Statistical Inference I
Homework 3

Magon Bowling

Due: Saturday, February 13th

Meyer, 30.11 A device has two components, and the lifetimes of these components are modeled by the random variables Y_1 and Y_2. The first component cannot fail before the second component fails, and the joint distribution of Y_1 and Y_2 is determined to be

$$f(y_1, y_2) = 2e^{-y_1}e^{-y_2}, \mathbf{1}\{0 < y_2 < y_1 < \infty\}.$$

The random variable $X_1 = Y_1 - Y_2$ can be interpreted as the time between failures, and $X_2 = Y_1 + Y_2$ can be interpreted as the lifetime of the device.

(a) Find the joint density of X_1 and X_2.

(b) Find the marginal density of X_1 and sketch it.

(c) Find the marginal density of X_2.

(a) To find the joint density of X_1 and X_2 we complete the following steps:
First write relations in little variables.

$$X_1 = Y_1 - Y_2 \Rightarrow x_1 = y_1 - y_2$$

$$X_2 = Y_1 + Y_2 \Rightarrow x_2 = y_1 + y_2$$

Second solve for y_1, y_2 in terms of x_1, x_2.

$$x_1 = y_1 - y_2 \ (+) \ x_2 = y_1 + y_2 \Rightarrow x_1 + x_2 = 2y_1 \Rightarrow y_1 = \frac{1}{2}(x_1 + x_2)$$

$$x_1 = y_1 + y_2 \ (-) \ x_2 = y_1 + y_2 \Rightarrow x_1 - x_2 = 2y_2 \Rightarrow y_2 = \frac{1}{2}(x_2 - x_1)$$

Third compute the **Jacobian**.

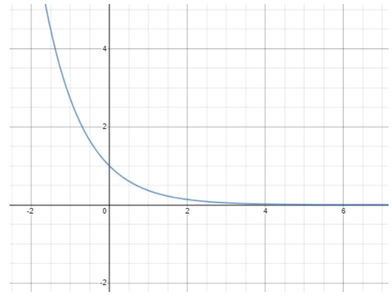
$$\mathbf{J} = \det\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \det\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{-1}{2}\right) = \frac{1}{4} - \left(\frac{-1}{4}\right) = \frac{1}{2}$$

Fourth then find the formula in terms of f_{Y_1,Y_2}.

$$\begin{aligned} f_{(X_1,X_2)}(x_1, x_2) &= f_{(Y_1,Y_2)}(y_1, y_2)|\mathbf{J}| \\ &= \left|\frac{1}{2}\right| 2e^{-\left(\frac{1}{2}(x_1+x_2)\right)}e^{-\left(\frac{1}{2}(x_2-x_1)\right)}, \mathbf{1}\left\{0 < \frac{1}{2}(x_2 - x_1) < \frac{1}{2}(x_1 + x_2) < \infty\right\} \\ &= exp\{-\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_2 + \frac{1}{2}x_1\}, \mathbf{1}\{0 < x_2 - x_1 < x_1 + x_2 < \infty\} \\ &= e^{-x_2}, \mathbf{1}\{0 < x_1 < x_2 < \infty\} \end{aligned}$$

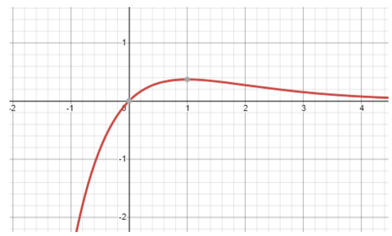
(b) To find the marginal density of X_1 we integrate out x_2:

$$\begin{aligned} f_{X_1}(x_1) &= \int_{x_1}^{\infty} e^{-x_2} dx_2 \\ &= -e^{-x_2}\Big|_{x_1}^{\infty} \\ &= 0 - (-e^{-x_1}) \\ &= e^{-x_1}, x_1 > 0 \end{aligned}$$



(c) To find the marginal density of X_2 we integrate out x_1:

$$\begin{aligned} f_{X_2}(x_2) &= \int_{0}^{x_2} e^{-x_2} dx_1 \\ &= e^{-x_2}x_1\Big|_{0}^{x_2} \\ &= x_2 e^{-x_2} - 0 \\ &= x_2 e^{-x_2}, x_2 > 0 \end{aligned}$$



Meyer, 30.9 The lifetimes Y_1 and Y_2 of two components of a device are jointly distributed as

$$f(y_1, y_2) = \frac{1}{8} y_1 e^{-(y_1+y_2)/2} \mathbf{1}\{y_1 > 0, y_2 > 0\}.$$

(a) Find $\mathbb{P}(Y_1 > 1, Y_2 > 1)$.

