Time Series Analysis
Homework 4

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Due: Friday, February 19th

Problem 13 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. Write a stationary solution of

$$x_i = 0.7x_{i-1} - 0.1x_{i-2} + \epsilon_i$$

in a casual form, i.e., an infinite sum of the $\{\epsilon_j, j \le i\}$.

Solution. The characteristic polynomial is

$$\phi(z) = 1 - 0.7z + 0.1z^2.$$

We need the inverse of $\phi(B)$ and we can show easily that the roots of $\phi(z) = (1-0.5z)(1-0.2z)$ are $z_1 = 2$ and $z_2 = 5$.

$$\frac{1}{\phi(z)} = \frac{1}{(1-0.5z)(1-0.2z)} = \frac{-1}{1-0.5z} + \frac{2}{1-0.2z}$$

Partial fractions

$$\frac{-1}{1-0.5z} = -\sum_{j=1}^{\infty}(0.5z)^{j-1}$$

$$\frac{2}{1-0.2z} = 2\sum_{j=1}^{\infty}(0.2z)^{j-1}$$

$$\frac{1}{\phi(z)} = \sum_{j=1}^{\infty}(-(0.5)^{j-1} + 2(0.2z)^{j-1})z^{j-1}$$

It can be shown that

$$\phi^{-1}(B) = \sum_{j=1}^{\infty}(-(0.5)^{j-1} + 2(0.2z)^{j-1})B^{j-1}.$$

Thus we get

$$x_i = \sum_{j=1}^{\infty}(-(0.5)^{j-1} + 2(0.2z)^{j-1})\epsilon_{i-(j-1)} = \sum_{k=0}^{\infty}(k+1)(-(0.5)^k + 2(0.2z)^k)\epsilon_{i-k}.$$

Problem 14 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. Write a stationary solution of

$$x_i = -0.25x_{i-2} + \epsilon_i$$

in a casual form, i.e., an infinite sum of the $\{\epsilon_j, j \le i\}$.

Solution. The characteristic polynomial is

$$\phi(z) = 1 + \frac{1}{4}z^2$$

the roots are complex number $2\mathbf{i}$ and $-2\mathbf{i}$, where $\mathbf{i}^2 = -1$

$$\frac{1}{\phi(z)} = \frac{1}{(1-0.5\mathbf{i}z)(1+0.5\mathbf{i}z)} = \frac{a}{1-0.5\mathbf{i}z} + \frac{b}{1+0.5\mathbf{i}z} = \frac{a+0.5\mathbf{i}az+b-0.5\mathbf{i}bz}{(1-0.5\mathbf{i}z)(1+0.5\mathbf{i}z)}$$

Choose $a = b$ and $a + b = 1$, so $a = b = \frac{1}{2}$. We have

$$\begin{aligned}
\frac{1}{(1-0.5\mathbf{i}z)(1+0.5\mathbf{i}z)} &= \frac{1}{2}\left(\frac{1}{1-0.5\mathbf{i}z} + \frac{1}{1+0.5\mathbf{i}z}\right) \\
&= \frac{1}{2}\left(\sum_{j=1}^{\infty}(-0.5\mathbf{i}z)^{j-1} + \sum_{j=1}^{\infty}(0.5\mathbf{i}z)^{j-1}\right) \\
&= \sum_{j=1}^{\infty}\frac{1}{2}\left((-0.5\mathbf{i})^{j-1} + (0.5\mathbf{i})^{j-1}\right)z^{j-1} \\
&= \sum_{j=1}^{\infty} c_{j-1}z^{j-1} \\
&= \sum_{k=0}^{\infty} c_k z^k
\end{aligned}$$

$$c_k = \begin{cases} 0, & \text{k is odd} \\ -(0.5)^k, & \text{k is even} \end{cases}$$

Hence

$$x_i = \sum_{k=0}^{\infty} c_k \epsilon_{i-k}.$$