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Contents

Note: This document is just a summary of what I have learnt. For Lect1UDec, I have listed my inference on studying the theorems and the proofs rather than the rigorous proofs themselves. Proofs have been added where they have seemed necessary though. The Lect1UDec can be accessed at the end of the document from the 'Sources' section.

1 Definitions and notations used:

1. If A is any set (collection, whose elements may be numbers or any other objects) we write $\underline{x \in A}$ to indicate that x is a member (or an element) of A.
   If x is not a member of A, we write $x \notin A$.

2. The set which contains no element is called the empty set, denoted by ϕ.

3. If A and B are sets, and if every element of A is an element of B, we say that A is a subset of B, and write $A \subset B$.
   In addition, if there is an element of B which is not in A, then A is a proper subset of B , $A \subsetneq B$.

4. If $A \subset B$ and $B \subset A$, we write $A = B$. Otherwise, $A \neq B$

5. Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

   (a) If $x \in S$ and $y \in S$, then one and only one of the statements
   $x < y, \;\; x = y, \;\; y < x$ is true.

   (b) If $x, y, z \in S$; if $x < y$ and $y < z$, then $x < z$

6. An ordered set is a set S in which an order is defined.
   Eg. Q is an ordered set if $r < s \Rightarrow s - r$ is a positive rational number(can be represented in the form $\frac{p}{q}$; $p, q \in \mathbb{Z}$, $q \neq 0$).

7. Let S be an ordered set, and $E \subset S$.

   (a) If there exists a $\alpha \in S$ such that $x \leq \alpha \; \forall x \in E$, we say that E is bounded above, and α an upper bound of E

   (b) If there exists a $\beta \in S$ such that $x \geq \beta \; \forall x \in E$, we say that E is bounded below, and β a lower bound of E

8. For an ordered set S and $e \subset S$, let E be bounded above. If there exists an α with the following properties:

   i) α is an upper bound of E.

   ii) If $\gamma < \alpha$, then γ is not an upper bound of E.

   Then α is called the least upper bound of E or the supremum of E, written as $\alpha = \sup E$
   Its lower-bound analogue is called the greatest lower bound or infimum.

9. An ordered set S is said to have the least-upper-bound property if the following is true:
   If $E \subset S$, E is not empty and it is bounded above, then $\sup E$ exists in S.

10. A field is a set with two operations, called "addition" and "multiplication", which satisfy certain field axioms. These axioms include the closure, commutative, associative, identity, inverse and disributive properties for addition and multiplication.Eg. $\mathbb{Q}$, the set of rational numbers.
    □ An ordered field is a field F which is also an ordered set such that:

    i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,

ii) $xy > 0$ if $x \in F$, $y \in F$ $x > 0$ and $y > 0$

11. The extended real number system consists of the real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$. We define as $\infty < x < +\infty, \quad \forall x \in \mathbb{R}$

    $+\infty$ can be taken as an upper bound of every subset of this system, and similarly for lower bound, we can take $-\infty$

12. For two sets A and B, if each element of A is associated with some element in B (denoted by $f(x)$), then f is said to be a function from A to B. A will be the domain, B the codomain and the set of the values of $f(x)$ will be the range.

13. By a sequence, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, we usually denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, \ldots$. The values of f, that is, the elements x_n are called terms of the sequence.

    If A is a set and if $x_n \in A$ $\forall n \in J$, then $\{x_n\}$ is said to be a sequence of elements in A.

14. Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 <, ...,,$ then the sequence $\{p_{n_i}\}$ is called a subsequence of $\{p_n\}$.

15. A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\varepsilon > 0$, there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N, m \geq N$.

16. A metric space in which every Cauchy sequence converges is said to complete.

17. A set X, whose elements we shall call points, is said to be a metric space if with any two points p and q of X, there is associated a real number $d(p, q)$ called the distance from p to q, such that:

    i) $d(p, q) > 0$ if $p \neq q$; $\quad d(p, p) = 0$
    ii) $d(p, q) = d(q, p)$
    iii) $d(p, q) \leq d(p, r) + d(r, q)$

    Eg. The Euclidean spaces $\mathbb{R}^k$
    Any function with these three properties is called a distance function or a metric.
    Distance in $\mathbb{R}^k$ is defined as $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|, \qquad (\vec{x}, \vec{y} \in \mathbb{R}^k)$

18. By the segment (a, b), we mean the set of all real numbers x such that $a < x < b$.
    Similarly, $[a, b]$ means x such that $a \leq x \leq b$

19. If X is a metric space, if $E \subset X$ and if E' denotes the set of all limit points of E in X, then the closure of E is the set $\bar{E} = E \cup E'$ A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

20. Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e. if no point of A lies in the closure of B and no point of B lies in the closure of A.

    A set $E \subset X$ is said to be connected if E is not a union of two non-empty sets.

21. Given a sequence $\{a_n\}$ and the notation $\sum\limits_{n=p}^{q} a_n = a_p + a_{p+1} + \ldots + a_q, \ \ (p \leq q)$,

we associate a sequence $\{s_n\}$ with $\{a_n\}$, where $s_n = \sum\limits_{k=1}^{n} a_k$.

We also say $\{s_n\} = \sum\limits_{n=1}^{\infty} a_n$ This is called an <u>infinite series</u> or just <u>series</u>.

The numbers s_n are called the <u>partial sums</u> of the series.

22. Let f be defined (real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \quad (a < t < b, t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t)$$

, provided this limit exists. f' is called the <u>derivative</u> of f.

2 Principles of Mathematical Analysis

2.1 Real and complex number system:

2.1.1 Theorems:

1. "Suppose S is an ordered set with the least upper bound property, $B \subset S$, B is not empty and it is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$ "
$\Longrightarrow$ This theorem implies a close relation between greatest lower bounds and the lowest upper bounds, and that every ordered set with the least upper bound property also has the greatest lower bound property. The proof employs the idea that if we take the set of lower bounds for B, that set must have a supremum because of the property of S. That supremum must be the infimum of B by definition. Hence, the infimum also exists.

2. "There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property.
. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield."
$\Longrightarrow$ This theorem establishes the existence of real numbers.

3. i) "If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.": *Archimedean property of* $\mathbb{R}$.
   ii) "If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.":($\mathbb{Q}$ *is dense in* $\mathbb{R}$*: Between any two real numbers, there is a rational one.*)

4. "For every real $x > 0$ and every integer $n > 0$, there is one and only one positive real y such that $y^n = x$ "
Then $y = x^{1/n}$ or $\sqrt[n]{x}$

Corollary: If a and b are positive real numbers and n is a positive integer, then

$$\sqrt[n]{ab} \equiv (ab)^{1/n} = a^{1/n}b^{1/n}$$

2.2 Euclidean Spaces:

- For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\vec{x} = (x_1, x_2, \ldots, x_k),$$

where $x_1, x_2, \ldots x_k$ are real numbers, called the coordinates of $\vec{x}$.
The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$.

- The zero element (origin or the null vector) is the point $\vec{0}$, whose all coordinates are 0.

- Important properties:

  i) $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$ (Vector addition)
  ii) $\alpha\vec{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_k)$, where $\alpha \in \mathbb{R}$ (Scalar multiplication)

iii) $\vec{x} \cdot \vec{y} = \sum\limits_{i=1}^{k} x_i y_i$ (Inner product/scalar product)

iv) $|\vec{x}| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}} = \left(\sum\limits_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}$ (*norm* of $\vec{x}$)

2.3 Finite, countable and uncountable sets

If there exists a 1-1 mapping of set A onto set B, then A and B have the same cardinal number, or that A and B are equivalent. ($A \sim B$). Equivalence implies reflexivity, symmetry and transitivity.

For any positive integer n, let J_n be the set whose elements are the integers 1,2, $\ldots n$; and let J be the set consisting all the positive integers. For any set A, we say:

i) $A \sim J_n$ for some $n \implies A$ is finite. If not, infinite.

ii) $A \sim J \implies A$ is countable. If not, uncountable.

iii) A is at most countable if A is finite or countable.

Note: Countable sets can be infinite , eg $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

□ A finite set cannot be equivalent to one of its proper subsets. That is, however, possible for infinite sets. Thus, we could define an infinite set A if it is equivalent to one of its proper subsets.

2.3.1 Theorems

1. 'Every infinite subset of a countable set A is countable.
   $\implies$ The proof for this arranges the distinct elements of the subset into a sequence and then attaches a 1-1 correspondence with J, as A is countable. Thus, the subset would also be countable.

