Time Series Analysis
Homework 3

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Due: Wednesday, February 17th

Problem 9 Let ϵ_i be a sequence of random variables with $E\epsilon_i = 0$, $E\epsilon_i^2 = \sigma^2$ if $i \neq j$. Let $x_0 = 0$ and $|\rho| > 1$. The sequence x_k defined by

$$x_i = \rho x_{i-1} + \epsilon_i,\ i = 1, 2, ...$$

Compute Ex_k and $Ex_k x_{k+2}$.

We start computing the elements of the recursion

$$\begin{aligned}
x_1 &= \rho x_0 + \epsilon_1 \\
x_2 &= \rho x_1 + \epsilon_2 = \rho(\rho x_0 + \epsilon_1) + \epsilon_2 = \rho^2 x_0 + \rho\epsilon_1 + \epsilon_2 \\
x_3 &= \rho x_2 + \epsilon_3 = \rho(\rho^2 x_0 + \rho\epsilon_1 + \epsilon_2) + \epsilon_3 = \rho^3 x_0 + \rho^2\epsilon_1 + \rho\epsilon_2 + \epsilon_3 \\
&\vdots \\
x_k &= \rho^k x_0 + \sum_{\ell=0}^{\infty} \rho^\ell \epsilon_{k-\ell}
\end{aligned}$$

If we let $x_0 = 0$ and $|\rho| > 1$ (an Explosive case) we write

$$\rho^{-k} x_k = x_0 + \rho^{-k} \sum_{\ell=0}^{k-1} \rho^\ell \epsilon_{k-\ell} = \sum_{\ell=0}^{k-1} \rho^{\ell-k} \epsilon_{k-\ell}.$$

We can perform an index shift with $j = k - \ell$,

$$\rho^{-k} x_k = \sum_{\ell=0}^{k-1} \rho^{-(k-\ell)} \epsilon_{k-\ell} = \sum_{j=1}^{k} \rho^{-j} \epsilon_j \Rightarrow x_k = \rho^k \sum_{j=1}^{k} \rho^{-j} \epsilon_j.$$

Now we compute Ex_k to get

$$Ex_k = E\left[\rho^k \sum_{j=1}^{k} \rho^{-j} \epsilon_j\right] = \rho^k \sum_{j=1}^{k} \rho^{-j} E\epsilon_j = 0.$$

Similarly,

$$\begin{aligned}
Ex_k x_{k+2} &= E\left[\rho^k \sum_{\ell=0}^{k-1} \rho^{-j} \epsilon_j\right]\left[\rho^{k+2} \sum_{j=0}^{k+1} \rho^{-i} \epsilon_{i+2}\right] \\
&= \rho^{2k+2} \sum_{\ell=0}^{k-1} \sum_{j=0}^{k+1} \rho^{-j-i} E(\epsilon_j \epsilon_{i+2})
\end{aligned}$$

The errors are uncorrelated so $E\epsilon_j\epsilon_{i+2} = 0$ except when $j = i+2 \Rightarrow i = j-2$ when the expected value is σ^2. Thus we get

$$\begin{aligned}
\rho^{2k+2}\sum_{\ell=0}^{k-1}\sum_{j=0}^{k+1}\rho^{-j-i}E(\epsilon_j\epsilon_{i+2}) &= \rho^{2k+2}\sum_{\ell=0}^{k-1}\rho^{-j}\rho^{-(j-2)}\sigma^2 \\
&= \rho^{2k+2}\sigma^2\sum_{\ell=0}^{k-1}\rho^{-2j+2} \\
&= \rho^{2k+4}\sigma^2\sum_{\ell=0}^{k-1}\rho^{-2j}
\end{aligned}$$

Thus we get

$$Ex_kx_{k+2} = \rho^{2k+4}\left(\frac{\sigma^2}{1-\rho^{-2}}\right).$$

Problem 10 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. Let x_i be the stationary solution of

$$x_i = \frac{1}{3}x_{i-1} + \epsilon_i, \quad -\infty < i < \infty.$$

If $\mathrm{Var}(x_0) = 100$, determine σ^2.

We have $\phi = \frac{1}{3}$ thus

$$Z_i = x_i + \frac{1}{3}x_{i-1} + \frac{1}{9}x_{i-2} + \ldots.$$

We know that

$$\begin{aligned}
E(Z_i) &= E\left(x_i + \frac{1}{3}x_{i-1} + \frac{1}{9}x_{i-2} + \ldots\right) \\
&= 0
\end{aligned}$$

so that $\{Z_i\}$ has a constant mean of zero. Also,

$$\begin{aligned}
\mathrm{Var}(Z_i) &= \mathrm{Var}\left(x_i + \frac{1}{3}x_{i-1} + \frac{1}{9}x_{i-2} + \ldots\right) \\
&= \mathrm{Var}(x_i) + \left(\frac{1}{3}\right)^2\mathrm{Var}(x_{i-1}) + \left(\frac{1}{3}\right)^4\mathrm{Var}(x_{i-2}) + \ldots \\
&= \sigma_x^2\left(1 + \frac{1}{9} + \frac{1}{81} + \ldots\right) \\
&= \frac{\sigma_x^2}{1-\frac{1}{9}}
\end{aligned}$$

Therefore

$$\mathrm{Var}(x_0) = 100 = \frac{\sigma_x^2}{1-\frac{1}{9}} \Rightarrow \sigma_x^2 = \frac{800}{9}.$$