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Abstract

We derive total mean curvature integration formulas of a three co-dimensional foliation $\mathcal{F}^n$ on a screen integrable half-lightlike submanifold, M^{n+1} in a semi-Riemannian manifold $\overline{M}^{n+3}$. We give generalized differential equations relating to mean curvatures of a totally umbilical half-lightlike submanifold admitting a totally umbilical screen distribution, and show that they are generalizations of those given by [K. L. Duggal and B. Sahin, Differential geometry of lightlike submanifolds, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2010].

1 Introduction

The rapidly growing importance of lightlike submanifolds in semi-Riemannian geometry, particularly Lorentzian geometry, and their applications to mathematical physics–like in general relativity and electromagnetism motivated the study of lightlike geometry in semi-Riemannian manifolds. More precisely, lightlike submanifolds have been shown to represent different black hole horizons (see [4] and [5] for details). Among other motivations for investing in lightlike geometry by many physicists is the idea that the universe we are living in can be viewed as a 4-dimensional hypersurface embedded in $(4 + m)$-dimensional spacetime manifold, where m is any arbitrary integer. There are significant differences between lightlike geometry and Riemannian geometry as shown in [4] and [5], and many more references therein. Some of the pioneering work on this topic is due to Duggal-Bejancu [4], Duggal-Sahin [5] and Kupeli [7]. It is upon those books that many other researchers, including but not limited to [3], [6], [8], [9], [10], [11], have extended their theories.

2 Integration formulas for $\mathcal{F}$

This section is devoted to derivation of integral formulas of foliation $\mathcal{F}$ of $S(TM)$ with a unit normal vector $\widehat{W}$. By the fact that $\overline{\nabla}$ is a metric connection then

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$\overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W},\widehat{W}) = 0$. This implies that the vector field $\overline{\nabla}_{\widehat{W}}\widehat{W}$ is always tangent to $\mathcal{F}$. Our main goal will be to compute the divergence of the vectors $T_r\overline{\nabla}_{\widehat{W}}\widehat{W}$ and $T_r\overline{\nabla}_{\widehat{W}}\widehat{W}+(-1)^r S_{r+1}\widehat{W}$. The following technical lemmas are fundamentally important to this paper. Let $\{E, Z_i, N, W\}$, for $i = 1,\cdots,n$ be a quasi-orthonormal field of frame of $T\overline{M}$, such that $S(TM) = \mathrm{span}\{Z_i\}$ and $\epsilon_i = \overline{g}(Z_i, Z_i)$.

Lemma 2.1. *Let M be a screen integrable half-lightlike submanifold of $\overline{M}^{n+3}$ and let M' be a foliation of $S(TM)$. Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where $\widehat{W}$ is a unit normal vector to $\mathcal{F}$. Then*

$$\overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y, Z) = \overline{g}(Y, (\nabla'_X\mathcal{A}_{\widehat{W}})Z),$$
$$\overline{g}((\nabla'_X T_r)Y, Z) = \overline{g}(Y, (\nabla'_X T_r)Z),$$

for all $X, Y, Z \in \Gamma(T\mathcal{F})$.

Proof. By simple calculations we have

$$\overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y, Z) = \overline{g}(\nabla'_X(\mathcal{A}_{\widehat{W}}Y), Z) - \overline{g}(\nabla'_X Y, \mathcal{A}_{\widehat{W}}Z). \tag{1}$$

Using the fact that ∇' is a metric connection and the symmetry of $\mathcal{A}_{\widehat{W}}$, (1) gives

$$\overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y, Z) = \overline{g}(Y, \nabla'_X(\mathcal{A}_{\widehat{W}}Z)) - \overline{g}(Y, \mathcal{A}_{\widehat{W}}(\nabla'_X Z)). \tag{2}$$

Then, from (2) we deduce the first relation of the lemma. A proof of the second relation follows in the same way, which completes the proof. $\square$