2. Let $\{E_n\}, n = 1, 2, 3, \ldots$ be a sequence of countable sets, and put $S = \bigcup\limits_{n=1}^{\infty} E_n$
   Then S is countable.
   Corollary: "Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable.
   Put $T = \bigcup\limits_{\alpha \in A} B_\alpha$. Then T is at most countable."

3. Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.
   $\implies$ The proof uses the method of induction. $B_1 = A$, which is countable, so B_1 is countable. Then every element in B_n can be represented as the tuple (b, a), where $b \in B_{n-1}$. Keeping b fixed for one instance, the varying 'a' will again represent A, hence countable.
   If we use B_2, each pair (p, q) can correspond to $\dfrac{p}{q}$, where $p, q \in \mathbb{Z}$; $q \neq 0$.
   This leads to the following:
   Corollary: "The set of all rational numbers is countable."

2.4 Metric spaces

Every subset Y of a metric space X is a metric space in its own right, with its own distance function.

2.4.1 Common terms used:

Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

a) A neighborhood of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$, for some $r > 0$. The number r is called the radius of $N_r(p)$.

b) A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
Note: A limit point of S may not belong to S. Eg. $S = (0,1)$, here 1 is a limit point.

c) If $p \in E$ and p is not a limit point of E, then p is called an isolated point of E.

d) E is closed if every limit point of E is a point of E.
Note: Every finite set is a closed set. Also, $\phi, \mathbb{N}$ and $\mathbb{Z}$ are closed.

e) A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$.

f) E is open if every point of E is an interior point of E.

g) The complement of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

h) E is perfect if E is closed and if every point of E is a limit point of E.

i) E is bounded if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

j) E is dense in X if every point of X is a limit point of E, or a point of E (or both).

2.4.2 Convexity

If $a_i < b_i$ for $i = 1, \dots k$, the set of the points $\vec{x} = (x_1, x_2, \dots, x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a k-cell.

We call a set $E \subset \mathbb{R}^k$ convex if

$$\lambda\vec{x} + (1-\lambda)\vec{y} \in E,$$

whenever $\vec{x} \in E, \vec{y} \in E$ and $0 < \lambda < 1$
Eg. Open and closed balls are convex.

2.4.3 Important theorems:

1. Every neighbourhood is an open set.

2. If p is a limit point of a set E, then every neighbourhood of p contains infinitely many points of E.
Corollary: A finite point set has no limit points.

3. Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$$

4. A set E is open iff its complement is closed.
Corollary: A set F is closed iff its complement is open.

5. a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup G_\alpha$ is open.
   b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap F_\alpha$ is closed.
   c) For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.
   d) For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.
6. If X is a metric space, if $E \subset X$ and if E' denotes the set of all limit points of E in X, then the closure of E is the set $\bar{E} = E \cup E'$
   Here, E' denotes the derived set, which is the set of all limit points of E. Eg.
   (a) $(0,1)' = [0,1]$
   (b) $\mathbb{N}' = \phi,\ \mathbb{Z}' = \phi,\ \phi` = \phi$
   (c) $\mathbb{Q}' = \mathbb{R}$
   (d) $\{\frac{1}{n}, n \in \mathbb{N}\}' = \{0\}$
7. *If X is a metric space and $E \subset X$, then*
   a) $\bar{E}$ is closed.
   b) $E = \bar{E}$ if and only if E is closed.
   c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.
8. Let E be a non-empty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Thus, $y \in E$ if E is closed.

   • Let $E \subset X \subset Y$. Then E may be an open set with respect to X without being an open set with respect to Y. Eg. the segment (a,b) is open with respect to $\mathbb{R}^1$, but not with respect to $\mathbb{R}^2$
9. Suppose $Y \subset X$. A subset E of $Y$is open relative to Y iff $E = Y \cap G$ for some open subset G of X.

2.5 Compact sets

•By an open cover of a set E in a metric space X, we mean a collection of $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup\limits_{\alpha} G_\alpha$

Eg. $C = \{(-n,n)|n \in \mathbb{N}\}$, then $\mathbb{R} \subset \bigcup\limits_{n\in\mathbb{N}} (-n,n)$. So, C can be an open cover for $\mathbb{R}$

•Subcover: A sub-collection of open covers of any set.
Eg. $C' = \{(-3n,3n)|n \in \mathbb{N}\}$ is a subcover of C, and both are open covers for $\mathbb{R}$
•A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.
More explicity, the requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$

i) Every finite set is compact.

ii) A finite subset of real numbers is compact.
Proof: Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite set and G be the family of open cover of S.
$\exists$ an open set containing each $a_i \in S$
$\therefore$ S is finite
$\Rightarrow$ open sets containing a_i are finite.
$\Rightarrow$ G has finite subcover of S.
Hence, S is compact.

iii) $\mathbb{R}$ is not compact.
Proof: We know that $\mathbb{R}$ has an open cover $C = \{(-n, n) : n \in \mathbb{N}\}$. But C has infinite subcovers, like $\{(-kn, kn) : n, k \in \mathbb{N}\}$
$\Rightarrow$ $\mathbb{R}$ is not compact.

2.5.1 Theorems:

1) Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.
$\Longrightarrow$ By virtue of this theorem, we are able, in many situations, to regard compact sets as metric spaces in their own right, without paying any attention to any embedding space.

2) Compact subsets of metric spaces are closed.

3) Closed subsets of compact sets are compact.
Corollary: If F is closed and K is compact, then $F \cap K$ is compact.

4) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\bigcap K_\alpha$ is non empty.
Corollary: If $\{K_\alpha\}$ is a sequence of non empty compact sets such that $K_n \supset K_{n+1}$, $(n = 1, 2, \ldots)$, then $\bigcap\limits_1^\infty K_n$ is not empty.

5) If E is an infinite subset of a compact set K, then E has a limit point in K.

6) IF $\{I_n\}$ is a sequence of intervals in $\mathbb{R}^1$, such that $I_n \supset I_{n+1}\,(n = 1, 2, 3, \ldots)$ then $\bigcap\limits_1^\infty I_n$ is not empty.

7) Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}(n = 1, 2, 3, \ldots)$, then $\bigcap\limits_1^\infty I_n$ is not empty.

8) Every k-cell is compact.

9) If a set E in $\mathbb{R}^k$ has one of the following properties, then it has the other two:
   i) E is closed and bounded.
   ii) E is compact.
   iii) Every infinite subset of E has a limit point in E.

10) Weierstrass: Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

11) Let P be a non-empty perfect set in $\mathbb{R}^k$. Then P is uncountable.
Corollary: Every interval $[a, b]$, $(a < b)$ is uncountable. In particular, the set of all real numbers is uncountable.

2.6 Connected sets

Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e. if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be connected if E is not a union of two non-empty sets.

• Separated sets are disjoint, but the converse may not be true.
Eg.$[0, 1]$ and $(1, 2)$ are not separated. However, $(0, 1)$ and $(1, 2)$ are separated.

Theorem:A subset E of the real line $\mathbb{R}^1$ is connected iff it has the following property:
If $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.

2.7 Sequence and series

By a sequence, we mean a function f defined on the set J of all positive integers.

If $f(n) = x_n$, for $n \in J$, we usually denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, \ldots$. The values of f, that is, the elements x_n are called terms of the sequence.

In other words, a sequence of real numbers is a function whose domain is the set $\mathbb{N}$ and the codomain is $\mathbb{R}$ If A is a set and if $x_n \in A$ $\forall n \in J$, then $\{x_n\}$ is said to be a sequence of elements in A.
Range of a sequence: The set of all distinct terms of a sequence. It may be finite or infinite, but it is always countable. The sequence is said to be bounded if its range is bounded.

Eg.
•$\{n^2\}$ is unbounded and thus, has infinite range.
•$\{i^n\}$ has finite range, and thus, is bounded. (But still not a convergent sequence)

However, the converse may not be true.
Eg.
•$\frac{1}{n}$ and $\left\{1 + \left[\frac{(-1)^n}{n}\right]\right\}$ have infinite range, but are bounded.

Constant sequence: $\{a_n\}$ is constant, if the range of the sequence is a singleton set.

•A sequence $\{s_n\}$ of real numbers is said to be :

i) monotonically increasing if $s_n \leq s_{n+1}$ $(n = 1, 2, 3, ...)$

ii) monotonically decreasing if $s_n \geq s_{n+1}$ $(n = 1, 2, 3, ...)$

Theorem:
Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Convergent sequence:A sequence $\{p_n\}$ in a metric space X is said to converge if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there exists an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$

We say that $\{p_n\}$ converges to p, or that p is the limit of the sequence; we denote it as

$$p_n \to p \text{ or } \lim_{n\to\infty} p_n = p$$

If $\{p_n\}$ does not converge, it is said to diverge.

