AM5390

AM25M009
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Advanced solid mechanics
Assignment -1

1. $S \Rightarrow$ ~~Skew~~ Sym. tensor

$W \Rightarrow$ skew " "

(a) since sym tensor's transpose it's same as before

$(S)^T = S$

or $S_{ij} = S^T_{ij} = S_{ji}$ — (1)

second order dot product

$\underbrace{S:T}_{\text{scalar}} \Rightarrow \underbrace{S_{ij}:T_{ij}}_{\text{In notation}} \Rightarrow \underbrace{S_{ji}:T_{ij}}_{\text{From (1)}} = \underbrace{S_{ji}:T_{ji}}_{\text{we can swap dummy variables}} = \underbrace{S_{ij}:T_{ji}}_{S:T^T}$

$S:T = S:T^T$ — (2)

we can split into 2 parts

$\frac{1}{2}(S:T) + \frac{1}{2}(S:T)$

or

$\frac{1}{2}(S:T) + \frac{1}{2}(S:T^T)$

$\frac{1}{2}(S:T + S:T^T) = \frac{1}{2}S:[T+T^T]$ (3)

(2) and (3)

$S:T = S:T^T = \frac{1}{2}S:[T+T^T]$

b) we can solve this similar to (a)

we know $W_{ij} = -W_{ji}$ — (4)

$$W:T = W_{ij}:T_{ij} = \underbrace{-W_{ji}:T_{ij}}_{\text{from (4)}} = \underbrace{-W_{ij}:T_{ji}}_{\text{we can swap dummy variables}} = -W:T^T$$

$W:T = -W:T^T$ — (5)

split into 2 parts

$\frac{1}{2} W:T + \frac{1}{2} W:T$

from (5)

$\frac{1}{2}(W:T) - \frac{1}{2} W:T^T$

$\frac{1}{2}[W:T - W:T^T]$

$\frac{1}{2} W:[T-T^T]$ — (6)

from (5) and (6)

$$W:T = -W:T^T = \frac{1}{2} W:[T-T^T]$$

c) $S:W=0$

Intuition $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ -12 & 0 \end{bmatrix} = 0$

$S_{ij}:W_{ij} \xleftrightarrow{\text{swapping}} S_{ji}\,W_{ji} \xleftrightarrow{\text{transpose}} S_{ij}:-W_{ij}$

we know
$S_{ij} = S_{ji}$
$W_{ij} = -W_{ji}$

$S_{ij}:W_{ij} = -[S_{ij}:W_{ij}]$

$2\,S_{ij}:W_{ij} = 0$

$S:W = 0$

~~2)~~ Tensors and principle invarient's relation

b) $[Tu, v, w] + [u, Tv, w] + [u, v, Tw] = I_1 [u, v, w]$

$[Tu, Tv, w] + [u, Tv, Tw] + [Tu, v, Tw] = I_2 [u, v, w]$

$① — [Tu, Tv, Tw] = I_3 [u, v, w]$

from ①

$[Tu, Tv, Tw] = I_3 [u, v, w]$

we know $I_3 = \det(T)$

so, ① becomes

$[Tu, Tv, Tw] = \det T [u, v, w]$

rewriting with $T = RS$

$= [RSu, RSv, RSw] = \det(RS)[u, v, w]$

$\det R [Su, Sv, Sw] = \det(RS)[u, v, w]$

basically some scalling factor

$(\det R) \det(S) [u, v, w] = \det(RS) [u, v, w]$

so it becomes $\det(RS) = (\det R) \det(S)$ — ②

d) To prove $\operatorname{cof}(RS) = \operatorname{cof}(R) \operatorname{cof}(S)$

