Vietnam National University, Ho Chi Minh City
University of Technology
Faculty of Computer Science and Engineering

MATHEMATICAL MODELING (CO2011)

Assignment (Semester: 222, Duration: 06 weeks)

"Dynamics of Love"

(Version 1.0)

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Contents

List of Symbols

i Unit imaginary number

Re λ The real part of the eigenvalue λ

Im λ The image part of the eigenvalue λ

t Time variable

t_0 The initial value of time t

h Time step

$\mathbf{u}$ A vector function dependent on t

$\dot{\mathbf{u}}$ The derivative of $\mathbf{u}$ with respect to t

$\mathbf{F}$ A vector function dependent on t, $\mathbf{u}$, and $\dot{\mathbf{u}}$

T The temperature of the surface of a certain object or the transpose operator

$\mathbb{N}$ Set of natural numbers

$\mathbb{R}$ Set of real numbers

$\mathbb{R}^+$ Set of positive real numbers

List of Acronyms

ODE (First-Order) Ordinary Differential Equation

IVP Initial-Value Problem

DL Deep Learning

ML Machine Learning

DS Dynamical System

Fig. Figure

Tab. Table

Sys. System of Equations

Eq. Equation

VF Vector Field

e.g. For Example

i.e. That Is

Ex. Exercise

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1 Introduction

This interesting assignment will show us how a real-life problem such as the love affair between two people can be solved just with a very simple system of (First-Order) Ordinary Differential Equations (ODEs). In general a system of ODEs has the form

$$\mathbf{F}(t, \dot{\mathbf{u}}, \mathbf{u}) = 0, \tag{1}$$

where $\mathbf{F}$ is a vector function dependent on t, $\mathbf{u}$, and $\dot{\mathbf{u}}$, t is the time variable, $\mathbf{u}$ is a vector function dependent on t, and $\dot{\mathbf{u}}$ is the derivative of $\mathbf{u}$ with respect to t. Such a system describes the evolution of a quantity $\mathbf{u}$ in time. For example, if the heat transfers from the body of a certain object to the surrounding environment at the constant rate 2 per second and the ambient temperature is 25 Kelvin degrees, the ODE

$$\dot{T} = -2(T - 25), \tag{2}$$

which is an instance of The Newton Law of Cooling, describes the evolution of the temperature (denoted by T) of the object surface in time. Moreover, by means of Eq. (2) and assume also that the initial value of T is 40 Kelvin degrees, we can easily see that $T(t) = 25 + 15e^{-2(t-t_0)}$ for $t \geq t_0 \geq 0$ (the initial time). This explicit formula of T allows us to predict the temperature of the object surface at any time $t \geq t_0 \geq 0$. An interesting thing is that $T(t) \to 25$ as $t \to +\infty$, i.e., the temperature of the object surface has a tendency to convert to 25 Kelvin degrees for large time. Unfortunately, not that many ODEs can be solved explicitly. However, we are going to study a system of ODEs that describes the love affair between two people. Moreover, we always consider $t_0 = 0$ for simplicity.

2 Love affairs and differential equations

This section is to study a system of ODEs that can describe the love affair between two people. For the sake of simplicity, let's consider the love of Romeo for Juliet and also the love of Juliet for Romeo. Romeo and Juliet are known as two famous characters in a tragedy written by William Shakespeare by the same name. Assume that the love of Romeo for Juliet can be measured as a differentiable function $R : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ and similarly differentiable $J : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ the love of Juliet for Romeo. The love affair between them can be described by the Initial-Value Problem (IVP) or also called the Dynamical System (DS)

$$\begin{cases} \dot{R} = aR + bJ, \\ \dot{J} = cR + dJ, \\ R(0) = R_0,\ J(0) = J_0. \end{cases} \tag{3}$$

Here R_0, $J_0 \in \mathbb{R}$ are the love of Romeo for Juliet and Juliet for Romeo at the initial time. The constant coefficients a, b, c, and $d \in \mathbb{R}$ describe the interaction of the love of one to the other. Due to [Str88], we can determine the romantic style of the love of Romeo for Juliet and the romantic style of the love of Juliet for Romeo by means of those coefficients. The author of [Str88] and his students came up with the following romantic styles of Romeo.