NOTE: The definition of "convergent sequence" depends not only on p_n but also on X.
Eg. $\left\{\frac{1}{n}\right\}$ converges in $\mathbb{R}^1$ (to 0) but fails to converge in the set of all positive real numbers.

2.7.1 Theorem

Let $\{p_n\}$ be a sequence in a metric space X.

a) $\{p_n\}$ converges to $p \in X$ iff every neighbourhood of p contains p_n for all but finitely many n.

b) If $p \in X$, $p' \in X$ and if $\{p_n\}$ converges to p and to p', then $p = p'$

   In other words, the limit is unique.

c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded. (The limit can itself act as the upper or lower bound for monotionically increasing or decreasing function).

d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim\limits_{n\to\infty} p_n$.

2.7.2 Theorems

1. Suppose $\{s_n\}, \{t_n\}$ are complex sequences, and $\lim\limits_{n\to\infty} s_n = s$, $\lim\limits_{n\to\infty} t_n = t$. Then:

   (a) $\lim\limits_{n\to\infty} (s_n + t_n) = s + t$

   (b) $\lim\limits_{n\to\infty} cs_n = cs$, $\lim\limits_{n\to\infty} (c + s_n) = c + s$, for any number c

   (c) $\lim\limits_{n\to\infty} s_n t_n = st$

   (d) $\lim\limits_{n\to\infty} \dfrac{1}{s_n} = \dfrac{1}{s}$ provided $s_n \neq 0$ $(n = 1, 2, ...)$, $s \neq 0$

2. i) Suppose $\vec{x}_n \in \mathbb{R}^k$ $(n = 1, 2, ...)$ and $\vec{x}_n = (\alpha_1, \alpha_2 ..., \alpha_k)$
   Then $\vec{x_n}$ converges to $\vec{x} = (\alpha_{1,n}, \alpha_{2,n} ..., \alpha_{k,n})$ iff

   $$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \qquad (1 \leq j \leq k)$$

   ii) Suppose $\{\vec{x}_n\}, \{\vec{y}_n\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $\vec{x}_n \to \vec{x}, \vec{y}_n \to \vec{y}, \beta_n \to \beta$. Then,

   $$\lim_{n\to\infty} (\vec{x}_n + \vec{y}_n) = \vec{x} + \vec{y}, \quad \lim_{n\to\infty} \vec{x}_n \cdot \vec{y}_n = \vec{x} \cdot \vec{y}, \quad \lim_{n\to\infty} \beta_n \vec{x}_n = \beta \vec{x}$$

2.7.3 Subsequences

Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 <, ...,$, then the sequence $\{p_{n_i}\}$ is called a subsequence of $\{p_n\}$.

If $\{p_n\}$ converges, its limit is called a <u>subsequential limit</u> of $\{p_n\}$.

Theorems

1. a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

   b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

2. The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

2.7.4 Cauchy sequences

A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\varepsilon > 0$, there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N, m \geq N$.

Let E be a non empty subset of a metric space X, and let S be the set of all real numbers of the form $d(p,q)$ with $p, q \in E$. The sup of S is called the diameter of E.

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N=2} \ldots,\ \ \{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} \text{diam } E_N = 0$$

A metric space in which every Cauchy sequence converges is said to complete.

Theorems:

1. a) If $\overline{E}$ is the closure of set E in a metric space X, then

$$\text{diam}\,\overline{E} = \text{diam}\,E$$

   b) If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1},\ (n = 1, 2, 3, \ldots)$, and if

$$\lim_{n\to\infty} \text{diam } K_n = 0$$

   then $\bigcap\limits_{n=1}^{\infty} K_n$ consists of exactly one point.

2. a) In any metric space X, every convergent sequence is a Cauchy sequence.

   b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point in X.

   c) In $\mathbb{R}^k$, every Cauchy sequence converges.

2.7.5 Upper and lower limits

Let $\{s_n\}$ be a sequence of real numbers with the following property:

For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$, then we can write $s_n \to +\infty$.

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $s_n \leq M$, we write $s_n \to -\infty$

•Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequences $\{s_{n_k}\}$. This set E contains all subsequential limits, plus possibly the numbers, $-\infty, +\infty$.

We put $s^* = \sup E$, and $\qquad s_* = \inf E$

The numbers s^*, s_* are called the upper and lower limits of $\{s_n\}$; we denote them as

$$\lim_{n\to\infty} \sup s_n = s^*, \qquad \lim_{n\to\infty} \inf s_n = s_*,$$

Theorem

1. Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* be as defined before. Then s^* has the following properties:

   a) $s^* \in E$

   b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.
      Moreover, s^* is the only number with these properties. Also, the analogous results for s_* are true.

2. If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

   i)
   $$\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \inf t_n$$

   ii)
   $$\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup t_n$$

2.7.6 Series

Given a sequence $\{a_n\}$ and the notation $\sum\limits_{n=p}^{q} a_n = a_p + a_{p+1} + \ldots + a_q, \;\; (p \leq q)$,

we associate a sequence $\{s_n\}$ with $\{a_n\}$, where $s_n = \sum\limits_{k=1}^{n} a_k$.

We also say $\{s_n\} = \sum\limits_{n=1}^{\infty} a_n$ This is called an <u>infinite series</u> or just <u>series</u>.

The numbers s_n are called the <u>partial sums</u> of the series.

If s_n converges to s, we say that the series converges. $\left(\sum\limits_{n=1}^{\infty} a_n = s\right)$

The number s is called the <u>sum of the series</u>. It is actually the limit of the sequence of sums, and not obtained simply by addition.

<u>Theorem(Cauchy criterion)</u>

•A sequence converges in $\mathbb{R}^k$ if and only if it is a Cauchy sequence.

•$\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$$

if $m \geq n \geq N$

In particular, by taking $m = n$, we have $|a_n| \leq \varepsilon \qquad (n \geq N)$. Thus,

If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$

<u>Theorems</u>

1. A series of **non negative terms** converges if and only if its partial sums form a bounded sequence.

2. a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

b) (only for series of non negative terms a_n)
   If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

3. If $0 \leq x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

4. Suppose $a_1 \geq a_2 \geq \ldots \geq 0$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 \ldots$$

   converges.

5. $\sum \frac{1}{n_p}$ converges if $p > 1$ and diverges if $p \leq 1$.

6. If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

   converges; if $p \leq 1$, the series diverges.

7. e, an irrational number is defined as

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

2.7.7 The root and ratio tests

Theorem(Root Test): Given $\sum a_n$, put $\alpha = \lim\limits_{n\to\infty} \sup \sqrt[n]{|a_n|}$. Then:

a) if $\alpha < 1$, then $\sum a_n$ converges.

b) if $\alpha > 1$, then $\sum a_n$ diverges.

c) if $\alpha = 1$, the test gives no information.

Theorem(Ratio Test): The series $\sum a_n$

a) *converges* if $\lim\limits_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$

b) *diverges* if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

The following **theorem** compares the root test and the ratio test:
For any sequence $\{c_n\}$ of positive numbers,

$$\lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} \leq \lim_{n\to\infty} \inf \sqrt[n]{c_n}$$

$$\lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n} \leq \lim_{n\to\infty} \sup \sqrt[n]{c_n}$$

2.7.8 Power series

Given a sequence $\{c_n\}$ of complex numbers, the series $\sum\limits_{n=0}^{\infty} c_n z^n$ is called a power series. (z is a complex number.)The numbers c_n are called the coefficients of the series.

With every power series, there is associated a circle of convergence. The series will converge if z is in the interior of the circle, it will diverge if z is in the exterior.

Theorem

Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}$$

(If $\alpha = 0, R = +\infty$; if $\alpha = +\infty, R = 0$)

Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

(The proof comes by considering $a_n = c_n z^n$ and then applying the root test.)

Eg. The series $\sum z^n$ has $R = 1$.

2.7.9 Summation by parts

Theorem: Given two sequences $\{a_n\}, \{b_n\}$, put

$$A_n = \sum_{k=0}^{n} a_k$$

If $n \geq 0$; put $A_{-1} = 0$. Then if $0 \leq p < q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Theorem: Suppose

a) the partial sums A_n of $\sum a_n$ form a bounded sequence.

b) $b_0 \geq b_1 \geq b_2 \ldots$

c) $\lim\limits_{n\to\infty} b_n = 0$ Then $\sum a_n b_n$ converges.

2.7.10 Absolute convergence

The series $\sum a_n$ is said to converge absolutely if the series $\sum |a_n|$ converges.

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say that it converges non-absolutely. Eg. $\sum \frac{(-1)^n}{n}$

The comparison test, as well as the root and ratio tests, are really tests for absolute convergence, and, therefore, cannot give any information about non-absolutely converging series.

