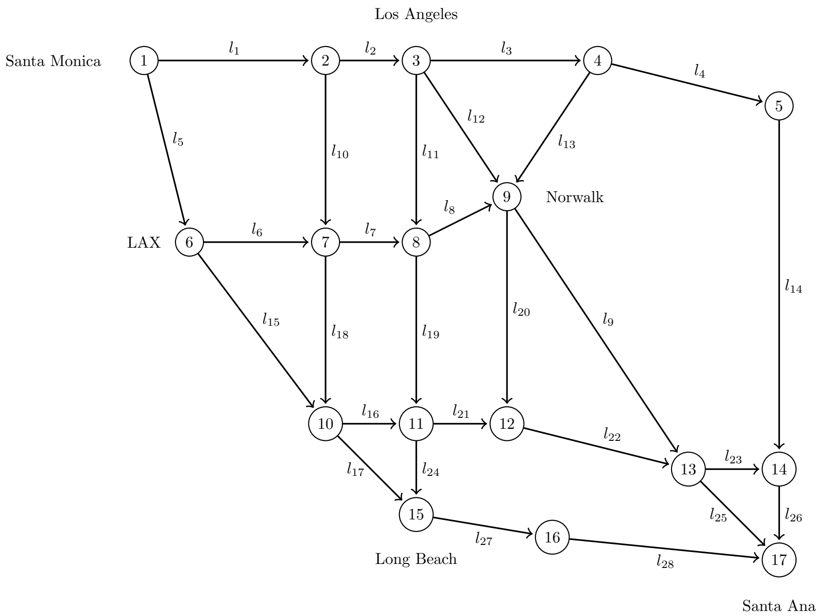

Figure 3: Some possible paths from Santa Monica (node 1) to Santa Ana (node 17).

Exercise 3. We are given the highway network in Los Angeles, see Figure 2. To simplify the problem, an approximate highway map is given in Figure 3, covering part of the real highway network. The node-link incidence matrix B, for this traffic network is given in the file *traffic.mat*. The rows of B are associated with the nodes of the network and the columns of B with the links. The i-th column of B has 1 in the row corresponding to the tail node of link e_i and (-1) in the row corresponding to the head node of link e_i. Each node represents an intersection between highways (and some of the area around).

Each link $e_i \in \{e_1, \ldots, e_{28}\}$, has a maximum flow capacity c_{e_i}. The capacities are given as a vector c_e in the file *capacities.mat*. Furthermore, each link has a minimum travelling time l_{e_i}, which the drivers experience when the road is empty. In the same manner as for the capacities, the minimum travelling times are given as a vector l_e in the file *traveltime.mat*. These values are simply retrieved by dividing the length of the highway segment with the assumed speed limit 60 miles/hour. For each link, we introduce the delay function

$$\tau_e(f_e) = \frac{l_e}{1 - f_e/c_e}, \quad 0 \leq f_e < c_e.$$

For $f_e \geq c_e$, the value of $\tau_e(f_e)$ is considered as $+\infty$.

If you use Python to solve the Exercise, you can load the .mat files by the following code:

```
f = scipy.io.loadmat('flow.mat')["flow"].reshape(28,)
C = scipy.io.loadmat('capacities.mat')["capacities"].reshape(28,)
B = scipy.io.loadmat('traffic.mat')["traffic"]
l = scipy.io.loadmat('traveltime.mat')["traveltime"].reshape(28,)
```

(a) Find the shortest path between node 1 and 17. This is equivalent to the fastest path (path with shortest traveling time) in an empty network.

(b) Find the maximum flow between node 1 and 17.

(c) Given the flow vector in *flow.mat*, compute the vector ν satisfying $Bf = \nu$.

In the following, we assume that the exogenous inflow is zero in all the nodes except for node 1, for which ν_1 has the same value computed in the point **(c)**, and node 17, for which $\nu_{17} = -\nu_1$.

(d) Find the social optimum f^* with respect to the delays on the different links $\tau_e(f_e)$. For this, minimize the cost function

$$\sum_{e \in \mathcal{E}} f_e \tau_e(f_e) = \sum_{e \in \mathcal{E}} \frac{f_e l_e}{1 - f_e/c_e} = \sum_{e \in \mathcal{E}} \left(\frac{l_e c_e}{1 - f_e/c_e} - l_e c_e \right)$$

subject to the flow constraints.

Hint: use the Tutorial (https://www.cvxpy.org/tutorial/functions/index.html) to learn how to code functions in cvxpy.

(e) Find the Wardrop equilibrium $f^{(0)}$. For this, use the cost function

$$\sum_{e \in \mathcal{E}} \int_0^{f_e} \tau_e(s) ds.$$

(f) Introduce tolls, such that the toll on link e is $\omega_e = \psi'_e(f^*_e) - \tau_e(f^*_e)$. For the considered $\psi_e(f_e)$, $\omega_e = f^*_e \tau'_e(f^*_e)$, where f^*_e is the flow at the system optimum. Now the delay on link e is given by $\tau_e(f_e) + \omega_e$. compute the new Wardrop equilibrium $f^{(\omega)}$. What do you observe?

(g) Instead of the total travel time, let the cost for the system be the total additional travel time compared to the total travel time in free flow, given by

$$\psi_e(f_e) = f_e(\tau_e(f_e) - l_e)$$

subject to the flow constraints. Compute the system optimum f^* for the costs above. Construct a toll vector ω^* such that the Wardrop equilibrium $f^{(\omega^*)}$ coincides with f^*. Compute the new Wardrop equilibrium with the constructed tolls $f^{(\omega^*)}$ to verify your result.