From ② $\det(RS) = (\det R)(\det S)$

replacing $R \Leftrightarrow R^{-1}$

$S \Leftrightarrow S^{-1}$

$\det(R^{-1} S^{-1}) = (\det R^{-1}) (\det S^{-1})$

we know $A^{-1} = \frac{1}{\det A} (\operatorname{cof} A)^T$

$= \det[(SR)^{-1}] = (\det R^{-1}) \det(S^{-1})$

$$\det\left[\frac{1}{\det(S\cancel{R})}\right]$$ we know

$$(RS)^{-1} = S^{-1}R^{-1}$$

$$\frac{\operatorname{cof}(RS)^T}{\det(RS)} = \frac{\operatorname{cof}(S)^T}{\det(S)} \times \frac{\operatorname{cof}(R)^T}{\det(R)}$$

↳ s from ②

$$\frac{\operatorname{cof}(RS)^T}{\det(R)\cancel{\det(S)}} = \frac{\operatorname{cof}(S)^T \operatorname{cof}(R)^T}{\det(R)\cancel{\det(S)}}$$

$$\operatorname{cof}(RS)^T = \operatorname{cof}(S)^T \operatorname{cof}(R)^T$$

Tak transpose

$$\underline{\operatorname{cof}(RS) = \operatorname{cof}(R)\operatorname{cof}(S)}$$

$\boxed{3}$ a) To prove $\operatorname{tr}(\operatorname{cof}(T)) = \frac{1}{2}(\operatorname{tr}(T)^2 - \operatorname{tr}(T^2))$

$$(\operatorname{cof} T)_{ij} \stackrel{\text{def}}{=} \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr} T_{jq} T_{kr}$$

taking trace. $i = p$

$$\operatorname{tr}[(\operatorname{cof} T)_{ij}] = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{iqr} T_{jq} T_{kr}$$

$$\frac{1}{2}(\delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq})(T_{jq} T_{kr})$$

$$\frac{1}{2}\left[\delta_{jq}\delta_{kr} T_{jq} T_{kr} - \delta_{jr}\delta_{kq} T_{jq} T_{kr}\right]$$

$$\frac{1}{2}\left[T_{qq} T_{rr} - T_{rq} T_{qr}\right]$$

$$\frac{1}{2}(\operatorname{tr}(T)^2 - \operatorname{tr}(T^2))$$

b) $cof(cof(T)) = det(T)T$

$cof T = \frac{1}{2} \varepsilon_{ijk} \varepsilon_{pqr} T_{jq} T_{kr}$

$\downarrow$

second order tensor

applying it again

$cof(cof(T)) = \frac{1}{2} \varepsilon_{ijk} \varepsilon_{pqr} \, cof(T)_{jq} \, cof(T)_{kr}$

$= \frac{1}{2 \times 2 \times 2} \varepsilon_{ijk} \varepsilon_{pqr} \varepsilon_{jab} \varepsilon_{qcd} T_{ac} T_{bd} \, \varepsilon_{kef} \varepsilon_{rgh} T_{eg} T_{fh}$

$\frac{1}{8} \varepsilon_{ijk} \varepsilon_{pqr} \varepsilon_{jab} \varepsilon_{pcd} \varepsilon_{kef} \varepsilon_{rgh} T_{ac} T_{bd} T_{eg} T_{fh}$

$\frac{1}{8} [\delta_{ia} \delta_{jb} - \delta_{ib} \delta_{ja}] [\delta_{pc} \delta_{qd} - \delta_{pd} \delta_{qc}] \varepsilon_{kef} \varepsilon_{rgh} T_{ac} T_{bd} T_{eg} T_{fh}$

$\frac{1}{8} [\delta_{pc} \delta_{qd} - \delta_{pd} \delta_{qc}] \varepsilon_{kef} \varepsilon_{rgh} [T_{ic} T_{jd} - T_{id} T_{jc}] T_{eg} T_{fh}$

$\frac{1}{8} [T_{ip} T_{jq} - T_{iq} T_{jp} - T_{iq} T_{jp} + T_{ip} T_{jq}] \varepsilon_{kef} \varepsilon_{rgh} T_{eg} T_{fh}$