2.7.11 Rearrangements

Let $\{k_n\}, n = 1, 2, 3, \ldots$ be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J). Putting $a'_n = a_{k_n}$, $(n = 1, 2, 3, \ldots)$, we say that $\sum a'_n$ is a rearrangement of $\sum a_n$

Theorem: Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty$$

Then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that

$$\lim_{n\to\infty} \inf s'_n = \alpha, \quad \lim_{n\to\infty} \sup s'_n = \beta$$

Theorem: If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

2.8 Continuity

Theorem: Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$. Then

a) $\lim_{x\to p} (f+g)(x) = A + B$

b) $\lim_{x\to p} (fg)(x) = AB$

c) $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$

Def: Suppose X and Y are metric spaces, $E \subset X, p \in E$ and f maps E into Y. Then f is said to be continuous at p if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.
If f is continuous at every point on E, then f is said to be continuous on E.

Theorems:

1. $\lim_{x\to p} f(x) = q$ iff $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n\to\infty} p_n = p$

   Corollary: If f has a limit at p, this limit is unique.

2. Let p be a limit point of E. Then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$

3. Suppose X, Y, Z are metric spaces, $E \subset X, f : E \mapsto Y, g$ maps the range of $f, f(E)$ into Z, and h is the mapping of E into Z defined by $h(x) = g(f(x)), (x \in E)$.
   If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

4. A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

   Corollary: A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.

5. Let f and g be complex continuous functions on a metric space X. Thus, $f+g, fg, f/g(g \neq 0)$ are continuous on X.
   One application is to show that every monomial of the form $x_1^{n_1} x_2^{n_2}, \dots x_k^{n_k}$, and its scalar multiple, and also rational functions are continuous.

6. a) Let $f_1, f_2, \dots, f_k$ be real functions on a metric space X, and let $\vec{f}$ be the mapping of X into $\mathbb{R}^k$defined by
   $$\vec{f}(x) = (f_1(x), \dots, f_k(x)) \qquad (x \in X)$$
   b) If $\vec{f}$ and $\vec{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\vec{f} + \vec{g}, \vec{f} \cdot \vec{g}$ are continuous on X.

2.8.1 Continuity and compactness

A mapping $\vec{f}$ of a set E into $\mathbb{R}^k$ is said to be bounded if there is a real number M such that $|\vec{f}(x)| \leq M$ for all $x \in E$

Theorems:

1. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

2. If $\vec{f}$ is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $\vec{f}(X)$ is closed and bounded. Thus, $\vec{f}$ is bounded.

3. Suppose f is a continuous real function on a compact metric space X, and
   $M = \sup_{p \in X} f(p), m = \inf_{p \in X} f(p)$

   Then there exist points $p, q \in X$ such that $f(p) = M, f(q) = m$

4. Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x,\ (x \in X)$ is a continuous mapping of Y onto X.

Def: Let f be a mapping of a metric space X into a metric space Y. We say that f is uniformly continuous on X if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \varepsilon$$

for all p and q in X for which $d_X(p, q) < \delta$

The major difference between 'uniformly continuous' and 'continuous' is that uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.
Moreover, if f is continuous on X, it is possible to find δ for each ε and each point. If f is however, uniformly continuous, we can find *one* $\delta < 0$ which will do for all points p of X. EVery uniformly continuous function is continuous.

Theorems:

1. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

2. Let E be a non compact set in $\mathbb{R}^1$. Then

   a) there exists a continuous function on E which is not bounded.

   b) there exists a continuous and bounded function on E which has no maximum.
   If, in addition, E is bounded, then

   c) there exists a continuous function on E which is not uniformly continuous.

2.8.2 Continuity and connectedness

2.8.3 Theorems

1. If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

2. Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$

   The converse may **not** be true: For any points $x_1 < x_2$ and for any number c between $f(x_1)$ and $f(x_2)$, there is a point x in (x_1, x_2) such that $f(x) = c$, then f is continuous.
   Eg.

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

2.8.4 Discontinuous

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is <u>discontinuous</u> at x. Two types of discontinuities—

i) First kind, or Simple discontinuity– Discontinuous at x, but left hand and right hand limits exist.

ii) Second discontinuity– other cases

2.8.5 Monotonic functions

Let f be a real function on (a, b). Then, f is said to be <u>monotonically increasing</u> on (a, b), if $a < x < y < b$ implies $f(x) \leq f(y)$. Its decreasing analogue will be <u>monotonically decreasing</u>.

Theorems

1. Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point x of (a, b). More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(y-)$$

Analogous results for monotonically decreasing functions also hold true.

2. Let f be monotonic on (a,b). Then the set of points (a,b) at which f is discontinuous is at most countable.

   Corollary: Monotonic functions don't have any discontinuity of the second kind.

2.9 Differentiation and Integration

Theorems

1. Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

2. Suppose f is continuous on $[a,b]$, $f'(x)$ exists and at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$. I $h(t) = g(f(t)), \quad (a \leq t \leq b)$, then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x)$$

3. Let f be defined on $[a,b]$; if f has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

4. If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

5. If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$f(b) - f(a) = (b-a)f'(x)$$

6. Suppose f is a real differentiable function on $[a,b]$and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$

   Corollary: If f is differentiable on $[a,b]$, then f' cannot have any simple discontinuities on $[a,b]$

7. **L'Hospital's rule**
   Suppose f and g are real and differentiable in (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$. Suppose $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$.

   If $f(x) \to 0$ and $g(x) \to 0$ (or) if $g(x) \to +\infty$ as $x \to a$, then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a$$

8. **Taylor's theorem:**
   Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$ and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

   Then there exists a point x in between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

Same rules apply for complex valued functions as well.

For vector valued functions, all rules are pretty similar. We, however, use inner product $\vec{f} \cdot \vec{g}$ instead of the scalar fg. The mean value theorems don't work with them though, and neither does L'Hospital's Rule. The following theorem is used for vector valued functions:
Theorem: Suppose $\vec{f}$ is a continuous mapping of $[a,b]$ into $\mathbb{R}^k$ and $\vec{f}$ is differentiable in (a,b). Then there exists $x \in (a,b)$ such that

$$|\vec{f}(b) - \vec{f}(a)| \leq (b-a)|\vec{f}'(x)|$$

2.9.1 Riemann-Stieltjes Integral

Let $[a,b]$ be a given interval. By a partition P of $[a,b]$ we mean a finite set of points $x_0, x_1, x_2, \ldots, x_n$ where $a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq x_n = b$,
we write $\triangle x_i = x_i - x_{i-1}$, $(i = 1, 2, \ldots, n)$

Now suppose f is a bounded and real function defined on $[a,b]$, we put

$M_i = \sup f(x)$, $(x_{i-1} \leq x \leq x_i)$; $\qquad m_i = \inf f(x)$, $(x_{i-1} \leq x \leq x_i)$

$U(P,f) = \sum_{i=1}^{n} M_i \triangle x_i$, $\qquad L(P,f) = \sum_{i=1}^{n} m_i \triangle x_i$

Finally,

$\overline{\int_a^b} f\,dx = \inf U(P,f)$, $\qquad$ (Upper Riemann integral)

$\underline{\int_a^b} f\,dx = \sup L(P,f)$ $\qquad$ (Lower Riemann integral)

If the Upper and Lower Riemann integrals are equal, we say that the function is Riemann integrable. ($f \in \mathscr{R}$). Obviously,

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

Def:(more general case, Riemann-Stieltjes integral)
Let α be a monotonically increasing function on $[a,b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a,b]$). Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

It is clear that $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a,b]$ we put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i, \qquad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

where M_i, m_i have the same meaning as in Definition 6.1, and we define

$$\overline{\int_a^b} f\, d\alpha = \inf U(P, f, \alpha), \tag{5}$$

$$\underline{\int_a^b} f\, d\alpha = \sup L(P, f, \alpha), \tag{6}$$

the inf and sup again being taken over all partitions.

If the left members of (5) and (6) are equal, we denote their common value by

$$\int_a^b f\, d\alpha \tag{7}$$

or sometimes by

$$\int_a^b f(x)\, d\alpha(x). \tag{8}$$

This is the *Riemann-Stieltjes integral* (or simply the *Stieltjes integral*) of f with respect to α, over $[a, b]$. (α need not be continuous here)

Def: We say that the partition P^* is a refinement of P if $P^* \supset P$. Given two partitions P_1 and P_2, we say that P^* is their common refinement if $P^* = P_1 \cup P_2$

Theorems

1. If P^* is a refinement of P, then
$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$
$$\text{and} \;\; U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

2. $$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$$

3. $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that
$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

4. (a) If the preceding condition holds for some P, then it will hold (with the same ε) for every refinement of P.

