TP1 TSIA 202

1) • $Z_t \sim WN(0, \sigma^2)$ donc $E(Z_t) = 0$ et $\text{cov}(Z_t, Z_s) = \sigma^2 \delta_{t=s}$

• $X_t = a + b Z_t + Z_{t-1}$. Par linéarité, $E(X_t) = a$ et

$\text{Cov}(X_t, X_s) = b^2 \text{Cov}(Z_t, Z_s) + b(\text{Cov}(Z_t, Z_{s-1}) + \text{Cov}(Z_s, Z_{t-1})) + \text{Cov}(X_{s-1}, X_{t-1})$

$$= \begin{cases} (1+b^2)\sigma^2 & \text{si } t = s \\ b\,\sigma^2 & \text{si } s-t = \pm 1 \\ 0 & \text{sinon} \end{cases}$$

• $X_t = \sum_{k=0}^{K} 2^{-k} Z_{t-k} + a \rightarrow E(X_t) = a$ et

$$\text{Cov}(X_s, X_t) = \text{Cov}\left(\sum_{k=0}^{K} \left(\frac{1}{2}\right)^k Z_{s-k}, \sum_{l=0}^{K} \left(\frac{1}{2}\right)^l Z_{t-l}\right)$$

$$= \sum_{0 \le k, l \le K} \left(\frac{1}{2}\right)^{k+l} \underbrace{\text{Cov}(Z_{s-k}, Z_{t-l})}_{\sigma^2 \mathbb{1}_{\{s-k=t-l\}}}$$

si $s-t \ge 0$ $\text{Cov}(X_s, X_t) = \sigma^2 \sum_{l=0}^{K} \left(\frac{1}{2}\right)^{2l+s-t}$

$$= \left(\frac{1}{2}\right)^{s-t} \sigma^2 \frac{4}{3} \quad \text{car } K \text{ très grand}$$

De même si $s-t \le 0$.

Donc $\gamma(t) = \text{autocov} = \frac{4}{3} \sigma^2 2^{-|t|}$

• $X_t = A_0 \cos(\lambda_0 t + \phi_0) + Z_t$ où $\lambda \in [0, \pi[$, $\phi_0 \sim \mathcal{U}([0; 2\pi])$

$E(X_t) = E(A_0 \cos(\lambda_0 t + \phi_0)) = A_0 \cos(\lambda_0 t) E(\cos(\phi_0)) - A_0 \sin(\lambda_0 t) E(\sin(\phi_0))$

$= 0$

car $E(\cos(\phi_0)) = \int_0^{2\pi} \frac{\cos(x)\,dx}{2\pi} = 0$

$\mathrm{Cov}(X_n, X_{n+h}) = E(X_{n+h} \times X_n) = A_0^2\, E(\cos(\lambda_0(2n+h) + 2\Phi_0) + \cos(\lambda_0 h)) + E(Z_{n+h} Z_n)$.

car $\cos(a) \times \cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$ et car

$\cos(\lambda_0 t + \Phi_0)$ et Z_t sont indépendants

Puis $\mathrm{Cov}(X_n, X_{n+h}) = \gamma(h) = \frac{A_0^2}{2} \cos(\lambda_0 h) + \sigma^2 \delta_{h=0}$