$\frac{1}{4} [T_{ip} T_{jq} - T_{iq} T_{jp}] \varepsilon_{kef} \varepsilon_{rgh} T_{eg} T_{fh}$

$\frac{1}{4} [T_{ip} T_{jq} - T_{iq} T_{jp}] [\delta_{eg} \delta_{fh} - \delta_{eh} \delta_{fg}] [T_{eg} T_{fh}]$

$\frac{1}{4} [T_{ip} T_{jq} - T_{iq} T_{jp}] [T_{ee} T_{ff} - \cancel{\delta_{hg}} \cancel{\delta_{gh}}]$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \downarrow \quad\quad \downarrow$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad tr(T)^2 - tr(T^2)$

trying to ~~make~~ sense of this.

we know ~~cot~~ $T^{-1} = \frac{1}{\cancel{det(T)}} (cof(T))^T$

Swap $p \to q$: $= T_{ip} T_{jq} + T_{ip} T_{jq})$

$= 2 \, T_{ip} T_{jq}$

$\frac{1}{2} T_{ip} T_{jq} [Tr(T^2) - Tr(T)^2]$

$= \det(T) T$

c) $\det(T) = \frac{1}{3} T : \mathrm{cof}(T)$

we know $T^{-1} = \frac{1}{\det T} (\mathrm{cof}\, T)^T$

$I (\det T) = (T \,\mathrm{cof}\, T)^T$

$tr(T \,\mathrm{cof}\, T)^T = 3 \det T$ $\quad [\because tr(\alpha A) = \alpha\, tr(A)$

$\downarrow$

and since $tr(A^T B) = A : B$

$\cancel{tr(\mathrm{cof}\, T^T)} = tr(T \,\mathrm{cof}\, T^T) = T : \mathrm{cof}\, T$

$\therefore 3 \det = T : \mathrm{cof}\, T$

$\det = \frac{1}{3} (T : \mathrm{cof}\, T)$

(5) To prove that

$$\det(R+S) = \det(R) + \det(S) + \mathrm{cof}(R):S + \mathrm{cof}(S):R$$

lets expand $\det(R+S)$

$$= \det(R+S) = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}(R+S)_{ip}(R+S)_{jq}(R+S)_{kr}$$

~~or~~

$$= \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}(R_{ip}+S_{ip})(R_{jq}+S_{jq})(R_{kr}+S_{kr})$$

$$\frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\left[R_{ip}R_{jq} + R_{ip}S_{jq} + S_{ip}R_{jq} + S_{ip}S_{jq}(R_{kr}+S_{kr})\right]$$

$$\frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\left[R_{ip}R_{jq}R_{kr} + R_{ip}R_{jq}S_{kr} + R_{ip}S_{jq}R_{kr} + R_{ip}S_{jq}S_{kr} + S_{ip}R_{jq}R_{kr} + S_{ip}R_{jq}S_{kr} + S_{ip}S_{jq}R_{kr} + S_{ip}S_{jq}S_{kr}\right]$$

$$\frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}[R_{ip}R_{jq}R_{kr}] + \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}S_{ip}S_{jq}S_{kr}$$

+ remaining terms

$$= \det(R) + \det(S) + \text{remaining terms} \quad — (4)$$

only thing to prove is that remaining terms $= \mathrm{cof}(R):S + \mathrm{cof}(S):R$

remaining terms

~~$\frac{1}{6}\varepsilon_{ij}\,[R$~~ $\frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}[R_{ip}R_{jq}S_{kr}] + \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}[R_{ip}R_{kr}S_{jq})$

$$+ \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}R_{ip}S_{jq}S_{kr} + \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}(S_{ip}R_{jq}R_{kr})$$

$$+ \frac{1}{6}S_{ip}R_{jq}S_{kr} + \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}S_{ip}S_{jq}R_{kr}$$

$$3\tfrac{1}{\cancel{6}}\left[\mathrm{cof}(R):S + \mathrm{cof}(R):S + \mathrm{cof}(S):R) + \mathrm{cof}(R):S + \mathrm{cof}(S):R + \mathrm{cof}(S):R\right.$$