a	b	**Style**
$+$	$+$	Eager Beaver
$+$	$-$	Narcissistic Nerd
$-$	$+$	Cautious Lover
$-$	$-$	Hermit

Tab. 1: *Romantic styles of Romeo*

By definition, an "eager beaver" is encouraged by his/her feelings and by the love of the other for him/her, i.e., $a > 0$ and $b > 0$ while a "cautious lover" is retreated from his/her feelings but is spurred on the love of the other for him/her, I.E., $a < 0$ and $b > 0$. Indeed, it can be explained by looking at SYS. (3). The rate change of the love of Romeo for Juliet is nothing but a weighted sum of the love of Romeo for Juliet and similarly in the case of Juliet. For example, if $a = -b = -1$ then there is a change in the love of Romeo for Juliet which is equal to $J - R$, the difference between the love of her for him and the love of him for her, for all the time. If at a fixed time t the love of Juliet for Romeo is greater than the love of him for her, it seems to us that the love of Romeo for Juliet increases as well at that time because $\dot{R}(t) > 0$. It means that Romeo has a tendency to love Juliet if she loves him more than he loves her. On the contrary, if he loves her more than she loves him, that love will decrease at that time because $\dot{R}(t) < 0$. In this case, Romeo is actually a very cautious person and therefore a "cautious lover" as described in [Str88].

Let's consider more concrete examples of the love between different types of lovers to understand more about the dynamics of love. The first example is the love between a cautious lover and a narcissistic nerd, which is given as the IVP

$$\begin{cases} \dot{R} = -3R + 3J, \\ \dot{J} = -2R + J, \\ R_0 = -4,\ J_0 = 2. \end{cases} \tag{4}$$

In this example, Romeo doesn't love Juliet and on the contrary, Juliet loves Romeo at the time we consider, I.E., $R_0 < 0$ and $J_0 > 0$. A narcissistic Juliet is a Juliet that is encouraged by her feelings and falls back from the love of Romeo for her. However, if he hates her more, she will love him more. There are at least two ways to investigate the love between Romeo and Juliet. Because of the linear form of SYS. (4), the explicit solution to it can be found. To obtain the solution, we need to transform SYS. (4) into the vector form

$$\begin{cases} \dot{\mathbf{u}} = A\mathbf{u}, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{5}$$

where $A = \begin{pmatrix} -3 & 3 \\ -2 & 1 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} R & J \end{pmatrix}^T$, and $\mathbf{u}_0 = \begin{pmatrix} -4 & 2 \end{pmatrix}^T$. In this case, T denotes the transpose operator. The matrix A has two complex "eigenvalues" $\lambda_1 = -1 + \sqrt{2}i$ and $\lambda_2 = -1 - \sqrt{2}i$. Hence the solution to the SYS. (4) is

$$\begin{cases} R(t) = e^{-t}\left[7\sqrt{2}\sin\left(\sqrt{2}t\right) - 4\cos\left(\sqrt{2}t\right)\right], \\ J(t) = e^{-t}\left[2\cos\left(\sqrt{2}t\right) + 6\sqrt{2}\sin\left(\sqrt{2}t\right)\right]. \end{cases} \tag{6}$$

We don't want to explain why we obtain the solution SYS. (6) because it will be an exercise for the readers to study general explicit solutions to the IVP SYS. (5) for every initial value $\mathbf{u}_0$ and coefficients a, b, c, and d. In this chapter we focus only on the behavior of the solutions in some specific cases for large time to see how beautiful Mathematics is in the explanation of everything even the love between two people of different romantic styles or of the same romantic style. By looking at the solution SYS. (6), because $\sin$ and $\cos$ are bounded functions, both $R(t)$ and $J(t)$ quickly converge to zeros for t large enough as showed in FIG. 1. The convergence rate is explicitly e^{-t}. Moreover, after a few simple calculation steps, from SYS. (6), we can see that both Romeo and Juliet have the same amount of love for the other, I.E., $R(t) = J(t)$ for $t = \dfrac{(\pi - 2\alpha)\sqrt{2}}{4} + k\pi$ for $k \in \mathbb{N}$, where $\alpha = \arctan\left(\dfrac{\sqrt{2}}{6}\right)$. Between those points, both Romeo and Juliet increase or decrease their love for the other with a small delay in time as we can see in FIG. 4. This is true in reality, for example, we observe the time interval $\left[0, \dfrac{(\pi - 2\alpha)\sqrt{2}}{4}\right]$, because Juliet in this case is a narcissistic nerd, the more Romeo hates her because $R_0 < 0$, the more she loves him. That is why starting from $t = 0$, Juliet increases her love for Romeo. On the other hand, Romeo is cautious, so because the love of Juliet for him increases, his love for Juliet therefore increases. However, when the love of Romeo for Juliet increases, the narcissistic Juliet at a certain point, she is going to be bored with his love for her and her love for him starts decreasing after then until both of their love for the other are equal to each other. Unfortunately, due to the damping term e^{-t}, their love for the other also decreases as time goes.

