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AM25M009

Linear Dynamic System

Assignment-2

1) a. $\lambda \to$ eigenvalue of A $\qquad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$x \to$ eigenvector of A

to show that x is eigen vector of $\underline{B = A - 7i}$

$Bx = \lambda x \leftarrow$ To show

~~$Ax = \lambda x$~~

$= \cancel{Bx}$ $\quad Bx = (A - 7i\cancel{I})x$

$Bx = Ax - 7ix$

$Bx = \lambda x - 7ix$

$Bx = (\lambda - 7i)x \leftarrow$ same eigen vector

new eigen value $= \lambda - 7i$

b. $Ax = \lambda x$

$A^{-1}Ax = A^{-1}\lambda x$

$x = A^{-1}(\lambda x) \quad \Rightarrow \quad \cancel{x\lambda = A}$

$\cancel{A^{-1}x = A\lambda x} \qquad \cancel{A^{-1} = x} \qquad A^{-1}x = \lambda^{-1}x$

same eigen vector

$\lambda^{-1} \Rightarrow$ new eigen value

2) Given in the question

$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$

Making clever choice of $\boxed{\lambda = 0}$

$\det(A - \lambda I) = \lambda_1 \lambda_2 \cdots \lambda_n$

$\det(A) = \prod_{i=1}^{n} \lambda_i$

(3) $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ → upper triangular matrix (3)

↳ eigen value of upper, lower triangular and diagonal matrix is the diagonal itself

$= \{1, 2, 3, 7, 8, 9\}$

(4) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ A has rank of 1, nullity = 3

so 1 non zero eigen value

~~a~~ non ~~z~~ero eigen value = $1+1+1+1 = \underline{4}$

~~a~~ ~~a~~ eigen value of $A = \lambda_1 = 4, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$

$\det(A - I) \Rightarrow$ ~~A I~~ eigenvalues of A I

$= \{3, -1, -1, -1\}$

$\det(A - AI) = 3 \times -1 \times -1 \times -1 = -3$

(5) A's ~~eg~~ eigen values = 0, 3, 5

eigen vectors: u, v, w

~~(2)~~ d. Basis for nullspace and column space

→ Null space (solution for $\underline{A}x = 0$)

~~on~~ only $Au = 0$

v, w, are ~~ho~~ $\neq 0$

span of u}

Basis for nullspace = {u}

~~a~~ → column space.

~~I can~~ show ~~similarity b/w vs w~~

~~3v = 5w~~

when we plug in v, we get $3v$

whe we plug in w, we get $5w$

any combination of $a\underset{\sim}{v} + b\underset{\sim}{w}$ will map $3a\underset{\sim}{v} + 5c\underset{\sim}{w}$

So range of A is spaned by $\{v, w\}$

b. Solve $A\underset{\sim}{x} = \underset{\sim}{u} + \underset{\sim}{w}$

$\cancel{A}\, x = au + bv + cw$

$Ax = aA\underset{\sim}{u} + bAv + cAw$

$a \cdot (0) + b(3v) + c(5w)$

$A\underset{\sim}{x} = 3b\underset{\sim}{v} + 5c\underset{\sim}{w} = \underbrace{u + w}$ → from question

$3bv + 5cw = u + w$

$5c = 1 \Rightarrow c = 1/5$

$3b\cancel{v} = 0 \Rightarrow v = 0$

$a(0)u = 1$ Not possible

No solution

~~Particular solution is~~ $\cancel{x = au + 0v + \frac{1}{5}w}$

~~for any value of a.~~

c. Show $Ax = u$ has no solution

$Ax = aAu + b\overset{A}{\underset{\sim}{v}} + c\overset{A}{\underset{\sim}{w}} = u$

$Ax = a(0) + 3bv + 5cw = u$

This is in span of v, w, not in u, not possible

$\cancel{tr(A) = a}$

(6) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\rightarrow$ I+ [1, 1] are eigen vector