$= \frac{1}{3}\left[3\,\mathrm{cof}(R):S + 3\,\mathrm{cof}(S):R\right]$

remaining term = $\mathrm{cof}(R):S + \mathrm{cof}(S):R$ — (5)

combining (4) and (5)

$$\det(R+S) = \underline{\det(R) + \det(S)} + \det(R):S + \det(S):R$$

(4) (a) we know skew symmetric matrix

$= W = \underset{\wedge}{a} \otimes \underset{\wedge}{b} - \underset{\wedge}{b} \otimes \underset{\wedge}{a}$ when a, b are vectors

or $W_{ij} = a_i b_j - b_j a_i$ — (1)

axial vector $w_k = -\frac{1}{2}\varepsilon_{ijk} W_{ij}$ — (2)

Putting (1) in (2)

$w_k = -\frac{1}{2}\varepsilon_{ijk}(a_i b_j - b_j a_i)$

$= w_k = -\frac{1}{2}\varepsilon_{ijk} a_i b_j + \frac{1}{2}\varepsilon_{ijk} b_j a_i$

$w_k = -\frac{1}{2}\varepsilon_{jik} \underset{\uparrow}{b_j a_i} + \frac{1}{2}\varepsilon_{ijk} b_j a_i$

swapping

$w_k = \varepsilon_{ijk} b_j a_i$

$\underline{\underline{w_k = a \times b}}$

(b) $(a \otimes b)^T = b \otimes a$ To prove

LHS	RHS
$LHS = (a\otimes b)_{ij} = a_i b_j$	$(b \otimes a)_{ij} = b_i a_j$
since second order tensor	
$(a\otimes b)^T_{ij} = a_j b_i$	$= a_j b_i$

$\underline{LHS = RHS}$

(c) $(a\otimes b)(c\otimes d) = a(b\cdot c)\otimes d$ (a)

$LHS = [(a\otimes b)(c\otimes d)]_{ij}$ ← since second order tensor

$(a\otimes b)_{ik}(c\otimes d)_{kj}$ ← matrix multiplication

$(a_i b_k)(c_k d_j)$

$a_i \underbrace{(b_k c_k)}_{\text{Dot product}} d_j = a_i (b\cdot c) d_j$

now $(b\cdot c)(a_i d_j)$

$= (b\cdot c)(a\otimes d) = RHS$

(d) $(a\otimes b) : (c\otimes d) = (b\cdot c)(a\cdot d)$

$LHS = (a\otimes b)_{ij} : (c\otimes d)_{ij}$

$(a\otimes b)_{ij}(c\otimes d)_{ij} = a_i b_j c_i d_j$

$= (a_i c_i)(b_j d_j) = (a\cdot c)(b\cdot d)$

Doubt

(e) $T(a\otimes b) = (Ta)\otimes b$

$LHS = T(a\otimes b)_{ij}$

$= T_{ik}(a\otimes b)_{kj} = T_{ik} a_k b_j = (Ta)_i b_j = \cancel{T a\otimes b}\ (Ta)\otimes b$

(d) $(a\otimes b)T = a\otimes(T^T b)$

$LHS: (a\otimes b)_{ij} T = (a\otimes b)_{ik} T_{kj} = a_i \underbrace{b_k T_{kj}}_{\text{Dot product}}$

$= a_i\,\cancel{(b\,T\cdot b)} \quad a_i \cancel{(Tb)}_j = a\otimes\cancel{(Tb)}$

$a_i\, T^T_{jk} b_k$

$a_i (Tb)_j$

$= a \otimes (Tb)$

(5) (a) $G(T) = \mathrm{cof}(T)$

To prove $[DG(T)[U]]_{ij} =$

$$[D(\mathrm{cof})(T)[U]]_{ij} = \varepsilon_{imn}\varepsilon_{jpq}(T_{mp}U_{nq} + U_{mp}T_{nq})$$

swapping.

