Time Series Analysis
Homework 1

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Math 5075

Due: Friday, January 29th

1 Problem 1

Let X and Y be independent and identically distributed exponential random variables with $EX = EY = 1$. Compute the distribution function of $Z = X + Y$.

We are given that $EX = EY = 1$, and we know that EX of an exponential random variable is $\frac{1}{\lambda}$. Thus, we have the random variables $X \sim exp(1)$ and $Y \sim exp(1)$. Since X and Y are non-negative, $X + Y$ is also non-negative. Let $Z = X + Y$, this means that $f_Z(z) = 0$ if $z < 0$. If $z \geq 0$ then

$$f_Z(z) = \iint f_X(x) f_Y(y) dx dy = \int_0^z \int_0^{z-x} f_X(x) f_Y(y) dx dy = \int_0^z f_X(x) f_Y(z-x) dx$$

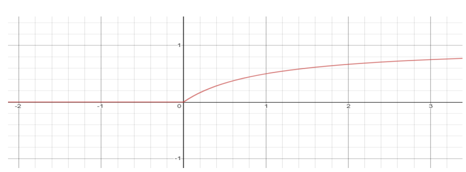
$$= \int_0^z e^{-x} e^{-(z-x)} dx = \int_0^z e^{-z} dx = x e^{-z} = z e^{-z}, z > 0$$

This identifies Z as a Gamma(2,1).

2 Problem 3

Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with distribution function

$$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - \frac{1}{1+x} & , x \geq 0. \end{cases}$$



Let $X_{n,n} = max_{1 \leq i \leq n} X_i$ and $Y_n = X_{n,n}/n$. Show that Y_n converges in distribution and determine the limit.

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\begin{aligned}
F_{Y_n}(x) &= F_{\frac{X_{n,n}}{n}}(x) \\
&= P\left(\frac{X_{n,n}}{n} \le x\right) \\
&= P(X_{n,n} \le nx) \\
&= P(max_{i \le n} X_i \le nx) \\
&= P(X_1 \le nx \cap X_2 \le nx \cap ... \cap X_n \le nx) \\
&= P\left(\cap_{i=1}^{n} X_i \le nx\right) \\
&= \prod_{i=1}^{n} P(X_i \le nx) \\
&= P(X_i \le nx)^n \\
&= F_1(nx)^n \\
&= \left(1 - \frac{1}{1+nx}\right)^n \\
&= \left(\frac{1+nx-1}{1+nx}\right)^n \\
&= \left(\frac{nx}{1+nx}\right)^n
\end{aligned}
$$

Now we need to show that Y_n converges in distribution by taking the limit.

$$\lim_{n \to \infty} F_{Y_n}(x) = e^{\frac{1}{x}}, x \ge 0.$$