   (b) If the condition holds for $P = x_0, x_1, \ldots, x_n$ and if s_i, t_i are arbitrary points in $\{x_{i-1}, x_i\}$, then
$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \triangle\alpha_i < \varepsilon$$

   (c) If $f \in \mathscr{R}$ and the hypotheses of (b) hold, then
$$\left| \sum_{i=1}^{n} f(t_i) \triangle\alpha_i - \int_a^b f\, d\alpha \right| < \varepsilon$$

5. If f is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

6. If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$.

7. Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}$

8. Suppose $f \in \mathscr{R}$ on $[a,b]$, $m \leq f \leq M$, ϕ is continuous on $[m,M]$ and $h(x) = \phi(f(x))$ on $[a,b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a,b]$

9. If $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a,b]$, then

   (a) $fg \in \mathscr{R}(\alpha)$

   (b) $|f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$

10. If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then

$$\int_a^b f\,d\alpha = f(s)$$

    Here, I is the unit step function

$$I(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$$

11. Suppose $c_n \geq 0$ for $1, 2, 3, \ldots, \sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a,b) and $\alpha(x) = \sum\limits_{n=1}^{\infty} c_n I(x - s_n)$.
    Let f be continuous on $[a,b]$. Then

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

12. Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a,b]$. Let f be a bounded real function on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$. In that case,

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$$

13. **Change of variable:**

    Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathscr{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by
    $\beta(y) = \alpha(\varphi(y))$, $\quad g(y) = f(\varphi(y))$. Then $g \in \mathscr{R}(\beta)$ and

$$\int_A^B g\,d\beta = \int_a^b f\,d\alpha$$

14. Let $f \in \mathscr{R}$ on $[a,b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t)\,dt$$

Then f is continuous on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0)$$

15. **Fundamental Theorem of Calculus:**
If $f \in \mathscr{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

16. **Integration by parts:**
Suppose F and G are differentiable functions on $[a,b], F' = f \in \mathscr{R}$ and $G' = g \in mathscrR$. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$

For **vector valued functions**, we define

$$\int_a^b \vec{f}\,d\alpha = \left(\int_a^b f_1\,d\alpha, \ldots, \int_a^b f_k\,d\alpha\right)$$

Theorems:

1. If $\vec{f}$ and $\vec{F}$ map $[a,b]$ into $\mathbb{R}^k$, if $\vec{f} \in \mathscr{R}$ on $[a,b]$, and if $\vec{F}' = \vec{f}$, then

$$\int_a^b \vec{f}(t)\,dt = \vec{F}(b) - \vec{F}(a)$$

2. If f maps $[a,b]$ into $\mathbb{R}^k$ and if $\vec{f} \in \mathscr{R}(\alpha)$ for some monotonically increasing function α on $[a,b]$, then $|\vec{f}| \in \mathscr{R}(\alpha)$ and

$$\left|\int_a^b \vec{f}\,d\alpha\right| \leq \int_a^b |\vec{f}|\,d\alpha$$

2.9.2 Rectifiable curves

Def: A continuous mapping γ of an interval $[a,b]$ into $\mathbb{R}^k$ is called a curve on $[a,b]$ in $\mathbb{R}^k$.

If γ is one-to-one, γ is called an arc.

If $\gamma(\alpha) = \gamma(\beta)$, then γ is said to be a closed curve.

We associate with each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and to each curve γ on $[a,b]$ the number

$$\Lambda(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

The i^{th} term in this sum is the distance between two points, and hence Λ is the length of the polygonal path with vertices at $\gamma(x_1), \ldots, \gamma(x_n)$. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely.

We define the length of γ as

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma)$$

, the supremum is taken over all partitions of $[a, b]$.

If $\Lambda(\gamma) < \infty$, we say that γ is rectifiable.

Theorem:

If γ' is continuous on $[a, b]$, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt$$

3 Lect1UDec

3.1 Definitions used:

1) Polygon, $\wp$: It is a simply connected closed region in $\mathbb{E}^2$ whose boundary $\partial\wp$ is a simple curve consisting of finitely many line segments.

2) Set S is said to be convex, if $\forall x, y \in S$ and $\forall\,\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)y \in S$$

3) Vertex: If $\partial\wp$ intersected with some disc with center 'p' consists of two line segments (with 'p' as common endpoint) which are not extensions of each other, it forms a vertex.

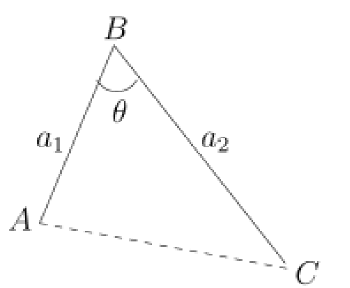
3.2 Part 1

"Among all polygons with n sides in $\mathbb{E}^2$ having area A, perimeter minimizer is the regular polygon."

3.2.1 Lemma

"Among all polygons with n sides in $\mathbb{E}^2$ whose all sides but one are given in length, maximum area is attained when the vertices lie on a circle whose center is the midpoint of the 'remaining side'."

$$A(T_\theta) = \frac{1}{2}\,a_1\,a_2\,\sin\theta \;\le\; \frac{1}{2}\,a_1\,a_2 = A(T_{\frac{\pi}{2}}).$$

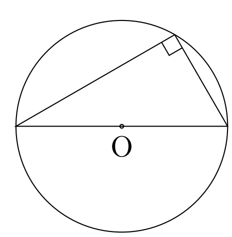


We can prove this by first proving it for a triangle and then, generalizing the result for other polygons.

For a triangle, we can simply use the formula for area i.e.

$$Area = \frac{1}{2}a_1 a_2 \sin\theta,$$

where θ is the angle between the two fixed sides a_1 and a_2. Then the angle for maximum area would be when $\sin\theta$ is highest, i.e. $\frac{\pi}{2}$ radians. But this is a semicircular angle, and we can draw a circle keeping the remaining side as the diameter and the vertex at the circumference. Hence, the theorem.

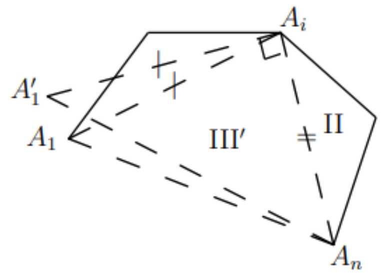


Semi circular angle

Now, for the **general case**,
We can fix one point of the polygon at (0,0) and then lay down the known sides, step-by-step, one after the other. The furthest it can go is the distance $(a_1 + a_2 + \ldots + a_{n-1})$. So, the set of all polygons of n-sides (whose all but one side-lenghts are known) are bounded by a circle of radius $(a_1 + a_2 + \ldots + a_{n-1})$ (and the origin as the centre). Also, limit of any converging sequence of polygons(the area maximizer) is still a valid polygon with the same side lengths.Thus, it is compact. Thus, by the compactness argument, we will have a polygon with maximum area.

Let $A_1, A_2, \ldots, A_n$ be the vertices of our area maximizer and $a_1, a_2, \ldots, a_{n-1}$ be the known side lengths. (A_1A_n) will be the unknown side.

If an angle $\angle A_1A_iA_n \neq \frac{\pi}{2}$, then we simply fix the point A_i and rotate the line $[A_1A_i]$ to a new position so that the angle becomes $\frac{\pi}{2}$. This increases the area by considering each 'i' as a triangle case, while keeping the known side lengths same. Thus, this new polygon becomes the new area maximizer.



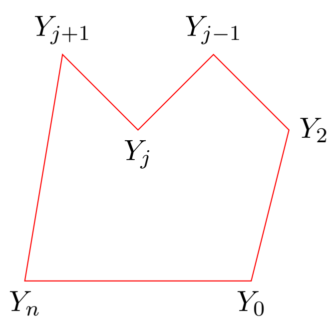
3.2.2 Theorem

Fix $A > 0$ and $n \geq 3$ in $\mathbb{N}$. Among all polygons with n sides in $\mathbb{E}^2$ having area A, the perimeter minimizer is the regular polygon.