$$= \varepsilon_{imn}\varepsilon_{jpq}(T_{mp}U_{nq})$$

(b) $\dfrac{\partial G_{ij}}{\partial T_{kj}} = \dfrac{\partial(\mathrm{cof}\,T_{ij})}{\partial T_{kj}} = \cancel{(1/2)}\ \varepsilon_{ikm}\varepsilon_{jln}\cancel{T}$

differentiation rule $\Rightarrow$ $\dfrac{\partial T_{an}}{\partial T_{kl}} = \delta_{ak}\delta_{nl}$

$$\frac{\partial(\mathrm{cof}\,T)_{ij}}{\partial T_{kj}} = \cancel{\frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}\delta_{jm}\delta_{kn}}$$

$$= \frac{\frac{1}{2}\varepsilon_{abc}\varepsilon_{lmn}T_{jm}T_{kn}}{\partial T_{kj}}$$

$$\frac{1}{2}\varepsilon_{abc}\varepsilon_{lmn}\left(\underline{T_{kn}\delta_{ak}\delta_{bj}} + \underline{T_{jm}\delta_{ck}\delta_{nj}}\right)$$

$$\frac{1}{2}\cancel{\varepsilon_{cjc}\varepsilon_{lmn}}^{\to 0}T_{kn}\ \ \&\ \ \cancel{\frac{1}{2}}$$

$$\cancel{\frac{1}{2}\varepsilon_{abc}\varepsilon_{ln}}\ \ \frac{1}{2}\varepsilon_{kjc}T_{kn} + \frac{1}{2}T_{jm}\varepsilon_{abk}\varepsilon_{lmj}$$

$$\frac{1}{2}\varepsilon_{lkj}\varepsilon_{lmn}T_{kn} + \frac{1}{2}T_{jm}\varepsilon_{abk}\varepsilon_{lmj}$$

$$\frac{1}{2}\varepsilon_{ckj}\varepsilon_{lmn}T_{kn} + \frac{1}{2}\varepsilon_{bka}\varepsilon_{lmj}T_{jm}$$

Changing indices and cyclic order $= \dfrac{1}{2}\varepsilon_{ikn}\varepsilon_{jlq}T_{nq} + \dfrac{1}{2}\varepsilon_{ikn}\varepsilon_{jlq}T_{nq} = \dfrac{1}{2}\varepsilon_{ikn}\varepsilon_{jlq}T_{nq}$ //

(c) if C is symmetric $\left[\frac{\partial \mathrm{cof}(C)}{\partial C}\right]_{ijkl}$.

we know that

$$\frac{\partial(\mathrm{cof})_{ij}}{\partial T_{kl}} = \varepsilon_{ikm}\varepsilon_{jln}T_{mn}$$

since C is symmetric, there will be 2 valid representation

$$\frac{\partial(\mathrm{cof})_{ij}}{\partial \cancel{C}_{kl}} = \frac{1}{2}\left(\frac{\partial(\mathrm{cof})_{ij}}{\partial C_{kl}} + \frac{\partial(\mathrm{cof})_{ij}}{\partial C_{lk}}\right)$$

$$= \frac{1}{2}\left[\varepsilon_{ikm}\varepsilon_{jln}T_{mn} + \varepsilon_{ilm}\varepsilon_{jkn}T_{mn}\right]$$

since C is symmetric $C_{mn} = \frac{1}{2}(C_{mn}+C_{nm})$

$$= \frac{1}{2}\times\frac{1}{2}\left[\varepsilon_{ikm}\varepsilon_{jlm}(C_{mn}+C_{nm}) + \varepsilon_{ilm}\varepsilon_{jkn}(C_{mn}+C_{nm})\right]$$

$$\frac{1}{4}\left[\varepsilon_{ikm}\varepsilon_{jln}C_{mn} + \varepsilon_{ikm}\varepsilon_{jln}C_{nm} + \varepsilon_{ilm}\varepsilon_{jkn}C_{mn} + \varepsilon_{ilm}\varepsilon_{jkn}C_{nm}\right]$$

or $\frac{1}{4}$ $m \to r$, $n \to s$

$$\frac{1}{4}\left[\varepsilon_{ikr}\varepsilon_{jls} + \varepsilon_{ikr}\varepsilon_{jls} + \varepsilon_{jlr}\varepsilon_{jks} + \varepsilon_{ilr}\varepsilon_{jks}\right]C_{rs}$$

