Statistical Inference I
Homework 5

Magon Bowling

Due: Saturday, February 27th

8.9 Supposes that $X \sim \chi^2(m)$, $S = X + Y \sim \chi^2(m+n)$, and X and Y are independent. Is $S - X \sim \chi^2(n)$?

We know $S = X + Y \sim \chi^2(m+n)$ and because X and Y are independent we have

$$M_S(t) = M_{X+Y}(t) = M_X(t)M_Y(t).$$

We can further say that because $X \sim \chi^2(m)$ and $X + Y \sim \chi^2(m+n)$ that

$$M_S(t) = (1-2t)^{\frac{-(m+n)}{2}} = (1-2t)^{\frac{-m}{2}}(1-2t)^{\frac{-n}{2}} = M_X(t)M_Y(t) = M_{X+Y}(t) = M_S(t).$$

We have shown that $Y \sim \chi^2(n)$ and we know

$$M_Y(t) = M_{X+Y-X}(t) = M_{S-X}(t),$$

therefore

$$S - X \sim \chi^2(n).$$

8.14 If $T \sim t(\nu)$, give the distribution of T^2.

We know that for $T \sim t(\nu)$ we have

$$T = \frac{Z}{\sqrt{V/\nu}}$$

where $Z \sim N(0,1)$ and $V \sim \chi^2(\nu)$ is independent of Z. Now if we square both sides we get

$$T^2 = \left(\frac{Z}{\sqrt{V/\nu}}\right)^2 = \frac{Z^2}{V/\nu}.$$

We also know that $Z^2 \sim \chi^2(1)$, thus we have

$$\frac{Z^2}{V/\nu} = \frac{\chi^2(1)}{V/\nu} = \frac{\chi^2(1)/1}{V/\nu}.$$

By definition of F distribution, $T^2 \sim F(1,\nu)$.

8.19 If $T \sim t(1)$ then show the following:

(a) The CDF of T is $F(t) = 1/2 + 1/\pi \arctan(t)$.

(b) The $100 \times \gamma$th percentile is $t_\gamma(1) = \tan[\pi(\gamma - 1/2)]$.

(a) To find the CDF of T, we integrate the pdf as follows:

$$\begin{aligned}
f_T(t;\nu) &= \int_{-\infty}^{t} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{t^2}{\nu}\right)^{-(v+1)/2} dt \\
f_T(t;1) &= \int_{-\infty}^{t} \frac{\Gamma\left(\frac{2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{\sqrt{\pi}} \left(1 + \frac{t^2}{1}\right)^{-2/2} dt \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \left(1 + t^2\right)^{-1} dt \\
&= \int_{-\infty}^{t} \left(\frac{1}{\sqrt{\pi}}\right)^2 \left(\frac{1}{1+t^2}\right) dt \\
&= \frac{1}{\pi} \int_{-\infty}^{t} \frac{1}{1+t^2} dt \\
&= \frac{1}{\pi} \left(\tan^{-1}\Big|_{-\infty}^{t}\right) \\
&= \frac{1}{\pi} \left(\tan^{-1}(t) - \tan^{-1}(-\infty)\right) \\
&= \frac{1}{\pi} \left(\tan^{-1}(t)\right) - \frac{1}{\pi}\left(-\frac{\pi}{2}\right) \\
&= \frac{1}{2} + \frac{1}{\pi}\arctan(t)
\end{aligned}$$

(b) Now that we have the CDF of T, we can use that to find the $100 \times \gamma$th percentile to be $t_\gamma(1) = \tan[\pi(\gamma - 1/2)]$. An arbitrary $100 \times \gamma$th percentile is defined by

$$\mathbb{P}\left(T \le t_\gamma(1)\right) = \gamma \implies F_T\left(t_\gamma(1)\right) = \gamma.$$

We can now substitute in the CDF of T and solve for the arbitrary $t_\gamma(1)$ as follows:

$$\begin{aligned}
\frac{1}{2} + \frac{1}{\pi}\arctan\left(t_\gamma(1)\right) &= \gamma \\
\frac{1}{\pi}\arctan\left(t_\gamma(1)\right) &= \gamma - \frac{1}{2} \\
\arctan\left(t_\gamma(1)\right) &= \pi\left(\gamma - \frac{1}{2}\right) \\
t_\gamma(1) &= \tan\left[\pi\left(\gamma - \frac{1}{2}\right)\right]
\end{aligned}$$

Thus, the $100 \times \gamma$th percentile to be $t_\gamma(1) = \tan[\pi(\gamma - 1/2)]$.