Lemma 2.2. *Let M be a screen integrable half-lightlike submanifold of $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where $\widehat{W}$ is a unit normal vector to $\mathcal{F}$. Denote by $\overline{R}$ the curvature tensor of $\overline{M}$. Then*

$$\mathrm{div}^{\nabla'}(T_0) = 0,$$
$$\mathrm{div}^{\nabla'}(T_r) = \mathcal{A}_{\widehat{W}}\mathrm{div}^{\nabla'}(T_{r-1}) + \sum_{i=1}^{n}\epsilon_i(\overline{R}(\widehat{W}, T_{r-1}Z_i)Z_i)',$$

where $(\overline{R}(\widehat{W}, X)Z)'$ denotes the tangential component of $\overline{R}(\widehat{W}, X)Z$ for $X, Z \in \Gamma(T\mathcal{F})$. Equivalently, for any $Y \in \Gamma(T\mathcal{F})$ then

$$\overline{g}(\mathrm{div}^{\nabla'}(T_r), Y) = \sum_{j=1}^{r}\sum_{i=1}^{n}\epsilon_i\overline{g}(\overline{R}(T_{r-1}Z_i, \widehat{W})(-\mathcal{A}_{\widehat{W}})^{j-1}Y, Z_i). \tag{3}$$

Proof. The first equation of the lemma is obvious since $T_0 = \mathbb{I}$. We turn to the second relation. By direct calculations using the recurrence relation we derive

$$\begin{aligned}\mathrm{div}^{\nabla'}(T_r) &= (-1)^r\mathrm{div}^{\nabla'}(S_r\mathbb{I}) + \mathrm{div}^{\nabla'}(\mathcal{A}_{\widehat{W}}\circ T_{r-1})\\ &= (-1)^r\nabla' S_r + \mathcal{A}_{\widehat{W}}\mathrm{div}^{\nabla'}(T_{r-1}) + \sum_{i=1}^{n}\epsilon_i(\nabla'_{Z_i}\mathcal{A}_{\widehat{W}})T_{r-1}Z_i. \end{aligned}\tag{4}$$

Using Codazzi equation

$$\overline{g}(\overline{R}(X,Y)Z,\widehat{W}) = \overline{g}((\nabla'_Y\mathcal{A}_{\widehat{W}})X,Z) - \overline{g}((\nabla'_X\mathcal{A}_{\widehat{W}})Y,Z),$$

for any $X,Y,Z \in \Gamma(T\mathcal{F})$ and Lemma 2.1, we have

$$\begin{aligned}\overline{g}((\nabla'_{Z_i}\mathcal{A}_{\widehat{W}})Y, T_{r-1}Z_i) &= \overline{g}((\nabla'_Y\mathcal{A}_{\widehat{W}})Z_i, T_{r-1}Z_i) + \overline{g}(\overline{R}(Y,Z_i)T_{r-1}Z_i,\widehat{W}) \\ &= \overline{g}(T_{r-1}(\nabla'_Y\mathcal{A}_{\widehat{W}})Z_i, Z_i) + \overline{g}(\overline{R}(\widehat{W},T_{r-1}Z_i)Z_i,Y),\end{aligned} \tag{5}$$

for any $Y \in \Gamma(T\mathcal{F})$. Then applying (4) and (5) we get

$$\begin{aligned}\overline{g}(\operatorname{div}^{\nabla'}(T_r),Y) &= (-1)^r\overline{g}(\nabla' S_r, Y) + \operatorname{tr}(T_{r-1}(\nabla'_Y\mathcal{A}_{\widehat{W}})) \\ &\quad + \overline{g}(\operatorname{div}^{\nabla'}(T_{r-1}),Y) + \overline{g}(Y,\sum_{i=1}^{n}\epsilon_i\overline{R}(\widehat{W},T_{r-1}Z_i)Z_i).\end{aligned} \tag{6}$$

Then we get the second equation of the lemma. Finally, (3) follows immediately by an induction argument. □

Notice that when the ambient manifold is a space form of constant sectional curvature, then $(\overline{R}(\widehat{W},X)Y)' = 0$, for each $X,Y \in \Gamma(T\mathcal{F})$. Hence, from Lemma (2.2) we have $\operatorname{div}^{\nabla'}(T_r) = 0$.