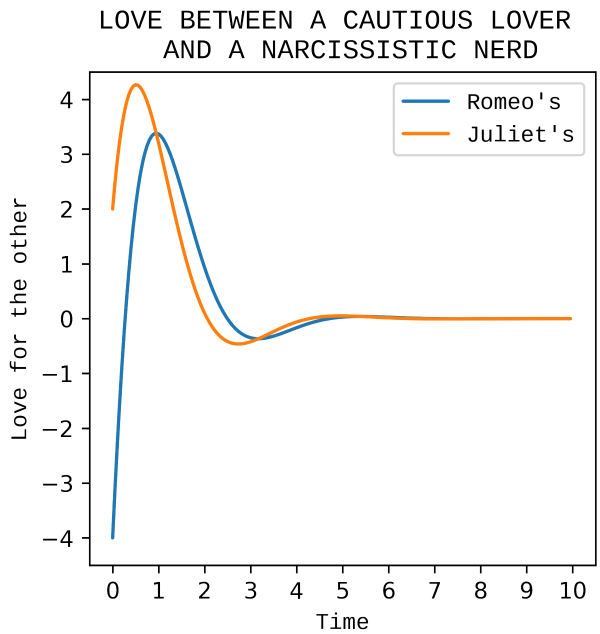

Fig. 1: *The love between a cautious lover and a narcissistic nerd*

Furthermore, we are also interested in the large-time behavior of solutions to IVP Sys. (5) without explicit formulae if they exist. That leads to the "qualitative study" of DSs. It can be done by observing the Vector Field (VF) caused by Sys. (5). The VF of Sys. (4) is sketched in Fig. 2. We can plot the VF by making a two-dimensional grid of R and J then at each point of the grid, we draw the vector $\left(\dot{R}, \dot{J}\right)$. The VFs help us understand the orbit of an object appearing in them as well as the large-time behavior of the orbit. On the other hand, there are "steady points" in the fields, which are defined by $\mathbf{u}$ so that $\dot{\mathbf{u}} = \mathbf{0}$, which is equivalent to $A\mathbf{u} = \mathbf{0}$. They are exactly the "eigenvectors" associated with the eigenvalue 0 of the matrix A and the vector $\mathbf{0}$. The matrix A in Sys. (5) doesn't have any eigenvector associated with the eigenvalue 0, in this case, the Sys. (4) has only one steady state which is $(0, 0)$. The steady states are interesting due to the fact that they do not change . A steady state can be either an "attractor" or a "repeller". If a steady state is a "local attractor", it attracts objects near it while a "global attractor" attracts every object appearing in the VF. Those things somehow describe the possibility of the existence of a solution to the IVP Sys. (5) for small initial value $\mathbf{u}_0$ or arbitrary initial value $\mathbf{u}_0$ and its large-time behavior. On the contrary, a "repeller" repels every object moving forward to it. In the case of the IVP Sys. (4), because $(0, 0)$ is actually a "global attractor", every trajectory in the VF is a "spiral" winding around and getting closer to the point $(0, 0)$ as time goes. The spiral pattern is indeed caused by "harmonic oscillations" of natural frequency $\sqrt{2}$ with "amplitudes" proportional to e^{-t} around $(0, 0)$ as we can see in the explicit formula Sys. (6) of the solution. The two terms are in fact corresponding to the "real parts" and the "imaginary parts" of the eigenvalues of the matrix A. Romeo and Juliet in this example no matter how wonderful or terrible their current love is, we just know that they neither love nor hate each other in the future.