$Ax = \lambda x$

$$A\begin{pmatrix}1\\1\end{pmatrix} = \lambda\begin{pmatrix}1\\1\end{pmatrix}$$

characteristic eqn

$$: \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = [(a-\lambda)(d-\lambda)] - bc = 0$$

given $a+b = c+d$

and $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$

$A\begin{pmatrix}1\\1\end{pmatrix} = \lambda\begin{pmatrix}1\\1\end{pmatrix} \longrightarrow \boxed{\lambda_1 = a+b}$ or $c+d$

$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$

trace $= a+d \Rightarrow \lambda_1+\lambda_2 \Rightarrow a+b+\lambda_2 = a+d$

det $= ad-bc \Rightarrow \lambda_1\lambda_2 \Rightarrow (a+b)\lambda_2 = ad-bc$

$\lambda_1+\lambda_2 = \cancel{a+b}\ a+d$

$\lambda_2 = a+d-a-b$

$\boxed{\lambda_2 = d-b}$

$\cancel{a+d-a-b = ad-bc}$

$\cancel{a+d-(a+b) = ad-bc}$

$$\boxed{\begin{aligned}\lambda_1 &= a+b \\ \lambda_2 &= d-b\end{aligned}}$$

(7) eigen value $= 1, 2, 4$

Trace$(A) = 1+2+4 = 7$

trace (A^2): $1+4+16 = 21$

eigen value $(A^{-1}) = 1 + \frac{1}{2} + \frac{1}{4}$

$\cancel{1+0.5+0.25}$

1, 0.5, 0.25 //

$\rightarrow$ eigen values remain unchanged by transpose. ⑤

$(A^{-1})^T = 1, 0.5, 0.25$

$\rightarrow$ det $= 1 \times 0.5 \times 0.25 = \underline{\underline{0.125}}$

⑧ Symmetry

$A = S\Lambda S^{-1}$ $\quad$ $S^T = S^{-1}$

Transpose

$A^T = S^{-T} \Lambda S^T$

$A^T = (S^{-1})^T \Lambda S^T$

$A^T = (S^T)^T \Lambda S^{-1}$

$A^T = S \Lambda S^{-1}$

Same, hence A is symmetric.

Orthogonal condition $Q^TQ = QQ^T = I$

$S^T = S^{-1}$ $\quad SS^T = SS^{-1}$
$= SS^T = I$

$S^T = S^{-1} \Rightarrow S^TS = S^{-1}S = I$
$S^TS = I$

$SS^T = S^TS = I$ $\quad$ when $\det(S) = \pm 1$

Hence orthogonal

⑨ $A = S\Lambda S^{-1}$ $\rightarrow$ eigen vectors, $\rightarrow$ eigen values.

$\hookrightarrow$ Diagnolizable

$A + 2I = S\Lambda S^{-1} + 2I$

adding $2I$ shifts eigen value (Λ) by $2I$

eigen vectors do not change

$\Rightarrow$ eigen values $(\Lambda + 2I)$

$\Rightarrow$ eigen vectors (S)

(10) $A = S\Lambda_1 S^{-1}$

$B = S\Lambda_2 S^{-1}$

same eigen vectors

different eigen valu.

$AB = S\Lambda_1 \underbrace{S^{-1} S}\Lambda_2 S^{-1}$

$S\Lambda_1 I \Lambda_2 S^{-1}$

~~$S A_1 A$~~

$S\underbrace{\Lambda_1 \Lambda_2} S^{-1}$

$BA = S\Lambda_2 \underbrace{S^{-1} S}\Lambda_1 S^{-1}$

$BA = S\Lambda_2 I \Lambda_1 S^{-1}$

$BA = S\underbrace{\Lambda_2 \Lambda_1} S^{-1}$

Diagnol matrix

so $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$

$\therefore$ ~~S~~ $AB = BA$