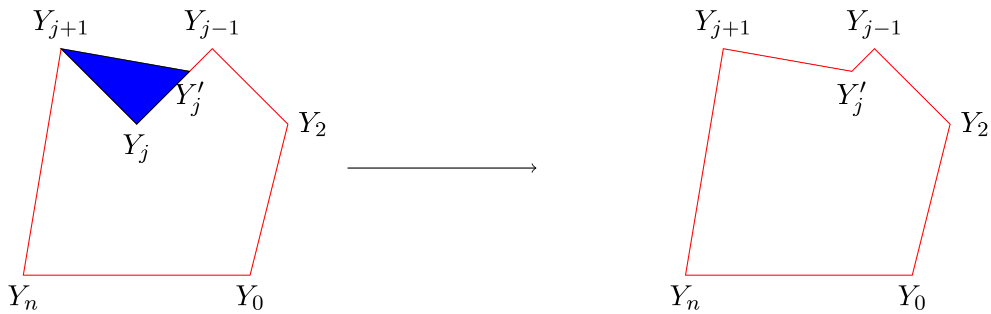
i. a perimeter minimizer exists, say $\wp_0$
$\implies$ First, we consider a set $\mathcal{F}$ which consists of n-sided polygons with **area**$\geq A$. Then, we take a sequence of polygons $\{\wp_k\}_{k\in\mathbb{N}}$ such that $\bar{0}$ is a vertex of $\wp_k$, $\forall\, k$ and whose perimeter approaches the value L. Thus, the set would be bound by a ball of finite radius, let's say $(L+1)$.
We can, therefore, find a converging subsequence (Bolzano-Weierstrass theorem) so that we get our perimeter minimizer.

ii. $\wp_0$ is convex, equilateral and equiangular.

__**Convex:**__ Let's say we have a polygon where one point is not locally convex.



Here, the polygon is not convex at Y_j. We can now choose point Y_j' on Y_{j-1}, Y_j such that $\triangle[Y_j'Y_jY_{j+1}]$ does not intersect the interior of the polygon.



To obtain the new polygon, we essentially replace two sides of the triangle with the third side(as shown). So, by using the triangle inequality, the perimeter of the new polygon must be less than our supposed 'perimeter minimizer'. Also, it has area $\geq A$ and thus, is included in our set $\mathcal{F}$. But this is a contradiction as we assumed that $\wp_0$ was our perimeter minimizer. So,to remove this contradiction, our perimeter minimizer must be convex.

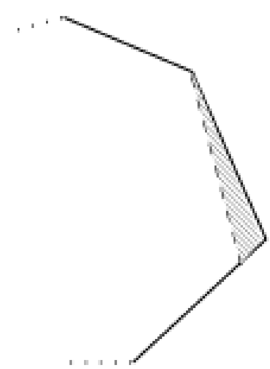
Area: By definition of $\wp_0$, we have $Ar(\wp_0) \geq 0$

Let us consider a strict inequality.

$A(\wp_0) > A \implies A(\wp_0) = A + \varepsilon, \qquad \varepsilon > 0$

Then we can always 'cut' a piece of area $< \varepsilon$ as shown. As the area will still be $\geq A$, it will belong to the set $\mathcal{F}$. But we used the convexity of the polygon to cut the piece as shown and thus, it will have less perimeter than the original polygon (i.e. our perimeter minimizer), which is not possible!

So, our perimeter minimizer must have area exactly equal to A, otherwise we could repeat the above process continuously to get a new polygon in the set with less perimeter.

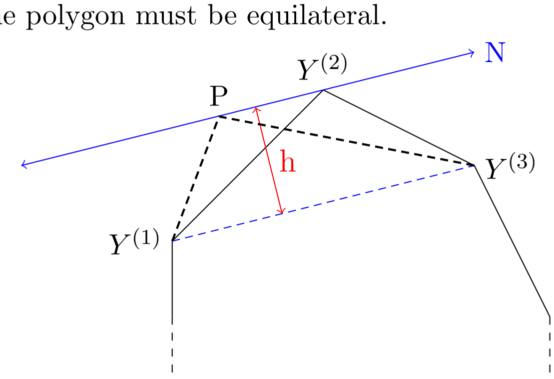


$\wp_0$

Equilateral: We need to prove

$$||Y^{(1)} - Y^{(2)}|| = ||Y^{(3)} - Y^{(2)}||$$

Then we can extend this process for all the sides, and prove that the polygon must be equilateral.



We draw a line N parallel to $Y^{(1)}Y^{(3)}$, passing through $Y^{(2)}$ and P is any point movable on the line N. Now, area of a triangle between two parallel lines and fixed base is constant. So, there is no change in area of $\triangle[Y^{(1)}PY^{(3)}]$ $(= area(\triangle[Y^{(1)}Y^{(2)}Y^{(3)}]))$

Let's consider the line $Y^{(1)}Y^{(3)}$ as our x-axis and its midpoint as the origin. Then $Y^{(1)}$ can be represented as $(-a, 0)$ in Cartesian coordinates and $Y^{(3)}$ as $(a, 0)$. The height is h (constant).

For $\triangle[Y^{(1)}PY^{(3)}]$,

$$\begin{aligned}
\text{Area} &= \frac{1}{2} \times 2a \times h = ah \qquad \text{(constant)}\\
\text{Perimeter} &= ||Y^{(1)}Y^{(3)}|| + ||Y^{(1)}P|| + ||Y^{(3)}P||\\
&= 2a + ||Y^{(1)}P|| + ||Y^{(3)}P||\\
&= 2a + \sqrt{(x+a)^2+h^2} + \sqrt{(x-a)^2+h^2}
\end{aligned}$$

For minimum perimeter,

$$\begin{aligned}
&\frac{d}{dx}(\text{Perimeter}) = 0\\
\implies &\frac{x+a}{\sqrt{(x+a)^2+h^2}} + \frac{x-a}{\sqrt{(x-a)^2+h^2}} = 0\\
\implies &(x+a)\sqrt{(x-a)^2+h^2} + (x-a)\sqrt{(x+a)^2+h^2} = 0
\end{aligned}$$

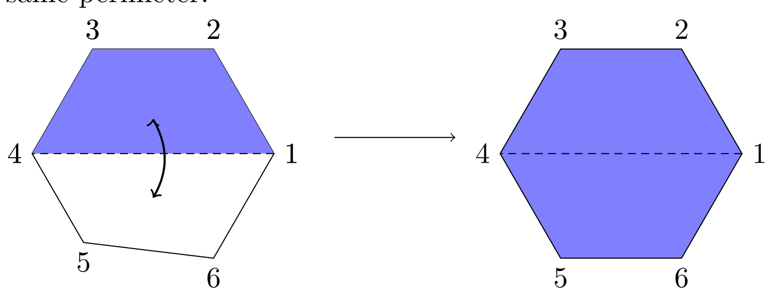
We can easily see that the equation is satisfied when $x = 0$. In that case, the length of both the variable sides become equal($\sqrt{a^2+h^2}$). In other words, the triangle is isosceles.

Using this for all the sides, we conclude that our perimeter minimizer must be equilateral.

Equiangular: We solve for two separate cases: when the number of sides is even vs when it is odd.

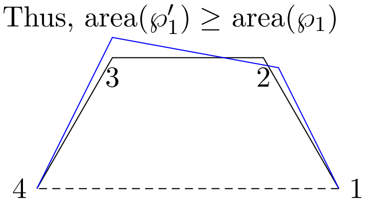
•Even:

We can divide such a polygon into two halves of equal areas. If one side has greater area (let), then we can just take a mirror image of the larger side and flip it to form a polygon with greater area but same perimeter.



Let us take the polygons with one side unequal and the rest equal (The dashed one is the unequal one in this case). This is the half of our perimeter minimizer, let $\wp_1$ with area $A/2$. Let us assume there exists another polygon $\wp_1'$ (blue) which has area more than A i.e. let it be the area maximizer.

Thus, area($\wp_1'$) $\geq$ area($\wp_1$)



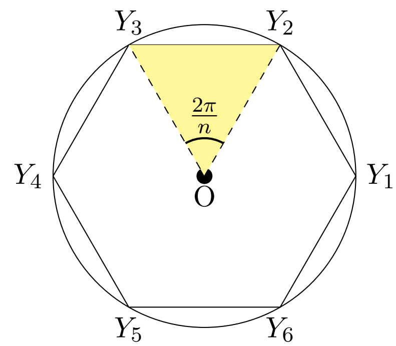
But if such a polygon existed, its area would be $> A/2$. And then we could just flip the larger polygon across the dashed line and get a new polygon with area greater than A, but with the same perimeter.

But that would contradict our theory that the area of our perimeter minimizer cannot be greater than A. So, such a polygon cannot exist.

By the Lemma proved before, we know that our perimeter minimizer must be such that in each half, the unequal side must be the diameter of the circle and the vertices must be on the circumference. Thus,

$$r = d(O, Y_i), \ \ \forall\, i = 1, 2, \ldots, 2k$$

We know that equal chords subtend equal angles at the centre. Each such isosceles triangle (as shown) is congruent. Furthermore, as the polygon is cyclic, each equal chord subtends equal angle at the centre. Thus, the polygon is equiangular (thereby, regular).



