Question 4: [20 marks] *Numerical integration.*

We wish to compute the definite integral

$$I = \int_a^b \sin\left(\frac{1}{x + \frac{1}{2}}\right)\, dx$$

numerically for $a = 0, b = 10$ and compare to the exact result, $I_{\text{exact}} = 2.74324739415100920$.

(a) Use the composite trapezium rule

$$\int_a^b f(x)dx \simeq \sum_{i=0}^{N} w_i f_i, \quad w_i = \begin{cases} \Delta x/2, & i = 0 \text{ or } i = N \\ \Delta x & 1 \leq i \leq N-1 \end{cases}, \quad \Delta x = \frac{b-a}{N},$$

to compute the integral I, using $N + 1 = 128$ equidistant points in $x \in [a, b]$. Use three instances of a `valarray<long double>` to store the values of the gridpoints x_i, function values $f_i = f(x_i)$ and weights w_i. [Hint: you may use the function from Question 2a to compute the dot product of the `valarrays` w_i and f_i.] Output to the screen (and list in your report) your numerical result $I_{\text{trapezium}}$ and the difference $I_{\text{trapezium}} - I_{\text{exact}}$.

(b) Use the composite Hermite rule

$$\int_a^b f(x)dx \simeq \sum_{i=0}^{N} w_i f_i + \frac{\Delta x^2}{12}[f'(a) - f'(b)]$$

with the derivatives $f'(x)$ at $x = a$ and $x = b$ evaluated analytically (and the weights w_i identical to those given above for the trapezium rule), to compute the integral I, using $N + 1 = 128$ equidistant points in $x \in [a, b]$. Output to the screen (and list in your report) your numerical result I_{Hermite} and the difference $I_{\text{Hermite}} - I_{\text{exact}}$.

(c) Use the Clenshaw-Curtis quadrature rule

$$\int_a^b f(x)dx \simeq \sum_{i=0}^{N} w_i f_i, \quad w_i = \frac{b-a}{2} * \begin{cases} \frac{1}{N^2}, & i = 0 \text{ or } i = N \\ \frac{2}{N}\left(1 - \sum_{k=1}^{(N-1)/2} \frac{2\cos(2k\theta_i)}{4k^2-1}\right) & 1 \leq i \leq N-1 \end{cases},$$

on a grid of $N + 1 = 128$ points $x_i = [(a + b) - (b - a)\cos\theta_i]/2$, where $\theta_i = i\pi/N$, $i = 0, 1, ..., N$ to compute the integral I. [Hint: First compute the values of θ_i, x_i, $f_i = f(x_i)$ and w_i and store them as `valarrays`. Then, you may use the function from Question 2a to compute the dot product of the `valarrays` w_i and f_i.] Output to the screen (and list in your report) your numerical result $I_{\text{ClenshawCurtis}}$ and the difference $I_{\text{ClenshawCurtis}} - I_{\text{exact}}$.
Remark: the above formula is valid only for odd N. Ensure your code does not use even N.

(d) Compute the integral I using a *Mean Value* Monte Carlo method with $N = 1000$, $N = 10000$ and $N = 100000$ samples. Output to the screen (and list in your report) your numerical results $I_{\text{MonteCarlo}}$ and the difference $I_{\text{MonteCarlo}} - I_{\text{exact}}$ for each N.