Math 5080
Homework 2

Magon Bowling

Due: Saturday, February 6th at 11:59 PM

6.17 Suppose that X_1 and X_2 are Gamma$(2, 1/2)$ random variables.

(a) Find the pdf of $Y = \sqrt{X_1 + X_2}$

To find the distribution of Y, we use the Moment Generating Function. Recall the MGF of a Gamma distribution is $M(t) = (1 - \theta t)^{-k}$ for $t < \frac{1}{\theta}$. If we let $Z = X_1 + X_2$, we get $Z \sim Gamma(2, 1)$ for $Z \in (0, \infty)$ as shown.

$$\begin{aligned} M_Z(t) &= M_{X_1}(t) M_{X_2}(t) \\ &= \left(\frac{2}{2-t}\right)^{\frac{1}{2}} \left(\frac{2}{2-t}\right)^{\frac{1}{2}} \\ &= \left(\frac{2}{2-t}\right) \end{aligned}$$

Now that we know the distribution, we can do a one-dimensional transformation to obtain the pdf of Y. This is done with $\phi(x) = \sqrt{x}$, which is one-to-one and is increasing on $(0, \infty)$, so it's inverse is $\phi^{-1}(x) = x^2$. Now,

$$f_{\phi(x)}(t) = f_X\big(\phi^{-1}(t)\big)\Big|(\phi^{-1)\prime}(t)\Big| \Rightarrow f_{\sqrt{X_1+X_2}}(t) = f_{X_1+X_2}(t^2)|2t| = \frac{e^{\frac{-t^2}{2}}}{2}(2t).$$

Therefore,

$$f_Y(y) = y \exp\big(-y^2/2\big) \text{ for } y > 0 \text{ and } 0 \text{ otherwise.}$$

(b) Find the pdf of $W = X_1/X_2$

I will let $V = X_1 + X_2$ and use a transformation of a joint pdf to solve for $f_W(w)$.

1. Rewrite relations in little variables: $w = \frac{x_1}{x_2}$ and $v = x_1 + x_2$
2. Solve for x_1, x_2 in terms of v, w:

$$w = \frac{x_1}{x_2} \Rightarrow x_1 = wx_2 \Rightarrow x_1 = \frac{vw}{1+w}$$

$$v = x_1 + x_2 \Rightarrow x_2 = v - x_1 \Rightarrow x_2 = v - wx_2 \Rightarrow x_2 = \frac{v}{1+w}$$

3. Compute the Jacobian:

$$J = \det\begin{pmatrix} \frac{\partial x_1}{\partial v} & \frac{\partial x_1}{\partial} \\ \frac{\partial x_2}{\partial v} & \frac{\partial x_2}{\partial w} \end{pmatrix} = \det\begin{bmatrix} \frac{w}{1+w} & \frac{v(1+w)-vw}{(1+w)^2} \\ \frac{1}{1+w} & \frac{-v}{(1+w)^2} \end{bmatrix} = \frac{-vw}{(1+w)^3} - \frac{v(1+w)-vw}{(1+w)^3} = \frac{v}{(1+w)^2}$$

4. Then the formula for $f_{(V,W)}$ in terms of $f_{(X_1,X_2)}$ is $f_{(V,W)}(v,w) = f_{(X_1,X_2)}(x_1,x_2)|J|$. Recall the following pdf's:

$$f_{X_1}(x_1) = \frac{e^{\frac{-x_1}{2}}}{\sqrt{2\pi x_1}} \text{ and } f_{X_2}(x_2) = \frac{e^{\frac{-x_2}{2}}}{\sqrt{2\pi x_2}} \Rightarrow f_{X_1,X_2}(x_1,x_2) = \frac{e^{\frac{-(x_1+x_2)}{2}}}{2\pi\sqrt{x_1 x_2}}$$

Now through substitution we have the following:

$$\begin{aligned} f_{(V,W)}(v,w) &= \frac{\exp\left[-\left(\frac{vw}{1+w}+\frac{v}{1+w}\right)/2\right]}{2\pi\sqrt{\frac{vw}{1+w}\cdot\frac{v}{1+w}}}\left|\frac{v}{(1+w)^2}\right| \\ &= \frac{\exp\left[-\left(\frac{v(1+w)}{1+w}\right)/2\right]}{2\pi\sqrt{\frac{v^2}{(1+w)^2}}\cdot\sqrt{w}}\left|\frac{v}{(1+w)^2}\right| \\ &= \frac{e^{\frac{-v}{2}}}{2\pi\frac{v}{1+w}\sqrt{w}}\left|\frac{v}{(1+w)^2}\right| \\ &= \frac{e^{\frac{-v}{2}}}{2\pi(1+w)\sqrt{w}}\ \mathbf{1}\{v>0, w>0\} \end{aligned}$$

5. The final step in solving for $f_W(w)$ is to integrate out v from the joint pdf as follows:

$$\begin{aligned} f_W(w) &= \int_0^\infty \frac{e^{\frac{-v}{2}}}{2\pi(1+w)\sqrt{w}}dv \\ &= \frac{-1}{\pi(1+w)\sqrt{w}}\int_0^\infty \frac{-1}{2}e^{\frac{-v}{2}}dv \\ &= \frac{-1}{\pi(1+w)\sqrt{w}}\left(e^{\frac{-v}{2}}\Big|_0^\infty\right) \\ &= \frac{-1}{\pi(1+w)\sqrt{w}}(0-1) \\ &= \frac{1}{\pi(1+w)\sqrt{w}}\ \mathbf{1}\{w>0\} \end{aligned}$$

6.26 Let X_1 and X_2 be independent negative binomial random variables $X_1 \sim NB(r_1,p)$ and $X_2 \sim NB(r_2,p)$.

(a) Find the MGF of $Y = X_1 + X_2$

The MGF for $X_1 \sim NB(r_1,p)$ is:

$$M_{X_1}(t) = \begin{cases} \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_1} & \text{for } t < -\log(1-p) \\ \infty & \text{otherwise.} \end{cases}$$

The MGF for $X_2 \sim NB(r_2, p)$ is:

$$M_{X_2}(t) = \begin{cases} \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_2} & \text{for } t < -\log(1-p) \\ \infty & \text{otherwise.} \end{cases}$$

Therefore, the MGF for $Y = X_1 + X_2$ is obtained by

$$\begin{aligned} M_Y(t) &= M_{X_1}(t) M_{X_2}(t) \\ &= \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_1} \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_2} \\ &= \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_1+r_2} \end{aligned}$$

The MGF of $Y = X_1 + X_2$ is:

$$M_Y(t) = \begin{cases} \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_1+r_2} & \text{for } t < -\log(1-p) \\ \infty & \text{otherwise.} \end{cases}$$

(b) What is the distribution of Y?

Thus we see that $Y \sim NB(r_1 + r_2, p)$.