Homework 1

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Math 5080

Due: Saturday, January 30th at 11:59 PM

6.03 The measured radius R of a circle has pdf $f(r) = 6r(1-r), 0 < r < 1$.

(a) Find the distribution of the circumference.

A circumference of a circle is $X = 2\pi R$. The function $\phi(x) = 2\pi x$ is one-to-one and is increasing on [0,1], so it maps [0,1] onto [0,2π]. It's inverse is $\phi^{-1}(r) = \frac{x}{2\pi}$, so

$$f_X(x) = f_X\big(\phi^{-1}(r)\big)\Big|(\phi^{-1})'(r)\Big| = f_X\left(\frac{x}{2\pi}\right)\left|\frac{1}{2\pi}\right| = \frac{6x}{2\pi}\left(2\pi - \frac{x}{2\pi}\right)\left(\frac{1}{2\pi}\right)$$

$$f_X(x) = \frac{6x(2\pi - x)}{(2\pi)^3}\{0 < x < 2\pi\}.$$

(b) Find the distribution of the area of the circle.

The area of a circle is $Y = \pi R^2$. The function $\phi(y) = \pi x^2$ is not one-to-one, so when mapping $x \to x^2\pi$ it maps [0,1] onto [0,π] with each point mapped twice. The function $\phi(y) = \pi x^2$ has an inverse of $\phi^{-1}(r) = \sqrt{\frac{y}{\pi}}$, so

$$f_Y(y) = f_Y\big(\phi^{-1}(r)\big)\Big|(\phi^{-1})'(r)\Big| = f_Y\left(\sqrt{\frac{y}{\pi}}\right)\left|\frac{1}{2\sqrt{\pi y}}\right| = 6\left(\sqrt{\frac{y}{\pi}}\right)\left(1 - \sqrt{\frac{y}{\pi}}\right)\left(\frac{1}{2\sqrt{\pi y}}\right)$$

$$f_Y(y) = \frac{3}{\pi}\left(1 - \sqrt{\frac{y}{\pi}}\right)\{0 < y < \pi\}$$

or

$$f_Y(y) = \frac{3(\sqrt{\pi} - \sqrt{y})}{\pi^{\frac{3}{2}}}\{0 < y < \pi\}.$$

6.13 Let X have pdf $f(x) = x^2/24$ for $-2 < x < 4$ and assume the pdf is zero otherwise. Find the pdf of $Y = X^2$.

The function $\phi(x) = x^2$ is not one-to-one on [-2,4]. We discover that when mapping $x \to x^2$ it maps [-2,4] onto [4,16]. It's inverse is $\phi^{-1}(x) = \sqrt{y}$. Thus we obtain,

$$f_Y(y) = f_Y\big(\phi^{-1}(x)\big)\Big|(\phi^{-1})'(x)\Big| = f_Y(\sqrt{y})\left|\frac{1}{\sqrt{y}}\right| = \left(\frac{(\sqrt{y})^2}{24}\right)\left(\frac{1}{2\sqrt{y}}\right)$$

$$f_Y(y) = \frac{\sqrt{y}}{48}\{4 \leq y < 16\}.$$