Statistical Inference I
Homework 4

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Due: Saturday, February 20th

7.9 Let $X_1, \ldots, X_{100}$ be iid Exponential(1) random variables and let $Y = X_1 + \ldots + X_{100}$.

(a) Use normal approximation to estimate $\mathbb{P}(Y > 110)$.

(b) Use normal approximation to estimate $\mathbb{P}(1.1 < \overline{X} < 1.2)$, where $\overline{X}$ is the sample mean.

(a) The sum S_{100} will be approximately normally distributed with mean $1 \cdot 100 = 100$ and variance $1 \cdot \sqrt{100} = 10$. Thus

$$\mathbb{P}(Y > 110) = \mathbb{P}(S_{100} > 110) = \mathbb{P}\left(\frac{S_{100} - n\mu}{\sigma\sqrt{n}} > \frac{110 - 100}{\sqrt{100}}\right) = \mathbb{P}(Z > 1)$$

where $Z \sim N(0, 1)$. We know that $\mathbb{P}(Z > 1) = 1 - \mathbb{P}(Z < 1)$, and using `pnorm(1)` in `R`, we get

$$1 - \mathbb{P}(Z < 1) = 1 - 0.8413 \approx 0.1587.$$

(b) To find the $\mathbb{P}(1.1 < \overline{X} < 1.2)$, we have

$$\mathbb{P}(1.1 < \overline{X} < 1.2) = \mathbb{P}\left(1.1 < \frac{Y}{100} < 1.2\right) = \mathbb{P}(110 < Y < 120).$$

We can then use the same calculations as we did in (a) and get

$$\mathbb{P}(110 < S_{100} < 120) = \mathbb{P}\left(\frac{110 - 100}{\sqrt{100}} < \frac{S_{100} - n\mu}{\sigma\sqrt{n}} < \frac{120 - 100}{\sqrt{100}}\right) = \mathbb{P}(1 < Z < 2)$$

where $Z \sim N(0, 1)$. In `R` we can simply calculate `pnorm(2)` $-$ `pnorm(1)` and get

$$0.97725 - 0.84134 \approx 0.1359.$$

7.11 Let $X_1, \ldots, X_{20}$ be iid Uniform$(0, 1)$. Find normal approximations for

(a) $\mathbb{P}\left(\sum_{i=1}^{20} X_i \leq 12\right)$,

(b) the 90th percentile of $\sum_{i=1}^{20} X_i$.

(a) From class, we discovered that $S_n \sim U\left(\frac{n}{2}, \frac{n}{12}\right)$. Since the sum is $S_{20} \sim U\left(10, \frac{5}{3}\right)$, we know $S_{20} \approx 10 + \sqrt{\frac{5}{3}}Z$ where $Z \sim N(0,1)$. Thus

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{20} X_i \leq 12\right) = \mathbb{P}\left(S_{20} \leq 12\right) &= \mathbb{P}\left(10 + \sqrt{\frac{5}{3}}Z \leq 12\right) \\
&= \mathbb{P}\left(Z \leq \frac{12-10}{\sqrt{\frac{5}{3}}}\right) \\
&= \mathbb{P}\left(Z \leq \frac{2}{\sqrt{\frac{5}{3}}}\right) \\
&= \mathbb{P}(Z \leq 1.54919)
\end{aligned}$$

Using `pnorm(1.54919)` in `R`, we get

$$\mathbb{P}(Z \leq 1.54919) \approx 0.9393$$

(b) Using the `qnorm(0.9)` in `R`, we know that the 90th percentile approximates to 1.281552. Let $Z_{0.9} = 1.281552$ and $P_{0.9}$ represent the value where the probability is the 90th percentile, we show

$$\begin{aligned}
\mathbb{P}\left(S_{20} \leq P_{0.9}\right) &= \mathbb{P}\left(10 + \sqrt{\frac{5}{3}}Z \leq P_{0.9}\right) \\
&= \mathbb{P}\left(Z \leq \frac{P_{0.9}-10}{\sqrt{\frac{5}{3}}}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{P_{0.9}-10}{\sqrt{\frac{5}{3}}} \approx Z_{0.9} \\
\Rightarrow &\frac{P_{0.9}-10}{\sqrt{\frac{5}{3}}} = 1.281552 \\
\Rightarrow &P_{0.9} - 10 \approx 1.65448 \\
\Rightarrow &P_{0.9} \approx 11.7.
\end{aligned}$$