To see a more general picture about the love dynamics, let's consider the love between an eager beaver and a hermit as follows.

$$\begin{cases} \dot{R} = 2R + 4J, \\ \dot{J} = -2R - 2J, \\ R_0 = \dfrac{5}{4},\ J_0 = \dfrac{5}{4}. \end{cases} \tag{7}$$

Because Juliet in this case is a hermit, she is always contrary to her feelings and the love of Romeo

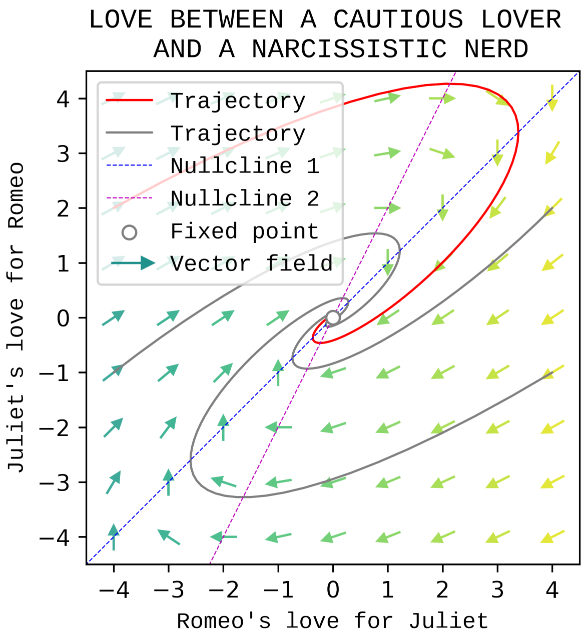

Fig. 2: *The phase portrait of the love between a cautious lover and a narcissistic nerd*

for her. The matrix A is $\begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$ and it has two complex eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. We thus obtain the exact solution

$$\begin{cases} R(t) = \frac{5}{4}\left[\cos(2t) + 3\sin(2t)\right], \\ J(t) = \frac{5}{4}\left[\cos(2t) - 2\sin(2t)\right] \end{cases} \tag{8}$$

for $t \geq 0$ to the IVP Sys. (7). Fig. 3 and Fig. 4 are respectively the plots of the exact solution in time and of the "phase portrait" of it. From the plots, we can see that the love is just harmonic oscillations around the origin. No damping effect occurs in this case. This is due to the fact that the real parts of the eigenvalues of the matrix A are actually zero. The love of those people is not steady, it always changes even sometimes they hate each other but after they increase their love. We thus call the origin a "center node". The loves are the same for all $t = \frac{k\pi}{2}$ for $k \in \mathbb{N}$.

We are going to consider a "strange" love between narcissistic nerds and the ending of the love is indeed dependent on how they start it. Let's begin with the IVP

$$\begin{cases} \dot{R} = R - 2J, \\ \dot{J} = -2R + J, \\ R_0,\ J_0. \end{cases} \tag{9}$$

If at the beginning the love of Romeo for Juliet is the same as her love for him, e.g., $R_0 = J_0 = 4$. The exact solution to the IVP Sys. (9) becomes

$$\begin{cases} R(t) = 4e^{-t}, \\ J(t) = 4e^{-t}. \end{cases} \tag{10}$$

It implies that $R(t) \to 0$ and $J(t) \to 0$ as $t \to +\infty$ (see Fig. 5). In this case, the origin is an attractor. In contradiction to the early example, the origin is not a global attractor. It is either an attractor or a repeller. Indeed, we consider the initial condition $R_0 = -J_0 = -\frac{1}{2}$ close to the origin to see what