(b) Are Y_1 and Y_2 independent random variables? Explain why or why not.

(c) Find the marginal density of Y_1.

(a) To find $\mathbb{P}(Y_1 > 1, Y_2 > 1)$ we integrate on the interval $[1, \infty]$.

$$\begin{aligned} \mathbb{P}(Y_1 > 1, Y_2 > 1) &= \int_1^{\infty}\int_1^{\infty} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 dy_2 \\ &= \int_1^{\infty}\int_1^{\infty} \frac{1}{8} y_1 e^{-y_1/2} e^{-y_2/2} dy_1 dy_2 \\ &= \int_1^{\infty} \frac{1}{4} y_1 e^{-y_1/2} dy_1 \int_1^{\infty} \frac{1}{2} e^{-y_2/2} dy_2 \\ &= \left(\frac{1}{2} y_1 e^{-y_1/2}\Big|_1^{\infty} - e^{-y_1/2}\Big|_1^{\infty}\right)\left(-e^{-y_2/2}\Big|_1^{\infty}\right) \\ &= \left[0 - \left(\frac{-1}{2} e^{\frac{-1}{2}}\right) - \left(0 - \left(e^{\frac{-1}{2}}\right)\right)\right]\left(0 - \left(-e^{\frac{-1}{2}}\right)\right) \\ &= \left(\frac{1}{2} e^{\frac{-1}{2}} + e^{\frac{-1}{2}}\right)\left(e^{\frac{-1}{2}}\right) \\ &= \left(\frac{3}{2} e^{\frac{-1}{2}}\right)\left(e^{\frac{-1}{2}}\right) \\ &= \frac{3}{2e} \end{aligned}$$

(b) To determine if Y_1 and Y_2 are independent, we need the product of the marginal densities to equal the joint density. I will prove this hereafter. We can also prove this by showing that the joint

density can factor into a function of two marginal densities as indicated above in our integration. Therefore, Y_1 and Y_2 are independent.
(c) To find the marginal of Y_1 we integrate out y_2:

$$\begin{aligned}
f_{Y_1}(y_1) &= \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_2 \\
&= \frac{1}{4} y_1 e^{-y_1/2} \int_0^\infty \frac{1}{2} e^{-y_2/2} dy_2 \\
&= \frac{1}{4} y_1 e^{-y_1/2} \left(-e^{-y_2/2} \Big|_0^\infty \right) \\
&= \frac{1}{4} y_1 e^{-y_1/2} (0 - (-1)) \\
&= \frac{1}{4} y_1 e^{-y_1/2}, \mathbf{1}\left\{y_1 > 0\right\}
\end{aligned}$$

To find the marginal of Y_2 we integrate out y_1:

$$\begin{aligned}
f_{Y_2}(y_2) &= \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 \\
&= \frac{1}{2} e^{-y_2/2} \int_0^\infty \frac{1}{4} y_1 e^{-y_1/2} dy_1 \\
&= \frac{1}{2} e^{-y_2/2} \left(\frac{1}{2} y_1 e^{-y_1/2} \Big|_0^\infty - e^{-y_1/2} \Big|_0^\infty \right) \\
&= \frac{1}{2} e^{-y_2/2} (0 - (0 - (1))) \\
&= \frac{1}{2} e^{-y_2/2}, \mathbf{1}\left\{y_2 > 0\right\}
\end{aligned}$$

To prove independence further,

$$\begin{aligned}
f_{Y_1,Y_2}(y_1, y_2) &= f_{Y_1}(y_1) f_{Y_2}(y_2) \\
\frac{1}{8} y_1 e^{-(y_1+y_2)/2} &= \left(\frac{1}{4} y_1 e^{-y_1/2} \right) \left(\frac{1}{2} e^{-y_2/2} \right) \\
\frac{1}{8} y_1 e^{-(y_1+y_2)/2} &= \frac{1}{8} y_1 e^{-y_1/2} e^{-y_2/2} \\
\frac{1}{8} y_1 e^{-(y_1+y_2)/2} &= \frac{1}{8} y_1 e^{-(y_1+y_2)/2}.
\end{aligned}$$