•<u>Odd</u>:

Let $\wp_R^n$ be the regular polygon with n sides having sides of length a. (This is uniquely determined upto congruence.) If we inscribe it in a circle of radius r, we have

$$a = 2r \sin\left(\frac{\pi}{n}\right)$$
$$\implies r = \frac{a}{2\sin\left(\frac{\pi}{n}\right)}$$

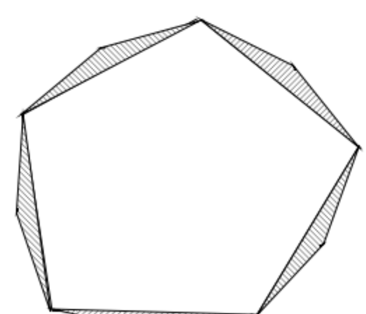
$$\begin{aligned} \therefore \text{Area} &= n \cdot \frac{1}{2} r \cdot r \sin\left(\frac{2\pi}{n}\right) \\ &= \frac{nr^2}{2} \sin\left(\frac{2\pi}{n}\right) && (1) \\ &= \frac{n}{2} \frac{a^2}{4\sin^2\left(\frac{\pi}{n}\right)} \cdot 2 \sin\frac{\pi}{n} \cos\frac{\pi}{n} \\ &= \frac{na^2}{4} \tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right) && (2) \end{aligned}$$

Suppose the perimeter minimizer is not regular. Then, we can put isosceles triangles on each side with the equal lengths s and small height h. Let's call this polygon $\wp_0^{2n}$.(As its parent polygon is not regular, this new polygon will also be not regular) We do a similar construction with $\wp_R^n$. Using our n-even case,

$$\begin{aligned} & A(\wp_0^{2n}) < A(\wp_R^{2n}) \\ \implies & A + \cancel{n \cdot A(T)} < A(\wp_R^n) + \cancel{n \cdot A(T)} \\ \implies & A < A(\wp_R^n) \end{aligned}$$

But this would mean that the perimeter minimizer has area greater than A, which is a contradiction!

Thus, the perimeter minimizer is a regular polygon.



$\wp_0^{2n}$

3.3 Part II

3.3.1 Theorem

Fix $A > 0$. There exists unique perimeter minimizer among all piece-wise smooth simple closed curves in $\mathbb{E}^2$ enclosing area A, and it is a circle of radius $\sqrt{\frac{A}{\pi}}$.

This is proved by the following two theorems.

3.3.2 Theorem(Proving the existence)

Fix $A > 0$. Among all piecewise smooth simple closed curves in $\mathbb{E}^2$ enclosing area A, circle of radius $\sqrt{\frac{A}{\pi}}$ has least perimeter.

$\Longrightarrow$

Step 1:

We take a collection $\mathcal{G}$ consisting of all piecewise smooth simple closed curves in $\mathbb{E}^2$ enclosing area at least A. Also, we define L as $L := \inf\{l(C) : C \in \mathcal{G}\}$

Then there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ in $\mathcal{G} \ni l(C_n) \to L$ and $A(C_n) \to A' \geq A$

We can now assume that $\bar{0} \in C_n$ and $l(C_n) \leq L+1$. So, it is bounded by an open ball of radius $L+1$

Thus, Area$(C_n) \leq \pi(L+1)^2, \ \forall n \in \mathbb{N}$

We can then extract a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $\{A(C_{n_k})\}$ is a converging subsequence. Let $A' := \lim\limits_{n\to\infty} A(C_{n_k})$

We will denote $\{C_{n_k}\}$ as $\{C_n\}$ from now.

We can approximate each $\{C_n\}$ by a polygon $\wp_q$ with $q(n) = q$ sides such that

$$q(n) > q, \lim_{n\to\infty} l(\wp_{q(n)}) = L \text{ and } \lim_{n\to\infty} A(\wp_{q(n)}) = A'$$

Next, we can form a unique regular polygon using A$(\wp_{q(n)})$ and $q(n)$, and name it as $\tilde{\wp}_{q(n)}$

By the theorem we proved in Part 1, $l(\tilde{\wp}_{q(n)}) \leq l(\wp_{q(n)})$

Step 2:

$$\lim_{n\to\infty} l(\tilde{\wp}_{q(n)}) = 2\sqrt{\pi A'}$$

$\therefore l(\tilde{\wp}_{q(n)}) \leq l(\wp_{q(n)}) \leq l(C_n) \ \ \forall n$ and $l(C_n) \to L$

$$\begin{aligned} \therefore L' := 2\sqrt{\pi A'} &= \lim_{n\to\infty} l(\tilde{\wp}_{q(n)}) \\ &\leq \lim_{n\to\infty} l(C_n) \\ &\leq L \end{aligned}$$

This regular polygon can then be inscribed in a circle with centre $\bar{0}$ and radius $\tilde{r}_{q(n)}$

Step 3:

$$\lim_{n\to\infty} \tilde{r}_{q(n)} = \sqrt{\frac{A'}{\pi}}$$

We put $r = d\sqrt{\frac{A'}{\pi}} = \lim\limits_{n\to\infty} \tilde{r}_{q(n)}$ Then, $A \leq A' = \pi r^2$ ...(1)

and

$L \geq L' = 2\sqrt{\pi A'} = 2\pi r$

Let C_r denote the circle of radius r with center $\bar{0}$. By (1), $C_r \in \mathcal{G}$ and hence,

$L' = l(C_r) = 2\pi r \geq L$

Thus, $L' = L = 2\pi r$ and C_r is a perimeter minimizer.

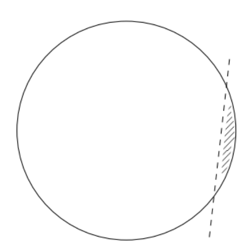
Step 4:

If $\pi r^2 > A$, then we can truncate C_r by using a line to produce a curve $\tilde{C}$ with

$l(\tilde{C}) < L = l(C_r)$ and $A < A(\tilde{C}) < A(C_r) = \pi r^2$

The area is such that $\tilde{C} \in \mathcal{G}$ with perimeter less than L— the infimum of the set! This is a contradiction.

Therefore, $A = A' = \pi r^2 = A(C_r)$



3.3.3 Theorem(The uniqueness part)

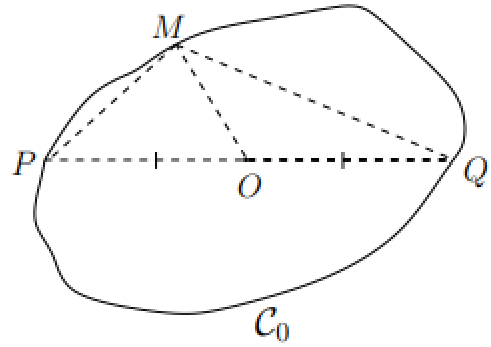
Fix $A > 0$. Among all piecewise smooth simple closed curves in $\mathbb{E}^2$ enclosing area A, any perimeter minimizer is a circle of radius $\sqrt{\frac{A}{\pi}}$.

$\Longrightarrow$ Let $\mathcal{G}$ and L be defined as before.

We know a perimeter minimizer exists over $\mathcal{G}$. Let it be C_0 and the region bounded by it be D_0. We already know the following:

•D_0 is convex and has Area$(C_0)A$ (using similar arguments as before)

•A line bisecting the perimeter of D_0 must bisect the area too.



We fix a point P on C_0. Another point Q divides C_0 into two arcs, C_0^+ and C_0^- of equal length. $\overline{PQ}$ divides the region D_0 into D_0^+ and D_0^- of equal area. Then we take $r_0 = \frac{d(P,Q)}{2}$('O' is the midpoint of $\overline{PQ}$)

To prove it a circle, we have to prove $d(O, M) = r_0, \ \forall M \in C_0$

Let there be a point M such that $d(O, M) \neq r_0$ As seen from the figure, D_0^+ is the region containing M with $C_0^+ \cup PQ$ as the boundary. As D_0 is convex, $\triangle PMQ$ is in the interior of D_0^+ We can generalize Lemma 1.1, and use it to construct a domain $\tilde{D}_0^+$ of area strictly greater than that of D_0^+ whose boundary consists of the curve $\tilde{C}_0^+$, which is congruent with C_0^+ and [P',Q'] (P',Q' are the endpoints of $\tilde{C}_0^+$). Finally, we can reflect this arc to get the polygon $\tilde{C}_0$ whose area is more than A, and perimeter L. The perimeter makes it eligible for the role of perimeter minimizer.