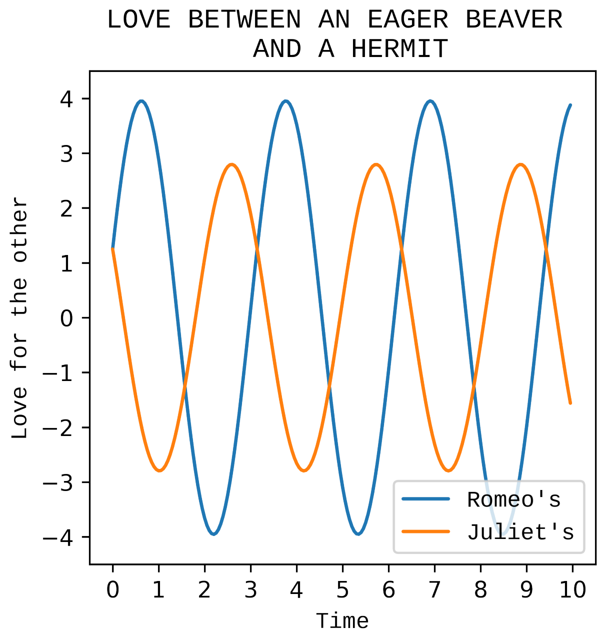

Fig. 3: *The love between an eager beaver and a hermit*

happens. The solution is

$$\begin{cases} R(t) = -\dfrac{e^{3t}}{2}, \\ J(t) = \dfrac{e^{3t}}{2}. \end{cases} \tag{11}$$

We can see that $R(t) \to -\infty$ and $J(t) \to +\infty$ as $t \to +\infty$ (see). This would be the same for every initial condition satisfying $R_0 = -J_0$ and in this case, the origin behaves as exactly a repeller. Due to the attractor-repeller property, trajectories in the VF of the Sys. (9) are branches of "hyperbolas" except for the ones lying on the two "asymptotes" $R = J$ and $R = -J$. We can verify that by looking at Fig. 8 and by observing the solution corresponding to the initial condition $R_0 = \dfrac{7}{3}$ and $J_0 = 3$. Moreover, the IVP Sys. (9) then has the solution

$$\begin{cases} R(t) = -\dfrac{e^{3t}}{3} + \dfrac{8e^{-t}}{3}, \\ J(t) = \dfrac{e^{3t}}{3} + \dfrac{8e^{-t}}{3}. \end{cases} \tag{12}$$

As $t \to +\infty$, $R(t) \to \infty$ and $J(t) \to -\infty$. The love of Romeo for Juliet just slightly decreases at the beginning but increases quickly after that while the love of Juliet for him never increases but decreases "exponentially" in time (poor Romeo!). In this case, we call the origin a "saddle node".

Let's see what conclusions we can make from the previous examples. Let λ_1 and λ_2 be the two eigenvalues of A. We have the following table.

Re λ_1	Re λ_2	$\lvert\text{Im } \lambda_1\rvert$	$\lvert\text{Im } \lambda_2\rvert$	**Type**
$-$	$-$	$+$	$+$	Spiral-In
0	0	$+$	$+$	Center
$+$	$-$	0	0	Saddle

Tab. 2: *(Incomplete) Phase-portrait classification*

For a good understanding of the subject, read [Lue79, HS74, Arn92].

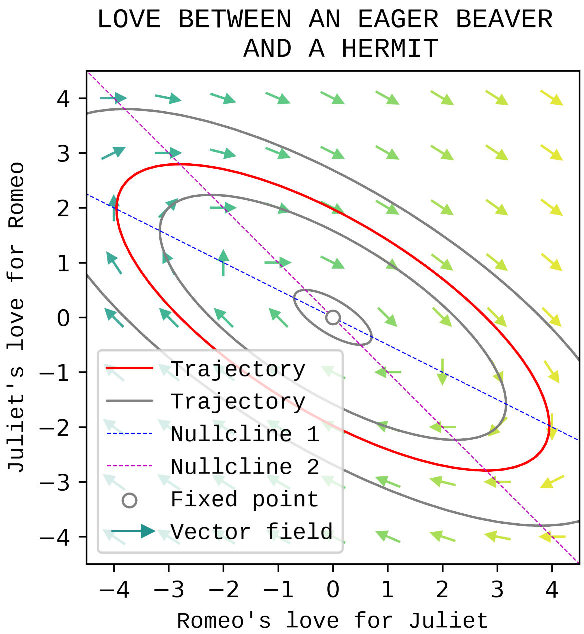

Fig. 4: *The phase portrait of the love between an eager beaver and a hermit*

3 Exercises

Use the answer form `https://tinyurl.com/2p85f7az` to do the following exercises.

Exercise 1 (2.0 points)**.** Answer these questions.

