(a) Write a C++ program that uses finite difference methods to numerically evaluate the first derivative of a function $f(x)$ whose values on a fixed grid of points are specified $f(x_i),\ i = 0, 1, ..., N$. Your code should use three instances of a `valarray<long double>` to store the values of x_i, $f(x_i)$ and $f'(x_i)$. Assume the grid-points are located at $x_i = a + i\,\Delta x$ with $\Delta x = (b - a)/N$. on the interval $x \in [a, b]$ and use 2nd order finite differencing to compute an approximation for $f'(x_i)$:

$$\begin{aligned} f'(x_0) &= \frac{-3f(x_0) + 4f(x_1) - f(x_2)}{2\Delta x} + O(\Delta x^2) \quad \text{for} \quad i = 0 \\ f'(x_i) &= \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x^2) \quad \text{for} \quad i = 1, 2, ..., N-1 \\ f'(x_N) &= \frac{f(x_{N-2}) - 4f(x_{N-1}) + 3f(x_N)}{2\Delta x} + O(\Delta x^2) \quad \text{for} \quad i = N \end{aligned}$$

Demonstrate that your program works by evaluating the derivatives of a known function, $f(x) = \sin 3x$, with $N + 1 = 32$ points on the interval $x \in [a, b] = [-1, 1]$. Compute the difference between your numerical derivatives and the known analytical ones:

$$e_i = f'_{\text{numerical}}(x_i) - f'_{\text{analytical}}(x_i)$$

at each grid-point. Output the values e_i of this `valarray<long double>` on the screen and tabulate (or plot) them in your report.

(b) For the same choice of $f(x)$, demonstrate 2nd-order convergence, by showing that, as N increases, the mean error $\langle e \rangle$ decreases proportionally to $\Delta x^2 \propto N^{-2}$. You may do so by tabulating the quantity $N^2 \langle e \rangle$ for different values of N (e.g. $N + 1 = 16,\ 32,\ 64,\ 128$) and checking if this quantity is roughly constant. Alternatively (optionally), you may plot $\log\langle e \rangle$ vs. $\log N$ and check if the dependence is linear and the slope is -2. Here, the mean error $\langle e \rangle$ is defined by

$$\langle e \rangle = \frac{1}{N+1} \sum_{i=0}^{N} |e_i| = \frac{1}{N+1} \ell_1(\vec{e}).$$