Time Series Analysis
Homework 2

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Due: Friday, February 5th

1 Problem 5

Let $X_1, X_2, ..., X_N$ be independent and identically distributed (iid) random vectors with $EX_i = 0$ and $EX_i^2 = 1$. Compute

$$E\left(\sum_{i=1}^{n} \frac{1}{i} X_i\right)^2$$

and show

$$\left|\sum_{i=1}^{n} \frac{1}{i} X_i\right| = Op(1).$$

We know $X_i \sim \text{iid}(0,1)$, let's let

$$Y_n = \sum_{i=1}^{n} \frac{1}{i} X_i = (1 \cdot X_1 + \frac{1}{2} \cdot X_2 + \frac{1}{3} \cdot X_3 + ... + \frac{1}{n} \cdot X_n).$$

We need to solve for $E[Y_n^2]$. We compute by the following:

$$Var(Y_n) = E(Y_n^2) - E(Y_n)^2 \Rightarrow E(Y_n^2) = E(Y_n)^2 + Var(Y_n)$$

$$E(Y_n) = E\left[\sum_{i=1}^{n} \frac{1}{i} X_i\right] \overset{\text{iid}}{=} \sum_{i=1}^{n} \frac{1}{i} E(X_i) = 0$$

We know that $E(X_i) = 0$, therefore, $\sum_{i=1}^{n} \frac{1}{i} E(X_i) = 0$. This leads to the understanding that $E(Y_n)^2 = 0$. Now we solve for $\text{Var}(Y_n)$ as follows:

$$E(Y_n^2) = 0 + \text{Var}(Y_n) = \text{Var}\left[\sum_{i=1}^{n} \frac{1}{i} X_i\right] \overset{\text{iid}}{=} \sum_{i=1}^{n} \frac{1}{i^2} \text{Var}(X_i) = \sum_{i=1}^{n} \frac{1}{i^2}$$

We know that $\text{Var}(V_i) = 1$, therefore, $\sum_{i=1}^{n} \frac{1}{i^2}$ is a harmonic series.
Now we show that Y_n is bounded in probability using the Chebishev's Inequality. Note: Used Wolfram Alpha calculation to find what the harmonic series converges to.

$$P\{|Y_n| > c\} \leq \frac{1}{c^2} E(Y_n^2)$$
$$P\{|Y_n| > c\} \leq \frac{1}{c^2}\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)$$

Now as $n \to \infty$, the series converges to $\frac{1}{c^2}\left(\frac{\pi^2}{6}\right)$, which means that Y_n is bounded by $\frac{\pi^2}{6c^2}$.

2 Problem 7

Let $X_1, X_2, ..., X_n$ be a sequence of random variables with $EX_i = 0$,

$$EX_iX_j = \begin{cases} \sigma^2, & i = j, \\ \rho, & |i-j| = 1, \\ 0, & |i-j| > 1. \end{cases}$$

Compute

$$E\left(\sum_{i=1}^{n} X_i\right)^2$$

and show that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \overset{P}{\to} 0.$$

If we let $Y_n = \sum_{i=1}^{n} X_i = (X_1 + X_2 + ... + X_n)$, we want to solve for $E(Y_n^2)$ again. Recall from above $\text{Var}(Y_n) = E(Y_n^2) - E(Y_n)^2$ and $E(Y_n^2) = E(Y_n)^2 + Var(Y_n)$. Using the same process,

$$E(Y_n) = E\left[\sum_{i=1}^{n} X_i\right] \overset{\text{iid}}{=} \sum_{i=1}^{n} E(X_i) = 0$$

Therefore, $E(Y_n)^2 = 0$.

$$E(Y_n^2) = 0 + \text{Var}(Y_n) = E\left[\sum_{i=1}^{n}\sum_{i=1}^{n} X_iX_j\right] \overset{\text{iid}}{=} \sum_{i=1}^{n}\sum_{i=1}^{n} E(X_iX_j)$$

We arrive at the following: $E(Y_n^2) = n\sigma^2 + 2(n-1)p$. It is important to note that if Y_n is a random sample from a distribution with a finite mean and variance, then the sequence of sample mean converges in probability to μ, as follows $\overline{Y}_n \overset{P}{\to} \mu$.

$$\text{We need to show: } \frac{1}{n}\sum_{i=1}^{n} X_i \overset{P}{\to} 0, \quad \text{hence we start with this: } \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{Y}_n.$$

Given finite mean 0 and variance (σ^2), we are going to use (1) stochastic convergence to a constant (c) theorem and (2) Chebishev's inequality.

(1) By definition, if $\lim_{n\to\infty} P\big[|Y_n - c| > \epsilon\big] = 0$ for all $\epsilon > 0$ the sequence has convergence in probability to c.

(2) Likewise by definition, $P\big(|\overline{Y}_n - \mu| \geq \epsilon\big) \leq \frac{\sigma^2}{\epsilon^2}$. Thus,

$$P\big[|Y_n - \mu| \geq \epsilon\big] \geq 1 - \frac{\sigma^2}{\epsilon^2} \text{ and } P\big[|\overline{Y}_n - \mu| \geq \epsilon\big] \geq 1 - \frac{\sigma^2}{\epsilon^2 n}, \text{ and } \overline{Y}_n \overset{P}{\to} \mu \text{ as } n \to \infty.$$

As $\mu = 0$ and $\text{Var}(\overline{Y_n}) = \left(\frac{1}{n^2}\right)(n\sigma^2 + 2(n-1)p)$,

$$P\big[|\overline{Y}_n| \geq \epsilon\big] \geq 1 - \frac{n\sigma^2 + 2(n-1)p}{\epsilon^2 n^2}.$$

As $\frac{\text{Var}(\overline{Y_n})}{n\epsilon^2} \to 0$ as $n \to \infty$, $\overline{Y}_n \overset{P}{\to} \mu$.