1) Find the formula of the exact solution to the IVP Sys. (5) for general 2×2 matrix $\mathbf{A}$ and initial condition $\mathbf{u}_0$ in terms of the eigenvalues λ_1 and λ_2 of $\mathbf{A}$ with explicit coefficients. (*1.0 point*)

2) Complete Tab. 2 for all possible cases of the signs of the eigenvalues of $\mathbf{A}$ ($+$ means nonnegative and $-$ means negative). (*1.0 point*)

Exercise 2 (4.0 points)**.** Answer these questions.

1) Give five examples of the IVP Sys. (3). Apply the formula in Ex. 1 to find the exact solutions to them. (*1.0 point*)

2) Assume that the love between Romeo and Juliet is perturbed by outer conditions, e.g., their families and social prejudices. In this case, the love is modeled by the IVP

$$\begin{cases} \dot{R} = aR + bJ + f(t), \\ \dot{J} = cR + dJ + g(t), \\ R(0) = R_0,\ J(0) = J_0. \end{cases} \tag{13}$$

Here f and g are two real functions dependent on t, e.g., $f(t) = t - 1$ and $g(t) = t^2$. Find the formula of the exact solution to the IVP Sys. (13), provided that f and g satisfy appropriate conditions so that the solution definitely exists. (*1.5 points*)

3) Find two examples of the IVP Sys. (13) and use the formula in 2) to find their exact solutions. *Hint: choose f and g so that the integrals related to f and g in the formula of the exact solution can be computed easily.* (*1.5 points*)

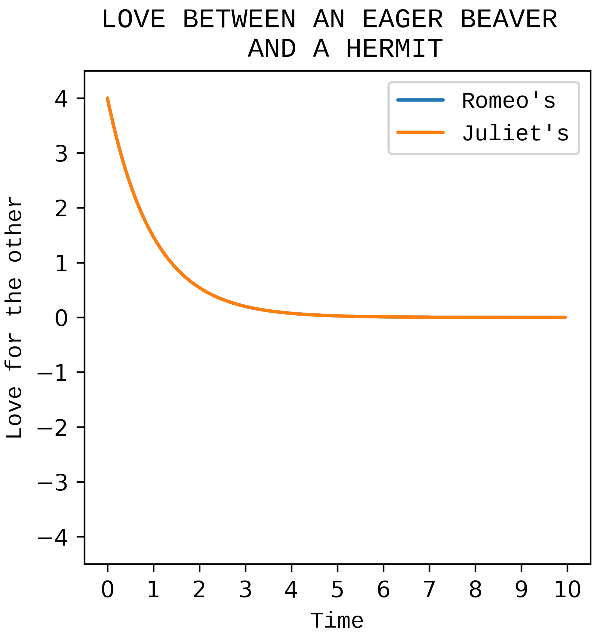

Fig. 5: *The love between two narcissistic nerds corresponding to* $R_0 = J_0 = 4$

Exercise 3 (4.0 points)**.** A more general and also complicated love between Romeo and Juliet is the IVP

$$\begin{cases} \dot{R} = f(t, R, J), \\ \dot{J} = g(t, R, J), \\ R(0) = R_0,\ J(0) = J_0, \end{cases} \tag{14}$$

where f and g are two real functions dependent on t, R, and J. Unfortunately, a solution to the IVP Sys. (14) does not always exist. Could you find some certain conditions on f and g so that a unique solution to the problem exists (maybe locally near the initial condition)? For example, a local solution exists for the IVP Sys. (14) where $f(R, J) = R(1 - J)$ and $g(R, J) = J(R - 1)$ (this is also known as the Lotka–Volterra equations in Biology to model the interaction between two species, often between preys and predators).

This exercise aims to solve the IVPs Sys. (13) and Sys. (14) "numerically", provided that the existence of a unique solution is guaranteed. Answer the following questions. Note that in this exercise, use only the libraries "numpy" and "matplotlib". Moreover, do not use functions provided in numpy that can solve this exercise directly.