Lemma 2.3. *Let M be a screen integrable half-lightlike submanifold of $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where $\widehat{W}$ is a unit normal vector to $\mathcal{F}$. Let $\{Z_i\}$ be a local field such $(\nabla'_X Z_i)p = 0$, for $i = 1,\cdots,n$ and any vector field $X \in \Gamma(T\overline{M})$. Then at $p \in \mathcal{F}$ we have*

$$\begin{aligned}g(\nabla'_{Z_i}\overline{\nabla}_{\widehat{W}}\widehat{W},Z_j) &= g(\mathcal{A}^2_{\widehat{W}}Z_i,Z_j) - \overline{g}(\overline{R}(Z_i,\widehat{W})Z_j,\widehat{W}) \\ &\quad - \overline{g}((\nabla'_{\widehat{W}}\mathcal{A}_{\widehat{W}})Z_i,Z_j) + g(\overline{\nabla}_{\widehat{W}}\widehat{W},Z_i)g(Z_j,\overline{\nabla}_{\widehat{W}}\widehat{W}).\end{aligned}$$

Proof. Applying $\overline{\nabla}_{Z_i}$ to $g(\overline{\nabla}_{\widehat{W}}\widehat{W},Z_j)$ and $\overline{g}(\widehat{W},\overline{\nabla}_{\widehat{W}}Z_j)$ in turn and then using the two resulting equations, we have

$$\begin{aligned}-\overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W},\overline{\nabla}_{Z_i}Z_j) &= g(\overline{\nabla}_{Z_i}\overline{\nabla}_{\widehat{W}}\widehat{W},Z_j) + \overline{g}(\overline{\nabla}_{Z_i}\widehat{W},\overline{\nabla}_{\widehat{W}}Z_j) \\ &\quad + \overline{g}(\widehat{W},\overline{\nabla}_{Z_i}\overline{\nabla}_{\widehat{W}}Z_j).\end{aligned} \tag{7}$$

Furthermore, by direct calculations using $(\nabla'_X Z_i)p = 0$ we have

$$\overline{g}((\nabla'_{\widehat{W}}\mathcal{A}_{\widehat{W}})Z_i,Z_j) = \overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W},\overline{Z_i}Z_j) + \overline{g}(\widehat{W},\overline{\nabla}_{\widehat{W}}\overline{Z_i}Z_j),$$

and hence

$$\begin{aligned}g(\mathcal{A}^2_{\widehat{W}}Z_i,Z_j) &- \overline{g}(\overline{R}(Z_i,\widehat{W})Z_j,\widehat{W}) - \overline{g}((\nabla'_{\widehat{W}}\mathcal{A}_{\widehat{W}})Z_i,Z_j) \\ &= g(\mathcal{A}^2_{\widehat{W}}Z_i,Z_j) - \overline{g}(\overline{R}(Z_i,\widehat{W})Z_j,\widehat{W}) \\ &\quad - \overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W},\overline{Z_i}Z_j) - \overline{g}(\widehat{W},\overline{\nabla}_{\widehat{W}}\overline{Z_i}Z_j) \\ &= g(\mathcal{A}^2_{\widehat{W}}Z_i,Z_j) - \overline{g}(\overline{\nabla}_{Z_i}Z_j,\overline{\nabla}_{\widehat{W}}\widehat{W}) \\ &\quad - \overline{g}(\overline{\nabla}_{Z_i}\overline{\nabla}_{\widehat{W}}Z_j,\widehat{W}) + \overline{g}(\overline{\nabla}_{[Z_i,\widehat{W}]}Z_j,\widehat{W}).\end{aligned} \tag{8}$$