(6) (i) $\nabla\cdot[(\nabla u)u]_i$

$$= \nabla\cdot\left(\varepsilon_{ijk}\frac{u_i}{\partial x_j}\right)u\right) = \nabla\cdot\left(\frac{u_i}{\partial x_j}u_j\right)$$

$$= u_j\nabla\left(\frac{u_i}{\partial x_j}\right) + \frac{u_i}{\partial x_j}\nabla(u_j) = u_j\frac{\partial}{\partial x_i}\frac{u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\ \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

$$u_j\left[\frac{\partial}{\partial x_j}(\nabla\cdot u)\right] + \frac{\partial u_j}{\partial x_j}\frac{\partial u_j}{\partial x_j}$$

$(u\cdot\nabla)(\nabla\cdot u) + \nabla u : (\nabla u)^T$

(ii) $\nabla u : (\nabla u)^T$

$= \cancel{\nabla} \left(\frac{\partial u_i}{\partial x_j}\right) : \cancel{\nabla}\left(\frac{\partial u_j}{\partial x_i}\right) \Rightarrow \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$

RHS $= \quad \nabla\cdot(\nabla u)u - \nabla\cdot(\nabla\cdot u)u + \nabla\cdot u^2$

$\nabla u : (\nabla u)^T + u\cdot(\nabla(\nabla\cdot u)) - (u\cdot\nabla)\cdot(\nabla\cdot u) + \nabla\cdot u^2$

$\nabla u : (\nabla u)^T + u\cdot(\nabla(\nabla\cdot u)) - (u\cdot\nabla)(\nabla\cdot u) \mp (\nabla\cdot u)^2 + \nabla\cdot u^2$

$= \nabla u : (\nabla u)^T - (\nabla\cdot u)^2$

$\nabla u : (\nabla u)^T = \nabla\cdot[(\nabla u)u - (\nabla\cdot u)u] + (\nabla\cdot u)^2$

(iii) $\nabla(\phi v)$

$= v\cancel{(\nabla\phi)} = \frac{\partial(\phi v)_i}{\partial x_j} = v_i\frac{\partial\phi}{\partial x_j} + \phi\frac{\partial v_i}{\partial x_i}$

$v\otimes\nabla\phi + \phi\nabla v$

(iv) $\nabla((u\cdot v)w) = (u\cdot v)\nabla w + w\otimes[(\nabla v)^T u] + w\otimes[(\nabla u)^T v]$

$w\frac{\partial(u\cdot v)_i}{\partial x_j} + (u\cdot v)\frac{\partial w_i}{\partial x_j} = w_i\frac{\partial(u_k v_k)}{\partial x_j} + u_k v_k\frac{\partial w_i}{\partial x_j}$

$w_i v_k\frac{\partial u_k}{\partial x_j} + w_i u_k\frac{\partial v_k}{\partial x_j} + u_k v_k\frac{\partial w_i}{\partial x_j}$

$(v\cdot u)\nabla w + w\otimes(\nabla v)^T u + [(u\cdot v)(\nabla w)]$

(v) $\nabla\cdot(u\otimes v)_{ij}$

$\nabla\cdot(u_i v_j) = u_i\frac{\partial v_j}{\partial x_j} + v_j\frac{\partial u_i}{\partial x_j} =$