1) The simplest numerical scheme is the explicit Euler method.

```
def ExplicitEuler(f, g, t0, R0, J0, h):
        R1 = R0 + f(t0, R0, J0) * h
        J1 = J0 + g(t0, R0, J0) * h
        return R1, J1
```

This function receives the values R_0 and J_0 of $R(t)$ and $J(t)$ at time t_0 and returns the approximate values R_1 and J_1 at $t_1 = t_0 + h$, where h is the time step. The "local truncation error" at t_1 is thus defined by

$$\mathcal{E}(t_1) := \sqrt{[R(t_1) - R_1]^2 + [J(t_1) - J_1]^2}. \tag{15}$$

It is well-known that $\mathcal{E}(t_1)$ is proportional to h^2. Could you prove it? Write Python codes to perform the explicit Euler method and 5 examples to test the codes (at least 3 of them are nonlinear). (*1.0 point*)

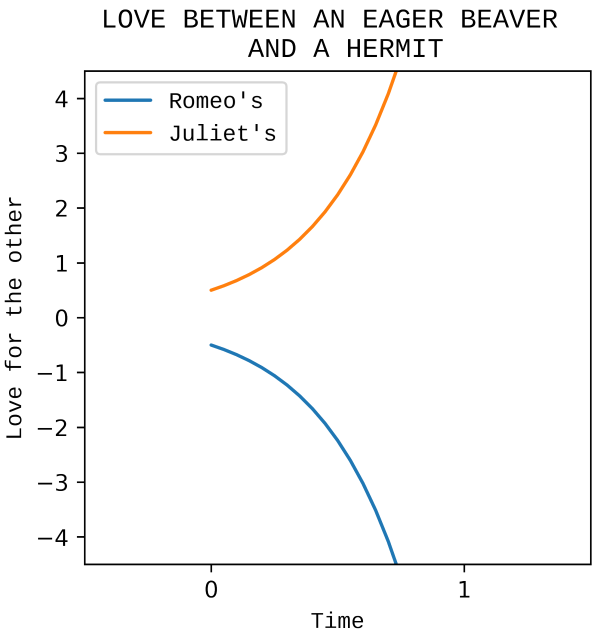

Fig. 6: *The love between two narcissistic nerds corresponding to* $R_0 = -J_0 = -\dfrac{1}{2}$

2) One of the advantages of this numerical scheme is the fast running time. However, this scheme is not "stable" for large time step h. For some problems, it requires $h < 1$. In this case, we must consider the "implicit" Euler method. Study and write Python codes to perform the implicit Euler method and use the 5 examples in 1) to test the codes. The implicit Euler method also has $\mathcal{E}(t_1)$ proportional to h^2. Could you prove it? What are the cons of the implicit Euler method? (*1.0 point*)

3) Try to obtain a plot similar to the plot `https://tinyurl.com/4fpt9xvk` of the example of IVP Sys. (5), where $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{u}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (*2.0 points*)

Exercise 4 (Optional + 2.0 points)**.** Consider 1000 data of Romeo's and Juliet's love for the other in the file `https://tinyurl.com/2cypybcw`. Those data are generated from the exact solution to the IVP Sys. (3) with initial condition $R_0 = -2$ and $J_0 = 3$ and time step $h = 0.001$. Some "noises" are also added to the data. Could you use those data to estimate the coefficients a, b, c, and d? What are they? *Hints: you might want to use Machine Learning (*ML*)/Deep Learning (*DL*) techniques, for example by constructing a simple neural network to learn from the data the coefficients* $a, b, c,$ *and* d *based on the idea in* [NVN+22] (Do you think the idea works in your case?).

4 Instructions and requirements

Students have to follow the instructions and comply with the requirements below. *Lecturers do not solve the cases arising due to the fact that students do not follow the instructions or do not comply with the requirements.*

4.1 Instructions

Students must work closely with the other members in their own team. Register and check your team at `https://tinyurl.com/y6w88rtr` (for CC0x students) and `https://tinyurl.com/yvpa86uz` (for L0x students).

All of the aspects related to this assignment will be quizzed (about 10 - 12 of about 25 multiple-choice questions) in the final exam of the subject. Therefore, team members must work together so

