Time Series Analysis
Midterm 1

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Due: Friday, February 12th

Problem 1 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. Define

$$x_i = i^\alpha \epsilon_i + i\epsilon_{i-1}, \text{ where } \alpha \geq 1.$$

(i) Compute Show that

$$E\left(\sum_{i=1}^{n} x_i\right)^2.$$

Show that

$$\frac{1}{n^{\alpha+1}}\sum_{i=1}^{n} x_i \xrightarrow{P} 0.$$

Solution. We observe,

$$E\sum_{i=1}^{n} x_i = E\sum_{i=1}^{n}\left(i^\alpha \epsilon_i + i\epsilon_{i-1}\right) = \sum_{i=1}^{n}\left(i^\alpha E\epsilon_i + iE\epsilon_{i-1}\right) = 0.$$

(i) We know that

$$\begin{aligned} E\left(\sum_{i=1}^{n} x_i\right)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left(i^\alpha \epsilon_i + i\epsilon_{i-1}\right)\left(j^\alpha \epsilon_j + j\epsilon_{j-1}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} i^\alpha j^\alpha E\epsilon_i\epsilon_j + \sum_{i=1}^{n}\sum_{j=1}^{n} i^\alpha jE\epsilon_i\epsilon_{j-1} + \sum_{i=1}^{n}\sum_{j=1}^{n} ij^\alpha E\epsilon_{i-1}\epsilon_j + \sum_{i=1}^{n}\sum_{j=1}^{n} ijE\epsilon_{i-1}\epsilon_{j-1} \\ &= 9n^{\alpha+1}i^{2\alpha}\sigma^2 - 4i^2\sigma^2 \end{aligned}$$

Hence

$$E\left(\frac{1}{n^{\alpha+1}}\sum_{i=1}^{n} x_i\right)^2 \leq \frac{9n^{\alpha+1}i^{2\alpha}\sigma^2 - 4i^2\sigma^2}{n^{2\alpha+2}}.$$

By Chebishev's inequality,

$$P\left\{\left|\frac{1}{n^{\alpha+1}}\sum_{i=1}^{n} x_i\right| > \epsilon\right\} \leq \frac{1}{\epsilon^2}\frac{9n^{\alpha+1}i^{2\alpha}\sigma^2 - 4i^2\sigma^2}{n^{2\alpha+2}}.$$

Problem 2 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. Define

$$x_i = \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \epsilon_{i-\ell}.$$

Compute $c(h), h \geq 0$, the autocovariance function.

Solution. The autocovariance function is

$$\begin{aligned} c(h) = cov(x_{i+h}, x_i) &= E\left[\left(\sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \epsilon_{i+h-\ell}\right)\left(\sum_{k=1}^{\infty} \frac{1}{k^2} \epsilon_{i-k}\right)\right] \\ &= E\left[\left(\epsilon_{i+h} + \ldots + \frac{1}{h^2}\epsilon_i + \frac{1}{(h+1)^2}\epsilon_{i-1} + \ldots\right)(\epsilon_i + \epsilon_{i-1} + \ldots)\right] \\ &= \sigma^2 \sum_{\ell=1}^{\infty} \frac{1}{(h+\ell)^2} \frac{1}{\ell^2} \\ &= \sigma^2 \frac{1}{h^2} \sum_{\ell=1}^{\infty} \frac{1}{\ell^4} \end{aligned}$$

Problem 3 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. Show that

$$x_i = -\frac{1}{4} x_{i-2} + \epsilon_i.$$

defines a stationary sequence and find c_ℓ such that

$$x_i = \sum_{\ell=0}^{\infty} c_\ell \epsilon_{i-\ell}.$$

Solution. The characteristic polynomial is

$$\phi(z) = 1 + \frac{1}{4} z^2$$

the roots are complex number $2\mathbf{i}$ and $-2\mathbf{i}$, where $\mathbf{i}^2 = -1$

$$\frac{1}{\phi(z)} = \frac{1}{(1-0.5\mathbf{i}z)(1+0.5\mathbf{i}z)} = \frac{a}{1-0.5\mathbf{i}z} + \frac{b}{1+0.5\mathbf{i}z} = \frac{a + 0.5\mathbf{i}az + b - 0.5\mathbf{i}bz}{(1-0.5\mathbf{i}z)(1+0.5\mathbf{i}z)}$$