$\nabla u\cdot v + v(\nabla u)$

(vii) $\nabla \cdot (T^T v)$

$= \nabla \cdot (T_{ji} v)$

$= v_i \frac{\partial T_{ji}}{\partial x_i} + T_{ji} \frac{\partial v_i}{\partial x_i}$ $\quad v_j \frac{\partial T_{ji}}{\partial x_j} + T_{ji} \frac{\partial v_j}{\partial x_j}$

$\nabla \cdot T$

$v \cdot (\nabla \cdot T)$ $\quad v \cdot (\nabla \cdot T) + T : \nabla v$

(viii) $\nabla(\phi T) = \phi \nabla T_{ijk} + T_{ij} \nabla \phi$

$\phi \frac{\partial T_{ij}}{\partial x_k} + T_{ij} \frac{\partial \phi}{\partial x_k}$

$\phi \nabla T + T_{ij} \otimes \nabla \phi$

(ix) $\nabla(\phi(Tv))$

$= \phi \frac{\partial (Tv)_{ij}}{\partial x_k} + Tv \frac{\partial \phi}{\partial x_k}$

$\phi \frac{\partial T_{ij} v_j}{\partial x_k} + (Tv)_{ij} \frac{\partial \phi}{\partial x_k}$

$\phi \nabla (Tv) + Tv \otimes \nabla \phi$

(x) $\nabla^2 (u \cdot v) = \frac{\partial \partial (u_i v_i)}{\partial x_j \partial_j} = \frac{\partial}{\partial x_j}\left[v_i \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial v_i}{\partial x_j} \right]$

$\underbrace{\frac{\partial v_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} + \frac{\partial^2 u_i}{\partial x_j^2}(v_i) + \underbrace{\frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}} + u_i \frac{\partial^2 v_i}{\partial x_j}$

$(v \cdot \nabla^2 u) + (u \cdot \nabla^2 v) + 2 \nabla u : \nabla v$

(7) we know

$$\omega_k = -\frac{1}{2}\varepsilon_{ijk} W_{ij}$$

and $W_{ij} = -\varepsilon_{ijk}\omega_k$ — (1)

$$\nabla \times W = \varepsilon_{jkl}\frac{\partial W_{il}}{\partial x_k} \quad (2)$$

: ~~(1) in (2)~~

(1) in (2)

$$\cancel{\varepsilon_{jkl}\frac{\partial(\varepsilon_{ilk}\omega_k)}{\partial x_k}} - \varepsilon_{jkl}\varepsilon_{ilm}\frac{\partial \omega_k}{\partial x_m}$$

$$= \left(-\delta_{ji}\delta_{km} + \delta_{jm}\delta_{ki}\right)\frac{\partial \omega_k}{\partial x_m}$$

$$-\delta_{ji}\frac{\partial \omega_k}{\partial x_k} + \frac{\partial \omega_i}{\partial x_j}$$

$$\downarrow$$

$$-(\nabla\cdot\omega)I + \nabla\omega$$

(Doubt) minus sign

(8) R~~HS term~~

LHS term

second term in LHS

$$\int_S u \otimes (T^T)$$

$$(u \otimes (\nabla\cdot T^T))_{ij}$$

$$(\nabla u)_{ik} T_{kj}$$

$$= \cancel{u_i \otimes T_{ji}}$$

$$= u_i(\nabla\cdot T^T)_j$$

$$\left(\frac{\partial u_i}{\partial x_k}\right) T_{kj}$$

$$\left(u_i \frac{\partial T_{kj}}{\partial x_j}\right)$$

Ls + add

$$= \int_V \left(u_i\frac{\partial T_{kj}}{\partial x_j}\right) + \left(\frac{\partial u_i}{\partial x_k} T_{kj}\right) dv$$

we know

$$\int_V \nabla F ds = \int_S F n ds$$

Product rule $\int_V \nabla(u_i T_{kj}) \longrightarrow \int_S n u_i T_{kj}\, ds$

$$\longrightarrow \int_S u \otimes (T^T n)\, ds.$$