Now, applying (7), the condition at p and the following relations

$$\overline{\nabla}_{Z_i}\widehat{W} = \sum_{k=1}^{n} \epsilon_k \overline{g}(\overline{\nabla}_{Z_i}\widehat{W}, Z_k)Z_k, \quad \overline{\nabla}_{\widehat{W}} Z_j = \overline{g}(\overline{\nabla}_{\widehat{W}} Z_j, \widehat{W})\widehat{W},$$

and $g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) = -\sum_{k=1}^{n} \epsilon_k \overline{g}(\overline{\nabla}_{Z_i}\widehat{W}, Z_k)\overline{g}(\overline{\nabla}_{Z_k} Z_j, \widehat{W})$ to the last line of (8) and the fact that $S(TM)$ is integrable we get

$$\begin{aligned} g(\mathcal{A}^2_{\widehat{W}} Z_i, Z_j) &- \overline{g}(\overline{R}(Z_i, \widehat{W})Z_j, \widehat{W}) - \overline{g}((\nabla'_{\widehat{W}}\mathcal{A}_{\widehat{W}})Z_i, Z_j) \\ &= g(\nabla'_{Z_i}\overline{\nabla}_{\widehat{W}}\widehat{W}, Z_j) - g(\overline{\nabla}_{\widehat{W}}\widehat{W}, Z_i)g(Z_j, \overline{\nabla}_{\widehat{W}}\widehat{W}), \end{aligned}$$

from which the lemma follows by rearrangement. □

Notice that, using parallel transport, we can always construct a frame field from the above lemma.

Proposition 2.4. *Let M be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a foliation of $S(TM)$. Then*

$$\begin{aligned} \mathrm{div}^{\nabla'}(T_r\overline{\nabla}_{\widehat{W}}\widehat{W}) &= \overline{g}(\mathrm{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}}\widehat{W}) + (-1)^{r+1}\widehat{W}(S_{r+1}) \\ &\quad + (-1)^{r+1}(-S_1S_{r+1} + (r+2)S_{r+2}) - \sum_{i=1}^{n}\epsilon_i \overline{g}(\overline{R}(Z_i, \widehat{W})T_rZ_i, \widehat{W}) \\ &\quad + \overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W}, T_r\overline{\nabla}_{\widehat{W}}\widehat{W}), \end{aligned}$$

where $\{Z_i\}$ is a field of frame tangent to the leaves of $\mathcal{F}$.

Proof. We deduce that

$$\mathrm{div}^{\nabla'}(T_rZ) = \overline{g}(\mathrm{div}^{\nabla'}(T_r), Z) + \sum_{i=1}^{n}\epsilon_i \overline{g}(\nabla'_{Z_i}Z, T_rZ_i), \tag{9}$$

for all $Z \in \Gamma(T\mathcal{F})$. Then using (9), Lemmas 2.2 and 2.3, we obtain the desired result. Hence the proof. □

From Proposition 2.4 we have

Theorem 2.5. *Let M be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then*

$$\begin{aligned} \mathrm{div}^{\overline{\nabla}}(T_r\overline{\nabla}_{\widehat{W}}\widehat{W}) &= \overline{g}(\mathrm{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}}\widehat{W}) + (-1)^{r+1}\widehat{W}(S_{r+1}) \\ &\quad + (-1)^{r+1}(-S_1S_{r+1} + (r+2)S_{r+2}) \\ &\quad - \sum_{i=1}^{n}\epsilon_i \overline{g}(\overline{R}(Z_i, \widehat{W})T_rZ_i, \widehat{W}). \end{aligned}$$

Proof. A proof follows easily from Proposition 2.4 by recognizing the fact that

$$\begin{aligned}\operatorname{div}^{\overline{\nabla}}(T_r\overline{\nabla}_{\widehat{W}}\widehat{W}) &= \operatorname{div}^{\nabla'}(T_r\overline{\nabla}_{\widehat{W}}\widehat{W}) \\ &\quad - \overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W}, T_r\overline{\nabla}_{\widehat{W}}\widehat{W}),\end{aligned}$$

which completes the proof. □

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