But the problem is that it contradicts our assumption that the area of our perimeter minimzer must not be greater than A. So, our perimeter minimizer is unique– a circle of radius $\dfrac{A}{\pi}$.

3.4 The Isoperimetric Inequality in $\mathbb{E}^2$

For any piecewise smooth simple closed curve C in $\mathbb{E}^2$ with arc length l and enclosing area $A > 0$, we have

$$l^2 \geq 4\pi A$$

The equality holds iff C is a circle of radius $\sqrt{\dfrac{A}{\pi}}$

Let us fix C and $A := A(C)$. Let $\mathcal{G}$ and L be as before. We know, the unique perimeter minimzer is a circle of radius $r_0 = \sqrt{A}\pi$.

$$\begin{aligned}
\therefore L^2 &= (2\pi r_0)^2 \\
&= 4\pi(\pi r_0^2) \\
&= 4\pi A \\
\therefore l^2 &\geq L^2 \\
&\geq 4\pi A
\end{aligned}$$

If $l^2 = 4\pi A$, then $l = \sqrt{4\pi A} = 2\sqrt{\pi A} = L$

4 Maximum principles for 1-D space $\mathbb{R}$ (functions of one variable)

•Let $a, b \in \mathbb{R}$. Let $g : (a,b) \xrightarrow{bdd} \mathbb{R}$. Let $u : [a,b] \xrightarrow{cont} \mathbb{R}$ .Let$u : (a,b) \to \mathbb{R}$ have cont. second order derivative and u satisfy:

$$L[u] \equiv u'' + g(x)u' > 0, \;\; \forall\, a < x < b \tag{1}$$

Then the max of u in the interval [a,b] cannot be attained anywhere except the endpoints a or b.

•Now, consider the non strict inequality:

$$L[u] = u'' + g(x)u' \geq 0 \qquad \forall\, a < x < b \tag{2}$$

Then u $\equiv$ const satisfies (2) and every pt of $[a,b]$ is the point of maximum.
We prove that this is the only exception possible:

4.1 Theorem 1: Maximum principle for 1D space

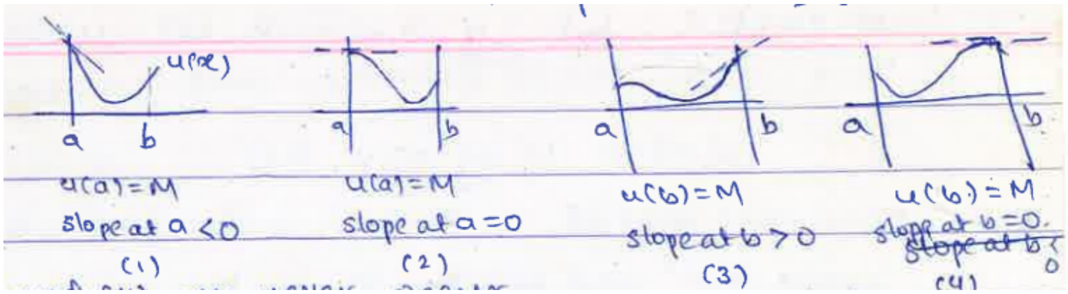
Suppose $u = u(x)$ satisfies the d.i.(2) on (a,b) with $g(x)$ bdd on (a,b).
If $u(x) \leq M$ $\forall x \in (a,b)$ and if $u(c) = M$ for some $c \in (a,b)$ then $u \equiv M$.

Proof: Let $u(c) = M$ for some $c \in (a,b)$ and suppose $\exists d \in (a,b)$ such that $u(d) < M$ (i.e. u is a non constant function)

- Assume $c < d$
- Since g is bdd on (a,b), $\exists \alpha > 0$ s.t. $(g(x)| < \alpha$ $\forall x \in (a,b)$
- Define $z(x) := e^{\alpha(x-c)} - 1 \qquad \forall x \in (a,b)$ Then
  - i. $z(c) = 0$
  - ii. $z > 0$ on (c,b)
  - iii. $z < 0$ on (a,c)
- Choose $0 < z < \dfrac{M - u(d)}{z(d)}$. Define $w(x) := u(x) + \epsilon z(x)$ Then
  - i. $w < M$ on (a,c) $[z < 0$ on $(a,c)]$
  - ii. $w(c) = M$
  - iii. $w(d) < M \qquad [w(d) = u(d) + \epsilon z(d)$
    $< u(d) + M - u(d)$
    $= M]$

$\implies \exists\, c' \in (c,d)\ s.t.\ w(c') = M' := max(w(x)), x \in (a,d)$
Clearly, $M' \geq M$
But $L[w] = L(u) + \epsilon\alpha[\alpha + g(x)]e^{\alpha(x-c)} > 0$ $\forall\, x \in (a,b)$ [on particular $\forall x \in (a,d)$]
This contradicts (1) $\square$
(For $c > d$, we take $z(x) := e^{-\alpha(x-c)} - 1$)

Note: If g is bdd on every closed sub-interval contained in (a,b) then Theorem 1 still holds. We need to apply the argument on any closed interval $[a', b'] \subset (a,b)$ such that $c, d \in [a', b']$
The function u satisfying (2) has one of the following forms:



Note:

- if $u(a)$ =Max then slope at $a \leq 0$
- if $u(b)$ =Max then slope at $b \geq 0$

We prove that situation (2) and (4) never occur.

4.2 Theorem 2

(Simply assume g is bdd on $[a, b]$)
Suppose u is a non constant function satisfying (2) with g bdd on every closed sub interval of (a, b) and suppose u has one-sided derivative at a and b.

1. If g is bdd below at $x = a$ and if $u(a) = \max$ then $u'(a) < 0$
2. If g is bdd above at $x = b$ and if $u(b) = \max$ then $u'(b) > 0$

<u>Proof:</u>

1. Suppose $u(a) = M, u(x) \leq M \ \ \forall x \in [a, b] \,\&\, u(d) < M$ for some $d \in (a, b)$
   Let ϵ, w be as in the previous proof. Then $L[w] > 0$ on $(a, d]$
   $\implies$ the max must occur at a or d. We have $w(a) = M > w(d) := u(d) + \epsilon z(d)$
   $\implies$ $w(a) = \max$ $\therefore$ The one sided derivative of w at a, $w'(a) \leq 0$ $(w'(a) = u'(a) + \epsilon z'(a) = u'(a) + \alpha\epsilon)$
   $\implies$ $u'(a) \leq -\epsilon\alpha < 0$
   $\implies$ $u'(a) < 0$

2. The case $u(b) = M$ is similar.

$\square$ The boundedness of u is very essential. Let

$$g(x) = \begin{cases} -\dfrac{3}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{3}$$

Then g is not bdd below on $(0, 1)$ $[\because g(\frac{1}{n}) = -3n, n \in \mathbb{N}]$
Then $u(x) = 1 - x^4$ satisfies $u'' + g(x)u' = 0$ But

- on $(-1, 1), u$ has max at 0. (Theorem 1 is violated)
- on $(0, 11), u'(0) = 0$ (Theorem 2 is violated)

Proposition: If u(x) satisfies the d.i. $(L+h)[u] \equiv u'' + g(x)u' + h(x)u \geq 0$ in an interval (a,b) with $h(x) \leq 0$ if $g \,\&\, h$ are bdd on (a,b) and if u assumes a non negative max value M at an interior point, then $u \equiv M$

Examples:

1. $u(x) = x^2$ on $(-1,1), g(x) \equiv 0, h(x) := \frac{-2}{(x+1)^2}$
2. $u(x) = -(1+x^2)$ on $(-1,1), h(x) \equiv -2, g(x) \equiv 0$
   $(L+h)[u] = -2 + 2(1+x^2) = 2x^2 \geq 0$

4.3 The Generalized Maximum Principle

Let $u(a,b) \mapsto \mathbb{R}$ satisfy

$$L + h[u] \equiv u'' + g(x)u' + h(x)u \geq 0$$

where $h(x)$ is bdd and $g(x)$ is bdd. from below. Then the following holds: ($h(x) \leq 0$ not assumed.)

i) For $[a,b]$ sufficiently short, a function w can be found such that

$$w > 0 \text{ on } [a,b]$$
$$(L+h)[w] \leq 0 \text{ in } (a,b)$$

ii) If the function $\dfrac{u}{w}$ assumes a non-negative max M at an interior point C, then $u \equiv M$.

5 Sources

- Principles of Mathematical Analysis-Rudin
- LectU1Dec
- Lecture: Maximum Principles
- Maximum Principles in Differential Equations— Murray H. Protter and Hans F. Weinberger