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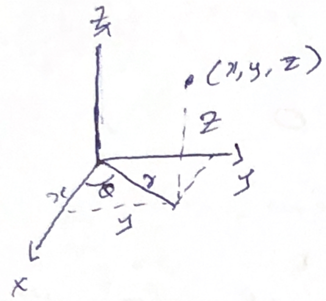
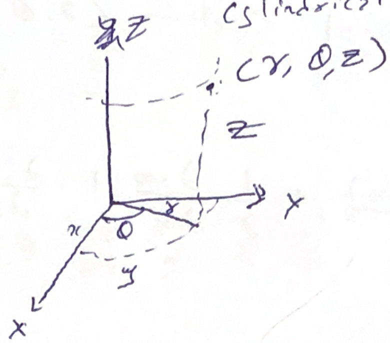
AM5530 - Assignment 1
Advanced Fluid Dynamics

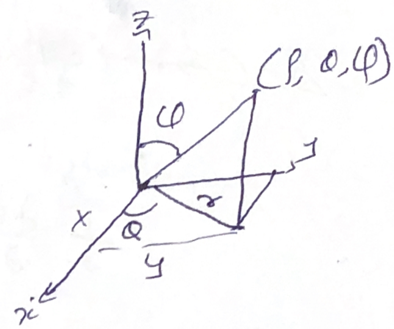
Rectangular



~~Sh Spherical~~ Cylindrical



Spherical



$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$Z = z$

$\cos\theta = \frac{x}{r} = x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

1. Cartesian Coordinates

i) Vector form

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = -\nabla p + \mu(\nabla^2 u) + \rho g$$

$$\& \quad \nabla\cdot u = 0$$

ii) Tensor form

$$\frac{\partial u_i}{\partial t} = 0$$

$$\rho\left[\left(\frac{\partial u_i}{\partial t}\right) + u_j\frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \rho g_i$$

iii) Cartesian form

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z$$

2. ~~Carte~~ Cylindrical System (r, θ, z) (u_r, u_θ, u_z)

Continuity:

$$\frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

eqn:

r-cmp $$\rho\left[\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right] = -\frac{\partial P}{\partial r} + \mu\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] + \rho g_r$$

θ-cmp: $$\rho\left[\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right] = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] + \rho g_\theta$$

z-cmp $$\rho\left[\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right] = -\frac{\partial P}{\partial z} + \mu\left[\nabla^2 u_z\right] + \rho g_z$$

where

$$\nabla^2 u_r = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r}$$

$$\nabla^2 u_\theta = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r}$$

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}$$

3. Spherical

$(r, \theta, \phi) \quad u = (u_r, u_\theta, u_\phi)$

continuity

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\theta^2 + u_\phi^2}{r}\right)$$

$$= -\frac{\partial p}{\partial r} + \mu(\nabla^2 u)_r + \rho g_r$$

$$\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{u_r u_\theta - u_\phi^2\cot\theta}{r}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu(\nabla^2 u)_\theta + \rho g_\theta$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r u_\phi + u_\theta u_\phi\cot\theta}{r} =$$

$$-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \mu(\nabla^2 u)_\phi + \rho g_\phi$$

where

$$(\nabla^2 u)_r = \frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

$$- \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}u_\theta - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$(\nabla^2 u)_\theta = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2} - \frac{u_\theta}{r^2\sin^2\theta}$$

$$+ \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$(\nabla^2 u)_\phi = \frac{\partial^2 u_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial u_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u_\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}$$

$$- \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}$$