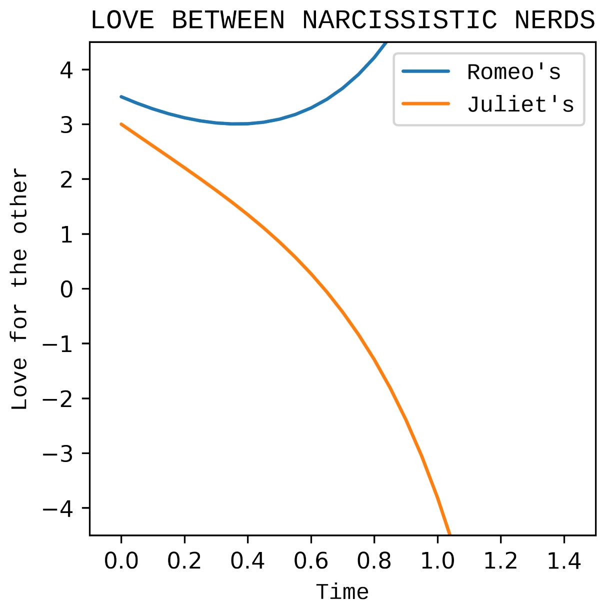

Fig. 7: *The love between two narcissistic nerds corresponding to* $R_0 = \frac{7}{3}$ *and* $J_0 = 3$

that all of you understand all of the aspects of the assignment. The team leader should organize the work team so that this requirement will be met.

During the work, if you have any question about the assignment, **please post that question on the forum** https://tinyurl.com/2dka5444 **(for CC0x and DTQ0x students) and** https://tinyurl.com/3zn4rtfm **(for L0x, DT0x, and CN0x students)**.

Regarding the background knowledge related to the topic, students are supposed to refer to all of the references. However, you have to put all of them in the reference section of the answer form.

4.2 Requirements

- Deadline for submission: **May 05, 2023**.
- Use the answer form https://tinyurl.com/2p85f7az to answer all of the exercises in Section 3.
- Programming language: Python.

4.3 Submission

- Students must submit the answer form of their own team after completing it via BK-eLearning system as a single IPython notebook named "*Assignment-CO2011-CSE222-Team_ name.ipynb*" on https://tinyurl.com/4t7btn7j (for CC0x and DTQ0x students) and on https://tinyurl.com/3djfptpr (for L0x, DT0x, and CN0x students).
- Noting that **only the leader of each team will submit the answer form of the team.**

5 Evaluation and cheating treatment

5.1 Evaluation

Each assignment will be evaluated as follows.

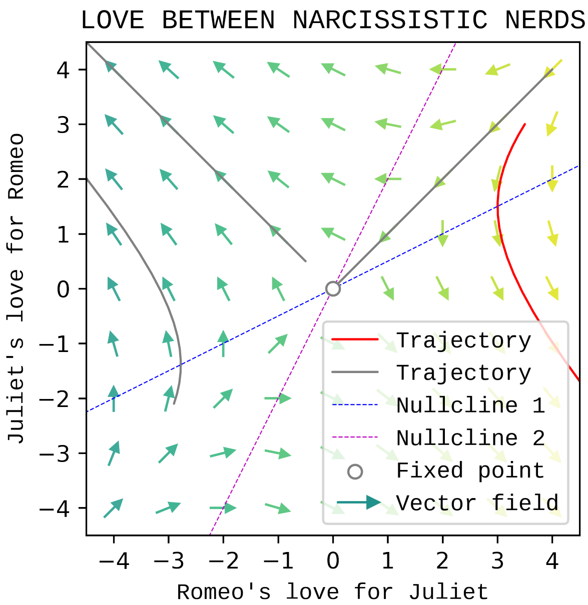

Fig. 8: *The phase portrait of the love between two narcissistic nerds*

Content	Score (%)
- The answers to non-coding questions focus on the goals of the questions, correct, and meet the requirements	40%
- For coding questions, the codes are executable without any error, meet the requirements, and give correct outputs	40%
- The answer form is clear and readable once completed	20%

Tab. 3: *Evaluation*

5.2 Cheating treatment

The assignment of a team will be considered as cheating if

- it is similar to the ones of other teams.
- the students of the team do not understand it. (You can consult from any source, but make sure that you understand everything that you will have written).

If an assignment is found as cheating, the students that did it will be judged according to the university's regulations.

References

[Arn92] Vladimir I Arnold. *Ordinary Differential Equations*. Springer Science & Business Media, 1992.

[HS74] Morris W Hirsch and Stephen Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, 1974.

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[NVN+22] Duc Q Nguyen, Nghia Q Vo, Thinh T Nguyen, Khuong Nguyen-An, Quang H Nguyen, Dang N Tran, and Tho T Quan. Becaked: An explainable artificial intelligence model for covid-19 forecasting. *Scientific Reports*, 12(1):1–26, 2022.

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