Choose $a = b$ and $a + b = 1$, so $a = b = \frac{1}{2}$. We have

$$\begin{aligned} \frac{1}{(1-0.5\mathbf{i}z)(1+0.5\mathbf{i}z)} &= \frac{1}{2}\left(\frac{1}{1-0.5\mathbf{i}z} + \frac{1}{1+0.5\mathbf{i}z}\right) \\ &= \frac{1}{2}\left(\sum_{\ell=0}^{\infty}(-0.5\mathbf{i}z)^\ell + \sum_{\ell=0}^{\infty}(0.5\mathbf{i}z)^\ell\right) \\ &= \sum_{\ell=0}^{\infty} \frac{1}{2}\left((-0.5\mathbf{i}z)^\ell + (0.5\mathbf{i}z)^\ell\right) \\ &= \sum_{\ell=0}^{\infty} c_\ell z^\ell \end{aligned}$$

$$c_\ell = \begin{cases} 0, & \ell \text{ is odd} \\ -(0.5)^\ell, & \ell \text{ is even} \end{cases}$$

Hence

$$x_i = \sum_{\ell=0}^{\infty} c_\ell \epsilon_{i-\ell}.$$

Problem 4 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. show that

$$x_i = \frac{3}{4}x_{i-1} - \frac{1}{8}x_{i-2} + \epsilon_i$$

defines a stationary sequence and compute the autocorrelation function.

Solution. The characteristic polynomial is

$$\phi(z) = 1 - \frac{3}{4}z + \frac{1}{8}z^2.$$

We solve the equation $\phi(z) =$ with roots 2 and 4 and therefore

$$\phi(z) = \left(1 - \frac{1}{2}z\right)\left(1 - \frac{1}{4}z\right).$$

By partial fractions

$$\begin{aligned} \frac{1}{\phi(z)} &= \frac{1}{(1-az)(1-bz)} \\ &= \frac{b}{b-a}\cdot\frac{1}{1-bz} - \frac{a}{b-a}\cdot\frac{1}{1-az} \\ &= \frac{1}{\left(1-\frac{1}{2}z\right)\left(1-\frac{1}{4}z\right)} \\ &= \frac{a}{1-\frac{1}{2}z} + \frac{b}{1-\frac{1}{4}z} \\ &= \frac{a(1-\frac{1}{4}z) + b(1-\frac{1}{2}z)}{(1-\frac{1}{2}z)(1-\frac{1}{4}z)} \\ &= \frac{a - \frac{a}{4}z + b - \frac{b}{2}z}{(1-\frac{1}{2}z)(1-\frac{1}{4}z)} \end{aligned}$$

so $a + b = 1$ and

$$\frac{a}{4} + \frac{b}{2} = 0.$$

We get $a = 4$ and $b = -2$,

$$\frac{1}{\phi(z)} = \frac{4}{1-\frac{1}{2}z} - \frac{2}{1-\frac{1}{4}z} = \sum_{\ell=0}^{\infty}\left(4\left(\frac{1}{2}\right)^\ell - 2\left(\frac{1}{4}\right)^\ell\right)z^\ell$$

so

$$c_\ell = 4\left(\frac{1}{2}\right)^\ell - 2\left(\frac{1}{4}\right)^\ell.$$

Problem 5 Let $\epsilon_i, -\infty < i < \infty$, be independent and identically distributed random variables with $E\epsilon_i = 0$ and $E\epsilon_i^2 = \sigma^2$. Define

$$x_i = \epsilon_i - \frac{3}{4}x_{i-1} + \frac{1}{8}x_{i-2}$$

Show that x_i is invertible and find d_ℓ such that

$$\epsilon_i = \sum_{\ell=0}^{\infty} d_\ell x_{i-\ell}.$$

Solution. The characteristic polynomial is

$$\phi(z) = 1 + \frac{3}{4}z - \frac{1}{8}z^2.$$

We can use the simple equation

$$\phi(z) = 1 - \phi_1(z) = 1 - \left(-\frac{3}{4}\right)z$$

$$\Rightarrow \frac{1}{\phi(z)} = \frac{1}{1 - \left(-\frac{3}{4}\right)z} = \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}z\right)^\ell = \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell z^\ell$$

Because $|\phi_1 z| < 1 \Rightarrow \left|-\frac{3}{4}\right| < 1$ we get the formal solution

$$\sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell B^\ell \epsilon_i = \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell \epsilon_{i-\ell}$$

We show

$$(1 - \phi_1 B)\left(\sum_{\ell=0}^{\infty}\phi_1^\ell B^\ell\right) = 1$$

Computing the product

$$\begin{aligned}
\left(1 - \left(-\frac{3}{4}\right)B\right)\left(\sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell B^\ell\right) &= \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell B^\ell - \left(-\frac{3}{4}\right)B\left(\sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell B^\ell\right) \\
&= \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell B^\ell - \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}B\right)^{\ell+1} \\
&= \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^\ell B^\ell - \sum_{\ell=0}^{\infty}\left(-\frac{3}{4}\right)^k B^k \\
&= 1
\end{aligned}$$

Now

$$d_\ell = O(\rho^\ell) \text{ where } 0 < \rho < 1.$$